```
init()
```

Fidget Spinner Lab

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Understanding Our System

The object of this lab is to gain an understanding of the behaviour of a fidget spinner over time and represent this behaviour through an ordinary differential equation. In this report, we assume that the fidget spinner does not have any vertical movement and that the material properties of the fidget spinner are consistent and perfect.

Figure 1. The fidget spinner, made of injection-molded ABS plastic, aluminum spokes and steel barrings.

In order to record the angular velocity of the fidget spinner, we recorded an overhead video using the setup seen in *Figure 2*. The video was recorded from after the fidget spinner begun spinning until it came to a complete and final stop.

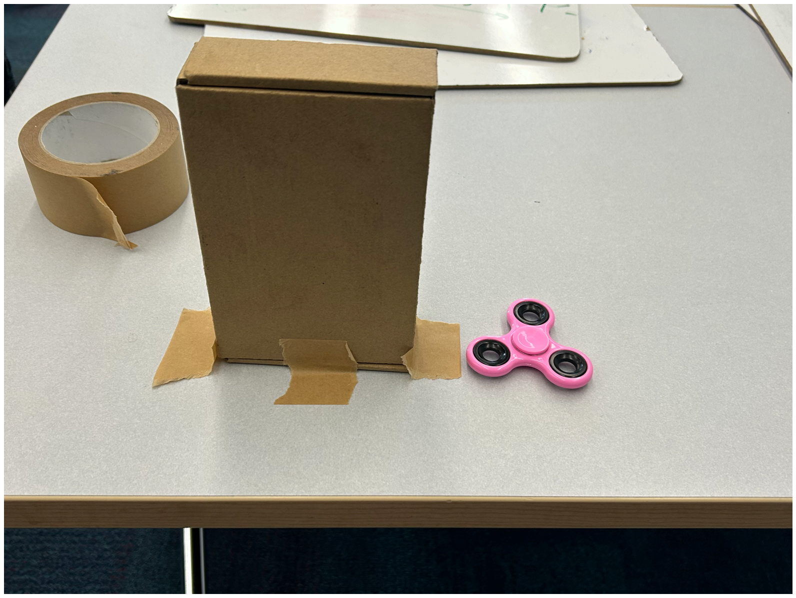

Figure 2. The setup has a taped box where the recording device sits on while the fidget spinner is spinning.

The video for our setup is included here.

Figure 3. Video of the fidget spinner spinning until rest.

When collecting data on the acceleration of the fidget spinner we aimed to remove as much external variability as possible. In order to prevent the fidget spinner from shifting on the table we applied a small amount of tape to secure it to the table. Additionally, we placed a small cardboard box next to the fidget spinner and taped it down to minimize any potential camera shaking during the data collection. A phone was positioned on top of the cardboard box, with its camera hanging over the edge to capture the motion of the fidget spinner. Lastly, we began recording and spun the fidget spinner. During post-processing, we cropped the video after the fidget spinner was spun until it came to a complete stop.

To determine the angular velocity of the fidget spinner, we detect the change in contrast between a spoke and the background. Each time a contrast change is detected we know that a spoke has passed through the detection area. By dividing this number by three, the number of spokes, we are able to calculate the amount of time it takes the fidget spinner to rotate one full time, in other words, find the angular velocity. As a result we can graph the angular velocity of the fidget spinner over time. As seen in *Figure 4*, the angular velocity appears to decrease at an decreasing rate as the time progresses until the fidget spinner comes to a complete stop.

```
initial_cond = frequency_plot("Angular Velocity vs. Time of Fidget Spinner", 1);
```

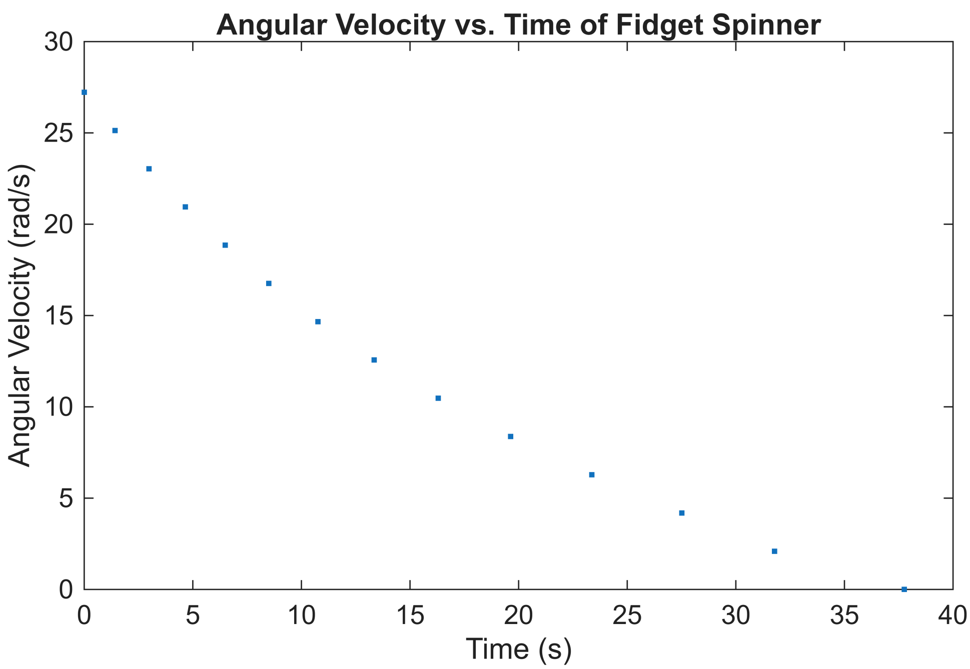

Figure 4. A graph of the what the angular velocity is at that time in the video. The angular velocity decreases over time, as represented in the video.

The angular acceleration of the fidget spinner can be represented as a quadratic equation (*Eq. 1*).

$$\dot{\omega}(t) = a\omega(t)^2 + b\omega(t) + c$$

Equation 1.

In *Equation 1*, (a) is the quadratic drag constant, (b) is the viscous damping constant, and (c) is the coulomb friction constant. The unit for angular velocity is radians per second $(\frac{rad}{s})$ and the unit for angular acceleration is radians per second squared $(\frac{rad}{s^2})$. As a result, we are able to evaluate the units of the constant in *Equation 1*.

Quantity Description	Symbol	Value	(units)
Estimated quadratic drag constant	a	$-.0007\,\frac{1}{rad}$	$\frac{1}{rad}$
Estimated viscous damping constant	b	$-.0253\,\frac{1}{s}$	$\frac{1}{s}$
Estimated Coulomb friction constant	c	$-.3507\,\frac{rad}{s^2}$	$\frac{rad}{s^2}$
Video frame rate	F_s	$60\,\frac{frames}{s}$	$\frac{frames}{s}$
Estimated initial angular velocity	$\omega(t=0)$	$27.2271\,\frac{rad}{s}$	$\frac{rad}{s}$

By taking the derivative of the angular velocity with respect to time, we can calculate the angular acceleration of the fidget spinner. *Equation 1*, which represents the angular acceleration of the fidget spinner model, is a quadratic equation. Therefore, as seen in *Figure 5*, we are able to fit the raw data graph of the angular acceleration and the angular velocity to a quadratic equation, allowing us to calculate the quadratic drag, viscous damping, and Coulomb friction constants.

```
gov_eq_comparison(1);
```

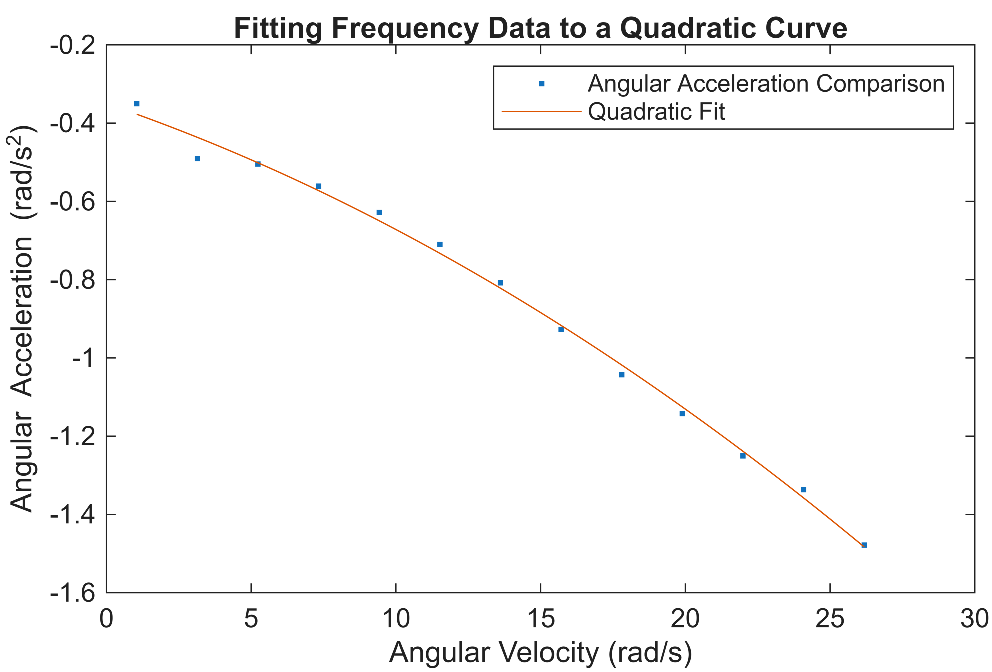

Figure 5. Comparing the angular velocity to it's derivative shows a quadratic relationship, as represented in this graph.

As seen in *Figure 5*, the fitted curve represents the data accurately. The decrease in acceleration follows a quadratic curve because the fidget spinner gradually loses angular momentum over time due to frictional forces.

This loss in momentum causes the acceleration to decrease at a nonlinear rate. Since we fit the curve using the quadratic equation (*Equation 1*), the coefficients account for the loss of momentum via the drag constant, damping constant, and friction constant, allowing the model to capture the overall decrease in acceleration over time. Based on our observations, our model is first order because, as seen in *Equation 1*, the derivative of $\omega(t)$, the dependent variable, is only taken to the first degree. Additionally, the model is nonlinear because the quadratic equation contains a nonlinear term with respect to the dependent variable ($a\omega(t)^2$). Lastly, the equation is forced because *Equation 1* contains the constant c, which does not involve the dependent variable. Since the equation is a first order derivative, we can simulate the solution to the differential equation numerically in MATLAB using `ode45`:

```
measured_vs_analytical();
```

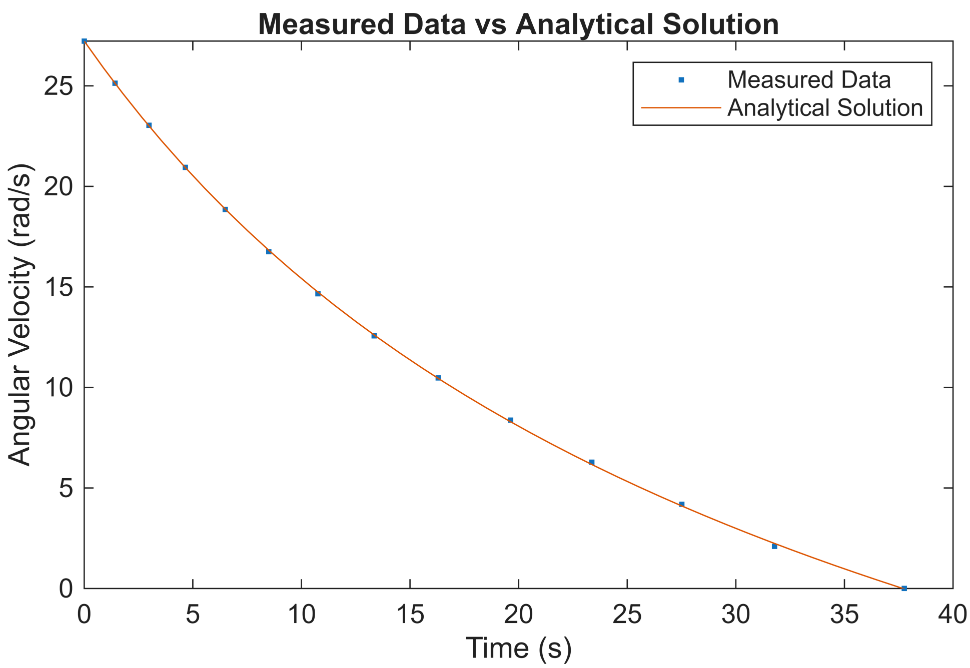

Figure 6: The numerical solution from `ode45` *matches well with the data that we measured for the angular velocity of the fidget spinner, with slight error.*

The model seems to accurately represent the original data; however, the solution assumes that the angular velocity can become negative but this is not representative of real life data because the fidget spinner will not rotate in the opposite direction. Using a vector field, we can demonstrate that the analytical numerical solution represents the general solution of the ODE. This can be verified by varying the initial conditions and comparing the resulting trajectories with the vector field of the ODE.

```
quiver_plot([20, 30, 40]);
```

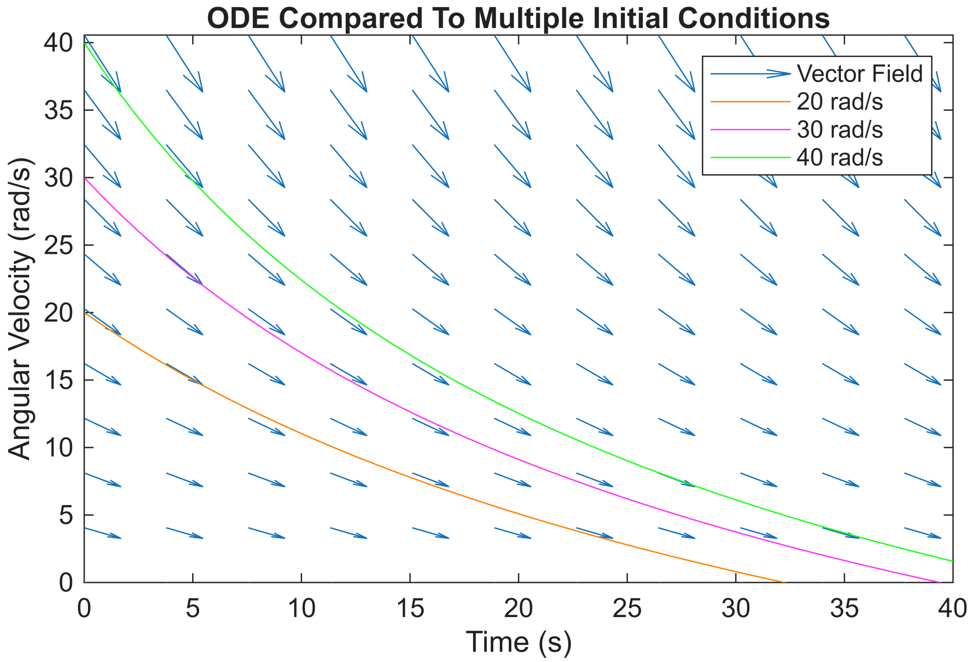

Figure 7. The solutions with ode45 all follow the same path in the vector field.

The angular velocity of the fidget spinner and the measured frequency of the video are related because the frame rate is only able to capture the changes in the position of the spokes when the frame rate is higher than the frequency of the fidget spinner. In this case, the camera would only be able to capture a blur or still appearing video of the fidget spinner due to the stroboscopic effect, prohibiting the detection of the wing rotation. However, since we are operating at a high enough frame rate when collecting data, the software is able to detect every time a spoke passes through a set point. Therefore, the frequency of the spokes can be converted to angular velocity.

$$\omega(t) = \frac{2\pi}{3} f$$

Equation 2. Where f is the measured angular frequency of the fidget spinner.

Equation 2 represents this relationship by converting the frequency into angular acceleration by multiplying by 2π and dividing by the number of spokes. The video to signal code detects the change between the spoke and the background. Since there are three spokes, the code will record every time a spoke passes through the frame. In order to find the measured frequency, which is the frequency of one full rotation, we must divide by the number of spokes (3).

The stroboscopic is when the frame rate of the camera aligns with the angular velocity of the spinning object. This means that the camera is essentially capturing a picture of the moving object in the same position. When these images are strung together to create a video, it will appear as if the object isn't moving or is barely moving (depending on the frequency and rate alignment) since all of the frames of the video will be of the object in the same position.

We did not capture this effect when taking data; however, by lowering the frame rate, spinning the fidget spinner at a high velocity, and waiting for it to slow to the frame rate we were able to capture this effect, as demonstrated in the linked video.

Figure 8. The fidget spinner seems like it's stopping as it is rotating.

In linear first order ordinary systems we assume the solution is an exponential function. In exponential decay solutions, as time increases indefinitely, the model will approach but never actually reach the final value of 0. However, in our case, the fidget spinner must eventually come to a complete stop and reach an angular velocity of 0. As a result, we represent this system using a nonlinear quadratic form. In addition, in a linear differential equation the angular acceleration and the angular velocity would have a linear relationship. In contrast, as seen in *Figure 6*, this relationship is not linear since the angular acceleration decreases at a decreasing rate relative to the angular velocity.

But What If...

To investigate the impact that the density of the fidget spinner material has on the time it spins before coming to a complete stop, we modified the system's ODE to include the ratio of the new material density to the original material density (*Eq. 3*).

$$\frac{\rho_{new}}{\rho_{old}}\dot{\omega}(t) = a\omega(t)^2 + b\omega(t) + c$$

Equation 3. Where $\frac{\rho_{new}}{\rho_{old}}$ *is the ratio of the density between the new and old density.*

By keeping the geometry and drag/friction coefficients the same, we can isolate the effect of density on the moment of inertia. Using the measured values of $\omega(t=0)$, a, b, and c, we applied the `ode45` to simulate the system for density ratios of .5, 1, 2, and 4. The results of this simulation are as follows:

```
density_ratios = [0.5, 1, 2, 4]; % Different density ratios between new and the previous material.
times = test_density_ratios(density_ratios); % Then returns a set of times for each of these density ratios.
```

```
Fidget Spinner Rest Values
Density Ratio: 0.5, Rest at Time = 18.7184 seconds
Density Ratio: 1, Rest at Time = 37.2846 seconds
Density Ratio: 2, Rest at Time = 70.8192 seconds
Density Ratio: 4, Rest at Time = 142.0692 seconds
```

Based on these times, we can conclude that there is some linear relationship for the time to rest and the density ratio, as proven by this *Figure 9*.

```
rest_vs_density_ratios(times, density_ratios);
```

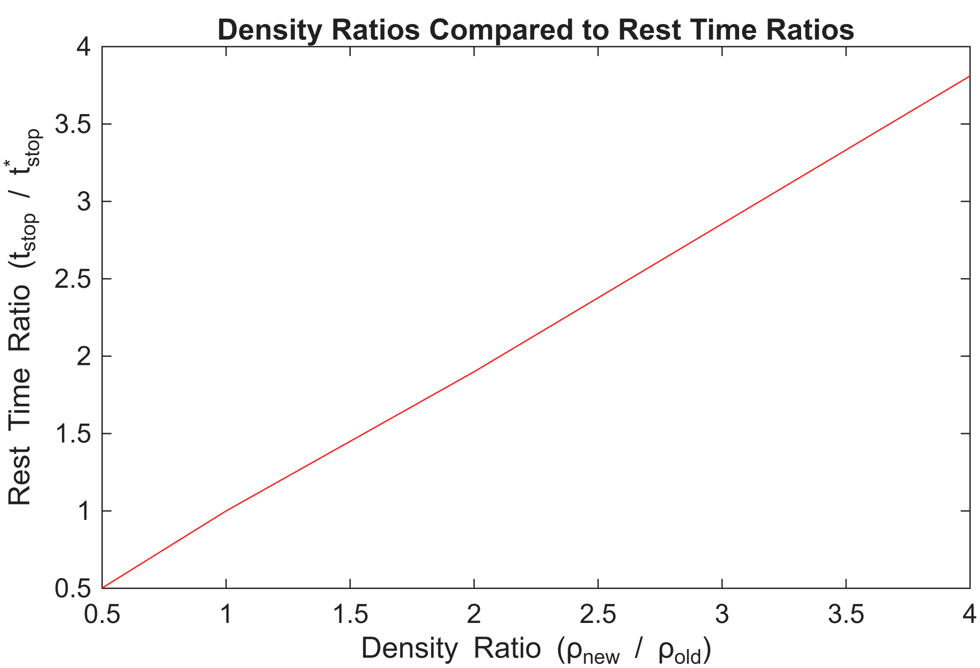

Figure 9. A clear linear relationship between the density ratio and the time to rest ratio.

There is a linear relationship between the ratio of the expected stop time and the actual stop time and the ratio of the new density and the actual density. This means that as the density of the object increases, the amount of time it takes the spinner to come to a complete stop will also increase.

Through the following derivation we can prove that our scaled ODE is equivalent to our original ODE where the angular velocity and angular acceleration are taken to time scaled by the density ratio rather than time.

$$\tilde{t} = \frac{t}{\frac{\rho_{new}}{\rho_{old}}}$$

$$\frac{d\omega(\tilde{t})}{d\tilde{t}} = a\omega\left(\frac{t}{\frac{\rho_{new}}{\rho_{old}}}\right)^2 + b\omega\left(\frac{t}{\frac{\rho_{new}}{\rho_{old}}}\right) + c$$

$$\dot{\omega}(t) = \frac{d\omega(t)}{dt} = \frac{d\omega(\tilde{t})}{d\tilde{t}} \times \frac{d\tilde{t}}{dt} = \frac{d\omega(\tilde{t})}{d\tilde{t}} \times \frac{\rho_{old}}{\rho_{new}}$$

Equation Set 1. This step taken here is proven by using the chain rule for derivatives.

$$\dot{\omega}(t) = \frac{d\omega(\tilde{t})}{d\tilde{t}} \times \frac{\rho_{old}}{\rho_{new}}$$

$$\frac{\rho_{new}}{\rho_{old}}\dot{\omega}(t) = \frac{d\omega(\tilde{t})}{d\tilde{t}}$$

Equation Set 2. This step proves that $\frac{d\omega(\tilde{t})}{d\tilde{t}}$ *is equal to the original equation.*

By converting these, we go from the t world to the $\tilde{t}$.

$$\frac{d\omega(\tilde{t})}{d\tilde{t}} = a\omega(t)^2 + b\omega(t) + c \rightarrow a\omega(\tilde{t})^2 + b\omega(\tilde{t}) + c$$

Equation 4. The scaled ODE.

The rest time ratio and the density ratio are linearly related. Therefore, when we take the derivative to the first degree of the time scaled by the density ratio, it is evident that the density ratio is a multiplicative scalar of the angular velocity ODE. So when you represent the equation in this way where time is scaled by the density ratio, it is clear that density ratios have a clear, direct multiplicative effect on the time to stop and the angular acceleration.

Provided the linear relationship between the spin time ratio and density ratio observed in *Figure 9*, we are able to predict the expected stop times of different materials provided the original density of the fidget spinner and the density of the new material.

```
fidget_spinner_density = 1.2; % abs plastic density, g / cm^3
fidget_spinner_time_to_rest = times(2); % seconds, 37.28
diamond_density = 3.5; % diamond density, g / cm^3
gold_density = 19.32; % gold density, g / cm^3

diamond_ratio = diamond_density / fidget_spinner_density;
gold_ratio = gold_density / fidget_spinner_density;


diamond_time_to_rest = fidget_spinner_time_to_rest * diamond_ratio
```

```
diamond_time_to_rest =
108.7468
```

```
gold_time_to_rest = fidget_spinner_time_to_rest * gold_ratio
```

```
gold_time_to_rest =
600.2823
```

As seen in the previous calculations, an increase in the density of the fidget spinner material will result in a longer spin time. Therefore, increasing density is an effective method of prolonging the spin time.

Appendix

Github Link: https://github.com/foxwithahdie/fidget-spinner

Setup Video: https://www.youtube.com/shorts/hN7-ArmZ5IU

Stroboscopic Effect Video: https://www.youtube.com/watch?v=p6CmFTCWpCA