Chapter 12

In-Class Activity Day 7: Basis Vectors, Dot Products, and Cross Products

Learning Objectives

We're going to start using some *vector operations* and concepts that will help us express our Neato motion model more precisely. At the end of this activity, you'll be able to use the following analytical tools:

- Basis vectors, a class of vectors which are used to connect physical quantities (*e.g.*, position, velocity) to a coordinate system
- Dot and cross products of vectors as a way to compare vectors with one another and to create combinations of vectors to describe other properties of a physical system
- Transformations from one coordinate system to another

12.1 Basis Vectors

Let's start off by thinking back to Day 4 when we began talking about vectors. We defined a *coordinate system* as a framework in which a vector can be represented by an arrow whose tail is located at one point in the coordinate system and whose head is located at another. In a Cartesian (or rectangular) coordinate system, the points are defined by an ordered pair of scalars (x, y) in a 2D system or an ordered triple (x, y, z) in a 3D system. Vectors encode two pieces of information, magnitude and direction, which are related to the length of the vector and the heading of the line going from the tail to head of the vector, respectively.

Now we'll see a another way to represent a general vector, $\vec{v}$, that more explicitly separates the magnitude and direction information. First, recall from Week 3 that a vector of length one is called a **unit vector**. We can think of unit vectors as a way to encode the direction information alone. Given an arbitrary vector $\vec{v}$, we can compute the unit vector that points in the same direction of $\vec{v}$ by dividing or normalizing$\vec{v}$ by its own magnitude,

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}. \tag{12.1}$$

The unit vectors that point in the positive x, y, and z directions are so useful that we give them special names:

$$\hat{\imath} = (1, 0, 0), \quad \hat{\jmath} = (0, 1, 0), \quad \hat{k} = (0, 0, 1). \tag{12.2}$$

We can use these unit vectors to elegantly describe any other 2D or 3D vector as a combination of scalar multiplication and vector addition (see Figs 12.1 and 12.2). Specifically, in 2D, the vector $\vec{v} = (v_x, v_y)$ can be rewritten as follows:

$$\vec{v} = (v_x, v_y) = (v_x, 0) + (0, v_y) = v_x(1, 0) + v_y(0, 1) = v_x\hat{\imath} + v_y\hat{\jmath}. \tag{12.3}$$

Similarly, in 3D, we get:

$$\vec{v} = (v_x, v_y, v_z) = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}. \tag{12.4}$$

In other words, *any* 2D or 3D vector $\vec{v}$ can be rewritten as a **linear combination** of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. The unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are a *basis* or a set of vectors that can be used to define any vector in the coordinate system. Furthermore, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are unit vectors and are mutually perpendicular. In this case, this basis is called an **orthonormal basis**. When doing vector operations, it is sometimes more convenient to use the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ representation instead of row/column vectors.

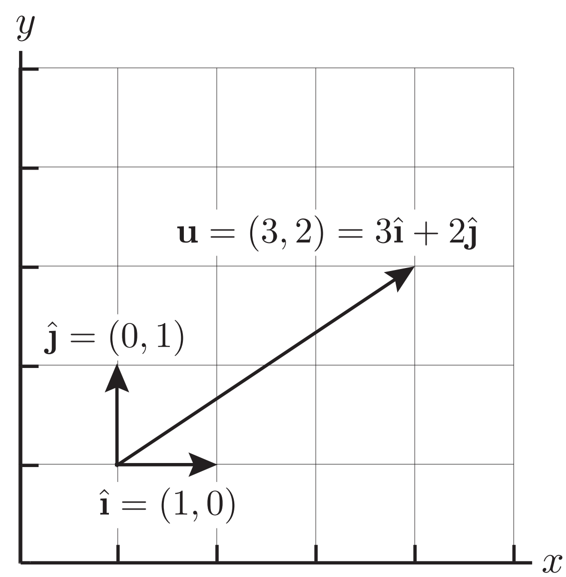

Figure 12.1: An example of how to represent a 2D vector as a linear combination of $\hat{\imath}$ and $\hat{\jmath}$.

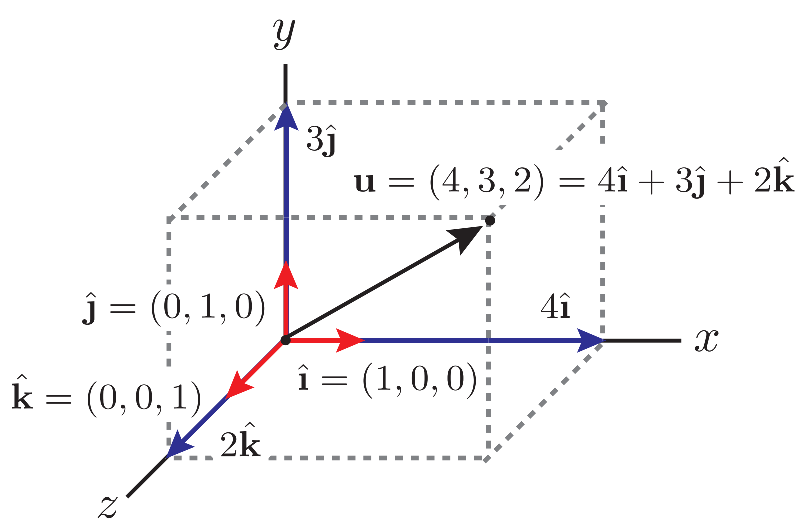

Figure 12.2: An example of how to represent a 3D vector as a linear combination of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

12.2 Vector Operations

12.2.1 Dot Products

The dot product (also called the inner product) of two vectors is the sum of the products of their coordinates. For example, in 2D, the dot product of vectors $\vec{v}$ and $\vec{u}$ is given by:

$$\vec{v} \cdot \vec{u} = v_x u_x + v_y u_y \tag{12.5}$$

More generally, if $\vec{v}$ and $\vec{u}$ are N-dimensional vectors, their dot product is given by:

$$\vec{v} \cdot \vec{u} = \sum_{i=1}^{N} v_i u_i \tag{12.6}$$

Geometrically, the dot product of two vectors is equal to the product of their respective magnitudes and the cosine of the angle between them (see Figure 12.3):

$$\vec{v} \cdot \vec{u} = ||\vec{v}||\ ||\vec{u}|| \cos\theta \tag{12.7}$$

As such, the dot product of $\vec{v}$ and $\vec{u}$ is a measure of how much $\vec{v}$ and $\vec{u}$ point in the same direction. More specifically:

1. The dot product $\vec{v} \cdot \vec{u}$ is **positive** when $\vec{v}$ and $\vec{u}$ point in the same general direction (i.e. when the angle between them is less than $90°$).
2. The dot product $\vec{v} \cdot \vec{u}$ is **negative** when $\vec{v}$ and $\vec{u}$ point away from each other (i.e. when the angle between them is greater than $90°$).
3. The dot product $\vec{v} \cdot \vec{u}$ is **zero** when $\vec{v}$ and $\vec{u}$ are perpendicular (i.e. when the angle between them is equal to $90°$).
4. The dot product $\vec{v} \cdot \vec{u}$ is exactly equal to the product of the magnitudes of $\vec{v}$ and $\vec{u}$ when they point in the exact same direction (i.e. when the angle between them is zero).

We can also think of the dot product as a way to measure the length of the shadow (projection) that the two vectors cast on each other. Specifically, $\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}$, is the length of the projection of $\vec{u}$ onto $\vec{v}$.

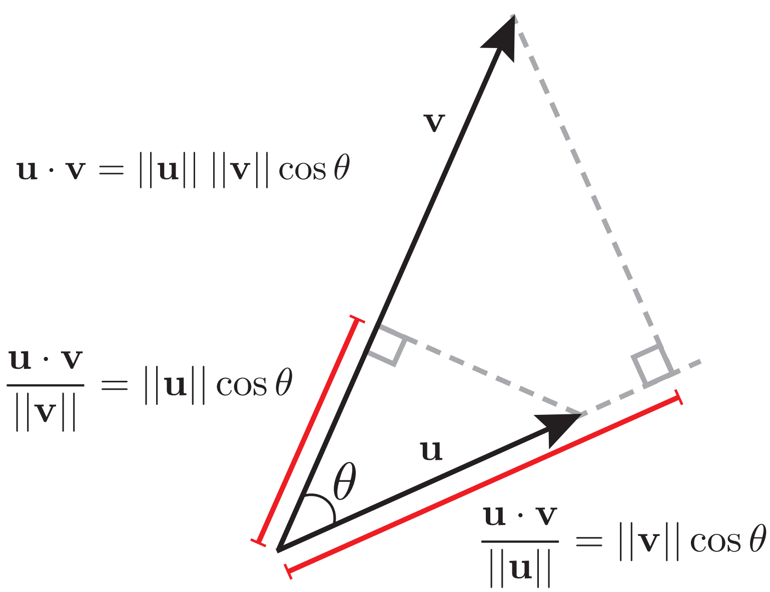

Figure 12.3: Geometrically, the dot product of two vectors is equal to the product of their respective magnitudes and the cosine of the angle between them.

Exercise 12.1

Conceptual Check When might taking a dot product between two vectors be useful? What quantities can you think of to compute using a dot product with respect to the Neato?

12.2.2 Cross Products

The cross product of $\vec{u}$ and $\vec{v}$, $(\vec{u} \times \vec{v})$, is a vector with the following three properties:

1. $(\vec{u} \times \vec{v})$ is perpendicular to both $\vec{u}$ and $\vec{v}$:

$$\vec{u} \perp (\vec{u} \times \vec{v}), \qquad \vec{v} \perp (\vec{u} \times \vec{v}) \tag{12.8}$$

2. The magnitude of $(\vec{u} \times \vec{v})$ is the product of the magnitudes of $\vec{u}$ and $\vec{v}$ and the sine of the angle between them (this is also the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$, see Figure 12.4).
$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin\theta \tag{12.9}$$
3. $(\vec{u} \times \vec{v})$ obeys the **right-hand rule**, which is a type of sign convention. A sign convention is a mutual agreement among mathematicians/physicists/engineers everywhere to be consistent about the choice of sign ($\pm$) of the value of a quantity. In this case, in defining the cross product, since there are two possible vectors that satisfy both of the previous two properties, we need to be consistent about which of those two to choose. Here, $(\vec{u} \times \vec{v})$ is the vector chosen such that if you point your right thumb along $\vec{u}$ and your right pointer finger along $\vec{v}$, then your right middle finger will point along $(\vec{u} \times \vec{v})$. Alternatively, if you curl the fingers of your right hand from the first to the second vector in the cross product, your thumb should point in the direction of the cross product.

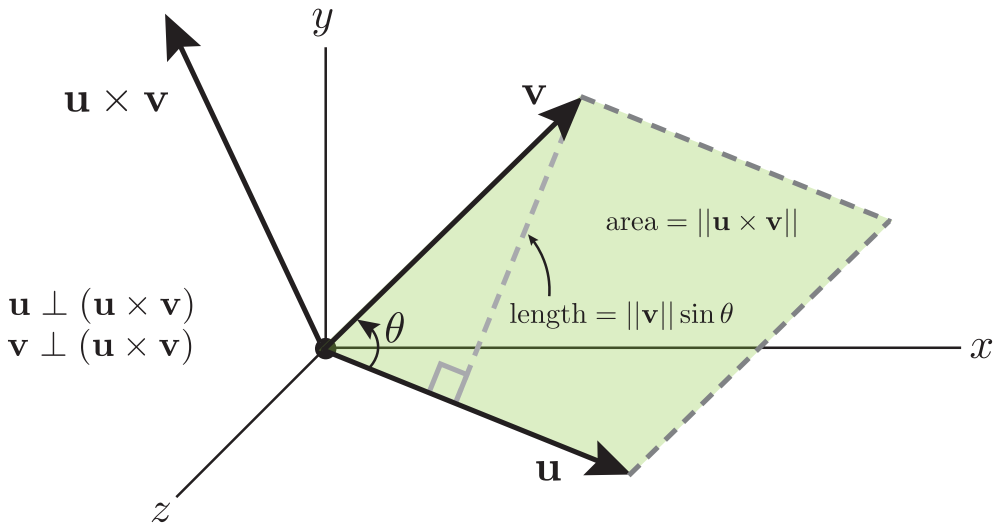

Figure 12.4: The cross product of $\vec{u}$ and $\vec{v}$, $(\vec{u} \times \vec{v})$, is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$, with length equal to the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.

We can compute the cross product of $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$ using the following formula:

$$\vec{u} \times \vec{v} = (u_y v_z - v_y u_z, \;\; u_z v_x - v_z u_x, \;\; u_x v_y - v_x u_y) \tag{12.10}$$

It should be noted that this is equivalent to computing the following determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \tag{12.11}$$

We can also rewrite the cross product as the following matrix multiplication:

$$\vec{u} \times \vec{v} = \{u_\times\}\vec{v} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{12.12}$$

Where $\{u_\times\}$ is a *skew-symmetric* matrix that encodes taking a cross product with $\vec{u}$.

12.2.3 Dot Products and Cross Products of $\hat{\imath}, \hat{\jmath}, \hat{k}$

Below are some useful dot products and cross products to keep in mind. Verify them for yourself.

$$\begin{aligned} \hat{\imath} \cdot \hat{\imath} = 1, \quad \hat{\imath} \cdot \hat{\jmath} = 0, \quad \hat{\imath} \cdot \hat{k} = 0 \\ \hat{\jmath} \cdot \hat{\imath} = 0, \quad \hat{\jmath} \cdot \hat{\jmath} = 1, \quad \hat{\jmath} \cdot \hat{k} = 0 \\ \hat{k} \cdot \hat{\imath} = 0, \quad \hat{k} \cdot \hat{\jmath} = 0, \quad \hat{k} \cdot \hat{k} = 1 \end{aligned} \tag{12.13}$$

$$\begin{aligned} \hat{\imath} \times \hat{\imath} = 0, \quad \hat{\imath} \times \hat{\jmath} = \hat{k}, \quad \hat{\imath} \times \hat{k} = -\hat{\jmath} \\ \hat{\jmath} \times \hat{\imath} = -\hat{k}, \quad \hat{\jmath} \times \hat{\jmath} = 0, \quad \hat{\jmath} \times \hat{k} = \hat{\imath} \\ \hat{k} \times \hat{\imath} = \hat{\jmath}, \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \quad \hat{k} \times \hat{k} = 0 \end{aligned} \tag{12.14}$$

Exercise 12.2

Vector Arithmetic

1. Compute the following dot and cross products:

   A. $\hat{\imath} \cdot (\hat{\jmath} \times \hat{k})$

   B. $\hat{\jmath} \times (\hat{k} \times \hat{\imath})$

2. Consider the following four vectors:

$$\vec{v}_1 = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}, \quad \vec{v}_2 = -\hat{\imath} + 4\hat{\jmath} + \hat{k}$$

$$\vec{v}_3 = 2\hat{\imath} + 2\hat{\jmath}, \quad \vec{v}_4 = \hat{\imath} - 2\hat{k}$$

   Compute the following quantities. Use MATLAB to verify your answers:

   A. $2\vec{v}_1 + 3\vec{v}_3$

   B. $||3\vec{v}_2||$

   C. $\vec{v}_1 \cdot \vec{v}_3$

   D. $\vec{v}_2 \cdot \vec{v}_4$

   E. $\vec{v}_1 \times \vec{v}_3$

   F. $\vec{v}_2 \times \vec{v}_4$

Exercise 12.3

The Right Hand Rule

1. For each vector $\vec{v}$ shown in Figure 12.5, sketch and label both $\hat{k} \times \vec{v}$ and $-\hat{k} \times \vec{v}$. Describe in words how the operations $\hat{k} \times \vec{v}$ and $-\hat{k} \times \vec{v}$ qualitatively transform any vector $\vec{v}$ on the page. Take $+\hat{k}$ to be coming perpendicularly out of the page.

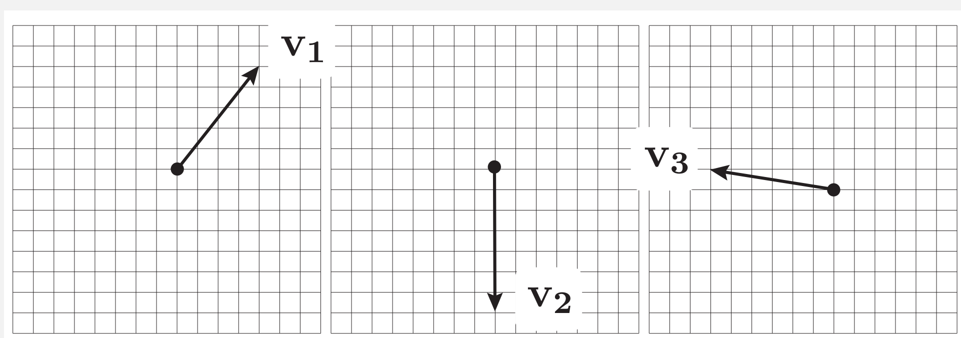

Figure 12.5

2. For each of the vector pairs, $(\vec{v}_1, \vec{v}_2)$ shown in Figure 12.6, determine whether the **cross product** $\vec{v}_1 \times \vec{v}_2$ points out of the page (the $+\hat{k}$ direction), into the page (the $-\hat{k}$ direction), or is zero. Please do not actually try to compute these cross products. You should be able to identify whether or not $\vec{v}_1 \times \vec{v}_2$ points into or out of the page by visually inspecting the two vectors.

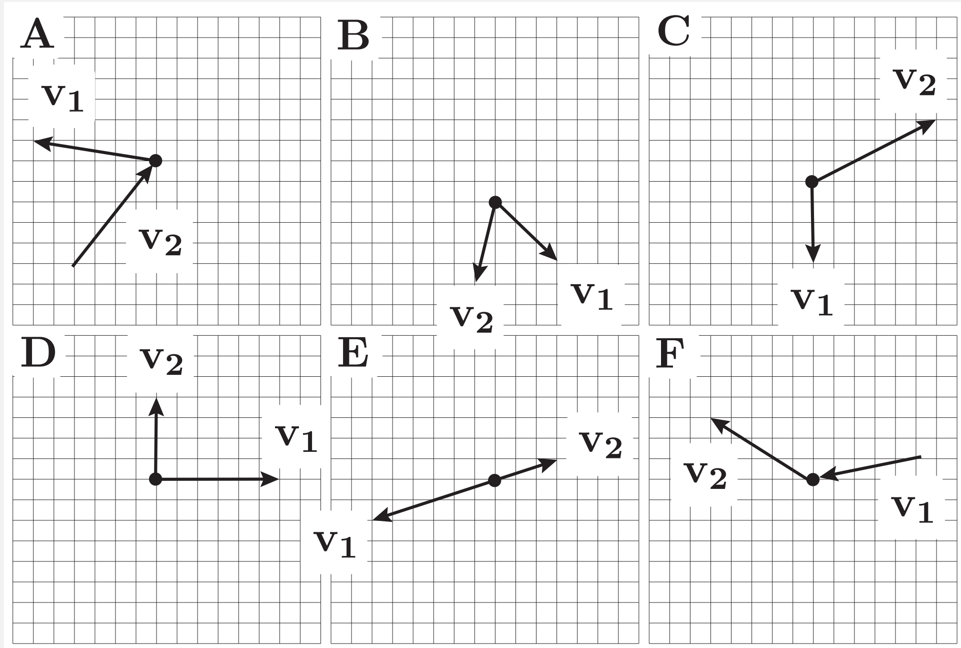

Figure 12.6

3. For each of the vector pairs, $(\vec{v}_1, \vec{v}_2)$ shown in Figure 12.6, determine whether the **dot product** $\vec{v}_1 \cdot \vec{v}_2$ is positive, negative, or zero. Please do not actually try to compute these dot products. You should be able to identify the sign of $\vec{v}_1 \cdot \vec{v}_2$ by visually inspecting the two vectors.

12.3 Transforming Planar (2D) Coordinate Systems

As you might expect, a vector can be written in terms of any coordinate system, *i.e.*, as different linear combinations of unit vectors from a different basis. The relationship between two coordinate systems can be characterized by a *linear transformation* or rotation of one coordinate system to another. Here we will focus on transforming planar (2D) coordinate systems.

12.3.1 Using Dot Products to Transform or Rotate Coordinate Systems

Consider the unit vectors $(\hat{e}_x, \hat{e}_y, \hat{e}_r, \hat{e}_\theta)$, which are illustrated in Figure 12.7. The unit vectors $(\hat{e}_x, \hat{e}_y)$ point horizontally and vertically respectively. The unit vectors $(\hat{e}_r, \hat{e}_\theta)$ are rotated counter-clockwise by an angle θ w/respect to the horizontal/vertical. Any vector $\vec{v}$ in the plane can be written in terms of either coordinate systems as

$$\vec{v} = a\hat{e}_x + b\hat{e}_y = c\hat{e}_r + d\hat{e}_\theta \tag{12.15}$$

where a, b, c, d are constants. In general, given an orthonormal basis $(\hat{e}_1, \ldots\hat{e}_n)$ and an arbitrary vector $\vec{v}$, all of which live in an n-dimensional vector space, it is possible to represent $\vec{v}$ in terms of the basis $(\hat{e}_1, \ldots\hat{e}_n)$ as follows:

$$\vec{v} = \sum_{i=1}^{n} (\vec{v} \cdot \hat{e}_i)\hat{e}_i \tag{12.16}$$

Here, the dot product $\vec{v} \cdot \hat{e}_i$ corresponds to the ith coordinate when writing out $\vec{v}$ in the coordinates described by the $(\hat{e}_1, \ldots\hat{e}_n)$ basis:

$$\vec{v} = (v_1, \ldots, v_n), \qquad v_i = \vec{v} \cdot \hat{e}_i \tag{12.17}$$

The sum, $\sum_{i=1}^{n} v_i\hat{e}_i$, is how we convert a set of coordinates into the corresponding vector.

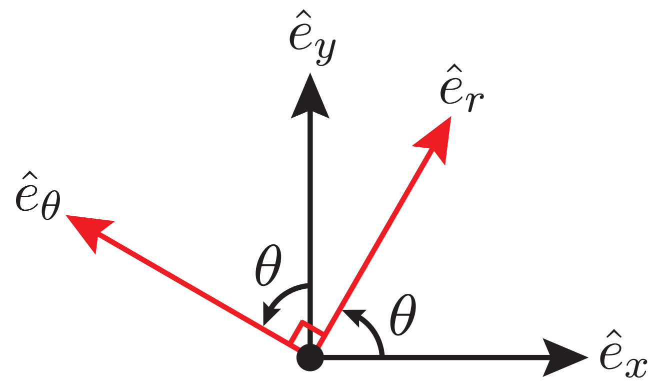

Figure 12.7

Exercise 12.4

1. Evaluate the following dot products:
   A. $\hat{e}_x \cdot \hat{e}_r$
   B. $\hat{e}_y \cdot \hat{e}_r$
   C. $\hat{e}_x \cdot \hat{e}_\theta$
   D. $\hat{e}_y \cdot \hat{e}_\theta$
2. A. Show that (12.16) holds true for the specific example of $\vec{v} = 5\hat{\imath} + 7\hat{\jmath}$ and $(\hat{\imath}, \hat{\jmath})$.
   B. Use (12.16) (and your answers to part 1.) to express the unit vectors $(\hat{e}_r, \hat{e}_\theta)$, in terms of the basis $(\hat{e}_x, \hat{e}_y)$.
   C. Use (12.16) (and your answers to part 1.) to express the unit vectors $(\hat{e}_x, \hat{e}_y)$, in terms of the basis $(\hat{e}_r, \hat{e}_\theta)$.
   D. Express $\vec{v} = 5\hat{e}_x + 7\hat{e}_y$ in terms of $(\hat{e}_r, \hat{e}_\theta)$.
   E. Given vector $\vec{v} = a\hat{e}_x + b\hat{e}_y$, how might we find values (c, d) such that $\vec{v} = c\hat{e}_r + d\hat{e}_\theta$. Relate (a, b) and (c, d) to one another via matrix multiplication. What kind of matrix are we using to map (a, b) to (c, d)?

12.4 2D Rotation Matrices

As you've just seen in the last exercise (and also back in QEA 1), a vector in a plane can be represented by the unit (basis) vectors in either coordinate system, *i.e.*,

$$\vec{v} = a\hat{e}_x + b\hat{e}_y = c\hat{e}_r + d\hat{e}_\theta \tag{12.18}$$

where a, b, c, and d are related by

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where θ is taken to be positive rotating counter-clockwise. The matrix

$$R_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is a *rotation matrix* which changes the vector written in the $(\hat{e}_x, \hat{e}_y)$ basis to one written in the $(\hat{e}_r, \hat{e}_\theta)$ basis. It's very important to remember that *the vector stays in the same place in space* but the coordinate system defined by $(\hat{e}_x, \hat{e}_y)$ have been changed (transformed or rotated) to the coordinate system defined by $(\hat{e}_r, \hat{e}_\theta)$. Note that the inverse transformation can be carried out by the matrix

$$R_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which is the inverse (and in the case of an orthonormal basis, the transpose) of R_1.