Oscillator Project

Hailee Gooden and Ramzey Burdette

In order to find the following values, critical to modeling the second order derivative of a mass-damper system we performed the following experimental procedures and analysis.

`estimated_values = 8×2 table`

	Key	Value
1	"Mass of Cart"	"0.731 kg"
2	"Average Stiffness Coefficient"	"51.0856 N * kg"
3	"Damping Coefficient"	"0.19244 (N * s) * m^-1"
4	"Exponential Decay Rate"	"0.13162 s^-1"
5	"Damped Frequency"	"9.8403 rad/s"
6	"Natural Frequency from Mass and Stiffness Coefficient"	"8.3597 rad/s"
7	"Natural Frequency from Data"	"9.8412 rad/s"
8	"Damping Ratio"	"0.013375 "

Table 1. Table of the estimated values.

Characterization of Springs

In order to find the natural length and the stiffness of the springs we attached each spring to the wall. Then, we applied various weights and measured the length that the spring was stretched.

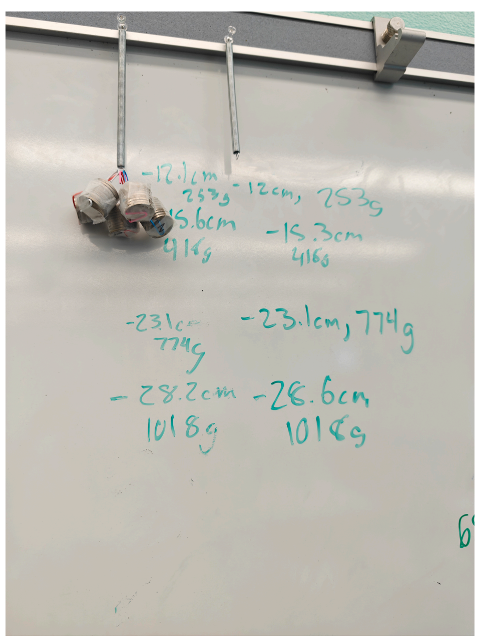

Figure 1. Attaching the weights to the springs, and measuring the distance that the spring stretches.

To calculate the amount of force applied to the spring, the added mass was multiplied by gravitational force. The corresponding forces and lengths were graphed and a linear relationship was fit to the line using linear regression

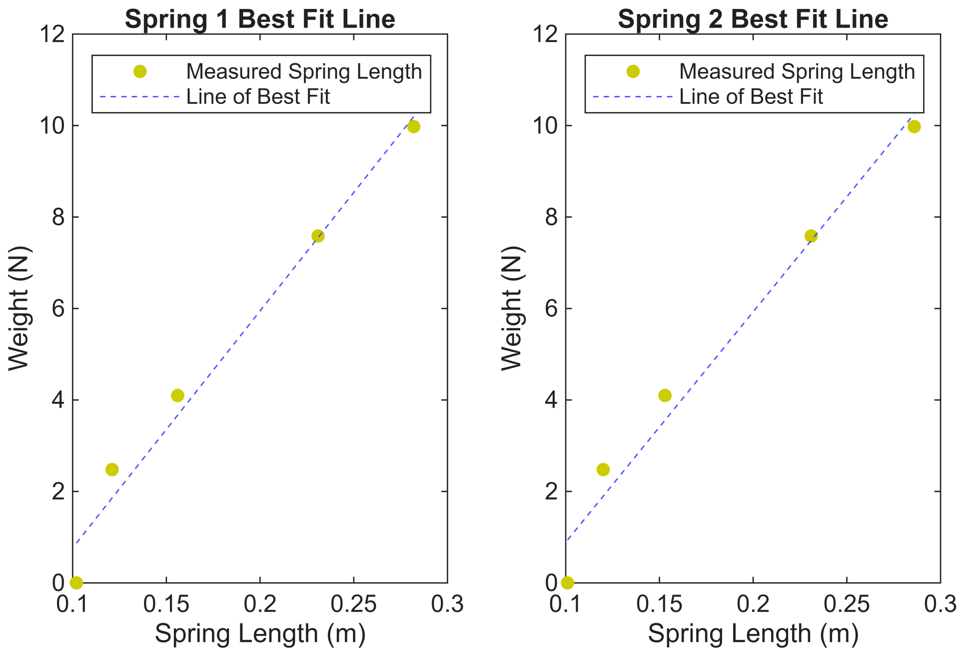

Figure 2. Line of best fit for each of the springs. We measured values, as seen in Figure 1, and created a line of best fit that follows them.

Through Hooke's law, we can derive the estimated natural length and stiffness of the springs with the following equations:

$$k = m$$

$$l_0 = -\frac{b}{m}$$

Where m is the slope of the line of best fit and b is the y-intercept of the line of best fit.

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Stiffness k_1: 51.81
Natural length l0_1: 0.09
Stiffness k_2: 50.36
Natural length l0_2: 0.08
```

Acceleration

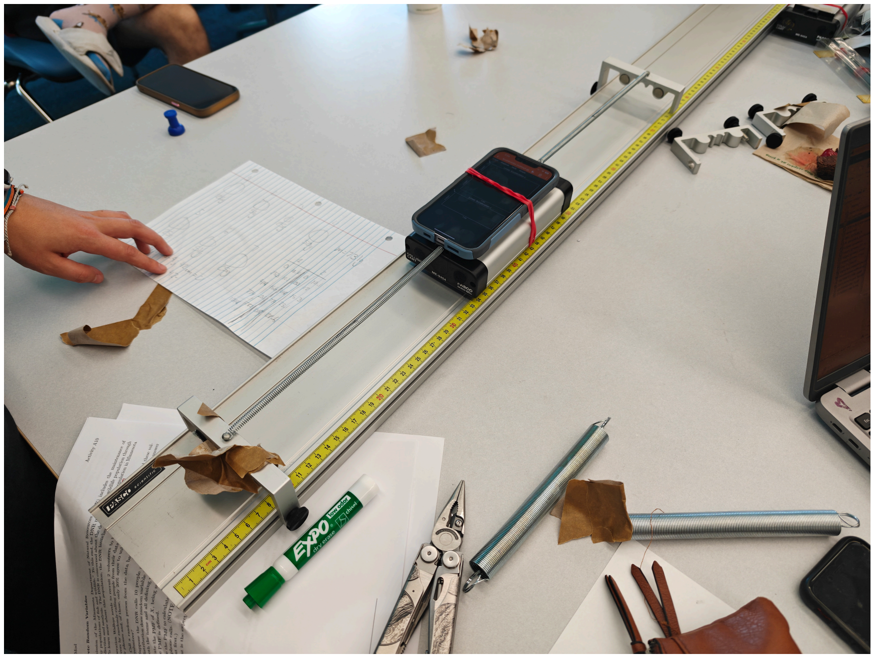

Figure 3. The mass-spring damper experiment setup.

To find the acceleration of a mass-damper system we placed a phone with an acceleration measuring app onto a cart placed on a horizontal track. On either end of the cart one of the previously characterized springs was placed. The cart was displaced and released resulting in an oscillating movement before the cart came to rest. In this scenario, the damper is all methods of loss of force including friction, air resistance, and internal damping within the springs.

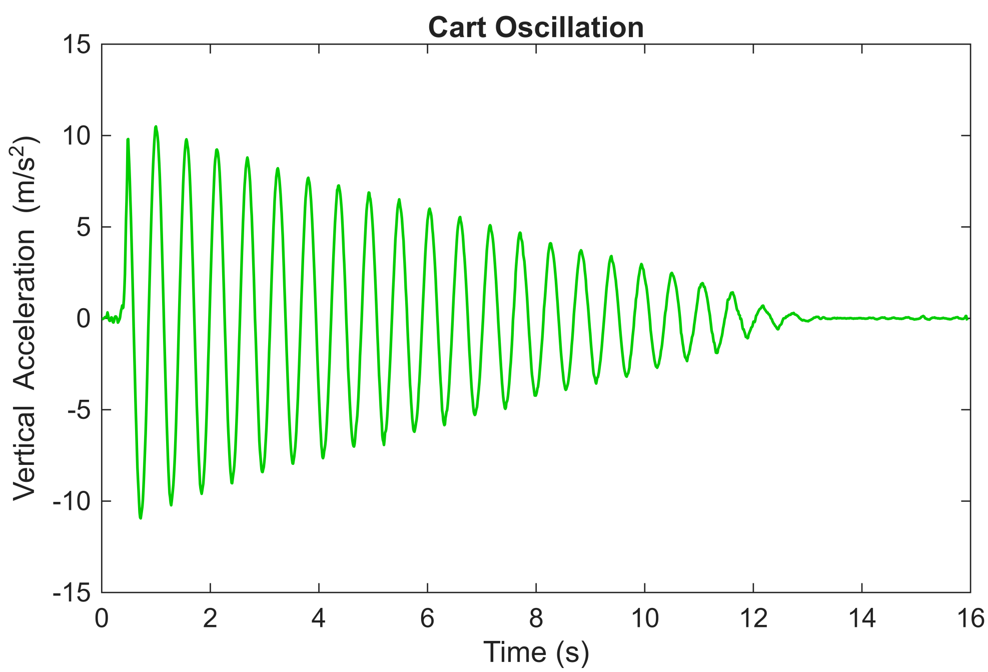

Figure 4. The vertical acceleration (going left and right) over time. The cart has X, Y and Z acceleration, but we only used Y for this project.

As seen in the acceleration graph, the cart oscillates around the point of equilibrium, with a decreasing amplitude, until it comes to rest. This is an example of an underdamped system.

Complex Experimental Coefficients

The equation for the complex experimental coefficients is:

$$\lambda = -\sigma \pm i\omega_d$$

As a result, we must calculate the exponential decay rate and damped frequency from the collected acceleration data. In order to calculate the damped frequency, we found the time period and the acceleration by finding the time difference between the peaks.

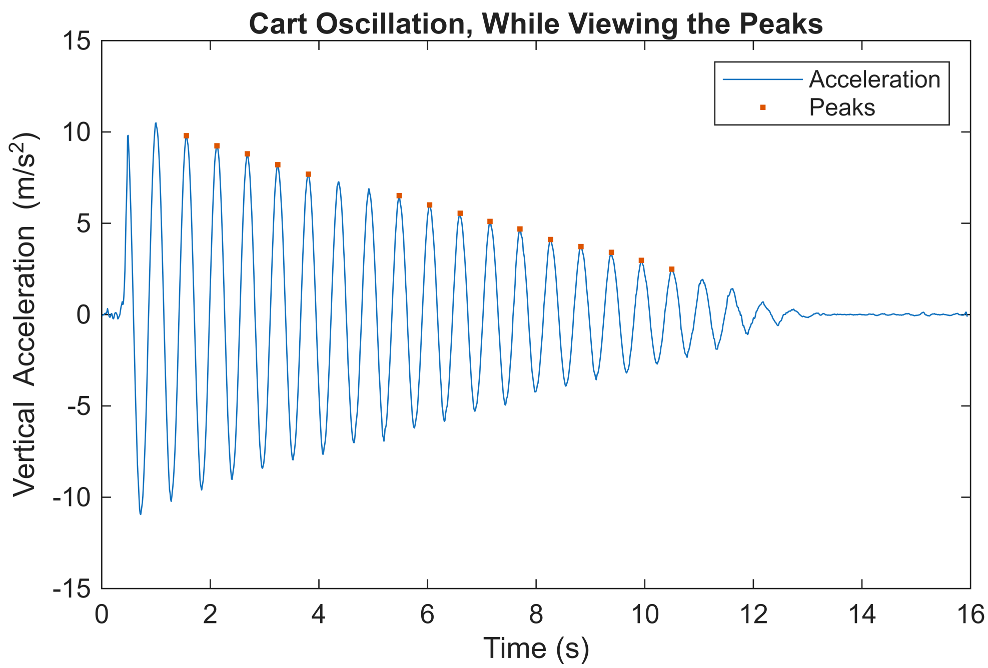

Figure 5. The peaks of the cart oscillation. We filtered most of the peaks out to get an accurate fit, as seen in the next figure.

From the time period we can calculate the damped frequency since $\tau = \frac{\omega_d}{2\pi}$.

```
w_d =
9.8403
```

In order to calculate the exponential decay rate we fit the peaks to an exponential line of the form $x(t) = ae^{bt}$, where σ is equivalent to `-b`.

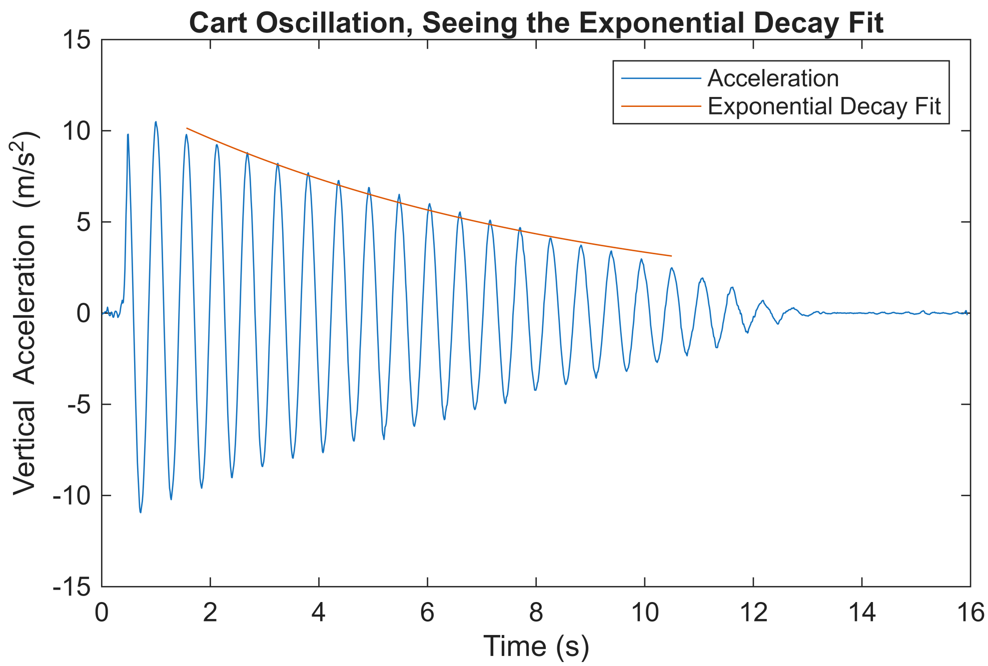

Figure 6. The exponential decay, fit to our data. It has some error, but it should generally match our data.

```
sigma =
0.1316
```

Since the system is underdamped the complex experimental coefficientals also take the form (*equation with zeta)* $\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$ where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ through the following derivation.

Since we have already calculated σ and ω_n, we can find natural frequency and damping ratio through the following equations:

$$\zeta = \frac{\sigma}{\omega_n}$$

$$\omega_n = \sqrt{\omega_d^2 + \sigma^2}$$

```
zeta =
0.0134

w_n =
9.8412
```

Analytical Solution

Using the analytical form of the equation of an underdamped system (x1 $x(t) = e^{-\sigma t}(X_1 cos(\omega_d t) + X_2 sin(\omega_d t)$, where $X_1 = X_0$ and $X_2 = \frac{V_o + \sigma X_0}{W_d}$) we can plug in the values calculated above to find that analytical solution.

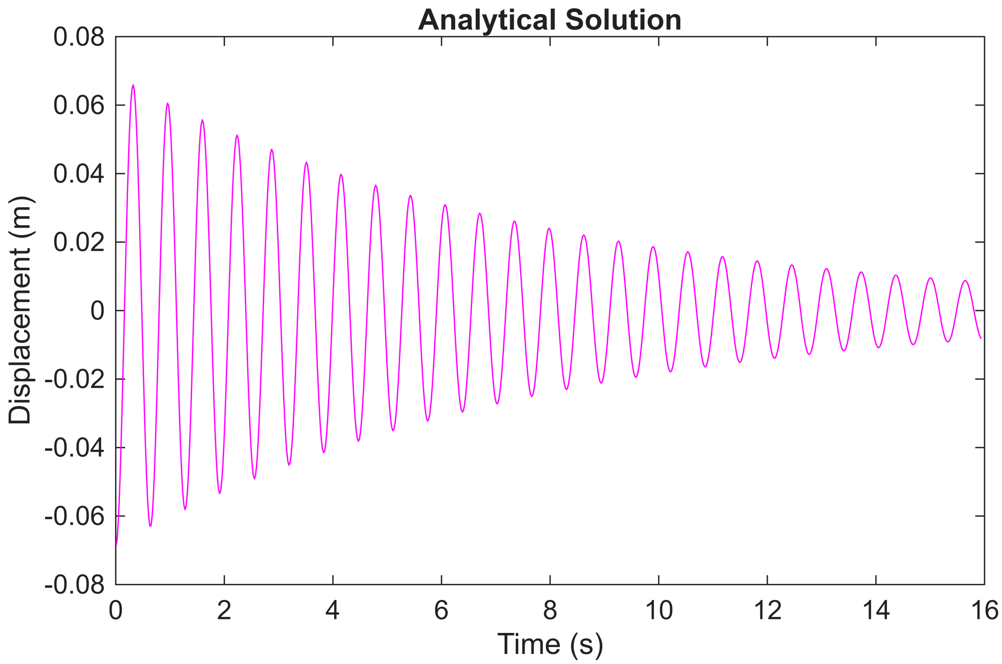

Figure 7. The analytical solution, solved by hand.

Numerical Solution

From the acceleration data, we can craft a similar numerical solution to our model by assuming that the system $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ can be represented instead in terms of acceleration instead of in terms of displacement, which looks like this:

$$\ddot{a} + 2\zeta\omega_n\dot{a} + \omega_n^2 a = 0$$

which is just the second derivative of the displacement equation. Using this equation and having initial acceleration data, we can easily plug this equation into `ode45`.

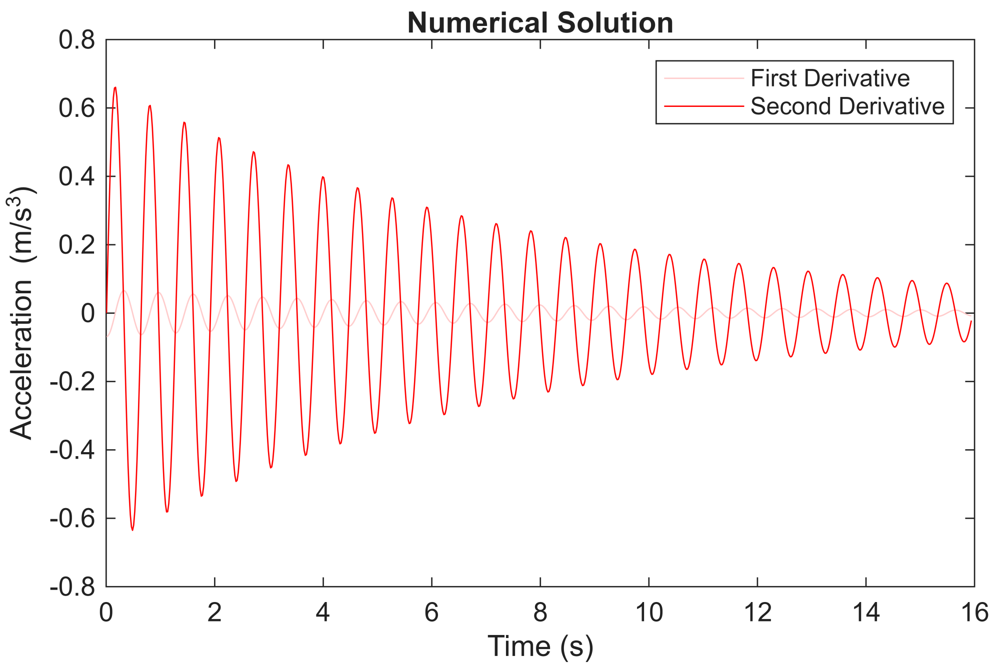

Figure 8: The numerical solution, solved with ode45.

Comparing Natural Frequency

From the acceleration data we estimated that the natural frequency was $\omega_{n,expected} = 9.84 Hz$. The natural frequency calculated using the measure stiffness of the spring, k, and mass of the cart, m, through the equation $\omega_n = \sqrt{\frac{k}{m}}$ is $\omega_{n,measured} = 8.36 Hz$. These values are reasonably close, indicating that the theoretical model provides a good approximation of the behaviour of the spring-damper system. The measured frequency is slightly lower than the estimated frequency, which could be due to experimental errors, unexpected energy losses from damping, or nonlinear motion of the cart.