Chapter 15

In-Class Activity Day 9: The Neato Motion Model

Learning Objectives

Today's agenda is as follows:

- We will use the angular velocity of the Neato to derive an equation that relates its translation and rotation to the velocities of its wheels.
- We will walk through the necessary mathematical steps to implement Rainbow Road, using the spiral path from the first day's assignment as an example.

Exercise 15.1

Before proceeding with this assignment, we have designated time for you to complete the previous assignment on derivatives in multiple frames and an introduction to angular velocity. Understanding these concepts will facilitate the completion of today's assignment.

15.1 Motion Model of the Neato

In the Rainbow Road challenge, you will be asked to drive the Neato along some path in space, whose geometry is described by a parametric curve, $\vec{r}(t)$. To accomplish this task, we've worked through the mathematics that describe parametric curves and their associated parameters. Additionally, we have developed an understanding of vector arithmetic (dot and cross products); learned how to take time derivatives of vectors with respect to a frame; and figured out how to use angular velocity ${}^{O}\vec{\omega}^{B}$ to relate the velocity of vector $\vec{A}$ in frame O to its velocity in frame B. The last puzzle that we need to be able to complete the Rainbow Road challenge is a model that can relate the parametric curve, $\vec{r}(t)$, to the scalar wheel velocities, $v_l(t)$ and $v_r(t)$.

Today's assignment will be split into two parts. In the first half, we will build on the previous assignment (which involved angular velocity) to derive the following equations that relate the forward speed and rotation rate of the Neato ($v_1(t)$ and $\dot{\theta}(t)$) to the scalar wheel velocities ($v_l(t)$ and $v_r(t)$):

$$v_l = v_1 - \frac{d}{2}\dot{\theta}, \qquad v_r = v_1 + \frac{d}{2}\dot{\theta} \tag{15.1}$$

$$v_1 = \frac{v_l + v_r}{2}, \qquad \dot{\theta} = \frac{v_r - v_l}{d} \tag{15.2}$$

The first pair of equations will allow us to extract the scalar wheel velocities of the Neato, given its forward speed and rotation rate (which can be derived as properties of the parametric curve, $\vec{r}(t)$). The second pair of equations are the formulas that we already used to reconstruct the Neato's speed and rotation rate from the encoder measurements in the "Back to the Starting Line" assignment.

In the second half of this assignment, we will walk through the mathematics of how to implement Rainbow Road, using the spiral path from the first day's assignment as an example. Do not worry if you are unable to complete this portion of the assignment. The main reason we have provided this final exercise is so that you can reference it (and its solutions) during the Rainbow Road project.

15.1.1 The Neato Frame

In order to build a model that describes the motion of the Neato, we need to first enumerate the positions, vectors, and variables that will define its kinematics. We'll begin by defining parameters in relation to a body-fixed frame (see Fig. 15.1):

- We define a body fixed frame, $R_B : \{B, \{\hat{b}_1, \hat{b}_2, \hat{b}_2\}$. The frame's origin, B, is located at the centroid of the two wheel positions. Vectors $(\hat{b}_1, \hat{b}_2, \hat{b}_2)$ point forward, left, and out of the page respectively.
- The left and right wheels are located at points E and D respectively.
- We define d as the distance between the two wheels.
- The vectors $\vec{U}_l = \vec{r}_{BE}$ and $\vec{U}_r = \vec{r}_{BD}$ denote the relative position of the left and right wheels with respect to B.
- Finally, the scalar wheel velocities v_l and v_r measure how fast each wheel is moving in the $\hat{b}_1$ (forward) direction with respect to the world frame.

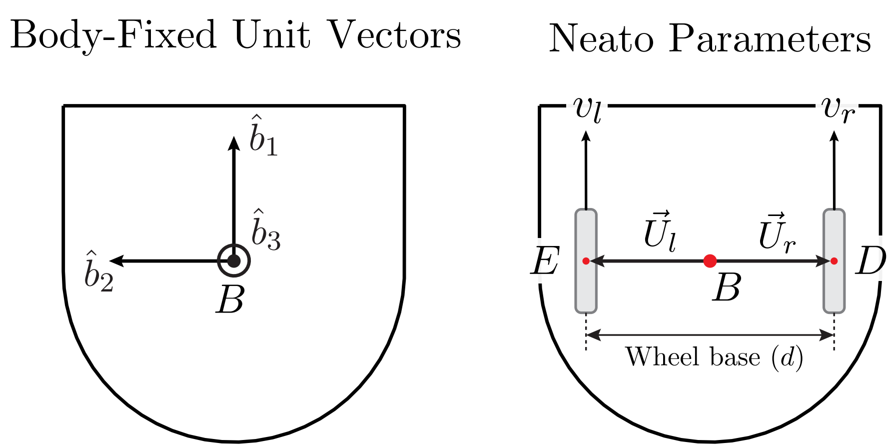

Figure 15.1: Illustration of relevant points and vectors in the Neato frame.

Exercise 15.2

Express $\vec{U}_l^B$ and $\vec{U}_r^B$ in terms of the Neato basis, $(\hat{b}_1, \hat{b}_2, \hat{b}_2)$ and the distance between the wheels, d.

15.1.2 The World Frame

Our next step is to look at the Neato's position and orientation in the world frame.

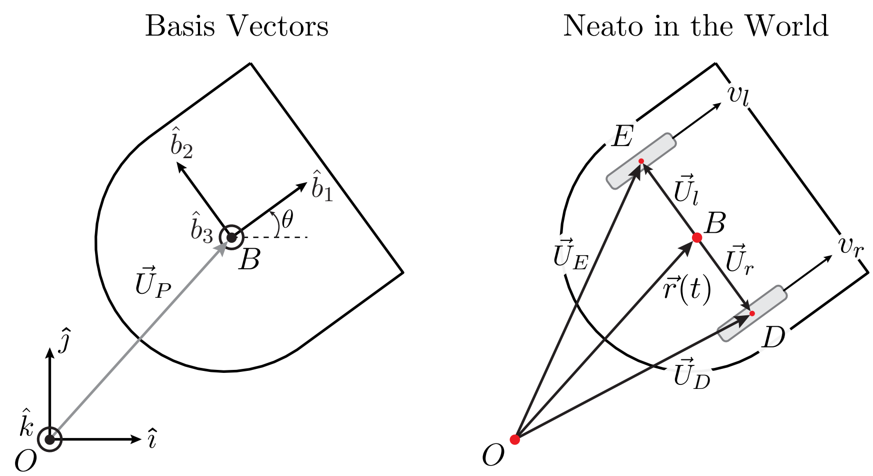

Figure 15.2: Illustration of relevant points and vectors in the world frame.

- We define the world frame $R_O = \{O, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$. We can place the frame's origin, O, in an arbitrary location. Vectors $(\hat{b}_1, \hat{b}_2, \hat{b}_2)$ point horizontally, vertically, and out of the page respectively.
- The position and orientation of the Neato are given by $\vec{r}(t)$ and $\theta(t)$ respectively, where $\vec{r}(t)$ is the vector from O to B, and $\theta(t)$ is the angle between $\hat{b}_1$ and $\hat{\imath}$.
- The positions of the left and right wheels (with respect to the world frame) are given by $\vec{U}_E = \vec{r}_{OE}$ and $\vec{U}_D = \vec{r}_{OD}$ respectively.

Equipped with these definitions, we have all the necessary information to derive equations 15.1 and 15.2:

Exercise 15.3

In this problem, we will work through the derivation of equations 15.1 and 15.2.

1. What is the rotation rate of the Neato (in the world frame)? What is its rotation axis?
2. Given your answer to the previous part, what is the angular velocity of the Neato, ${}^{O}\vec{\omega}^{B}$?
3. What are the derivatives of $\vec{U}_l^B$ and $\vec{U}_r^B$ with respect to the body-fixed frame, i.e.:
$$\left.\frac{d\vec{U}_l^B}{dt}\right|_B, \quad \left.\frac{d\vec{U}_r^B}{dt}\right|_B \tag{15.3}$$
4. Compute the derivatives of $\vec{U}_l^B$ and $\vec{U}_r^B$ with respect to the world frame, i.e.:
$$\left.\frac{d\vec{U}_l^B}{dt}\right|_O, \quad \left.\frac{d\vec{U}_r^B}{dt}\right|_O \tag{15.4}$$
by applying the angular velocity formula we derived in the previous assignment:
$$\left.\frac{d\vec{A}^B}{dt}\right|_O = \underbrace{\left.\frac{d\vec{A}^B}{dt}\right|_B}_{\text{change within } B} + \underbrace{{}^{O}\vec{\omega}^{B} \times \vec{A}^B}_{\text{direction change of } B} \tag{15.5}$$

If we assume that the Neato can only travel forwards or backwards (the $\pm\hat{b}_1$ direction), we can express its velocity (in the world frame) as follows:

$$\vec{v}^B(t) = \left.\frac{d\vec{r}^B(t)}{dt}\right|_O = v_1\hat{b}_1 + 0\hat{b}_2 = v_1\hat{b}_1 \tag{15.6}$$

where $v_1 =$ is the $\hat{b}_1$ component of the Neato's velocity:

$$v_1 = \left(\frac{d\vec{r}^B(t)}{dt}\bigg|_O \right) \cdot \hat{b}_1 \tag{15.7}$$

5. Compute the velocities of the left and right wheels with respect to the world frame, expressed in terms of the Neato basis i.e.:

$$\frac{d\vec{U}_E^B}{dt}\bigg|_O, \qquad \frac{d\vec{U}_D^B}{dt}\bigg|_O \tag{15.8}$$

   Your answers should be in terms of d, $\dot{\theta}$, v_1, and $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$. **Hint:** Remember that:

$$\vec{U}_E^B = \vec{r}^B(t) + \vec{U}_l^B, \qquad \vec{U}_D^B = \vec{r}^B(t) + \vec{U}_r^B \tag{15.9}$$

6. The scalar wheel velocities, v_l and v_r, are the $\hat{b}_1$ component of the world-frame velocities of the left and right wheels:

$$v_l = \left(\frac{d\vec{U}_E^B}{dt}\bigg|_O \right) \cdot \hat{b}_1, \qquad v_r = \left(\frac{d\vec{U}_D^B}{dt}\bigg|_O \right) \cdot \hat{b}_1 \tag{15.10}$$

   Use your solution to the previous part to show that:

$$v_l = v_1 - \frac{d}{2}\dot{\theta}, \qquad v_r = v_1 + \frac{d}{2}\dot{\theta} \tag{15.11}$$

   We can use these formulas to extract the scalar wheel velocities of the Neato, given its forward speed and rotation rate.

7. Show that:

$$v_1 = \frac{v_l + v_r}{2}, \qquad \dot{\theta} = \frac{v_r - v_l}{d} \tag{15.12}$$

   Hint: look at the sum and difference of the previous equations the relate v_l and v_r to v_1 and $\dot{\theta}$. These are the formulas that we used to reconstruct the Neato's speed and rotation rate from the encoder measurements.

15.2 Rainbow Road: Practice Trial

We now have all the necessary mathematical cools to implement Rainbow Road! In the next exercise, we will do a practice trial. You may not have enough time to complete this exercise in class. Regardless, you are encouraged to use this exercise (and the provided solutions) as a reference as you work on Rainbow Road with your teammate.

Exercise 15.4

In this problem, we will program the Neato to travel along an Archimedean spiral. An Archimdean spiral increases by a fixed radius each loop, and has a parametric equation of the form:

$$\vec{r}(t) = hu(t)\cos(u(t))\hat{\imath} + hu(t)\sin(u(t))\hat{\jmath} \tag{15.13}$$

or alternatively:

$$\vec{r}(t) = (x(t), y(t))\,, \qquad x(t) = \frac{h}{2\pi}u(t)\cos(u(t)), \qquad y(t) = \frac{h}{2\pi}u(t)\sin(u(t)) \tag{15.14}$$

where h is the change in radius between each loop, and $u(t)$ is some function of time (ideally a strictly increasing function of time). For the purposes of this problem, we consider the following $u(t)$ and value of h:

$$u(t) = 3\sqrt{\frac{t}{1\,\text{sec}} + 10}, \qquad h = .24\text{m}, \qquad t \in [0\,\text{sec}, 60\,\text{sec}] \tag{15.15}$$

1. In MATLAB, create symbolic expressions for $u(t)$, $x(t)$, and $y(t)$. Generate a plot of the path that the robot travels during the time interval $t \in [0\,\text{sec}, 60\,\text{sec}]$.
2. Using the diff command, create symbolic expressions for the x and y components of the Neato's velocity (in the world frame), as well as its speed.
$$v_x(t) = \dot{x}(t), \qquad v_y(t) = \dot{y}(t), \qquad ||\vec{v}(t)|| = \sqrt{\dot{x}^2 + \dot{y}^2} \tag{15.16}$$
Plot the speed of the Neato as a function of time.
3. Compute symbolic expressions for the x and y components of the tangent vector:
$$T_x = \frac{v_x}{||\vec{v}(t)||}, \qquad T_y = \frac{v_y}{||\vec{v}(t)||} \tag{15.17}$$
Plot the tangent vector $\hat{T}$ on top of the Neato's path for a few different values of t.
4. Create symbolic expressions for the x and y components of the derivative of the tangent vector (in the world frame):
$$q_x = \frac{dT_x}{dt}, \qquad q_y = \frac{dT_y}{dt} \tag{15.18}$$

In the last exercise of the previous assignment, we showed that the angular velocity of the Neato can be found via the formula:

$$^O\vec{\omega}^B = \hat{T} \times \left.\frac{d\hat{T}}{dt}\right|_O \tag{15.19}$$

Since the Neato is restricted to move in the plane, the tangent vector and its derivative can be expressed as follows:

$$\hat{T}^O = T_x\hat{\imath} + T_y\hat{\jmath}, \qquad \left.\frac{d\hat{T}^O}{dt}\right|_O = q_x\hat{\imath} + q_y\hat{\jmath} \tag{15.20}$$

Note that there is no $\hat{k}$ component because the Neato is restricted to planar motion. Plugging these into our angular velocity formula, we get:

$$^O\vec{\omega}^B = (T_x\hat{\imath} + T_y\hat{\jmath}) \times (q_x\hat{\imath} + q_y) = (T_x q_y - T_y q_x)\,\hat{k} \tag{15.21}$$

Since the Neato rotates at rate $\dot{\theta}$ about the $\hat{k}$ axis, we see that its angular velocity is given by:

$$^{O}\vec{\omega}^{B} = \dot{\theta}\hat{k} \tag{15.22}$$

Substituting this expression into our previous equation, we get:

$$\dot{\theta}\hat{k} = (T_x q_y - T_y q_x)\,\hat{k} \tag{15.23}$$

If we take the dot product of both sides of the equation with $\hat{k}$, we see that:

$$\left(\dot{\theta}\hat{k}\right)\cdot\hat{k} = \left((T_x q_y - T_y q_x)\,\hat{k}\right)\cdot\hat{k} \tag{15.24}$$

$$\dot{\theta} = T_x q_y - T_y q_x \tag{15.25}$$

5. Apply the formula $\dot{\theta} = T_x q_y - T_y q_x$ to create a symbolic expression for $\dot{\theta}$ (in MATLAB).

6. Remember that we can relate the Neato's speed, $v_1 = ||\vec{v}(t)||$, and rotation rate, $\dot{\theta}(t)$, to the scalar wheel velocities ($v_l(t)$ and $v_r(t)$) with the following formula:

$$v_l(t) = ||\vec{v}(t)|| - .5d\dot{\theta}, \qquad v_r(t) = ||\vec{v}(t)|| + .5d\dot{\theta} \tag{15.26}$$

   where $d = .24$m for the Neato. Use these formulas to make symbolic expressions of $v_l(t)$ and $v_r(t)$ (in MATLAB). Add the plots of $v_l(t)$ and $v_r(t)$ vs. time to your plot of the Neato's speed as a function of time.

7. How does our choice of $u(t)$ affect the shape of the path that the robot travels? How does $u(t)$ affect the speed at which the robot travels along the path.

8. Write a script/function that drives the Neato along the spiral path for 60 seconds. You are encouraged to use the return-to-home routine that you previously wrote as a starting point.

   It should be noted that using subs and double to numerically evaluate a symbolic expression can be quite time consuming, which is problematic, since we need to be sending commands to the Neato relatively frequently. As such, we recommend that you pre-compute $v_l(t)$ and $v_r(t)$ for a range of time values, and then using MATLAB's linear interpolation function (interp1) to use the pre-computed lists to approximate $v_l(t)$ and $v_r(t)$ in real time (when sending commands to the robot). This implementation is demonstrated in the example below:

```
%this example shows how to precompute vl and vr for
%a range of time values, and then use those precompute values
%to approximate vl(t) and vr(t) in real time

%vl and vr are symbolic expressions for vl(t) and vr(t)
%that were computed beforehand

%define a list of time values from 0 to 70 (seconds)
%to evaluate vl(t) and vr(t) beforehand
t_range = 0:.2:70;

%evaluate vl(t) and vr(t) for t = t_range
%result should be two lists
vl_range = double(subs(vl,t_range));
vr_range = double(subs(vr,t_range));

%start the timer
tic;

%loop until 60 seconds have passed
while toc<=60
    %set t_in to the amount of time that has elapsed
    t_in = toc;
```

```
    %use MATLAB's linear interpolation to approximate
    %vl(t) and vr(t) at t=t_in
    vl_out = interp1(t_range,vl_range,t_in);
    vr_out = interp1(t_range,vr_range,t_in);

    %do something with vl_out and vr_out
    %your code here...
end
```