Chapter 14

In-Class Activity Day 8: Vector Derivatives with Respect to Rotating Reference Frames

Learning Objectives

This module is an introduction to reference frames. We will cover:

- Expressing the same vector in multiple reference frames
- Inertial vs. body-fixed reference frames
- The derivatives of vectors w/respect to difference reference frames
- The use of notation to clearly indicate which reference frame a vector is expressed in and in which frame the derivative is taken, without ambiguity
- How to use reference frames to compute the velocity of points of a rigid body

14.1 Representing Vectors Using Different Reference Frames

So far in this course, we have spent a lot of time defining various vectors that describe the motion of a particle along a path in space (i.e. position, velocity, and acceleration), as well as several vector specifically related to the geometry of that path (i.e. the unit tangent, normal, and binormal vectors).

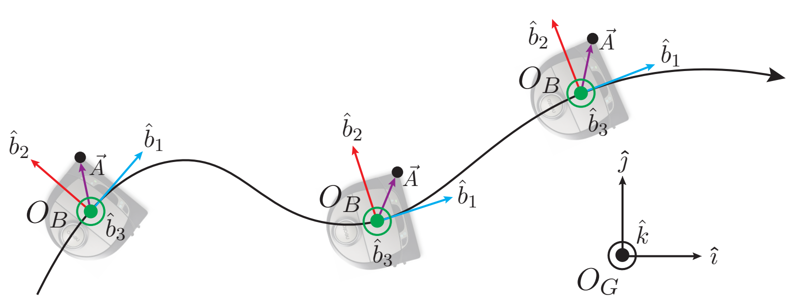

Figure 14.1: If we wanted to compute the velocity or acceleration of a particle that is attached to the Neato (here described by the vector A), it would be useful to consider the derivative of that particle from both he perspective of a stationary observer (ex. frame $\{O_G, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$) *and* from the perspective of the Neato itself (frame $\{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$).

Though we have taken time derivatives of several of these vector quantities, it has always been under the assumption that "observations" are being made from a fixed/static perspective. In other words, we have assumed that our coordinate axes is not capable of moving. In order to properly lay the mathematical groundwork for Rainbow Road, it would be useful to generalize our understanding of the derivative to include the possibility of dynamic coordinate axes. For instance, if we wanted to compute the velocity or acceleration of a particle that is attached to the Neato it would be useful to consider both the derivative of that particle from the both from the perspective of a stationary observer in the world *and* from the perspective of the Neato itself (see Fig. 14.1). To this end, let's introduce the concept of a reference frame.

A reference frame is a set of coordinate axes attached to an origin point. Reference frames are the means by which we are able to assign numerical values to vectors. Reference frames are usually parametrized by their origin, and a set of mutually orthogonal (perpendicular) unit vectors (see Fig. 14.2 for an example).

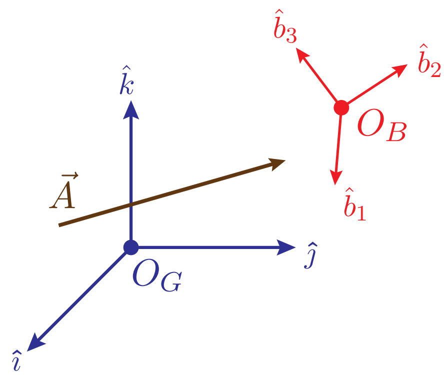

Figure 14.2: The blue frame has origin O_G and basis vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$. The red frame has origin O_B and basis vectors $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$. A generic vector $\vec{A}$ can be represented in terms of either frame.

Take a vector $\vec{A}$ in space, which could represent position, velocity, force, etc. Given frames $G : \{O_G, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$ and $B : \{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$, we can express $\vec{A}$ in terms of the basis (unit) vectors of either frame as:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} = A_1\hat{b}_1 + A_2\hat{b}_2 + A_3\hat{b}_3 \tag{14.1}$$

Depending on the reference frame that we are looking at, the numeric representation of A (i.e. the entries in the row or column vector) could change:

$$\vec{A}^G = [A_x, A_y, A_z], \qquad \vec{A}^B = [A_1, A_2, A_3] \tag{14.2}$$

even though these are both just representations of the same underlying object (the vector $\vec{A}$). This is what we mean by the idea that frames are means of assigning numerical values to vectors.

Exercise 14.1

In this exercise, we will practice representing vectors using different reference frames.

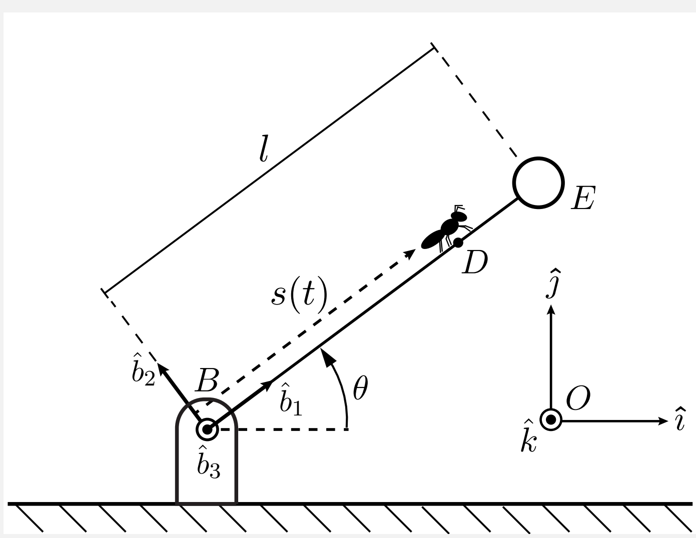

Figure 14.3

The rod BE rotates around the stationary point B at a constant rotation rate $\Omega = \frac{d\theta}{dt}$. At the same time, there is an ant (located at point D) traveling up the rod at a distance $s(t)$ from point B. The angle $\theta(t)$ tracks the orientation of the rod w/respect to the horizontal. For the purposes of tracking the ant, we have define an **stationary reference frame**, $R_O : \{O, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$, and a **moving reference frame** $R_B : \{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$. The moving reference frame, R_B is fixed to the bar such that the unit vector $\hat{b}_1$ is always parallel to BD.

1. Let l denote the length of the rod. Express vector $\vec{r}_{BE}$ (the vector from B to E) as a linear combination of one or more of the stationary basis vectors, $(\hat{\imath}, \hat{\jmath}, \hat{k})$.
2. Express $\vec{r}_{BE}$ as a linear combination of one or more of the rotating basis vectors $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$.
3. Let $s(t)$ denote the distance of the ant from B. Express vector $\hat{r}_{BD}$ as a linear combination of one or more of the stationary basis vectors, $(\hat{\imath}, \hat{\jmath}, \hat{k})$.
4. Express $\vec{r}_{BD}$ as a linear combination of one or more of the rotating basis vectors $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$.

14.2 Reference Frames: Inertial vs. Body

Reference frames can be classified into two different categories: inertial (also called fixed, room, or ground) and body (also called body-fixed, local, or rotating) frames. An inertial frame is not accelerating (therefore Newton's Laws apply in it). Any reference frame that is fixed in space is inertial (and this is the most common type of inertial reference frame that you will encounter in physics problems); however, any non-rotating reference frame with a constant velocity is inertial (less common). A frame is *body-fixed* if it is rigidly attached to a moving object (and thus undergoes the same rotations and translations as the body that it is fixed to). Body-fixed frames are usually assumed to be non-inertial (accelerating). In order to relate the derivative of a vector from the perspective of one frame to another, we'll need to characterize the motion of the two frames with respect to each other. This motion can be expressed as a combination of *translation* and *rotation*. For the purposes of relating the derivatives of a vector in two different frames, we will find that only *rotation* or change of direction/orientation matters.

Exercise 14.2

1. Referring back to the previous exercise with the ant, which of the two reference frames (R_O and R_B) is **inertial**? Which is non-inertial? Which frame is body-fixed, and why?
2. Describe the motion of frames R_O and R_B w/respect to one another.
3. Suppose we were to describe the vector $\vec{r}_{BE}$ in terms of the R_O frame:

$$\vec{r}_{BE} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \quad \rightarrow \quad [x, y, z] \tag{14.3}$$

(where x, y, and z are determined by your answer to the previous exercise). What is the derivative of the above row vector?

$$\frac{d}{dt}\left([x, y, z]\right) \tag{14.4}$$

4. Suppose we were to describe the vector $\vec{r}_{BE}$ in terms of the R_B frame:

$$\vec{r}_{BE} = q_1\hat{b}_1 + q_2\hat{b}_2 + q_3\hat{b}_3 \quad \rightarrow \quad [q_1, q_2, q_3] \tag{14.5}$$

(where q_1, q_2, and q_3 are determined by your answer to the previous exercise). What is the derivative of the above row vector:

$$\frac{d}{dt}\left([q_1, q_2, q_3]\right) \tag{14.6}$$

5. Compare your answers to your previous two parts. Are they the same, or different? Explain your result in terms of the two reference frames.

14.3 Derivative Taken with Respect to One Frame of a Vector Written in Terms of Another Frame

Given frames $G : \{O_G, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$ and $B : \{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$, the time derivative of a general vector, $\vec{A}$ can be taken with respect to either frame, B or G. As we demonstrated in the previous exercise, the result of these derivatives depend on the reference frame, and *are therefore not necessarily equal*! To help formalize the definition of a derivative we will introduce the superscript notation $\vec{V}^F$ to indicate that we are representing vector $\vec{V}$ using the basis vector of frame F. For example, $\vec{A}^G$ and $\vec{A}^B$ correspond to the following linear combinations/row vectors:

$$\vec{A}^G = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \quad \rightarrow \quad [A_x, A_y, A_z] \tag{14.7}$$

$$\vec{A}^B = A_1\hat{b}_1 + A_2\hat{b}_2 + A_3\hat{b}_3 \quad \rightarrow \quad [A_1, A_2, A_3] \tag{14.8}$$

Mathematically, we can express the derivative of a vector w/respect to a particular frame as:

$$\left.\frac{d\vec{V}}{dt}\right|_F \tag{14.9}$$

which can be read as "the derivative of vector $\vec{V}$ taken with respect to frame F". When taking the derivative of a vector w/respect to a particular reference frame, we treat the basis vectors of that reference frame as being stationary, and therefore having derivative zero. To illustrate how this works, let's take the derivative of vector A with respect to frame G. First, suppose we were to represent A in terms of the basis vectors of G:

$$\vec{A}^{G} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}, \qquad \left.\frac{d\vec{A}^{G}}{dt}\right|_{G} = \frac{d}{dt}\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\Bigg|_{G} \tag{14.10}$$

To take the derivative of this quantity, we must apply the product rule (which applies to vectors!):

$$\frac{d}{dt}(uv) = \frac{du}{dt}v + u\frac{dv}{dt} \tag{14.11}$$

Doing so, we get:

$$\left.\frac{d\vec{A}^{G}}{dt}\right|_{G} = \underbrace{\dot{A}_x\hat{\imath} + \dot{A}_y\hat{\jmath} + \dot{A}_z\hat{k}}_{\text{change } \textbf{within } G} + \underbrace{A_x\left.\frac{d\hat{\imath}}{dt}\right|_{G} + A_y\left.\frac{d\hat{\jmath}}{dt}\right|_{G} + A_z\left.\frac{d\hat{k}}{dt}\right|_{G}}_{\textbf{direction change of } G \textbf{ wrt } G=0} \tag{14.12}$$

Since the unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$ are the basis vectors of G, we treat them as stationary, so their derivative w/respect to G is zero:

$$\left.\frac{d\hat{\imath}}{dt}\right|_{G} = \left.\frac{d\hat{\jmath}}{dt}\right|_{G} = \left.\frac{d\hat{k}}{dt}\right|_{G} = 0 \tag{14.13}$$

Plugging this into the previous equation, we get:

$$\left.\frac{d\vec{A}^{G}}{dt}\right|_{G} = \dot{A}_x\hat{\imath} + \dot{A}_y\hat{\jmath} + \dot{A}_z\hat{k} \tag{14.14}$$

Now, let's instead represent A in terms of the basis vectors of B:

$$\vec{A}^{B} = A_1\hat{b}_1 + A_2\hat{b}_2 + A_3\hat{b}_3, \qquad \left.\frac{d\vec{A}^{B}}{dt}\right|_{G} = \frac{d}{dt}\left(A_1\hat{b}_1 + A_2\hat{b}_2 + A_3\hat{b}_3\right)\Bigg|_{G} \tag{14.15}$$

Applying the chain rule once again, we get:

$$\left.\frac{d\vec{A}^{B}}{dt}\right|_{G} = \underbrace{\dot{A}_1\hat{b}_1 + \dot{A}_2\hat{b}_2 + \dot{A}_3\hat{b}_3}_{\text{change } \textbf{within } B} + \underbrace{A_1\left.\frac{d\hat{b}_1}{dt}\right|_{G} + A_2\left.\frac{d\hat{b}_2}{dt}\right|_{G} + A_3\left.\frac{d\hat{b}_3}{dt}\right|_{G}}_{\textbf{direction change of } B \textbf{ wrt } G\neq 0}$$

Unlike the previous representation, since $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ are not stationary w/respect to frame G, we cannot assume that their derivative is zero:

$$\left.\frac{d\hat{b}_1}{dt}\right|_{G} \neq 0, \qquad \left.\frac{d\hat{b}_2}{dt}\right|_{G} \neq 0, \qquad \left.\frac{d\hat{b}_3}{dt}\right|_{G} \neq 0 \tag{14.16}$$

meaning that we cannot simplify our expression for the derivative as we did before (it must be left as is):

$$\left.\frac{d\vec{A}^B}{dt}\right|_G = \underbrace{\dot{A}_1\hat{b}_1 + \dot{A}_2\hat{b}_2 + \dot{A}_3\hat{b}_3}_{\text{change } \textbf{within } B} + \underbrace{A_1\left.\frac{d\hat{b}_1}{dt}\right|_G + A_2\left.\frac{d\hat{b}_2}{dt}\right|_G + A_3\left.\frac{d\hat{b}_3}{dt}\right|_G}_{\textbf{direction change of } B \textbf{ wrt } G\neq 0}$$

Exercise 14.3

1. Using the above example as a guide, express the vector derivative $\left.\frac{d\vec{A}^B}{dt}\right|_B$ in component form and indicate which terms are zero (if any) and which are non-zero.

2. Using the above example as a guide, express the vector derivative $\left.\frac{d\vec{A}^G}{dt}\right|_B$ in component form and indicate which terms are zero (if any) and which are non-zero.

Remember if the derivative is being taken with respect to the *same* frame as that of the basis vectors, the time derivatives of those basis vectors will be zero. *This is the key concept of the day.* Make sure you understand this. If not talk to your table mates or to an instructor before moving on.

14.4 Derivatives of the Basis Vectors for a Rotating Reference Frame Taken with Respect to an Inertial Frame

Last class we came up with relationships between two sets of basis vectors, $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$, that are aligned as drawn in the figure below. Their respective centers, O_G and O_B coincide at the same point. The angle θ defines the angle between $\hat{\imath}$ and $\hat{b}_1$. Note, in the this case, $\theta(t)$ is a function of time.

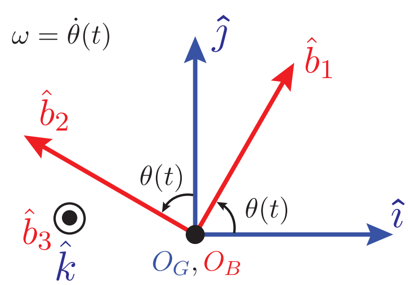

Figure 14.4: A body frame (red) is rotating relative to an inertial frame (blue) with an angular speed $\omega = \dot{\theta}(t)$. The basis vectors, $\hat{b}_3$ and $\hat{k}$, point out of the page.

Relationships (transformations) between the basis vectors were found as:

$$\hat{b}_1 = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \quad \hat{b}_2 = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}, \quad \text{and} \quad \hat{b}_3 = \hat{k}. \tag{14.17}$$

Now, let's say we want to take the time derivatives of $\hat{b}_1$ and $\hat{b}_2$ with respect to the inertial $\hat{\imath} - \hat{\jmath}$ frame. We need to evaluate

$$\left.\frac{d\hat{b}_1}{dt}\right|_G \tag{14.18}$$

where the vertical line with the G subscript indicates the derivative was taken with respect to the O_G or inertial reference frame. This derivative is itself a vector which describes how $\hat{b}_1$is changing in time. Let's try to reason out what is the magnitude and direction of vector (derivative). We can see that $\hat{b}_1$ is rotating like a clock hand counter-clockwise and we're told that the rate $\omega = \dot{\theta}$. That would be the magnitude. As to the direction, imagine $\hat{b}_1$ as drawn in Fig. 14.4 rotates the slightest amount (*i.e.*, $\Delta\theta$ is infinitesimal). In which direction does the point of the arrow head go? It moves perpendicularly to the left of $\hat{b}_1$ which is in the $\hat{b}_2$ direction. Therefore,

$$\left.\frac{d\hat{b}_1}{dt}\right|_G = \dot{\theta}\,\hat{b}_2. \tag{14.19}$$

This can be verified by transforming $\hat{b}_1$ into the inertial frame ($\hat{\imath} - \hat{\jmath}$) and taking the time derivative as follows:

$$\left.\frac{d\hat{b}_1}{dt}\right|_G = \frac{d}{dt}(\cos\theta\ \hat{\imath} + \sin\theta\ \hat{\jmath}) \tag{14.20}$$

Applying the product rule, we get:

$$\left.\frac{d\hat{b}_1}{dt}\right|_G = \frac{d}{dt}(\cos\theta)\ \hat{\imath} + \cos\theta\left.\frac{d\hat{\imath}}{dt}\right|_G + \frac{d}{dt}(\sin\theta)\ \hat{\jmath} + \sin\theta\left.\frac{d\hat{\jmath}}{dt}\right|_G \tag{14.21}$$

Note in the inertial frame $\hat{\imath}$ and $\hat{\jmath}$ do not change length (magnitude) and *do not change direction*. As a result, we see that:

$$\left.\frac{d\hat{\imath}}{dt}\right|_G = \left.\frac{d\hat{\jmath}}{dt}\right|_G = 0 \tag{14.22}$$

Plugging this into the previous equation, we get:

$$\left.\frac{d\hat{b}_1}{dt}\right|_G = -\dot{\theta}\sin\theta\ \hat{\imath} + \dot{\theta}\cos\theta\ \hat{\jmath} = \dot{\theta}\left(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\right) = \dot{\theta}\ \hat{b}_2 \tag{14.23}$$

Exercise 14.4

Using the previous example as a guide, evaluate the following derivative:

$$\left.\frac{d\hat{b}_2}{dt}\right|_G \tag{14.24}$$

14.5 Introduction to Angular Velocity

The relative rotation of one reference frame with respect to another can be described using an *angular velocity vector*, ${}^G\vec{\omega}^B$, which represents the rate of rotation of reference frame, B, with respect to (or as seen in) reference frame, G; see Fig. 14.5. Positive rotations follow the right-hand rule (RHR). If you point your thumb in the positive direction of the instantaneous axis of rotation, your fingers will curl in the direction of positive rotation. In general, ${}^G\vec{\omega}^B$ changes in *both* magnitude and direction as a function of time. It can be written in terms of a unit vector ($\hat{e}_{axis}$) that is aligned with the instantaneous axis of rotation as

$${}^G\vec{\omega}^B(t) = \omega\hat{e}_{axis} = \dot{\theta}\hat{e}_{axis} \tag{14.25}$$

where the magnitude is the angular speed (rad/s) which is called ω or is the time derivative of the angle of rotation, $\dot{\theta}$ in (rad/s). The fact that the angular velocity is a vector quantity (as opposed to the matrix representation of finite rotations) is a very important idea, with many implications that extend beyond the scope of this course/module.

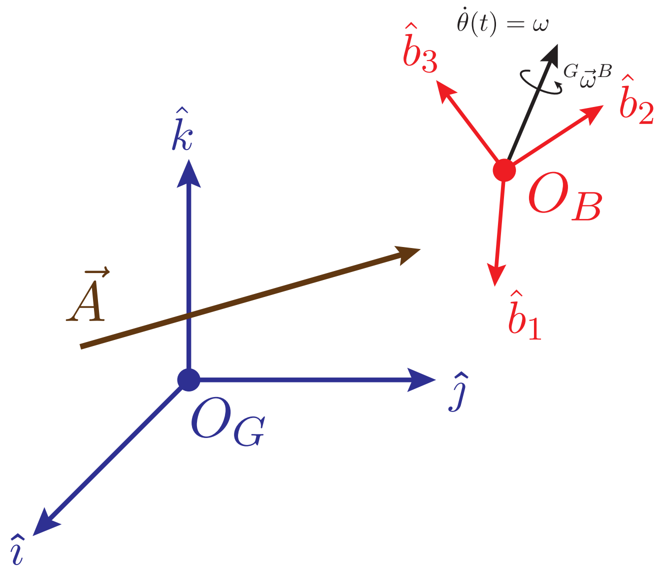

Figure 14.5: A body frame $B : \{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$ (red) is rotating relative to an inertial frame $O : \{O_G, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$ (blue). This rotation is characterized by the **angular** velocity, ${}^G\vec{\omega}^B$, which can be read as "the instantaneous rotation of frame B taken w/respect to frame G.

As with any vector, angular velocity can be written in terms of any reference frame, *e.g.*,

$$ {}^G\vec{\omega}^B(t) = \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3 = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}. \tag{14.26}$$

Remember: angular velocity tells you how one reference frame is rotating with respect to another. In other words, it tells you how a vector in one frame *changes direction* with respect to the other frame. This implies ${}^G\vec{\omega}^B = -{}^B\vec{\omega}^G$. So it's not surprising that it's related to time derivatives. Note, for 2D motion in a plane, the axis of rotation must *always* be perpendicular to the plane

$$ {}^G\vec{\omega}^B = \omega\hat{k} = \dot{\theta}\hat{k}. \tag{14.27}$$

This is *not* the case in 3D which leads to a lot of interesting behavior. **Author's note:** you are advised to pay attention to the slight change in notation that is specific to angular velocity. When writing out ${}^G\vec{\omega}^B$, the superscript B in the top right indicates the frame that the angular velocity vector $\vec{\omega}$ belongs to, not the set of basis vectors used to represent $\vec{\omega}$.

14.6 Vector Derivatives as a Cross-Product of Angular Velocity

Now we're going to see something that's pretty amazing–how a time derivative of a vector can be expressed as a cross-product of the angular velocity and itself. To illustrate this, let's return to our previous example:

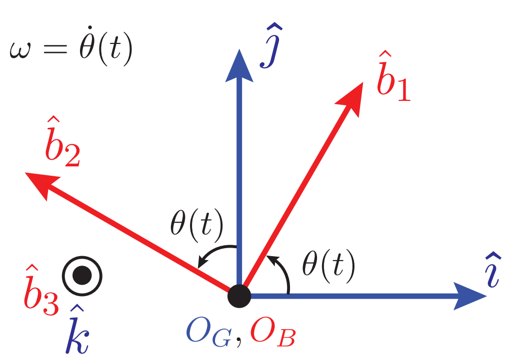

Figure 14.6: A body frame (red) is rotating relative to an inertial frame (blue) with an angular speed $\omega = \dot{\theta}(t)$. The basis vectors, $\hat{b}_3$ and $\hat{k}$, point out of the page.

The body frame $B : \{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$ rotates at rate $\dot{\theta}(t)$ about the $\hat{k}$ (out-of-page) axis with respect to the inertial frame $O : \{O_G, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$. Thus, the angular velocity *vector* of B with respect to G is given by:

$$ {}^{G}\vec{\omega}^{B}(t) = \dot{\theta}(t)\hat{k}. \tag{14.28} $$

Note that we chose $+\hat{k}$ instead of $-\hat{k}$ as the angular velocity direction because we must obey the right-hand-rule (increasing θ corresponds to a **counterclockwise** rotation, which corresponds to a $+\hat{k}$ rotation axis). Additionally, it should be noted that in 2D $\hat{k} = \hat{b}_3$ because all rotations in plane must be about an axis perpendicular to the plane. In the previous section, showed that the time derivatives of $\hat{b}_1$ and $\hat{b}_2$ w/respect to the inertial frame are given by:

$$ \left.\frac{d\hat{b}_1}{dt}\right|_G = \dot{\theta}\hat{b}_2 \;\text{ and }\; \left.\frac{d\hat{b}_2}{dt}\right|_G = -\dot{\theta}\hat{b}_1. \tag{14.29} $$

If you evaluate the cross products of the angular velocity vector and the two unit vector, you'll see that:

$$ {}^{G}\vec{\omega}^{B} \times \hat{b}_1 = \dot{\theta}\hat{k} \times \hat{b}_1 = \dot{\theta}\hat{b}_2 = \left.\frac{d\hat{b}_1}{dt}\right|_G \tag{14.30} $$

$$ {}^{G}\vec{\omega}^{B} \times \hat{b}_2 = \dot{\theta}\hat{k} \times \hat{b}_2 = -\dot{\theta}\hat{b}_1 = \left.\frac{d\hat{b}_2}{dt}\right|_G \tag{14.31} $$

This can be generalized to any unit vector ($\hat{u}$) as

$$ {}^{G}\vec{\omega}^{B} \times \hat{u} = \left.\frac{d\hat{u}}{dt}\right|_G \tag{14.32} $$

So *the derivative of a unit vector is the cross-product of the angular velocity and the unit vector itself.* This make sense as the unit vector is changing direction in time. That's pretty amazing, right? It means that we don't have to evaluate time derivatives of unit vectors when differentiating a vector. We can simply take its cross-product with the angular velocity vector!

Now let's see how this works with a generic vector, $\vec{A}$ which is written in terms of the body frame B as $\vec{A}^B = A_1\hat{b}_1 + A_2\hat{b}_2 + A_3\hat{b}_3$. Its derivative with respect to the inertial frame G is given by

$$ \left.\frac{d\vec{A}^{B}}{dt}\right|_{G} = \underbrace{\dot{A}_1\hat{b}_1 + \dot{A}_2\hat{b}_2 + \dot{A}_3\hat{b}_3}_{\text{change }\textbf{within}\ B} + \underbrace{A_1\left.\frac{d\hat{b}_1}{dt}\right|_{G} + A_2\left.\frac{d\hat{b}_2}{dt}\right|_{G} + A_3\left.\frac{d\hat{b}_3}{dt}\right|_{G}}_{\textbf{direction change of } B \textbf{ wrt } G \neq 0} $$

Applying our new angular velocity cross-product equation, we see that:

$$ A_1\left.\frac{d\hat{b}_1}{dt}\right|_{G} + A_2\left.\frac{d\hat{b}_2}{dt}\right|_{G} + A_3\left.\frac{d\hat{b}_3}{dt}\right|_{G} = A_1({}^{G}\vec{\omega}^{B} \times \hat{b}_1) + A_2({}^{G}\vec{\omega}^{B} \times \hat{b}_2) + A_3({}^{G}\vec{\omega}^{B} \times \hat{b}_3) \tag{14.33} $$

Since cross-products distribute across addition, we can factor out the angular velocity term from the sum:

$$ A_1\left.\frac{d\hat{b}_1}{dt}\right|_{G} + A_2\left.\frac{d\hat{b}_2}{dt}\right|_{G} + A_3\left.\frac{d\hat{b}_3}{dt}\right|_{G} = {}^{G}\vec{\omega}^{B} \times (A_1\hat{b}_1 + A_2\hat{b}_2 + A_3\hat{b}_3) = {}^{G}\vec{\omega}^{B} \times \vec{A}^{B} \tag{14.34} $$

Substituting this back into our expression for the derivative of $\vec{A}$ w/respect to frame G, we get:

$$ \left.\frac{d\vec{A}^{B}}{dt}\right|_{G} = \underbrace{\left.\frac{d\vec{A}^{B}}{dt}\right|_{B}}_{\text{change }\textbf{within}\ B} + \underbrace{{}^{G}\vec{\omega}^{B} \times \vec{A}^{B}}_{\text{direction change of } B}. \tag{14.35} $$

The change **within** B is what you see $\vec{A}^B$ doing when you are standing inside (are fixed) to the B reference frame. You don't notice that B is rotating. You only see what is happening **within** the reference frame. The direction change of B accounts for the rotation of the reference frame B with respect to G.

If $\vec{A}$ is position, then its derivative is velocity and can be written as

$$ \left.\frac{d\vec{r}^{B}}{dt}\right|_{G} = \left.\frac{d\vec{r}^{B}}{dt}\right|_{B} + {}^{G}\vec{\omega}^{B} \times \vec{r}^{B} \tag{14.36} $$

or

$$ \vec{v}^{B}|_{G} = \vec{v}^{B}|_{B} + {}^{G}\vec{\omega}^{B} \times \vec{r}^{B}. \tag{14.37} $$

Now we have some tools to calculate a rigid body's (*e.g.*, Neato's) velocity and angular velocity. By tying in what we know about curves in space, we can start thinking about driving our mobile robots along a path.

Exercise 14.5

Let's return to our ant on a rotating rod example from before.

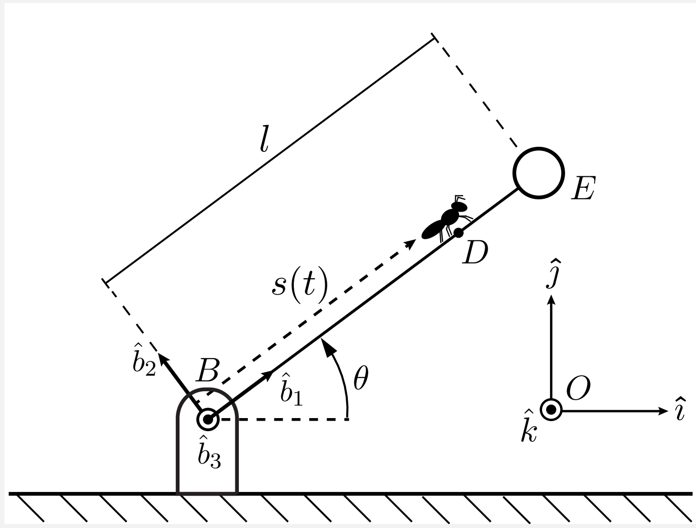

Figure 14.7

1. What is the angular velocity of frame R_B w/respect to frame R_O?

$$ {}^{O}\vec{\omega}^{B} = ? \tag{14.38}$$

2. Use your new understanding of frames to compute the velocity of the end of the rod, E (w/respect to frame R_O):

$$\vec{v}_E\Big|_O = ? \tag{14.39}$$

3. Use your new understanding of frames to compute the velocity of the ant, D (w/respect to frame R_O):

$$\vec{v}_D\Big|_O = ? \tag{14.40}$$

Exercise 14.6

Let's return to the problem of computing the motion parameters of the Neato (see Fig. 14.1). Here, we define an inertial frame $G : \{O_G, \{\hat{\imath}, \hat{\jmath}, \hat{k}\}\}$ and a body-fixed frame $B : \{O_B, \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}\}$, where the body fixed basis vectors are equal to the unit tangent/normal/binormal vectors of the path:

$$\hat{b}_1 = \hat{T}^B, \qquad \hat{b}_2 = \hat{N}^B \tag{14.41}$$

If the angular velocity of the Neato w/respect to the inertial frame is given by ${}^{G}\vec{\omega}^{B}$, then it can be shown that:

$$ {}^{G}\vec{\omega}^{B} = \hat{T}^B \times \frac{d\hat{T}^B}{dt}\Big|_G \tag{14.42}$$

This equation is useful (for the purposes of Rainbow Road) because it allows us compute the angular velocity of the Neato, given the unit tangent vector and its time derivative.
Starting from Eqn. 14.35, derive Eqn. 14.42 using vector operations. You may find the following vector triple product identity useful:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{14.43}$$

Hint: If two vectors are perpendicular, then their dot product is zero. Additionally, the dot product of a unit vector with itself is one.