

Diss. ETH No. 20089

Ramsey games in random graphs

A dissertation submitted to

ETH ZÜRICH

for the degree of

DOCTOR OF SCIENCES

presented by

TORSTEN MÜTZE

Dipl.-Ing. für Informationssystemtechnik, TU Dresden

born 05.02.1981

citizen of Germany

accepted on the recommendation of

Prof. Dr. Angelika Steger, examiner

Prof. Dr. József Balogh, co-examiner

2011

Contents

Abstract	iv
Zusammenfassung	vi
Acknowledgments	ix
List of symbols	x
Chapter 1. Introduction	1
1.1. Ramsey theory	2
1.2. The chromatic number problem	2
1.3. Average case analysis and random graphs	3
1.4. Probabilistic and deterministic graph coloring games	3
1.5. Results	4
1.6. Outlook	5
1.7. Organization of this thesis	6
Chapter 2. Background and related work	7
2.1. From Ramsey numbers to Ramsey densities	7
2.2. Vertex-Ramsey properties	8
2.3. Online Ramsey games	9
2.4. Random graphs	9
2.5. Ramsey properties of random graphs	10
Chapter 3. The edge-coloring setting	12
3.1. Introduction	12
3.2. Proof of Theorem 3.4 and Theorem 3.5	17
3.3. Examples	20
3.4. Computation of $k^*(F, r)$	23
3.5. Path-avoidance games	24
3.6. Open questions	27
Chapter 4. The vertex-coloring setting	29
4.1. Introduction	29
4.2. Proof ideas	34
4.3. Computing the online vertex-Ramsey density	39
4.4. Builder in the deterministic game	66
4.5. Painter in the deterministic game	75
4.6. Proof of Theorem 4.3	94
Chapter 5. The path-avoidance vertex-coloring game	99
5.1. Introduction	99
5.2. Basic observations	102

5.3. A general recursion	103
5.4. Analyzing the recursion	109
Bibliography	120

Abstract

Playing games has ever since been an important part of humans' social interaction, and over the last centuries the theoretical analysis of games has fascinated and stimulated researchers from different scientific backgrounds. Somewhat surprisingly, for certain combinatorial games the outcome of the game between two optimal players ('clever' vs. 'clever') is the same as if both players played randomly ('random' vs. 'random'), and insights from the probabilistic analysis can be turned into efficient deterministic winning strategies. The main contribution of this thesis is building a similar bridge between two previously disconnected areas of research: the world of probabilistic one-player games ('clever' vs. 'random') in which the goal is to avoid some given local substructure, and the afore-mentioned world of deterministic two-player games ('clever' vs. 'clever'). This link is established by replacing randomness by a deterministic adversary who is subject to certain restrictions inherited from the random setting. Exploiting this connection allows us to transfer insights and techniques between the two worlds and to derive new results in each of them.

We demonstrate the full strength of our approach by considering a class of games where the edges or vertices of a graph have to be colored online (i.e., one after the other without seeing the entire graph in advance), where we have fixed number of colors at our disposal and our goal is to avoid a monochromatic copy of some fixed graph (this is the forbidden local substructure). The edge-coloring version of this notion of graph colorability is one of the central topics of *Ramsey theory*, and the vertex-coloring version generalizes the *chromatic number problem*, one of the 21 algorithmic problems shown to be \mathcal{NP} -complete by Karp in his 1972 landmark paper.

In the world of *probabilistic* games, the above coloring problem was first considered by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali, who introduced the following *one-player game* ('clever' vs. 'random'): To an initially empty graph on n vertices randomly chosen edges are added one after the other. The player, henceforth called Painter, has to color them immediately and irrevocably with one of $r \geq 2$ available colors with the goal of avoiding a monochromatic copy of some fixed graph F . The authors determined the typical number of steps Painter can 'survive' this process with an optimal strategy for the special case where F is a triangle and $r = 2$ colors are available — this typical number of steps is formalized in the notion of a *threshold function*. This problem was further investigated by Marcinišzyn, Spöhel and Steger, who proved explicit threshold functions for a large class of graphs F , including cliques and cycles, and $r = 2$ colors. Marcinišzyn and Spöhel also introduced the vertex-coloring version of this problem, proving threshold results for a similarly large class of graphs, and $r \geq 2$ colors. As it turns out, in all cases where the online threshold is known (in the edge- and the vertex-coloring setting), a simple greedy strategy is optimal for Painter.

In the world of *deterministic* games, Beck and independently Kurek and Ruciński introduced the following *two-player game* ('clever' vs. 'clever'): The two players are called Painter and Builder. Builder in each step adds an edge to an initially empty graph and as before Painter has to color these edges with the goal of avoiding a monochromatic copy of F , while Builder's goal is to

enforce such a copy. The question for the minimum number of steps Builder needs to win this game and the question whether Builder can still win if we impose certain additional restrictions on him received considerable attention in the literature.

In this thesis we continue the study of the online F -avoidance game in random graphs, both in the edge- and the vertex-coloring version. Our main contribution is establishing a link between the probabilistic one-player game Painter vs. the random graph ('clever' vs. 'random') and a suitably defined deterministic two-player game Painter vs. Builder ('clever' vs. 'clever'), where Builder is subject to certain restrictions inherited from the random setting. We emphasize here that apart from these specific applications to graph coloring games our idea to replace randomness by a deterministic adversary can also be fruitfully applied to other scenarios like random hypergraphs or random subsets of integers, and to other random processes involving choices like e.g. the well-studied Achlioptas process.

In the deterministic two-player game we consider, we impose the following *density restriction* on Painter's adversary Builder: We require that throughout the game, the ratio of edges to vertices in every subgraph of the evolving graph is bounded by d , for some fixed real number $d > 0$ known to both players.

Concerning the edge-coloring problem, our first main result is that for any F and r , the existence of a winning strategy for Builder in the deterministic F -avoidance game with r colors and density restriction d implies an upper bound of $n^{2-1/d}$ on the threshold of the probabilistic game. We thus obtain a new general approach to derive upper bounds for the online threshold. Recently, Balogh and Butterfield successfully applied our result and proved the first nontrivial upper bounds for the case where F is a triangle and $r \geq 3$ colors are available.

For the vertex-coloring problem, we obtain the following general threshold result, which is our second main result: For any F and r , the threshold of the probabilistic online problem is given by $n^{2-1/m_1^*(F,r)}$, where the parameter $m_1^*(F,r)$, referred to as the *online vertex-Ramsey density*, is defined as the infimum over all d for which Builder has a winning strategy in (the vertex-coloring version of) the deterministic F -avoidance game with r colors and density restriction d . As our third main result we prove that for any F and r , the parameter $m_1^*(F,r)$ is a computable rational number, and the infimum in its definition is attained as a minimum. Our lower bound proof of the online threshold is algorithmic, i.e., we obtain polynomial-time algorithms for computing colorings of the random graph with no monochromatic copies of F online in the entire regime below the corresponding thresholds (while coloring above the threshold is impossible).

Still concerning the vertex-coloring problem, it is known that for a large class of graphs F , including cliques, cycles, complete bipartite graphs, hypercubes, wheels and stars of arbitrary size, a simple greedy strategy is optimal for Painter, implying simple closed formulas for the parameter $m_1^*(F,r)$ in these cases. As our last main result we show that in the innocent-looking case where $F = P_\ell$ is a (long) path, the greedy strategy fails quite badly and the parameter $m_1^*(P_\ell,r)$ exhibits a surprisingly complex behavior, giving some evidence that a general closed formula for the parameter $m_1^*(F,r)$ does not exist.

Zusammenfassung

Spiele war schon immer ein wichtiger Teil menschlicher sozialer Interaktion, und über die letzten Jahrhunderte hinweg hat die theoretische Analyse von Spielen Forscher mit verschiedensten wissenschaftlichen Hintergründen fasziniert und stimuliert. Überraschenderweise ist für gewisse kombinatorische Spiele der Ausgang des Spiels zwischen zwei optimalen Spielern ('intelligent' gegen 'intelligent') derselbe wie wenn beide Spieler zufällig spielen ('zufällig' gegen 'zufällig'), und Einsichten aus der wahrscheinlichkeitstheoretischen Analyse können in effiziente deterministische Gewinnstrategien übersetzt werden. Der Hauptbeitrag dieser Arbeit ist das Schlagen einer ähnlichen Brücke zwischen zwei vorher nicht verbundenen Forschungsgebieten: dies ist zum einen die Welt der probabilistischen Ein-Personen-Spiele ('intelligent' gegen 'zufällig'), bei denen eine gegebene lokale Teilstruktur vermieden werden soll, und zum anderen die vorher genannte Welt der deterministischen Zwei-Personen-Spiele ('intelligent' gegen 'intelligent'). Diese Verbindung wird hergestellt, indem die Zufälligkeit durch einen deterministischen Gegner ersetzt wird, der gewissen aus dem probabilistischen Spiel herrührenden Einschränkungen unterliegt. Indem wir diese Verbindung ausnutzen, können wir Einsichten und Techniken zwischen beiden Welten transferieren und in jeder von ihnen neue Resultate ableiten.

Wir demonstrieren die ganze Stärke unserer Herangehensweise, indem wir eine Klasse von Spielen betrachten, bei denen die Kanten oder Knoten eines Graphen online gefärbt werden müssen (online heisst hier nach und nach, d.h. ohne vorherige Kenntnis des gesamten Graphen), wobei eine feste Anzahl an Farben zur Verfügung steht und wir einfarbige Kopien eines festen Graphen vermeiden wollen (dies ist die verbotene lokale Teilstruktur). Die Kantenfärbungsversion dieses Begriffs von Graphenfärbbarkeit ist eines der zentralen Inhalte der *Ramsey-Theorie*, und die Knotenfärbungsversion verallgemeinert das *Problem der chromatischen Zahl*, eines der 21 algorithmischen Probleme, die Karp in seiner bahnbrechenden Arbeit aus dem Jahre 1972 als \mathcal{NP} -vollständig nachwies.

In der Welt der *probabilistischen* Spiele wurde das obige Färbungsproblem zuerst von Friedgut, Kohayakawa, Rödl, Ruciński und Tetali betrachtet, die das folgende *Ein-Personen-Spiel* ('intelligent' gegen 'zufällig') einführten: Zu einem anfangs leeren Graphen auf n Knoten werden nach und nach zufällig ausgewählte Kanten hinzugefügt. Der Spieler, den wir im Weiteren Painter nennen, muss diese sofort und unwiderruflich mit einer von $r \geq 2$ verfügbaren Farben färben mit dem Ziel, einfarbige Kopien eines festen Graphen F zu vermeiden. Die Autoren bestimmten die typische Anzahl an Schritten, für die Painter diesen Prozess mit einer optimalen Strategie 'überleben' kann für den Spezialfall, bei dem F ein Dreieck ist und $r = 2$ Farben zur Verfügung stehen — diese typische Anzahl an Schritten wird durch den Begriff der *Schwellenwertfunktion* formalisiert. Dieses Problem wurde weiter von Marciniszyn, Spöhel und Steger untersucht, die explizite Schwellenwertfunktionen für eine grosse Klasse von Graphen F , einschliesslich Cliques und Kreise, und für $r = 2$ Farben bewiesen. Marciniszyn und Spöhel führten auch die Knotenfärbungsversion dieses Problems ein, für die sie Schwellenwertresultate für eine ähnlich grosse Klasse von Graphen und für $r \geq 2$ Farben bewiesen. Es stellt sich heraus, dass in allen Fällen,

wo der Online-Schwellenwert bekannt ist (im Kanten- und im Knotenfärbungs-Szenario), eine einfache Greedy-Strategie für Painter optimal ist.

In der Welt der *deterministischen* Spiele haben Beck und unabhängig auch Kurek und Ruciński das folgende *Zwei-Personen-Spiel* (‘intelligent’ gegen ‘intelligent’) eingeführt: Die beiden Spieler heissen Painter und Builder. Builder fügt in jedem Schritt eine Kante zu einem anfangs leeren Graphen hinzu, und wie vorher muss Painter diese Kanten mit dem Ziel färben, einfarbige Kopien von F zu vermeiden, während Builders Ziel das Erzwingen einer solchen Kopie ist. Die Frage nach der minimalen Anzahl an Schritten, die Builder benötigt, um dieses Spiel zu gewinnen, und die Frage ob Builder das Spiel noch gewinnen kann, wenn wir ihm gewisse zusätzliche Beschränkungen auferlegen, hat seither in der Literatur beachtliche Aufmerksamkeit erlangt.

In dieser Arbeit führen wir die Untersuchung des Online- F -Vermeidungsspiels in Zufallsgraphen fort, sowohl in der Kanten- als auch der Knotenfärbungsversion. Unser Hauptbeitrag ist das Herstellen einer Verbindung zwischen dem probabilistischen Ein-Personen-Spiel Painter gegen den Zufallsgraphen (‘intelligent’ gegen ‘zufällig’) und einem geeignet definierten deterministischen Zwei-Personen-Spiel Painter gegen Builder (‘intelligent’ gegen ‘intelligent’), wobei Builder gewissen aus dem probabilistischen Spiel herrührenden Einschränkungen unterliegt. Wir betonen an dieser Stelle, dass unsere Idee, die Zufälligkeit durch einen deterministischen Gegner zu ersetzen, abgesehen von diesen konkreten Anwendungen auf Graphenfärbungsspiele auch fruchtbringend auf andere Szenarien wie beispielsweise zufällige Hypergraphen oder zufällige Teilmengen der ganzen Zahlen angewendet werden kann, sowie auch auf andere Zufallsprozesse, die Auswahlmöglichkeiten beinhalten, wie zum Beispiel der gut untersuchte Achlioptas-Prozess.

In dem deterministischen Zwei-Personen-Spiel, das wir betrachten, erlegen wir Painters Gegner Builder die folgende *Dichtebeschränkung* auf: Wir verlangen, dass während des gesamten Spiels das Verhältnis von Kanten zu Knoten in jedem Teilgraphen des wachsenden Graphen durch d beschränkt ist, wobei $d > 0$ eine feste reelle Zahl ist, die beiden Spielern bekannt ist.

Bezüglich des Kantenfärbungsproblems ist unser erstes Hauptresultat, dass für jedes F und r die Existenz einer Gewinnstrategie für Builder im deterministischen F -Vermeidungsspiel mit r Farben und Dichtebeschränkung d eine obere Schranke von $n^{2-1/d}$ für den Schwellenwert des probabilistischen Spiels impliziert. Wir erhalten so eine neue allgemeine Methode, um obere Schranken für den Online-Schwellenwert abzuleiten. Kürzlich haben Balogh und Butterfield unser Resultat erfolgreich verwendet und damit die ersten nichttrivialen oberen Schranken bewiesen für den Fall, bei dem F ein Dreieck ist und $r \geq 3$ Farben verfügbar sind.

Für das Knotenfärbungsproblem erhalten wir das folgende allgemeine Schwellenwertresultat, welches unser zweites Hauptresultat ist: Für jedes F und r ist der Schwellenwert des probabilistischen Online-Problems durch $n^{2-1/m_1^*(F,r)}$ gegeben, wobei der Parameter $m_1^*(F, r)$, genannt die *Online-Knoten-Ramsey-Dichte*, definiert ist als Infimum über alle d , für die Builder eine Gewinnstrategie im deterministischen F -Vermeidungsspiel (genauer, der Knotenfärbungsversion davon) mit r Farben und Dichtebeschränkung d besitzt. Als unser drittes Hauptresultat beweisen wir, dass der Parameter $m_1^*(F, r)$ für jedes F und r eine berechenbare rationale Zahl ist, und dass das Infimum in der Definition als Minimum angenommen wird. Unser Beweis für die untere Schranke des Online-Schwellenwerts ist algorithmisch, d.h. wir erhalten polynomielle Algorithmen, die im gesamten Regime unterhalb der entsprechenden Schwellenwerte Färbungen des Zufallsgraphen online berechnen, die keine einfarbigen Kopien von F enthalten (während das Färben oberhalb des Schwellenwerts unmöglich ist).

Für das Knotenfärbungsproblem ist weiterhin bekannt, dass für eine grosse Klasse von Graphen F , einschliesslich Cliques, Kreise, vollständige bipartite Graphen, Hyperwürfel, Räder und Sterne beliebiger Grösse, eine einfache Greedy-Strategie für Painter optimal ist, woraus einfache geschlossene Formeln für den Parameter $m_1^*(F, r)$ in diesen Fällen folgen. Als unser letztes Hauptresultat zeigen wir, dass im unschuldig erscheinenden Fall, bei dem $F = P_\ell$ ein (langer) Pfad ist, die Greedy-Strategie ziemlich gravierend fehlschlägt, und dass der Parameter $m_1^*(F, r)$ ein überraschend komplexes Verhalten offenbart, ein Indiz dafür, dass eine allgemeine geschlossene Formel für den Parameter $m_1^*(F, r)$ nicht existiert.

Acknowledgments

First of all, I would like to thank my supervisor Angelika Steger for her guidance and support during the last five years. She gave me the opportunity to work in this field of ‘discrete’ problems which I very much loved to work on. I greatly benefitted from her mentorship and from the excellent research environment in her group. She introduced me to the customs of academia, and gave advice on various subjects. I am also very thankful to her for giving me a lot of freedom in pursuing research interests on my own and with other co-authors.

I am very grateful to József Balogh for agreeing to act as the co-examiner of this thesis. It was great seeing him and his student Jane Butterfield applying and building on our own results.

I also thank my co-authors, who have all been very pleasant to work with — through our collaboration I learned a lot from each of them: Gustavo Alonso, Michael Belfrage, Dan Hefetz, Fabian Kuhn, Thomas Rast, Justus Schwartz, Reto Spöhel, Patrick Stüdi, Henning Thomas, Ueli Peter and Franziska Weber. In particular, I am grateful to Reto Spöhel for our continuous and fruitful collaboration on Ramsey games — I am glad we share the interest and enthusiasm in those problems, and the view on how proofs and papers should be written in general.

I thank Reto Spöhel and Konstantinos Panagiotou for inviting me to the Max-Planck-Institut für Informatik — I very much enjoyed their hospitality during this week in Saarbrücken.

I am very thankful to the past and present members of the CSA group for creating such a great working atmosphere through interesting discussions, enjoyable lunch and cookie breaks, challenging table soccer matches, exhausting cycling trips and long Dominion game evenings. So, thank you Nicla Bernasconi, Stefanie Gerke, Luca Gugelmann, Dan Hefetz, Florian Jug, Christoph Krautz, Fabian Kuhn, Johannes Lengler, Julian Lorenz, Martin Marciniszyn, Konstantinos Panagiotou, Ueli Peter, Thomas Rast, Jan Remy, Justus Schwartz, Reto Spöhel, Henning Thomas and Andreas Weißl. I also thank Beat Gfeller, Rastislav Šrámek and Anna Zych from Peter Widmayer’s group for sharing their offices with me and/or adventurous sledge rides in the Swiss alps.

I am indebted to Reto Spöhel and Ueli Peter for proofreading parts of my thesis, and to Marianna Berger, Barbara Heller and Andrea Salow for help with administrative matters.

Last but not least I want to thank my parents for their love and support, my sister Annekathrin for the great time we had when she lived in Zurich, and also my dear wife Regula for her love and encouragement, her interest in what I was working on, and the joyful years we had in the past and will spend together in the future.

List of symbols

The following table summarizes some frequently appearing symbols and gives references to where the definitions can be found.

Symbol	Definition or reference
$e(G)$	number of edges of G
$v(G)$	number of vertices of G
K_ℓ	clique on ℓ vertices
C_ℓ	cycle on ℓ vertices
S_ℓ	star with ℓ rays
P_ℓ	path on ℓ edges (in Chapter 3) / ℓ vertices (in Chapter 5)
F	the forbidden graph
r	the number of colors
$R(F, r), R(F) := R(F, 2)$	Ramsey number of F and r , (2.1)
$R_e(F, r)$	size Ramsey number of F and r , (2.2)
$R_e^*(F, r)$	online size Ramsey number of F and r , Section 2.3
$m_2^o(F, r)$	Ramsey density of F and r , (2.3)
$m_1^o(F, r)$	vertex-Ramsey density of F and r , (2.4)
$G_{n,m}, G_{n,p}$ a.a.s.	random graph models, Section 2.4 asymptotically almost surely, i.e., with probability $1 - o(1)$ as $n \rightarrow \infty$
$f(n) \ll g(n)$	$f(n) = o(g(n))$
$f(n) \gg g(n)$	$f(n) = \omega(g(n))$
$f(n) \asymp g(n)$	$f(n) = \Theta(g(n))$
$m(F)$	(2.5)
$m_2(F)$	(2.6)
$m_1(F)$	(2.7)
$\overline{m}_2(F, r)$	greedy density for F and r in the edge case, (3.2)
$m_2^*(F, r)$	online Ramsey density of F and r , (3.12)
$k^*(F, r)$	minimum tree size restriction (number of edges), Section 3.1.2 $m_2^*(F, r) = \frac{k^*(F, r)}{k^*(F, r) + 1}$
$\overline{m}_1(F, r)$	greedy density for F and r in the vertex case, (4.1)
$m_1^*(F, r)$	online vertex-Ramsey density of F and r , (4.3)
$k^*(F, r)$	minimum tree size restriction (number of vertices), Section 5.1.1 $m_1^*(F, r) = \frac{k^*(F, r) - 1}{k^*(F, r)}, (5.2)$

CHAPTER 1

Introduction

Playing games has ever since been an important part of humans' social interaction, and over the last centuries the theoretical analysis of games has fascinated and stimulated researchers from different scientific backgrounds. Game theory in the classical sense was largely influenced by the works of von Neumann and Morgenstern [vN28, vNM44] and Nash [Nas51]. It mainly deals with games where players are lacking information because they move simultaneously (unaware of the actions of the other players) or because one player knows something the others do not (such as a secret hand of cards in Poker). On the other hand, games of perfect and complete information (such as Chess, Tic-Tac-Toe or Nim), also called combinatorial games, whose outcome can in principle be predicted by exhaustively enumerating all possible ways the game may evolve, can be roughly classified into two main categories: games that can be viewed as the sum of several small games ('Nim-like games'), for which a beautiful algebraic theory has been developed (see the works of Bouton [Bou02], Sprague [Spr36], Grundy [Gru39], Berlekamp, Conway and Guy [BCG04]), and games for which such a decomposition is not possible and we are facing 'combinatorial chaos'. This latter class includes most positional games where two players alternately claim elements of some board (Beck [Bec08] refers to them as 'Tic-Tac-Toe-like games'). As it turns out, probabilistic methods are a very powerful tool in dealing with 'combinatorial chaos' in positional games. A particularly striking phenomenon, first pointed out by Chvátal and Erdős [CE78] and later covered in great detail by Beck [Bec93, Bec94, Bec96], is that for certain positional games the outcome of the game between two optimal players ('clever' vs. 'clever') is the same as if both players played randomly ('random' vs. 'random'), and insights from the probabilistic analysis can be turned into efficient deterministic winning strategies.

The main contribution of this thesis is building a similar bridge between two previously disconnected areas of research: the world of probabilistic one-player games ('clever' vs. 'random') in which the goal is to avoid some given local substructure, and the afore-mentioned world of deterministic two-player games ('clever' vs. 'clever'). This link is established by replacing randomness by a deterministic adversary who is subject to certain restrictions inherited from the random setting. Exploiting this connection allows us to transfer insights and techniques between the two worlds and to derive new results in each of them. Already our earlier work [MST11], where we disproved a conjecture of Krivelevich, Loh and Sudakov [KLS09], can be seen as a variant of this approach — in this thesis we develop it further and demonstrate its full strength by considering a class of graph coloring games. In these games, the edges or vertices of a graph have to be colored online (i.e., one after the other without seeing the entire graph in advance), where we have a fixed number $r \geq 2$ of colors at our disposal and our goal is to avoid a monochromatic copy of some fixed graph F (this is the forbidden local substructure). Before presenting our results we retrace the most important lines of research that motivated these problems.

1.1. Ramsey theory

Coloring the *edges* of a graph with no monochromatic subgraphs, i.e., with no subgraphs whose edges all receive the same color, is one of the classical problems in *Ramsey theory*, which itself is a fundamental branch of extremal combinatorics. The most basic theorem in this field, proved by the British mathematician Frank P. Ramsey in his fundamental paper [Ram30] and marking the hour of birth of the whole theory, states that for any integer ℓ , there is an integer $N = N(\ell)$ such that in *any* two-coloring of the edges of a complete graph on N vertices, a monochromatic clique K_ℓ on ℓ vertices can be found. The smallest such integer N is defined as the *Ramsey number* $R(K_\ell)$. Even with the help of today's supercomputers, researchers have not been able to determine the exact value of $R(K_5)$, and quantifying the growth of the Ramsey number $R(K_\ell)$ as ℓ tends to infinity is still one of the most notorious open questions. An abundance of work has been devoted to several variations and generalizations of this problem, e.g., if the forbidden subgraph is not a clique, but a cycle, say, or even more general, any fixed graph F , and if not only two, but any fixed number r of colors at our disposal. There is also a rich account of beautiful results concerning other extremal properties of graphs that have the Ramsey property with respect to some F and r (i.e., which have the property that any r -coloring of their edges contains a monochromatic copy of F). Here one is not only interested in how few vertices such a graph can possibly have (this is the question for the Ramsey number), but also in how few edges it can possibly have, or how locally or globally sparse it can possibly be. In Chapter 2 we will present a few selected results and more references in this direction.

1.2. The chromatic number problem

Coloring the *vertices* of a graph subject to certain restrictions is one of the archetypical problems in computer science, with a wide range of practical applications. The *chromatic number* of a graph G is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices receive the same color. Such a coloring is usually called a *proper* coloring of G . The chromatic number is one of the most studied graph parameters in the literature, and variations of the graph coloring problem arise in many algorithmic problems [Bol98, JT95, Tho01].

The problem to decide whether a given graph G is properly colorable with a given number r of colors is one of the 21 problems shown to be \mathcal{NP} -complete by Karp in his landmark paper [Kar72]. In fact, the problem is \mathcal{NP} -complete even for any fixed integer $r \geq 3$. Approximation algorithms cannot be of much help either — it was shown by Feige and Kilian [FK98] that for any fixed $\epsilon > 0$ it is impossible to approximate the chromatic number of a graph on n vertices up to a factor of $n^{1-\epsilon}$ in polynomial time, unless $\mathcal{NP} \subseteq \mathcal{ZPP}$. (\mathcal{ZPP} is the class of languages decidable by a randomized expected polynomial-time algorithm that makes no errors.)

In this work we are concerned with the following generalized notion of colorability: We say that a coloring of a graph G is *valid* with respect to some given graph F if it contains no monochromatic copy of F , i.e., if there is no copy of F in G whose vertices all receive the same color. Note that a proper coloring in the usual sense is a coloring that is valid with respect to a single edge ($F = K_2$). The problem of deciding whether a given graph G has a valid coloring w.r.t. some F with a given number r of colors is therefore clearly \mathcal{NP} -complete in general (see also [Rut86]).

One motivation for studying these generalized colorability properties comes of course from Ramsey theory. In fact, like in the edge-coloring case mentioned above also the vertex-coloring setting

offers many interesting results concerning extremal properties of graphs that have the vertex-Ramsey property w.r.t. some F and r , i.e., graphs that have *no* valid coloring (we will present some of those results in Chapter 2). We shall see that there is in fact a rich interplay between edge- and vertex-coloring problems of this kind.

1.3. Average case analysis and random graphs

We have seen that from a worst-case perspective, the chromatic number problem and many related coloring problems are intractable. A more promising approach to address these problems from a theoretical point of view is therefore to study their *average-case complexity*, i.e., their typical difficulty on random graphs sampled from an appropriate distribution.

The systematic study of random graphs was initiated by two Hungarian mathematicians, Paul Erdős and Alfred Rényi, in a series of papers in the 1960's (see [KR97] for an account of these eight fundamental papers). Since then the subject has grown into an independent and fast-growing field of research. A recurring theme in the theory of random graphs is that of a *phase transition*: With an increasing edge density, at a certain point called the *threshold*, some property of the random graph changes abruptly (e.g., the property of being connected, containing a Hamilton cycle or being not properly 4-colorable). Today, explicit threshold functions are known for many natural graph properties.

We have argued before that edge- and vertex-coloring problems where a monochromatic copy of some fixed graph F should be avoided are in general very difficult, and many questions in this area are still wide open (even for special cases like $F = K_\ell$ and $r = 2$ colors). For random graphs, however, both problems have been solved in full generality. Rödl and Ruciński [RR93, RR95] proved explicit threshold functions, valid for any F and r , for the random graph having the Ramsey property with respect to F and r (i.e., for edge densities below the threshold, the edges of the random graph can be r -colored without a monochromatic copy of F , and for edge densities above the threshold this is impossible). An analogous result for the vertex-coloring problem was obtained by Łuczak, Ruciński and Voigt [LRV92]. Somewhat surprisingly, these thresholds do not depend on the number r of colors.

1.4. Probabilistic and deterministic graph coloring games

More recently, the above coloring problems have also been studied in an online setting, where the edges or vertices of a graph have to be colored one after the other without knowing the entire graph in advance. The main motivation for considering the online setting is to understand and quantify how much harder coloring gets, if we are restricting the global view of the problem instance to a more local view.

In the *probabilistic* world, Friedgut, Kohayakawa, Rödl, Ruciński and Tetali [FKR⁺03] introduced the following *one-player game* ('clever' vs. 'random'): To an initially empty graph on n vertices randomly chosen edges are added one after the other. The player, henceforth called Painter, has to color them immediately and irrevocably with one of $r \geq 2$ available colors with the goal of avoiding a monochromatic copy of some fixed graph F . The authors proved that for the case $F = K_3$ and $r = 2$, the typical number of steps Painter can 'survive' this process with an optimal strategy is $n^{4/3} = n^{1.333\dots}$, much less than the corresponding offline threshold $n^{3/2} = n^{1.5}$ given by the Rödl/Ruciński result [RR93, RR95] mentioned above. For comparison, the first (not necessarily monochromatic) triangle appears already after roughly n random

edges have been added. This problem was further investigated by Marciniszyn, Spöhel and Steger [MSS09a, MSS09b], who proved explicit threshold functions for a large class of graphs F , including cliques and cycles of arbitrary fixed size, and $r = 2$ colors. As it turns out, for those graphs a simple greedy strategy is optimal for Painter. Already the lower bound given by the greedy strategy shows that for *any* F , the order of magnitude of the online threshold approaches the offline threshold as the number r of colors increases (in particular, the online thresholds depend on r). E.g., for $F = K_3$, a greedy Painter can survive for $n^{1.444\dots}$ steps with $r = 3$ colors, for $n^{1.481\dots}$ steps with $r = 4$ colors and for $n^{1.499\dots}$ steps with $r = 10$ colors.

The corresponding vertex-coloring problem, where the vertices of an initially hidden random graph are revealed one after the other and Painter has to color them, was introduced by Marciniszyn and Spöhel in [MS10], where explicit threshold functions were proved for a large class of graphs F , including cliques and cycles, and any fixed $r \geq 2$. Also in those cases a simple greedy strategy is optimal for Painter, and a similar convergence of the online threshold towards the offline threshold can be observed.

In the *deterministic* world, Beck [Bec83b] and independently Kurek and Ruciński [KR05] introduced the following *two-player game* (‘clever’ vs. ‘clever’): The two players are called Painter and Builder. Builder in each step adds an edge to an initially empty graph (which has an infinite number of vertices) and as before Painter has to color these edges with the goal of avoiding a monochromatic copy of F , while Builder’s goal is to enforce such a copy. By Ramsey’s theorem, Builder can of course win by simply building a huge complete graph, but can he win more quickly, or can he still win if we impose certain restrictions on him? We will present several results and references in this direction in Chapter 2. Probably the most interesting question in this area is whether or not Builder can win the game where $F = K_\ell$ and $r = 2$ colors are available within substantially less than $\binom{R(K_\ell)}{2}$ steps (Conlon [Con10] recently showed that the answer is ‘yes’ for infinitely many values of ℓ).

1.5. Results

In this thesis we continue the study of the online F -avoidance game in random graphs, both in the edge- and the vertex-coloring version. Our main contribution is establishing a link between the probabilistic one-player game Painter vs. the random graph (‘clever’ vs. ‘random’) and a suitably defined deterministic two-player game Painter vs. Builder (‘clever’ vs. ‘clever’), where Builder is subject to certain restrictions inherited from the random setting. As already mentioned, our idea to replace randomness by a deterministic adversary can be exploited also for other random processes in which the goal is to avoid some given local substructure. We will briefly discuss the potential and limitations of our approach in the next section, after summarizing our results for the above-mentioned graph coloring games.

In the deterministic two-player game we consider, we impose the following *density restriction* on Painter’s adversary Builder (the restriction arises naturally, if one thinks of the original probabilistic one-player game): We require that throughout the game, the ratio of edges to vertices in every subgraph of the evolving graph is bounded by d , for some fixed real number $d > 0$ known to both players.

1.5.1. A new upper bound approach for the edge-coloring problem. For the edge-coloring setting, our first main result is that for any F and r , the existence of a winning strategy for Builder in the deterministic F -avoidance game with r colors and density restriction d implies

an upper bound of $n^{2-1/d}$ on the threshold of the probabilistic game. Unlike previous approaches this result covers the game with an arbitrary number r of colors and also extends to graphs F for which the greedy strategy is not optimal. We thus obtain a new general approach to derive upper bounds for the online threshold.

Recently, our result has been successfully applied by Balogh and Butterfield [BB10], who proved the first nontrivial upper bounds for the case where $F = K_3$ and $r \geq 3$ colors are available.

We also show that for the case of cycles and two colors, and for the case of forests and an arbitrary number of colors, the best upper bound that can be derived in this way is indeed the threshold of the probabilistic game. For the general case this question remains open.

The results for the edge-coloring problem are joint work with Michael Belfrage and Reto Spöhel [BMS11⁺].

1.5.2. A general threshold result for the vertex-coloring problem. For the vertex-coloring problem we obtain a threshold result of the same generality as the already mentioned offline result by Łuczak, Ruciński and Voigt [LRV92]: Our second main result is that for any F and r , the threshold of the probabilistic online problem is given by $n^{2-1/m_1^*(F,r)}$, where the parameter $m_1^*(F,r)$, referred to as the *online vertex-Ramsey density*, is defined as the infimum over all d for which Builder has a winning strategy in (the vertex-coloring version of) the deterministic F -avoidance game with r colors and density restriction d .

As our third main result we prove that for any F and r , the parameter $m_1^*(F,r)$ is a computable rational number, and the infimum in its definition is attained as a minimum. We note here that this is the only instance where such a general statement about Ramsey densities is known (in the various online/offline, edge-/vertex-coloring settings). With the help of a computer we determined $m_1^*(F,2)$ for all graphs F with at most 9 vertices (recall that the Ramsey number $R(K_5)$ is still unknown!).

While the upper bound approach discussed in the previous section can be easily transferred to the vertex-coloring setting, the main contribution here is a matching lower bound statement (i.e., the proof that certain optimal strategies for *Painter* can be transferred from the deterministic to the probabilistic game). Our lower bound proof is algorithmic, i.e., we obtain polynomial-time algorithms for computing valid colorings of the random graph online in the entire regime below the corresponding thresholds (while coloring above the threshold is impossible).

For a large class of graphs F , including cliques, cycles, complete bipartite graphs, hypercubes, wheels and stars of arbitrary size, the greedy strategy analyzed in [MS10] is optimal for *Painter*, implying simple closed formulas for the parameter $m_1^*(F,r)$ in these cases. As our last main result we show that in the innocent-looking case where $F = P_\ell$ is a (long) path, the greedy strategy fails quite badly and the parameter $m_1^*(P_\ell,r)$ exhibits a surprisingly complex behavior, giving some evidence that a general closed formula for the parameter $m_1^*(F,r)$ does not exist.

The results for the vertex-coloring problem are joint work with Thomas Rast and Reto Spöhel [MRS11⁺, MS11].

1.6. Outlook

We conclude this chapter by a short discussion of the potential and the limitations of the techniques presented in this thesis.

We believe that the upper bound proof technique discussed above (replacing the random graph by an adversary) has the potential for further interesting applications, as the underlying argument is fairly generic and does not depend much on the specifics of the problem studied. In fact, it can be adapted straightforwardly to other scenarios like random hypergraphs or random subsets of integers, and to other random processes involving choices like e.g. the so-called *Achlioptas process* introduced in [BF01]. In the latter, at each step one is presented two, or more generally r , random edges, and is allowed to choose exactly one of them for inclusion into the evolving graph (this can be seen as a probabilistic one-player game, Chooser vs. the random graph). As already mentioned, our solution of the small subgraph avoidance-problem in the Achlioptas process [MST11] can be seen as a variant of the approach presented here. Implicit in that paper is the analysis of a deterministic two-player game Chooser vs. Presenter with an appropriate notion of density restriction. With very similar techniques one can also tackle *balanced* graph coloring games [MMS07, PST09, SG11], where in each step r random edges (or vertices) appear, and Painter has to use each of the r available colors for exactly one of them (again her goal is to avoid a monochromatic copy of some fixed graph F).

An inherent limitation of our approach is that it essentially only applies to settings in which the goal is to avoid local, i.e., *constant-sized*, substructures. The main reason for this is that our calculations and arguments contain several factors that are unimportant constants as long as the size of the substructures in question is bounded, but would spoil our arguments for, say, substructures of logarithmic size.

Our algorithmic result for the vertex-coloring problem bounding the online threshold from below is certainly much harder to generalize or adapt than the upper bound proof. The key issue here is that the random graph (or any other similar structure) satisfies appropriate density restrictions only locally. Thus simply ‘playing like Painter in the deterministic game’ does not automatically yield a good lower bound strategy or algorithm for the probabilistic setting. Proving this matching lower bound statement required a lot of problem-specific technical work, and in particular a thorough understanding of the corresponding deterministic two-player game. We suspect that this will be the norm rather than the exception when proving similar results for other settings.

1.7. Organization of this thesis

In Chapter 2 we present some relevant background and related work on Ramsey-theoretical properties of graphs in various settings. This chapter also introduces some notation which is summarized in the list of symbols at the beginning of this thesis. Each of the following chapters has its own introduction, summarizing the known results which are the immediate starting point of our own contributions. In Chapter 3 and Chapter 4 we present our results for the F -avoidance game in random graphs in the edge- and vertex-coloring version, respectively. Chapter 5 picks up on some questions left open in Chapter 4; it contains our results about the path-avoidance vertex-coloring game.

CHAPTER 2

Background and related work

In this chapter we survey some of the most important results concerning Ramsey-theoretical properties of graphs. Basically each of the problems studied in this field can be classified according to three orthogonal criteria: Whether the setting is *deterministic* or *probabilistic*, whether the problem is about coloring the *edges* or the *vertices* of a graph, and whether it is an *offline* or *online* coloring problem. Understanding the relations between the different variants is very fruitful, and we will try to highlight existing connections whenever possible. The known results about the probabilistic online problems (in the edge- and the vertex-coloring version), which are the immediate starting point of our own work, will be presented at the beginning of Chapter 3 and Chapter 4, respectively.

2.1. From Ramsey numbers to Ramsey densities

In presenting some classical results from Ramsey theory we focus mainly on extremal properties of Ramsey graphs (number of vertices, edges, degrees, local and global sparseness) that are in some sense related to the random graph setting discussed later on. For a more comprehensive treatment we refer the reader to the book [GRS90] (with several historical notes and also covering Ramsey properties of progressions and equations) and the surveys [GR87, Neš95] (with mainly asymptotic results) and [CG83, Rad94] (focusing mainly on exact results and small cases).

For some fixed graph F and some fixed integer $r \geq 2$, we say that a graph G is (F, r) -Ramsey, if any r -coloring of the edges of G contains a monochromatic copy of F . We denote the number of edges and vertices of a graph G by $e(G)$ and $v(G)$, respectively. The celebrated result of Ramsey [Ram30] states that for any F and r the quantity

$$R(F, r) := \min\{v(G) \mid G \text{ is } (F, r)\text{-Ramsey}\} , \quad (2.1)$$

today referred to as the *Ramsey number of F and r* , is finite. Probably the most notorious open problem in this area is to quantify the growth of the Ramsey number $R(K_\ell, 2)$ as ℓ tends to infinity. It is relatively easy to show that

$$\sqrt{2}^\ell \leq R(K_\ell, 2) \leq 4^\ell$$

[ES35, Erd47], but despite considerable efforts [Spe75, GR87, Tho88, Con09], so far nobody has been able to improve the base of the exponents in the lower or upper bound even by a small $\varepsilon > 0$.

Pursuing the question how few edges a Ramsey graph can possibly have, Erdős, Faudree, Rousseau and Schelp [EFRS78] introduced the so-called *size Ramsey number of F and r* , defined as

$$R_e(F, r) := \min\{e(G) \mid G \text{ is } (F, r)\text{-Ramsey}\} . \quad (2.2)$$

Note that we clearly have $R_e(F, r) \leq \binom{R(F, r)}{2}$. In fact, as was shown in [EFRS78] (with a proof attributed to Chvátal) this upper bound is attained for cliques $F = K_\ell$ and any $r \geq 2$.

The authors also raised the question whether for paths $F = P_\ell$ on ℓ edges the size Ramsey number is substantially smaller than this trivial upper bound. This question was answered in the affirmative by Beck [Bec83a], who proved that $R_e(P_\ell, r) \leq c \cdot \ell$ for some constant $c = c(r)$. As was shown by Haxell, Kohayakawa and Łuczak [HKL95], also the size Ramsey number of cycles $F = C_\ell$ is linear in ℓ , i.e., we have $R_e(C_\ell, r) \leq c' \cdot \ell$ for some constant $c' = c'(r)$. For more results concerning size Ramsey numbers, we refer the reader to the survey [FS02].

These results show that we do not always need a just huge enough (and therefore very dense) complete graph to find a monochromatic copy of some F in any of its edge-colorings (as guaranteed by Ramsey's theorem). Folkman [Fol70] proved the surprising fact that a Ramsey graph does not have to be *locally dense* at all (just as dense as absolutely necessary): Answering a question raised by Erdős, Hajnal and Galvin [EH67], he proved that there are $(K_\ell, 2)$ -Ramsey graphs that do not contain a $K_{\ell+1}$ as a subgraph. This result was later generalized by Nešetřil and Rödl [NR76] to the case of more than 2 colors. The smallest currently known graph that is $(K_3, 2)$ -Ramsey and K_4 -free has 941 vertices [DR08].

The question how *globally sparse* Ramsey graphs can possibly be was raised by Rödl and Ruciński [RR93], who introduced the *Ramsey density of F and r* , defined as

$$m_2^o(F, r) := \inf\{m(G) \mid G \text{ is } (F, r)\text{-Ramsey}\} , \quad (2.3)$$

where $m(G) := \max_{H \subseteq G} \frac{e(H)}{v(H)}$. We will see later that the parameter $m(G)$ (and variations of it) arises naturally in the theory of random graphs. Kurek and Ruciński [KR05] proved the somewhat surprising fact that the sparsest (K_ℓ, r) -Ramsey graph in this sense is a complete graph on $R(K_\ell, r)$ many vertices, i.e., we have $m_2^o(K_\ell, r) = (R(K_\ell, r) - 1)/2$. Apart from cliques, the only graphs for which the Ramsey density is known are the trivial cases of stars and the path on 3 edges.

Further variants of this theme with respect to the chromatic number of Ramsey graphs and their minimum and maximum degree can be found in [BEL76, FL07, SZZ10].

2.2. Vertex-Ramsey properties

We say that a graph G is (F, r) -*vertex-Ramsey* if any r -coloring of the *vertices* of G contains a monochromatic copy of F . In the already mentioned paper [Fol70], Folkman proved that there exist (F, r) -vertex-Ramsey graphs that contain no larger cliques than F itself (i.e., no larger than absolutely necessary). His result raised the question how large such graphs need to be: We define the *vertex-Folkman number* $f(\ell, r)$ as the minimum number of vertices of a graph that is (K_ℓ, r) -vertex-Ramsey but contains no clique $K_{\ell+1}$. The currently best bound $f(\ell, r) \leq c\ell^2 \log^4(\ell)$ for some constant $c = c(r)$ is due to Dudek and Rödl [DR10]. We refer the reader to the survey [DFR10] for more results concerning Folkman numbers.

Concerning the global sparseness of vertex-Ramsey graphs, Rödl and Ruciński [RR93] introduced the *vertex-Ramsey density of F and r* , defined as

$$m_1^o(F, r) := \inf\{m(G) \mid G \text{ is } (F, r)\text{-vertex-Ramsey}\} . \quad (2.4)$$

Kurek and Ruciński [KR94] showed that, similarly to the edge-coloring case described above, the sparsest (K_ℓ, r) -vertex-Ramsey graph is a complete graph on $r(\ell - 1) + 1$ vertices (this number comes from the pigeonhole principle), i.e., we have $m_1^o(K_\ell, r) = r(\ell - 1)/2$. Apart from cliques, the vertex-Ramsey density is not known for any other graph, and the authors of [KR94] offer

400,000 złoty (Polish currency in 1993) for the exact determination of $m_1^q(P_3, 2)$ (here P_3 denotes the path on three vertices).

2.3. Online Ramsey games

Beck [Bec83b] and independently Kurek and Ruciński [KR05] introduced the following Ramsey-type game, played by two players called Painter and Builder. The board is an infinite set of vertices, and initially no edges are present. In each step, Builder adds an edge and Painter immediately and irrevocably colors it with one of r available colors. Painter's goal is to avoid a monochromatic copy of some fixed graph F , and Builder's goal is to enforce such a copy. The minimum number of steps needed for Builder to win if Painter plays optimally is called the *online size Ramsey number of F and r* , denoted by $R_e^*(F, r)$. We clearly have $R_e^*(F, r) \leq R_e(F, r)$ for any F and r . As we have seen in Section 2.1, the size Ramsey number of cliques is known to be $R_e(K_\ell, r) = \binom{R(K_\ell, r)}{2}$. The question whether $R_e^*(K_\ell, r) = o(\binom{R(K_\ell, r)}{2})$ as ℓ tends to infinity, attributed by Kurek and Ruciński [KR05] to Rödl, is one of the most interesting problems in this area. Recently, Conlon [Con10] made a big step towards an answer of this question by showing that there exists a constant $c > 1$ such that $R_e^*(K_\ell, 2) \leq c^{-\ell} \binom{R(K_\ell, 2)}{2}$ holds for infinitely many values of ℓ .

Another result we want to mention here is that of Grytczuk, Kierstead and Prałat [GKP08], who proved that $R_e^*(P_\ell, 2) \leq 4\ell$, where the constant 4 is much smaller than the best known constant in the linear bound for the ordinary size Ramsey number $R_e(P_\ell, 2)$ mentioned in Section 2.1, which is still in the hundreds (cf. [Bol01]).

Grytczuk, Hałuszczak, Kierstead and Konjevod [GHK04, KK09] proved the surprising fact that if we restrict Builder to constructing only graphs with chromatic number at most ℓ , then he can still enforce a monochromatic copy of K_ℓ in the game with r colors. To compare this with the corresponding offline problem discussed above, note that any (K_ℓ, r) -Ramsey graph has chromatic number at least $R(K_\ell, r)$ (see also [BEL76]). It was also shown in [GHK04] that if we restrict Builder to not creating any cycles, then he can still enforce a monochromatic copy of any forest in the game with r colors.

An interesting question raised in [BGK⁺11] is whether there exists a constant $d(\Delta)$ such that Builder can enforce any graph F with maximum degree Δ by building only graphs with all degrees bounded by $d(\Delta)$.

2.4. Random graphs

The systematic study of random graphs was initiated by Erdős and Rényi in a series of papers in the 1960's [ER59, ER60, ER61a, ER61b, ER63, ER66, ER64, ER68] (see also [KR97]). Since then the subject has grown into an independent and fast-growing field of research, and today an abundance of beautiful results about random graphs is known (see the monographs [JLR00, Bol01] for an overview of the field).

The most commonly used random graph models are $G_{n,m}$, the graph drawn uniformly at random from all graphs with vertex set $\{1, \dots, n\}$ and exactly m edges, and the binomial random graph $G_{n,p}$ with vertex set $\{1, \dots, n\}$, where each edge is present with probability p , independently from all other edges. The two models behave in fact very similar for $p = m/\binom{n}{2}$ and via this relation threshold results can be routinely transferred between them (see [JLR00, Bol01]). When studying these models, one is mainly interested in the asymptotic behavior when the

number of vertices n tends to infinity, and therefore $m = m(n)$ and $p = p(n)$ are considered functions of n . We write $f(n) \ll g(n)$ if $f(n) = o(g(n))$, $f(n) \gg g(n)$ if $f(n) = \omega(g(n))$, and $f(n) \asymp g(n)$ if $f(n) = \Theta(g(n))$ as n tends to infinity.

We say that a function $m_0 = m_0(n)$ is a *threshold function* for some graph property if for any $m \ll m_0$, $G_{n,m}$ satisfies this property a.a.s. (asymptotically almost surely, i.e., with probability $1 - o(1)$ as n tends to infinity), and if for any $m \gg m_0$, a.a.s. $G_{n,m}$ does not satisfy this property. The definition of a threshold function $p_0 = p_0(n)$ for the model $G_{n,p}$ is analogous. A well-known result due to Bollobás and Thomason [BT87] states that such a threshold function indeed exists for every monotone graph property (a property that is preserved under the addition of edges). Today, explicit threshold functions are known for many natural graph properties.

One important result in this direction, relating directly to the problems discussed in this thesis is due to Bollobás, generalizing a result already found in [ER60]. It determines the threshold for the appearance of any fixed graph F in the random graph $G_{n,m}$ (containing a copy of F is clearly a monotone graph property). (Note that this is not actually a Ramsey property, but it can be seen as the special case of it where only $r = 1$ color is available.)

THEOREM 2.1 ([Bol81]). *Let F be a fixed graph with at least one edge. Then*

$$\lim_{n \rightarrow \infty} \mathbb{P}[G_{n,m} \text{ contains a copy of } F] = \begin{cases} 0 & \text{if } m \ll n^{2-1/m(F)} , \\ 1 & \text{if } m \gg n^{2-1/m(F)} , \end{cases}$$

where

$$m(F) := \max_{H \subseteq F} \frac{e(H)}{v(H)} . \quad (2.5)$$

2.5. Ramsey properties of random graphs

Despite the fact that many questions about (F, r) -Ramsey and (F, r) -vertex-Ramsey graphs are still wide open (for general F and r , but even for special cases like $F = K_\ell$ and $r = 2$ colors), for random graphs both problems have been solved in full generality (note that being (F, r) -Ramsey or (F, r) -vertex-Ramsey is also a monotone graph property). Extending earlier work on the special case $F = K_3$ [FR86, LRV92, RR94], Rödl and Ruciński proved the following general threshold result for the edge-coloring problem.

THEOREM 2.2 ([RR93, RR95]). *Let $r \geq 2$ be a fixed integer, and let F be a fixed graph that is not a star forest and in the case $r = 2$ not a forest of stars and paths on 3 edges. Then there exist positive constants $c = c(F, r)$ and $C = C(F, r)$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}[G_{n,m} \text{ is } (F, r)\text{-Ramsey}] = \begin{cases} 0 & \text{if } m \leq cn^{2-1/m_2(F)} , \\ 1 & \text{if } m \geq Cn^{2-1/m_2(F)} , \end{cases}$$

where

$$m_2(F) := \max_{H \subseteq F: v(H) \geq 3} \frac{e(H) - 1}{v(H) - 2} . \quad (2.6)$$

The exceptional cases in Theorem 2.2 behave somewhat differently, but are all well-understood (see [JLR00]). Note that the threshold behavior stated in Theorem 2.2 is sharper than that of Theorem 2.1. It is widely believed (see e.g. [Fri05]) that the threshold behavior is even sharper than stated in Theorem 2.2 and that the constants c and C can be replaced by $(1 - \varepsilon)c'$ and $(1 + \varepsilon)c'$ for some c' , respectively. So far this conjecture could only be verified for trees [FK00] and for the case $F = K_3$ and $r = 2$ [FRRT06].

The following equally general result for the vertex-coloring setting was obtained by Łuczak, Ruciński and Voigt [LRV92]. We say that a graph is a *matching* if its maximum degree is 1.

THEOREM 2.3 ([LRV92]). *Let $r \geq 2$ be a fixed integer and F a fixed graph with at least one edge that in the case $r = 2$ is not a matching. Then there exist positive constants $c = c(F, r)$ and $C = C(F, r)$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}[G_{n,p} \text{ is } (F, r)\text{-vertex-Ramsey}] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_1(F)} , \\ 1 & \text{if } p \geq Cn^{-1/m_1(F)} , \end{cases}$$

where

$$m_1(F) := \max_{H \subseteq F: v(H) \geq 2} \frac{e(H)}{v(H) - 1} . \quad (2.7)$$

Also in the vertex-coloring setting it is believed that the ‘semi-sharp’ threshold in Theorem 2.3 can be strengthened to a ‘sharp’ threshold, a conjecture that has been verified for the class of strictly 1-balanced graphs, i.e. for graphs F for which $e(H)/(v(H) - 1) < e(F)/(v(F) - 1)$ for all proper subgraphs $H \subsetneq F$ with $v(H) \geq 2$.

CHAPTER 3

The edge-coloring setting

In this chapter we present our new approach to the online edge-coloring game in random graphs via a suitably defined deterministic two-player game. These results are joint work with Michael Belfrage and Reto Spöhel; the corresponding paper [BMS11⁺] is currently under review.

3.1. Introduction

Consider the following probabilistic one-player game: The board is a graph with n vertices, which initially contains no edges. In each step, a new edge is drawn uniformly at random from all non-edges and is presented to the player, henceforth called Painter. Painter must assign one of r available colors to each edge immediately, where $r \geq 2$ is a fixed integer. The game is over as soon as a monochromatic copy of some fixed graph F has been created, and Painter's goal is to 'survive' for as many steps as possible before this happens. We refer to this as the *online F -avoidance game with r colors*. This game was introduced by Friedgut, Kohayakawa, Rödl, Ruciński, and Tetali [FKR⁺03] for the case $F = K_3$ and $r = 2$, and further investigated in [MSS09a, MSS09b].

For any graph F and any number r of colors, this game has a threshold $N_0 = N_0(F, r, n)$ in the following sense [MSS09a, Lemma 2.1]: For any $N = o(N_0)$, there exists a coloring strategy that a.a.s. does not create a monochromatic copy of F in the first N steps of the process. On the other hand, if $N = \omega(N_0)$ then *any* online strategy will a.a.s. create a monochromatic copy of F within the first N steps.

Let us point out two bounds on the threshold of the online game that follow from well-known offline results. Clearly, Painter can only lose the online F -avoidance game once the evolving random graph contains a copy of F . A well-known result of Bollobás [Bol81] gives a threshold of $n^{2-1/m(F)}$ for the latter property (Theorem 2.1). On the other hand, Painter can only survive in the online game as long as the evolving random graph is not (F, r) -Ramsey, i.e., does not have the property that every r -coloring of its edges contains a monochromatic copy of F . Rödl and Ruciński [RR93, RR95] proved a threshold of $n^{2-1/m_2(F)}$ for this property (Theorem 2.2). Thus it is clear from the outset that for any F and r the threshold of the online F -avoidance game with r colors satisfies

$$n^{2-1/m(F)} \leq N_0(F, r, n) \leq n^{2-1/m_2(F)}, \quad (3.1)$$

where in fact the lower bound can be interpreted as the threshold of the 'game' with $r = 1$ colors.

In [MSS09a], the following lower bound approach was analyzed completely: Assuming that the colors are numbered from 1 to r , the *greedy strategy* fixes a sequence of subgraphs $H_1, \dots, H_r \subseteq F$, and at each step uses the highest-numbered color $s \in [r]$ that does not complete a monochromatic copy of H_s (or color 1 if no such color exists).

The subgraphs of F that are relevant for the greedy strategy are determined as follows: For any graph F and any integer $r \geq 1$ we define the parameter $\overline{m}_2(F, r)$ recursively by

$$\overline{m}_2(F, r) := \begin{cases} \max_{H \subseteq F} \frac{e(H)}{v(H)} & \text{if } r = 1, \\ \max_{H \subseteq F} \frac{e(H)}{v(H) - 2 + 1/\overline{m}_2(F, r-1)} & \text{if } r \geq 2, \end{cases} \quad (3.2)$$

and the subgraphs $H_1, \dots, H_r \subseteq F$ considered by the greedy strategy are given by graphs $H \subseteq F$ for which the maximum in the definition (3.2) is attained.

The results of [MSS09a] can be stated as follows. We say that a strategy *attains* some lower bound $N'(F, r, n)$ on the threshold $N_0(F, r, n)$ if for any $N = o(N')$, a.a.s. it does not create a monochromatic copy of F in the first N steps of the game.

THEOREM 3.1 ([MSS09a]). *Let F be a graph that is not a forest, and let $r \geq 2$. Then the threshold of the online F -avoidance game with r colors satisfies*

$$N_0(F, r, n) \geq n^{2-1/\overline{m}_2(F, r)},$$

where $\overline{m}_2(F, r)$ is defined in (3.2). This lower bound is attained by the greedy strategy.

From a qualitative point of view, the main interest of this lower bound is the fact that for every graph F we have

$$\lim_{r \rightarrow \infty} \overline{m}_2(F, r) = m_2(F).$$

Thus the threshold of the online game approaches the threshold of the offline setting as the number r of colors increases, cf. (3.1).

As was also pointed out in [MSS09a], in general the greedy strategy is *not* optimal, i.e., there exist non-forests F for which the threshold is strictly higher than $n^{2-1/\overline{m}_2(F, r)}$. We will encounter such an example below.

For the game with two colors and F satisfying a certain precondition, an upper bound matching the lower bound given by Theorem 3.1 was proved in [MSS09b], making crucial use of the already mentioned results by Rödl and Ruciński about offline colorings of random graphs [RR95]. In particular, the following explicit threshold results for cliques K_ℓ and cycles C_ℓ were obtained. (For cliques and cycles, the sequence of subgraphs $H_1, \dots, H_r \subseteq F$ considered by the greedy strategy is simply $H_1 = \dots = H_r = F$.)

THEOREM 3.2 ([MSS09a, MSS09b]). *For any $\ell \geq 3$, the threshold of the online K_ℓ -avoidance game with $r = 2$ colors is*

$$N_0(K_\ell, 2, n) = n^{(2-\frac{2}{\ell+1})(1-\binom{\ell}{2}^{-2})}.$$

The threshold is attained by the greedy strategy.

THEOREM 3.3 ([MSS09a, MSS09b]). *For any $\ell \geq 3$, the threshold of the online C_ℓ -avoidance game with $r = 2$ colors is*

$$N_0(C_\ell, 2, n) = n^{1+1/\ell}.$$

The threshold is attained by the greedy strategy.

3.1.1. A new upper bound approach. In the following we present a new approach to proving upper bounds on the threshold of the online F -avoidance game. In contrast to the approach pursued in [MSS09b], the ideas presented here cover the game with an arbitrary number of colors and extend to graphs for which the greedy strategy is not optimal. In particular, the technique proposed here has been used by Balogh and Butterfield [BB10] to derive the first nontrivial upper bounds for the game where F is a triangle and $r \geq 3$ colors are available. On the other hand, there seems to be no easy way of recovering all the results of [MSS09b] by our methods, so (at least for the time being) the two approaches should be considered complementary to each other.

Our key idea is to study the deterministic two-player version of the game, which is played by two players called Builder and Painter on a board with some large number a of vertices. In each step, Builder presents an edge, which Painter has to color immediately and irrevocably with one of r available colors. As before, Painter loses as soon as she creates a monochromatic copy of F . So far this is exactly the same game as before, except that we replaced ‘randomness’ by the second player Builder. However, we now impose the restriction that Builder is not allowed to present an edge that would create a (not necessarily monochromatic) subgraph H with $e(H)/v(H) > d$ on the board, for some fixed real number d known to both players. In other words, Builder must adhere to the restriction that the evolving board B satisfies $m(B) = \max_{H \subseteq B} e(H)/v(H) \leq d$ at all times. We will refer to this as the *deterministic F -avoidance game with r colors and density restriction d (on a board with a vertices)*.

We say that *Builder has a winning strategy* in this game (for a fixed graph F , a fixed number of colors r , and a fixed density restriction d) if he can force Painter to create a monochromatic copy of F on a board with a vertices for some large enough integer a . Conversely, we say that *Painter has a winning strategy* if she can avoid creating a copy of F on any finite board. (Note that we can think of such a winning strategy as a countably infinite collection of explicit winning strategies, one for every possible board size a .)

Our approach is based on the following theorem, which relates the original (probabilistic one-player) online F -avoidance game to the deterministic two-player game we just introduced.

THEOREM 3.4. *Let F be a graph with at least one edge, and let $r \geq 2$. If $d > 0$ is such that Builder has a winning strategy in the deterministic F -avoidance game with r colors and density restriction d , then the threshold of the online F -avoidance game with r colors satisfies*

$$N_0(F, r, n) \leq n^{2-1/d}.$$

The key insight behind Theorem 3.4 is the following: By the Bollobás result [Bol81] mentioned above, for a fixed graph G with $m(G) \leq d$ and for any $N = \omega(n^{2-1/d})$, a.a.s. after N steps the evolving random graph contains a copy of G . Adapting the standard textbook proof of this fact (see e.g. [JLR00, Section 3]), we show that for a fixed Builder strategy that respects a density restriction of d and for any $N = \omega(n^{2-1/d})$, a.a.s. the random process exactly reproduces the given Builder strategy somewhere on the board within the first N steps. Thus if Builder has a winning strategy for some graph F and some given density restriction d , then Painter will a.a.s. lose the online F -avoidance game within any $N = \omega(n^{2-1/d})$ steps; i.e., the threshold of the online F -avoidance game satisfies $N_0(F, r, n) \leq n^{2-1/d}$.

Let us illustrate our approach with two examples.

Example 1. Consider the case where $F = C_\ell$ is a cycle of length $\ell \geq 3$ and $r = 2$ colors are available. We will describe an explicit winning strategy for Builder that respects the density

restriction $d := \overline{m}_2(C_\ell, 2) = \ell/(\ell - 1)$ corresponding to the lower bound given by Theorem 3.1. Applying Theorem 3.4, this yields a new elementary proof of Theorem 3.3 that does not resort to offline coloring results.

Example 2. Consider $r = 2$ and F the ‘bowtie’ graph consisting of two triangles that are joined by an edge. This is one of the simplest cases in which the greedy strategy is *not* optimal: According to Theorem 3.1, the greedy strategy achieves a lower bound of $n^{29/21} = n^{1.380\dots}$, which by an ad hoc Painter strategy can be improved to $n^{60/43} = n^{1.395\dots}$. The results of [MSS09b] do not yield any nontrivial upper bound for this example (the trivial one being $n^{2-1/m_2(F)} = n^{1.5}$, cf. (3.1)), but using Theorem 3.4 we can derive an upper bound of $n^{86/61} = n^{1.409\dots}$. The threshold of this example therefore satisfies $n^{1.395\dots} \leq N_0(F, 2, n) \leq n^{1.409\dots}$.

As a third example, we present the result by Balogh and Butterfield [BB10] we already mentioned.

Example 3. Consider the case where $F = K_3$ is a triangle and $r = 3$ colors are available. According to Theorem 3.1 the greedy strategy yields a lower bound of $n^{13/9} = n^{1.444\dots}$. It was conjectured in [MSS09a] that this is indeed the threshold for this case. By an application of Theorem 3.4, Balogh and Butterfield proved an upper bound of $n^{31/21} = n^{1.476\dots}$, showing that the online threshold is strictly lower than the trivial upper bound $n^{2-1/m_2(F)} = n^{1.5}$ coming from the corresponding offline problem. Qualitatively similar results were also obtained for larger values of r , although the resulting improvement over the offline threshold seems to be much smaller in those cases.

Our result raises the question whether the best possible upper bound that can be derived from Theorem 3.4 is indeed the threshold of the probabilistic game. Similarly, one may ask whether a lower bound counterpart of Theorem 3.4 holds, i.e., whether the existence of a winning strategy for *Painter* in the deterministic game with density restriction d implies a *lower* bound of $n^{2-1/d}$ on the threshold of the probabilistic game. (An affirmative answer to the first question would imply that this is indeed the case.)

In Chapter 4 we settle these questions in the affirmative for the *vertex*-coloring analogue of the problem studied here. The proof of a matching lower bound for the vertex-coloring setting is much more technical than the relatively elementary upper bound arguments given here (which generalize straightforwardly to the vertex-coloring setting).

We are unable to prove such a matching lower bound result for the edge-coloring problem studied here. (We will briefly come back to this at the end of this chapter.) However, as we shall see next, the answer to the above questions is ‘yes’ if F is an arbitrary forest.

3.1.2. A threshold result for forests. Suppose d is of the form $d = k/(k+1)$ for some integer $k \geq 1$. Then the restriction that Builder must not create a subgraph of density more than d is equivalent to requiring that Builder creates no cycles and no components (=trees) with more than k edges. We call this game the *deterministic F -avoidance game with r colors and tree size restriction k* .

An elementary proof shows that for any forest F and any integer r , Builder has a winning strategy in this game if k is chosen large enough (see [GHK04, Prop. 1]; the result there is stated for $r = 2$ but generalizes straightforwardly to any $r \geq 2$). It follows that there is a unique smallest integer k for which Builder has a winning strategy. The next theorem states that the threshold of the probabilistic game for some fixed forest F and integer r is indeed given by this smallest integer.

ℓ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\bar{k}(P_\ell, 2)$	1	3	7	10	17	21	31	36	49	55	71	78	97
$k^*(P_\ell, 2)$	1	3	7	10	17	21	31	39	49	55	71	79	97

TABLE 1. Exact values of $\bar{k}(P_\ell, 2)$ and $k^*(P_\ell, 2)$ for $\ell \leq 13$.

THEOREM 3.5. *Let F be a forest with at least one edge, and let $r \geq 2$. Then the threshold of the online F -avoidance game with r colors is*

$$N_0(F, r, n) = n^{1-1/k^*(F, r)} ,$$

where $k^*(F, r)$ is the smallest integer k for which Builder has a winning strategy in the deterministic F -avoidance game with r colors and tree size restriction k . The threshold is attained by any winning strategy for Painter in the deterministic game with tree size restriction $k^*(F, r) - 1$.

Note that Theorem 3.5 implies that for the case of forests, the probabilistic aspect of the problem is fully understood, and in order to find the threshold of the probabilistic game for some given F and r it remains to solve the purely deterministic combinatorial problem of determining $k^*(F, r)$. We will see that, in principle, this can be achieved by a finite procedure. However, this computation becomes intractable already for quite small examples.

In view of Theorem 3.2 and Theorem 3.3, one might expect that $k^*(F, r)$ has a simple closed form if F is a ‘nice’ forest like a star or a path. For $F = S_\ell$ a star with ℓ rays, this is indeed the case: For $k \leq r(\ell - 1)$ Painter can win the game with tree size restriction k by playing greedily, and for $k \geq r(\ell - 1) + 1$ Builder easily wins the game by the pigeonhole principle. Thus for $\ell \geq 1$ and $r \geq 2$ we have

$$k^*(S_\ell, r) = r(\ell - 1) + 1 .$$

As we shall see next, already for the case where F is a long path the situation turns out to be more complicated.

3.1.3. Exact values and lower bounds for paths. We focus on the P_ℓ -avoidance game with $r = 2$ colors, where P_ℓ denotes the path with ℓ edges. It was shown in [MSS09a] that the greedy strategy (with $H_1 = H_2 = P_\ell$) yields a lower bound of

$$k^*(P_\ell, 2) \geq \ell + \lceil \ell/2 \rceil (\ell - 1) =: \bar{k}(P_\ell, 2) . \quad (3.3)$$

Table 1 lists the exact values of $\bar{k}(P_\ell, 2)$ and $k^*(P_\ell, 2)$ (as defined in Theorem 3.5) for all $\ell \leq 13$. The values $k^*(P_\ell, 2)$ were determined with the help of a computer, using various branch-and-bound heuristics (cf. Section 3.5.1). As we can see, the threshold of the game coincides with the lower bound given by the greedy strategy for $\ell \in \{1, \dots, 13\} \setminus \{8, 12\}$, but not for $\ell = 8$ and $\ell = 12$. This shows in particular that the greedy strategy is not always optimal for trees, answering a question left open in [MSS09a].

The observation that for some values of ℓ we have $k^*(P_\ell, 2) > \bar{k}(P_\ell, 2)$ raises the question by how much better strategies can improve on the greedy lower bound asymptotically as $\ell \rightarrow \infty$. Here we show that the improvement is at least by a constant factor; note that $\bar{k}(P_\ell, 2) = (1/2 + o(1)) \cdot \ell^2$.

THEOREM 3.6. *We have*

$$k^*(P_\ell, 2) \geq (8/15 + o(1)) \cdot \ell^2$$

as $\ell \rightarrow \infty$.

The constant $8/15$ is not best possible, and it is entirely conceivable that in fact $k^*(P_\ell, 2) = \Omega(\ell^{2+\varepsilon})$ for some $\varepsilon > 0$. (We can show such an improvement by a polynomial factor for the vertex-coloring variant of the problem studied here, see Chapter 5.) On the other hand, the improvement cannot be more than a polynomial factor: one can show that $k^*(P_\ell, 2) = \mathcal{O}(\ell^{2.9})$ (the proof can be found in the preprint version of [BMS11⁺] on arXiv). The question of the order of magnitude of $k^*(P_\ell, 2)$ remains an intriguing open problem.

3.1.4. Organization of this chapter. We prove Theorem 3.4 and derive Theorem 3.5 as a corollary in Section 3.2. In Section 3.3, we present our examples for non-forests, reproving Theorem 3.3 in an elementary way and deriving new bounds for the bowtie example. After outlining in Section 3.4 how for any forest F and any $r \geq 2$ the parameter $k^*(F, r)$ can be determined by a finite procedure, we focus on the special case of path-avoidance games in Section 3.5. We discuss how the values in Table 1 were found, and prove the asymptotic lower bound stated in Theorem 3.6. We conclude this chapter by outlining some open questions in Section 3.6.

3.2. Proof of Theorem 3.4 and Theorem 3.5

In order to prove Theorem 3.4, we identify Builder's strategies in the deterministic two-player game with r colors (on a board with some fixed number a of vertices) with r -ary rooted trees \mathcal{T} , where each node of such a tree corresponds to an intermediate stage of the game. Specifically, the tree \mathcal{T} representing a given Builder strategy is constructed as follows: The root of \mathcal{T} is the empty graph on a vertices. Its r children are the graphs obtained by inserting the first edge of Builder's strategy and coloring it with one of the r available colors. The r children of each of these nodes are in turn obtained by inserting the second edge of Builder's strategy and coloring it with one of the r colors. (Note that the second edge of Builder's strategy may depend on Painter's decision how to color the first edge, i.e., in general the second edge will be a different one in different branches of \mathcal{T} .) Continuing like this, we construct \mathcal{T} , representing any situation in which Builder stops playing by a leaf of \mathcal{T} . Thus a node at depth k in \mathcal{T} is an r -colored graph B on a vertices with exactly k edges representing the board of the deterministic game after Painter's k -th move if Builder plays according to \mathcal{T} .

Note that in this formalization, a given tree \mathcal{T} represents a generic strategy for Builder (in the deterministic game with r colors on a board with a vertices) that may or may not satisfy a given density restriction d , and that can be thought of as a strategy for the ' F -avoidance' game for any given graph F . We say that \mathcal{T} is a *winning strategy* for Builder in a specific F -avoidance game if and only if every leaf of \mathcal{T} contains a monochromatic copy of F . We say that \mathcal{T} is a *legal strategy* in the game with density restriction d if and only if $m(B) \leq d$ for (the underlying uncolored graph of) every node B in \mathcal{T} .

Going back to the probabilistic one-player game, we denote the board of the probabilistic game after N moves by G_N . Thus G_N is an r -colored graph on n vertices with exactly N edges, and the underlying uncolored graph of G_N is uniformly distributed over all graphs on n vertices with N edges. When we say that G_N contains a copy of some r -colored graph B (e.g. a node of some Builder strategy \mathcal{T}) we mean that there is a subgraph of G_N that is isomorphic to B as a *colored* graph.

Theorem 3.4 and Theorem 3.5 are immediate consequences of the following lemma.

LEMMA 3.7 (Random process reproduces Builder strategy). *Let $r \geq 2$ and $a \geq 1$ be fixed integers, let $d > 0$ be a fixed real number, and let \mathcal{T} represent an arbitrary legal strategy for Builder in the deterministic game with r colors and density restriction d on a board with a vertices.*

If $N \gg n^{2-1/d}$, then regardless of how Painter plays, a.a.s. G_N contains a copy of a leaf of \mathcal{T} .

PROOF OF THEOREM 3.4. By assumption there exists an integer $a = a(F, r, d)$ such that Builder has a winning strategy \mathcal{T} for the deterministic F -avoidance game with r colors and density restriction d on a board with a vertices. As each leaf of \mathcal{T} contains a monochromatic copy of F , applying Lemma 3.7 to \mathcal{T} yields that if $N \gg n^{2-1/d}$, a.a.s. G_N contains a monochromatic copy of F regardless of how Painter plays, which is exactly the statement of Theorem 3.4. \square

PROOF OF THEOREM 3.5. Applying Theorem 3.4 with $d = k^*(F, r)/(k^*(F, r) + 1)$ immediately yields that $N_0(F, r, n) \leq n^{1-1/k^*(F, r)}$, i.e., Painter will be forced to create a monochromatic copy of F a.a.s. if $N \gg n^{1-1/k^*(F, r)}$. On the other hand, as long as $N \ll n^{1-1/k^*(F, r)}$, by standard first moment calculations (see for example [JLR00, Section 3]) a.a.s. G_N contains no cycle, and no tree of size $k^*(F, r)$. In other words, a.a.s. all components of G_N are trees of size at most $k^*(F, r) - 1$. Hence by following a winning strategy for the deterministic game with tree size restriction $k^*(F, r) - 1$ on a board with n vertices (such a strategy exists by definition of k^*), Painter can ensure she will a.a.s. not create a monochromatic copy of F . \square

REMARK 3.8. Note that after any $N \gg n$ steps we have $m(G_N) \geq N/n \gg 1$, i.e., the density of the board of the probabilistic game is larger than any constant d . In other words, the random process Painter faces in the probabilistic one-player game only behaves like Builder in the deterministic two-player game (with some constant density restriction d) as long as $N = \mathcal{O}(n)$, which is equal to or less than the trivial lower bound $n^{2-1/m(F)}$ if F is a non-forest (cf. (3.1)). Therefore ‘playing just as in the deterministic game’ without any additional assumptions only yields a useful guarantee to Painter if F is a forest.

In order to prove Lemma 3.7, we shall show the following more technical statement by induction on k .

LEMMA 3.9 (Random process reproduces Builder strategy step by step). *Let $r \geq 2$ and $a \geq 1$ be fixed integers, let $d > 0$ be a fixed real number, and let \mathcal{T} represent an arbitrary legal strategy for Builder in the deterministic game with r colors and density restriction d on a board with a vertices.*

If $n^{2-1/d} \ll N \ll n^2$, then for any integer $k \geq 0$ the following holds. Regardless of how Painter plays, a.a.s. G_N satisfies one of the following two properties:

- G_N contains a copy of a leaf of \mathcal{T} , or
- there is a node B at depth k in \mathcal{T} such that G_N contains $\Omega(n^a(Nn^{-2})^k)$ many copies of B .

The second property of Lemma 3.9 is meaningful since, due to the assumption that \mathcal{T} is a legal strategy for Builder in the game with density restriction d , we have

$$k/a = e(B)/v(B) \leq m(B) \leq d ,$$

which yields with $N \gg n^{2-1/d} \geq n^{2-a/k}$ that

$$n^a(Nn^{-2})^k \gg 1 .$$

PROOF OF LEMMA 3.7. Since \mathcal{T} has depth at most $\binom{a}{2}$, Lemma 3.7 follows by setting $k := \binom{a}{2} + 1$ in Lemma 3.9. \square

It remains to prove Lemma 3.9.

PROOF OF LEMMA 3.9. We proceed by induction on k . Clearly, G_N contains $\Theta(n^a)$ copies of the root of \mathcal{T} , i.e., vertex sets of size a . This takes care of the induction base.

For the induction step we employ a two-round approach. That is, we divide the process into two rounds of equal length $N/2$ (w.l.o.g. we assume N to be even) and analyze these two rounds separately. Specifically, we apply the induction hypothesis and some standard random graph arguments to the edges of the first round, and then show by a variance calculation that, conditional on a ‘good’ first round, the second round turns out as claimed.

By the induction hypothesis, if the graph $G_{N/2}$ does not contain a copy of a leaf of \mathcal{T} (in which case we are done), a.a.s. it contains a family of

$$M \asymp n^a (Nn^{-2})^{k-1} \quad (3.4)$$

copies of some graph B^- corresponding to a non-leaf node at depth $k-1$ in \mathcal{T} . We label these copies B_i^- , $1 \leq i \leq M$. For a given copy B_i^- , consider a vertex pair corresponding to Builder’s next move as specified by \mathcal{T} , and call this vertex pair e_i . (If this does not define e_i uniquely, we simply fix one possible choice of e_i .) If e_i is an edge of $G_{N/2}$, then clearly B_i^- and e_i form a copy of one of the children of B^- in \mathcal{T} . If this is the case for $n^a (Nn^{-2})^k$ many indices i , then by the pigeonhole principle the color used for the majority of these indices yields a child of B^- for which the inductive claim holds, and we are done with the proof. For the remainder of the proof we assume that this is not the case. Due to $n^a (Nn^{-2})^k \ll M$, we can safely ignore indices for which e_i is in $G_{N/2}$; therefore, we assume w.l.o.g. that none of the e_i is in $G_{N/2}$.

For our analysis of the second round we condition on a fixed outcome of $G_{N/2}$. For $1 \leq i \leq M$, let Z_i be the indicator variable for the event that e_i is among the $N/2$ edges drawn in the second round. Let

$$Z := \sum_{i=1}^M Z_i ,$$

and note that by the pigeonhole principle at least Z/r many copies of one of the children of B^- in \mathcal{T} are created. Thus the existence of B as claimed follows if we show that a.a.s.

$$Z \asymp n^a (Nn^{-2})^k = n^{v(B)} (Nn^{-2})^{e(B)} . \quad (3.5)$$

We will do so by the methods of first and second moment.

Using that $N \ll n^2$ we have

$$\Pr[Z_i = 1] = \frac{\binom{\binom{n}{2} - N/2 - 1}{N/2 - 1}}{\binom{\binom{n}{2} - N/2}{N/2}} \asymp Nn^{-2} , \quad (3.6)$$

and, conditioning on the first round satisfying the induction hypothesis,

$$\mathbb{E}[Z] \asymp M \cdot Nn^{-2} \stackrel{(3.4)}{\asymp} n^a (Nn^{-2})^k = n^{v(B)} (Nn^{-2})^{e(B)} . \quad (3.7)$$

In the following we slightly abuse notation and write B for the *uncolored* graph formed by B^- and the next edge of Builder’s strategy \mathcal{T} . Let \mathcal{D} denote the family of all (uncolored) graphs D that can be constructed by considering the union of two edge-intersecting copies of B and removing one edge from the intersection of these two copies. To calculate the variance of Z ,

observe that for pairs with $e_i \neq e_j$ the variables Z_i and Z_j are negatively correlated. Hence such pairs can be omitted, and we have

$$\begin{aligned} \text{Var}[Z] &= \sum_{i,j=1}^M (\mathbb{E}[Z_i Z_j] - \mathbb{E}[Z_i] \mathbb{E}[Z_j]) \leq \sum_{(i,j): e_i=e_j} \Pr[Z_i = 1 \wedge Z_j = 1] \\ &\stackrel{(3.6)}{\asymp} \sum_{(i,j): e_i=e_j} N n^{-2} \leq M_{\mathcal{D}} \cdot \Theta(1) \cdot N n^{-2} , \end{aligned} \quad (3.8)$$

where $M_{\mathcal{D}}$ denotes the total number of copies of graphs $D \in \mathcal{D}$ in (the underlying uncolored graph of) $G_{N/2}$. Here the factor $\Theta(1)$ is due to fact that in general a given copy of a given graph $D \in \mathcal{D}$ corresponds to a constant number of pairs of copies of B^- . By definition of \mathcal{D} , each such graph satisfies

$$\begin{aligned} v(D) &= 2v(B) - v(J) , \\ e(D) &= 2e(B) - e(J) - 1 \end{aligned} \quad (3.9)$$

for some subgraph $J \subseteq B$. Moreover, since we assumed that \mathcal{T} is a legal strategy for Builder in the game with density restriction d , we have

$$e(J)/v(J) \leq m(B) \leq d ,$$

which yields with $N \gg n^{2-1/d} \geq n^{2-v(J)/e(J)}$ that

$$n^{v(J)} (N n^{-2})^{e(J)} \gg 1 . \quad (3.10)$$

Thus, going back to analyzing the first round, we obtain that the (unconditional) expected number of copies of D in (the underlying uncolored graph of) $G_{N/2}$ is

$$\begin{aligned} &\binom{n}{v(D)} \cdot \Theta(1) \cdot \frac{\binom{\binom{n}{2}-e(D)}{N/2-e(D)}}{\binom{\binom{n}{2}}{N/2}} \asymp n^{v(D)} (N n^{-2})^{e(D)} \\ &\stackrel{(3.9)}{=} n^{2v(B)-v(J)} (N n^{-2})^{2e(B)-e(J)-1} \stackrel{(3.10)}{\ll} n^{2v(B)} (N n^{-2})^{2e(B)-1} . \end{aligned}$$

As the number of graphs in \mathcal{D} is bounded by a constant depending only on a , it follows with Markov's inequality that

$$M_{\mathcal{D}} \ll n^{2v(B)} (N n^{-2})^{2e(B)-1} \quad (3.11)$$

a.a.s. Thus, conditioning on the first round satisfying the induction hypothesis (cf. (3.4) and (3.7)) and (3.11), we obtain from (3.8) that

$$\text{Var}[Z] \stackrel{(3.11)}{\ll} (n^{v(B)} (N n^{-2})^{e(B)})^2 \stackrel{(3.7)}{\asymp} \mathbb{E}[Z]^2 .$$

Chebyshev's inequality now yields that a.a.s. the second round satisfies (3.5). This implies that there is at least the claimed number of copies of one of the children of B_- in G_N , as discussed. \square

3.3. Examples

In this section we present our examples illustrating how Theorem 3.4 can be applied to derive explicit upper bounds on the threshold of the original (probabilistic one-player) online F -avoidance game.

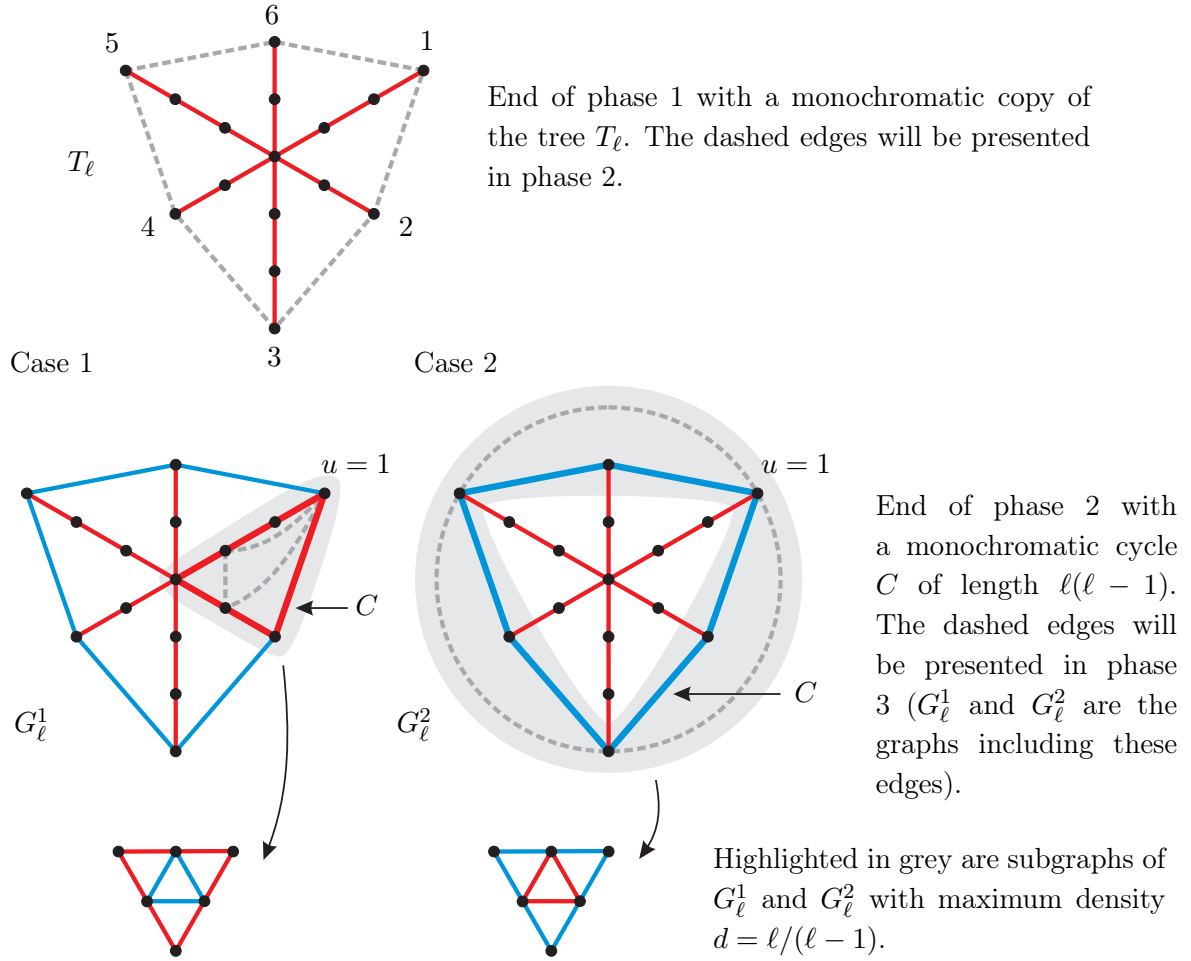


FIGURE 3.1. Builder strategy to enforce a monochromatic copy of C_ℓ in the deterministic game with $r = 2$ colors ($\ell = 3$).

3.3.1. Elementary proof of Theorem 3.3. In the following we rederive the threshold of the cycle-avoidance game with $r = 2$ colors in a more elementary way. Specifically, we replace the upper bound proof given in [MSS09b] by an application of Theorem 3.4.

PROOF OF THEOREM 3.3. Applying Theorem 3.1 with $F = C_\ell$ and $r = 2$ yields $\overline{m}_2(C_\ell, 2) = \ell/(\ell - 1)$ and establishes $n^{2-1/\overline{m}_2(C_\ell, 2)} = n^{1+1/\ell}$ as a lower bound for the threshold of the probabilistic C_ℓ -avoidance game with $r = 2$ colors. To show that this lower bound given by the greedy strategy is tight, we apply Theorem 3.4 and specify a winning strategy for Builder in the deterministic C_ℓ -avoidance game with $r = 2$ colors and density restriction $d := \overline{m}_2(C_\ell, 2) = \ell/(\ell - 1) > 1$. To define Builder's strategy, let T_ℓ denote the tree that is obtained by replacing half of the edges of a star with $\ell(\ell - 1)$ rays by paths of length $\ell(\ell - 1)/2$, and the other half by paths of length $\ell(\ell - 1)/2 - 1$.

Builder's strategy consists of three phases (cf. Figure 3.1). In the first phase he enforces a monochromatic copy of T_ℓ without creating any cycles on the board. As already mentioned in Section 3.1.2, it has been proved in [GHK04, Prop. 1] that Builder can enforce a monochromatic copy of *any* tree without creating cycles on the board. At the end of the first phase, the monochromatic copy of T_ℓ (w.l.o.g. we assume that it is a red copy) is contained in some larger

tree T' . This phase is noncritical as far as the density restriction $d > 1$ is concerned, as for any tree T we have $m(T) = e(T)/v(T) = 1 - 1/v(T) < 1$.

In the second and third phase Builder joins vertices of the red copy of T_ℓ to enforce a monochromatic copy of C_ℓ . Note that a subgraph H of any non-forest G that maximizes $e(H)/v(H)$ has the property that each of its edges is contained in a cycle of G . As Builder joins only vertices of the red copy of T_ℓ in the second and third phase, we can neglect the edges of T' that do not belong to this copy when checking whether the density restriction d is respected. For simplicity we will therefore describe how Builder proceeds with an *isolated* red copy of T_ℓ in the second phase (and neglect the embedding of this copy into the larger tree T'). Referring to the vertex of maximum degree $\ell(\ell - 1)$ of the copy of T_ℓ as the root, we label the leaves of this copy with distance $\ell(\ell - 1)/2$ from the root by $1, 3, 5, \dots, \ell(\ell - 1) - 1$, and the leaves with distance $\ell(\ell - 1)/2 - 1$ from the root by $2, 4, 6, \dots, \ell(\ell - 1)$. In the second phase Builder adds $\ell(\ell - 1)$ edges, each edge connecting two vertices with successive labels (where the labels $\ell(\ell - 1)$ and 1 are defined to be successive as well). If Painter uses red for one of these edges (Case 1), then a red cycle of length $\ell(\ell - 1)$ is created (containing exactly one edge from the second phase). On the other hand, if Painter always uses blue (Case 2), then a blue cycle of length $\ell(\ell - 1)$ is created (consisting only of edges from the second phase). In any case, Builder has enforced a monochromatic cycle C of length $\ell(\ell - 1)$. Let u denote a vertex with an odd label in this cycle.

In the third phase, Builder connects any two consecutive vertices along C whose distance from u is an integer multiple of $\ell - 1$ with a new edge (thus adding ℓ edges in total). Clearly, Painter cannot avoid creating a monochromatic copy of C_ℓ by the end of this phase.

Depending on the outcome of the second phase (Case 1 or Case 2), the resulting graphs after the third phase, denoted by G_ℓ^1 and G_ℓ^2 , respectively, are different, and it remains to check that $m(G_\ell^1) \leq d$ and $m(G_\ell^2) \leq d$. Straightforward calculations show that the maximum density of both graphs is indeed bounded by d ; in fact it is exactly d (cf. the bottom part of Figure 3.1). \square

3.3.2. Bounds for the bowtie example. In the following, F denotes the ‘bowtie’ graph consisting of two triangles that are joined by an edge. We prove that the threshold of the online F -avoidance game with $r = 2$ colors satisfies $n^{60/43} \leq N_0(F, 2, n) \leq n^{86/61}$.

LOWER BOUND PROOF. To prove the claimed lower bound, we consider the following Painter strategy: Color an edge blue if and only if it does not close a blue copy of F and coloring it red would close a red triangle. We will perform a backward analysis, showing that if Painter plays according to this strategy and loses the game with a monochromatic copy of F , then the board contains as a subgraph one of a finite family \mathcal{W} of ‘witness’ graphs, where each graph $W \in \mathcal{W}$ has a density of $m(W) \geq 43/26$. A standard first moment calculation (see for example [JLR00, Section 3]) yields that for any $N \ll n^{2-26/43} = n^{60/43}$, a.a.s. the board G_N contains no graph from \mathcal{W} (here we use that the family \mathcal{W} is finite), which implies the claim.

By definition of the strategy, the game ends with a red copy of F . When the last edge in each of the two triangles of this copy was colored red, the alternative for Painter must have been to complete a blue copy of F . This implies that six blue edges are adjacent to each red triangle. Each of these blue edges in turn was colored blue only because the alternative was to close a red triangle. Figure 3.2 shows two ‘nice’ possible witness graphs W_1 and W_2 resulting from this analysis. Of course, a witness graph resulting from the above argument is not necessarily as nicely symmetric as W_1 and W_2 . Moreover, some of the blue or red edges could in fact coincide (the graph on the right hand side of Figure 3.2 is such an example). Our analysis therefore yields

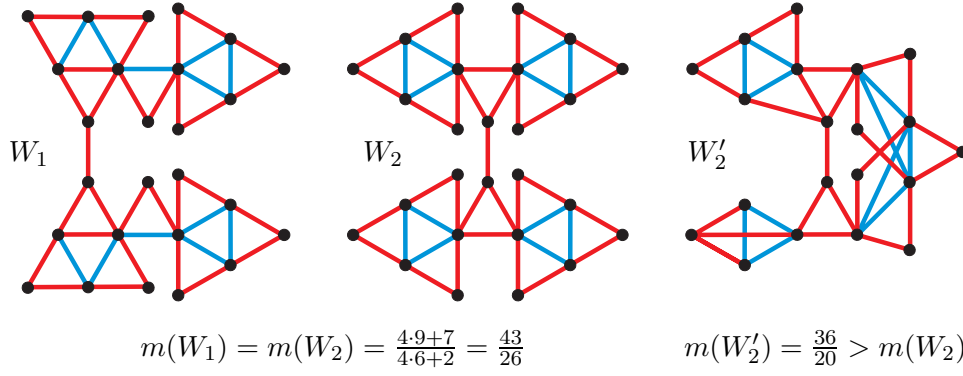
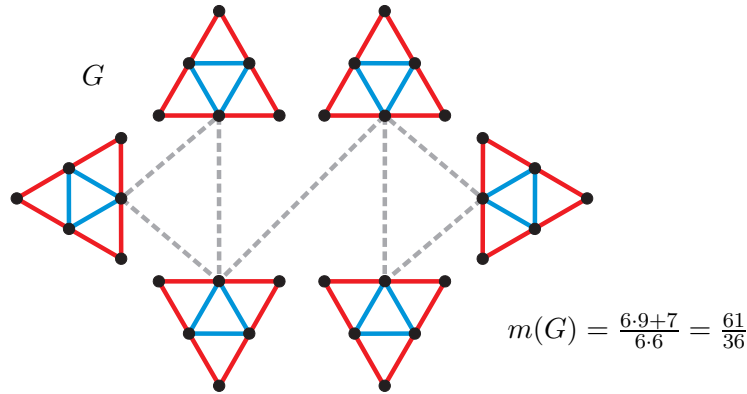
FIGURE 3.2. Three graphs from the family of witness graphs \mathcal{W} .

FIGURE 3.3. Builder strategy to enforce a monochromatic copy of the ‘bowtie’ graph.

a fairly large (but finite) family \mathcal{W} of witness graphs. A straightforward but rather tedious case analysis shows that all graphs $W \in \mathcal{W}$ satisfy $m(W) \geq 43/26 = m(W_1) = m(W_2)$, concluding the proof. \square

UPPER BOUND PROOF. To prove an upper bound of $n^{86/61}$, we describe a strategy for Builder to enforce a monochromatic copy of F in the deterministic F -avoidance game with density restriction $d = 61/36$. Then Theorem 3.4 implies an upper bound of $n^{2-1/d} = n^{86/61}$ for the threshold of the probabilistic game. Builder’s strategy consists of two phases. In the first phase he enforces six triangles of the same color (w.l.o.g. in blue) in the same way as described in the proof of Theorem 3.3 (cf. Figure 3.1). In the second phase Builder selects one vertex from each blue triangle and joins those vertices to a copy of F . Clearly, Painter cannot avoid creating a monochromatic copy of F by the end of the second phase. Builder’s strategy is illustrated in Figure 3.3 (the dashed edges are presented in the second phase), where some parts of the board that were necessary for Builder to enforce the blue triangles are hidden. It is readily checked that the graph G shown in the figure is in fact the densest subgraph of the board (even if the hidden edges are taken into account). \square

3.4. Computation of $k^*(F, r)$

We outline how the integer $k^*(F, r)$, defined in Theorem 3.5 as the smallest integer k for which Builder has a winning strategy in the deterministic F -avoidance with r colors and tree size

restriction k , can be found by a finite computation. First observe that if Builder confronts Painter several times with the decision on how to color a new edge between copies of the same two r -edge-colored trees (rooted trees, to be more precise, the root representing the connection vertex), then by the pigeonhole principle, Painter's decision will be the same in at least a $(1/r)$ -fraction of the cases. As a consequence we can assume w.l.o.g. that Painter plays *consistently* in the sense that her strategy is determined by a strategy function π that maps unordered pairs of r -edge-colored rooted trees to the set of available colors $\{1, \dots, r\}$. For each such strategy function π we define the family $\mathcal{T}_{r,k}^\pi$ as the set of all r -edge-colored trees on exactly k edges that Builder can enforce if Painter plays as specified by π . Observe also that $\mathcal{T}_{r,k}^\pi$ can be computed recursively, by joining for each $i = 0, \dots, k-1$ every tree in the family $\mathcal{T}_{r,i}^\pi$ with every tree in the family $\mathcal{T}_{r,k-i-1}^\pi$ with a new edge in all possible ways. (The basis for the recursion is the set $\mathcal{T}_{r,0}^\pi$, which contains only an isolated vertex.) Note that so far this formalism is completely generic, i.e., independent of the specific forest F Painter tries to avoid.

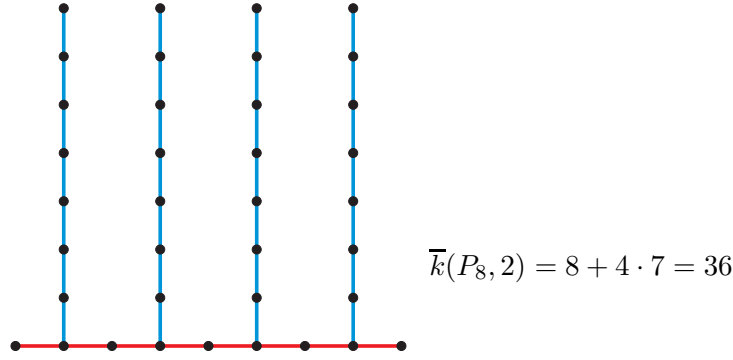
For a given forest F and a given strategy function π , it is straightforward to determine the smallest integer $k^*(F, r, \pi)$ such that all components (=trees) of F appear monochromatically in the same color in $\bigcup_{i=0}^{k^*(F, r, \pi)} \mathcal{T}_{r,i}^\pi$. This value $k^*(F, r, \pi)$ is exactly the lower bound on $k^*(F, r)$ that the strategy function π guarantees to Painter. As mentioned in Section 3.1.2, an argument given in [GHK04] yields an explicit $k_0 = k_0(F, r)$ such that for any size restriction $k \geq k_0$ Builder wins the deterministic F -avoidance game. Thus we have $k^*(F, r, \pi) \leq k_0$ for all strategy functions π , and consequently there are only finitely many strategy functions that have to be taken into account (as only the coloring decisions on pairs of rooted trees on at most k_0 many edges are relevant). Therefore, $k^*(F, r)$ can be computed as the maximum of $k^*(F, r, \pi)$ over finitely many strategy functions π .

3.5. Path-avoidance games

Throughout this section, we study the deterministic game with $F = P_\ell$ (the path with ℓ edges) and $r = 2$ colors. Our main goal here is to show that Painter can do significantly better than what is guaranteed by the greedy lower bound.

3.5.1. Exact values for $\ell \leq 13$. Let us sketch briefly how the exact values of $k^*(P_\ell, 2)$ given in Table 1 were determined with the help of a computer. The approach from Section 3.4 can be easily adapted to compute upper bounds on $k^*(F, r)$ (instead of exact values) by computing only small subsets of the families $\mathcal{T}_{r,k}^\pi$. The hope is that considering these suffices to enforce a monochromatic copy of F effectively, and that therefore we do not need to branch on too many decisions of Painter (i.e., values of the strategy function π). If suitable heuristics are used, this approach turns out to be much faster than a full exhaustive enumeration. Moreover, it has the advantage that any resulting upper bound comes with an explicit Builder strategy. Our computer-generated Builder strategies establishing the values given in Table 1 as upper bounds on $k^*(P_\ell, 2)$, $\ell \leq 13$ are available on the authors' website [Müt], along with a verification routine.

It remains to prove the matching lower bounds. As discussed in Section 3.1.3, for $\ell \in \{1, \dots, 13\} \setminus \{8, 12\}$ such lower bounds are provided by the greedy strategy, and it remains to consider the cases $\ell = 8$ and $\ell = 12$. Here the greedy strategy yields a lower bound of $\bar{k}(P_8, 2) = 36$ and $\bar{k}(P_{12}, 2) = 78$, respectively; indeed Figure 3.4 shows an easy way for Builder to win against a greedy Painter in the P_8 -avoidance game with tree size restriction 36 (where Painter greedily colored blue whenever this did not close a blue P_8 , and all blue edges appeared before all red edges). In the following we propose and analyze a Painter strategy that outperforms the greedy

FIGURE 3.4. Analysis of the greedy strategy for the P_8 -avoidance game.

strategy and yields $k^*(P_8, 2) \geq 39$ and $k^*(P_{12}, 2) \geq 79$, thus establishing matching lower bounds for these remaining cases.

PROOF OF $k^*(P_8, 2) \geq 39$ AND $k^*(P_{12}, 2) \geq 79$. Consider the following generic Painter strategy for the P_ℓ -avoidance game: Color an edge blue if and only if it is adjacent to a red edge and does not close a blue P_ℓ . Let us focus on the case $\ell = 8$ first. We will perform a backward analysis, showing that if Painter plays according to this strategy and loses the deterministic P_8 -avoidance game with a monochromatic P_8 , then Builder must have closed a component (=tree) with at least 39 edges, thus establishing the claimed lower bound for the smallest tree size restriction that guarantees a win for Builder.

By definition of the strategy, the game ends with a red P_8 . Each edge of this P_8 was colored red either because it was not adjacent to any red edge, or because coloring it blue would have closed a blue P_8 . In the latter case we call a red edge *heavy*. Note that among any two adjacent edges on the red P_8 , the one that appeared last must be heavy. We now mark all heavy edges on the red P_8 . Observe that by unmarking some of these edges again we can always guarantee a pattern of four marked edges where either all of them are disjoint or exactly two of them are adjacent (see Figure 3.5).

In the remainder of the proof, we will argue that each heavy edge implies 10 additional edges, and that two adjacent heavy edges imply 11 additional edges. This then proves our claim as we have counted at least $8 + \min(4 \cdot 10, 2 \cdot 10 + 11) = 39$ edges in each component (=tree) that might have forced Painter to close a monochromatic P_8 .

By definition, each heavy edge is adjacent to two blue paths of length k and $7 - k$ for some $0 \leq k \leq 7$. For any such k the two blue paths contain at least three disjoint blue edges (indicated by dotted ellipses in Figure 3.5) that are not adjacent to the central red P_8 . Again by definition of the strategy, at least one extra red edge must be adjacent to each of these blue edges. We have thus counted at least $7 + 3 = 10$ additional edges for each heavy edge.

Assume that two adjacent heavy edges are given, say (u, v) and (v, w) . As argued in the previous paragraph, the edge (u, v) guarantees at least 10 additional edges. If the length of the blue path adjacent to v is strictly smaller than 7, then at least one additional blue edge must be adjacent to w (cf. the right hand side of Figure 3.5). Otherwise, a blue P_7 is adjacent to v . But this blue path must have been completed *before* both heavy edges appeared, hence not only three but at least four additional red edges are adjacent to this blue P_7 . In both cases we are guaranteed at least 11 additional edges for each pair of adjacent heavy edges. This concludes the proof for the case $\ell = 8$.

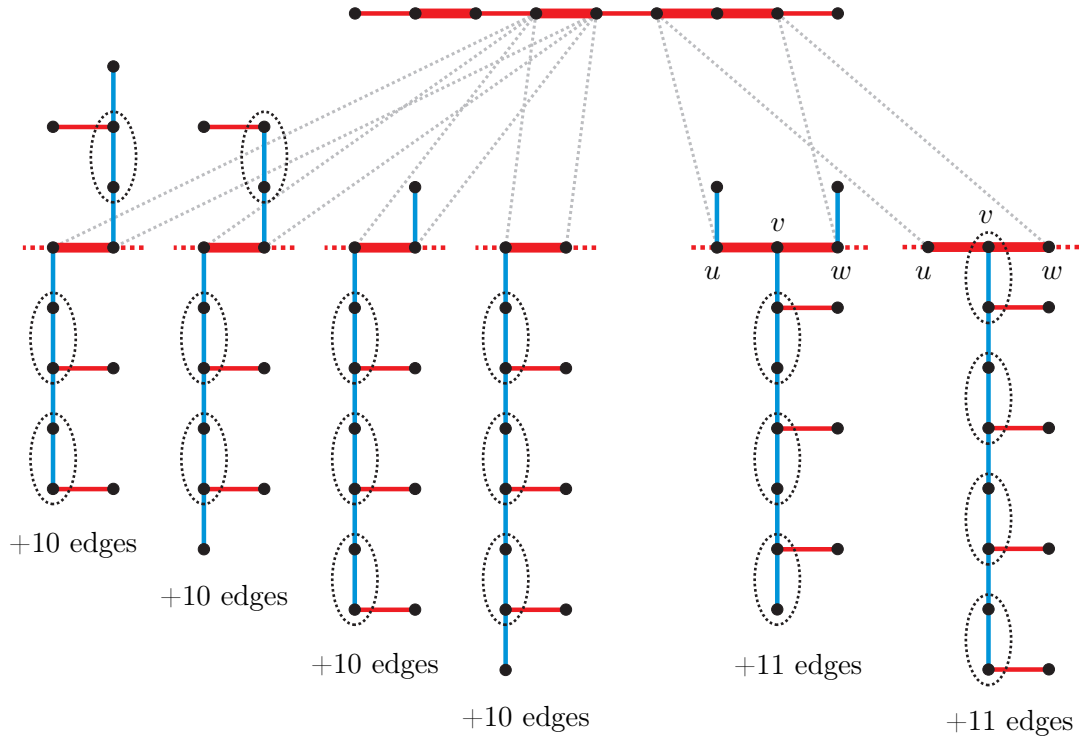


FIGURE 3.5. Analysis of an optimal strategy for the P_8 -avoidance game (heavy edges are indicated by bold lines).

A very similar analysis of the same strategy for $\ell = 12$ yields a lower bound of $12 + 16 + 3 \cdot 17 = 79$ for $k^*(P_{12}, 2)$, where in the extremal example we have 12 edges coming from the central red P_{12} , 16 additional edges implied by an isolated heavy edge, and $3 \cdot 17$ additional edges implied by three pairs of adjacent heavy edges. \square

3.5.2. Asymptotic lower bound. In this section, we prove the lower bound on $k^*(P_\ell, 2)$ claimed in Theorem 3.6 by describing and analyzing an explicit Painter strategy.

PROOF OF THEOREM 3.6. Consider the following Painter strategy: Color an edge red if and only if (at least) one of the following two conditions holds: (a) the edge is adjacent to a blue edge and it does not close a red P_4 , (b) coloring the edge blue would close a blue P_ℓ .

As before, we will perform a backward analysis, showing that if Painter plays according to this strategy and loses the deterministic P_ℓ -avoidance game with a monochromatic P_ℓ , then Builder must have closed a component (=tree) with at least $\frac{8}{15}\ell^2 + \mathcal{O}(\ell)$ many edges, thus establishing the claimed lower bound for the smallest tree size restriction that guarantees a win for Builder.

By definition of the strategy, Painter loses with a red P_ℓ . This P_ℓ can be partitioned into at least $\lfloor \ell/5 \rfloor$ disjoint red paths of length 4. For each of those P_4 's, when the last edge was inserted (and colored red), the alternative for Painter must have been to complete a blue P_ℓ . Hence $\ell - 1$ blue edges are adjacent to the last edge of each red P_4 . These form two disjoint blue paths of length k and $\ell - k - 1$ for some $k \geq 0$.

In the remainder of the proof, we will argue that a blue path of length k guarantees at least $\frac{5}{3}k + \mathcal{O}(1)$ additional edges that are adjacent to it. This then proves our claim as we have counted

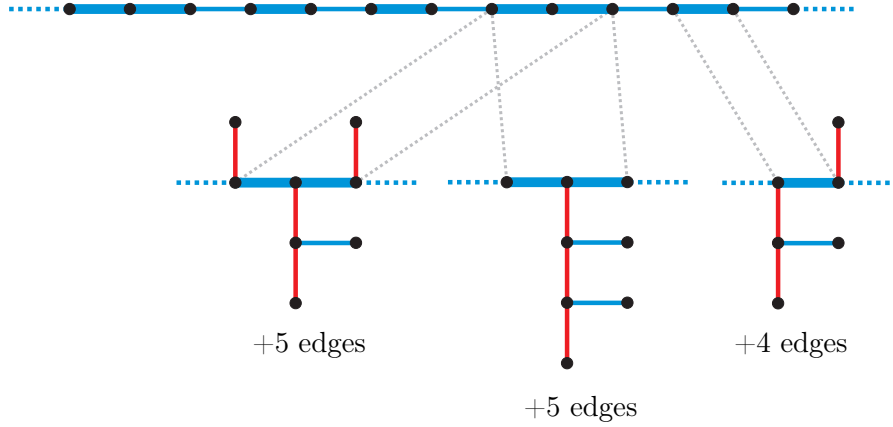


FIGURE 3.6. Additional edges implied by heavy edges (heavy edges are indicated by bold lines).

at least

$$\underbrace{\ell}_{\text{central red } P_\ell} + \underbrace{\left\lfloor \frac{\ell}{5} \right\rfloor \cdot (\ell - 1)}_{\text{blue 'almost-copies' of } P_\ell} + \underbrace{\left\lfloor \frac{\ell}{5} \right\rfloor \cdot \left(\frac{5}{3}\ell + \mathcal{O}(1) \right)}_{\text{additional edges}} = \frac{8}{15}\ell^2 + \mathcal{O}(\ell)$$

edges in each component (=tree) that might have forced Painter to close a monochromatic P_ℓ .

So consider a blue path of length k , where $k < \ell$. Each of its edges was colored blue because condition (a) in the strategy definition was violated, either because the edge was not adjacent to any blue edge, or because coloring it red would have closed a red P_4 . In the latter case we call a blue edge *heavy*. Note that among any two adjacent edges on the blue P_k , the one that appeared last must be heavy. We now mark all heavy edges on the blue P_k . If three or more marked edges appear consecutively, then we unmark one or more of the interior edges again such that the resulting pattern only consists of marked paths of length one or two with exactly one unmarked edge in between (see Figure 3.6). Similarly to the proof for the case $\ell = 8$ in the previous section, unrolling the history of the heavy edges according to the given Painter strategy yields that each heavy edge implies at least 4 additional edges (3 red and 1 blue), and two adjacent heavy edges imply at least 5 additional edges (4 red and 1 blue, or 3 red and 2 blue). We can thus partition the blue P_k (minus a constant number of border edges) according to the marked edge pattern into segments of length 2 and 3 such that there are at least 4 additional edges for each blue segment of length 2, and at least 5 additional edges for each blue segment of length 3. Hence, in total the Painter strategy guarantees at least $\frac{5}{3}k + \mathcal{O}(1)$ additional edges for any blue path of length k , as required. \square

3.6. Open questions

Let us conclude this chapter by stating some open questions. We see two main directions for possible future work. On the one hand, it would be interesting to further investigate the relation between the probabilistic one-player and the deterministic two-player game, with the goal of deriving ‘abstract’ results in the vein of Theorem 3.4 and Theorem 3.5.

Theorem 3.4 suggests to define, for any graph F and any integer r , the *online Ramsey density* $m_2^*(F, r)$ as

$$m_2^*(F, r) := \inf \left\{ d \in \mathbb{R} \mid \text{Builder has a winning strategy in the deterministic } F\text{-avoidance game with } r \text{ colors and density restriction } d \right\}, \quad (3.12)$$

i.e., as the infimum over all d for which the theorem is applicable. The parameter $m_2^*(F, r)$ can be seen as a natural combination of the two well-established concepts of Ramsey densities and online size Ramsey numbers discussed in Section 2.1 and Section 2.3, respectively. In view of the many open questions revolving around these notions, it is not so surprising that also the online Ramsey densities studied here do not seem to be easily tractable.

In our view, the main open question is the following.

QUESTION 3.10. Is it true that for any graph F and any integer $r \geq 2$, the threshold of the online F -avoidance game with r colors is

$$N_0(F, r, n) = n^{2-1/m_2^*(F, r)} \quad ?$$

As we shall see in the next chapter, for the vertex-coloring variant of the game studied here, the answer to the analogous question turns out to be ‘yes’. For the edge-coloring problem studied here however, Question 3.10 is wide open.

The results in this thesis answer Question 3.10 in the affirmative for the case of cycles and two colors, and for the case of forests and an arbitrary number of colors. In both cases the infimum in (3.12) is attained as a minimum, which in particular implies that $m_2^*(F, r)$ is rational. In general, we do not even know whether for all F and r the threshold is of the form $N_0(F, r, n) = n^{2-1/x}$ for some rational number $x = x(F, r)$. (Note that a threshold that is not of this form would necessarily have to be sharp, in contrast to what is the case in all known examples; cf. the remarks after Theorem 2.1 of [Fri05].) Let us also point out that Question 3.10 remains open for cliques and two colors, even though an explicit threshold function is known for this case (cf. Theorem 3.2).

A second goal for further research would be to derive further explicit threshold formulas for some specific graphs, either by applying Theorem 3.4 or by coming up with new proof techniques. In [MSS09b], it was conjectured that for cliques and cycles and any number of colors, the lower bound given by Theorem 3.1 is in fact the threshold of the probabilistic game. Can a matching upper bound be derived from Theorem 3.4? The result by Balogh and Butterfield can be seen as a first step in that direction (recall Example 3 in Section 3.1.1). Similarly, it would be interesting to either improve upon the lower bound in Theorem 3.6 by deriving better strategies for Painter, or to show that no improvement by more than a constant factor is possible.

CHAPTER 4

The vertex-coloring setting

In this chapter we present our solution to the online vertex-coloring problem in random graphs via a suitably defined deterministic two-player game. These results are joint work with Thomas Rast and Reto Spöhel; the corresponding paper [MRS11⁺] is currently under review.

4.1. Introduction

The study of colorability properties of random graphs has a rich history and has spurred many important developments in random graph theory. Thanks to the efforts of many researchers (e.g., [GM75, Mat87, Bol88, Łuc91a, Łuc91b, McD90, AK97, AF99, COPS08]), very precise bounds on the chromatic number of the random graph are known by now. More recently, also several related coloring notions and their associated ‘chromatic numbers’ have been investigated for the random graph (e.g., [LRV92, Sch92, BT95, AKS99, BFS08, LS08, KP09, KM10]).

Here we are concerned with the following generalized notion of colorability: We say that a coloring of a graph G is *valid* with respect to some given graph F if it contains no monochromatic copy of F , i.e., if there is no copy of F in G whose vertices all receive the same color. Note that a proper coloring in the usual sense is a coloring that is valid with respect to a single edge ($F = K_2$). More generally, a coloring that is valid with respect to the star with ℓ rays is a coloring in which each color class induces a graph with maximum degree at most $\ell - 1$ (this is sometimes called an $(\ell - 1)$ -improper coloring, see [KM10] and references therein).

The main motivation for studying this notion of colorability comes from Ramsey theory, where one usually considers similarly defined *edge*-colorings. The threshold for the existence of a valid vertex-coloring of the random graph with respect to some given fixed graph F was determined by Łuczak, Ruciński, and Voigt [LRV92]; their result is stated in Theorem 2.3. Recall that we call a graph (F, r) -*vertex-Ramsey* if it does *not* allow a valid r -coloring with respect to F .

Implicit in the lower bound proof of Theorem 2.3 is the existence of a polynomial-time algorithm that a.a.s. succeeds in finding a valid coloring of $G_{n,p}$ for $p(n) \leq cn^{-1/m_1(F)}$, where polynomial here and throughout means polynomial in n for F and r fixed.

In this chapter, we study the same coloring problem in an *online setting*, and derive results of the same generality as those stated in Theorem 2.3 for the offline case.

4.1.1. The online setting. We consider the following online problem: The vertices of an initially hidden instance of $G_{n,p}$ are revealed one by one in increasing order, and at each step of the process only the edges induced by the vertices revealed so far are visible. Alternatively, one can think of the random edges leading from each vertex to previous vertices as being generated at the moment the vertex is revealed (each edge being inserted with probability p independently from all other edges). Each vertex has to be colored immediately and irrevocably with one of

r available colors as soon as it is revealed, with the goal of avoiding monochromatic copies of a fixed graph F as before.

It follows from standard arguments (see [MSS09a, Lemma 2.1]) that this online problem has a threshold $p_0(F, r, n)$ in the following sense: For any function $p(n) = o(p_0)$ there is a strategy that a.a.s. finds an r -coloring of $G_{n,p}$ that is valid with respect to F online, and for any function $p(n) = \omega(p_0)$ any online strategy will a.a.s. fail to do so. (Observe that no computational restrictions are imposed in this definition, i.e., the coloring strategy is *not* required to be an efficient algorithm.)

Note that this is a weaker threshold behavior than the one stated in Theorem 2.3. A closer inspection of the arguments in this chapter shows that the online thresholds are indeed coarser than the offline thresholds given by Theorem 2.3: the limiting probability that a valid coloring can be found online is a constant bounded away from 0 and 1 whenever $p(n)$ has the same order of magnitude as the threshold $p_0(F, r, n)$. This is a consequence of the fact that the online thresholds turn out to be determined by *local* substructures (see [JLR00, Theorem 3.9]).

The online problem was first studied in [MS10], where the following simple strategy was analyzed. Assuming that the colors are numbered from 1 to r , the *greedy strategy* fixes a sequence of subgraphs $H_1, \dots, H_r \subseteq F$, and at each step uses the highest-numbered color $s \in [r]$ that does not complete a monochromatic copy of H_s (or color 1 if no such color exists). Note that this strategy can easily be implemented in polynomial time.

The subgraphs of F that are relevant for the greedy strategy are determined as follows: For any graph F and any integer $r \geq 1$ we define the parameter $\overline{m}_1(F, r)$ recursively by

$$\overline{m}_1(F, r) := \begin{cases} \max_{H \subseteq F} \frac{e(H)}{v(H)} & \text{if } r = 1, \\ \max_{H \subseteq F} \frac{e(H) + \overline{m}_1(F, r-1)}{v(H)} & \text{if } r \geq 2, \end{cases} \quad (4.1)$$

and the subgraphs $H_1, \dots, H_r \subseteq F$ considered by the greedy strategy are given by graphs $H \subseteq F$ for which the maximum in the definition (4.1) is attained.

The results of [MS10] can be stated as follows.

THEOREM 4.1 ([MS10]). *For any fixed graph F with at least one edge and any fixed integer $r \geq 2$, the threshold for finding an r -coloring of $G_{n,p}$ that is valid with respect to F online satisfies*

$$p_0(F, r, n) \geq n^{-1/\overline{m}_1(F, r)},$$

where $\overline{m}_1(F, r)$ is defined in (4.1). A polynomial-time algorithm that succeeds a.a.s. for any $p(n) = o(n^{-1/\overline{m}_1(F, r)})$ is given by the greedy strategy.

If F has an induced subgraph $F^\circ \subsetneq F$ on $v(F) - 1$ vertices satisfying

$$m_1(F^\circ) \leq \overline{m}_1(F, 2), \quad (4.2)$$

where $m_1(F^\circ)$ is defined in (2.7), the greedy strategy is best possible, i.e., the threshold is

$$p_0(F, r, n) = n^{-1/\overline{m}_1(F, r)}.$$

In Chapter 5 we list several natural graph families (cliques, cycles, complete bipartite graphs, hypercubes, wheels and stars of arbitrary fixed size) which satisfy the condition in the second part of Theorem 4.1 and for which therefore explicit threshold functions are known.

It was also pointed out in [MS10] that the greedy strategy is *not* best possible in general — as we shall see in Chapter 5, the innocent-looking case where F is a (long) path is an example where the threshold is significantly higher than what is guaranteed by Theorem 4.1.

Our main result of this chapter is a combinatorial characterization of the online threshold that allows us to compute, for any F and r , a value $\gamma = \gamma(F, r)$ such that the threshold is given by $p_0(F, r, n) = n^{-\gamma}$. We also obtain polynomial-time coloring algorithms that a.a.s. find valid colorings of $G_{n,p}$ online in the entire regime below the respective thresholds, i.e., for any $p(n) = o(p_0)$.

4.1.2. A general characterization of the online threshold. Our main result characterizes the general threshold for the online problem in terms of a *deterministic two-player game*, which we describe in the following. The two players are called *Builder* and *Painter*, and the board is a graph that grows in each step of the game. Painter wants to maintain a valid coloring of the board, and her opponent Builder tries to prevent her from doing so by forcing her to create a monochromatic copy of F .

The game starts with an empty board, i.e., no vertices are present at the beginning of the game. In each step, Builder presents a new vertex and a number of edges leading from previous vertices to this new vertex. Painter has to color the new vertex immediately and irrevocably with one of r available colors, and as before she loses as soon as she creates a monochromatic copy of F . Note that so far this is the same setting as before, except that we replaced ‘randomness’ by the second player Builder. (Put differently, if Builder presents every possible edge with probability p independently, this is exactly the online process introduced above.) However, we additionally impose the restriction that Builder is not allowed to present an edge that would create a (not necessarily monochromatic) subgraph H with $e(H)/v(H) > d$ on the board, for some fixed real number d known to both players. In other words, Builder must adhere to the restriction that the evolving board B satisfies $m(B) = \max_{H \subseteq B} e(H)/v(H) \leq d$ at all times. We will refer to this game as the *deterministic F -avoidance game with r colors and density restriction d* .

We say that *Builder has a winning strategy* in this game (for a fixed graph F , a fixed number of colors r , and a fixed density restriction d) if he can force Painter to create a monochromatic copy of F within a finite number of steps. Conversely, we say that *Painter has a winning strategy* if she can avoid creating a monochromatic copy of F for an arbitrary number of steps. Note that if for some fixed F and r , Builder has a winning strategy for some density restriction d , then he also has a winning strategy for every density restriction $d' \geq d$. We say that a Painter or Builder strategy is *optimal* if it is a winning strategy simultaneously for all d for which the respective player has a winning strategy.

For any graph F and any integer $r \geq 2$ we define the *online vertex-Ramsey density* $m_1^*(F, r)$ as

$$m_1^*(F, r) := \inf \left\{ d \in \mathbb{R} \mid \begin{array}{l} \text{Builder has a winning strategy in the deterministic} \\ \text{\textit{F}-avoidance game with } r \text{ colors and density restriction } d \end{array} \right\}. \quad (4.3)$$

It is not hard to see that $m_1^*(F, r)$ is indeed well-defined for any F and r . With these definitions in hand, our results can be stated as follows.

THEOREM 4.2. *For any graph F with at least one edge and any integer $r \geq 2$, the online vertex-Ramsey density $m_1^*(F, r)$ is a computable rational number, and the infimum in (4.3) is attained as a minimum.*

THEOREM 4.3. *For any fixed graph F with at least one edge and any fixed integer $r \geq 2$, the threshold for finding an r -coloring of $G_{n,p}$ that is valid with respect to F online is*

$$p_0(F, r, n) = n^{-1/m_1^*(F, r)} , \quad (4.4)$$

where $m_1^*(F, r)$ is defined in (4.3). A polynomial-time algorithm that succeeds a.a.s. for any $p(n) = o(p_0)$ can be derived from one of Painter's optimal strategies in the deterministic two-player game.

Theorem 4.3 reduces the problem of determining the threshold of the original probabilistic problem to the purely deterministic combinatorial problem of computing $m_1^*(F, r)$ or, informally speaking, of 'solving' the deterministic two-player game. According to Theorem 4.2, the latter is possible by a finite computation; note that in the asymptotic setting of Theorem 4.3, this is in fact a constant-size computation.

It follows from the results of [MS10] that for any graph F we have

$$\lim_{r \rightarrow \infty} m_1^*(F, r) = m_1(F) .$$

Thus the online thresholds approach the offline threshold given by Theorem 2.3 as the number of colors increases. It also follows from an example presented in [MS10] that if F is the disjoint union of two graphs H_1 and H_2 , the parameter $m_1^*(F, r)$ may be strictly higher than both $m_1^*(H_1, r)$ and $m_1^*(H_2, r)$. (Such a behavior cannot occur for the parameter $m_1(F)$ appearing in Theorem 2.3.)

4.1.3. Remarks on Theorem 4.2. To put Theorem 4.2 into perspective, we mention that none of its three statements (computable, rational, infimum attained as minimum) is known to hold for the offline counterpart of $m_1^*(F, r)$, i.e., for the vertex-Ramsey density $m_1^o(F, r)$ introduced in [RR93] and further studied in [KR94] (recall the definition of this parameter in (2.4)). It is also not known whether such statements are true for the two analogous parameters $m_2^o(F, r)$ and $m_2^*(F, r)$ related to *edge*-colorings (recall their definitions in (2.3) and (3.12), respectively).

As is the case for many parameters in Ramsey theory, computing the online vertex-Ramsey density $m_1^*(F, r)$ becomes intractable already for moderately large graphs F . We have an implementation that computes $m_1^*(F, 2)$ for all graphs F with at most 7 vertices in under 10 minutes on an ordinary desktop computer. Using more computational power, we managed to determine $m_1^*(F, 2)$ exactly for all non-forests on at most 9 vertices. As it turns out, for forests our general procedure for computing $m_1^*(F, r)$ can be simplified considerably, an insight that allowed us to determine $m_1^*(F, 2)$ for all forests on at most 10 vertices (see [Häl11] and [Wac11] for details). As we shall see in Chapter 5, for paths our general procedure can be replaced altogether by a simple recursion, allowing us to compute $m_1^*(F, 2)$ for all paths with at most 45 vertices. Those programs and the values for $m_1^*(F, r)$ we found can be downloaded from the authors' website [Müt] (the values for paths are given in Chapter 5). There might be some room for improvement here, but it seems unrealistic to compute $m_1^*(F, 2)$ for, say, general graphs with 20 vertices with our approach in reasonable time. (Recall that the Ramsey number $R(K_5)$ is still unknown!)

4.1.4. Remarks on Theorem 4.3. The intuition behind Theorem 4.3 is the following: It is well-known that for any fixed graph G , a.a.s. the random graph $G_{n,p}$ with $p(n) = \omega(n^{-1/m(G)})$ contains a copy of G (this is the 1-statement of the Bollobás result [Bol81] stated in Theorem 2.1). The textbook second moment method proof of this fact (see e.g. [JLR00, Section 3])

can be adapted to show that for $p(n) = \omega(n^{-1/d})$ and any fixed finite Builder strategy for the deterministic two-player game that respects the density restriction d , a.a.s. the random process will exactly reproduce the given Builder strategy somewhere on the board. Thus if Builder has a winning strategy for a graph F and some given density restriction d , then in the probabilistic process with $p(n) = \omega(n^{-1/d})$, any online algorithm will a.a.s. be forced to create a monochromatic copy of F somewhere on the board. Consequently, the threshold of the probabilistic problem satisfies $p_0(F, r, n) \leq n^{-1/d}$. This argument is completely generic in the sense that it does not require any assumptions on the structure of Builder's winning strategy. The underlying connection between the probabilistic and a deterministic variant of the same problem was already exploited in the previous chapter for the edge-coloring version of the problem studied here (see Theorem 3.4). The main contribution here is that for the vertex-coloring problem, the best upper bound on the online threshold resulting from this approach is tight (recall (4.3) and (4.4)).

By the argument we just sketched, every winning strategy for Builder in the deterministic game translates to an upper bound on the threshold of the probabilistic problem. It seems to be much harder to prove an equally general statement translating *Painter's* winning strategies in the deterministic game to *lower* bounds on the threshold of the probabilistic problem. The reason for this is that the probabilistic process satisfies the density restriction imposed on Builder only *locally*: Even though the random graph $G_{n,p}$ with $p(n) = \Theta(n^{-1/d})$ a.a.s. contains no *constant-sized* graphs G with $m(G) > d$, the density of larger subgraphs is unbounded — in particular, the expected density of the entire random graph is $\binom{n}{2}p/n = \Theta(n^{1-1/d})$, which is unbounded for $d > 1$. Consequently, winning strategies for Painter in the deterministic game do not automatically give rise to successful coloring strategies for the probabilistic problem. In order to nevertheless establish the desired lower bound result we will need a quite detailed understanding of the *structure* of Painter's and Builder's optimal strategies in the deterministic game.

Theorem 4.3 establishes a general correspondence between the original probabilistic problem and the deterministic two-player game. We are not aware of any other results that establish a similar correspondence between probabilistic and deterministic variants of the same problem. In particular, such a correspondence does not hold for the offline version of the problem studied here: according to Theorem 2.3, the threshold for the existence of a valid r -coloring w.r.t. F in the probabilistic setting is determined by the parameter $m_1(F)$, and not by the parameter $m_1^*(F, r)$ defined in (2.4) coming from the corresponding deterministic problem (in general we have $m_1(F) \neq m_1^*(F, r)$).

4.1.5. Algorithms for efficiently coloring random graphs online. We now describe the structure of the coloring algorithms that arise from our approach, and their relation to Painter's optimal strategies in the deterministic game.

We use the concept of *ordered graphs*. An ordered graph is a graph with an associated ordering of its vertices, where this ordering is interpreted as the order in which these vertices appeared in the probabilistic process or the deterministic game. We will see that for any graph F and any integer r , there exists an optimal Painter strategy (i.e., a strategy that is a winning strategy for any density restriction $d < m_1^*(F, r)$) that can be represented as a *priority list over ordered monochromatic subgraphs of F* . Such a priority list is computed along with $m_1^*(F, r)$ in our approach, and encodes the relative 'level of danger' Painter associates with copies of a given ordered subgraph of F in a given color. In the asymptotic setting of the probabilistic process, determining such a priority list is a constant-size computation.

Given this priority list, Painter's strategy is the following: Whenever Builder presents a new vertex, Painter determines for each color the most dangerous ordered graph that would be completed if the new vertex were assigned this color, and then selects the color for which this most dangerous graph is least dangerous among all colors. (Observe that this requires Painter to memorize the order in which the vertices on the board arrived.) Note that this strategy based on a priority list can be easily implemented in polynomial time.

As we shall see, for any F and r we can compute a priority list such that the strategy represented by it is not only (i) a winning strategy for Painter in the deterministic game with density restriction d for any $d < m_1^*(F, r)$, but also (ii) a (polynomial-time) algorithm that succeeds a.a.s. in finding a valid coloring of $G_{n,p}$ online for any $p(n) = o(n^{-1/m_1^*(F, r)})$. (Recall from Section 4.1.4 that (i) does not automatically imply (ii)!)

4.1.6. Is there an explicit formula for $m_1^*(F, r)$? From Theorem 4.1 and Theorem 4.3 it follows that for any graph F and any $r \geq 2$, the greedy strategy guarantees a lower bound for the online vertex-Ramsey density of $m_1^*(F, r) \geq \overline{m}_1(F, r)$ with $\overline{m}_1(F, r)$ defined in (4.1). Furthermore, if F has an induced subgraph $F^\circ \subsetneq F$ on $v(F) - 1$ vertices satisfying (4.2), then the greedy strategy is optimal, i.e., we have $m_1^*(F, r) = \overline{m}_1(F, r)$. (Of course, this can also be proved directly by considering only the deterministic game.)

The question arises whether also for general graphs F the abstract definition of $m_1^*(F, r)$ in (4.3) can be replaced by an explicit formula, perhaps by suitably generalizing the definition (4.1). In Chapter 5 we present some evidence that such a general closed formula does not exist by showing that the parameter $m_1^*(F, r)$ exhibits a surprisingly complex behavior already in the innocent-looking case when F is a (long) path.

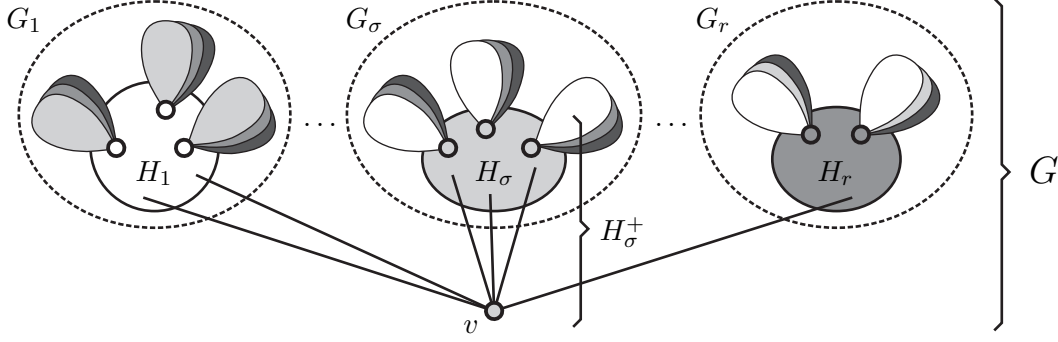
4.1.7. Organization of this chapter. Before actually proving Theorem 4.2 and Theorem 4.3, we informally present the main ideas behind our proofs in Section 4.2. In Section 4.3 we describe our procedure to compute the online vertex-Ramsey density $m_1^*(F, r)$ for any F and r . In this section we formulate two central propositions (Proposition 4.4 and Proposition 4.5 below) which together show that an optimal Builder strategy and an optimal Painter strategy for the deterministic game can be derived from this procedure. The proof of Theorem 4.2 is based on these two propositions and is also presented in Section 4.3.

In Section 4.4 we prove Proposition 4.4 by deriving an explicit Builder strategy from the procedure presented in Section 4.3, and in Section 4.5 we prove Proposition 4.5 by deriving an explicit Painter strategy from the same procedure. These two sections can be read independently from each other.

In Section 4.6 we finally turn to the original probabilistic problem and present the proof of Theorem 4.3. While the upper bound proof is completely self-contained and can be read independently from all other proofs, the lower bound proof relies very much on our analysis of the deterministic game in the preceding sections.

4.2. Proof ideas

In this section we aim to give an informal description of the main ideas behind our proofs. We will first focus on our procedure for computing the online vertex-Ramsey density, and then briefly comment on the proofs of Theorem 4.2 and Theorem 4.3.

FIGURE 4.1. Builder enforces larger monochromatic subgraphs of F from smaller ones.

4.2.1. Computing the online vertex-Ramsey density. Throughout this section, we focus on the deterministic game and sketch the underlying ideas in our procedure for computing the online vertex-Ramsey density $m_1^*(F, r)$ for given F and r . Note that the following is *not* a proof sketch of Theorem 4.2 — rather, our goal in this section is to develop some intuition for how one arrives at the key definitions which stand at the very beginning of our formal arguments.

4.2.1.1. Basic observations. Consider a family $\{G_1, \dots, G_f\}$ of disjoint copies of the same graph G on the board, and suppose that Builder adds vertices v_1, \dots, v_f to the board connecting v_i to G_i in exactly the same way for all $1 \leq i \leq f$. Then, by the pigeonhole principle, for a $(1/r)$ -fraction of the new vertices, Painter's coloring decision will be the same and result in copies of the same r -colored graph G^+ . By performing this pigeonholing in each step of his strategy, Builder can thus force Painter to always create *many* copies of one of the r graphs G^+ that Painter may choose from. Consequently, in the following we may assume w.l.o.g. that whenever Builder manages to enforce an r -colored graph G^+ on the board, he has as many such copies available as he needs in further steps.

As it turns out, the only type of move that is useful for Builder is of the following form: Assume that for each of the colors $s \in [r]$ the board contains a monochromatic copy of some subgraph H_s of F in color s . Then Builder can force Painter to extend one of these copies to a monochromatic copy of a subgraph H_σ^+ of F with $v(H_\sigma^+) = v(H_\sigma) + 1$ for a color $\sigma \in [r]$ by presenting a new vertex v and connecting it appropriately to the already existing monochromatic copies of H_1, \dots, H_r (see Figure 4.1). Furthermore, w.l.o.g. Builder will always perform such a step using monochromatic copies of the graphs H_1, \dots, H_r that have evolved independently from each other so far, and that are therefore contained in distinct components of the board (playing like this throughout will not increase the density restriction d for which Builder's strategy is legal). Proceeding in this fashion, Builder step by step enforces larger monochromatic subgraphs of F from smaller ones, and eventually a monochromatic copy of F (if the density restriction allows it).

Each monochromatic copy of some subgraph H of F created in this way is contained in a larger 'history graph' G that encodes all of Builder's construction steps that lead to the monochromatic copy of H . Using the notation from the preceding paragraph, the history graph G of H_σ^+ arises as the union of the history graphs G_1, \dots, G_r of the copies of H_1, \dots, H_r (due to our assumption on how Builder plays, these r history graphs are disjoint from each other), the vertex v and the edges that connect v to the copies of H_1, \dots, H_r in G_1, \dots, G_r .

4.2.1.2. Exploring Builder's options. The key ingredient in our approach is a systematic exploration from Builder's point of view which monochromatic subgraphs of F he can enforce against

a *fixed* Painter strategy. Our final procedure for computing $m_1^*(F, r)$ will have to branch on different coloring decisions of Painter, each branching corresponding to a different Painter strategy, but these branchings do not interfere with the ideas we want to present here. In the following we therefore assume that Painter plays according to a fixed strategy, and explain on an intuitive level how Builder can determine the smallest density restriction d for which he can enforce a monochromatic copy of F against the given Painter strategy.

As a first approach to such a systematic exploration, Builder could maintain for each color $s \in [r]$ a list \mathcal{H}_s of all subgraphs H_s of F for which he has already enforced a monochromatic copy in color s against the given Painter strategy, and also record the specific way in which the graph H_s can be enforced by storing the corresponding history graph G_s . Builder can then use entries $(H_s, G_s) \in \mathcal{H}_s$, one for every color $s \in [r]$, to create new entries $(H_\sigma^+, G) \in \mathcal{H}_\sigma$ in the manner described above (see Figure 4.1), and compute for each such step the smallest density restriction d for which this step is legal. (Recall that by appropriate pigeonholing in each step, Builder can create as many copies of each entry as he needs on the board.) There is no obvious termination criterion for this procedure, i.e., without further arguments Builder can never be sure that he found the *smallest* possible density restriction d for which he can enforce a monochromatic copy of F against Painter's fixed strategy (it could be that by building larger and larger graphs he discovers new ways to enforce F that are compliant with smaller and smaller density restrictions). In the following we will sketch how this approach can be refined to eventually yield a procedure which is guaranteed to find the smallest such d in a finite number of steps.

4.2.1.3. A generalized density restriction. Note that each new history graph G arising in a given step of Builder has a *recursive* structure. Unfortunately, for computing the smallest admissible density restriction for which this step is legal the recursive structure of G does not help. However, by suitably generalizing our concept of density restriction the recursive structure of G can indeed be exploited.

For a fixed real number $\theta > 0$ and any graph H we define

$$\mu_\theta(G) := v(H) - e(H) \cdot \theta, \quad (4.5)$$

and consider the following generalization of the deterministic F -avoidance game with r colors and density restriction d : For fixed real parameters $\theta > 0$ and β we require that Builder adheres to the restriction that every subgraph H of the evolving board B with $v(H) \geq 1$ satisfies

$$\mu_\theta(H) \geq \beta. \quad (4.6)$$

We refer to this game as the *deterministic F -avoidance game with r colors and generalized density restriction (θ, β)* . For any graph F with at least one edge, any integer $r \geq 2$ and any real number $\theta > 0$ we define the parameter

$$\beta^*(F, r, \theta) := \sup \left\{ \beta \in \mathbb{R} \left| \begin{array}{l} \text{Builder has a winning strategy in the deter-} \\ \text{ministic } F\text{-avoidance game with } r \text{ colors and} \\ \text{generalized density restriction } (\theta, \beta) \end{array} \right. \right\}. \quad (4.7)$$

Before discussing how this generalized game allows us to exploit the recursive structure of Builder's construction steps, let us explain how it relates to the original game with density restriction d that we are actually interested in.

Note that for any $\theta > 0$, the game with generalized density restriction $(\theta, 0)$ is equivalent to the game with density restriction $d = 1/\theta$. Together with the definition in (4.7) it follows that if for a given $\theta > 0$ we have $\beta^*(F, r, \theta) < 0$, then Painter has a winning strategy in the game with density restriction $d = 1/\theta$, and if $\beta^*(F, r, \theta) > 0$, then Builder has a winning strategy in the game with

density restriction $d = 1/\theta$. So intuitively speaking, computing the online vertex-Ramsey density $m_1^*(F, r)$ is equivalent to determining the root of $\beta^*(F, r, \theta)$, although it is not clear yet whether such a root exists and whether it is unique.

As it turns out, $\beta^*(F, r, \theta)$ does indeed have a unique root $\theta^* = \theta^*(F, r)$, and the online vertex-Ramsey density $m_1^*(F, r)$ satisfies $m_1^*(F, r) = 1/\theta^*$. Furthermore, we can show that the root θ^* lies in an explicitly given finite set $Q = Q(F, r)$ of rational numbers. Therefore, it is straightforward to compute $m_1^*(F, r)$ provided we can compute $\beta^*(F, r, \theta)$ for given rational values of θ . We describe a procedure that does essentially that, with one major caveat: Observe that if $\beta \geq 0$, then the condition (4.6) holds for all subgraphs of the board if and only if it holds for all *connected* subgraphs. Our approach makes crucial use of this observation, and consequently our procedure computes $\beta^*(F, r, \theta)$ exactly for any input parameters F, r, θ for which $\beta^*(F, r, \theta) \geq 0$, but returns meaningless negative values on input parameters for which $\beta^*(F, r, \theta) < 0$. This makes no difference for our purposes since in order to find the root of $\beta^*(F, r, \theta)$ it suffices to check whether $\beta^*(F, r, \theta)$ equals zero for given values of $\theta \in Q$.

In the following we explain how the generalized density restriction allows us to exploit the recursive structure of the history graphs arising in the game. As before our viewpoint is that we are exploring Builder's options against a fixed strategy of Painter. More precisely, we consider a fixed value of $\theta > 0$, and our goal now is to determine the largest value of β for which Builder can enforce a monochromatic copy of F in the game with generalized density restriction (θ, β) against the given Painter strategy. Combining this with the already mentioned branching on different strategies of Painter allows us to compute $\beta^*(F, r, \theta)$ as defined in (4.7).

4.2.1.4. From history graphs to vertex weights. We return to considering Builder's construction step in which monochromatic copies of subgraphs H_1, \dots, H_r of F with the corresponding history graphs G_1, \dots, G_r are connected to a new vertex v , and Painter's decision to assign color σ to v creates a copy of H_σ^+ in color σ with history graph G (see Figure 4.1). In order to find the largest $\beta \geq 0$ for which this step is legal in the game with generalized density restriction (θ, β) , we need to find the minimal value $\mu_\theta(J)$ among all connected subgraphs J of G that contain v (recall that we may assume that J is connected due to the assumption that $\beta \geq 0$). As $\mu_\theta(J) = v(J) - e(J) \cdot \theta$ as defined in (4.5) is linear in $e(J)$ and $v(J)$, a connected subgraph J of G containing v that minimizes $\mu_\theta(J)$ can be found recursively as follows: determine *independently* for each $s \in [r]$ the connected subgraph J_s of $G_s + v$ containing v that minimizes $\mu_\theta(J_s)$, where $G_s + v$ denotes the subgraph of G induced by v and all vertices of the copy of G_s in G . The graph J we are interested in is then given by the union of the graphs J_s for all $s \in [r]$. (Note that the subgraph J' of G that contains v and maximizes $e(J')/v(J')$ can *not* be found by independently considering each of the graphs $G_s + v$, $s \in [r]$ — this is precisely why we introduced the generalized notion of density restriction.) This independence allows us to compute the subgraph J of G that minimizes $\mu_\theta(J)$ *recursively* without remembering the actual structure of the history graphs G_s , $s \in [r]$. All the information that is necessary to do the same minimization in future steps (when the copy of H_σ^+ is extended to form larger subgraphs of F in color σ) can be stored by assigning the value $\sum_{s \in [r] \setminus \{\sigma\}} (\mu_\theta(J_s) - 1)$ to the vertex v in H_σ^+ (the -1 in the sum accounts for the fact that all the graphs $G_s + v$, $s \in [r]$, share the vertex v). In other words, we can condense the ‘history’ behind each of the vertices of a monochromatic copy of some subgraph of F into a single number. (Recall that we consider $\theta > 0$ to be fixed — this is crucial in all of the above.)

As a consequence, when maintaining the lists \mathcal{H}_s , $s \in [r]$, Builder no longer needs to store the entire history graph associated with some monochromatic subgraph H of F on one of these lists, but can store all the necessary information as a simple vertex-weighting of H . This greatly

reduces the amount of information Builder needs to keep track of, but does not yet solve the issue that there is no obvious termination criterion for Builder's exploration (Builder might still keep constructing new non-trivial entries forever).

4.2.1.5. Unique vertex weights via vertex orderings. In general, it may and will happen that the same subgraph H of F appears several times on one of Builder's lists with different vertex-weightings in such a way that none of these entries is redundant — depending on how H is used in future steps, different vertex-weightings of the same graph might be desirable from Builder's view. In other words, there is no unique best way of enforcing a copy of H in a given color for Builder.

It turns out, however, that different useful vertex-weightings can only arise if Builder presents the vertices of H in different orders (there are $v(H)!$ many different orders). For a fixed such order, there is a well-defined best vertex-weighting that Builder can achieve when enforcing H in that particular order. Thus to explore his options completely Builder only needs to compute *finite* lists \mathcal{H}_s , one for every color $s \in [r]$, which contain one entry for each vertex-ordering of every subgraph H of F .

This does not quite solve the issues we mentioned yet — it could still occur that Builder needs to recompute the vertex-weighting for a given entry many times because he finds better and better ways to enforce a given graph H in a particular order. To prevent this from happening, we need to be quite careful about the order in which we compute the entries of the lists \mathcal{H}_s — essentially we start by considering the game with generalized density restriction (θ, β) for the given fixed $\theta > 0$ and a very large β , and then successively lower β by the minimal amount that makes new options available to Builder. In each step we compute the weights for all graphs that Builder can create respecting the current generalized density restriction (θ, β) . This guarantees that we need to compute the weights for each graph only once, and therefore finally allows Builder to explore his options completely by a *finite* procedure.

4.2.1.6. Tying it all together. Along the lines sketched in the previous sections, we can compute $\beta^*(F, r, \theta)$ by dynamic programming over vertex-ordered subgraphs of F (provided that $\beta^*(F, r, \theta)$ is non-negative for the given $\theta > 0$, see the remarks in Section 4.2.1.3), branching on Painter's decisions as appropriate. The online vertex-Ramsey density $m_1^*(F, r)$ can then be derived from $\beta^*(F, r, \theta)$ as explained in Section 4.2.1.3. As this is now a finite procedure, it also follows that the supremum in (4.7) is attained as a maximum, which with some further arguments also implies that the infimum in (4.3) is attained as a minimum.

4.2.2. About the proof of Theorem 4.2. For any graph F and any integer r , let $\tilde{m}(F, r)$ denote the value computed by the procedure outlined in Section 4.2.1. We prove that $\tilde{m}(F, r)$ equals $m_1^*(F, r)$ as defined in (4.3) by constructing explicit winning strategies for Builder and Painter, for arbitrary density restrictions $d \geq \tilde{m}(F, r)$ and $d < \tilde{m}(F, r)$, respectively.

For Builder such a strategy follows from the general principles underlying the procedure sketched in Section 4.2.1: all steps of the dynamic program which is at the heart of our approach can be interpreted as actual construction steps on the board of the deterministic game.

For Painter, such a strategy can be recovered from the branching on Painter's decisions performed in our procedure — we show that the decisions corresponding to a 'worst' path in the branching tree (viewed from Builder's perspective) give rise to a Painter strategy that succeeds in avoiding a monochromatic copy of F against *any* Builder strategy. This Painter strategy can be encoded by a priority list as described in Section 4.1.5.

To prove the success of this strategy, we use a *witness graph argument*: Essentially, we show inductively that whenever a monochromatic copy of some ordered subgraph H of F in some color $s \in [r]$ appears on the board, then this copy is contained in a graph that is at least as dense as indicated by the weights computed for H and the color s by the dynamic program in our procedure. (Recall from Section 4.2.1 that these weights basically encode the density of the history graph corresponding to the best way for Builder to enforce a monochromatic copy of H in color s .) This invariant holds in particular for all vertex-orderings of the graph F and all colors $s \in [r]$, and implies that whenever a monochromatic copy of F is completed, the board contains a graph that violates the density restriction imposed on Builder.

The proof of Theorem 4.2 we just sketched also shows that there exists an integer $a_{\max} = a_{\max}(F, r)$ such that for any given density restriction d Builder never needs more than a_{\max} steps to enforce a monochromatic copy of F , if he is able to do so at all. Note that this statement alone directly implies all three assertions of Theorem 4.2, as it shows that $m_1^*(F, r)$ can also be computed trivially by exhaustive search over the finitely many possible ways Builder and Painter can play in a_{\max} steps of the game.

4.2.3. About the proof of Theorem 4.3. We have already discussed the proof of the upper bound part of Theorem 4.3 in Section 4.1.4; as mentioned this proof is self-contained and does not depend on the rest of this chapter. The proof of the lower bound part is much more involved and relies on the same witness graph approach as the argument for Painter's success in the deterministic game described in the previous section. However, there is the additional issue that, as explained in Section 4.1.4, the random graph $G_{n,p}$ with $p(n) = o(n^{-1/m_1^*(F,r)})$ satisfies a density restriction of $d = m_1^*(F, r)$ only locally and *not* globally. Consequently, in order to apply the witness graph argument outlined above to the probabilistic setting of Theorem 4.3, we also need to show that the size of the witness graphs resulting from our arguments is bounded by some constant $v_{\max} = v_{\max}(F, r)$ (and not, say, linear in n). Unfortunately, we cannot show this for *all* priority lists that represent optimal strategies for Painter in the deterministic game. However, by applying a number of further technical refinements to the procedure described in Section 4.2.1, we can guarantee that it only computes priority lists for which a constant v_{\max} as desired indeed exists. It follows with the same witness graph argument as before that these priority lists represent polynomial-time coloring algorithms that a.a.s. succeed in finding a valid coloring of $G_{n,p}$ online for any $p(n) = o(n^{-1/m_1^*(F,r)})$.

4.3. Computing the online vertex-Ramsey density

4.3.1. Proof of Theorem 4.2. Recall the definition of the deterministic F -avoidance game with r colors and generalized density restriction (θ, β) from Section 4.2.1.3, and recall further that, at least intuitively, computing the online vertex-Ramsey density $m_1^*(F, r)$ is equivalent to determining the root of $\beta^*(F, r, \theta)$ as defined in (4.7) (where existence and uniqueness of this root are not clear yet).

As already mentioned, we are going to derive a procedure that returns $\beta^*(F, r, \theta)$ for any $\theta > 0$ for which $\beta^*(F, r, \theta) \geq 0$, and a meaningless negative value for any $\theta > 0$ for which $\beta^*(F, r, \theta) < 0$. This procedure will be described in Section 4.3.3, and its output will be denoted by $\Lambda_\theta(F, r)$. We will see that the function $\Lambda_\theta(F, r)$ is well-defined for any real number $\theta > 0$, and for rational values of θ it can be computed using only integer arithmetic. Most of the remainder of this chapter will be devoted to the proofs of the following two key statements.

PROPOSITION 4.4 (Builder strategy from $\Lambda_\theta(F, r)$). *Let F be a graph with at least one edge and $r \geq 2$ an integer. There is a constant $a_{\max} = a_{\max}(F, r)$ such that the following holds: For any real numbers $\theta > 0$ and $\beta \geq 0$ with*

$$\Lambda_\theta(F, r) \geq \beta , \quad (4.8)$$

where $\Lambda_\theta()$ is defined in (4.23) below, Builder can enforce a monochromatic copy of F in the deterministic F -avoidance game with r colors and generalized density restriction (θ, β) in at most a_{\max} steps, regardless of how Painter plays.

PROPOSITION 4.5 (Painter strategy from $\Lambda_\theta(F, r)$). *Let F be a graph with at least one edge, $r \geq 2$ an integer, and $\theta > 0$ and $\beta \geq 0$ real numbers such that*

$$\Lambda_\theta(F, r) < \beta , \quad (4.9)$$

where $\Lambda_\theta()$ is defined in (4.23) below.

Then Painter can avoid creating a monochromatic copy of F in the deterministic F -avoidance game with r colors and generalized density restriction (θ, β) , regardless of how Builder plays.

Before going into any details about the procedure that defines $\Lambda_\theta(F, r)$, we show how Proposition 4.4 and Proposition 4.5 imply Theorem 4.2.

For technical reasons, our formal arguments do not rely on the parameter $\beta^*()$ defined in (4.7), but on a related parameter that we introduce now. For any graph F with at least one edge, any integer $r \geq 2$, any real number $\theta > 0$ and any integer $a \geq a_{\min} := r(v(F) - 1) + 1$, we define

$$\beta'(F, r, \theta, a) := \sup \left\{ \beta \in \mathbb{R} \left| \begin{array}{l} \text{Builder has a winning strategy in the deterministic} \\ F\text{-avoidance game with } r \text{ colors and generalized den-} \\ \text{density restriction } (\theta, \beta) \text{ in at most } a \text{ steps} \end{array} \right. \right\} . \quad (4.10)$$

Here the supremum is over a nonempty set of values because presenting the complete graph on a_{\min} vertices sequentially is a winning strategy for Builder that satisfies the generalized density restriction (θ, β) for any $\beta \leq \min\{k - \binom{k}{2} \cdot \theta \mid 1 \leq k \leq a_{\min}\}$. Note that for all F , r , and θ as before we have

$$\beta^*(F, r, \theta) = \sup_{a \geq a_{\min}} \beta'(F, r, \theta, a) = \lim_{a \rightarrow \infty} \beta'(F, r, \theta, a) . \quad (4.11)$$

As in the definition of $\beta'()$ in (4.10) there is only a finite number of possible Builder strategies to consider, it is not hard to derive the following properties of $\beta'()$.

LEMMA 4.6 (Properties of $\beta'(F, r, \theta, a)$). *For any graph F with at least one edge, any integer $r \geq 2$, any real number $\theta > 0$ and any integer $a \geq a_{\min}$, the supremum in (4.10) is attained as a maximum. For fixed F , r , and a as before, $\beta'(F, r, \theta, a)$ viewed as a function of $\theta > 0$ is continuous, non-increasing, piecewise linear, and has a unique root, which is contained in the set*

$$Q(a) := \{ 0 < \frac{v}{e} < 2 \mid v, e \in \mathbb{N} \wedge 1 \leq v \leq a \wedge 1 \leq e \leq \binom{v}{2} \} . \quad (4.12)$$

PROOF. We identify Builder's strategies in the deterministic two-player game with r colors with finite r -ary rooted trees, where each node at depth k of such a tree is an r -colored graph on k vertices, representing the board after the k -th step of the game. Specifically, the tree \mathcal{T} representing a given Builder strategy is constructed as follows: The root of \mathcal{T} is the null graph (the graph whose vertex set is empty). The r children of any node B at depth k of \mathcal{T} are obtained by adding the $(k+1)$ -th vertex of Builder's strategy to B (together with the edges that connect this vertex to previously added vertices according to Builder's strategy) and coloring it with one of the r available colors. Continuing like this, we construct \mathcal{T} , representing any situation in which Builder stops playing by a leaf of \mathcal{T} .

Note that in this formalization, a given tree \mathcal{T} represents a generic strategy for Builder (in the deterministic game with r colors) that may or may not satisfy a given generalized density restriction (θ, β) , and that can be thought of as a strategy for the ‘ F -avoidance’ game for any given graph F . We say that \mathcal{T} is a *winning strategy* for Builder in a specific F -avoidance game if and only if every leaf of \mathcal{T} contains a monochromatic copy of F . We say that a Builder strategy \mathcal{T} is a *legal strategy* in the game with generalized density restriction (θ, β) if and only if (4.6) is satisfied for every subgraph H with $v(H) \geq 1$ of every node B in \mathcal{T} .

Let F , r and $a \geq a_{\min}$ be given. As the number of steps of the game is bounded by a , there is only a finite family $\mathfrak{T} = \mathfrak{T}(r, a)$ of different Builder strategies, obtained by exhaustive enumeration of all possible ways to add a new vertex to the board. Let $\mathfrak{W} = \mathfrak{W}(F, r, a) \subseteq \mathfrak{T}$ denote the set of winning strategies for Builder for the given F , and recall that for $a \geq a_{\min}$ the family \mathfrak{W} is nonempty.

Note that for any winning strategy $\mathcal{T} \in \mathfrak{W}$ and for any fixed $\theta > 0$,

$$f_{\mathcal{T}}(\theta) := \min_{\substack{B \in \mathcal{T} \\ H \subseteq B: v(H) \geq 1}} \mu_{\theta}(H) \quad (4.13)$$

is the maximal value of β such that \mathcal{T} is a legal strategy in the game with generalized density restriction (θ, β) . Optimizing over the (finite and nonempty) set of winning strategies, we obtain $\beta'(F, r, \theta, a)$ as defined in (4.10) as

$$\beta'(F, r, \theta, a) = \max_{\mathcal{T} \in \mathfrak{W}} f_{\mathcal{T}}(\theta) . \quad (4.14)$$

We conclude that the supremum in (4.10) is attained as a maximum. In the following we derive the claimed properties of $\beta'(F, r, \theta, a)$ as a function of $\theta > 0$ by considering the functions $f_{\mathcal{T}}(\theta)$, $\mathcal{T} \in \mathfrak{W}$.

Using (4.13) and combining the properties of the linear functions $\mu_{\theta}(H)$ for all $H \subseteq B$ with $v(H) \geq 1$ and all $B \in \mathcal{T}$ it is not hard to see that for any $\mathcal{T} \in \mathfrak{W}$ the function $f_{\mathcal{T}}(\theta)$ satisfies the following properties:

- $f_{\mathcal{T}}(\theta)$ is continuous and piecewise linear.
- There is an $\varepsilon = \varepsilon(\mathcal{T}) > 0$ such that $f_{\mathcal{T}}(\theta) = 1$ for all $0 < \theta \leq \varepsilon$ and $f_{\mathcal{T}}(\theta)$ is strictly decreasing for all $\theta \geq \varepsilon$.
- $f_{\mathcal{T}}(\theta)$ has a unique root in the set $\{\frac{v}{e} \mid v, e \in \mathbb{N} \wedge 1 \leq v \leq a \wedge 1 \leq e \leq \binom{v}{2}\}$.

Note that the root of $f_{\mathcal{T}}(\theta)$ is strictly smaller than 2: For any winning strategy $\mathcal{T} \in \mathfrak{W}$, there is a leaf B in \mathcal{T} that contains a (not necessarily monochromatic) copy of P_3 as a subgraph. This is trivially true if $P_3 \subseteq F$ (as every leaf of \mathcal{T} contains a monochromatic copy of F). If $P_3 \not\subseteq F$, then F is a matching, and any strategy where Painter colors endpoints of isolated edges on the board with different colors corresponds to a root-leaf path in \mathcal{T} that does not end with a matching (as otherwise \mathcal{T} would not be a winning strategy for Builder). Thus in either case the graph $H = P_3$ is a subgraph of some node B of \mathcal{T} , and consequently the minimization in (4.13) includes the function $\mu_{\theta}(H) = 3 - 2 \cdot \theta$, whose root is strictly smaller than 2.

It follows with (4.14) that also $\beta'(F, r, \theta, a)$ satisfies the three properties listed above, and that its root is strictly smaller than 2. Combining those properties shows that $\beta'(F, r, \theta, a)$ satisfies the conditions claimed in the lemma. \square

Using Proposition 4.4, Proposition 4.5 and Lemma 4.6, we will prove the following explicit version of Theorem 4.2.

THEOREM 4.7 (Explicit Version of Theorem 4.2). *For any graph F with at least one edge and any integer $r \geq 2$, the online vertex-Ramsey density $m_1^*(F, r)$ defined in (4.3) satisfies*

$$m_1^*(F, r) = 1/\theta^*, \quad (4.15)$$

where $\theta^* = \theta^*(F, r)$ is the unique solution of

$$\Lambda_\theta(F, r) \stackrel{!}{=} 0 \quad (4.16)$$

and $\Lambda_\theta()$ is defined in (4.23) below.

Moreover, θ^* is a rational number from the set $Q(a_{\max})$, where $Q()$ is defined in (4.12) and a_{\max} is the constant guaranteed by Proposition 4.4. Furthermore, the infimum in (4.3) is attained as a minimum.

Theorem 4.2 is an immediate consequence of Theorem 4.7, observing that the solution of the equation (4.16) can be computed by evaluating $\Lambda_\theta(F, r)$ for all (finitely many) rational $\theta \in Q(a_{\max})$ (the constant a_{\max} is given explicitly in the proof of Proposition 4.4, see (4.96) below).

PROOF OF THEOREM 4.7. Throughout the proof we consider F and r fixed and let $a_{\max} = a_{\max}(F, r)$ denote the constant guaranteed by Proposition 4.4.

Proposition 4.4 and Proposition 4.5 imply that for any given $\theta > 0$ for which $\Lambda_\theta(F, r) \geq 0$, the parameter $\Lambda_\theta(F, r)$ is the maximal value of β for which Builder can win the deterministic game with generalized density restriction (θ, β) , and if he can win then he needs at most a_{\max} steps to enforce a monochromatic copy of F , i.e., $\Lambda_\theta(F, r)$ coincides with $\beta'(F, r, \theta, a_{\max})$ as defined in (4.10).

Recall that according to Lemma 4.6 the supremum in (4.10) is always attained as a maximum, i.e., for any $\theta > 0$ Builder has a winning strategy in the game with generalized density restriction $(\theta, \beta'(F, r, \theta, a_{\max}))$. Thus if $\beta'(F, r, \theta, a_{\max}) \geq 0$ we must have that $\Lambda_\theta(F, r) \geq \beta'(F, r, \theta, a_{\max})$ as otherwise we could apply Proposition 4.5 with $\beta = \beta'$ to obtain a contradiction. Hence also $\Lambda_\theta(F, r)$ is non-negative in that case.

It follows that for any $\theta > 0$ the two functions $\Lambda_\theta(F, r)$ and $\beta'(F, r, \theta, a_{\max})$ either coincide or are both negative. Thus in particular they have the same set of roots, which by Lemma 4.6 consists of a single rational number $\theta^* = \theta^*(F, r)$ from the set $Q(a_{\max})$.

Applying Proposition 4.4 with $\theta = \theta^*$ and $\beta = 0$ yields that Builder has a winning strategy in the game with generalized density restriction $(\theta^*, 0)$ (in at most a_{\max} steps). Conversely, for any $\theta > \theta^*$ we obtain with Lemma 4.6 that $\beta'(F, r, \theta, a_{\max})$ is negative which, as discussed above, implies that also $\Lambda_\theta(F, r)$ is negative. Consequently we may apply Proposition 4.5 with $\beta = 0$ to infer that Painter has a winning strategy in the game with generalized density restriction $(\theta, 0)$.

Recalling that for any $\theta > 0$ the game with generalized density restriction $(\theta, 0)$ is equivalent to the original deterministic game with density restriction $d = 1/\theta$, we may restate our findings as follows: Builder has a winning strategy in the game with density restriction $d = 1/\theta^*$ (in at most a_{\max} steps), and for any $d < 1/\theta^*$ Painter has a winning strategy in the game with density restriction d . I.e., the online vertex-Ramsey density defined in (4.3) satisfies $m_1^*(F, r) = 1/\theta^*$, and the infimum in (4.3) is attained as a minimum. \square

REMARK 4.8. Analogously to the second paragraph of the preceding proof it follows that for any given $\theta > 0$ for which $\Lambda_\theta(F, r) \geq 0$, also $\beta^*(F, r, \theta)$ as defined in (4.7) coincides with $\Lambda_\theta(F, r) = \beta'(F, r, \theta, a_{\max})$. Thus the unique root θ^* of $\Lambda_\theta(F, r) = \beta'(F, r, \theta, a_{\max})$ is also a root of $\beta^*(F, r, \theta)$.

Furthermore, the observation that the non-increasing functions $\beta'(F, r, \theta, a)$, $a \geq a_{\min}$, have a slope of at most -1 around their respective roots implies with (4.11) that the pointwise limit $\beta^*(F, r, \theta)$ has at most one root. Thus θ^* is indeed also the unique root of $\beta^*(F, r, \theta)$, as claimed in Section 4.2.1.3.

4.3.2. Definitions and notations. In order to present our procedure for computing the values $\Lambda_\theta(F, r)$ satisfying Proposition 4.4 and Proposition 4.5, we need to introduce a number of definitions and notations. Along with the definitions we give some intuition how those formal objects implement the ideas outlined in Section 4.2.1.

To simplify notation, for a graph H and any vertex v of H we abbreviate $v \in V(H)$ to $v \in H$. For a graph H and any set of vertices $U \subseteq V(H)$, we denote by $H \setminus U$ the graph obtained from H by removing all vertices in U and all edges incident to them. To indicate removal of a single vertex $v \in H$ we abbreviate $H \setminus \{v\}$ to $H \setminus v$.

4.3.2.1. Weighted graphs. A *vertex-weighted graph* is a graph H with a weight function $w : V(H) \rightarrow \mathbb{R}$. We refer to the values $w(u)$, $u \in H$, as *vertex weights*. Throughout this chapter, these vertex weights represent contributions to the linear function $\mu_\theta()$ defined in (4.5) that are obtained from ‘condensing’ history graphs as outlined in Section 4.2.1.4. They will always be non-positive.

For a fixed real number $\theta > 0$, any graph H , any vertex $v \in H$ and any weight function $w : V(H) \setminus \{v\} \rightarrow \mathbb{R}$ we define the value

$$d_\theta(H, v, w) := \min_{J \subseteq H : v \in J} \left(\sum_{u \in J \setminus v} (1 + w(u)) - e(J) \cdot \theta \right), \quad (4.17)$$

where the minimization is over all subgraphs J of H that contain the vertex v . As this minimization includes the graph J that consists only of the isolated vertex v , we always have $d_\theta(H, v, w) \leq 0$. Note that the minimum in (4.17) is always attained by an *induced* subgraph $J \subseteq H$. For convenience we will also use this notation for weight functions w whose domain is strictly larger than the set $V(H) \setminus \{v\}$. Of course, for the value of $d_\theta(H, v, w)$ only the values $w(u)$ of vertices $u \in H \setminus v$ are relevant.

The intuition behind the value $d_\theta(H, v, w)$ is the following: Assume that a copy of $H \setminus v$ is used as one of the graphs H_s in Figure 4.1, and Painter selects a color $\sigma \in [r]$ such that a copy of some other graph H_σ is extended to a copy of H_σ^+ . Then H becomes part of the history graph G of H_σ^+ , and the recursive contribution to the value $\mu_\theta(J)$ (as defined in (4.5)) of a subgraph $J \subseteq G$ minimizing $\mu_\theta(J)$ is exactly $d_\theta(H, v, w)$ if v is included in J . In our dynamic program, this will be recorded by adding a term of $d_\theta(H, v, w)$ to the vertex weight of v in H_σ^+ (and this is also how the vertex weights w of $H \setminus v$ were computed in earlier steps).

For a fixed real number $\theta > 0$, any graph H and any weight function $w : V(H) \rightarrow \mathbb{R} \cup \{-\infty\}$ we define

$$\lambda_\theta(H, w) := \sum_{u \in H} (1 + w(u)) - e(H) \cdot \theta. \quad (4.18)$$

As it is the case for the definition of $d_\theta()$ in (4.17), it is also convenient here to allow weight functions w whose domain is strictly larger than the set $V(H)$. Of course, for the value of $\lambda_\theta(H, w)$ only the values $w(u)$ of vertices $u \in H$ are relevant. Observe that $\lambda_\theta(H, w)$ defined in (4.18) can be written recursively for every vertex $v \in H$ as

$$\lambda_\theta(H, w) = \lambda_\theta(H \setminus v, w) + 1 + w(v) - \deg_H(v) \cdot \theta, \quad (4.19)$$

where $\deg_H(v)$ denotes the degree of v in H . This will be used several times in our arguments.

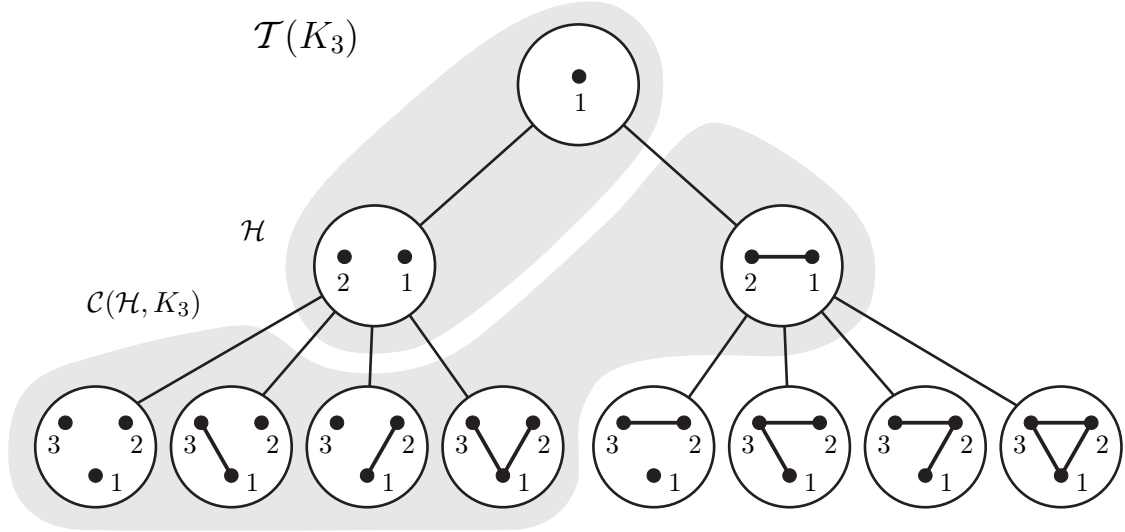


FIGURE 4.2. Illustration of the tree $\mathcal{T}(K_3)$ and the definition in (4.21) (the shaded regions represent subsets of nodes of $\mathcal{T}(K_3)$).

Using (4.5), $\lambda_\theta(H, w)$ defined in (4.18) can also be written as $\lambda_\theta(H, w) = \mu_\theta(H) + \sum_{u \in H} w(u)$, which intuitively means the following: If we imagine H to be at the center of a large history graph G , the parameter $\lambda_\theta(H, w)$ corresponds to the value $\mu_\theta(J)$ of the graph J obtained by attaching to each vertex $v \in H$ the $r - 1$ subgraphs that minimize $\mu_\theta(J_s)$ among all subgraphs J_s containing v in each of the $r - 1$ branches of the history graph G .

4.3.2.2. Ordered graphs. For any graph H , $h := v(H)$, a *vertex ordering* is a bijective mapping $\pi : V(H) \rightarrow \{1, \dots, h\}$, conveniently denoted by its preimages, $\pi = (\pi^{-1}(1), \dots, \pi^{-1}(h))$. An *ordered graph* is a pair (H, π) , where H is a graph and π is an ordering of its vertices. In the context of the F -avoidance game we interpret the ordering $\pi = (v_1, \dots, v_h)$ as the order in which the vertices of H appeared in the game, where v_h is the vertex that appeared first (we refer to it as the *oldest* vertex) and v_1 is the vertex that appeared last (we refer to it as the *youngest* vertex). We use $\Pi(V(H))$ to denote the set of all vertex orderings π of H .

For an ordered graph (H, π) and any subgraph $J \subseteq H$, we denote by $\pi|_J$ the order on the vertices of J induced by π . For any set $U \subseteq V(H)$ we use $\pi \setminus U$ as a shorthand notation for $\pi|_{H \setminus U}$. To indicate removal of a single vertex $v \in H$ we abbreviate $\pi \setminus \{v\}$ to $\pi \setminus v$.

Moreover, we define

$$\mathcal{S}(F) := \{[(H, \pi)]_\sim \mid H \subseteq F \text{ with } v(H) \geq 1 \text{ and } \pi \in \Pi(V(H))\} \quad (4.20)$$

as the family of all isomorphism classes of ordered subgraphs of F , where we write $(H, \pi) \sim (H', \pi')$ if (H, π) and (H', π') are isomorphic as ordered graphs. For simplicity we refer to the elements $[(H, \pi)]_\sim \in \mathcal{S}(F)$ in the following always as graphs $(H, \pi) \in \mathcal{S}(F)$. It is convenient to think of the graphs in $\mathcal{S}(F)$ as nodes of a rooted tree $\mathcal{T}(F)$ with root node $(K_1, (v_1))$ (an isolated vertex), where for each node $(H, \pi) \in \mathcal{S}(F)$, $\pi = (v_1, \dots, v_h)$, with $v(H) \geq 2$ the parent node is given by $(H \setminus v_1, \pi \setminus v_1)$. For any subset $\mathcal{H} \subseteq \mathcal{S}(F)$ we define the set $\mathcal{C}(\mathcal{H}, F) \subseteq \mathcal{S}(F)$ as

$$\mathcal{C}(\mathcal{H}, F) := \begin{cases} \{(K_1, (v_1))\} & \text{if } \mathcal{H} = \emptyset \\ \{(H, \pi = (v_1, \dots, v_h)) \in \mathcal{S}(F) \setminus \mathcal{H} \mid (H \setminus v_1, \pi \setminus v_1) \in \mathcal{H}\} & \text{otherwise} \end{cases} \quad (4.21)$$

Note that $\mathcal{C}(\mathcal{H}, F)$ is exactly the set of nodes of $\mathcal{T}(F)$ that are children of some node in \mathcal{H} , but that are not contained in \mathcal{H} . Figure 4.2 shows the tree $\mathcal{T}(F)$ for $F = K_3$ and illustrates the definition in (4.21).

REMARK 4.9. Note that for two graphs $H_1 \subsetneq H_2$ with $v(H_1) = v(H_2)$, a monochromatic copy of H_1 on the board can never evolve into a copy of H_2 later in the game, as new edges appear only incident to newly added vertices. As a consequence, we could restrict our attention to *induced* subgraphs of F in all of our arguments. While changing the definition of $\mathcal{S}(F)$ in (4.20) accordingly would indeed lead to some algorithmic savings (see Section 4.3.4), for our formal arguments we find it more convenient to include *all* subgraphs of F in the definition (4.20). Otherwise, unnecessary distraction would arise everytime an *induced* subgraph is mentioned in a proof.

4.3.3. The algorithm. In the following we present an algorithm `COMPUTEWEIGHTS()`, whose output is then used to define the function $\Lambda_\theta(F, r)$ that is referred to in Proposition 4.4 and Proposition 4.5.

Beside the graph F and the number of colors r , the algorithm has two more input parameters: the parameter θ from the generalized density restriction (see Section 4.2.1.3), and a finite sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ with the following interpretation: As indicated in Section 4.2.1, the underlying idea of the algorithm is to explore systematically from Builder's point of view which monochromatic ordered subgraphs of F he can enforce if Painter plays according to a fixed strategy. Step by step Builder enforces larger monochromatic subgraphs from smaller ones, and the appropriate vertex weights for these graphs are computed by dynamic programming. The sequence α encodes Painter's coloring decisions in the order they occur in the course of the algorithm (i.e., it represents a fixed Painter strategy), where an entry of this sequence may correspond to several coloring decisions of Painter for which she uses the same color. (Our proofs show that she would not gain anything by using different colors for these decisions.)

The algorithm maintains for each color $s \in [r]$ a family $\mathcal{H}_s \subseteq \mathcal{S}(F)$ of ordered subgraphs of F and a function $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$. The families \mathcal{H}_s correspond to the ordered subgraphs of F for which Builder has already enforced a monochromatic copy in color s . In the course of the algorithm, the families \mathcal{H}_s are successively enlarged. Initially, we have $\mathcal{H}_s = \emptyset$ for all $s \in [r]$, and at each step the candidate graphs to be added to the families \mathcal{H}_s are given by the sets $\mathcal{C}(\mathcal{H}_s, F)$ defined in (4.21); these correspond to the graphs that Builder can construct by adding a single vertex to a graph he has already enforced. Consequently, throughout the algorithm the families \mathcal{H}_s , $s \in [r]$, viewed as subsets of nodes of the tree $\mathcal{T}(F)$ defined after (4.20), grow downwards from the root.

For each $s \in [r]$ and each ordered graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, the function $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$ maintained by the algorithm induces a weight function $w_{(H, \pi, s)} : V(H) \rightarrow \mathbb{R}$ as follows: The weight $w_{(H, \pi, s)}(v_1)$ of the youngest vertex v_1 is given directly by $w_s(H, \pi)$; the weight $w_{(H, \pi, s)}(v_2)$ of the second-youngest vertex v_2 is given by $w_s(H \setminus v_1, \pi \setminus v_1)$, i.e., by the value of w_s for the parent of (H, π) in $\mathcal{T}(F)$; and so on. The full weight function $w_{(H, \pi, s)} : V(H) \rightarrow \mathbb{R}$ is therefore obtained by considering the value of w_s for all graphs on the path from (H, π) to the root $(K_1, (v_1))$ of the tree $\mathcal{T}(F)$, and each graph $(H, \pi) \in \mathcal{H}_s$ inherits all vertex weights except that of the youngest vertex from his ancestors in $\mathcal{T}(F)$.

More formally, and extending this construction to all graphs $(H, \pi) \in \mathcal{S}(F)$, we define for each $s \in [r]$ and each $(H, \pi) \in \mathcal{S}(F)$, $\pi = (v_1, \dots, v_h)$, the weight function

$$w_{(H, \pi, s)}(v_i) := \begin{cases} w_s(H \setminus \{v_1, \dots, v_{i-1}\}, \pi \setminus \{v_1, \dots, v_{i-1}\}) & \text{if } (H \setminus \{v_1, \dots, v_{i-1}\}, \pi \setminus \{v_1, \dots, v_{i-1}\}) \in \mathcal{H}_s, \\ -\infty & \text{otherwise.} \end{cases} \quad (4.22)$$

This notation will also be used in the formulation of `COMPUTEWEIGHTS()` in Algorithm 1 below. (We shall see that the algorithm never encounters the value $-\infty$ during its execution.) Note that an ordered graph $(H, \pi) \in \mathcal{S}(F)$ has vertices of weight $-\infty$ if and only if $(H, \pi) \in \mathcal{S}(F) \setminus \mathcal{H}_s$ for the corresponding $s \in [r]$, which intuitively means that Builder has not yet enforced a monochromatic copy of (H, π) in color s .

The families $\mathcal{H}_s \subseteq \mathcal{S}(F)$ and the functions $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$, $s \in [r]$, are extended step by step in the course of the algorithm, and their final values are returned when the algorithm terminates.

Consider now the algorithm `COMPUTEWEIGHTS()` as given in Algorithm 1. In the following we will try to convey an intuitive understanding of its operation, building on the informal remarks given in Section 4.2.1.

The algorithm works in rounds, and each round corresponds to relaxing the generalized density restriction (θ, β) by slightly lowering β , and then fully exploring Builder's options that become available as a consequence. Each iteration of the repeat-loop (*) is one such round.

At the beginning of the i -th round, for every color $s \in [r]$ the maximal $d_\theta(\cdot)$ -value among all graphs in $\mathcal{C}(\mathcal{H}_s, F)$, denoted by d_s^i , is determined (lines 7–8). Here the sets $\mathcal{C}(\mathcal{H}_s, F)$ correspond to all graphs in color s that Builder could try to enforce next, and considering for each color a graph that *maximizes* $d_\theta(\cdot)$ yields a new construction step for which β needs to be lowered *least* in order for that step to be compliant with the generalized density restriction (θ, β) . (Specifically, β needs to be lowered to $\beta_i := 1 + \sum_{s \in [r]} d_s^i$; note however that this successive lowering of β is not done explicitly in the algorithm.)

The i -th entry of the sequence α is then used to determine Painter's coloring decision $\sigma := \alpha_i$ for this construction step (line 9), and the rest of the round consists of updating the families \mathcal{H}_s and the functions w_s with all the information that can be extracted from that decision. In fact, only the family \mathcal{H}_σ grows; the families \mathcal{H}_s , $s \in [r] \setminus \{\sigma\}$, do not change.

The value $w^i := \sum_{s \in [r] \setminus \{\sigma\}} d_s^i$ defined in line 10 corresponds to the weight that needs to be assigned to the youngest vertex of every graph that is completed in color σ as a direct consequence of Painter's coloring decision. When the repeat-loop (**) is executed for the first time, those graphs are added to \mathcal{H}_σ via the set $\mathcal{C}^{i,1} \subseteq \mathcal{S}(F)$, and the function $w_\sigma : \mathcal{H}_\sigma \rightarrow \mathbb{R}$ is updated by assigning the value w^i to the newly created graphs (lines 14–19). For technical reasons, these graphs are also stored separately in a set $\mathcal{C}_\sigma(d_\sigma^i)$ that will be relevant later in the algorithm.

The remainder of the i -th round explores options that became available to Builder as a result of the graphs in $\mathcal{C}^{i,1}$ being added to \mathcal{H}_σ . These graphs can now be used themselves for further construction steps, and the graphs created in those construction steps can be used even further, etc. Some of these new potential construction steps are not legal for the current generalized density restriction (θ, β_i) , and will therefore only be explored in later rounds when (intuitively) β is lowered further. However, some of these are indeed legal for the current value of β , and it turns out that the previous decisions of Painter already imply which colors Painter should use in each of those construction steps (!). These indirect consequences of Painter's decision to use color $\sigma = \alpha_i$ in round i are explored in the repeat-loop (***), and in the repeat-loop (**) when

Algorithm 1: COMPUTEWEIGHTS(F, r, θ, α)

Input: a graph F with at least one edge, an integer $r \geq 2$, a real number $\theta > 0$, a sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$

Output: an r -tuple $((\mathcal{H}_s, w_s))_{s \in [r]}$, where $\mathcal{H}_s \subseteq \mathcal{S}(F)$ and $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$ for all $s \in [r]$

```

1 foreach  $s \in [r]$  do
2    $\mathcal{H}_s := \emptyset$ 
3    $\forall d \in \mathbb{R} : \mathcal{C}_s(d) := \emptyset$ 
4  $i := 0$ 
5 repeat (*)
6    $i := i + 1$ 
7   foreach  $s \in [r]$  do
8      $d_s^i := \max_{(H, \pi = (v_1, \dots, v_h)) \in \mathcal{C}(\mathcal{H}_s, F)} d_\theta(H, v_1, w_{(H, \pi, s)})$ 
9      $\sigma := \alpha_i$ 
10     $w^i := \sum_{s \in [r] \setminus \{\sigma\}} d_s^i$ 
11     $j := 0$ 
12    repeat (**)
13       $j := j + 1$ 
14       $\mathcal{C}^{i,j} := \{(H, \pi = (v_1, \dots, v_h)) \in \mathcal{C}(\mathcal{H}_\sigma, F) \mid d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = d_\sigma^i\}$ 
15      if  $j = 1$  then
16         $\mathcal{C}_\sigma(d_\sigma^i) := \mathcal{C}^{i,1}$ 
17      foreach  $(H, \pi) \in \mathcal{C}^{i,j}$  do
18         $w_\sigma(H, \pi) := w^i$ 
19       $\mathcal{H}_\sigma := \mathcal{H}_\sigma \cup \mathcal{C}^{i,j}$ 
20       $k := 0$ 
21      repeat (***)
22         $k := k + 1$ 
23         $\mathcal{T}^{i,j,k} := \{(H, \pi = (v_1, \dots, v_h)) \in \mathcal{C}(\mathcal{H}_\sigma, F) \mid d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \geq d_\sigma^i\}$ 
24         $\mathcal{C}^{i,j,k} := \emptyset$ 
25        foreach  $(H, \pi) \in \mathcal{T}^{i,j,k}, \pi = (v_1, \dots, v_h),$  do
26          if  $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) > d_\sigma^i$  or  $\nexists J \subseteq H : v_1 \in J \wedge (J, \pi|_J) \in \mathcal{C}_\sigma(d_\sigma^i)$  then
27            if  $\nexists J \subseteq H : v_1 \in J \wedge (J, \pi|_J) \in \mathcal{C}_\sigma(d_\theta(H, v_1, w_{(H, \pi, \sigma)}))$  then
28               $\hat{i} := \max\{1 \leq \bar{i} \leq i \mid \alpha_{\bar{i}} = \sigma \wedge d_\theta(H, v_1, w_{(H, \pi, \sigma)}) < d_{\sigma}^{\bar{i}}\}$ 
29            else
30               $\hat{i} := \max\{1 \leq \bar{i} \leq i \mid \alpha_{\bar{i}} = \sigma \wedge d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \leq d_{\sigma}^{\bar{i}}\}$ 
31             $w_\sigma(H, \pi) := w^{\hat{i}}$ 
32             $\mathcal{C}^{i,j,k} := \mathcal{C}^{i,j,k} \cup \{(H, \pi)\}$ 
33         $\mathcal{H}_\sigma := \mathcal{H}_\sigma \cup \mathcal{C}^{i,j,k}$ 
34      until  $\mathcal{C}^{i,j,k} = \emptyset$ 
35    until all  $(H, \pi) \in \mathcal{C}(\mathcal{H}_\sigma, F), \pi = (v_1, \dots, v_h),$  satisfy  $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) < d_\sigma^i$ 
36  until  $\mathcal{H}_s = \mathcal{S}(F)$  for some  $s \in [r]$ 
37 return  $((\mathcal{H}_s, w_s))_{s \in [r]}$ 

```

it is executed for $j \geq 2$. The resulting graphs are added to \mathcal{H}_σ via the sets $\mathcal{T}^{i,j,k}, \mathcal{C}^{i,j,k} \subseteq \mathcal{S}(F)$ in the repeat-loop (**), and via the sets $\mathcal{C}^{i,j} \subseteq \mathcal{S}(F)$, $j \geq 2$, in the repeat-loop (*). This exploration of indirect consequences involves some technicalities for which we cannot give much intuition; see however the remarks in the first three paragraphs of Section 4.3.4 below. Note that the sets $\mathcal{C}_\sigma(d_\sigma^i)$ defined in line 16 (in this or an earlier round) come back into play in lines 26–27.

The i -th round terminates as soon as all ordered graphs $(H, \pi) \in \mathcal{C}(\mathcal{H}_\sigma, F)$, $\pi = (v_1, \dots, v_h)$, satisfy $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) < d_\sigma^i$ (line 35). This corresponds to Builder having exhausted all his legal options in the game with generalized density restriction (θ, β_i) (recall that $\beta_i = 1 + \sum_{s \in [r]} d_s^i$). The $(i+1)$ -th round of the algorithm will then consider the game with generalized density restriction (θ, β_{i+1}) for some $\beta_{i+1} < \beta_i$.

The algorithm terminates as soon as one of the families \mathcal{H}_s , $s \in [r]$, contains all ordered subgraphs of F , i.e., $\mathcal{H}_s = \mathcal{S}(F)$ (line 36). This corresponds to Builder having enforced copies of all ordered subgraphs of F in color s (in particular, monochromatic copies of F in all possible vertex orderings).

We defer the formal arguments that COMPUTEWEIGHTS() is a well-defined algorithm and terminates correctly to Section 4.3.5, where we will prove the following claim.

LEMMA 4.10 (Well-definedness and termination of algorithm). *All expressions that occur in the algorithm COMPUTEWEIGHTS() are well-defined, all numerical values and all sets that occur are finite, and on any input as specified the algorithm terminates correctly after at most $r \cdot |\mathcal{S}(F)|$ iterations of the repeat-loop (*).*

With Algorithm 1 in hand, we now define the parameter $\Lambda_\theta(F, r)$ for which we will prove Proposition 4.4 and Proposition 4.5.

For a fixed real number $\theta > 0$, any graph F with at least one edge and any integer $r \geq 2$ we define

$$\Lambda_\theta(F, r) := \min_{\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}} \max_{\substack{s \in [r] \\ \pi \in \Pi(V(F))}} \min_{H \subseteq F: v(H) \geq 1} \lambda_\theta(H, w_{(H, \pi|_{H,s})}) , \quad (4.23)$$

where $\lambda_\theta()$ is defined in (4.18), and $w_{(H, \pi, s)}()$ is defined for all $(H, \pi) \in \mathcal{S}(F)$ and all $s \in [r]$ in (4.22) using the results $((\mathcal{H}_s, w_s))_{s \in [r]} := \text{COMPUTEWEIGHTS}(F, r, \theta, \alpha)$ of Algorithm 1.

We defer the formal arguments that $\Lambda_\theta(F, r)$ is well-defined to Section 4.3.5, where we will prove the following claim.

LEMMA 4.11 (Well-definedness of $\Lambda_\theta(F, r)$). *For any real number $\theta > 0$, any graph F with at least one edge and any integer $r \geq 2$, the parameter $\Lambda_\theta(F, r)$ defined in (4.23) is a well-defined finite value.*

Note that for rational values of $\theta > 0$, the parameter $\Lambda_\theta(F, r)$ can be computed using only integer arithmetic.

Before we begin with the technical analysis of the algorithm COMPUTEWEIGHTS() in Sections 4.3.5–4.3.7, we give a few remarks about its implementation in the next section.

4.3.4. Simplifications and implementation of the algorithm. By Theorem 4.7, we can compute the online vertex-Ramsey density $m_1^*(F, r)$ as the inverse of the root of the parameter $\Lambda_\theta(F, r)$ defined in (4.23), where this definition involves the return values of the algorithm COMPUTEWEIGHTS(). As it turns out, the algorithm COMPUTEWEIGHTS() can be simplified considerably if one is only interested in computing the online vertex-Ramsey density $m_1^*(F, r)$

for given F and r (and not in proving Theorem 4.2 or Theorem 4.3, or in computing explicit winning strategies for Builder and Painter). The program to compute $m_1^*(F, r)$ that is available from the authors' website [Müt] uses such a simplified version of the pseudocode above. In the following we outline the most important steps in this simplification.

First of all, the sets $\mathcal{C}_s(d)$ defined in line 16 and the case distinctions inside the repeat-loop (***) whether certain subgraphs are contained in those sets or not can be omitted, as they are only used for proving the lower bound part of Theorem 4.3, our result for the probabilistic problem. Specifically, these extra technicalities are needed to bound the size of the witness graphs for certain coloring strategies that are derived from the algorithm COMPUTEWEIGHTS() — recall from Section 4.2.2 and Section 4.2.3 that such a bound is unimportant for the deterministic game, but crucial for the original probabilistic problem (see also Lemma 4.33 and Remark 4.34 below).

In a second step the algorithm can be simplified even further: As it turns out, the entire repeat-loop (***) can be omitted; i.e., we do not need to compute any vertex weights for graphs $(H, \pi) \in \mathcal{C}(\mathcal{H}_\sigma, F)$, $\pi = (v_1, \dots, v_h)$, that satisfy $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) > d_\sigma^i$ for the current value of d_σ^i , and we do not need to add such graphs to the corresponding family \mathcal{H}_σ (thus for the color σ the algorithm will ignore the entire subtree of $\mathcal{T}(F)$ rooted at (H, π)). The reason for this is that such graphs are essentially useless for Builder, and therefore we do not need to consider them in our systematic exploration of Builder's options (see Lemma 4.21 and Algorithm 2 below).

Yet another simplification follows from Lemma 4.26 below: Combining (4.23) and (4.86) shows that we can change the return value of the algorithm COMPUTEWEIGHTS() to the sum on the right hand side of (4.86) (note that this sum is exactly β_i , as used in our informal description of the algorithm). Thus we may stop the algorithm as soon as for some $\pi \in \Pi(V(F))$ the graph (F, π) is added to one of the families \mathcal{H}_s , $s \in [r]$, which may happen considerably earlier than the termination condition in line 36.

Further major savings are achieved by considering only induced subgraphs in the definition (4.20), as pointed out in Remark 4.9.

Some of these modifications might change the values returned by the algorithm COMPUTEWEIGHTS(F, r, θ, α) for a specific sequence α (as the families \mathcal{H}_s , $s \in [r]$, may evolve differently in the course of the algorithm, the entries of α get a different semantic), but not the value of $\Lambda_\theta(F, r)$ as defined in (4.23).

We conclude this section by sketching some ideas to further speed up the computation of $m_1^*(F, r)$ that do not directly relate to the pseudocode given in Algorithm 1.

When evaluating $\Lambda_\theta(F, r)$ for a given $\theta \in (0, 2)$, rather than calling the algorithm COMPUTEWEIGHTS() for each possible input sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ separately, we call it only once and, in each iteration, branch on all r values the variable σ can assume in line 9. Since most of the branches of the resulting recursion tree end after much fewer than $r \cdot |\mathcal{S}(F)|$ iterations, this allows us to evaluate the minimization in (4.23) and hence the value of $\Lambda_\theta(F, r)$ much more efficiently.

By Theorem 4.7 we have $m_1^*(F, r) = 1/\theta^*$, where $\theta^* = \theta^*(F, r)$ is the unique root of $\Lambda_\theta(F, r)$ defined in (4.23), which is guaranteed to be in the finite set $Q(a_{\max})$. In order to efficiently search for θ^* in $Q(a_{\max})$, we can exploit that the function $\Lambda_\theta(F, r)$ changes its sign from positive to negative at θ^* . Specifically, in order to compute θ^* , we alternate between shrinking the possible interval for the root θ^* by binary search (starting with the interval $(0, 2)$), and evaluating $\Lambda_\theta(F, r)$ for all rational values of θ inside the current interval up to a certain size of the denominator.

4.3.5. Basic properties of the algorithm. In this section we establish a number of basic properties of the algorithm `COMPUTEWEIGHTS()`, including several important monotonicity properties. We also provide the proofs for Lemma 4.10 and Lemma 4.11.

We begin by proving that the families \mathcal{H}_s grow downward from the root in the tree $\mathcal{T}(F)$ throughout the algorithm, as already mentioned.

LEMMA 4.12 (Closure property of families \mathcal{H}_s). *Throughout the algorithm `COMPUTEWEIGHTS()` and for each $s \in [r]$ we have that if (H, π) , $\pi = (v_1, \dots, v_h)$, $h \geq 2$, is in \mathcal{H}_s , then $(H \setminus v_1, \pi \setminus v_1)$ is also in \mathcal{H}_s . In particular, if $\mathcal{H}_s \neq \emptyset$ then $(K_1, (v_1)) \in \mathcal{H}_s$.*

PROOF. Observe that graphs are only added to \mathcal{H}_s in lines 19 and 33 of iterations for which $\alpha_i = s$, via the sets $\mathcal{C}^{i,j}$ and $\mathcal{C}^{i,j,k} \subseteq \mathcal{T}^{i,j,k}$. Thus by the definition of $\mathcal{C}^{i,j}$ in line 14 and of $\mathcal{T}^{i,j,k}$ in line 23, only graphs that are currently in $\mathcal{C}(\mathcal{H}_s, F)$ are added to \mathcal{H}_s . The claim thus follows from the definition of $\mathcal{C}(\mathcal{H}_s, F)$ in (4.21). \square

Next we prove the first part of Lemma 4.10, which states that all expressions that occur in the algorithm `COMPUTEWEIGHTS()` are well-defined and that all numerical values and all sets that occur are finite. (We ignore the assignment $\sigma := \alpha_i$ in line 9 for the time being — well-definedness of that assignment is immediate once we have proven the second part of Lemma 4.10, namely that the algorithm `COMPUTEWEIGHTS()` terminates correctly after at most $r \cdot |\mathcal{S}(F)|$ rounds.)

PROOF OF LEMMA 4.10 (WELL-DEFINEDNESS OF ALGORITHM). We need to argue that the expression $d_\theta(H, v_1, w_{(H, \pi, s)})$ (wherever it occurs) is well-defined, and that the maximum in line 8, line 28 and line 30 is always over a nonempty set.

Note that $w_\sigma(H, \pi)$ is defined in line 18 or line 31, just before (H, π) is added to \mathcal{H}_σ in line 19 or line 33, respectively. Combining this with Lemma 4.12 and using the definition in (4.22), it follows that the function $w_{(H, \pi, s)}$ defines finite vertex weights for all vertices of every graph $(H, \pi) \in \mathcal{H}_s$. Consequently, for every graph $(H, \pi) \in \mathcal{C}(\mathcal{H}_s, F)$, $\pi = (v_1, \dots, v_h)$, the function $w_{(H, \pi, s)}$ defines finite weights for *all but the youngest vertex* v_1 . As throughout the algorithm the function $d_\theta(H, v_1, w_{(H, \pi, s)})$ is evaluated only for graphs from the corresponding set $\mathcal{C}(\mathcal{H}_s, F)$, it follows that this expression (wherever it occurs) is indeed well-defined (recall the definition of $d_\theta()$ in (4.17)).

As long as none of the families \mathcal{H}_s , $s \in [r]$, contains all graphs from $\mathcal{S}(F)$, the sets $\mathcal{C}(\mathcal{H}_s, F)$ are nonempty. Thus the termination condition in line 36 ensures that the maximum in line 8 is always over a nonempty set and therefore well-defined.

For any $\sigma \in [r]$, let \tilde{i} be the smallest integer $i \geq 1$ for which $\alpha_i = \sigma$ and note that the \tilde{i} -th iteration of the repeat-loop (*) is the first one where graphs are added to the family \mathcal{H}_σ (initially, we have $\mathcal{H}_\sigma = \emptyset$). As $\mathcal{C}(\emptyset, F) = \{(K_1, (v_1))\}$ and $d_\theta(K_1, v_1, w_{(K_1, (v_1), \sigma)}) = 0$, by the definitions in lines 8, 14, and 16 we have $d_\sigma^{\tilde{i}} = 0$, and $(K_1, (v_1))$ is contained in $\mathcal{C}^{\tilde{i}, 1} = \mathcal{C}_\sigma(0)$. By the definition of $d_\theta()$ in (4.17), $d_\theta(H, v_1, w_{(H, \pi, \sigma)})$ on the right hand side of line 30 is always non-positive, i.e. less than or equal to $d_\sigma^{\tilde{i}} = 0$, implying that the maximum in line 30 is always over a nonempty set and therefore well-defined (this set contains at least the integer \tilde{i}). We can argue analogously for the set on the right hand side of line 28 if $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) < 0$. On the other hand, if $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = 0$ then by what we said before the subgraph $J \subseteq H$ that consists only of the vertex v_1 is contained in $\mathcal{C}_\sigma(0) = \mathcal{C}_\sigma(d_\theta(H, v_1, w_{(H, \pi, \sigma)}))$, and thus the condition in line 27 is violated.

For this last argument we used that no graphs are ever removed from the sets $\mathcal{C}_s(d)$, $d \in \mathbb{R}$. We have not shown this yet formally; however, it follows from Lemma 4.14 below that after the initialization in line 3, each set $\mathcal{C}_s(d)$ is modified at most once, namely in line 16 of the unique iteration i for which $\alpha_i = s$ and $d_s^i = d$. (If the reader is worried about this forward reference, he is welcome to substitute line 16 by $\mathcal{C}_\sigma(d_\sigma^i) := \mathcal{C}_\sigma(d_\sigma^i) \cup \mathcal{C}^{i,1}$ for the time being, i.e., until Lemma 4.14 is proven.) \square

Having established that all numerical values assigned in the algorithm COMPUTEWEIGHTS() are finite, we state the following observations for further reference.

LEMMA 4.13 (Finite and non-positive weights). *Throughout the algorithm COMPUTEWEIGHTS(), for each $s \in [r]$ we have that for each $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, all vertex weights $w_{(H, \pi, s)}(v_i)$ defined in (4.22) are finite non-positive values, while for each $(H, \pi) \in \mathcal{S}(F) \setminus \mathcal{H}_s$, at least one of the vertex weights is $-\infty$.*

Consequently, for all $(H, \pi) \in \mathcal{H}_s$, $\lambda_\theta(H, w_{(H, \pi, s)})$ defined in (4.18) is a finite value bounded by $v(F)$, while for all $(H, \pi) \in \mathcal{S}(F) \setminus \mathcal{H}_s$, we have $\lambda_\theta(H, w_{(H, \pi, s)}) = -\infty$.

PROOF. By the definition of $d_\theta()$ in (4.17), $d_\theta(H, v_1, w_{(H, \pi, s)})$ on the right hand side of line 8 is always non-positive and finite, from which we conclude, using the definitions in line 10, 18 and 31, that $w_s(H, \pi)$ is a non-positive finite value for all $s \in [r]$ and all $(H, \pi) \in \mathcal{H}_s$. The first part of the statement now follows from the definition in (4.22) and Lemma 4.12. The second part follows from the first part using the definition in (4.18). \square

The next lemma establishes two important monotonicity properties, which will be used in many of the upcoming proofs.

LEMMA 4.14 (Monotonicity of d_s^i and w^i in i). *Let $\sigma \in [r]$ and $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ be the input sequence of the algorithm COMPUTEWEIGHTS(). Throughout the algorithm, if $\alpha_i = \sigma$, then the variables d_s^i, d_s^{i+1} , $s \in [r]$, defined in line 8 satisfy*

$$d_\sigma^{i+1} < d_\sigma^i \quad \text{and} \quad d_s^{i+1} = d_s^i \quad \text{for all } s \in [r] \setminus \{\sigma\}.$$

Moreover, if $\alpha_{\bar{i}} = \alpha_i = \sigma$ for some $\bar{i} < i$, then the variables $w^{\bar{i}}, w^i$ defined in line 10 satisfy

$$w^i \leq w^{\bar{i}},$$

with equality if and only if $\alpha_{\bar{i}} = \alpha_{\bar{i}+1} = \dots = \alpha_i = \sigma$.

Of course, the variables d_s^i and d_s^{i+1} referred to in Lemma 4.14 are defined only if the number of iterations of the repeat-loop (*) is at least $i + 1$. Similarly, w^i and $w^{\bar{i}}$ are defined only if the number of iterations is at least i . Otherwise the statement of the lemma is void.

PROOF. The first part of the lemma follows from the definition of d_σ^i in line 8, the termination condition in line 35, and the fact that none of the families \mathcal{H}_s , $s \in [r] \setminus \{\sigma\}$, is modified within the i -th iteration of the repeat-loop (*). The second part of the lemma follows from the first part and the definition of w^i in line 10. \square

For the proof that COMPUTEWEIGHTS() terminates correctly after at most $r \cdot |\mathcal{S}(F)|$ iterations of the repeat-loop (*) we will need the following auxiliary statement.

LEMMA 4.15 (End of repeat-loop (**)). *Throughout the algorithm COMPUTEWEIGHTS(), at the end of each iteration of the repeat-loop (**), all graphs $(H, \pi) \in \mathcal{C}(\mathcal{H}_\sigma, F)$, $\pi = (v_1, \dots, v_h)$, satisfy*

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \leq d_\sigma^i. \quad (4.24)$$

For those graphs satisfying (4.24) with equality there is a subgraph $J \subseteq H$ with $v_1 \in J$ and $(J, \pi|_J) \in \mathcal{C}_\sigma(d_\sigma^i)$.

PROOF. The j -th iteration of the repeat-loop (**) ends as soon as the repeat-loop (***) terminates. Due to the condition in line 34 this happens in the first iteration k for which the set $\mathcal{C}^{i,j,k} \subseteq \mathcal{T}^{i,j,k}$ is empty, which means that all graphs currently in $\mathcal{C}(\mathcal{H}_\sigma, F)$ violate the condition in the definition of $\mathcal{T}^{i,j,k}$ in line 23, or the condition in line 26. This implies the claim. \square

We now prove the second part of Lemma 4.10, namely that the algorithm COMPUTEWEIGHTS() terminates correctly after at most $r \cdot |\mathcal{S}(F)|$ rounds.

PROOF OF LEMMA 4.10 (TERMINATION OF ALGORITHM). Before bounding the number of iterations of the repeat-loop (*) we need to argue that the inner two repeat-loops always terminate.

Let $\sigma \in [r]$ and suppose that we have $\alpha_i = \sigma$ in the current iteration i of the repeat-loop (*). Note that in each iteration of the repeat-loop (***) except the last one, at least one element is added to the family \mathcal{H}_σ via the set $\mathcal{C}^{i,j,k}$ in line 33. Since throughout the algorithm, \mathcal{H}_σ is a subfamily of $\mathcal{S}(F)$, and since no graphs are ever deleted from \mathcal{H}_σ , the repeat-loop (***) terminates after at most $|\mathcal{S}(F)| + 1$ iterations.

It follows directly from the definition of d_σ^i in line 8 that in the first iteration $j = 1$ of the repeat-loop (**), the set $\mathcal{C}^{i,1}$ defined in line 14 is nonempty. As a consequence of the first part of Lemma 4.15 and the termination condition in line 35, the set $\mathcal{C}^{i,j}$ is also nonempty in all later iterations $j > 1$. Therefore, in each iteration of the repeat-loop (**), at least one element is added to the family \mathcal{H}_σ via the set $\mathcal{C}^{i,j}$ in line 19, and thus similarly to above the repeat-loop (**) terminates after at most $|\mathcal{S}(F)|$ iterations.

The above also implies that in each iteration of the repeat-loop (*), the size of exactly one of the families \mathcal{H}_s , $s \in [r]$, increases by at least one. Considering the condition in line 36, we conclude that the algorithm terminates after at most $r \cdot |\mathcal{S}(F)|$ iterations of the repeat-loop (*). \square

We proceed by proving Lemma 4.11, which states that $\Lambda_\theta()$ defined in (4.23) is a well-defined finite value.

PROOF OF LEMMA 4.11. Due to the termination condition in line 36 of COMPUTEWEIGHTS(), for each possible input sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ there is an $s \in [r]$ for which the family \mathcal{H}_s returned by COMPUTEWEIGHTS() equals $\mathcal{S}(F)$. By Lemma 4.13, for this s the parameter $\lambda_\theta(H, w_{(H, \pi, s)})$ is a finite value for all $(H, \pi) \in \mathcal{S}(F)$. Thus for any fixed $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ the maximization over all $s \in [r]$ in (4.23) yields a finite value, and consequently also the outer minimization over all (finitely many) sequences $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ in (4.23) yields a finite value. \square

We continue with a simple invariant that holds at the beginning of each iteration of the main loop of COMPUTEWEIGHTS().

LEMMA 4.16 (Beginning of repeat-loop (*)). *Throughout the algorithm COMPUTEWEIGHTS(), at the beginning of the i -th iteration of the repeat-loop (*), for every $s \in [r]$ all graphs $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, satisfy $d_\theta(H, v_1, w_{(H, \pi, s)}) > d_s^i$.*

PROOF. Fix $i, s \in [r]$, and $(H, \pi) \in \mathcal{H}_s$ as in the lemma, and let $\bar{i} < i$ denote the iteration in which (H, π) was added to \mathcal{H}_s . We must have $\alpha_{\bar{i}} = s$, and (H, π) was added to \mathcal{H}_s either via one of the sets $\mathcal{C}^{\bar{i},j}$ in line 19, or via one of the sets $\mathcal{C}^{\bar{i},j,k} \subseteq \mathcal{T}^{\bar{i},j,k}$ in line 33. By the conditions in the definition of $\mathcal{C}^{\bar{i},j}$ in line 14 and of $\mathcal{T}^{\bar{i},j,k}$ in line 23 we have $d_\theta(H, v_1, w_{(H,\pi,s)}) \geq d_s^{\bar{i}} > d_s^{\bar{i}+1} \geq d_s^{\bar{i}}$, where the last two inequalities follow from the first part of Lemma 4.14. \square

The next lemma takes a closer look at the $d_\theta()$ -value of graphs that are added to the families \mathcal{H}_s via one of the sets $\mathcal{C}^{i,j,k}$ in the repeat-loop (***) .

LEMMA 4.17 (Sandwiched $d_\theta()$ -values for graphs in $\mathcal{C}^{i,j,k}$). *Let $\sigma \in [r]$ and $\alpha \in [r]^{r \cdot |S(F)|}$ be the input sequence of the algorithm COMPUTEWEIGHTS(). Throughout the algorithm, if $\alpha_i = \sigma$, then for any graph (H, π) , $\pi = (v_1, \dots, v_h)$, that is added to the set $\mathcal{C}^{i,j,k}$ in line 32 (for this graph $w_\sigma(H, \pi)$ is defined in line 31 by setting it to $w^{\hat{i}}$) the following holds: If \hat{i} is defined in line 28, we have*

$$d_\sigma^{\hat{i}+1} \leq d_\theta(H, v_1, w_{(H,\pi,\sigma)}) < d_\sigma^{\hat{i}} . \quad (4.25a)$$

Otherwise, i.e. if \hat{i} is defined in line 30, we have

$$d_\sigma^{\hat{i}+1} < d_\theta(H, v_1, w_{(H,\pi,\sigma)}) \leq d_\sigma^{\hat{i}} . \quad (4.25b)$$

PROOF. Clearly, the definition of \hat{i} in line 28 implies

$$d_\theta(H, v_1, w_{(H,\pi,\sigma)}) < d_\sigma^{\hat{i}} . \quad (4.26)$$

As (H, π) is in $\mathcal{C}^{i,j,k} \subseteq \mathcal{T}^{i,j,k}$, by the definition in line 23 we have $d_\theta(H, v_1, w_{(H,\pi,\sigma)}) \geq d_\sigma^i$, and consequently $\hat{i} < i$. Consider the smallest integer $\nu \geq 1$ for which $\alpha_{i+\nu} = \alpha_{\hat{i}} = \sigma$, and note that $\hat{i} + \nu \leq i$. Again by the definition of \hat{i} in line 28 we have

$$d_\sigma^{\hat{i}+\nu} \leq d_\theta(H, v_1, w_{(H,\pi,\sigma)}) . \quad (4.27)$$

Moreover, by the choice of ν and the first part of Lemma 4.14 we have

$$d_\sigma^{\hat{i}+\nu} = d_\sigma^{\hat{i}+1} . \quad (4.28)$$

Combining (4.27) and (4.28) yields the first inequality in (4.25a), and together with (4.26) proves the first part of the lemma. The second part follows similarly by using the definition of \hat{i} in line 30 and by interchanging $<$ with \leq in (4.26) and (4.27) (also in this case we must have $\hat{i} < i$, as otherwise we would have $d_\theta(H, v_1, w_{(H,\pi,\sigma)}) = d_\sigma^{\hat{i}}$, a contradiction to the condition in line 26 and the negation of the condition in line 27). \square

The next lemma captures another important monotonicity condition: the lower the $d_\theta()$ -value of an ordered graph in a particular color is, the smaller is the weight assigned to its youngest vertex.

LEMMA 4.18 ($d_\theta()$ -value vs. weight monotonicity). *Throughout the algorithm COMPUTEWEIGHTS(), for every $s \in [r]$ and any two graphs $(H, \pi), (J, \tau) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, $\tau = (u_1, \dots, u_c)$, the following two properties hold:*

- If

$$d_\theta(H, v_1, w_{(H,\pi,s)}) < d_\theta(J, u_1, w_{(J,\tau,s)})$$

then we have

$$w_{(H,\pi,s)}(v_1) \leq w_{(J,\tau,s)}(u_1) .$$

• If

$$d_\theta(H, v_1, w_{(H, \pi, s)}) = d_\theta(J, u_1, w_{(J, \tau, s)}) \quad \text{and} \quad w_{(H, \pi, s)}(v_1) < w_{(J, \tau, s)}(u_1)$$

then $w_{(H, \pi, s)}(v_1) \stackrel{(4.22)}{=} w_s(H, \pi)$ is defined either in line 18 or in line 31 with \hat{i} defined in line 30, and $w_{(J, \tau, s)}(u_1) = w_s(J, \tau)$ is defined in line 31 with \hat{i} defined in line 28.

For the second part of Lemma 4.18 note that $w_s(H, \pi)$ and $w_s(J, \tau)$ are defined exactly once in the course of the algorithm (each in some iteration of the various repeat-loops), just before the corresponding graph (H, π) or (J, τ) is added to the family \mathcal{H}_s .

PROOF. Let i_{\max} denote the total number of iterations of the repeat-loop (*). Fix some $\sigma \in [r]$ and consider the set $R_\sigma \in \mathbb{R}^2$ defined by

$$R_\sigma := \bigcup_{1 \leq i \leq i_{\max} : \alpha_i = \sigma} \left\{ ((1-t) \cdot d_\sigma^i + t \cdot d_\sigma^{i+1}, w^i) \mid t \in [0, 1] \right\}, \quad (4.29)$$

where we use the convention $d_\sigma^{i_{\max}+1} := d_\sigma^{i_{\max}}$ if $\alpha_{i_{\max}} = \sigma$. By Lemma 4.14 we have for any two pairs $(x, y), (x', y') \in R_\sigma$

$$x < x' \implies y \leq y'. \quad (4.30)$$

Now fix some graph $(H, \pi) \in \mathcal{H}_\sigma$, $\pi = (v_1, \dots, v_h)$, and consider the iteration i of the repeat-loop (*) where $\alpha_i = \sigma$ and where (H, π) is added to the family \mathcal{H}_σ . If (H, π) is added to \mathcal{H}_σ in line 19 then we have

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = d_\sigma^i \quad \text{and} \quad w_{(H, \pi, \sigma)}(v_1) \stackrel{(4.22)}{=} w_\sigma(H, \pi) = w^i \quad (4.31)$$

by the definitions in line 14 and line 18. If on the other hand (H, π) is added to \mathcal{H}_σ in line 33 then we obtain with Lemma 4.17 that

$$d_\sigma^{i+1} \leq d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \leq d_\sigma^{\hat{i}} \quad \text{and} \quad w_{(H, \pi, \sigma)}(v_1) \stackrel{(4.22)}{=} w_\sigma(H, \pi) = w^{\hat{i}} \quad (4.32)$$

for some $\hat{i} \leq i$ with $\alpha_{\hat{i}} = \sigma$ (defined either in line 28 or in line 30).

Combining (4.29), (4.31) and (4.32) shows that any graph $(H, \pi) \in \mathcal{H}_\sigma$, $\pi = (v_1, \dots, v_h)$, satisfies

$$(d_\theta(H, v_1, w_{(H, \pi, \sigma)}), w_{(H, \pi, \sigma)}(v_1)) \in R_\sigma. \quad (4.33)$$

The first part of the claim now follows from (4.30) and (4.33).

The second part of the claim follows from (4.29), (4.33) and Lemma 4.14 by examining where in the set R_σ the point $(d_\theta(H, v_1, w_{(H, \pi, \sigma)}), w_{(H, \pi, \sigma)}(v_1))$ can possibly be located, depending on whether (H, π) is added to \mathcal{H}_σ in line 19 (then $w_\sigma(H, \pi)$ is defined in line 18) or in line 33 (then $w_\sigma(H, \pi)$ is defined in line 31), where the cases whether \hat{i} is defined in line 28 or in line 30 have to be distinguished using (4.25a) and (4.25b) from Lemma 4.17. \square

In the following we will repeatedly use the following auxiliary statement, which is an immediate consequence of the definition in (4.17).

LEMMA 4.19 (Weights vs. $d_\theta()$ -value monotonicity). *Let H be a graph, $v \in H$ and $J \subseteq H$ with $v \in J$. Moreover, let $w_H : V(H) \setminus \{v\} \rightarrow \mathbb{R}$ and $w_J : V(J) \setminus \{v\} \rightarrow \mathbb{R}$ with $w_H(u) \leq w_J(u)$ for all $u \in J \setminus v$. Then we have $d_\theta(H, v, w_H) \leq d_\theta(J, v, w_J)$.*

The next lemma establishes an important monotonicity condition for the vertex weights with respect to taking (ordered) subgraphs: the weights of a subgraph are always at least as high as the weights of the entire graph.

LEMMA 4.20 (Subgraph weight monotonicity). *Throughout the algorithm COMPUTEWEIGHTS(), for every $s \in [r]$, if $(H, \pi) \in \mathcal{H}_s$, then for every subgraph $J \subseteq H$ we have $(J, \pi|_J) \in \mathcal{H}_s$ and $w_{(H, \pi, s)}(u) \leq w_{(J, \pi|_J, s)}(u)$ for all $u \in J$.*

PROOF. We will prove the following auxiliary claim: for every $s \in [r]$, if $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, then for every subgraph $J \subseteq H$ with $v_1 \in J$ we have $(J, \pi|_J) \in \mathcal{H}_s$ and $w_{(H, \pi, s)}(u) \leq w_{(J, \pi|_J, s)}(u)$ for all $u \in J$. This implies the original claim, where the subgraphs $J \subseteq H$ are not required to contain the youngest vertex v_1 , as follows: if $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, and $J \subseteq H$ is *any* subgraph of H , then by Lemma 4.12 we also have $(H^{-c}, \pi^{-c}) \in \mathcal{H}_s$ where $c := \min\{i \mid v_i \in J\} - 1$ and $(H^{-c}, \pi^{-c}) := (H \setminus \{v_1, \dots, v_c\}, \pi \setminus \{v_1, \dots, v_c\})$. Moreover, $(J, \pi|_J)$ contains the youngest vertex v_{c+1} of (H^{-c}, π^{-c}) . Therefore, applying the auxiliary claim to (H^{-c}, π^{-c}) and $(J, \pi|_J)$, together with the observation that $w_{(H, \pi, s)}(u) = w_{(H^{-c}, \pi^{-c}, s)}(u)$ for all $u \in H^{-c}$ completes the argument.

To prove the auxiliary claim we argue by induction over the number of vertices of H . The claim clearly holds if H consists only of a single vertex, as then $J = H$ is the only subgraph of H . For the induction step let $\sigma \in [r]$ and consider a graph $(H, \pi) \in \mathcal{H}_\sigma$, $\pi = (v_1, \dots, v_h)$, with at least two vertices. We consider the iteration i of the repeat-loop (*) where $\alpha_i = \sigma$ and where (H, π) is added to the family \mathcal{H}_σ . Let J be a subgraph of H with $v_1 \in J$. By Lemma 4.12 we have that $(H \setminus v_1, \pi \setminus v_1) \in \mathcal{H}_\sigma$ at this point, and thus we know by induction that

$$(J \setminus v_1, \pi|_{J \setminus v_1}) \in \mathcal{H}_\sigma \quad (4.34)$$

and that

$$w_{(H, \pi, \sigma)}(u) \leq w_{(J, \pi|_J, \sigma)}(u) \quad \text{for all } u \in J \setminus v_1. \quad (4.35)$$

To complete the proof we only need to show two things: Firstly, that $(J, \pi|_J)$ is either already contained in \mathcal{H}_σ or added to this set together with (H, π) at the latest, and secondly, that $w_{(H, \pi, \sigma)}(v_1) \leq w_{(J, \pi|_J, \sigma)}(v_1)$ for the last vertex v_1 .

Recall that graphs are only added to \mathcal{H}_σ via one of the sets $\mathcal{C}^{i,j}$ in line 19, or via one of the sets $\mathcal{C}^{i,j,k} \subseteq \mathcal{T}^{i,j,k}$ in line 33. If (H, π) is contained in one of the sets $\mathcal{C}^{i,j}$, then by the definition in line 14 we have

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = d_\sigma^i, \quad (4.36a)$$

whereas if (H, π) is contained in one of the sets $\mathcal{C}^{i,j,k}$, then by the definition in line 23 we have

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \geq d_\sigma^i. \quad (4.36b)$$

Applying Lemma 4.19 using (4.35) shows that

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \leq d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) . \quad (4.37)$$

We will distinguish the cases where the inequality (4.37) is strict,

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) < d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) , \quad (4.38a)$$

and where it is tight,

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) . \quad (4.38b)$$

Altogether we distinguish four cases: whether (H, π) is contained in one of the sets $\mathcal{C}^{i,j}$ or $\mathcal{C}^{i,j,k}$, and whether the inequality (4.37) is strict or tight.

- $(H, \pi) \in \mathcal{C}^{i,j}$ and inequality (4.37) is strict. Combining (4.36a) and (4.38a) yields

$$d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) > d_\sigma^i . \quad (4.39)$$

By the definition of d_σ^i in line 8, it follows from (4.39) that if $(J, \pi|_J)$ was not already contained in \mathcal{H}_σ at the beginning of the repeat-loop (**) (in the i -th iteration of the repeat-loop (*)), then at this point $(J \setminus v_1, \pi|_{J \setminus v_1})$ was not contained in \mathcal{H}_σ either. By (4.34) there must then be some $j' < j$ such that $(J \setminus v_1, \pi|_{J \setminus v_1})$ was added to \mathcal{H}_σ in the j' -th iteration of the repeat-loop (**). Combining the first part of Lemma 4.15 and (4.39) shows that also $(J, \pi|_J)$ was added to \mathcal{H}_σ in the j' -th iteration of the repeat-loop (**). In any case $(J, \pi|_J)$ is already contained in \mathcal{H}_σ when (H, π) is added to this set. Applying the first part of Lemma 4.18 using (4.38a) yields that $w_{(H, \pi, \sigma)}(v_1) \leq w_{(J, \pi|_J, \sigma)}(v_1)$, completing the inductive step in this case.

- $(H, \pi) \in \mathcal{C}^{i,j,k}$ and inequality (4.37) is strict. Combining (4.36b) and (4.38a) yields $d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) > d_\sigma^i$. Therefore, if $(J, \pi|_J)$ is not already contained in \mathcal{H}_σ when (H, π) is added to this set via the set $\mathcal{C}^{i,j,k}$, then $(J, \pi|_J)$ is contained in $\mathcal{C}^{i,j,k}$ as well (recall (4.34) and the definition in line 23 and note that $(J, \pi|_J)$ satisfies the first condition in line 26), and added to \mathcal{H}_σ together with (H, π) . To complete the inductive step apply again the first part of Lemma 4.18 using (4.38a).
- $(H, \pi) \in \mathcal{C}^{i,j}$ and inequality (4.37) is tight. Combining (4.36a) and (4.38b) yields $d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) = d_\sigma^i$. Therefore, if $(J, \pi|_J)$ is not already contained in \mathcal{H}_σ when (H, π) is added to this set via the set $\mathcal{C}^{i,j}$, then $(J, \pi|_J)$ is contained in $\mathcal{C}^{i,j}$ as well (recall (4.34) and the definition in line 14), and added to \mathcal{H}_σ together with (H, π) . The definitions in line 18, 28, 30 and 31 show that in any case

$$w_{(H, \pi, \sigma)}(v_1) \stackrel{(4.22)}{=} w_\sigma(H, \pi) = w^i \quad \text{and} \quad w_{(J, \pi|_J, \sigma)}(v_1) \stackrel{(4.22)}{=} w_\sigma(J, \pi|_J) = w^{\bar{i}} \quad (4.40)$$

for some $\bar{i} \leq i$ with $\alpha_{\bar{i}} = \sigma$. Applying the second part of Lemma 4.14 using (4.40) yields that $w_{(H, \pi, \sigma)}(v_1) \leq w_{(J, \pi|_J, \sigma)}(v_1)$, completing the inductive step in this case.

- $(H, \pi) \in \mathcal{C}^{i,j,k}$ and inequality (4.37) is tight. Combining (4.36b) and (4.38b) yields

$$d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) \geq d_\sigma^i. \quad (4.41)$$

Therefore, if $(J, \pi|_J)$ is not already contained in \mathcal{H}_σ when (H, π) is added to this set via the set $\mathcal{C}^{i,j,k}$, then $(J, \pi|_J)$ is contained in $\mathcal{T}^{i,j,k}$ as well (recall (4.34) and the definition in line 23). Suppose for the sake of contradiction that $(J, \pi|_J)$ was not transferred from $\mathcal{T}^{i,j,k}$ to $\mathcal{C}^{i,j,k}$, i.e., that it violated the condition in line 26. By (4.41) and the first condition in line 26 we have

$$d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) = d_\sigma^i, \quad (4.42)$$

and by the second condition in line 26 there is a subgraph $\bar{J} \subseteq J$ with $v_1 \in \bar{J}$ and

$$(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_\sigma(d_\sigma^i). \quad (4.43)$$

Combining (4.36b), (4.37) and (4.42) shows that

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = d_\sigma^i. \quad (4.44)$$

Clearly \bar{J} is a subgraph of H that contains the youngest vertex v_1 , which combined with (4.43) and (4.44) contradicts the fact that (H, π) satisfies the condition in line 26 (only graphs satisfying this condition are transferred from $\mathcal{T}^{i,j,k}$ to $\mathcal{C}^{i,j,k}$). Hence the graph $(J, \pi|_J)$ is either already contained in \mathcal{H}_σ or added to this set together with (H, π) via the set $\mathcal{C}^{i,j,k}$ at the latest.

It remains to show that $w_{(H, \pi, \sigma)}(v_1) \leq w_{(J, \pi|_J, \sigma)}(v_1)$. Suppose for the sake of contradiction that this inequality is violated. Using (4.38b) and the second part of Lemma 4.18 this implies that $w_\sigma(H, \pi)$ is defined in line 31 with \hat{i} defined in line 28, and that $w_\sigma(J, \pi|_J)$ is defined either in line 18 or in line 31 with \hat{i} defined in line 30. In the following we show

that none of those cases can occur, as in each case, similarly to above, the existence of a graph $\bar{J} \subseteq H$ with $v_1 \in \bar{J}$ and $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_\sigma(d_\theta(H, v_1, w_{(H, \pi, \sigma)}))$ causes (H, π) to violate the condition in line 27 (thus causing a contradiction). First consider the case that $w_\sigma(J, \pi|_J)$ is defined in line 18, i.e., $(J, \pi|_J)$ is an element of one of the sets $\mathcal{C}^{\bar{i}, \bar{J}}$ defined in some iteration $\bar{i} \leq i$. From the definition in line 14 we know that

$$d_\sigma^{\bar{J}} = d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}) \stackrel{(4.38b)}{=} d_\theta(H, v_1, w_{(H, \pi, \sigma)}) . \quad (4.45)$$

We distinguish the subcases $\bar{j} = 1$ and $\bar{j} \geq 2$. If $\bar{j} = 1$ then $(J, \pi|_J)$ was added to the set $\mathcal{C}_\sigma(d_\sigma^{\bar{J}})$ in line 16. Using (4.45) it follows that $(J, \pi|_J) \in \mathcal{C}_\sigma(d_\theta(H, v_1, w_{(H, \pi, \sigma)}))$, showing that the graph $\bar{J} = J$ itself causes (H, π) to violate the condition in line 27. Similarly, if $\bar{j} \geq 2$, then by the second part of Lemma 4.15 there is a subgraph $\bar{J} \subseteq J$ with $v_1 \in \bar{J}$ and $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_\sigma(d_\sigma^{\bar{J}})$. Using again (4.45) it follows that $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_\sigma(d_\theta(H, v_1, w_{(H, \pi, \sigma)}))$, causing a contradiction also in this case. Now consider the case that $w_\sigma(J, \pi|_J)$ is defined in line 31 with \hat{i} defined in line 30. Then by the condition in line 27 there is a subgraph $\bar{J} \subseteq J$ with $v_1 \in \bar{J}$ and $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_\sigma(d_\theta(J, v_1, w_{(J, \pi|_J, \sigma)}))$. Using (4.38b) it follows that $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_\sigma(d_\theta(H, v_1, w_{(H, \pi, \sigma)}))$, causing (H, π) to violate the condition in line 27. This completes the proof. \square

4.3.6. Graphs in $\mathcal{C}^{i,j,k}$ irrelevant for Builder. When proving Proposition 4.4 in Section 4.4 we develop a Builder strategy along the lines of the algorithm COMPUTEWEIGHTS(). The following lemma will be crucial in this: It shows that Builder does not need to enforce any ordered graph $(H, \pi) \in \mathcal{S}(F)$ that is added to one of the families \mathcal{H}_s via one of the sets $\mathcal{C}^{i,j,k}$, as for each such graph there is an alternative ordering $\pi' \in \Pi(V(H))$ of its vertices such that (H, π') has already been added to \mathcal{H}_s via the set $\mathcal{C}^{i,j}$ with weights that are at least as good for Builder.

LEMMA 4.21 (Partners between $\mathcal{C}^{i,j}$ and $\mathcal{C}^{i,j,k}$). *Let $\sigma \in [r]$ and consider some iteration i of the repeat-loop (*) with $\alpha_i = \sigma$ of the algorithm COMPUTEWEIGHTS(). In every iteration $j \geq 1$ of the repeat-loop (**), the set $\mathcal{C}^{i,j}$ defined in line 14 and the sets $\mathcal{C}^{i,j,k}$, $k \geq 1$, defined in line 24 and 32 during each iteration of the repeat-loop (***), satisfy the following: For any graph $(H, \pi) \in \mathcal{C}^{i,j,k}$, $\pi = (v_1, \dots, v_h)$, the graph (H, π') , defined by $\pi' := (v_{k+1}, v_1, v_2, \dots, v_k, v_{k+2}, \dots, v_h)$, is contained in $\mathcal{C}^{i,j}$ and satisfies $w_{(H, \pi, \sigma)}(u) \leq w_{(H, \pi', \sigma)}(u)$ for all $u \in H$.*

PROOF. For the reader's convenience, Figure 4.3 illustrates the notations used throughout the proof.

We shall prove the following more technical claim: Let $\sigma \in [r]$ and consider some iteration i of the repeat-loop (*) with $\alpha_i = \sigma$ and some iteration j of the repeat-loop (**). For $k = 0$ and any graph $(H, \pi) \in \mathcal{C}^{i,j}$, $\pi = (v_1, \dots, v_h)$, and for $k \geq 1$ and any graph $(H, \pi) \in \mathcal{C}^{i,j,k}$, $\pi = (v_1, \dots, v_h)$, the graph (H, π') , defined by $\pi' := (v_{k+1}, v_1, v_2, \dots, v_k, v_{k+2}, \dots, v_h)$, is contained in $\mathcal{C}^{i,j}$ and satisfies $w_{(H, \pi, \sigma)}(u) \leq w_{(H, \pi', \sigma)}(u)$ for all $u \in H$.

We will argue at the end of the proof that any graph (H, π) contained in one of the sets $\mathcal{C}^{i,j,k}$ has at least three vertices, ensuring that all subgraphs used in the following arguments have at least one vertex.

To prove the auxiliary claim we consider a fixed iteration $j \geq 1$ of the repeat-loop (**) and argue by induction over k , the number of iterations of the repeat-loop (***). We choose the state

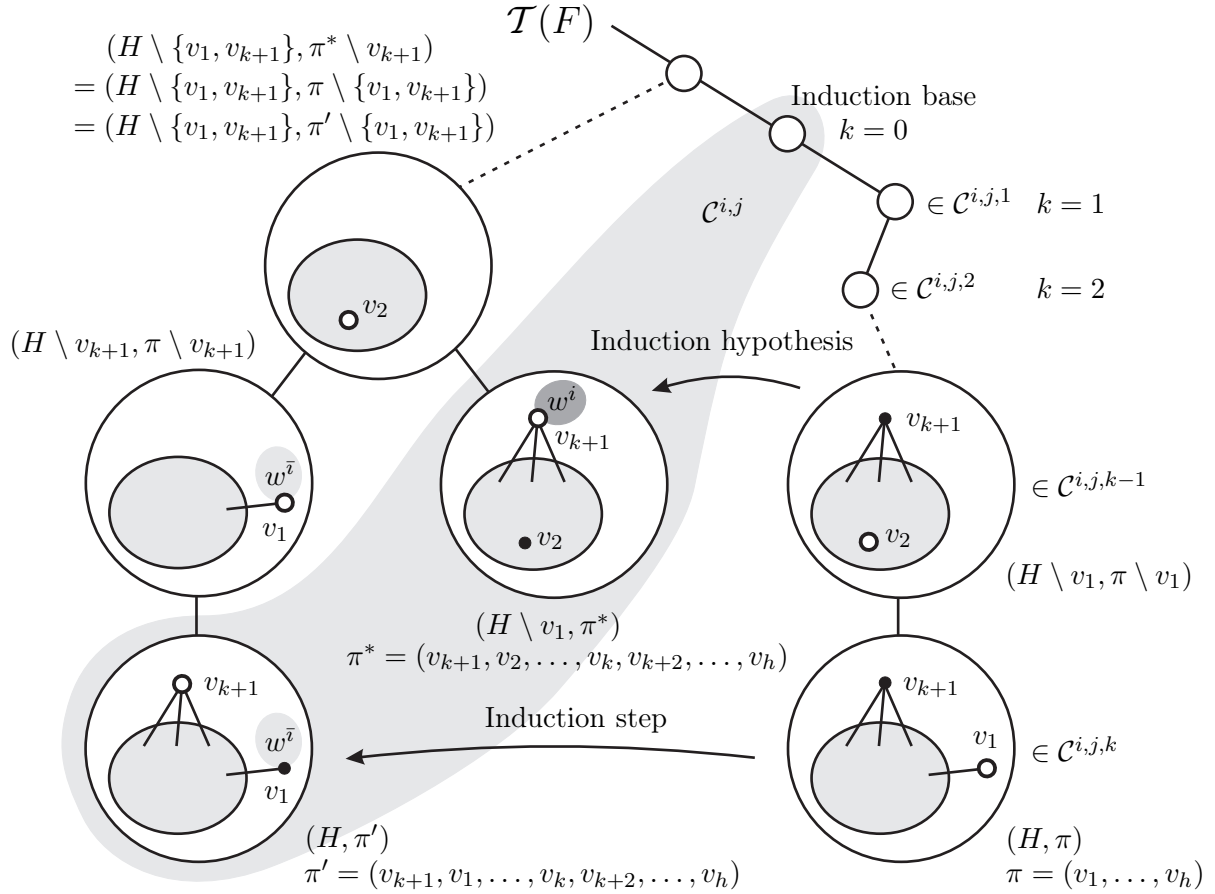


FIGURE 4.3. Notations used in the proof of Lemma 4.21. For each ordered graph, the respective youngest vertex is emphasized.

before the beginning of the first iteration ($k = 0$) as our induction base. In this case $\pi' = \pi$ and the claim is trivially true.

For the induction step consider a graph $(H, \pi) \in \mathcal{C}^{i,j,k}$, $\pi = (v_1, \dots, v_h)$, that is added to \mathcal{H}_σ in the k -th iteration of the repeat-loop (***) ($k \geq 1$). By the definition of $\mathcal{T}^{i,j,k}$ in line 23 (recall that $\mathcal{C}^{i,j,k} \subseteq \mathcal{T}^{i,j,k}$) we clearly have

$$d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \geq d_\sigma^i. \quad (4.46)$$

As (H, π) is obtained from $(H \setminus v_1, \pi \setminus v_1)$ by adding v_1 as the youngest vertex and all edges incident to it, we have

$$w_{(H, \pi, \sigma)}(u) = w_{(H \setminus v_1, \pi \setminus v_1, \sigma)}(u) \quad \text{for all } u \in H \setminus v_1. \quad (4.47)$$

If $j = 1$, the definition of d_σ^i in line 8 ensures that at the beginning of the j -th iteration of the repeat-loop (**) we have $d_\theta(J, u_1, w_{(J, \tau, \sigma)}) \leq d_\sigma^i$ for all $(J, \tau) \in \mathcal{C}(\mathcal{H}_\sigma, F)$, $\tau = (u_1, \dots, u_c)$. If $j > 1$, the same statement is true by the first part of Lemma 4.15. Thus if the inequality (4.46) is strict, then (H, π) was not in $\mathcal{C}(\mathcal{H}_\sigma, F)$ at the beginning of the j -th iteration of the repeat-loop (**), and thus $(H \setminus v_1, \pi \setminus v_1)$ was not in \mathcal{H}_σ at this point. If $k = 1$ this means that $(H \setminus v_1, \pi \setminus v_1)$ must have been added to \mathcal{H}_σ via the set $\mathcal{C}^{i,j}$. The same conclusion holds if $k = 1$ and the inequality (4.46) is tight, as otherwise (H, π) would have qualified for inclusion in $\mathcal{C}^{i,j}$.

Note that the conditions for inclusion into $\mathcal{T}^{i,j,k}$ and $\mathcal{C}^{i,j,k}$ in lines 23 and 26 do not change during the entire repeat-loop (***) except for the requirement that (H, π) is in the current set

$\mathcal{C}(\mathcal{H}_\sigma, F)$. Thus if $k \geq 2$ then $(H \setminus v_1, \pi \setminus v_1)$ must have been added to \mathcal{H}_σ via $\mathcal{C}^{i,j,k-1}$ in the $(k-1)$ -th iteration of the repeat-loop (**).

In all cases we can apply the induction hypothesis and conclude that the graph $(H \setminus v_1, \pi^*)$, defined by $\pi^* := (v_{k+1}, v_2, v_3, \dots, v_k, v_{k+2}, \dots, v_h)$ (to be understood as $\pi \setminus v_1$ if $k = 1$) is contained in $\mathcal{C}^{i,j}$ and satisfies

$$w_{(H \setminus v_1, \pi \setminus v_1, \sigma)}(u) \leq w_{(H \setminus v_1, \pi^*, \sigma)}(u) \quad \text{for all } u \in H \setminus v_1. \quad (4.48)$$

By the definition of $\mathcal{C}^{i,j}$ in line 14, $(H \setminus v_1, \pi^*)$ satisfies

$$d_\theta(H \setminus v_1, v_{k+1}, w_{(H \setminus v_1, \pi^*, \sigma)}) = d_\sigma^i, \quad (4.49)$$

and the weight of its youngest vertex v_{k+1} is set to

$$w_{(H \setminus v_1, \pi^*, \sigma)}(v_{k+1}) \stackrel{(4.22)}{=} w_\sigma(H \setminus v_1, \pi^*) = w^i \quad (4.50)$$

in line 18.

Consider now the graph $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ and observe that it is a subgraph of (H, π) . Applying Lemma 4.19 and Lemma 4.20 hence yields

$$d_\theta(H \setminus v_{k+1}, v_1, w_{(H \setminus v_{k+1}, \pi \setminus v_{k+1}, \sigma)}) \geq d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \stackrel{(4.46)}{\geq} d_\sigma^i. \quad (4.51)$$

As $(H \setminus v_1, \pi^*) \in \mathcal{C}^{i,j}$ we must have had

$$(H \setminus \{v_1, v_{k+1}\}, \pi^* \setminus v_{k+1}) = (H \setminus \{v_1, v_{k+1}\}, \pi \setminus \{v_1, v_{k+1}\}) \in \mathcal{H}_\sigma \quad (4.52)$$

at the beginning of the j -th iteration of the repeat-loop (**).

As our next intermediate step we will show that the graph $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ (which is obtained from the graph in (4.52) by adding v_1 as the youngest vertex and all edges incident to it) is contained in \mathcal{H}_σ at the beginning of the j -th iteration of the repeat-loop (**) as well. We first consider the case that one of the inequalities in (4.51) is strict, i.e., we have

$$d_\theta(H \setminus v_{k+1}, v_1, w_{(H \setminus v_{k+1}, \pi \setminus v_{k+1}, \sigma)}) > d_\sigma^i. \quad (4.53)$$

If $j = 1$, it follows from (4.52) and (4.53) that $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ is indeed contained in \mathcal{H}_σ at the beginning of the j -th iteration of the repeat-loop (**), as otherwise we would obtain a contradiction to the definition of d_σ^i in line 8. The same conclusion holds for $j > 1$ by using (4.52), (4.53) and the first part of Lemma 4.15. Now consider the case that all inequalities in (4.51) are tight, i.e., we have

$$d_\theta(H \setminus v_{k+1}, v_1, w_{(H \setminus v_{k+1}, \pi \setminus v_{k+1}, \sigma)}) = d_\theta(H, v_1, w_{(H, \pi, \sigma)}) = d_\sigma^i. \quad (4.54)$$

Suppose that $j = 1$ and that $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ was not already contained in \mathcal{H}_σ at the beginning of the first iteration of the repeat-loop (**). Then by (4.52) and (4.54), this graph would be added to $\mathcal{C}^{i,1} = \mathcal{C}_\sigma(d_\sigma^i)$ in line 14. As $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ is a subgraph of (H, π) that contains the vertex v_1 , this observation together with the second equality in (4.54) contradicts the fact (H, π) satisfies the condition in line 26. The remaining subcase $j > 1$ can be proven analogously by using the second part of Lemma 4.15.

Therefore, we indeed have

$$(H \setminus v_{k+1}, \pi \setminus v_{k+1}) \in \mathcal{H}_\sigma \quad (4.55)$$

at the beginning of the j -th iteration of the repeat-loop (**). The weight assigned to the youngest vertex v_1 of this graph is

$$w_{(H \setminus v_{k+1}, \pi \setminus v_{k+1}, \sigma)}(v_1) = w^{\bar{i}} \geq w^i \quad (4.56)$$

for some $\bar{i} \leq i$ with $\alpha_{\bar{i}} = \sigma$, where the last inequality follows from the second part of Lemma 4.14.

We will show that the graph (H, π') , defined by $\pi' := (v_{k+1}, v_1, \dots, v_k, v_{k+2}, \dots, v_h)$ (which is obtained from $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ by adding v_{k+1} as the youngest vertex and all edges incident to it), satisfies the inductive claim: We first demonstrate that this graph is contained in $\mathcal{C}^{i,j}$ and then prove the claimed inequality between the vertex weights of (H, π) and (H, π') .

As a first step towards this goal we show that $d_\theta(H, v_{k+1}, w_{(H, \pi', \sigma)}) = d_\sigma^i$.

Clearly we have $(H \setminus \{v_1, v_{k+1}\}, \pi^* \setminus v_{k+1}) = (H \setminus \{v_1, v_{k+1}\}, \pi' \setminus \{v_1, v_{k+1}\})$, implying that

$$w_{(H \setminus v_1, \pi^*, \sigma)}(u) = w_{(H, \pi', \sigma)}(u) \quad \text{for all } u \in H \setminus \{v_1, v_{k+1}\}. \quad (4.57)$$

As (H, π') is obtained from $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ by adding v_{k+1} as the youngest vertex and all edges incident to it, we have

$$w_{(H, \pi', \sigma)}(v_1) = w_{(H \setminus v_{k+1}, \pi \setminus v_{k+1}, \sigma)}(v_1) \stackrel{(4.56)}{\geq} w^i. \quad (4.58)$$

Observe that (H, π') is a supergraph of $(H \setminus v_1, \pi^*)$, so applying Lemma 4.19 and Lemma 4.20 yields

$$d_\theta(H, v_{k+1}, w_{(H, \pi', \sigma)}) \leq d_\theta(H \setminus v_1, v_{k+1}, w_{(H \setminus v_1, \pi^*, \sigma)}) \stackrel{(4.49)}{=} d_\sigma^i. \quad (4.59)$$

Now suppose for the sake of contradiction that the first inequality in (4.59) is strict, i.e., we have

$$d_\theta(H, v_{k+1}, w_{(H, \pi', \sigma)}) < d_\sigma^i. \quad (4.60)$$

By (4.17) and (4.57) this implies that some

$$J' \in \arg \min_{J \subseteq H: v_{k+1} \in J} \left(\sum_{u \in J \setminus v_{k+1}} (1 + w_{(H, \pi', \sigma)}(u)) - e(J) \cdot \theta \right) \quad (4.61)$$

includes the vertex v_1 . But then we would have

$$\begin{aligned} d_\theta(H, v_1, w_{(H, \pi, \sigma)}) &\stackrel{(4.17)}{\leq} \sum_{u \in J' \setminus v_1} (1 + w_{(H, \pi, \sigma)}(u)) - e(J') \cdot \theta \\ &\stackrel{(4.47), (4.48)}{\leq} \sum_{u \in J' \setminus v_1} (1 + w_{(H \setminus v_1, \pi^*, \sigma)}(u)) - e(J') \cdot \theta \\ &= \sum_{u \in J' \setminus \{v_1, v_{k+1}\}} (1 + w_{(H \setminus v_1, \pi^*, \sigma)}(u)) + \underbrace{(1 + w_{(H \setminus v_1, \pi^*, \sigma)}(v_{k+1}))}_{\stackrel{(4.50)}{=} w^i} - e(J') \cdot \theta \\ &\stackrel{(4.57), (4.58)}{\leq} \sum_{u \in J' \setminus v_{k+1}} (1 + w_{(H, \pi', \sigma)}(u)) - e(J') \cdot \theta \\ &\stackrel{(4.17), (4.61)}{=} d_\theta(H, v_{k+1}, w_{(H, \pi', \sigma)}) \stackrel{(4.60)}{<} d_\sigma^i, \end{aligned}$$

contradicting (4.46). Hence (4.59) holds with equality and we have indeed

$$d_\theta(H, v_{k+1}, w_{(H, \pi', \sigma)}) = d_\sigma^i. \quad (4.62)$$

Clearly, as $(H \setminus v_1, \pi^*)$ is contained in $\mathcal{C}^{i,j}$, it follows that this graph is not contained in \mathcal{H}_σ at the beginning of the j -th iteration of the repeat-loop (*). As (H, π') is a supergraph of $(H \setminus v_1, \pi^*)$, it follows from Lemma 4.20 that (H, π') is not contained in \mathcal{H}_σ at this point either. Hence, by (4.55), (4.62) and the definition of $\mathcal{C}^{i,j}$ in line 14 we have $(H, \pi') \in \mathcal{C}^{i,j}$.

It remains to check the claimed inequality between the vertex weights of (H, π) and (H, π') . Note that $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ is a subgraph of (H, π) . We can hence apply Lemma 4.20 and, observing that (H, π') is obtained from $(H \setminus v_{k+1}, \pi \setminus v_{k+1})$ by adding v_{k+1} as the youngest vertex and all edges incident to it, obtain the desired inequality for all vertices in the set $\{v_1, \dots, v_h\} \setminus \{v_{k+1}\}$.

For the vertex v_{k+1} , note that (4.47), (4.48) and (4.50) together yield $w_{(H,\pi,\sigma)}(v_{k+1}) \leq w^i$ and that $w_{(H,\pi',\sigma)}(v_{k+1}) \stackrel{(4.22)}{=} w_\sigma(H, \pi') = w^i$ by the definition in line 18.

This completes the inductive proof of Lemma 4.21.

It remains to show that every graph $(H, \pi) \in \mathcal{C}^{i,j,k}$ has at least three vertices, and that therefore all graphs used in the above arguments are well-defined and have at least one vertex. We show that all graphs $(H, \pi) \in \mathcal{S}(F)$ on at most two vertices that are ever added to the family \mathcal{H}_σ in the course of the algorithm are added to it via one of the sets $\mathcal{C}^{i,j}$ defined in line 14: As argued in the proof of Lemma 4.10 on page 50, this is true for the graph $(K_1, (v_1))$ (an isolated vertex), which is added via the set $\mathcal{C}^{i,1}$ in the first iteration i for which $\alpha_i = \sigma$. Once we have $\mathcal{H}_\sigma = \{(K_1, (v_1))\}$, the only two ordered subgraphs of F on two vertices, a single edge and two isolated vertices, are contained in $\mathcal{C}(\mathcal{H}_\sigma, F)$. Using this fact together with the observation that $(K_1, (v_1))$ is contained in $\mathcal{C}_\sigma(0)$ and is a subgraph of both of them, it is easy to check that each of those two graphs can only be added to \mathcal{H}_σ via one of the sets $\mathcal{C}^{i,j}$: If the $d_\theta(\cdot)$ -value of one of these graphs is equal to 0, then it is added via $\mathcal{C}^{i,2}$. Otherwise its $d_\theta(\cdot)$ -value is strictly smaller than 0 and it is added via $\mathcal{C}^{i,1}$ for some $i > \tilde{i}$. \square

4.3.7. Further properties of the algorithm. The next lemma implies in particular that for a graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$ and a subgraph $J \subseteq H$, $v_1 \in J$, minimizing the right hand side of the definition of $d_\theta(H, v_1, w_{(H,\pi,s)})$ in (4.17), the inequality stated in Lemma 4.20 is in fact an equality. As a consequence, in all situations where the vertex weights of a subgraph $J \subseteq H$ are relevant, these weights only depend on $(J, \pi|_J)$ and not on the ‘context’ H . This is far from clear a priori, and in fact not true for arbitrary subgraphs $J \subseteq H$.

LEMMA 4.22 (Irrelevant context of $d_\theta(\cdot)$ -minimizing subgraphs). *Throughout the algorithm COMPUTEWEIGHTS(), for every $s \in [r]$, any graph $(H, \pi) \in \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$, $\pi = (v_1, \dots, v_h)$, and any graph \hat{J} from the family*

$$\arg \min_{J \subseteq H: v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w_{(H,\pi,s)}(u)) - e(J) \cdot \theta \right) \quad (4.63)$$

we have

$$w_{(H,\pi,s)}(u) = w_{(\hat{J}, \pi|_{\hat{J}}, s)}(u) \quad \text{for all } u \in \hat{J} \setminus v_1. \quad (4.64a)$$

Moreover, if $(H, \pi) \in \mathcal{H}_s$, then we have

$$w_{(H,\pi,s)}(v_1) = w_{(\hat{J}, \pi|_{\hat{J}}, s)}(v_1). \quad (4.64b)$$

Note that by Lemma 4.13 and Lemma 4.20, all vertex weights referred to in the formulation of Lemma 4.22 are finite values. We will not mention this explicitly again in the following.

The following two auxiliary statements are only used for proving Lemma 4.22.

LEMMA 4.23. *Let H be a graph with $V(H) = \{v_1, \dots, v_h\}$, $w : V(H) \setminus \{v_1\} \rightarrow \mathbb{R}$ an arbitrary weight function, $\theta > 0$ a real number, and let \hat{J} be a graph from the family*

$$\arg \min_{J \subseteq H: v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w(u)) - e(J) \cdot \theta \right). \quad (4.65)$$

Moreover, let v_k be a vertex contained in \hat{J} and \tilde{J} a graph from the family

$$\arg \min_{J \subseteq H \setminus \{v_1, \dots, v_{k-1}\}: v_k \in J} \left(\sum_{u \in J \setminus v_k} (1 + w(u)) - e(J) \cdot \theta \right). \quad (4.66)$$

Then the graph $\tilde{J} \cap \hat{J}$ is also contained in the family (4.66). In particular, for two graphs J', J'' from the family (4.65) the graph $J' \cap J''$ is also contained in (4.65).

PROOF. In order to simplify notation, we introduce for a real number $\theta > 0$, any graph H , any vertex $v \in H$ and any weight function $w : V(H) \setminus \{v\} \rightarrow \mathbb{R}$, the function

$$\lambda_{\theta}^{-}(H, v, w) := \sum_{u \in H \setminus v} (1 + w(u)) - e(H) \cdot \theta . \quad (4.67)$$

As for the definition of $d_{\theta}()$ in (4.17), it is also convenient here to allow functions w in the third argument of $\lambda_{\theta}^{-}()$ whose domain is strictly larger than the set $V(H) \setminus \{v\}$. Of course, for the value of $\lambda_{\theta}^{-}(H, v, w)$ only the values $w(u)$ for all $u \in H \setminus v$ are relevant.

By the choice of \tilde{J} in (4.66) and by (4.67), we have

$$\lambda_{\theta}^{-}(\tilde{J}, v_k, w) \leq \lambda_{\theta}^{-}(\tilde{J} \cap \hat{J}, v_k, w) . \quad (4.68)$$

This inequality, however, must be tight, as otherwise the second inequality in

$$\lambda_{\theta}^{-}(\hat{J} \cup \tilde{J}, v_1, w) \stackrel{(4.67)}{=} \lambda_{\theta}^{-}(\hat{J}, v_1, w) + \lambda_{\theta}^{-}(\tilde{J}, v_k, w) - \lambda_{\theta}^{-}(\tilde{J} \cap \hat{J}, v_k, w) \stackrel{(4.68)}{\leq} \lambda_{\theta}^{-}(\hat{J}, v_1, w)$$

would be strict, contradicting the choice of \hat{J} in (4.65). This proves the lemma. \square

The next auxiliary statement will be used to prove Lemma 4.22 by induction.

LEMMA 4.24. *The following invariant holds throughout the algorithm COMPUTEWEIGHTS(). Let $s \in [r]$ and let (H, π) , $\pi = (v_1, \dots, v_h)$, be a graph in \mathcal{H}_s , and suppose that every graph J' from the family*

$$\arg \min_{J \subseteq H : v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w_{(H, \pi, s)}(u)) - e(J) \cdot \theta \right) \quad (4.69)$$

satisfies

$$w_{(H, \pi, s)}(u) = w_{(J', \pi|_{J'}, s)}(u) \quad \text{for all } u \in J' \setminus v_1 . \quad (4.70)$$

Then every such graph J' satisfies

$$w_{(H, \pi, s)}(v_1) = w_{(J', \pi|_{J'}, s)}(v_1) . \quad (4.71)$$

PROOF. Let J' be a graph from the family (4.69) and note that

$$d_{\theta}(H, v_1, w_{(H, \pi, s)}) \stackrel{(4.17)}{=} \stackrel{(4.69)}{=} d_{\theta}(J', v_1, w_{(H, \pi, s)}) \stackrel{(4.70)}{=} d_{\theta}(J', v_1, w_{(J', \pi|_{J'}, s)}) . \quad (4.72)$$

By Lemma 4.20 we clearly have $w_{(H, \pi, s)}(v_1) \leq w_{(J', \pi|_{J'}, s)}(v_1)$. Consequently, using (4.72) and applying the second part of Lemma 4.18, the only way that $w_{(H, \pi, s)}(v_1)$ can be different from $w_{(J', \pi|_{J'}, s)}(v_1)$ is if $w_s(H, \pi)$ is defined either in line 18 or in line 31 with \hat{i} defined in line 30, and $w_s(J', \pi|_{J'})$ is defined in line 31 with \hat{i} defined in line 28. We will show that none of those cases can occur.

- First consider the case that $w_s(H, \pi)$ is defined in line 18 in some iteration i of the repeat-loop (*) (for which $\alpha_i = s$) and the first iteration $j = 1$ of the repeat-loop (**), i.e., (H, π) is contained in $\mathcal{C}^{i,1}$ and satisfies $d_{\theta}(H, v_1, w_{(H, \pi, s)}) = d_s^i$ and $w_s(H, \pi) = w^i$. Then by (4.72) and by Lemma 4.16 and Lemma 4.20 the graph $(J', \pi|_{J'})$ must be contained in $\mathcal{C}^{i,1}$ as well and is added to \mathcal{H}_{σ} together with (H, π) . Hence we have $w_s(J', \pi|_{J'}) = w^i$ by the definition in line 18, proving (4.71) in this case.

- Now consider the case that $w_s(H, \pi)$ is defined in line 18 in some iteration i of the repeat-loop (*) (for which $\alpha_i = s$) and some iteration $j > 1$ of the repeat-loop (**), i.e., (H, π) is contained in $\mathcal{C}^{i,j}$ and satisfies

$$d_\theta(H, v_1, w_{(H, \pi, s)}) = d_s^i . \quad (4.73)$$

Then (H, π) must have been in $\mathcal{C}(\mathcal{H}_s, F)$ at the end of the previous iteration of the repeat-loop (**), and by the second part of Lemma 4.15 there is a subgraph $\bar{J} \subseteq H$ with $v_1 \in \bar{J}$ and $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_s(d_s^i)$. From the definitions in line 14 and line 16 it follows that

$$d_\theta(\bar{J}, v_1, w_{(\bar{J}, \pi|_{\bar{J}}, s)}) = d_s^i . \quad (4.74)$$

Fix some graph J'' from the family

$$\arg \min_{J \subseteq \bar{J}: v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w_{(\bar{J}, \pi|_{\bar{J}}, s)}(u)) - e(J) \cdot \theta \right) \quad (4.75)$$

and note that

$$d_\theta(\bar{J}, v_1, w_{(\bar{J}, \pi|_{\bar{J}}, s)}) \stackrel{(4.17), (4.75)}{=} \sum_{u \in J'' \setminus v_1} (1 + w_{(\bar{J}, \pi|_{\bar{J}}, s)}(u)) - e(J'') \cdot \theta . \quad (4.76)$$

By Lemma 4.20 we have

$$w_{(H, \pi, s)}(u) \leq w_{(\bar{J}, \pi|_{\bar{J}}, s)}(u) \quad \text{for all } u \in \bar{J} . \quad (4.77)$$

We hence have

$$d_\theta(H, v_1, w_{(H, \pi, s)}) \stackrel{(4.17)}{\leq} \sum_{u \in J'' \setminus v_1} (1 + w_{(H, \pi, s)}(u)) - e(J'') \cdot \theta \stackrel{(4.76), (4.77)}{\leq} d_\theta(\bar{J}, v_1, w_{(\bar{J}, \pi|_{\bar{J}}, s)}) ,$$

which combined with (4.73) and (4.74) shows that the graph $(J'', \pi|_{J''})$ is contained in the family (4.69). Applying Lemma 4.23 yields that the graph $(J' \cap J'', \pi|_{J' \cap J''})$ is also contained in the family (4.69). Analogously to (4.72) we have

$$d_\theta(H, v_1, w_{(H, \pi, s)}) = d_\theta(J' \cap J'', v_1, w_{(J' \cap J'', \pi|_{J' \cap J''}, s)}) ,$$

which combined with (4.73) shows that

$$d_\theta(J' \cap J'', v_1, w_{(J' \cap J'', \pi|_{J' \cap J''}, s)}) = d_s^i . \quad (4.78)$$

Clearly $J' \cap J''$ is a subgraph of \bar{J} that contains the youngest vertex v_1 . Using this observation and (4.78) and applying Lemma 4.16 and Lemma 4.20, the fact that $(\bar{J}, \pi|_{\bar{J}})$ is contained in $\mathcal{C}_s(d_s^i)$ implies that $(J' \cap J'', \pi|_{J' \cap J''})$ is contained in $\mathcal{C}_s(d_s^i)$ as well. But as $J' \cap J''$ is also a subgraph of J' (that contains the youngest vertex v_1), this implies with (4.72) and (4.73) that the graph $(J', \pi|_{J'})$ violates the condition in line 27, which yields the desired contradiction.

- Finally consider the case that $w_s(H, \pi)$ is defined in line 31 (in some iteration i of the repeat-loop (**)) with \hat{i} defined in line 30. By the conditions in line 26 and line 27 we have $d_\theta(H, v_1, w_{(H, \pi, s)}) > d_s^i$ and there is a graph $\bar{J} \subseteq H$ with $v_1 \in \bar{J}$ and $(\bar{J}, \pi|_{\bar{J}}) \in \mathcal{C}_s(d_\theta(H, v_1, w_{(H, \pi, s)}))$. Using the definitions in line 14 and line 16, as well as the first part of Lemma 4.14, it follows that

$$d_\theta(H, v_1, w_{(H, \pi, s)}) = d_s^{\bar{i}}$$

and

$$d_\theta(\bar{J}, v_1, w_{(\bar{J}, \pi|_{\bar{J}}, s)}) = d_s^{\bar{i}}$$

for some $\bar{i} < i$. From here the proof continues analogously to the second case (where d_s^i needs to be replaced by $d_s^{\bar{i}}$), concluding that $(J', \pi|_{J'})$ must violate the condition in line 27, which again yields the desired contradiction. \square

PROOF OF LEMMA 4.22. We argue by induction over the number of vertices of H . The claim clearly holds if H consists only of a single vertex, as then $\hat{J} = H$ is the only graph contained in the family (4.63). This settles the base of the induction.

For the induction step suppose that H has at least two vertices, and let \hat{J} be a graph from the family (4.63). Clearly, $(H \setminus v_1, \pi \setminus v_1)$ is contained in \mathcal{H}_s (recall the definition of $\mathcal{C}()$ in (4.21)). Applying Lemma 4.20 we obtain that $(\hat{J} \setminus v_1, \pi|_{\hat{J} \setminus v_1})$ is contained in \mathcal{H}_s as well and that

$$w_{(H, \pi, s)}(v_i) \leq w_{(\hat{J}, \pi|_{\hat{J}}, s)}(v_i) \quad \text{for all } v_i \in \hat{J} \setminus v_1. \quad (4.79)$$

We will first show that this inequality is tight for all $v_i \in \hat{J} \setminus v_1$ (which is exactly the statement of (4.64a)). Suppose for the sake of contradiction that the inequality in (4.79) is strict for some $v_i \in \hat{J} \setminus v_1$, and choose the largest index k for which this is the case, i.e.

$$w_{(H, \pi, s)}(v_k) < w_{(\hat{J}, \pi|_{\hat{J}}, s)}(v_k) \quad (4.80)$$

and

$$w_{(H, \pi, s)}(v_i) = w_{(\hat{J}, \pi|_{\hat{J}}, s)}(v_i) \quad \text{for all } v_i \in \hat{J} \setminus \{v_1, \dots, v_k\} \quad (4.81)$$

(we clearly have $k \geq 2$). Fix a graph \tilde{J} from the family

$$\arg \min_{J \subseteq H \setminus \{v_1, \dots, v_{k-1}\} : v_k \in J} \left(\sum_{u \in J \setminus v_k} (1 + w_{(H, \pi, s)}(u)) - e(J) \cdot \theta \right) \quad (4.82)$$

and observe that by Lemma 4.23, also the graph $\tilde{J} \cap \hat{J}$ is contained in the family (4.82). By (4.81) the same graph $\tilde{J} \cap \hat{J}$ is also contained in the family

$$\arg \min_{J \subseteq \hat{J} \setminus \{v_1, \dots, v_{k-1}\} : v_k \in J} \left(\sum_{u \in J \setminus v_k} (1 + w_{(\hat{J}, \pi|_{\hat{J}}, s)}(u)) - e(J) \cdot \theta \right).$$

By induction, we therefore have $w_{(\tilde{J} \cap \hat{J}, \pi|_{\tilde{J} \cap \hat{J}}, s)}(v_k) = w_{(H, \pi, s)}(v_k)$ and $w_{(\tilde{J} \cap \hat{J}, \pi|_{\tilde{J} \cap \hat{J}}, s)}(v_k) = w_{(\hat{J}, \pi|_{\hat{J}}, s)}(v_k)$, which together contradicts (4.80) and shows that (4.79) holds with equality for all $v_i \in \hat{J} \setminus v_1$, thus proving (4.64a).

The relation (4.64b) follows from (4.64a) by applying Lemma 4.24. \square

Lemma 4.22 allows us to derive the next statement, which is similar in spirit but considers $\lambda_\theta()$ -values instead of $d_\theta()$ -values.

LEMMA 4.25 (Irrelevant context of $\lambda_\theta()$ -minimizing subgraphs). *For every $s \in [r]$ and any graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, we have*

$$\min_{J \subseteq H} \lambda_\theta(J, w_{(H, \pi, s)}) = \min_{J \subseteq H} \lambda_\theta(J, w_{(J, \pi|_J, s)}) .$$

PROOF. Using the definition of $\lambda_\theta()$ in (4.18), we obtain from Lemma 4.20 that

$$\min_{J \subseteq H : v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s)}) \leq \min_{J \subseteq H : v_1 \in J} \lambda_\theta(J, w_{(J, \pi|_J, s)}) . \quad (4.83)$$

From

$$\min_{J \subseteq H : v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s)}) \stackrel{(4.18)}{=} \min_{J \subseteq H : v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w_{(H, \pi, s)}(u)) - e(J) \cdot \theta \right) + 1 + w_{(H, \pi, s)}(v_1) \quad (4.84)$$

it follows that the minimum on the left hand side of (4.83) is attained for some graph

$$\hat{J} \in \arg \min_{J \subseteq H: v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w_{(H, \pi, s)}(u)) - e(J) \cdot \theta \right). \quad (4.85)$$

We can hence apply Lemma 4.22 and obtain

$$\min_{J \subseteq H: v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s)}) \stackrel{(4.84), (4.85)}{=} \sum_{u \in \hat{J}} (1 + w_{(H, \pi, s)}(u)) - e(\hat{J}) \cdot \theta \stackrel{(4.18), (4.64)}{=} \lambda_\theta(\hat{J}, w_{(\hat{J}, \pi|_{\hat{J}}, s)}) ,$$

which shows that the inequality in (4.83) is tight. The lemma now follows by combining the resulting identity with the identity

$$\min_{J \subseteq H} \lambda_\theta(J, w_{(H, \pi, s)}) = \min_{\substack{1 \leq i \leq h \\ J \subseteq H \setminus \{v_1, \dots, v_{i-1}\}: v_i \in J}} \lambda_\theta(J, w_{(H, \pi, s)}) .$$

□

We are now ready to state and prove the relation between $\Lambda_\theta(F, r)$ as defined in (4.23) and the parameter $\beta_i = 1 + \sum_{s \in [r]} d_s^i$ used in our informal explanation of the algorithm COMPUTEWEIGHTS() in Section 4.3.3.

LEMMA 4.26 (Relation between $\Lambda_\theta(F, r)$ and d_s^i). *For any input sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ of the algorithm COMPUTEWEIGHTS() we have*

$$\max_{\substack{s \in [r] \\ \pi \in \Pi(V(F))}} \min_{H \subseteq F} \lambda_\theta(H, w_{(H, \pi|_H, s)}) = 1 + \sum_{s \in [r]} d_s^{\check{i}} , \quad (4.86)$$

where \check{i} is the smallest integer i for which $(F, \pi) \in \mathcal{C}^{i, j}$ for some $\pi \in \Pi(V(F))$ and some integer $j \geq 1$, and d_s^i and $\mathcal{C}^{i, j}$ are defined in line 8 and line 14.

PROOF. Throughout the proof, we will repeatedly use that, as a consequence of the first part of Lemma 4.14, each of the values d_t^i , $t \in [r]$, is non-increasing with i , and that the sum

$$1 + \sum_{t \in [r]} d_t^i \quad (4.87)$$

is decreasing with i .

For every $s \in [r]$ and any graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, we have

$$\min_{J \subseteq H: v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s)}) \stackrel{(4.17), (4.18)}{=} d_\theta(H, v_1, w_{(H, \pi, s)}) + 1 + w_{(H, \pi, s)}(v_1) . \quad (4.88)$$

If $(H, \pi) \in \mathcal{C}^{i, j}$ for some integers $i, j \geq 1$ with $\alpha_i = s$, then by using the definition in line 14 and by combining (4.22) with the definitions in line 10 and 18 we obtain from (4.88) that

$$\min_{J \subseteq H: v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s)}) = 1 + \sum_{t \in [r]} d_t^i . \quad (4.89a)$$

Similarly, if $(H, \pi) \in \mathcal{C}^{i, j, k}$ for some integers $i, j, k \geq 1$ with $\alpha_i = s$, then by using the definition in line 23 (recall that $\mathcal{C}^{i, j, k} \subseteq \mathcal{T}^{i, j, k}$) and by combining (4.22) with the definitions in line 10 and 31 we obtain from (4.88), using the monotonicity of the values d_t^i in i , that

$$\min_{J \subseteq H: v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s)}) \geq 1 + \sum_{t \in [r]} d_t^i . \quad (4.89b)$$

By Lemma 4.13, in the maximization in (4.86) it suffices to consider those $s \in [r]$ and vertex orderings $\pi \in \Pi(V(F))$, $\pi = (v_1, \dots, v_f)$, for which $(F, \pi) \in \mathcal{H}_s$. We clearly have

$$\min_{H \subseteq F} \lambda_\theta(H, w_{(F, \pi, s)}) = \min_{\substack{H \subseteq F \setminus \{v_1, \dots, v_{c-1}\} : v_c \in H \\ 1 \leq c \leq f}} \lambda_\theta(H, w_{(F, \pi, s)}) . \quad (4.90)$$

If $(F, \pi) \in \mathcal{C}^{i,j}$ for some integers $i, j \geq 1$ with $\alpha_i = s$, then by (4.89) and the monotonicity of the sum (4.87) in i , the minimum on the right hand side of (4.90) is attained for $c = 1$, yielding

$$\min_{H \subseteq F} \lambda_\theta(H, w_{(F, \pi, s)}) = 1 + \sum_{t \in [r]} d_t^i . \quad (4.91)$$

If $(F, \pi) \in \mathcal{C}^{i,j,k}$ for some integers $i, j, k \geq 1$ with $\alpha_i = s$, then by Lemma 4.21, the graph (F, π') , defined by $\pi' := (v_{k+1}, v_1, v_2, \dots, v_k, v_{k+2}, \dots, v_f)$, is contained in $\mathcal{C}^{i,j}$ and satisfies

$$w_{(F, \pi, s)}(u) \leq w_{(F, \pi', s)}(u) \quad \text{for all } u \in F , \quad (4.92)$$

implying that

$$\min_{H \subseteq F} \lambda_\theta(H, w_{(F, \pi, s)}) \stackrel{(4.92)}{\leq} \min_{H \subseteq F} \lambda_\theta(H, w_{(F, \pi', s)}) \stackrel{(4.91)}{=} 1 + \sum_{t \in [r]} d_t^i . \quad (4.93)$$

By Lemma 4.25 we can replace the weight function $w_{(F, \pi, s)}$ on the left hand side of (4.91) by $w_{(H, \pi|_H, s)}$ and the weight functions $w_{(F, \pi, s)}$ and $w_{(F, \pi', s)}$ in (4.93) by $w_{(H, \pi|_H, s)}$ and $w_{(H, \pi'|_H, s)}$, respectively. From the two modified equations the claim follows immediately, observing that as a consequence of the monotonicity of the sum (4.87) in i , their respective right hand sides are maximized for $i = \tilde{i}$ as defined in the lemma. \square

4.4. Builder in the deterministic game

In this section we prove Proposition 4.4 by explicitly constructing, for F , r , θ and β as in the proposition, a Builder strategy that enforces a monochromatic copy of F in the deterministic game with r colors in at most a_{\max} steps (where $a_{\max} = a_{\max}(F, r)$ is defined in (4.96) below), and that respects the generalized density restriction (θ, β) .

4.4.1. The pigeonholing. We will derive Builder's strategy from the algorithm COMPUTE-WEIGHTS() (Algorithm 1 on page 47) in two steps, using an abstract version of the game as an intermediate step. The reader should not be put off by our introducing yet another game — this abstract game is merely a convenience to separate a conceptually simple but important pigeonholing argument from the more interesting part of the proof. We give the pigeonholing argument in detail because there are some subtleties involved, and also because we want to derive an *explicit* upper bound $a_{\max} = a_{\max}(F, r)$ (that in particular does not depend on θ) on the number of steps that Builder needs to enforce a copy of F in the original deterministic game.

The abstract game is played by two players AbstractBuilder and AbstractPainter that correspond to the players of the original deterministic game. The state of the abstract game after t steps is described by a list (G^1, \dots, G^t) of r -colored graphs G^i , $1 \leq i \leq t$, where the same r -colored graph may appear several times in the list. (Intuitively, these entries represent r -colored graphs of which Builder can enforce isolated copies on the board of the actual deterministic game, where by an isolated copy of some (r -colored) graph G on the board we mean a copy of G that is the union of one or several components.) In each step $t + 1$ of the abstract game, AbstractBuilder constructs a new graph by choosing an arbitrary subset \mathcal{X} of the index set $\{1, \dots, t\}$, and connecting an additional vertex v in an arbitrary way to the disjoint union of the graphs G^i ,

$i \in \mathcal{X}$. AbstractPainter then chooses a color $s \in [r]$ for v , and the resulting r -colored graph G^{t+1} is added to the list. The game starts with the empty list (and thus the first graph G^1 constructed is simply an isolated vertex), and AbstractBuilder's goal is to create an r -colored graph G^t that contains a monochromatic copy of F . Similarly to before we say that an AbstractBuilder strategy satisfies the generalized density restriction (θ, β) for given values $\theta > 0$ and β if, at all times, all subgraphs H with $v(H) \geq 1$ of all graphs G^i in AbstractBuilder's list satisfy $\mu_\theta(H) \geq \beta$ (recall (4.6)).

The following lemma relates the abstract game to the original deterministic game.

LEMMA 4.27 (Link between abstract and original deterministic game). *Let ABSTRACTSTRATEGY be an arbitrary AbstractBuilder strategy for the abstract game with r colors. If ABSTRACTSTRATEGY enforces a monochromatic copy of F in at most t_{\max} steps, then it gives rise to a Builder strategy STRATEGY for the original deterministic game with r colors that enforces a monochromatic copy of F in at most $(r+1)^{t_{\max}}$ steps. Furthermore, if ABSTRACTSTRATEGY satisfies the generalized density restriction (θ, β) for given values $\theta > 0$ and $\beta \geq 0$, then also STRATEGY satisfies the generalized density restriction (θ, β) .*

PROOF. We simultaneously capture all possible ways the abstract game may evolve if AbstractBuilder plays according to ABSTRACTSTRATEGY by an r -ary rooted tree \mathcal{T} in which a node at depth t is a list $b = (G^1, \dots, G^t)$ of r -colored graphs G^i , $1 \leq i \leq t$, and has as its r children the nodes $b_s = (G^1, \dots, G^t, G_s^{t+1})$, $s \in [r]$, where G_s^{t+1} is obtained from G^1, \dots, G^t by applying the next construction step of ABSTRACTSTRATEGY and coloring the new vertex with color s . Thus the graphs G_s^{t+1} differ only in the color assigned to the new vertex.

We assume w.l.o.g. that AbstractBuilder stops playing as soon as a monochromatic copy of F is created. Thus if $b = (G^1, \dots, G^t)$ is a leaf of \mathcal{T} , the graph G^t (the last graph constructed) contains a monochromatic copy of F . Furthermore, by the assumption of the lemma, the depth of the strategy tree \mathcal{T} is bounded by t_{\max} . In the following we assume w.l.o.g. that the depth of \mathcal{T} is exactly t_{\max} .

To derive STRATEGY from ABSTRACTSTRATEGY, we compute for each node $b = (G^1, \dots, G^t)$ of \mathcal{T} a function $f_b : \{G^1, \dots, G^t\} \rightarrow \mathbb{N}_0$ that specifies for each of the graphs G^i the number of isolated copies of G^i that are needed to implement the strategy ABSTRACTSTRATEGY in the original deterministic game. This can be done recursively as follows.

If $b = (G^1, \dots, G^t)$ is a leaf of \mathcal{T} , we set

$$f_b(G^t) := 1 \tag{4.94a}$$

and

$$f_b(G^i) := 0, \quad 1 \leq i \leq t-1. \tag{4.94b}$$

If $b = (G^1, \dots, G^t)$ is an internal node of \mathcal{T} , then letting $\mathcal{X}_b \subseteq \{1, \dots, t\}$ denote the index set of the graphs that are used in the construction step corresponding to b , and denoting the descendants of b by $b_s = (G^1, \dots, G^t, G_s^{t+1})$, $s \in [r]$, as before, we define for $1 \leq i \leq t$,

$$f_b(G^i) := \begin{cases} \max_{s \in [r]} f_{b_s}(G^i), & \text{if } i \notin \mathcal{X}_b, \\ \max_{s \in [r]} f_{b_s}(G^i) + \sum_{s \in [r]} f_{b_s}(G_s^{t+1}), & \text{if } i \in \mathcal{X}_b. \end{cases} \tag{4.94c}$$

With these definitions, STRATEGY is obtained from ABSTRACTSTRATEGY by proceeding as described by the strategy tree \mathcal{T} , and repeating every construction step corresponding to a given node b exactly $\sum_{s \in [r]} f_{b_s}(G_s^{t+1})$ times, each time connecting a new vertex to (previously unused)

isolated copies of the graphs G^i , $i \in \mathcal{X}_b$, on the board as specified by the corresponding step of the abstract game. By the pigeonhole principle, this guarantees that regardless of how Painter plays there is a color σ such that at least $f_{b_\sigma}(G_\sigma^{t+1})$ isolated copies of G_σ^{t+1} are created, and by our recursive definition in (4.94c) it also follows that at least $f_{b_\sigma}(G_\sigma^i)$ isolated copies of each graph G_σ^i , $1 \leq i \leq t$, are left unused. Thus Builder may continue with the construction step corresponding to the node b_σ . This shows that at every node $b = (G^1, \dots, G^t)$ of \mathcal{T} , Builder has at least $f_b(G^i)$ isolated copies of every graph G^i available. In particular, when he reaches a leaf of \mathcal{T} , due to (4.94a) he will have created at least one copy of a graph G^t containing a monochromatic copy of F .

This shows that STRATEGY indeed creates a monochromatic copy of F in the original deterministic game, and it remains to bound the number of steps it needs to do so. For every $t = 1, \dots, t_{\max}$ we denote by c_t the maximum of $f_b(G^i)$ over all nodes $b = (G^1, \dots, G^t)$ at depth t in \mathcal{T} and all $1 \leq i \leq t$. It follows from (4.94c) that

$$c_t \leq (r+1)c_{t+1} \quad , \quad (4.95a)$$

and by (4.94a) and (4.94b) we have

$$c_{t_{\max}} = 1 \quad . \quad (4.95b)$$

By definition of the rule how often to repeat each step of ABSTRACTSTRATEGY in STRATEGY, the number of repetitions of a step that corresponds to a node b at depth t in \mathcal{T} is bounded by $r \cdot c_{t+1}$. It follows that the total number of Builder steps when executing STRATEGY is bounded by

$$\sum_{t=0}^{t_{\max}-1} r \cdot c_{t+1} \stackrel{(4.95)}{\leq} r \sum_{t=0}^{t_{\max}-1} (r+1)^t \leq (r+1)^{t_{\max}} \quad ,$$

as claimed.

Furthermore, as the strategy STRATEGY differs from ABSTRACTSTRATEGY merely in how often (Abstract)Builder's construction steps are repeated, and because for $\beta \geq 0$ it suffices to check the condition (4.6) for all *connected* subgraphs H of the board, it follows that with ABSTRACTSTRATEGY also STRATEGY satisfies the generalized density restriction (θ, β) . \square

4.4.2. Builder's strategy and proof of Proposition 4.4. We now present AbstractBuilder's strategy ABSTRACTBUILD(F, r, θ) that will yield our final Builder strategy BUILD(F, r, θ) via Lemma 4.27. Throughout this section, F , r , and θ are fixed, and we usually omit these arguments when we refer to ABSTRACTBUILD(F, r, θ) or COMPUTEWEIGHTS(F, r, θ, α).

The strategy ABSTRACTBUILD() proceeds in rounds along the lines of the algorithm COMPUTEWEIGHTS(). (As before, the term 'round' refers to one iteration of the repeat-loop (*) of COMPUTEWEIGHTS().) ABSTRACTBUILD() maintains, for each color $s \in [r]$, a family $\mathcal{G}_s \subseteq \mathcal{S}(F)$ and a mapping G_s from \mathcal{G}_s to the r -colored graphs in AbstractBuilder's list. For any $(H, \pi) \in \mathcal{G}_s$ the graph $G_s(H, \pi)$ will always contain a distinguished monochromatic copy of H in color s to which we will refer as the *central copy of H in $G_s(H, \pi)$* ; it is however possible that this copy was constructed in an order different from π (this is where we make crucial use of Lemma 4.21 proved in Section 4.3.6).

At the same time, ABSTRACTBUILD() extracts a sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ from AbstractPainter's coloring decisions such that the following holds: After each round, the families \mathcal{G}_s contain all graphs from the families \mathcal{H}_s occurring after the same number of rounds of COMPUTEWEIGHTS() with input sequence α . We will also see that for each graph $(H, \pi) \in \mathcal{H}_s$, the graph $G_s(H, \pi)$ on AbstractBuilder's list can indeed be used in further construction steps (without violating some

given generalized density restriction) as indicated by the weight function $w_{(H,\pi,s)}$ computed by `COMPUTEWEIGHTS()` with input sequence α .

In order to construct a sequence α for which the above statements hold, `ABSTRACTBUILD()` uses variables defined by the algorithm `COMPUTEWEIGHTS()` for several different input sequences. We will use the following notations: For any sequence $\alpha \in [r]^{i-1}$ and any $s \in [r]$ we let $\alpha \circ s \in [r]^i$ denote the concatenation of α with s . When we refer to the algorithm `COMPUTEWEIGHTS()` with some input sequence $\alpha \in [r]^i$, we tacitly assume that α is extended arbitrarily to a sequence $\alpha' \in [r]^{r \cdot |\mathcal{S}(F)|}$ with prefix α . As we will only use this convention when we refer to variables defined in the first i iterations of the repeat-loop (*) of `COMPUTEWEIGHTS()`, the values of α' beyond the prefix α are irrelevant (recall that the i -th iteration reads exactly the i -th element of the input sequence α').

A key ingredient in the construction of the sequence α is the following lemma. Recall that for any set X and any integer $r \geq 1$, an r -coloring of X is simply a mapping $f : X \rightarrow [r]$.

LEMMA 4.28 (Dominating color). *Let $r \geq 1$ be an integer and X_1, \dots, X_r finite, nonempty sets. For any r -coloring f of $X_1 \times \dots \times X_r$ there is a color $\sigma \in [r]$ such that for every $x_\sigma \in X_\sigma$ there are elements $x_s \in X_s$, $s \in [r] \setminus \{\sigma\}$, with $f(x_1, \dots, x_r) = \sigma$.*

We defer the proof of Lemma 4.28 to the next section.

Consider now the pseudocode description of `ABSTRACTBUILD()` in Algorithm 2. Note that its loop structure mirrors the structure of `COMPUTEWEIGHTS()`, with the crucial difference that while the loop (**) of `COMPUTEWEIGHTS()` simply focuses on one color $\sigma \in [r]$ (as indicated by the i -th entry of the input sequence α), for the strategy `ABSTRACTBUILD()` the ‘right’ color σ depends on the individual decisions of `AbstractPainter` occurring during the loop (++), and is therefore not known until this loop terminates.

In the next section we will prove the following two properties of `ABSTRACTBUILD(F, r, θ)`.

LEMMA 4.29 (Well-definedness and duration of `AbstractBuilder` strategy). *For F , r , and θ as in Proposition 4.4, the strategy `ABSTRACTBUILD(F, r, θ)` enforces a monochromatic copy of F in at most $r^2 \cdot |\mathcal{S}(F)|^{r+2}$ steps of the abstract game.*

LEMMA 4.30 (`AbstractBuilder` strategy is legal). *For F , r , θ , and β as in Proposition 4.4, the strategy `ABSTRACTBUILD(F, r, θ)` satisfies the generalized density restriction (θ, β) .*

Together with Lemma 4.27, the preceding statements about `ABSTRACTBUILD(F, r, θ)` imply Proposition 4.4 straightforwardly.

PROOF OF PROPOSITION 4.4. Using Lemma 4.29 and Lemma 4.30, we may apply Lemma 4.27 to `ABSTRACTBUILD(F, r, θ)` to obtain a strategy `BUILD(F, r, θ)` which enforces a monochromatic copy of F in the deterministic game with r colors in at most

$$a_{\max} = a_{\max}(F, r) := (r+1)^{r^2 \cdot |\mathcal{S}(F)|^{r+2}} \quad (4.96)$$

steps, and satisfies the generalized density restriction (θ, β) . \square

It remains to prove Lemma 4.28, Lemma 4.29, and Lemma 4.30, which we will do in the next section.

Algorithm 2: AbstractBuilder strategy $\text{ABSTRACTBUILD}(F, r, \theta)$ **Input:** a graph F with at least one edge, an integer $r \geq 2$, a real number $\theta > 0$

```

B1  $\alpha := ()$ 
B2 foreach  $s \in [r]$  do
B3    $\mathcal{G}_s := \emptyset$ 
B4  $i := 0$ 
B5 repeat (+)
B6    $i := i + 1$ 
B7   foreach  $s \in [r]$  do
B8     Let  $j_{\max,s}$  denote the total number of iterations of the repeat-loop (**) in the  $i$ -th
      iteration of the repeat-loop (*) of the algorithm  $\text{COMPUTEWEIGHTS}()$  with input
      sequence  $\alpha \circ s$ .
B9     For  $1 \leq j \leq j_{\max,s}$ , let  $\mathcal{C}_s^{i,j}$  and  $\mathcal{C}_s^{i,j,k}$ ,  $k \geq 1$ , denote the sets defined in the algorithm
       $\text{COMPUTEWEIGHTS}()$  with input sequence  $\alpha \circ s$  in the corresponding iterations of the
      repeat-loops (**) and (***) in line 14, or line 24 and 32, respectively.
B10     $j_s := 1$ 
B11  repeat (++)
B12    foreach  $((H_1, \pi_1), \dots, (H_r, \pi_r)) \in \mathcal{C}_1^{i,j_1} \times \dots \times \mathcal{C}_r^{i,j_r}$  do
B13      For each color  $s \in [r]$  let  $v_{s1}$  denote the youngest vertex of  $(H_s, \pi_s)$ . AbstractBuilder
      constructs a new graph by taking the disjoint union of all graphs
       $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ ,  $s \in [r]$ , for which  $v(H_s) \geq 2$ , and by adding a new vertex  $v$  in
      such a way that for each  $s \in [r]$ , coloring  $v$  in color  $s$  will extend the central copy of
       $H_s \setminus v_{s1}$  in  $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$  to a copy of  $H_s$ . AbstractPainter chooses a color
       $\bar{\sigma} \in [r]$  for  $v$ , the resulting new  $r$ -colored graph  $G$  is added to AbstractBuilder's list,
      and the newly created copy of  $H_{\bar{\sigma}}$  is designated as the central copy of  $G$ .
B14    By Lemma 4.28, for any combination of colors AbstractPainter chooses in the previous
      loop (line B12 and B13), there is a color  $\hat{\sigma} \in [r]$  such that for each graph  $(H, \pi) \in \mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}}}$ ,
      in at least one construction step an  $r$ -colored graph with a central copy of  $H$  in color  $\hat{\sigma}$ 
      is created. Fix such a color  $\hat{\sigma}$ .
B15    foreach  $(H, \pi) \in \mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}}}$  do
B16      Define  $G_{\hat{\sigma}}(H, \pi)$  to be the  $r$ -colored graph on AbstractBuilder's list resulting from an
      arbitrary construction step in line B13 that created a central copy of  $H$  in color  $\hat{\sigma}$ .
B17     $\mathcal{G}_{\hat{\sigma}} := \mathcal{G}_{\hat{\sigma}} \cup \mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}}}$ 
B18     $k := 0$ 
B19    repeat (+++)
B20       $k := k + 1$ 
B21      foreach  $(H, \pi) \in \mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}},k}$ ,  $\pi = (v_1, \dots, v_h)$ , do
B22        By Lemma 4.21, the graph  $(H, \pi')$ , defined by
         $\pi' := (v_{k+1}, v_1, \dots, v_k, v_{k+2}, \dots, v_h)$  is contained in  $\mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}}}$ , and consequently
         $G_{\hat{\sigma}}(H, \pi')$  was just defined in line B16. Let  $G_{\hat{\sigma}}(H, \pi) := G_{\hat{\sigma}}(H, \pi')$ .
B23       $\mathcal{G}_{\hat{\sigma}} := \mathcal{G}_{\hat{\sigma}} \cup \mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}},k}$ 
B24    until  $\mathcal{C}_{\hat{\sigma}}^{i,j_{\hat{\sigma}},k} = \emptyset$ 
B25     $j_{\hat{\sigma}} := j_{\hat{\sigma}} + 1$ 
B26  until  $j_{\sigma} > j_{\max,\sigma}$  for some  $\sigma \in [r]$ 
B27   $\alpha := \alpha \circ \sigma$  for this  $\sigma$ 
B28 until  $(F, \pi) \in \mathcal{G}_s$  for some  $s \in [r]$  and  $\pi \in \Pi(V(F))$ 

```

4.4.3. Analysis of ABSTRACTBUILD().

PROOF OF LEMMA 4.28. We refer to a color $\sigma \in [r]$ that satisfies the conditions of the lemma as a *color that is dominating in* $X_1 \times \cdots \times X_r$.

We argue by double induction over r and $\sum_{s \in [r]} |X_s|$. To settle the induction base note that the claim is trivially true for $r = 1$ and any finite set X_1 , and also for $r \geq 2$ and $|X_1| = \cdots = |X_r| = 1$. For the induction step let $r \geq 2$ and suppose that one of the sets X_s , $s \in [r]$, contains at least two elements. We assume w.l.o.g. that it is X_1 , and fix an element $x \in X_1$.

By induction (over the sum of the cardinalities of the sets X_s) we know that for the restriction of f to the set $(X_1 \setminus \{x\}) \times X_2 \times \cdots \times X_r$ there is a dominating color $\sigma \in [r]$. If $\sigma \neq 1$, then σ is also dominating in $X_1 \times \cdots \times X_r$ and we are done. Otherwise we have $\sigma = 1$, i.e. for all $x_1 \in X_1 \setminus \{x\}$ there are elements $x_s \in X_s$, $2 \leq s \leq r$, with $f(x_1, \dots, x_r) = 1$. Therefore, if f assigns color 1 to any of the elements in $\{x\} \times X_2 \times \cdots \times X_r$, then $\sigma = 1$ is dominating in $X_1 \times \cdots \times X_r$ and we are done as well. The only remaining case is that $f(x, \bullet, \dots, \bullet)$ never uses color 1, and therefore is an $(r - 1)$ -coloring of $X_2 \times \cdots \times X_r$. By induction (over r), there is a color $\sigma' \in [r] \setminus \{1\}$ that is dominating in $X_2 \times \cdots \times X_r$, and therefore also in $X_1 \times \cdots \times X_r$. This settles the last remaining case. \square

In order to prove Lemma 4.29 and Lemma 4.30 we will make use of the following technical lemma, which relates the evolution of the families \mathcal{G}_s occurring in ABSTRACTBUILD() to the evolution of the families \mathcal{H}_s occurring in COMPUTEWEIGHTS().

LEMMA 4.31 (Evolution of the families \mathcal{G}_s). *At the end of each iteration of the repeat-loop (++) during some iteration i of the repeat-loop (+) in ABSTRACTBUILD(), for each $s \in [r]$ we have $\mathcal{G}_s \supseteq \mathcal{H}_s$, where \mathcal{H}_s denotes the value of \mathcal{H}_s after $j_s - 1$ iterations of the repeat-loop (**) during iteration i of the repeat-loop (*) in COMPUTEWEIGHTS() for the input sequence $\alpha \circ s$, for the current value of j_s . Here $\alpha \in [r]^{i-1}$ denotes the sequence that has been constructed in previous rounds of ABSTRACTBUILD() as a result of AbstractPainter's coloring decisions.*

PROOF. Consider the i -th iteration of the repeat-loop (+), and let $\hat{\sigma}$ be the color defined in line B14 in some iteration of the repeat-loop (++) . Note that the set of graphs (H, π) that are added to $\mathcal{G}_{\hat{\sigma}}$ in line B17 and line B23 in this iteration is

$$\mathcal{H}_{\hat{\sigma}}^{i, j_{\hat{\sigma}}} := \mathcal{C}_{\hat{\sigma}}^{i, j_{\hat{\sigma}}} \cup \bigcup_{k \geq 1} \mathcal{C}_{\hat{\sigma}}^{i, j_{\hat{\sigma}}, k} \quad (4.97)$$

for the current value of $j_{\hat{\sigma}}$, where the union in (4.97) is over all $k \geq 1$ for which $\mathcal{C}_{\hat{\sigma}}^{i, j_{\hat{\sigma}}, k}$ is defined in line B9. By the definitions in line B9 these are exactly the graphs that are added to the family $\mathcal{H}_{\hat{\sigma}}$ in the $j_{\hat{\sigma}}$ -th iteration of the repeat-loop (**) in the i -th iteration of the repeat-loop (*) of the algorithm COMPUTEWEIGHTS() for the input sequence $\alpha \circ \hat{\sigma}$. It follows inductively that at the end of the i -th iteration of the repeat-loop (+), the set of graphs (H, π) that have been added to \mathcal{G}_{σ} for the color $\sigma \in [r]$ satisfying the termination condition in line B26 is

$$\mathcal{H}_{\sigma}^i := \bigcup_{j_{\sigma}=1}^{j_{\max, \sigma}} \mathcal{H}_{\sigma}^{i, j_{\sigma}} .$$

By the definition of $j_{\max, s}$ in line B8, these are exactly the graphs added to the family \mathcal{H}_{σ} in the i -th iteration of the repeat-loop (*) of the algorithm COMPUTEWEIGHTS() for the input sequence $\alpha \circ \sigma$. As moreover no graphs are added to the families \mathcal{H}_s , $s \in [r] \setminus \{\sigma\}$, during the i -th

round of COMPUTEWEIGHTS() with input sequence $\alpha \circ \sigma$, it follows inductively that throughout, the families \mathcal{G}_s occurring in ABSTRACTBUILD() are related to the families \mathcal{H}_s occurring in COMPUTEWEIGHTS() as claimed. \square

PROOF OF LEMMA 4.29. We will first argue that ABSTRACTBUILD() is a well-defined winning strategy.

Note that whenever a graph $(H, \pi) \in \mathcal{S}(F)$ is added to one of the families \mathcal{G}_s , $s \in [r]$, in line B17 or B23, then $G_s(H, \pi)$ is defined in B16 or in line B22, respectively, and in either case this r -colored graph was added to AbstractBuilder's list in line B13. Thus throughout the strategy ABSTRACTBUILD(), for every $s \in [r]$ and all graphs $(H, \pi) \in \mathcal{G}_s$, the graph $G_s(H, \pi)$ is well-defined and exists on AbstractBuilder's list. With the termination condition in line B28 this implies in particular that when ABSTRACTBUILD() terminates, AbstractBuilder's list indeed contains a graph containing a monochromatic copy of F .

Next we show that whenever the construction step in line B13 is executed, all involved graphs $(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ are in the respective families \mathcal{G}_s at this point, and thus the graphs $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ used for the construction step are indeed on AbstractBuilder's list. It follows from the definition of $\mathcal{C}^{i,j}$ as a subset of $\mathcal{C}(\mathcal{H}_s, F)$ (recall (4.21)) in line 14 of COMPUTEWEIGHTS() that for each $s \in [r]$ and each $(H_s, \pi_s) \in \mathcal{C}_s^{i,j_s}$ with $v(H_s) \geq 2$ in line B12, the graph $(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ is contained in \mathcal{H}_s as defined in the i -th iteration of the repeat-loop (*) at the beginning of the j_s -th iteration of the repeat-loop (**) of COMPUTEWEIGHTS() for the input sequence $\alpha \circ s$. Thus by Lemma 4.31, the graph $(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ is in the corresponding family \mathcal{G}_s , as claimed.

Also note that whenever the construction step in line B13 is executed, the involved entries $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$, $s \in [r]$, on AbstractBuilder's list are different from each other, as the corresponding central copies of $H_s \setminus v_{s1}$, $s \in [r]$, are all in different colors.

Together the above arguments show that ABSTRACTBUILD() is indeed a well-defined strategy for the abstract game, and it remains to bound the number of construction steps ABSTRACTBUILD() needs to enforce a monochromatic copy of F . By Lemma 4.31 and the termination condition in line B28, the number of iterations of the repeat-loop (+) until ABSTRACTBUILD() terminates is bounded by the number of iterations of the repeat-loop (*) in COMPUTEWEIGHTS() until the first of the families \mathcal{H}_s , $s \in [r]$, contains the graph (F, π) for some vertex-ordering $\pi \in \Pi(V(F))$. The termination condition in line 36 and Lemma 4.10 therefore show that ABSTRACTBUILD() terminates after at most $r \cdot |\mathcal{S}(F)|$ iterations of the repeat-loop (+). In each iteration i , the number of iterations of the repeat-loop (++) is at most $\sum_{s \in [r]} j_{\max, s} \leq r \cdot |\mathcal{S}(F)|$, as the values $j_{\max, s}$ are bounded by $|\mathcal{S}(F)|$ as argued in the proof of Lemma 4.10 on page 52. Lastly, the number of iterations of the loop in line B12 is $|\mathcal{C}_1^{i,j_1}| \cdots |\mathcal{C}_r^{i,j_r}| \leq |\mathcal{S}(F)|^r$, as the sets \mathcal{C}_s^{i,j_s} are subsets of $\mathcal{S}(F)$. Multiplying those numbers yields the claimed bound on the total number of construction steps throughout the strategy. \square

PROOF OF LEMMA 4.30. For the reader's convenience, Figure 4.4 illustrates the notations used throughout the proof.

We prove inductively that for each construction step in line B13 in some round i of ABSTRACTBUILD() the following holds: Recall the notations from the algorithm, and let H' , $v(H') \geq 1$, be a subgraph of the newly constructed graph G such that each component of H' shares at least one vertex with the central copy of $H_{\bar{\sigma}}$ in G . Letting $J_{\bar{\sigma}}$ denote the intersection of H' with this central copy, we have

$$\mu_\theta(H') \geq \lambda_\theta(J_{\bar{\sigma}}, w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma})}) , \quad (4.98)$$

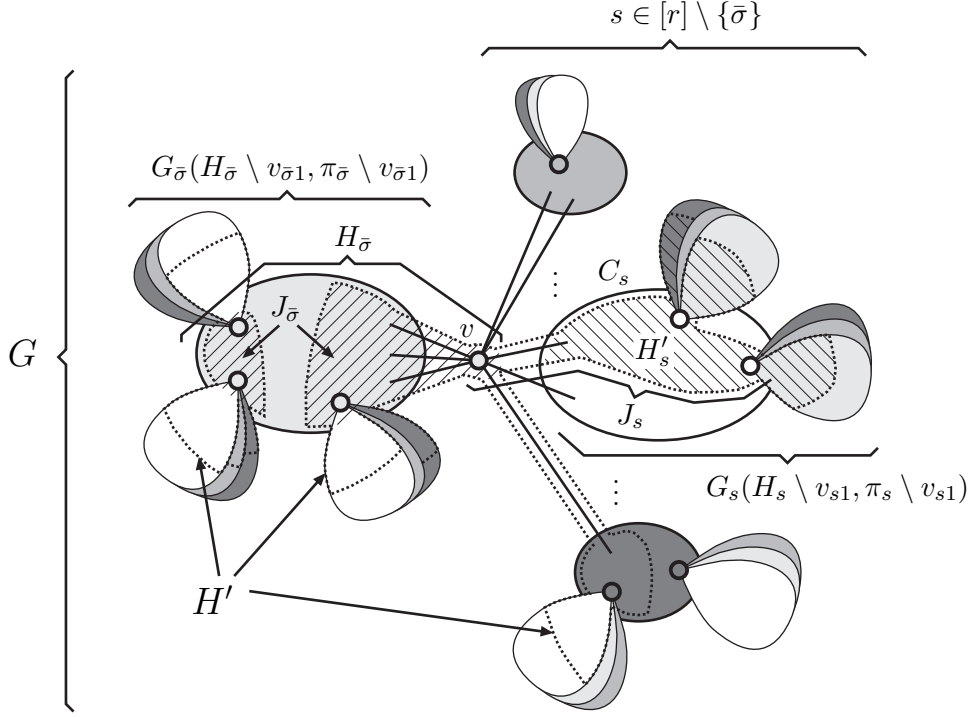


FIGURE 4.4. Notations used in the proof of Lemma 4.30.

where here and throughout we denote for any $s \in [r]$ by $w_{(H, \pi, s), s}$ the weight function $w_{(H, \pi, s), s}$ computed by `COMPUTEWEIGHTS()` for the input sequence $\alpha \circ s$. (Recall the definitions in (4.5), (4.18) and (4.22), and that during the i -th round of `ABSTRACTBUILD()`, the sequence α has length $i - 1$.)

For subgraphs $H' \subseteq G$ that do not contain the new vertex v , the claim follows by induction if $G_{\bar{\sigma}}(H_{\bar{\sigma}} \setminus v_{\bar{\sigma}1}, \pi_{\bar{\sigma}} \setminus v_{\bar{\sigma}1})$ was defined in line B16 (either in the same or in an earlier iteration of the repeat-loop (+)), or by induction and by Lemma 4.21 if $G_{\bar{\sigma}}(H_{\bar{\sigma}} \setminus v_{\bar{\sigma}1}, \pi_{\bar{\sigma}} \setminus v_{\bar{\sigma}1})$ was defined in line B22 (either in the same or in an earlier iteration of the repeat-loop (+)), recalling the definition (4.18) and the fact that the functions $w_{(H_{\bar{\sigma}} \setminus v_{\bar{\sigma}1}, \pi_{\bar{\sigma}} \setminus v_{\bar{\sigma}1}, \bar{\sigma}), \bar{\sigma}}$ and $w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma}), \bar{\sigma}}$ assign the same weight to all vertices of $H_{\bar{\sigma}}$ different from $v_{\bar{\sigma}1}$ (recall (4.22)).

It remains to prove (4.98) for subgraphs H' that contain the vertex v . Note that throughout the i -th iteration of the repeat-loop (+), whenever the construction step in line B13 is executed, by the definition of the sets \mathcal{C}_s^{i, j_s} (see line B9 of `ABSTRACTBUILD()` and line 14 of `COMPUTEWEIGHTS()`), all graphs $(H_s, \pi_s) \in \mathcal{C}_s^{i, j_s}$ used for the construction step satisfy

$$d_\theta(H_s, v_{s1}, w_{(H_s, \pi_s, s), s}) = d_s^i, \quad s \in [r], \quad (4.99)$$

where the values d_s^i , $s \in [r]$, are defined in line 8 of `COMPUTEWEIGHTS()` with input sequence α . Note that these values depend only on the first $i - 1$ entries of α , i.e., they are the same during the i -th iteration of the repeat-loop (*) of `COMPUTEWEIGHTS()` for each input sequence $\alpha \circ s$, $s \in [r]$.

For the weight assigned to the youngest vertex $v_{\bar{\sigma}1}$ of $(H_{\bar{\sigma}}, \pi_{\bar{\sigma}})$ by `COMPUTEWEIGHTS()` with input sequence $\alpha \circ \bar{\sigma}$, we thus obtain by combining (4.22) with the definitions in line 10 and 18 that

$$w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma}), \bar{\sigma}}(v_{\bar{\sigma}1}) = \sum_{s \in [r] \setminus \{\bar{\sigma}\}} d_s^i \stackrel{(4.99)}{=} \sum_{s \in [r] \setminus \{\bar{\sigma}\}} d_\theta(H_s, v_{s1}, w_{(H_s, \pi_s, s), s}). \quad (4.100)$$

For each $s \in [r]$ with $v(H_s) \geq 2$ we define the graph H'_s as the intersection of H' with the copy of $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ used for the construction of G . Furthermore, for each such $s \in [r]$ we define a subgraph $J_s \subseteq H_s$ with $v_{s1} \in J_s$ as follows: Let C_s denote the central copy of $H_s \setminus v_{s1}$ in the copy of $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ used for the construction of G , and recall that the new vertex v completes C_s to a copy of H_s . Let $J_s \subseteq H_s$ denote the graph that is isomorphic to the intersection of H' with this copy of H_s , and note that H'_s intersects C_s in a copy of $J_s \setminus v_{s1}$. For all $s \in [r]$ with $v(H_s) = 1$ (i.e., H_s consists only of an isolated vertex) we define H'_s as the null graph (the graph whose vertex set is empty) and set $J_s := H_s$. Using these definitions we obtain

$$v(H') = \sum_{s \in [r]} v(H'_s) + 1, \quad (4.101a)$$

$$e(H') = \sum_{s \in [r]} (e(H'_s) + \deg_{J_s}(v_{s1})) . \quad (4.101b)$$

Furthermore, for every $s \in [r]$ we have

$$\mu_\theta(H'_s) \geq \lambda_\theta(J_s \setminus v_{s1}, w_{(H_s, \pi_s, s), s}) \quad (4.102)$$

(this holds trivially if $v(H_s) = 1$; otherwise, similarly to before, if $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ was defined in line B16 then this follows by induction, whereas if $G_s(H_s \setminus v_{s1}, \pi_s \setminus v_{s1})$ was defined in line B22 then this follows by induction and by Lemma 4.21).

Combining our previous observations we obtain

$$\begin{aligned} \mu_\theta(H') &\stackrel{(4.5), (4.101)}{=} \sum_{s \in [r]} (\mu_\theta(H'_s) - \deg_{J_s}(v_{s1}) \cdot \theta) + 1 \\ &\stackrel{(4.102)}{\geq} \lambda_\theta(J_{\bar{\sigma}} \setminus v_{\bar{\sigma}1}, w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma}), \bar{\sigma}}) - \deg_{J_{\bar{\sigma}}}(v_{\bar{\sigma}1}) \cdot \theta \\ &\quad + \sum_{s \in [r] \setminus \{\bar{\sigma}\}} \underbrace{(\lambda_\theta(J_s \setminus v_{s1}, w_{(H_s, \pi_s, s), s}) - \deg_{J_s}(v_{s1}) \cdot \theta)}_{\stackrel{(4.17), (4.18)}{\geq} d_\theta(H_s, v_{s1}, w_{(H_s, \pi_s, s), s})} + 1 \\ &\stackrel{(4.100)}{\geq} \lambda_\theta(J_{\bar{\sigma}} \setminus v_{\bar{\sigma}1}, w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma}), \bar{\sigma}}) - \deg_{J_{\bar{\sigma}}}(v_{\bar{\sigma}1}) \cdot \theta + w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma}), \bar{\sigma}}(v_{\bar{\sigma}1}) + 1 \\ &\stackrel{(4.19)}{=} \lambda_\theta(J_{\bar{\sigma}}, w_{(H_{\bar{\sigma}}, \pi_{\bar{\sigma}}, \bar{\sigma}), \bar{\sigma}}) , \end{aligned}$$

completing the inductive proof of (4.98).

From (4.98) it follows in particular that for every graph G that is added to AbstractBuilder's list during the i -th iteration of the repeat-loop (+), every connected subgraph $H' \subseteq G$ containing the last added vertex v satisfies

$$\mu_\theta(H') \geq \min_{J \subseteq H: v_1 \in J} \lambda_\theta(J, w_{(H, \pi, s), s}) \quad (4.103)$$

for some $s \in [r]$ and some (H, π) , $\pi = (v_1, \dots, v_h)$, from one of the sets $\mathcal{C}^{i,j}$ defined in the i -th iteration of COMPUTEWEIGHTS() when called with input sequence $\alpha \circ s$.

As argued in the proof of Lemma 4.26 (see (4.89a)), the right hand side of (4.103) equals $1 + \sum_{s \in [r]} d_s^i$ for the values d_s^i defined by COMPUTEWEIGHTS() with input sequence α . Regardless of how the sequence α constructed by ABSTRACTBUILD() evolves in further iterations of the repeat-loop (+), this quantity is decreasing in i by the first part of Lemma 4.14. Moreover, by Lemma 4.31 and the termination condition in line B28, ABSTRACTBUILD() terminates after at most \tilde{i} iterations, for \tilde{i} as defined in Lemma 4.26. It follows that all connected subgraphs H' with

$v(H') \geq 1$ of all graphs G added to AbstractBuilder's list in the course of ABSTRACTBUILD() satisfy

$$\mu_\theta(H') \geq 1 + \sum_{s \in [r]} d_s^i, \quad (4.104)$$

where i and d_s^i , $s \in [r]$, are defined in COMPUTEWEIGHTS() for the input sequence α constructed by ABSTRACTBUILD().

By Lemma 4.26 and the definition of $\Lambda_\theta()$ in (4.23), we thus obtain from (4.104) that

$$\mu_\theta(H') \geq \Lambda_\theta(F, r) \stackrel{(4.8)}{\geq} \beta$$

for all connected subgraphs H' with $v(H') \geq 1$ of all graphs G appearing on AbstractBuilder's list. Due to the assumption that $\beta \geq 0$, the same statement also follows for all disconnected subgraphs $H' \subseteq G$ with $v(H') \geq 1$, concluding the proof that the strategy ABSTRACTBUILD(F, r, θ) respects the generalized density restriction (θ, β) throughout. \square

4.5. Painter in the deterministic game

In this section we prove Proposition 4.5 by explicitly constructing, for F , r , θ and β as in the proposition, a Painter strategy that avoids creating a monochromatic copy of F in the deterministic game with r colors and generalized density restriction (θ, β) .

4.5.1. Painter's strategy and proof of Proposition 4.5. Consider the following Painter strategy, which has four parameters: a graph F with at least one edge, an integer $r \geq 2$, a real number $\theta > 0$ and a sequence $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$. The strategy uses the output of Algorithm 1: $((\mathcal{H}_s, w_s))_{s \in [r]} := \text{COMPUTEWEIGHTS}(F, r, \theta, \alpha)$. In each step of the game, Painter picks a color as follows: Let v denote the vertex added in the current step, and for each $s \in [r]$, define

$$\mathcal{D}_s := \left\{ (H, \pi) \in \mathcal{S}(F) \left| \begin{array}{l} \text{assigning color } s \text{ to } v \text{ would create a copy} \\ \text{of } (H, \pi) \text{ in color } s \text{ on the board} \end{array} \right. \right\}. \quad (4.105)$$

(Note that this requires Painter to memorize the order in which the vertices on the board arrived.) Calculate for each color $s \in [r]$ the value

$$d(s) := \min_{(H, \pi) \in \mathcal{D}_s} \lambda_\theta(H, w_{(H, \pi, s)}) , \quad (4.106)$$

where $\lambda_\theta()$ is defined in (4.18), and $w_{(H, \pi, s)}()$ is defined in (4.22) using $\mathcal{H}_s \subseteq \mathcal{S}(F)$ and $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$ as returned by Algorithm 1. (It is possible that $d(s) = -\infty$ for some colors $s \in [r]$.) Then select $\sigma \in [r]$ as the color for which this value is maximal, and assign color σ to the vertex v .

Intuitively, the parameter $\lambda_\theta(H, w_{(H, \pi, s)})$ measures the 'level of danger' that the Painter strategy encoded by α assigns to copies of the ordered graph (H, π) in color s , where a graph is considered the more dangerous the smaller its $\lambda_\theta()$ -value is. Thus the definition of $d(s)$ in (4.106) corresponds to determining the most dangerous graph in color s that would be created by assigning color s to v , and our strategy selects σ as the color for which this most dangerous graph is least dangerous.

If several colors have the same maximal value of $d(s)$, the above rule does not determine a color σ uniquely. Such ties are broken as follows: Consider the families

$$\mathcal{H}'_s = \mathcal{H}'_s(F, r, \theta, \alpha) := \bigcup_{\alpha_i = s \wedge (i=1 \vee \alpha_i \neq \alpha_{i-1})} \mathcal{C}^{i,1}, \quad (4.107)$$

$s \in r$, where $\mathcal{C}^{i,j}$ are the sets defined in line 14 of the algorithm COMPUTEWEIGHTS(F, r, θ, α). Note that these families are fixed throughout Painter's strategy. In Lemma 4.48 below we will

show that ties can arise only between *two* different colors, and that whenever such a tie arises, then for exactly one of the two colors the set

$$\mathcal{J}_s := \arg \min_{(H, \pi) \in \mathcal{D}_s} \lambda_\theta(H, w_{(H, \pi, s)}) \quad (4.108)$$

contains an ordered graph from the corresponding family \mathcal{H}'_s . Our tie-breaking rule is to then pick the *other* color, i.e., the color $\sigma \in [r]$ for which \mathcal{J}_σ contains no graph from \mathcal{H}'_σ . (Intuitively, Painter considers the ordered graphs in the families \mathcal{H}'_s , $s \in [r]$, as slightly more dangerous than other ordered graphs with the same $\lambda_\theta(\cdot)$ -value.)

In the following we denote the Painter strategy defined above by $\text{PAINT}(F, r, \theta, \alpha)$. Note that this strategy can be employed both in the deterministic two-player game and in the original probabilistic process.

REMARK 4.32. Note that the actual $\lambda_\theta(\cdot)$ -values of monochromatic ordered subgraphs of F are not relevant in the above strategy — all that matters is the partial order on the set $\mathcal{S}(F) \times [r]$ induced by the $\lambda_\theta(\cdot)$ -values and our tie-breaking rule. This partial order can be extended arbitrarily to a total order by defining an arbitrary order among all elements of $\mathcal{S}(F) \times [r]$ that have the same $\lambda_\theta(\cdot)$ -value and are in one of the sets \mathcal{H}'_s , and among all elements that have the same $\lambda_\theta(\cdot)$ -value and are *not* in one of the sets \mathcal{H}'_s . Thus the strategy $\text{PAINT}(F, r, \theta, \alpha)$ can indeed be represented as a priority list of ordered monochromatic subgraphs of F , as described in Section 4.1.5.

A careful analysis of the strategy $\text{PAINT}(F, r, \theta, \alpha)$ will eventually yield the following key lemma. As its statement is purely deterministic, it is applicable to both the deterministic game and the probabilistic process. Note that the lemma does not assume any density restrictions for the evolving board.

LEMMA 4.33 (Witness graph invariant). *For F , r , θ , and α as specified in Algorithm 1 there is a constant $v_{\max} = v_{\max}(F, r, \theta, \alpha)$ such that if Painter plays according to the strategy $\text{PAINT}(F, r, \theta, \alpha)$ then the following invariant is maintained throughout:*

The board contains a graph K' with $v(K') \leq v_{\max}$ and

$$\mu_\theta(K') < 0 \quad , \quad (4.109)$$

or for every $s \in [r]$ and every $(H, \pi) \in \mathcal{S}(F)$ we have that every copy of (H, π) in color s on the board is contained in a graph H' with $v(H') \leq v_{\max}$ and

$$\mu_\theta(H') \leq \lambda_\theta(H, w_{(H, \pi, s)}) \quad , \quad (4.110)$$

where $\mu_\theta(\cdot)$, $\lambda_\theta(\cdot)$, and $w_{(H, \pi, s)}$ are defined in (4.5), (4.18), and (4.22) (using $\mathcal{H}_s \subseteq \mathcal{S}(F)$ and $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$ as returned by Algorithm 1), respectively.

REMARK 4.34. As we shall see shortly, the statement that the size of the graphs K' and H' in Lemma 4.33 is bounded by some constant $v_{\max} = v_{\max}(F, r, \theta, \alpha)$ is not needed to prove Proposition 4.5. However, it will be crucial for proving the lower bound part of Theorem 4.3 in Section 4.6.1 below (recall the remarks in Section 4.2.2 and Section 4.2.3). In fact, the proof of the existence of the bound v_{\max} relies primarily on our tie-breaking rule described above; a version of Lemma 4.33 *without* a bound on the size of the graphs K' and H' (which suffices to infer Proposition 4.5) can also be proven if ties are broken arbitrarily. In this case the proof of Lemma 4.33 can be simplified considerably; in particular, Lemma 4.45, Lemma 4.46, Lemma 4.48 and the second part of Lemma 4.49 below are not needed. The reader might want to skip those parts on his first read-through, or if he is only interested in the deterministic game.

With Lemma 4.33 in hand, the proof of Proposition 4.5 is straightforward.

PROOF OF PROPOSITION 4.5. Let $\alpha \in [r]^{r \cdot |\mathcal{S}(F)|}$ be a sequence for which the minimum in the definition of $\Lambda_\theta(F, r)$ in (4.23) is attained. By the definition in (4.23), for all colors $s \in [r]$ and all vertex orderings $\pi \in \Pi(V(F))$ there is a subgraph $H \subseteq F$ with

$$\lambda_\theta(H, w_{(H, \pi|_H, s)}) \leq \Lambda_\theta(F, r) \stackrel{(4.9)}{<} \beta. \quad (4.111)$$

Suppose now that Painter plays according to the strategy $\text{PAINT}(F, r, \theta, \alpha)$ and that for some $\pi \in \Pi(V(F))$ a copy of (F, π) in some color $s \in [r]$ appears on the board. Choose $H \subseteq F$ such that (4.111) holds. Then, by Lemma 4.33, the board contains a graph K' with

$$\mu_\theta(K') \stackrel{(4.109)}{<} 0 \leq \beta,$$

or the copy of $(H, \pi|_H)$ in color s that is contained in the copy of (F, π) is contained in a graph H' with

$$\mu_\theta(H') \stackrel{(4.110)}{\leq} \lambda_\theta(H, w_{(H, \pi|_H, s)}) \stackrel{(4.111)}{<} \beta.$$

None of the two cases can occur if Builder adheres to the generalized density restriction (θ, β) , and consequently Painter can avoid creating a monochromatic copy of F in the deterministic F -avoidance game with r colors and generalized density restriction (θ, β) by playing according to the strategy $\text{PAINT}(F, r, \theta, \alpha)$. \square

The rest of this section is devoted to proving Lemma 4.33. To do so we will need a number of technical lemmas. Throughout the following, F , r , θ and α are fixed, and we usually omit these arguments when we refer to $\text{COMPUTEWEIGHTS}(F, r, \theta, \alpha)$ or $\text{PAINT}(F, r, \theta, \alpha)$. We let $\mathcal{H}_s \subseteq \mathcal{S}(F)$ and $w_s : \mathcal{H}_s \rightarrow \mathbb{R}$ denote the return values of $\text{COMPUTEWEIGHTS}()$, and $w_{(H, \pi, s)}$ the weight function defined in (4.22) with respect to these return values.

4.5.2. A geometric viewpoint. We begin by relating the strategy $\text{PAINT}()$ and many of the quantities defined in previous parts of this chapter to a simple geometric object. This geometric viewpoint will be a key ingredient in our proof of Lemma 4.33.

DEFINITION 4.35 (Axis-parallel decreasing walk). We say that $(x_\nu)_{1 \leq \nu \leq k}$, $x_\nu \in \mathbb{R}^r$, is a *decreasing axis-parallel walk in \mathbb{R}^r* if for any two subsequent elements x_ν and $x_{\nu+1}$ there is a coordinate $s \in [r]$ such that $x_{\nu+1, s} < x_{\nu, s}$ and $x_{\nu+1, t} = x_{\nu, t}$ for all $t \in [r] \setminus \{s\}$.

The following lemma is an immediate consequence of this definition.

LEMMA 4.36 (Order on the walk). Let $(x_\nu)_{1 \leq \nu \leq k}$, $x_\nu \in \mathbb{R}^r$, be a decreasing axis-parallel walk in \mathbb{R}^r . For any two elements x_μ , x_ν we have $1 + \sum_{t \in [r]} x_{\nu, t} \leq 1 + \sum_{t \in [r]} x_{\mu, t}$ if and only if $x_{\nu, t} \leq x_{\mu, t}$ for all $t \in [r]$.

We can think of the points $(x_\nu)_{1 \leq \nu \leq k}$ of a decreasing axis-parallel walk as lying on a sequence of consecutive axis-parallel line segments. For technical reasons we also specify a direction $\sigma \in [r]$ in which, intuitively, the walk continues beyond the point x_k . Moreover, we sometimes allow the last segment of this walk to degenerate into the single point x_k . This is made precise in the following definition.

DEFINITION 4.37 (Extended walk, turning point, starting point/endpoint, segment, order). Given some decreasing axis-parallel walk $(x_\nu)_{1 \leq \nu \leq k}$, $x_\nu \in \mathbb{R}^r$, and some $\sigma \in [r]$, we say that the pair $((x_\nu)_{1 \leq \nu \leq k}, \sigma)$ is an *extended decreasing axis-parallel walk in \mathbb{R}^r* . We refer to a point

x_ν , $2 \leq \nu \leq k-1$, on this extended walk as a *turning point* if the coordinate in which x_ν differs from $x_{\nu-1}$ is different from the coordinate in which x_ν differs from $x_{\nu+1}$. We also define x_1 to be a turning point. Moreover, x_k is a turning point if and only if the coordinate in which x_k differs from x_{k-1} is different from σ .

Given two consecutive turning points x_μ and x_ν , $\mu < \nu$, that differ in some coordinate $s \in [r]$, we refer to the line segment that connects them as an *s-segment*, and we call x_μ the *starting point* and x_ν the *endpoint* of this segment. Furthermore, we refer to the line segment that connects the last turning point x_μ to the last point x_k of the walk as a σ -segment (if $\mu = k$, this segment degenerates to a single point x_k). We call x_μ the starting point and x_k the endpoint of this segment.

For two points x_μ and x_ν , we say that x_μ is *higher on the walk than* x_ν (or equivalently, x_ν is lower on the walk than x_μ) if $\mu < \nu$. We extend this notion to segments on the extended walk by saying that an *s*-segment Γ is higher on the walk than an *s'*-segment Γ' (or equivalently, Γ' is lower than Γ) if the starting point of Γ is higher on the walk than the starting point of Γ' .

Note that according to Lemma 4.14, the points (d_1^i, \dots, d_r^i) , $i \geq 1$, form a decreasing axis-parallel walk in \mathbb{R}^r . In the following we define, for every $s \in [r]$ and every $(H, \pi) \in \mathcal{H}_s$, a point $x_{(H, \pi, s)} \in \mathbb{R}^r$ on one of the line segments of this walk.

Let d_s^i , $s \in [r]$, denote the values defined in line 8 of the algorithm COMPUTEWEIGHTS(), and $\mathcal{C}^{i,j}$ and $\mathcal{C}^{i,j,k}$ the sets defined in line 14, or line 24 and 32, respectively. Recall from Section 4.3.3 that for each $s \in [r]$, the sets $\mathcal{C}^{i,j}$ and $\mathcal{C}^{i,j,k}$ for which $\alpha_i = s$ form a partition of the family \mathcal{H}_s .

DEFINITION 4.38 (*x*-points). For every $s \in [r]$ and every graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, we define a point $x_{(H, \pi, s)} \in \mathbb{R}^r$ as follows:

(a) If $(H, \pi) \in \mathcal{C}^{i,j}$ for some $i, j \geq 1$ with $\alpha_i = s$, then we define

$$x_{(H, \pi, s)} := (d_1^i, \dots, d_{s-1}^i, d_\theta(H, v_1, w_{(H, \pi, s)}) = d_s^i, d_{s+1}^i, \dots, d_r^i) \in \mathbb{R}^r. \quad (4.112a)$$

(b) If $(H, \pi) \in \mathcal{C}^{i,j,k}$ for some $i, j, k \geq 1$ with $\alpha_i = s$, then we define

$$x_{(H, \pi, s)} := (\hat{d}_1^i, \dots, \hat{d}_{s-1}^i, d_\theta(H, v_1, w_{(H, \pi, s)}) = \hat{d}_s^i, \hat{d}_{s+1}^i, \dots, \hat{d}_r^i) \in \mathbb{R}^r, \quad (4.112b)$$

where \hat{i} is the value defined either in line 28 or in line 30 for this (H, π) .

The fact that $d_\theta(H, v_1, w_{(H, \pi, s)}) = d_s^i$ in part (a) of the previous definition follows directly from the definition in line 14 of the algorithm COMPUTEWEIGHTS().

The following lemma states that the points $x_{(H, \pi, s)}$ defined above indeed form an (extended) decreasing axis-parallel walk. We point out that the order in which the points $x_{(H, \pi, s)}$ appear on this walk is not necessarily the order in which the corresponding graphs (H, π) are added to the families \mathcal{H}_s in the course of the algorithm COMPUTEWEIGHTS(). To be more precise, such a statement is true for the graphs that are added via one of the sets $\mathcal{C}^{i,j}$, but not for the graphs that are added via one of the sets $\mathcal{C}^{i,j,k}$.

Of particular importance are the points $x_{(H, \pi, s)}$ for the graphs $(H, \pi) \in \mathcal{H}'_s$, where $\mathcal{H}'_s \subseteq \mathcal{H}_s$ are the families defined in (4.107) for our tie-breaking rule. As it turns out, those points are always turning points of the walk. Figure 4.5 illustrates Definition 4.38 and the different statements of Lemma 4.39.

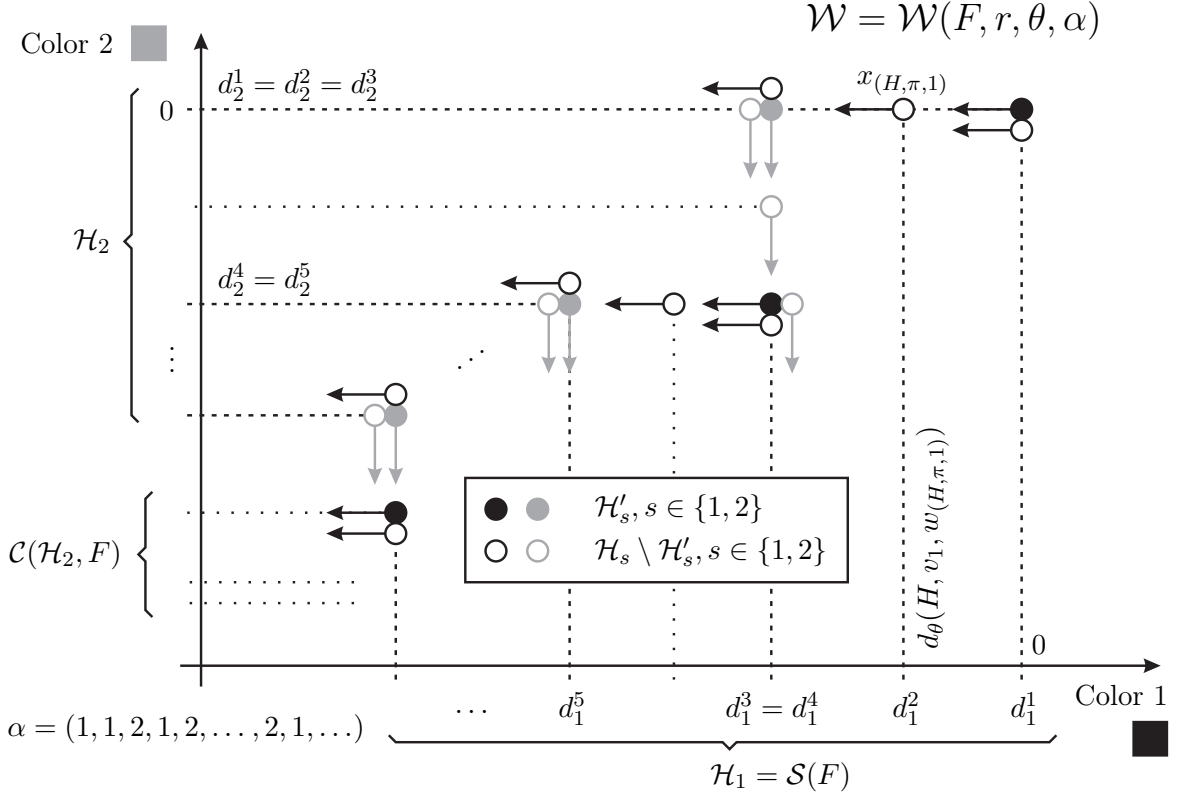


FIGURE 4.5. Illustration of Definition 4.38 and Lemma 4.39. The figure shows how certain variables of the algorithm $\text{COMPUTEWEIGHTS}(F, r, \theta, \alpha)$ might evolve for some graph F , some value of θ , $r = 2$ and $\alpha = (1, 1, 2, 1, 2, \dots, 2, 1, \dots)$ (where the algorithm terminates in the round corresponding to the last 1-entry shown). For any graph $(H, \pi) \in \mathcal{H}_s$, $s \in \{1, 2\}$, a bullet shows the location of the point $x_{(H, \pi, s)}$, and the arrow attached to the bullet points along the s -axis. One bullet may represent multiple points at the same location if the corresponding graphs are of the same type (see legend). If graphs with the same associated point are of different type, then the bullets are drawn directly adjacent to each other (instead of on top of each other) to maintain readability.

LEMMA 4.39 (Walk formed by x -points). *Let i_{\max} denote the total number of iterations of the repeat-loop (*) of $\text{COMPUTEWEIGHTS}()$. The elements in the set $\{x_{(H, \pi, s)} \mid s \in [r] \wedge (H, \pi) \in \mathcal{H}_s\}$ can be ordered to form a decreasing axis-parallel walk $W = W(F, r, \theta, \alpha)$ in \mathbb{R}^r such that the extended walk $\mathcal{W} = \mathcal{W}(F, r, \theta, \alpha) := (W, \alpha_{i_{\max}})$ satisfies the following properties:*

- (i) *For any $s \in [r]$ and any graph $(H, \pi) \in \mathcal{H}_s$, the point $x_{(H, \pi, s)}$ is contained in an s -segment.*
- (ii) *For any s -segment Γ , there is a graph $(H, \pi) \in \mathcal{H}_s$ such that $x_{(H, \pi, s)}$ is the starting point of Γ .*
- (iii) *For any s -segment Γ , if $x_{(H, \pi, s)}$, $\pi = (v_1, \dots, v_h)$, is the starting point of Γ , then there is some $J \subseteq H$ with $v_1 \in J$ such that $x_{(J, \pi|_J, s)} = x_{(H, \pi, s)}$ and $(J, \pi|_J) \in \mathcal{H}'_s$.*
- (iv) *For any s -segment Γ , if $x_{(H, \pi, s)} \in \Gamma$ is not the starting point of Γ , then $(H, \pi) \notin \mathcal{H}'_s$.*
- (v) *The lowest segment of \mathcal{W} is an $\alpha_{i_{\max}}$ -segment and $\mathcal{H}_{\alpha_{i_{\max}}} = \mathcal{S}(F)$.*

PROOF. By the first part of Lemma 4.14, the sequence $\overline{W} := (\overline{W}^i)_{1 \leq i \leq i_{\max}}$, $\overline{W}^i := (d_1^i, \dots, d_r^i)$, is a decreasing axis-parallel walk with the property that \overline{W}^i and \overline{W}^{i+1} differ exactly in the

coordinate α_i . We first show that the set

$$\{x_{(H,\pi,s)} \mid s \in [r] \wedge (H,\pi) \in \mathcal{C}^{i,j} \text{ for some } i,j \geq 1 \text{ with } \alpha_i = s\} \quad (4.113)$$

coincides exactly with the set of elements of this walk, and that the extended walk $\overline{\mathcal{W}} := (\overline{\mathcal{W}}, \alpha_{i_{\max}})$ satisfies the properties of the lemma. In the second part of the proof we argue that for any $s \in [r]$ and any graph $(H,\pi) \in \mathcal{C}^{i,j,k}$ for some $i,j,k \geq 1$ with $\alpha_i = s$, the point $x_{(H,\pi,s)}$ lies on one of the segments of $\overline{\mathcal{W}}$, and that the subdivided walk obtained by inserting all those points into the walk $\overline{\mathcal{W}}$ still satisfies the claimed properties.

First note that on the extended walk $\overline{\mathcal{W}}$, every point $\overline{\mathcal{W}}^i$, $1 \leq i \leq i_{\max}$, is contained in an α_i -segment.

For any $s \in [r]$, any $i,j \geq 1$ with $\alpha_i = s$ and any graph $(H,\pi) \in \mathcal{C}^{i,j}$, by the definition in (4.112a) we have

$$x_{(H,\pi,s)} = (d_1^i, \dots, d_r^i) = \overline{\mathcal{W}}^i \quad (4.114)$$

(independently of j). Thus property (i) is satisfied for the elements in the set (4.113) and the walk $\overline{\mathcal{W}}$.

Recall that for each $1 \leq i \leq i_{\max}$ the set $\mathcal{C}^{i,1}$ is nonempty (see the definitions in line 8 and line 14), implying that there is a graph $(H,\pi) \in \mathcal{C}^{i,1}$ which for $s := \alpha_i$ satisfies $x_{(H,\pi,s)} = \overline{\mathcal{W}}^i$, proving in particular property (ii) for the walk $\overline{\mathcal{W}}$.

To prove properties (iii) and (iv), we fix some $s \in [r]$, some $i,j \geq 1$ with $\alpha_i = s$ and some graph $(H,\pi) \in \mathcal{C}^{i,j}$, $\pi = (v_1, \dots, v_h)$. Let Γ denote the s -segment of $\overline{\mathcal{W}}$ containing the point $x_{(H,\pi,s)}$. We distinguish two cases depending on whether $x_{(H,\pi,s)}$ is the starting point of Γ or not. Note that by the first part of Lemma 4.14, $x_{(H,\pi,s)} = \overline{\mathcal{W}}^i$ is the starting point of Γ if and only if $i = 1$ or $\alpha_i \neq \alpha_{i-1}$.

We first consider the case that $x_{(H,\pi,s)}$ is the starting point of Γ , i.e., we have

$$i = 1 \text{ or } \alpha_i \neq \alpha_{i-1} . \quad (4.115)$$

If $j = 1$, then by (4.107) and (4.115) we have $(H,\pi) \in \mathcal{H}'_s$. If $j > 1$, then by the second part of Lemma 4.15 (recall that by the definition in line 14 we have $d_\theta(H, v_1, w_{(H,\pi,s)}) = d_s^i$) there is a subgraph $J \subseteq H$ with $v_1 \in J$ for which $(J, \pi|_J)$ is contained in $\mathcal{C}_s(d_s^i)$. By the definition in line 16 we have $\mathcal{C}_s(d_s^i) = \mathcal{C}^{i,1}$, implying that $(J, \pi|_J) \in \mathcal{C}^{i,1}$ and

$$x_{(J,\pi|_J,s)} \stackrel{(4.114)}{=} x_{(H,\pi,s)} .$$

Furthermore, using (4.115) it follows from the definition in (4.107) that $(J, \pi|_J) \in \mathcal{H}'_s$, proving that property (iii) holds for the walk $\overline{\mathcal{W}}$.

If on the other hand $x_{(H,\pi,s)}$ is not the starting point of Γ , i.e., $i > 1$ and $\alpha_i = \alpha_{i-1}$, then by the definition in (4.107) we have $(H,\pi) \notin \mathcal{H}'_s$, proving property (iv) for the walk $\overline{\mathcal{W}}$.

Note that by the definition of $\overline{\mathcal{W}}$, the lowest segment of this walk is indeed an $\alpha_{i_{\max}}$ -segment. By the termination condition in line 36 and the observation that during the i -th iteration of the repeat-loop (*), none of the families \mathcal{H}_s , $s \in [r] \setminus \{\alpha_i\}$, is modified, we have $\mathcal{H}_{\alpha_{i_{\max}}} = \mathcal{S}(F)$. Together this proves property (v) for the walk $\overline{\mathcal{W}}$.

To complete the proof of the lemma we fix some $s \in [r]$, some $i,j,k \geq 1$ with $\alpha_i = s$ and some graph $(H,\pi) \in \mathcal{C}^{i,j,k}$, $\pi = (v_1, \dots, v_h)$, and show that the point $x_{(H,\pi,s)}$ lies on some s -segment of the walk $\overline{\mathcal{W}}$ (possibly in between two points $\overline{\mathcal{W}}^i$ and $\overline{\mathcal{W}}^{i+1}$), and that by including all such points $x_{(H,\pi,s)}$ into the walk $\overline{\mathcal{W}}$ we obtain a subdivided walk \mathcal{W} that still satisfies the claimed properties (note that beside (i) we only need to verify that properties (iii) and (iv) are maintained).

Note that by the definitions in line 28 and line 30 we have $\alpha_{\hat{i}} = \alpha_i = s$ for \hat{i} as in part (b) of Definition 4.38. Using this relation, the definition in (4.112b), and Lemma 4.17 we obtain that $x_{(H,\pi,s)}$ lies on the s -segment Γ that contains $\overline{W}^{\hat{i}}$ and $\overline{W}^{\hat{i}+1}$ on the walk \overline{W} , showing that the walk \mathcal{W} satisfies property (i).

By the strict inequality in (4.25a), $x_{(H,\pi,s)}$ can not be the starting point of Γ if \hat{i} was defined in line 28. Moreover, by (4.25b) the point $x_{(H,\pi,s)}$ is the starting point of Γ if and only if

$$d_{\theta}(H, v_1, w_{(H,\pi,s)}) = d_s^{\hat{i}} \quad (4.116)$$

and

$$\hat{i} = 1 \text{ or } \alpha_{\hat{i}} \neq \alpha_{\hat{i}-1} . \quad (4.117)$$

In this case, by the condition in line 27 there is a subgraph $J \subseteq H$ with $v_1 \in J$ such that $(J, \pi|_J)$ is contained in $\mathcal{C}_s(d_{\theta}(H, v_1, w_{(H,\pi,s)}))$. Using (4.116) and the definition in line 16 shows that $(J, \pi|_J) \in \mathcal{C}^{\hat{i},1}$, implying that

$$x_{(J,\pi|_J,s)} \stackrel{(4.114)}{=} (d_1^{\hat{i}}, \dots, d_r^{\hat{i}}) \stackrel{(4.112b),(4.116)}{=} x_{(H,\pi,s)} .$$

Furthermore, using (4.117) it follows from the definition in (4.107) that $(J, \pi|_J) \in \mathcal{H}'_s$, proving that property (iii) holds for the walk \mathcal{W} .

As none of the graphs in the sets $\mathcal{C}^{i,j,k}$ with $\alpha_i = s$ is contained in \mathcal{H}'_s (recall (4.107)), the walk \mathcal{W} trivially satisfies property (iv). This completes the proof. \square

4.5.3. Relation of the walk to other quantities. In the following lemmas we establish several relations between the walk \mathcal{W} defined in Lemma 4.39, the parameters $d_{\theta}()$ and $w_{(H,\pi,s)}$ used in the algorithm COMPUTEWEIGHTS(), and the parameter $\lambda_{\theta}()$ and the families \mathcal{H}'_s used in the definition of the strategy PAINT(). We will see that for the ordered monochromatic subgraphs of F that are relevant for the strategy PAINT(), the order of the corresponding x -points on the walk \mathcal{W} coincides with the ordering given by the $\lambda_{\theta}()$ -values — the lower on the walk the point $x_{(H,\pi,s)}$ appears, the lower the value $\lambda(H, w_{(H,\pi,s)})$, i.e., the more dangerous a copy of (H, π) in color s is considered (see Lemma 4.43 below).

LEMMA 4.40 ($d_{\theta}()$ -value and weight from x -point). *For any $s \in [r]$ and any graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, we have*

$$d_{\theta}(H, v_1, w_{(H,\pi,s)}) = x_{(H,\pi,s),s}$$

and

$$w_{(H,\pi,s)}(v_1) = \sum_{t \in [r] \setminus \{s\}} x_{(H,\pi,s),t} . \quad (4.118)$$

PROOF. The first part of the lemma is an immediate consequence of the definition in (4.112).

For any $s \in [r]$ and any graph $(H, \pi) \in \mathcal{H}_s$ as in part (a) of Definition 4.38 we obtain, using the definitions in line 10 and line 18,

$$w_{(H,\pi,s)}(v_1) \stackrel{(4.22)}{=} w_s(H, \pi) = w^i = \sum_{t \in [r] \setminus \{s\}} d_t^i \stackrel{(4.112a)}{=} \sum_{t \in [r] \setminus \{s\}} x_{(H,\pi,s),t} . \quad (4.119)$$

For any $s \in [r]$ and any graph $(H, \pi) \in \mathcal{H}_s$ as in part (b) of Definition 4.38 we obtain, using the definitions in line 10 and line 31,

$$w_{(H,\pi,s)}(v_1) \stackrel{(4.22)}{=} w_s(H, \pi) = w^{\hat{i}} = \sum_{t \in [r] \setminus \{s\}} d_t^{\hat{i}} \stackrel{(4.112b)}{=} \sum_{t \in [r] \setminus \{s\}} x_{(H,\pi,s),t} . \quad (4.120)$$

Together (4.119) and (4.120) prove the second part of the lemma. \square

For the next lemma, recall the definition of $\mathcal{C}(\mathcal{H}, F)$ in (4.21).

LEMMA 4.41 (Graphs in $\mathcal{C}(\mathcal{H}_s, F)$ have smallest $d_\theta(\cdot)$ -value). *Let $\sigma \in [r]$ and $(H, \pi) \in \mathcal{H}_\sigma$, $\pi = (v_1, \dots, v_h)$, and let $s \in [r]$ and $(J, \tau) \in \mathcal{C}(\mathcal{H}_s, F)$, $\tau = (u_1, \dots, u_c)$. If $s = \sigma$, then we have*

$$d_\theta(J, u_1, w_{(J, \tau, \sigma)}) < x_{(H, \pi, \sigma), \sigma} = d_\theta(H, v_1, w_{(H, \pi, \sigma)}) .$$

If $s \neq \sigma$, then we have

$$d_\theta(J, u_1, w_{(J, \tau, s)}) \leq x_{(H, \pi, \sigma), s} .$$

PROOF. Let i_{\max} denote the total number of iterations of the repeat-loop (*) in COMPUTE-WEIGHTS().

First suppose that $s = \sigma$. Denoting by \hat{i} the largest index $\bar{i} \leq i_{\max}$ for which $\alpha_{\bar{i}} = \sigma$, the first part of Lemma 4.14 and the definitions in line 14 and line 23 show that $d_\theta(H, v_1, w_{(H, \pi, \sigma)}) \geq d_\sigma^{\hat{i}}$. By the termination condition in line 35 we also have $d_\theta(J, u_1, w_{(J, \tau, \sigma)}) < d_\sigma^{\hat{i}}$. We thus obtain $d_\theta(J, u_1, w_{(J, \tau, \sigma)}) < d_\theta(H, v_1, w_{(H, \pi, \sigma)})$. By the definition in (4.112) the right hand side of this last inequality equals $x_{(H, \pi, \sigma), \sigma}$, proving the first part of the lemma.

Now suppose that $s \neq \sigma$. By the definition in (4.112) we have

$$x_{(H, \pi, \sigma), s} = d_s^i \tag{4.121}$$

for some $1 \leq i \leq i_{\max}$. By the termination condition in line 36 and the observation that during the \bar{i} -th iteration of the repeat-loop (*), none of the families \mathcal{H}_t , $t \in [r] \setminus \{\alpha_{\bar{i}}\}$, is modified, we must have $\alpha_{i_{\max}} \neq s$, as we would have $\mathcal{H}_s = \mathcal{S}(F)$ otherwise, implying that $\mathcal{C}(\mathcal{H}_s, F)$ would be empty. So let \hat{i} be the largest index $\bar{i} \leq i_{\max} - 1$ for which $\alpha_{\bar{i}} = s$. By the definition in line 8 we have

$$d_\theta(J, u_1, w_{(J, \tau, s)}) \leq d_s^{\hat{i}+1} . \tag{4.122}$$

Using the first part of Lemma 4.14 twice we obtain

$$d_s^{\hat{i}+1} = \dots = d_s^{i_{\max}} \quad \text{and} \quad d_s^{i_{\max}} \leq d_s^i ,$$

which together with (4.121) and (4.122) yields the second part of the lemma. \square

LEMMA 4.42 (Relation between $\lambda_\theta(\cdot)$ -value and x -point). *Let $s \in [r]$. For any graph $(H, \pi) \in \mathcal{H}_s$, $\pi = (v_1, \dots, v_h)$, we have*

$$\lambda_\theta(H, w_{(H, \pi, s)}) \geq 1 + \sum_{t \in [r]} x_{(H, \pi, s), t} .$$

Moreover, there is a subgraph $\hat{J} \subseteq H$ with $v_1 \in \hat{J}$ satisfying

$$\lambda_\theta(\hat{J}, w_{(\hat{J}, \pi|_{\hat{J}}, s)}) = 1 + \sum_{t \in [r]} x_{(\hat{J}, \pi|_{\hat{J}}, s), t} = 1 + \sum_{t \in [r]} x_{(H, \pi, s), t} .$$

PROOF. We clearly have

$$\lambda_\theta(H, w_{(H, \pi, s)}) \stackrel{(4.18)}{=} \sum_{u \in H} (1 + w_{(H, \pi, s)}(u)) - e(H) \cdot \theta \stackrel{(4.17)}{\geq} 1 + w_{(H, \pi, s)}(v_1) + d_\theta(H, v_1, w_{(H, \pi, s)}) . \tag{4.123}$$

By Lemma 4.40 the right hand side of (4.123) equals $1 + \sum_{t \in [r]} x_{(H, \pi, s), t}$, proving the first part of the lemma.

Now consider a graph \hat{J} from the family

$$\arg \min_{J \subseteq H: v_1 \in J} \left(\sum_{u \in J \setminus v_1} (1 + w_{(H, \pi, s)}(u)) - e(J) \cdot \theta \right) . \quad (4.124)$$

Using the definition of $d_\theta()$ in (4.17) we obtain

$$d_\theta(H, v_1, w_{(H, \pi, s)}) \stackrel{(4.17), (4.124)}{=} \sum_{u \in \hat{J} \setminus v_1} (1 + w_{(H, \pi, s)}(u)) - e(\hat{J}) \cdot \theta \stackrel{(4.17), (4.124)}{=} d_\theta(\hat{J}, v_1, w_{(H, \pi, s)}) . \quad (4.125)$$

Furthermore, Lemma 4.22 yields that

$$w_{(H, \pi, s)}(u) = w_{(\hat{J}, \pi|_{\hat{J}}, s)}(u) \quad \text{for all } u \in \hat{J} . \quad (4.126)$$

Recall from the first part of the proof that the right hand side of (4.123) equals $1 + \sum_{t \in [r]} x_{(H, \pi, s), t}$. Applying (4.125), (4.126) and Lemma 4.40 shows that the right hand side of (4.123) also equals $1 + \sum_{t \in [r]} x_{(\hat{J}, \pi|_{\hat{J}}, s), t}$. Furthermore, applying the first equality in (4.125), (4.126) and the definition of $\lambda_\theta()$ in (4.18) shows that the right hand side of (4.123) equals $\lambda_\theta(\hat{J}, w_{(\hat{J}, \pi|_{\hat{J}}, s)})$, completing the proof of the second part of the lemma. \square

We say that a family \mathcal{D} of ordered graphs is *closed under taking subgraphs that contain the youngest vertex* if for any $(H, \pi) \in \mathcal{D}$, $\pi = (v_1, \dots, v_h)$, we have that for every $J \subseteq H$ with $v_1 \in J$ the ordered graph $(J, \pi|_J)$ is also contained in \mathcal{D} .

Note that the families \mathcal{D}_s , $s \in [r]$, used by the strategy `PAINT()` and defined in (4.105) are nonempty and closed under taking subgraphs that contain the youngest vertex.

LEMMA 4.43 (x -point of $\lambda_\theta()$ -minimizing graphs). *Let $s \in [r]$ and $\mathcal{D}_s \subseteq \mathcal{H}_s$ a nonempty family of ordered graphs that is closed under taking subgraphs that contain the youngest vertex. For any graph (J, τ) from the family*

$$\arg \min_{(H, \pi) \in \mathcal{D}_s} \lambda_\theta(H, w_{(H, \pi, s)})$$

we have

$$\lambda_\theta(J, w_{(J, \tau, s)}) = 1 + \sum_{t \in [r]} x_{(J, \tau, s), t} .$$

PROOF. The claim follows immediately from Lemma 4.42, using the closure property of the family \mathcal{D}_s and the choice of (J, τ) . \square

LEMMA 4.44 ($d_\theta()$ -value of $\lambda_\theta()$ -minimizing graphs). *Let $s \in [r]$ and let $\mathcal{D}_s \subseteq \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$ be a nonempty family of ordered graphs that is closed under taking subgraphs that contain the youngest vertex. Furthermore, let (J, τ) , $\tau = (u_1, \dots, u_c)$, be an inclusion-minimal graph from the family*

$$\arg \min_{(H, \pi) \in \mathcal{D}_s} \lambda_\theta(H, w_{(H, \pi, s)}) .$$

Then we have

$$\lambda_\theta(J \setminus u_1, w_{(J, \tau, s)}) - \deg_J(u_1) \cdot \theta = d_\theta(J, u_1, w_{(J, \tau, s)}) . \quad (4.127)$$

PROOF. We distinguish two cases depending on whether $(J, \tau) \in \mathcal{D}_s \subseteq \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$ is contained in \mathcal{H}_s or in $\mathcal{C}(\mathcal{H}_s, F)$.

If $(J, \tau) \in \mathcal{H}_s$, then by Lemma 4.43 we have

$$\lambda_\theta(J, w_{(J, \tau, s)}) = 1 + \sum_{t \in [r]} x_{(J, \tau, s), t} . \quad (4.128)$$

Rewriting the left hand side of (4.128) according to (4.19) and the right hand side according to Lemma 4.40 yields the desired equality (4.127).

We now consider the case $(J, \tau) \in \mathcal{C}(\mathcal{H}_s, F)$ (in this case we have $\lambda_\theta(J, w_{(J, \tau, s)}) = -\infty$ by Lemma 4.13). We clearly have

$$\lambda_\theta(J \setminus u_1, w_{(J, \tau, s)}) - \deg_J(u_1) \cdot \theta \stackrel{(4.18)}{=} \sum_{u \in J \setminus u_1} (1 + w_{(J, \tau, s)}(u)) - e(J) \cdot \theta \stackrel{(4.17)}{\geq} d_\theta(J, u_1, w_{(J, \tau, s)}) , \quad (4.129)$$

and it remains to show that this inequality is in fact an equality. If the last inequality in (4.129) were strict, then, as in the proof of Lemma 4.42 (cf. (4.124), (4.125) and (4.126)), there would be a proper subgraph $\hat{J} \subsetneq J$ with $u_1 \in \hat{J}$ satisfying

$$d_\theta(J, u_1, w_{(J, \tau, s)}) = d_\theta(\hat{J}, u_1, w_{(\hat{J}, \tau|_{\hat{J}}, s)}) . \quad (4.130)$$

As $(J, \tau) \in \mathcal{C}(\mathcal{H}_s, F)$ we have $(J \setminus u_1, \tau \setminus u_1) \in \mathcal{H}_s$, which by Lemma 4.20 implies that $(\hat{J} \setminus u_1, \tau|_{\hat{J} \setminus u_1}) \in \mathcal{H}_s$ as well. Hence $(\hat{J}, \tau|_{\hat{J}})$ must be in $\mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$. Using (4.130) and the first part of Lemma 4.41 shows that $(\hat{J}, \tau|_{\hat{J}})$ must be contained in $\mathcal{C}(\mathcal{H}_s, F)$. But then we have $\lambda_\theta(\hat{J}, w_{(\hat{J}, \tau|_{\hat{J}}, s)}) = -\infty$ by Lemma 4.13, a contradiction to the inclusion-minimality of (J, τ) (here we used again the closure property of the family \mathcal{D}_s). Therefore the last inequality in (4.129) holds with equality, proving the lemma also in this case. \square

LEMMA 4.45 ($\lambda_\theta(\cdot)$ -minimizing graphs in \mathcal{H}'_s). *Let $s \in [r]$ and let $\mathcal{D}_s \subseteq \mathcal{H}_s$ be a nonempty family of ordered graphs that is closed under taking subgraphs that contain the youngest vertex. Furthermore, let (J, τ) be an inclusion-minimal graph from the family*

$$\arg \min_{(H, \pi) \in \mathcal{D}_s} \lambda_\theta(H, w_{(H, \pi, s)})$$

and suppose that $x_{(J, \tau, s)}$ is the starting point of some s -segment of the walk \mathcal{W} defined in Lemma 4.39. Then (J, τ) is contained in \mathcal{H}'_s .

PROOF. By Lemma 4.43 we have

$$\lambda_\theta(J, w_{(J, \tau, s)}) = 1 + \sum_{t \in [r]} x_{(J, \tau, s), t} . \quad (4.131)$$

Let u_1 denote the youngest vertex of (J, τ) and let $\tilde{J} \subsetneq J$ be any proper subgraph of J with $u_1 \in \tilde{J}$. By Lemma 4.42 there is a subgraph $\hat{J} \subseteq \tilde{J}$ with $u_1 \in \hat{J}$ satisfying

$$\lambda_\theta(\hat{J}, w_{(\hat{J}, \tau|_{\hat{J}}, s)}) = 1 + \sum_{t \in [r]} x_{(\tilde{J}, \tau|_{\tilde{J}}, s), t} . \quad (4.132)$$

By the inclusion-minimal choice of (J, τ) , (4.132) must be strictly larger than (4.131), i.e., we have

$$1 + \sum_{t \in [r]} x_{(J, \tau, s), t} < 1 + \sum_{t \in [r]} x_{(\tilde{J}, \tau|_{\tilde{J}}, s), t} ,$$

in particular

$$x_{(J, \tau, s)} \neq x_{(\tilde{J}, \tau|_{\tilde{J}}, s)} .$$

Using this observation together with the assumption that $x_{(J, \tau, s)}$ is the starting point of some s -segment of \mathcal{W} , it follows from property (iii) in Lemma 4.39 that (J, τ) must be contained in \mathcal{H}'_s . \square

LEMMA 4.46 (*x*-points of graphs from \mathcal{H}'_s on the walk). *Let $s \in [r]$ and $(J, \tau) \in \mathcal{H}'_s$, $\tau = (u_1, \dots, u_c)$. Moreover, let $1 \leq b \leq c - 1$ and define $(J^{-b}, \tau^{-b}) := (J \setminus \{u_1, \dots, u_b\}, \tau \setminus \{u_1, \dots, u_b\})$.*

Then $x_{(J, \tau, s)}$ is lower than $x_{(J^{-b}, \tau^{-b}, s)}$ on the walk \mathcal{W} defined in Lemma 4.39 and both points are contained in different s -segments of this walk.

PROOF. By the definition of \mathcal{H}'_s in (4.107) we have $(J, \tau) \in \mathcal{C}^{i,1}$ for some $i \geq 1$ with $\alpha_i = s$, i.e., (J, τ) was added to the family \mathcal{H}_s in the first iteration of the repeat-loop (**) in the i -th iteration of the repeat-loop (*) of COMPUTEWEIGHTS(). By the definition in (4.112a) we have

$$x_{(J, \tau, s)} = d_\theta(J, u_1, w_{(J, \tau, s)}) = d_s^i. \quad (4.133)$$

By Lemma 4.20, the graph (J^{-b}, τ^{-b}) was added to the family \mathcal{H}_s either before the graph (J, τ) or together with it. But as (J^{-b}, τ^{-b}) is a predecessor of (J, τ) in the tree $\mathcal{T}(F)$ defined after (4.20), it follows from the definition in line 14 that (J^{-b}, τ^{-b}) must have already been contained in \mathcal{H}_s at the beginning of the i -th iteration of the repeat-loop (*). Applying Lemma 4.16 yields

$$x_{(J^{-b}, \tau^{-b}, s)} \stackrel{(4.112)}{=} d_\theta(J^{-b}, u_{b+1}, w_{(J^{-b}, \tau^{-b}, s)}) > d_s^i. \quad (4.134)$$

Combining (4.133) and (4.134) shows that $x_{(J, \tau, s)}$ is lower than $x_{(J^{-b}, \tau^{-b}, s)}$ on the walk \mathcal{W} . As by the assumption $(J, \tau) \in \mathcal{H}'_s$ and property (iv) from Lemma 4.39 the point $x_{(J, \tau, s)}$ is the starting point of an s -segment of \mathcal{W} , this implies that both points must be contained in different s -segments of this walk. \square

4.5.4. Analysis of PAINT(). We are now in a position to actually analyze our Painter strategy PAINT(). Recall from Section 4.5.1 that the parameter $d(s)$ defined in (4.106) might be equal to $-\infty$ for some colors $s \in [r]$ (intuitively, Painter considers such a color extremely dangerous). The following lemma shows that PAINT() never chooses such a color.

LEMMA 4.47 (Painter strategy creates only graphs from \mathcal{H}_s). *Consider a fixed step of the game, and let the families $\mathcal{D}_s \subseteq \mathcal{S}(F)$, $s \in [r]$, and the values $d(s) \in \mathbb{R} \cup \{-\infty\}$ be defined as in (4.105) and (4.106), respectively. For any $\sigma \in \arg \max_{s \in [r]} d(s)$ the value $d(\sigma)$ is finite and we have $\mathcal{D}_\sigma \subseteq \mathcal{H}_\sigma$.*

Consequently, playing according to the strategy PAINT() throughout ensures that for all $s \in [r]$ we always have $\mathcal{D}_s \subseteq \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$ (even if ties are broken arbitrarily).

PROOF. By the definition in (4.106) and Lemma 4.13 the value $d(s)$ is finite if and only if $\mathcal{D}_s \subseteq \mathcal{H}_s$. By the termination condition in line 36 there is some color $s \in [r]$ for which $\mathcal{H}_s = \mathcal{S}(F)$. For this color we therefore have $\mathcal{D}_s \subseteq \mathcal{H}_s$, implying that the corresponding value $d(s)$ is finite. This shows that for any $\sigma \in \arg \max_{s \in [r]} d(s)$, the value $d(\sigma)$ is finite and therefore $\mathcal{D}_\sigma \subseteq \mathcal{H}_\sigma$, proving the first part of the lemma. The second part follows inductively by observing that the strategy PAINT() in each step picks a color $\sigma \in \arg \max_{s \in [r]} d(s)$ (regardless of the tie-breaking rule), showing that in this step only graphs from the family $\mathcal{D}_\sigma \subseteq \mathcal{H}_\sigma$ in color σ are created on the board. \square

The following lemma shows that the tie-breaking rule of the strategy PAINT(), which uses the families \mathcal{H}'_s and \mathcal{J}_s defined in (4.107) and (4.108), is indeed well-defined.

LEMMA 4.48 (Well-definedness of Painter strategy). *Ties in the strategy PAINT() can arise only between two different colors, and if they arise then for exactly one of the two colors (say σ) we have $\mathcal{J}_\sigma \cap \mathcal{H}'_\sigma = \emptyset$, and for the other color (say s) we have $\mathcal{J}_s \cap \mathcal{H}'_s \neq \emptyset$. (Thus the tie-breaking rule will decide for color σ .)*

If such a tie arises, the walk \mathcal{W} defined in Lemma 4.39 contains a σ -segment Γ whose endpoint $x \in \mathbb{R}^r$ is also the starting point of an s -segment Γ' , and for any $(J_\sigma, \tau_\sigma) \in \mathcal{J}_\sigma$ and any $(J_s, \tau_s) \in \mathcal{J}_s$ we have $x_{(J_\sigma, \tau_\sigma, \sigma)} = x_{(J_s, \tau_s, s)} = x$.

PROOF. Recall that the families \mathcal{D}_s , $s \in [r]$, defined in (4.105) are nonempty and closed under taking subgraphs that contain the youngest vertex. Fix some color $\sigma \in [r]$ such that

$$d(s) \leq d(\sigma) \quad , \quad s \in [r] \setminus \{\sigma\} \quad , \quad (4.135)$$

for the values $d(s)$, $s \in [r]$, defined in (4.106). The tie-breaking rule of the strategy PAINT() is only considered if the inequality in (4.135) is tight for some color different from σ . We fix such a color $s \in [r] \setminus \{\sigma\}$ for which

$$d(s) = d(\sigma) \quad . \quad (4.136)$$

The first part of Lemma 4.47 yields with (4.135) and (4.136) that $\mathcal{D}_s \subseteq \mathcal{H}_s$ and $\mathcal{D}_\sigma \subseteq \mathcal{H}_\sigma$ (and that $d(s)$ and $d(\sigma)$ are finite values). Thus by the definition in (4.108) we also have $\mathcal{J}_s \subseteq \mathcal{H}_s$ and $\mathcal{J}_\sigma \subseteq \mathcal{H}_\sigma$. Fix some $(J_s, \tau_s) \in \mathcal{J}_s$ and some $(J_\sigma, \tau_\sigma) \in \mathcal{J}_\sigma$. By the definition in (4.108) and Lemma 4.43 we have

$$\lambda_\theta(J_s, w_{(J_s, \tau_s, s)}) = 1 + \sum_{t \in [r]} x_{(J_s, \tau_s, s), t} \quad \text{and} \quad \lambda_\theta(J_\sigma, w_{(J_\sigma, \tau_\sigma, \sigma)}) = 1 + \sum_{t \in [r]} x_{(J_\sigma, \tau_\sigma, \sigma), t} \quad . \quad (4.137)$$

Furthermore, using (4.136) and the definitions in (4.106) and (4.108) shows that

$$\lambda_\theta(J_s, w_{(J_s, \tau_s, s)}) = \lambda_\theta(J_\sigma, w_{(J_\sigma, \tau_\sigma, \sigma)}) \quad . \quad (4.138)$$

Combining (4.137) and (4.138) we obtain

$$1 + \sum_{t \in [r]} x_{(J_s, \tau_s, s), t} = 1 + \sum_{t \in [r]} x_{(J_\sigma, \tau_\sigma, \sigma), t} \quad . \quad (4.139)$$

Note that the points $x_{(J_s, \tau_s, s)}$ and $x_{(J_\sigma, \tau_\sigma, \sigma)}$ are elements of the walk \mathcal{W} defined in Lemma 4.39. By Lemma 4.36 the relation (4.139) implies that $x_{(J_s, \tau_s, s)} = x_{(J_\sigma, \tau_\sigma, \sigma)}$, i.e., the graphs (J_s, τ_s) and (J_σ, τ_σ) (and all other graphs in the families \mathcal{J}_s and \mathcal{J}_σ) have the same associated point on the walk \mathcal{W} . By property (i) in Lemma 4.39, $x_{(J_s, \tau_s, s)}$ is contained in an s -segment and $x_{(J_\sigma, \tau_\sigma, \sigma)}$ in a σ -segment of \mathcal{W} , implying that $x_{(J_s, \tau_s, s)} = x_{(J_\sigma, \tau_\sigma, \sigma)}$ must be the endpoint of some segment and the starting point of the next lower segment. As on the walk \mathcal{W} , only pairs of consecutive segments have a point in common, this shows that the inequality (4.135) can be tight for at most one color different from σ , proving that ties can arise only between two different colors.

Assume w.l.o.g. that for all $(J_\sigma, \tau_\sigma) \in \mathcal{J}_\sigma$, the point $x_{(J_\sigma, \tau_\sigma, \sigma)}$ is the endpoint of some σ -segment Γ and for all $(J_s, \tau_s) \in \mathcal{J}_s$ the point $x_{(J_s, \tau_s, s)}$ is the starting point of the next lower s -segment Γ' . By property (iv) from Lemma 4.39 we have $\mathcal{J}_\sigma \cap \mathcal{H}'_\sigma = \emptyset$, and by Lemma 4.45 we have $\mathcal{J}_s \cap \mathcal{H}'_s \neq \emptyset$. This proves the first part of the lemma and shows that our tie-breaking rule is well-defined.

Note that the segment Γ is higher on the walk \mathcal{W} than Γ' , and that the tie-breaking rule decides for the color σ corresponding to the higher of the two segments. Together with our previous observations about the location of the points $x_{(J_\sigma, \tau_\sigma, \sigma)}$ for all $(J_\sigma, \tau_\sigma) \in \mathcal{J}_\sigma$ and $x_{(J_s, \tau_s, s)}$ for all $(J_s, \tau_s) \in \mathcal{J}_s$ this proves the second part of the lemma. \square

The following lemma will be the key to proving Lemma 4.33, our main strategy invariant based on witness graphs.

LEMMA 4.49 (Painter strategy ensures sufficient weight). *There is a constant $\varepsilon = \varepsilon(F, r, \theta, \alpha) > 0$ such that the following holds: Let $\sigma \in [r]$ denote the color selected by the strategy PAINT() in a certain step of the game given the families \mathcal{D}_s , $s \in [r]$, defined in (4.105). For every $s \in [r] \setminus \{\sigma\}$, let (J_s, τ_s) be an inclusion-minimal graph from the family*

$$\mathcal{J}_s \stackrel{(4.108)}{=} \arg \min_{(H, \pi) \in \mathcal{D}_s} \lambda_\theta(H, w_{(H, \pi, s)}) ,$$

and let u_{s1} denote the youngest vertex of (J_s, τ_s) .

Then for any graph $(H, \pi) \in \mathcal{D}_\sigma$, $\pi = (v_1, \dots, v_h)$, we have

$$\sum_{s \in [r] \setminus \{\sigma\}} (\lambda_\theta(J_s \setminus u_{s1}, w_{(J_s, \tau_s, s)}) - \deg_{J_s}(u_{s1})) \leq w_{(H, \pi, \sigma)}(v_1) . \quad (4.140)$$

If the inequality (4.140) is strict, then the difference between the right and left hand side is at least ε .

If on the other hand the inequality (4.140) is tight, then for every $s \in [r] \setminus \{\sigma\}$ we have $(J_s, \tau_s) \in \mathcal{H}'_s \cup \mathcal{C}(\mathcal{H}_s, F)$. Moreover, denoting by \mathcal{W} the walk defined in Lemma 4.39 and by Γ the σ -segment containing $x_{(H, \pi, \sigma)}$ on this walk, we have the following: if $(J_s, \tau_s) \in \mathcal{H}'_s$, then $x_{(J_s, \tau_s, s)}$ is the starting point of the next s -segment on \mathcal{W} that is lower than Γ , whereas if $(J_s, \tau_s) \in \mathcal{C}(\mathcal{H}_s, F)$, then there is no s -segment on \mathcal{W} lower than Γ .

PROOF. Let

$$\begin{aligned} \varepsilon = \varepsilon(F, r, \theta, \alpha) := \min \{ & |d_\theta(H, v_1, w_{(H, \pi, s)}) - d_\theta(J, u_1, w_{(J, \tau, s)})| \mid s \in [r] \wedge \\ & (H, \pi = (v_1, \dots, v_h)), (J, \tau = (u_1, \dots, u_c)) \in \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F) \wedge \\ & d_\theta(H, v_1, w_{(H, \pi, s)}) \neq d_\theta(J, u_1, w_{(J, \tau, s)}) \} > 0 \end{aligned} \quad (4.141)$$

(recall that for all $s \in [r]$ the $d_\theta(\cdot)$ -value of all graphs in $\mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$ is a finite real number). Note that in the boundary case that for all $s \in [r]$ and all $(H, \pi) \in \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$, $\pi = (v_1, \dots, v_h)$, the value $d_\theta(H, v_1, w_{(H, \pi, s)})$ is the same (i.e., the walk \mathcal{W} defined in Lemma 4.39 degenerates to a single point), the minimum in (4.141) is over an empty set. We will see that in this case the inequality (4.140) is never strict. Therefore we may set ε to an arbitrary positive constant in this case, $\varepsilon := 1$, say.

Recall that the families \mathcal{D}_s , $s \in [r]$, are nonempty and closed under taking subgraphs that contain the youngest vertex. By the second part of Lemma 4.47 we have $\mathcal{D}_s \subseteq \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$ for all $s \in [r]$.

We first prove that (4.140) holds. By the definition of the strategy, the selected color $\sigma \in [r]$ satisfies

$$d(s) \leq d(\sigma) \quad , \quad s \in [r] \setminus \{\sigma\} \quad (4.142)$$

for the values $d(s)$, $s \in [r]$, defined in (4.106). Let (J_σ, τ_σ) be an arbitrary graph from the family

$$\mathcal{J}_\sigma \stackrel{(4.108)}{=} \arg \min_{(H, \pi) \in \mathcal{D}_\sigma} \lambda_\theta(H, w_{(H, \pi, \sigma)}) .$$

By the definition in (4.106) and the choice of (J_s, τ_s) , $s \in [r]$, we have

$$\lambda_\theta(J_s, w_{(J_s, \tau_s, s)}) = d(s) \quad , \quad s \in [r] . \quad (4.143)$$

Combining (4.142) and (4.143) yields

$$\lambda_\theta(J_s, w_{(J_s, \tau_s, s)}) \leq \lambda_\theta(J_\sigma, w_{(J_\sigma, \tau_\sigma, \sigma)}) \quad , \quad s \in [r] \setminus \{\sigma\} . \quad (4.144)$$

By (4.142) and the first part of Lemma 4.47 we have $\mathcal{D}_\sigma \subseteq \mathcal{H}_\sigma$. We fix some graph $(H, \pi) \in \mathcal{D}_\sigma$, $\pi = (v_1, \dots, v_h)$. By Lemma 4.42 there is a subgraph $\hat{J} \subseteq H$ with $v_1 \in \hat{J}$ satisfying

$$\lambda_\theta(\hat{J}, w_{(\hat{J}, \pi|_{\hat{J}}, \sigma)}) = 1 + \sum_{t \in [r]} x_{(H, \pi, \sigma), t} . \quad (4.145)$$

By the closure property of the family \mathcal{D}_σ and the choice of (J_σ, τ_σ) we have

$$\lambda_\theta(J_\sigma, w_{(J_\sigma, \tau_\sigma, \sigma)}) \leq \lambda_\theta(\hat{J}, w_{(\hat{J}, \pi|_{\hat{J}}, \sigma)}) . \quad (4.146)$$

By Lemma 4.43 we have for every $s \in [r] \setminus \{\sigma\}$ for which (J_s, τ_s) is contained in \mathcal{H}_s that

$$\lambda_\theta(J_s, w_{(J_s, \tau_s, s)}) = 1 + \sum_{t \in [r]} x_{(J_s, \tau_s, s), t} . \quad (4.147)$$

For those $s \in [r] \setminus \{\sigma\}$ we thus obtain

$$1 + \sum_{t \in [r]} x_{(J_s, \tau_s, s), t} \stackrel{(4.147)}{=} \lambda_\theta(J_s, w_{(J_s, \tau_s, s)}) \stackrel{(4.144)}{\leq} \lambda_\theta(J_\sigma, w_{(J_\sigma, \tau_\sigma, \sigma)}) \stackrel{(4.145), (4.146)}{\leq} 1 + \sum_{t \in [r]} x_{(H, \pi, \sigma), t} . \quad (4.148)$$

Note that if $(J_s, \tau_s) \in \mathcal{H}_s$, then $x_{(J_s, \tau_s, s), t}$ is an element of the walk \mathcal{W} defined in Lemma 4.39 (the point $x_{(H, \pi, \sigma), t}$ is clearly also an element of this walk as $\mathcal{D}_\sigma \subseteq \mathcal{H}_\sigma$). Using (4.148) and Lemma 4.36 yields that for every $s \in [r] \setminus \{\sigma\}$ for which (J_s, τ_s) is contained in \mathcal{H}_s we have

$$x_{(J_s, \tau_s, s), t} \leq x_{(H, \pi, \sigma), t} \quad \text{for all } t \in [r] , \quad (4.149)$$

from which we conclude using

$$x_{(J_s, \tau_s, s), s} \stackrel{(4.112)}{=} d_\theta(J_s, u_{s1}, w_{(J_s, \tau_s, s)}) \quad (4.150)$$

that

$$d_\theta(J_s, u_{s1}, w_{(J_s, \tau_s, s)}) \leq x_{(H, \pi, \sigma), s} . \quad (4.151)$$

For all $s \in [r] \setminus \{\sigma\}$ for which (J_s, τ_s) is not contained in \mathcal{H}_s but in $\mathcal{C}(\mathcal{H}_s, F)$, the relation (4.151) follows from the second part of Lemma 4.41.

Combining our previous observations and applying Lemma 4.40 and Lemma 4.44, we thus obtain

$$\begin{aligned} \sum_{s \in [r] \setminus \{\sigma\}} (\lambda_\theta(J_s \setminus u_{s1}, w_{(J_s, \tau_s, s)}) - \deg_{J_s}(u_{s1})) &\stackrel{(4.127)}{=} \sum_{s \in [r] \setminus \{\sigma\}} d_\theta(J_s, u_{s1}, w_{(J_s, \tau_s, s)}) \\ &\stackrel{(4.151)}{\leq} \sum_{s \in [r] \setminus \{\sigma\}} x_{(H, \pi, \sigma), s} \stackrel{(4.118)}{=} w_{(H, \pi, \sigma)}(v_1) , \end{aligned} \quad (4.152)$$

proving (4.140).

If the inequality (4.152) is strict, then by (4.151) we have

$$d_\theta(J_s, u_{s1}, w_{(J_s, \tau_s, s)}) < x_{(H, \pi, \sigma), s} \quad \text{for some } s \in [r] \setminus \{\sigma\} . \quad (4.153)$$

By the definition in (4.112) and the definition in line 8, the right hand side of (4.153) equals $d_\theta(\bar{J}, \bar{u}_1, w_{(\bar{J}, \bar{\tau}, s)})$ for some $(\bar{J}, \bar{\tau}) \in \mathcal{H}_s \cup \mathcal{C}(\mathcal{H}_s, F)$, where \bar{u}_1 denotes the youngest vertex of $(\bar{J}, \bar{\tau})$. With the definition in (4.141) it follows that the difference between the right and left hand side of (4.153) and therefore also the difference between the right and left hand side of (4.152) is at least ε .

If the inequality in (4.152) is tight, then by (4.151) we have

$$d_\theta(J_s, u_{s1}, w_{(J_s, \tau_s, s)}) = x_{(H, \pi, \sigma), s} \quad \text{for all } s \in [r] \setminus \{\sigma\} . \quad (4.154)$$

Let Γ denote the σ -segment on the walk \mathcal{W} that contains the point $x_{(H, \pi, \sigma)}$. We fix some $s \in [r] \setminus \{\sigma\}$ and distinguish the cases whether (J_s, τ_s) is contained in \mathcal{H}_s or in $\mathcal{C}(\mathcal{H}_s, F)$.

We first consider the case that $(J_s, \tau_s) \in \mathcal{H}_s$. We claim that the s -segment Γ' on the walk \mathcal{W} that contains the point $x_{(J_s, \tau_s, s)}$ is lower on the walk \mathcal{W} than Γ : This is trivially true if one of the inequalities in (4.149) is strict. If on the other hand (4.149) holds with equality for all $t \in [r]$, then also all inequalities in (4.148) are tight, from which we conclude using (4.143) that $d(s) = d(\sigma)$, i.e., we have a tie between the colors s and σ . In this case, by the second part of Lemma 4.48, our tie-breaking rule ensures that Γ' is lower on the walk \mathcal{W} than Γ . From (4.150) and (4.154) it follows that Γ' must be the next s -segment on \mathcal{W} that is lower than Γ and that $x_{(J_s, \tau_s, s)}$ must be the starting point of Γ' . Applying Lemma 4.45 shows that $(J_s, \tau_s) \in \mathcal{H}'_s$ (note the inclusion-minimal choice of (J_s, τ_s)), completing the proof in this case.

It remains to consider the case $(J_s, \tau_s) \in \mathcal{C}(\mathcal{H}_s, F)$. Suppose for the sake of contradiction that there was some s -segment Γ' that is lower than Γ on the walk \mathcal{W} . By property (v) in Lemma 4.39 the segment Γ' can not be lowest segment of \mathcal{W} , as we would otherwise have $\mathcal{H}_s = \mathcal{S}(F)$ and therefore $\mathcal{C}(\mathcal{H}_s, F) = \emptyset$. So Γ' has an endpoint $x \in \mathbb{R}^r$ which clearly satisfies

$$x_s < x_{(H, \pi, \sigma), s} \quad (4.155)$$

and which is also the starting point of the next lower segment $\bar{\Gamma}$ (the segment $\bar{\Gamma}$ is an \bar{s} -segment for some $\bar{s} \in [r] \setminus \{s\}$). By property (ii) of Lemma 4.39 there is some $(\bar{J}, \bar{\tau}) \in \mathcal{H}_{\bar{s}}$ such that

$$x_{(\bar{J}, \bar{\tau}, \bar{s})} = x . \quad (4.156)$$

Applying the second part of Lemma 4.41 we thus obtain

$$d_\theta(J_s, u_{s1}, w_{(J_s, \tau_s, s)}) \leq x_{(\bar{J}, \bar{\tau}, \bar{s}), s} \stackrel{(4.156)}{=} x_s \stackrel{(4.155)}{<} x_{(H, \pi, \sigma), s} ,$$

contradicting (4.154). This completes the proof also in this case. \square

4.5.5. Proof of Lemma 4.33. We are now ready to prove Lemma 4.33, our main strategy invariant.

PROOF OF LEMMA 4.33. For the reader's convenience, Figure 4.6 illustrates the notations used in the first part of the proof.

Let

$$v_{\max} = v_{\max}(F, r, \theta, \alpha) := r^{v(F)/\varepsilon + (v(F)/\varepsilon + 1)(r \cdot |S(F)| + 1)(v(F) + 1) + 2} \cdot v(F) + 1 , \quad (4.157)$$

where $\varepsilon = \varepsilon(F, r, \theta, \alpha)$ is the constant guaranteed by Lemma 4.49 (and explicitly defined in (4.141)).

We argue by induction over the number of vertices of the board. For the induction base consider the board at the beginning of the game when no vertex is added yet. It is convenient for the proof to extend the statement of the lemma to (H, π) being the null graph (the graph whose vertex set is empty). For this graph we define H' to be the null graph as well. Clearly, for every $s \in [r]$, every copy of the null graph (H, π) ‘in color s ’ on the board is contained in this subgraph H' of the board, and we have $\mu_\theta(H') = 0 = \lambda_\theta(H, w_{(H, \pi, s)})$ and $v(H') = 0 \leq v_{\max}$. This shows that the second condition of the lemma holds at the beginning of the game and settles the induction base.

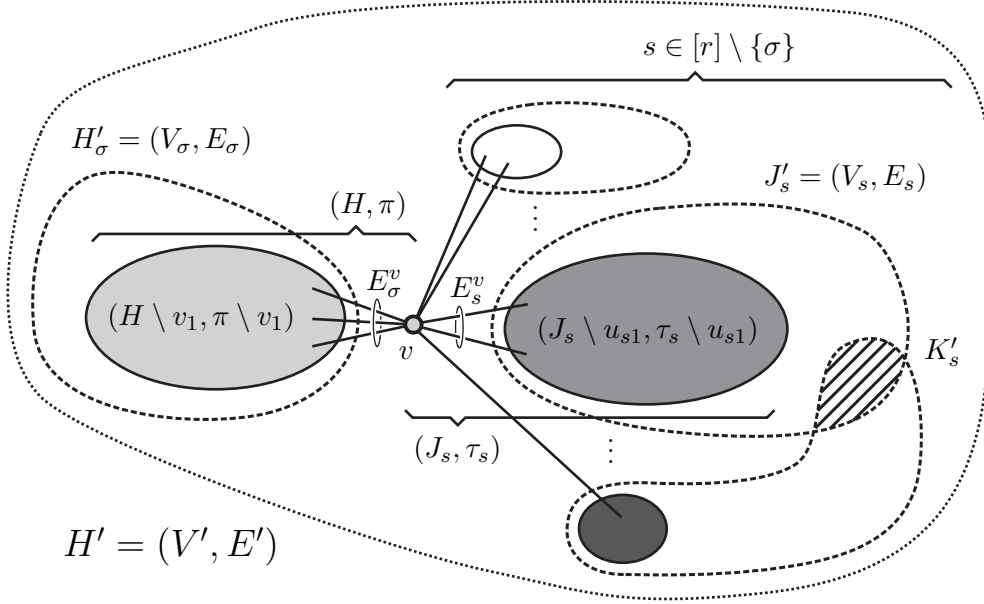


FIGURE 4.6. Notations used in the first part of the proof of Lemma 4.33.

For the induction step, let v denote the vertex added in the current step of the game, \mathcal{D}_s , $s \in [r]$, the families defined in (4.105), and σ the color the strategy $\text{PAINT}()$ assigns to the vertex v . By the first part of Lemma 4.47, we have $\mathcal{D}_{\sigma} \subseteq \mathcal{H}_{\sigma}$.

For a fixed graph $(H, \pi) \in \mathcal{D}_{\sigma}$, $\pi = (v_1, \dots, v_h)$, we consider a fixed copy of $(H \setminus v_1, \pi \setminus v_1)$ in color σ that is completed by v to a copy of (H, π) in this color. Denoting by E_{σ}^v the corresponding set of edges incident to v , we clearly have

$$|E_{\sigma}^v| = \deg_H(v_1) . \quad (4.158)$$

By induction, we know that this copy of $(H \setminus v_1, \pi \setminus v_1)$ is contained in a graph $H'_{\sigma} = (V_{\sigma}, E_{\sigma})$ with

$$\mu_{\theta}(H'_{\sigma}) \leq \lambda_{\theta}(H \setminus v_1, w_{(H, \pi, \sigma)}) \quad (4.159)$$

(recall from (4.22) that $w_{(H, \pi, \sigma)}(u) = w_{(H \setminus v_1, \pi \setminus v_1, \sigma)}(u)$ for all $u \in H \setminus v_1$) and

$$v(H'_{\sigma}) \leq v_{\max} . \quad (4.160)$$

For every $s \in [r] \setminus \{\sigma\}$, let (J_s, τ_s) be an inclusion-minimal graph from the family $\mathcal{J}_s \subseteq \mathcal{D}_s$ defined in (4.108), and let u_{s1} denotes the youngest vertex of (J_s, τ_s) . For each $s \in [r] \setminus \{\sigma\}$ we consider a fixed copy of $(J_s \setminus u_{s1}, \tau_s \setminus u_{s1})$ in color s that is completed by v to a copy of (J_s, τ_s) (the vertex v has the color $\sigma \neq s$, so the resulting copy is not monochromatic). Denoting by E_s^v the corresponding set of edges incident to v , we clearly have

$$|E_s^v| = \deg_{J_s}(u_{s1}) \quad , \quad s \in [r] \setminus \{\sigma\} . \quad (4.161)$$

By induction, those copies of $(J_s \setminus u_{s1}, \tau_s \setminus u_{s1})$ are contained in graphs $J'_s = (V_s, E_s)$ with

$$\mu_{\theta}(J'_s) \leq \lambda_{\theta}(J_s \setminus u_{s1}, w_{(J_s, \tau_s, s)}) \quad , \quad s \in [r] \setminus \{\sigma\} , \quad (4.162)$$

and

$$v(J'_s) \leq v_{\max} \quad , \quad s \in [r] \setminus \{\sigma\} . \quad (4.163)$$

Applying Lemma 4.49 shows that the graphs (H, π) and (J_s, τ_s) , $s \in [r] \setminus \{\sigma\}$, satisfy

$$\sum_{s \in [r] \setminus \{\sigma\}} (\lambda_\theta(J_s \setminus u_{s1}, w_{(J_s, \tau_s, s)}) - \deg_{J_s}(u_{s1})) \leq w_{(H, \pi, \sigma)}(v_1) . \quad (4.164)$$

If $\mu_\theta(H'_\sigma) < 0$ or $\mu_\theta(J'_s) < 0$ for some $s \in [r] \setminus \{\sigma\}$, we have found a graph K' with $\mu_\theta(K') < 0$ and $v(K') \leq v_{\max}$ (see (4.160) and (4.163)). Otherwise we have $\mu_\theta(H'_\sigma) \geq 0$ and $\mu_\theta(J'_s) \geq 0$ for all $s \in [r] \setminus \{\sigma\}$. We will argue later that this implies even stronger bounds on the number of vertices of H'_σ and J'_s , namely

$$\begin{aligned} v(H'_\sigma) &\leq (v_{\max} - 1)/r , \\ v(J'_s) &\leq (v_{\max} - 1)/r , \quad s \in [r] \setminus \{\sigma\} . \end{aligned} \quad (4.165)$$

We define the graph $H' = (V', E')$ as

$$\begin{aligned} V' &:= \{v\} \cup \bigcup_{s \in [r]} V_s , \\ E' &:= \bigcup_{s \in [r]} (E_s \cup E_s^v) \end{aligned} \quad (4.166)$$

(see Figure 4.6). This graph clearly contains the copy of (H, π) in color σ we are considering.

Furthermore, we define for $2 \leq s \leq r$ the graphs

$$K'_s := \left(V_s \cap \bigcup_{1 \leq t \leq s-1} V_t, E_s \cap \bigcup_{1 \leq t \leq s-1} E_t \right) . \quad (4.167)$$

From (4.165) and (4.167) we conclude that $v(K'_s) \leq (v_{\max} - 1)/r \leq v_{\max}$, $2 \leq s \leq r$. Therefore, if $\mu_\theta(K'_s) < 0$ for some $2 \leq s \leq r$, then we have found a graph K' with $\mu_\theta(K') < 0$ and $v(K') \leq v_{\max}$. Otherwise we have

$$\mu_\theta(K'_s) \geq 0 , \quad 2 \leq s \leq r . \quad (4.168)$$

With (4.158) and (4.161) we obtain from (4.166) and (4.167) that

$$\begin{aligned} v(H') &= 1 + v(H'_\sigma) + \sum_{s \in [r] \setminus \{\sigma\}} v(J'_s) - \sum_{2 \leq s \leq r} v(K'_s) , \\ e(H') &= e(H'_\sigma) + \deg_H(v_1) + \sum_{s \in [r] \setminus \{\sigma\}} (e(J'_s) + \deg_{J_s}(u_{s1})) - \sum_{2 \leq s \leq r} e(K'_s) . \end{aligned} \quad (4.169)$$

Combining our previous observations yields

$$\begin{aligned} \mu_\theta(H') &\stackrel{(4.5), (4.169)}{=} 1 + \mu_\theta(H'_\sigma) - \deg_H(v_1) \cdot \theta + \sum_{s \in [r] \setminus \{\sigma\}} (\mu_\theta(J'_s) - \deg_{J_s}(u_{s1}) \cdot \theta) - \sum_{2 \leq s \leq r} \mu_\theta(K'_s) \\ &\stackrel{(4.159), (4.162), (4.168)}{\leq} 1 + \lambda_\theta(H \setminus v_1, w_{(H, \pi, \sigma)}) - \deg_H(v_1) \cdot \theta \\ &\quad + \underbrace{\sum_{s \in [r] \setminus \{\sigma\}} (\lambda_\theta(J_s \setminus u_{s1}, w_{(J_s, \tau_s, s)}) - \deg_{J_s}(u_{s1}) \cdot \theta)}_{\stackrel{(4.164)}{\leq} w_{(H, \pi, \sigma)}(v_1)} \\ &\leq \lambda_\theta(H \setminus v_1, w_{(H, \pi, \sigma)}) + 1 + w_{(H, \pi, \sigma)}(v_1) - \deg_H(v_1) \cdot \theta \stackrel{(4.19)}{=} \lambda_\theta(H, w_{(H, \pi, \sigma)}) \end{aligned} \quad (4.170)$$

which proves (4.110). From (4.165) and (4.166) we conclude that $v(H') \leq v_{\max}$.

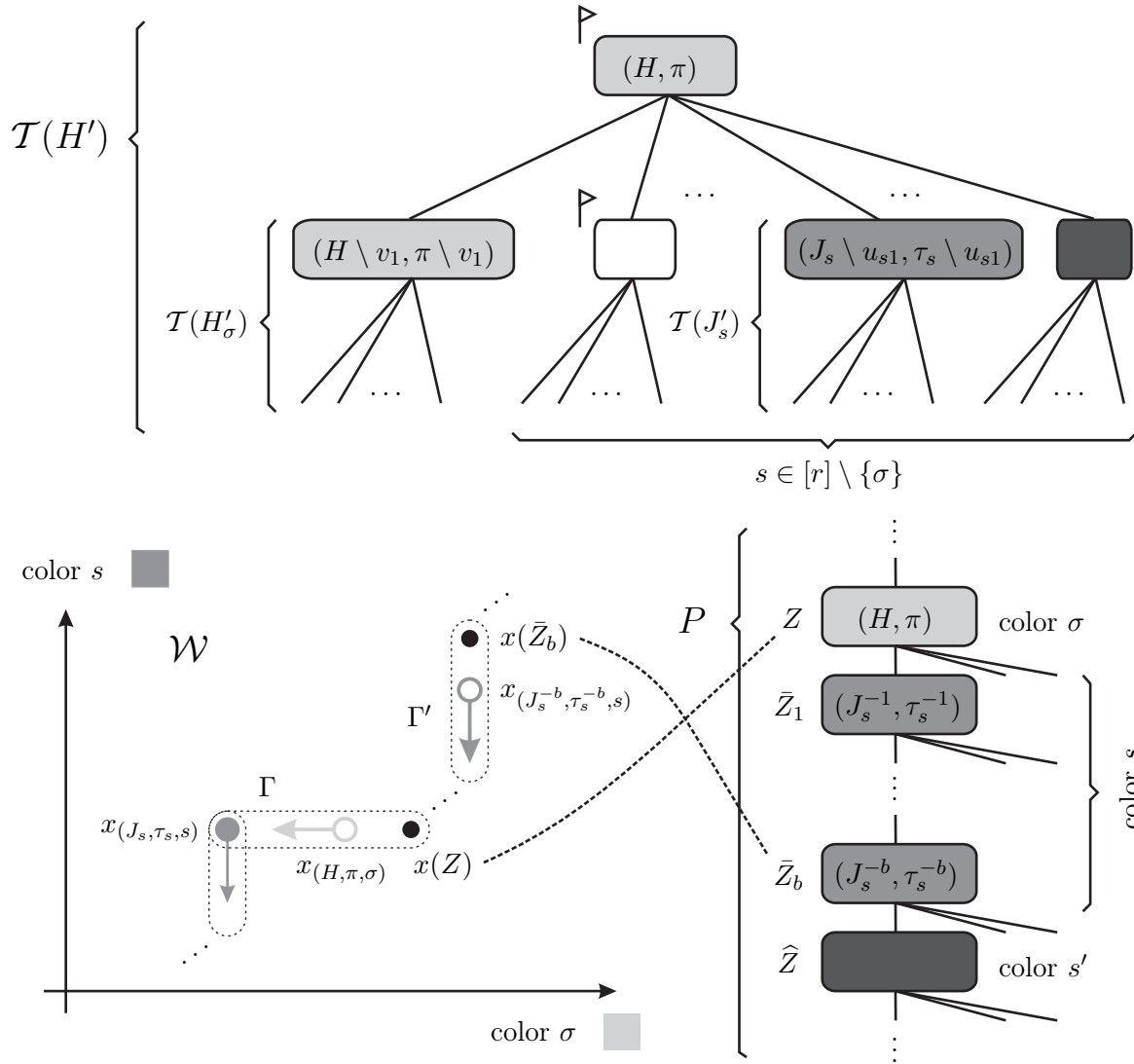


FIGURE 4.7. Notations used in the second part of the proof of Lemma 4.33.

It remains to show (4.165), i.e. that for every graph H' as defined in (4.166) with $\mu_\theta(H') \geq 0$ we have $v(H') \leq (v_{\max} - 1)/r$. For the reader's convenience, the notations used in this part of the proof are illustrated in Figure 4.7.

In the above argument we constructed the graph H' containing the copy of (H, π) in color σ inductively from the graph H'_σ containing the copy of $(H \setminus v_1, \pi \setminus v_1)$ in color σ and the graphs J'_s containing the copies of $(J_s \setminus u_{s1}, \tau_s \setminus u_{s1})$ in color s , $s \in [r] \setminus \{\sigma\}$. We associate this inductive construction with a node-colored rooted tree $\mathcal{T}(H')$, some of whose non-leaf nodes receive a special marking (we refer to it as a *flag*), as follows (see the upper part of Figure 4.7): The nodes of $\mathcal{T}(H')$ correspond to monochromatic copies of graphs from $\mathcal{S}(F)$ on the board (the same copy may appear as a node multiple times). If (H, π) is the null graph ‘in color σ ’ (recall that in this case H' is the null graph as well), $\mathcal{T}(H')$ consists only of this copy of (H, π) as an isolated node which receives the color σ . Otherwise $\mathcal{T}(H')$ consists of the copy of (H, π) as the root node joined to r subtrees, $\mathcal{T}(H'_\sigma)$ and $\mathcal{T}(J'_s)$ for all $s \in [r] \setminus \{\sigma\}$. The root node receives the color σ , and it is flagged if and only if the instance of the inequality (4.164) corresponding to this induction step is strict. Note that the tree $\mathcal{T}(H')$ captures only the logical structure of the

inductive history of H' . Overlappings (captured by the graphs K'_s , $2 \leq s \leq r$) are completely neglected.

Every flagged node of $\mathcal{T}(H')$ corresponds to a strict inequality in (4.164). In this case, inequality (4.170) is strict as well, with a difference of at least ε between the right and left hand side, where ε is the constant guaranteed by Lemma 4.49. Consequently, each flagged node contributes a term of $-\varepsilon$ to the right hand side of (4.170) in the corresponding induction step. Accumulating these terms along the induction yields that

$$\mu_\theta(H') \leq \lambda_\theta(H, w_{(H, \pi, \sigma)}) - f(H') \cdot \varepsilon, \quad (4.171)$$

where $f(H')$ denotes the number of flagged nodes in $\mathcal{T}(H')$.

By Lemma 4.13 we have $\lambda_\theta(H, w_{(H, \pi, \sigma)}) \leq v(F)$. Thus if $\mu_\theta(H') \geq 0$, then by (4.171) the tree $\mathcal{T}(H')$ has at most $\lambda_\theta(H, w_{(H, \pi, \sigma)})/\varepsilon \leq v(F)/\varepsilon$ many flagged nodes. We will show that this bound on the number of flagged nodes of $\mathcal{T}(H')$ implies the claimed bound of $(v_{\max} - 1)/r$ on the number of vertices of H' . To that end, we first show that the length of any descending path in $\mathcal{T}(H')$ that consists only of non-flagged nodes is bounded by a constant depending only on F and r . We will do so by showing that any descending sequence of non-flagged nodes in $\mathcal{T}(H')$ corresponds to an ascending sequence of points on the walk \mathcal{W} defined in Lemma 4.39.

Specifically, we assign to every non-leaf node Z in $\mathcal{T}(H')$ a point $x(Z) \in \mathbb{R}^r$ on the walk \mathcal{W} as follows: Let σ denote the color of Z and (H, π) the graph for which a copy in color σ is represented by the node Z . We define $x(Z)$ as the starting point of the σ -segment that contains the point $x_{(H, \pi, \sigma)}$ on the walk \mathcal{W} .

Consider a descending sequence $Z, \bar{Z}_1, \dots, \bar{Z}_b, \hat{Z}$ of consecutive non-flagged nodes in $\mathcal{T}(H')$, where Z has some color $\sigma \in [r]$, all nodes $\bar{Z}_1, \dots, \bar{Z}_b$ have the same color $s \in [r] \setminus \{\sigma\}$ and \hat{Z} has some color $s' \in [r] \setminus \{s\}$ (see the lower right part of Figure 4.7). Let (H, π) be the graph for which a copy in color σ is represented by the node Z , and (J_s, τ_s) , $\tau_s = (u_{s1}, \dots, u_{sc})$, the graph for which a copy of $(J_s \setminus u_{s1}, \tau_s \setminus u_{s1})$ in color s is represented by the node \bar{Z}_1 . Using these definitions, clearly the nodes $\bar{Z}_1, \dots, \bar{Z}_b$ represent a sequence of nested copies of (J_s^{-a}, τ_s^{-a}) , $a = 1, \dots, b$, in color s , where $(J_s^{-a}, \tau_s^{-a}) := (J_s \setminus \{u_{s1}, \dots, u_{sa}\}, \tau_s \setminus \{u_{s1}, \dots, u_{sa}\})$. Let Γ denote the σ -segment of the walk \mathcal{W} that contains the point $x_{(H, \pi, \sigma)}$ and Γ' the s -segment that contains the point $x_{(J_s^{-b}, \tau_s^{-b}, s)}$ (the starting points of these segments are $x(Z)$ and $x(\bar{Z}_b)$, respectively; see the lower left part of Figure 4.7). As the node Z is not flagged, the corresponding instance of the inequality (4.164) is tight. Hence, by Lemma 4.49, we have $(J_s, \tau_s) \in \mathcal{H}'_s \cup \mathcal{C}(\mathcal{H}_s, F)$, and if $(J_s, \tau_s) \in \mathcal{H}'_s$, then $x_{(J_s, \tau_s, s)}$ is the starting point of the next s -segment on \mathcal{W} that is lower than Γ , whereas if $(J_s, \tau_s) \in \mathcal{C}(\mathcal{H}_s, F)$, then there is no s -segment on \mathcal{W} lower than Γ . In the first case we apply Lemma 4.46 to conclude that Γ is lower than Γ' on the walk \mathcal{W} , in the second case this conclusion is trivially true. It follows that in any case $x(Z)$ is lower on the walk than $x(\bar{Z}_b)$.

Let now P be a descending path in $\mathcal{T}(H')$ that consists only of non-flagged nodes, and recall that our goal is to bound the length of P by a constant depending only on F and r . We refer to a maximal sequence of consecutive nodes of the same color along P as a *section* in this color (in the above argument, $\bar{Z}_1, \dots, \bar{Z}_b$ is a section in color s). Moreover, we call a section of P *internal* if it is neither the first nor the last section on P . By the argument above, for the last node \bar{Z} of an internal section and the last node Z of the preceding section on P , $x(\bar{Z})$ is higher on the walk \mathcal{W} than $x(Z)$. As the walk \mathcal{W} contains at most $r \cdot |\mathcal{S}(F)|$ different elements, the path P can have at most $r \cdot |\mathcal{S}(F)| - 1$ internal sections, and at most $r \cdot |\mathcal{S}(F)| + 1$ sections in total (including the

first and last section). As each section consists of at most $v(F) + 1$ nodes, P consists of at most

$$(r \cdot |\mathcal{S}(F)| + 1)(v(F) + 1) =: p \quad (4.172)$$

nodes, proving that the length of P is indeed bounded by a constant depending only on F and r .

Since in total there are at most $v(F)/\varepsilon$ many flagged nodes in $\mathcal{T}(H')$, the depth of $\mathcal{T}(H')$ is bounded by

$$v(F)/\varepsilon + (v(F)/\varepsilon + 1)p =: t, \quad (4.173)$$

where the first term bounds the number of flagged nodes and the second term the number of non-flagged nodes. Consequently, we have

$$v(\mathcal{T}(H')) \leq 1 + r + r^2 + \dots + r^t \leq r^{t+1}.$$

Observing that every node of $\mathcal{T}(H')$ corresponds to at most $v(F)$ vertices of H' , we finally obtain that

$$v(H') \leq r^{t+1} \cdot v(F) \stackrel{(4.157), (4.172), (4.173)}{=} (v_{\max} - 1)/r.$$

This justifies (4.165) and concludes the proof. \square

4.6. Proof of Theorem 4.3

We denote the board of the probabilistic process after i steps by G_i , where $0 \leq i \leq n$. We take the alternative view mentioned in Section 4.1.1, in which the random edges leading from a newly added vertex to previous vertices are generated at the moment this vertex is revealed instead of at the beginning of the process. (Recall that each edge is inserted with probability $p = p(n)$ independently from all other edges.) Thus G_i is an r -colored graph on i vertices, and the underlying uncolored graph is distributed as $G_{i,p}$.

Recall that all our asymptotic results are with respect to n , the number of vertices of G_n or $G_{n,p}$.

4.6.1. Lower bound. The crucial ingredient for the proof of the lower bound part of Theorem 4.3 is Lemma 4.33 from Section 4.5.

PROOF OF THEOREM 4.3 (LOWER BOUND). Let $\theta^* = \theta^*(F, r)$ be defined as in Theorem 4.7, and let $\alpha^* = \alpha^*(F, r)$ be a sequence from the set $[r]^{r \cdot |\mathcal{S}(F)|}$ for which the minimum in the definition of $\Lambda_{\theta^*}(F, r)$ in (4.23) is attained. We show that the strategy $\text{PAINT}(F, r, \theta^*, \alpha^*)$ defined in Section 4.5.1 a.a.s. avoids F for all n steps of the process if

$$p \ll p_0(F, r, n) = n^{-1/m_1^*(F, r)} \stackrel{(4.15)}{=} n^{-\theta^*}. \quad (4.174)$$

By the choice of α^* and the definition in (4.23) we have that for all colors $s \in [r]$ and all vertex orderings $\pi \in \Pi(V(F))$ there is a subgraph $H \subseteq F$ such that

$$\lambda_{\theta^*}(H, w_{(H, \pi|_{H, s})}) \leq \Lambda_{\theta^*}(F, r) \stackrel{(4.16)}{=} 0. \quad (4.175)$$

According to Lemma 4.33 we then have for each such $(H, \pi|_H)$: if G_n contains a copy of $(H, \pi|_H)$ in color s , then it contains a graph K' with $v(K') \leq v_{\max}$ and $\mu_{\theta^*}(K') < 0$, or a graph H' with $v(H') \leq v_{\max}$ and

$$\mu_{\theta^*}(H') \stackrel{(4.110)}{\leq} \lambda_{\theta^*}(H, w_{(H, \pi|_{H, s})}) \stackrel{(4.175)}{\leq} 0.$$

This yields a family $\mathcal{W} = \mathcal{W}(F, \pi, s, r)$ of graphs W' satisfying $\mu_{\theta^*}(W') \leq 0$ and $v(W') \leq v_{\max}$ such that, deterministically, G_n contains a graph from \mathcal{W} if it contains a copy of (F, π) in color s . It follows that G_n contains a graph from

$$\mathcal{W}^* = \mathcal{W}^*(F, r) := \bigcup_{\substack{s \in [r] \\ \pi \in \Pi(V(F))}} \mathcal{W}(F, \pi, s, r)$$

if it contains a monochromatic copy of F .

Moreover, since no graph in \mathcal{W}^* has more than $v_{\max}(F, r, \theta^*(F, r), \alpha^*(F, r))$ vertices, the size of \mathcal{W}^* is bounded by a constant only depending on F and r . By the definition of $\mu_{\theta^*}(\cdot)$ in (4.5) and the fact that $\mu_{\theta^*}(W') \leq 0$ for all $W' \in \mathcal{W}^*$, the expected number of copies of the (underlying uncolored) graphs from \mathcal{W}^* in $G_{n,p}$ is of order

$$\sum_{W' \in \mathcal{W}^*} n^{v(W')} p^{e(W')} \stackrel{(4.174)}{\ll} \sum_{W' \in \mathcal{W}^*} n^{\mu_{\theta^*}(W')} \leq |\mathcal{W}^*| \cdot n^0 = \Theta(1) .$$

It follows with Markov's inequality that a.a.s. $G_{n,p}$ contains no copy of any of the (underlying uncolored) graphs from \mathcal{W}^* . Consequently, a.a.s. G_n contains no copy of any of the graphs from \mathcal{W}^* and hence no monochromatic copy of F . This proves the claimed lower bound on the threshold of the probabilistic process.

To prove the second part of Theorem 4.3 it suffices to show that the strategy $\text{PAINT}(F, r, \theta^*, \alpha^*)$ is an optimal strategy for Painter in the deterministic two-player game, i.e., that it is a winning strategy in the game with density restriction d for any $d < m_1^*(F, r) = 1/\theta^*$ (we have already argued that this strategy can be implemented as a polynomial-time algorithm in Section 4.1.5 and Remark 4.32). Fix some $0 < d < 1/\theta^*$ and define $\theta := 1/d > \theta^*$. Suppose Painter plays according to the strategy $\text{PAINT}(F, r, \theta^*, \alpha^*)$ in the game with density restriction d and suppose for the sake of contradiction that the game ends with a monochromatic copy of F . Then as before it follows from Lemma 4.33 that the board contains a graph K' with

$$\mu_{\theta^*}(K') < 0 \tag{4.176}$$

or a graph H' with

$$\mu_{\theta^*}(H') \leq 0 \tag{4.177}$$

(note that H' contains at least one vertex and as a consequence of (4.177) and the definition in (4.5) also at least one edge; similarly, K' contains at least one edge as a consequence of (4.176)). Using that $\theta > \theta^*$ it follows from (4.176), (4.177) and the definition in (4.5) that in any case the board contains a graph W' (with $v(W') \geq 1$) satisfying $\mu_{\theta}(W') < 0$, or equivalently, $e(W')/v(W') > 1/\theta = d$, violating the given density restriction. \square

4.6.2. Upper bound. As in the proof of Lemma 4.6 we identify Builder's strategies in the deterministic two-player game with r colors with finite r -ary rooted trees, where each node at depth k of such a tree is an r -colored graph on k vertices, representing the board after the k -th step of the game.

Note that in this formalization, a given tree \mathcal{T} represents a generic strategy for Builder (in the deterministic game with r colors) that may or may not satisfy a given density restriction d , and that can be thought of as a strategy for the ' F -avoidance' game for any given graph F . We say that \mathcal{T} is a *winning strategy* for Builder in a specific F -avoidance game if and only if every leaf of \mathcal{T} contains a monochromatic copy of F . We say that a Builder strategy \mathcal{T} is a *legal strategy* in the game with density restriction d if and only if $e(H)/v(H) \leq d$ for every subgraph H of every node B in \mathcal{T} .

When we say that G_i , the board of the probabilistic process after i steps, contains a copy of some r -colored graph B (e.g. a node of some Builder strategy \mathcal{T}) we mean that there is a subgraph of G_i that is isomorphic to B as a *colored* graph.

The upper bound part of Theorem 4.3 is an immediate consequence of the following lemma.

LEMMA 4.50 (Random process reproduces Builder strategy). *Let $r \geq 2$ be a fixed integer, let $d > 0$ be a fixed real number, and let \mathcal{T} represent an arbitrary legal strategy for Builder in the deterministic game with r colors and density restriction d .*

If $p \gg n^{-1/d}$, then regardless of the online coloring strategy employed, a.a.s. G_n contains a copy of a leaf of \mathcal{T} .

PROOF OF THEOREM 4.3 (UPPER BOUND). By Theorem 4.2 there exists a legal winning strategy \mathcal{T} for Builder in the deterministic F -avoidance game with r colors and density restriction $d = m_1^*(F, r)$. As each leaf of \mathcal{T} contains a monochromatic copy of F , applying Lemma 4.50 to \mathcal{T} yields that if $p \gg p_0(F, r, n) = n^{-1/m_1^*(F, r)}$, then a.a.s. G_n contains a monochromatic copy of F , regardless of the online coloring strategy employed, which is exactly the upper bound statement of Theorem 4.3. \square

In order to prove Lemma 4.50, we shall show the following more technical statement by induction on k .

LEMMA 4.51 (Random process reproduces Builder strategy step by step). *Let $r \geq 2$ be a fixed integer, let $d > 0$ be a fixed real number, and let \mathcal{T} represent an arbitrary legal strategy for Builder in the deterministic game with r colors and density restriction d .*

If $p \gg n^{-1/d}$, then for any integer $k \geq 1$ the following is true. Regardless of the online coloring strategy employed, a.a.s. one of the following two statements holds:

- G_n contains a copy of a leaf of \mathcal{T} , or
- there is a node B at depth k in \mathcal{T} such that G_n contains $\Omega(n^{v(B)}p^{e(B)})$ many copies of B .

The second property of Lemma 4.51 is meaningful since, due to the assumption that \mathcal{T} is a legal strategy for Builder in the game with density restriction d , we have

$$e(B)/v(B) \leq m(B) \leq d ,$$

which yields with $p \gg n^{-1/d} \geq n^{-v(B)/e(B)}$ that

$$n^{v(B)}p^{e(B)} \gg 1 .$$

PROOF OF LEMMA 4.50. Set $k := \text{depth}(\mathcal{T}) + 1$ in Lemma 4.51. \square

It remains to prove Lemma 4.51.

PROOF OF LEMMA 4.51. We proceed by induction on k . For the induction base $k = 1$, note that each of the r nodes B at depth 1 in \mathcal{T} consists simply of an isolated vertex, colored in one of the r available colors. Clearly, G_n contains at least $n/r = \Omega(n)$ copies of one of these by the pigeonhole principle.

For the induction step we employ a two-round approach. That is, we divide the process into two rounds of equal length $n/2$ (w.l.o.g. we assume n to be even) and analyze these two rounds separately. Denoting the vertices added throughout the process by v_1, \dots, v_n , the first round consists of adding the vertices $v_1, \dots, v_{n/2}$ together with the corresponding random edges. At

the end of the first round, we thus obtain a graph $G_{n/2}$, to which we can apply the induction hypothesis and some standard random graph arguments. The second round consists of adding the vertices $v_{n/2+1}, \dots, v_n$ (together with the corresponding random edges). Using a variance calculation, we show that conditional on a ‘good’ first round, the second round turns out as claimed. (In fact, our argument does not make use of any edges added between vertices of the set $\{v_{n/2+1}, \dots, v_n\}$.)

By the induction hypothesis, if the graph $G_{n/2}$ does not contain a copy of a leaf of \mathcal{T} (in which case we are done), a.a.s. it contains a family of

$$M \asymp n^{v(B^\circ)} p^{e(B^\circ)} \quad (4.178)$$

copies of some graph B° corresponding to a non-leaf node at depth $k-1$ in \mathcal{T} . We label these copies B_i° , $1 \leq i \leq M$. Let B denote the graph obtained from B° by adding a new vertex v to it together with edges connecting v to B° as prescribed by Builder’s next move specified by \mathcal{T} (so v is uncolored in B , but assigning it one of the r available colors yields exactly one of the children of B° in \mathcal{T}).

For each copy B_i° , $1 \leq i \leq M$, and each vertex v_ℓ , $n/2 + 1 \leq \ell \leq n$, we fix a set $E_{i,\ell}$ of $\deg_B(v)$ many vertex pairs such that if the elements of $E_{i,\ell}$ are actual edges generated in the second round, then v_ℓ together with those edges completes B_i° to a copy of B . We let $Z_{i,\ell}$ be the indicator variable for the event that the elements of $E_{i,\ell}$ are generated as edges in the second round. Let

$$Z := \sum_{i=1}^M \sum_{\ell=n/2+1}^n Z_{i,\ell} ,$$

and note that by the pigeonhole principle at least Z/r many copies of one of the children of B° in \mathcal{T} are created. Thus the second condition of the lemma is satisfied if we show that a.a.s.

$$Z \asymp n^{v(B)} p^{e(B)} . \quad (4.179)$$

We will do so by the methods of first and second moment.

We clearly have

$$\Pr[Z_{i,\ell} = 1] = p^{|E_{i,\ell}|} = p^{\deg_B(v)} ,$$

and, conditioning on the first round satisfying the induction hypothesis,

$$\mathbb{E}[Z] = M \cdot n/2 \cdot p^{\deg_B(v)} \stackrel{(4.178)}{\asymp} n^{v(B)} p^{e(B)} . \quad (4.180)$$

In the following, we slightly abuse notation and write B also for the *uncolored* graph underlying B . Let \mathcal{D} denote the family of all (uncolored) graphs D that can be constructed by considering the union of two copies of B intersecting in at least two vertices, one of which must be the vertex v (we again slightly abuse notation in the following and refer to the corresponding vertex in each such graph D as v). For any $D \in \mathcal{D}$, we denote by D° the graph obtained by removing v from D .

To calculate the variance of Z , observe that the variables $Z_{i,\ell}$ and $Z_{j,\ell'}$ are independent whenever $\ell \neq \ell'$ or $B_i^\circ \cap B_j^\circ = \emptyset$. Hence such pairs can be omitted, and we have

$$\begin{aligned}
\text{Var}[Z] &= \sum_{i,j=1}^M \sum_{\ell,\ell'=n/2+1}^n (\mathbb{E}[Z_{i,\ell}Z_{j,\ell'}] - \mathbb{E}[Z_{i,\ell}]\mathbb{E}[Z_{j,\ell'}]) \\
&\leq \sum_{\substack{i,j=1,\dots,M: \\ B_i^\circ \cap B_j^\circ \neq \emptyset}} \sum_{\ell=n/2+1}^n \Pr[Z_{i,\ell} = 1 \wedge Z_{j,\ell} = 1] \\
&= \sum_{D \in \mathcal{D}} \sum_{\substack{i,j=1,\dots,M: \\ B_i^\circ \cap B_j^\circ = D^\circ}} \sum_{\ell=n/2+1}^n p^{|E_{i,\ell} \cup E_{j,\ell}|} \\
&\leq \sum_{D \in \mathcal{D}} M_{D^\circ} \cdot \Theta(1) \cdot np^{\deg_D(v)} ,
\end{aligned} \tag{4.181}$$

where M_{D° denotes the total number of copies of D° in (the underlying uncolored graph of) $G_{n/2}$. By definition of \mathcal{D} , each $D \in \mathcal{D}$ satisfies

$$\begin{aligned}
v(D^\circ) &= 2v(B) - v(J) - 1 , \\
e(D^\circ) &= 2e(B) - e(J) - \deg_D(v)
\end{aligned} \tag{4.182}$$

for some subgraph $J \subseteq B$. Moreover, since we assumed that \mathcal{T} is a legal strategy for Builder in the game with density restriction d , we have

$$e(J)/v(J) \leq m(B) \leq d ,$$

which yields with $p \gg n^{-1/d} \geq n^{-v(J)/e(J)}$ that

$$n^{v(J)}p^{e(J)} \gg 1 . \tag{4.183}$$

Thus the expected number of copies of D° in (the underlying uncolored graph of) $G_{n/2}$ is

$$\binom{n}{v(D^\circ)} \cdot \Theta(1) \cdot p^{e(D^\circ)} \stackrel{(4.182)}{\asymp} n^{2v(B)-v(J)-1} p^{2e(B)-e(J)-\deg_D(v)} \stackrel{(4.183)}{\ll} n^{2v(B)-1} p^{2e(B)-\deg_D(v)} .$$

and Markov's inequality implies that

$$M_{D^\circ} \ll n^{2v(B)-1} p^{2e(B)-\deg_D(v)} \tag{4.184}$$

a.a.s. As moreover the number of graphs in \mathcal{D} is bounded by a constant depending only on \mathcal{T} , a.a.s. (4.184) holds for all $D \in \mathcal{D}$ simultaneously.

Thus, conditioning on the first round satisfying the induction hypothesis (cf. (4.178)), and (4.184) for all $D \in \mathcal{D}$, we obtain from (4.181) that

$$\text{Var}[Z] \stackrel{(4.184)}{\ll} \sum_{D \in \mathcal{D}} \left(n^{v(B)} p^{e(B)} \right)^2 \stackrel{(4.180)}{\asymp} \mathbb{E}[Z]^2 .$$

Chebyshev's inequality now yields that a.a.s. the second round satisfies (4.179). This implies that there is at least the claimed number of copies of one of the children of B° in G_n , as discussed. \square

CHAPTER 5

The path-avoidance vertex-coloring game

In this chapter we present our results for the F -avoidance vertex-coloring game discussed in the previous chapter for the case where $F = P_\ell$ is a path on ℓ vertices. These results are joint work with Reto Spöhel and they have been published in [MS11].

5.1. Introduction

The algorithm presented in the previous chapter to compute the online vertex-Ramsey density $m_1^*(F, r)$ for general F and r is rather complex and gives no hint as to how this quantity behaves for natural graph families (recall the definition of $m_1^*(F, r)$ via the deterministic two-player game with density restriction in (4.3)). However, for a large class of graphs F , a simple closed formula for the parameter $m_1^*(F, r)$ follows from the results in [MS10]: As already mentioned in Section 4.1.6, the greedy strategy defined in Section 4.1.1 guarantees a lower bound of $m_1^*(F, r) \geq \overline{m}_1(F, r)$ with $\overline{m}_1(F, r)$ defined in (4.1). Furthermore, if F has an induced subgraph $F^\circ \subsetneq F$ on $v(F) - 1$ vertices satisfying (4.2), then the greedy strategy is optimal, i.e., we have $m_1^*(F, r) = \overline{m}_1(F, r)$.

This condition is indeed satisfied for the cases where F is a clique K_ℓ , a cycle C_ℓ , a complete bipartite graph $K_{s,t}$, a d -dimensional hypercube Q_d , a wheel W_ℓ with ℓ spokes, or a star S_ℓ with ℓ rays. Using (4.1) it follows that in all those cases the online vertex-Ramsey density is given by $m_1^*(F, r) = \frac{e(F)(1-v(F)^{-r})}{v(F)-1}$ (in those cases the sequence of subgraphs $H_1, \dots, H_r \subseteq F$ considered by the greedy strategy is simply $H_1 = \dots = H_r = F$), i.e., we have

$$\begin{aligned} m_1^*(K_\ell, r) &= \frac{\ell(1-\ell^{-r})}{2} \quad , & m_1^*(C_\ell, r) &= \frac{\ell(1-\ell^{-r})}{\ell-1} \quad , \\ m_1^*(K_{s,t}, r) &= \frac{st(1-(s+t)^{-r})}{s+t-1} \quad , & m_1^*(Q_d, r) &= \frac{d2^{d-1}(1-2^{-dr})}{2^d-1} \quad , \\ m_1^*(W_\ell, r) &= 2(1-(\ell+1)^{-r}) \quad , & m_1^*(S_\ell, r) &= 1-(\ell+1)^{-r} \quad . \end{aligned} \tag{5.1}$$

In this chapter we show that the situation is much more complicated in the innocent-looking case where $F = P_\ell$ is a path on ℓ vertices. As it turns out, for this family of graphs the greedy strategy fails quite badly, and the parameter $m_1^*(P_\ell, r)$ exhibits a much more complex behavior than one might expect in view of the previous examples, giving some evidence that a general closed formula for the parameter $m_1^*(F, r)$ does not exist.

5.1.1. Forests. We first introduce a more convenient way to express $m_1^*(F, r)$ for the case where F is an arbitrary forest. Note that a density restriction of the form $d = (k-1)/k$ for some integer $k \geq 2$ is equivalent to requiring that Builder creates no cycles and no components (=trees) with more than k vertices. We call this game the *F -avoidance game with r colors and tree size restriction k* .

It is not hard to see that for any forest F and any integer r , Builder has a winning strategy in the F -avoidance game with r colors and tree size restriction k for large enough k . We denote by $k^*(F, r)$ the smallest such integer k for which Builder has a winning strategy in this game.

ℓ	2, ..., 27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$k^*(P_\ell, 2)$	$2^2, \dots, 27^2$	791	841	902	961	1040	1089	1156	1225	1323	1376	1449	1521	1641	1699	1796	1856	1991	2057
$k^*(P_\ell, 2) - \ell^2$	0	7	0	2	0	16	0	0	0	27	7	5	0	41	18	32	7	55	32

TABLE 1. Exact values of $k^*(P_\ell, 2)$ for $\ell \leq 45$.

Noting that for any forest F we have

$$m_1^*(F, r) = \frac{k^*(F, r) - 1}{k^*(F, r)} , \quad (5.2)$$

we obtain the following corollary to Theorem 4.3.

COROLLARY 5.1. *For any fixed forest F with at least one edge and any fixed integer $r \geq 2$, the threshold for finding an r -coloring of $G_{n,p}$ that is valid with respect to F online is*

$$p_0(F, r, n) = n^{-1-1/(k^*(F, r)-1)} .$$

For the rest of this chapter, we restrict our attention to forests and focus on the parameter $k^*(F, r)$. From the greedy lower bound $m_1^*(F, r) \geq \overline{m}_1(F, r)$ and the definitions in (4.1) and (5.2) it follows that for any tree F and any integer $r \geq 2$ the greedy strategy guarantees a lower bound of $k^*(F, r) \geq v(F)^r$ (for the sake of completeness we give the argument explicitly in Lemma 5.6 below). Note that if F is a tree, the sequence of subgraphs $H_1, \dots, H_r \subseteq F$ considered by the greedy strategy is always $H_1 = \dots = H_r = F$, a fact we will from now on not mention explicitly anymore when referring to the greedy strategy.

5.1.2. Our results. For the rest of this introduction we focus on the case where $F = P_\ell$ and $r = 2$ colors are available. Table 1 shows the exact values of $k^*(P_\ell, 2)$ for $\ell \leq 45$. These were determined with the help of a computer, based on the insights of this chapter and using some extra tweaks to improve running times (see Section 5.3.3 below). The bottom row shows the difference $k^*(P_\ell, 2) - \ell^2$, i.e., by how much optimal Painter strategies can improve on the greedy lower bound $v(P_\ell)^2 = \ell^2$.

In stark contrast to the formulas in (5.1), the values in Table 1 (and the corresponding optimal Painter strategies) exhibit a rather irregular behavior and seem to follow no discernible pattern. In particular, the greedy strategy turns out to be optimal for $\ell \in \{2, \dots, 27\} \cup \{29, 31, 33, 34, 35, 39\}$, but not for the other values of $\ell \leq 45$. (In fact, for all $\ell \geq 46$ we have $k^*(P_\ell, 2) > \ell^2$, so the listed values are the only ones for which the greedy strategy is optimal.)

These numerical findings raise the question whether and by how much optimal Painter strategies can improve on the greedy lower bound asymptotically as $\ell \rightarrow \infty$. Our main result of this chapter shows that there exist Painter strategies that improve on the greedy lower bound by a factor polynomial in ℓ , and that no superpolynomial improvement is possible (cf. Theorem 3.6 and the remarks thereafter for the edge-coloring variant of the problem studied here).

THEOREM 5.2. *We have*

$$\Theta(\ell^{2.01}) \leq k^*(P_\ell, 2) \leq \Theta(\ell^{2.59}) .$$

We prove the bounds in Theorem 5.2 by analyzing a more general asymmetric version of the path-avoidance game, where Painter's goal is to avoid a path on ℓ vertices in color 1, and a path on c vertices in color 2. We denote by $k^*(P_\ell, P_c)$ the smallest integer k for which Builder has a winning strategy in this asymmetric (P_ℓ, P_c) -avoidance game with tree size restriction k .

In the following we present our results for this asymmetric game. The next theorem shows in particular that for any fixed value of c , the parameter $k^*(P_\ell, P_c)$ grows linearly with ℓ .

THEOREM 5.3. *For any $c \geq 1$ there is a constant $\delta(c)$ such that for any $\ell \geq 1$ we have*

$$k^*(P_\ell, P_c) = (\delta(c) - o(1)) \cdot \ell ,$$

where $o(1)$ stands for a non-negative function of c and ℓ that tends to 0 for c fixed and $\ell \rightarrow \infty$.

Note that Theorem 5.3 does *not* imply that $k^*(P_\ell, 2) = (\delta(\ell) - o(1)) \cdot \ell$ as $\ell \rightarrow \infty$.

Similarly to the symmetric game, the greedy strategy guarantees a lower bound of $k^*(P_\ell, P_c) \geq c \cdot \ell$, and it is not hard to see that this is an exact equality for $c \in \{1, 2, 3\}$ (i.e., the greedy strategy is optimal, see Lemmas 5.6 and 5.7 below). Thus the constant $\delta(c)$ from Theorem 5.3 satisfies $\delta(c) = c$ for $c \in \{1, 2, 3\}$. The next theorem states the exact value of $\delta(c)$ for $c \in \{4, 5, 6\}$. Perhaps surprisingly, these values turn out to be irrational.

THEOREM 5.4. *For the constant $\delta(c)$ from Theorem 5.3 we have*

$$\delta(4) = \frac{1}{2}(\sqrt{13} + 5) = 4.302\dots ,$$

$$\delta(5) = \frac{1}{2}(\sqrt{24} + 6) = 5.449\dots ,$$

$$\delta(6) = \frac{1}{2}(\sqrt{37} + 7) = 6.541\dots .$$

Our last result bounds the asymptotic growth of the constant $\delta(c)$ from Theorem 5.3.

THEOREM 5.5. *As a function of c , the constant $\delta(c)$ from Theorem 5.3 satisfies*

$$\Theta(c^{1.05}) \leq \delta(c) \leq \Theta(c^{1.59}) .$$

Note that the upper bound in Theorem 5.2 follows immediately by combining Theorem 5.3 with the upper bound on $\delta(c)$ stated in Theorem 5.5, using the non-negativity of the $o(1)$ term in Theorem 5.3.

5.1.3. About the proofs. We conclude this introduction by highlighting some of the key features in our proofs in an informal way.

As it turns out, the family of all ‘reasonable’ Painter strategies in the P_ℓ -avoidance game with $r = 2$ colors is in one-to-one correspondence with monotone walks from $(1, 1)$ to (ℓ, ℓ) in the integer lattice \mathbb{Z}^2 . Such a walk is interpreted as follows: If the walk goes from (x, y) to $(x + 1, y)$, Painter will use color 1 when faced with the decision of either creating a P_x in color 1 or a P_y in color 2. Conversely, a step from (x, y) to $(x, y + 1)$ indicates that Painter uses color 2 in the same situation. (The greedy strategy corresponds to the walk that goes from $(1, 1)$ first to $(1, \ell)$ and then to (ℓ, ℓ) .) Note that there are $\binom{2(\ell-1)}{\ell-1} = 4^{(1+o(1))\ell}$ such walks, and thus the same number of ‘candidate strategies’ for Painter.

For any fixed such walk, we can compute the smallest tree size restriction that allows Builder to enforce a monochromatic copy of P_ℓ against this particular Painter strategy by a recursive computation along the walk. This recursion involves only integers and no complicated tree structures. We can then compute the parameter $k^*(P_\ell, 2)$ by performing this recursive computation for all (exponentially many) walks of the described form, and taking the maximum. (This entire procedure can be seen as a highly specialized form of the general algorithm for computing $m_1^*(F, r)$ presented in the previous chapter.) With these insights in hand, understanding the vertex-coloring path-avoidance game reduces to the algebraic problem of understanding this recursion along lattice walks.

The lattice walks (i.e. Painter strategies) yielding the lower bounds in Theorem 5.2 and Theorem 5.5 have an interesting self-similar structure: essentially, they are obtained by nesting a large number of copies of a nearly-optimal walk for the asymmetric (P_ℓ, P_4) -avoidance game at different scales into each other (see Figure 5.3 below).

5.1.4. Organization of this chapter. In Section 5.2 we collect a few general observations about the F -avoidance game for the case where F is a forest. In Section 5.3 we turn to the case of paths and present the recursion that allows us to compute the parameter $k^*(P_\ell, 2)$ (or more generally, the parameter $k^*(P_\ell, P_c)$). This recursion is analyzed in Section 5.4 to derive Theorems 5.2–5.5.

5.2. Basic observations

For our proofs we will consider the general asymmetric (F_1, \dots, F_r) -avoidance game, where Painter's goal is to avoid a (possibly different) forest F_s in each color $s \in [r]$. We denote by $k^*(F_1, \dots, F_r)$ the smallest integer k for which Builder has a winning strategy in this asymmetric (F_1, \dots, F_r) -avoidance game with tree size restriction k .

In this section we prove straightforward lower and upper bounds for this parameter (Lemma 5.6 and Lemma 5.7 below). These lemmas show that the constant $\delta(c)$ from Theorem 5.3 satisfies $\delta(c) = c$ for $c \in \{1, 2, 3\}$, and their proofs also serve as a warm-up for the reader to get familiar with the type of reasoning that is used in later sections.

The definition of the greedy strategy extends in a straightforward manner to the general asymmetric (F_1, \dots, F_r) -avoidance game: This strategy in each step uses the highest-numbered color $s \in [r]$ that does not complete a monochromatic copy of F_s (or color 1 if no such color exists).

LEMMA 5.6 (Greedy lower bound). *For any trees F_1, \dots, F_r , we have $k^*(F_1, \dots, F_r) \geq v(F_1) \cdots v(F_r)$.*

PROOF. We show that the greedy strategy is a winning strategy for Painter in the game with tree size restriction $v(F_1) \cdots v(F_r) - 1$. Suppose for the sake of contradiction that Painter loses this game when playing the greedy strategy. Then, by the definition of the strategy, the board contains a copy of F_1 in color 1. Moreover, each vertex v in color 1 in this copy is adjacent to a set of trees in color 2 which together with v form a copy of F_2 , so the board contains a tree on $v(F_1) \cdot v(F_2)$ vertices in the colors 1 or 2. Continuing this argument inductively, we obtain that for all $k = 2, \dots, r$ each vertex v in one of the colors $\{1, \dots, k-1\}$ is adjacent to a set of trees in color k which together with v form a copy of F_k , and that consequently the board contains a tree on $v(F_1) \cdots v(F_k)$ vertices in colors from $\{1, \dots, k\}$. For $k = r$ this yields the desired contradiction. \square

Observe that if Builder confronts Painter several times with the decision on how to color a new vertex that connects in the same way to copies of the same r -colored trees, then by the pigeonhole principle, Painter's decision will be the same in at least a $(1/r)$ -fraction of the cases. As a consequence, we can assume w.l.o.g. that Painter plays *consistently* in the sense that her strategy is determined by a function that maps unordered tuples of r -colored rooted trees to the set of available colors $\{1, \dots, r\}$ (with the obvious interpretation that Painter uses the corresponding color whenever a new vertex connects exactly to the roots of copies of the trees in such a tuple). This assumption is very useful when proving upper bounds for $k^*(F_1, \dots, F_r)$ by describing explicit strategies for Builder, as it implies that if Builder has enforced a copy of some tree on

the board, then he can enforce as many additional copies of this tree as he needs. We thus avoid the hassle of making the repetitive pigeonholing steps for Builder explicit.

For the following lemma recall that we denote by S_ℓ the star with ℓ rays.

LEMMA 5.7 (Tree versus star upper bound). *For any tree F and any $\ell \geq 1$ we have $k^*(F, S_\ell) \leq v(F) \cdot v(S_\ell) = v(F) \cdot (\ell + 1)$.*

Note that this bound matches the greedy lower bound given by the previous lemma. It follows in particular that $k^*(P_\ell, P_c) = c \cdot \ell$ for any $\ell \geq 1$ and $c \in \{1, 2, 3\}$.

For the proof of Lemma 5.7 we use the following auxiliary lemma (for a proof see e.g. [Vyg11]).

LEMMA 5.8 (Tree splitting). *For any tree F and any integer $s \geq 1$ there is a subset $S \subseteq V(F)$ with $|S| \leq \lfloor \frac{v(F)}{s} \rfloor$ such that when removing the vertices of S from F all remaining components (=trees) have at most $s - 1$ vertices.*

PROOF OF LEMMA 5.7. We describe a winning strategy for Builder in the (F, S_ℓ) -avoidance game with tree size restriction $v(F) \cdot v(S_\ell)$. We may and will assume w.l.o.g. that Painter plays consistently as defined above, implying that if Builder has enforced a copy of some tree on the board, then he can enforce as many additional copies of this tree as he needs.

Builder's strategy works in two phases. The first phase lasts as long as Painter continues using color 1, and ends when she uses color 2 for the first time. In the first phase, for $n = 1, 2, \dots$ Builder enforces copies of *all* trees with exactly n vertices in color 1 (first all trees with one vertex, then all trees with two vertices and so on; all those copies are isolated, i.e., they are not connected to other parts of the board). Let s denote the value of n when Painter uses color 2 for the first time. At this point Builder has enforced, for each $n \leq s - 1$, a copy of every tree on n vertices in color 1, and a single vertex in color 2 that is contained in a tree T with $v(T) = s$ vertices.

For the second phase, apply Lemma 5.8 and fix a subset $S \subseteq V(F)$ with $|S| \leq \lfloor \frac{v(F)}{s} \rfloor$ such that when removing the vertices of S from F all remaining components (=trees) have at most $s - 1$ vertices. In this phase Builder uses copies of the components in $F \setminus S$ in color 1 from the first phase and connects them with $|S|$ many new vertices in such a way that assigning color 1 to all of these new vertices would create a copy of F in color 1. At the same time, Builder also connects each of these new vertices to the vertex in color 2 of ℓ separate copies of T , such that assigning color 2 to any of the new vertices would create a copy of S_ℓ in color 2. (In total Builder uses $\ell \cdot |S|$ many copies of T .) Hence the game ends either with a copy of F in color 1 or a copy of S_ℓ in color 2, and the number of vertices of the largest component (=tree) Builder constructs during the game is

$$v(F) + \ell \cdot |S| \cdot v(T) \leq v(F) + \ell \cdot \left\lfloor \frac{v(F)}{s} \right\rfloor \cdot s \leq v(F) \cdot (\ell + 1) = v(F) \cdot v(S_\ell) ,$$

proving the lemma. □

5.3. A general recursion

In this section we will derive a general recursion that allows us to compute the parameter $k^*(P_{\ell_1}, \dots, P_{\ell_r})$ for arbitrary values $\ell_1, \dots, \ell_r \geq 1$ (see Proposition 5.10 below). This turns the problem of analyzing the $(P_{\ell_1}, \dots, P_{\ell_r})$ -avoidance game into the algebraic problem of analyzing this recursion. As innocent as this recursion may look, it generates surprisingly complex patterns, which surface only for relatively large values of ℓ_1, \dots, ℓ_r (recall Table 1 for the special

case $r = 2$, $\ell_1 = \ell_2 = \ell$). Understanding the asymptotic features of this recursion will be the key to proving Theorems 5.2–5.5.

Throughout this section we include the case with more than two colors. There is little overhead for doing so, and it is notationally convenient to distinguish indices $s \in [r]$ referring to colors from certain indices 1 and 2 that appear otherwise.

5.3.1. A recursion along lattice walks. Let $\alpha = (\alpha_i)_{i \geq 1}$ be an infinite sequence with entries from the set $[r]$. For any $i \geq 0$ and any $s \in [r]$ we define

$$\nu_{i,s} := 1 + |\{1 \leq j \leq i \mid \alpha_j = s\}| . \quad (5.3a)$$

It is convenient to think of α as an increasing axis-parallel walk in the r -dimensional integer lattice \mathbb{Z}^r with starting point $(1, 1, \dots, 1)$, where in the i -th step of the walk the current position changes by $+1$ in the coordinate direction α_i . Note that $\nu_i = (\nu_{i,1}, \dots, \nu_{i,r})$ as defined in (5.3a) denotes the position of the walk after the first i steps.

The recursion defined below is parametrized by such a sequence $\alpha = (\alpha_i)_{i \geq 1}$, $\alpha_i \in [r]$, where this sequence can be interpreted as a strategy for Painter in some $(P_{\ell_1}, \dots, P_{\ell_r})$ -avoidance game as follows: For any point ν_i , $i \geq 0$, on the walk corresponding to α , whenever the longest path that would be created by assigning color s to a new vertex on the board is $\nu_{i,s}$ for each color $s \in [r]$, Painter chooses color $\sigma := \alpha_{i+1}$ (i.e., she prefers completing a path on $\nu_{i,\sigma}$ vertices in color σ over the other alternatives). To obtain a fully defined Painter strategy we will extend this criterion using certain natural monotonicity conditions: If e.g. Painter prefers a P_5 in color 1 over a P_7 in color 2, she will also prefer a P_5 in color 1 over a P_8 in color 2. The precise strategy definition is given below in the proof of Proposition 5.10. The recursion defined in the following evaluates the performance of the strategy corresponding to the given sequence α .

For a given sequence $\alpha = (\alpha_i)_{i \geq 1}$, $\alpha_i \in [r]$, the recursion computes an infinite sequence of integers $(k_i)_{i \geq 0}$. As auxiliary variables it maintains sequences of integers x_1, x_2, \dots, x_r , where for each $s \in [r]$ we write $x_s = (x_{s,0}, x_{s,1}, \dots)$. (To simplify notation we suppress the dependence of the values k_i , of the sequences x_s and of the values $\nu_{i,s}$ defined in (5.3a) from the parameter α .)

For each $i \geq 0$, first k_i is computed, and then this value is appended to exactly one of the sequences x_1, \dots, x_r , namely to the sequence specified by α_{i+1} . Specifically, for each $s \in [r]$ we define

$$x_{s,0} := 0 , \quad (5.3b)$$

and for any $i \geq 0$ we define

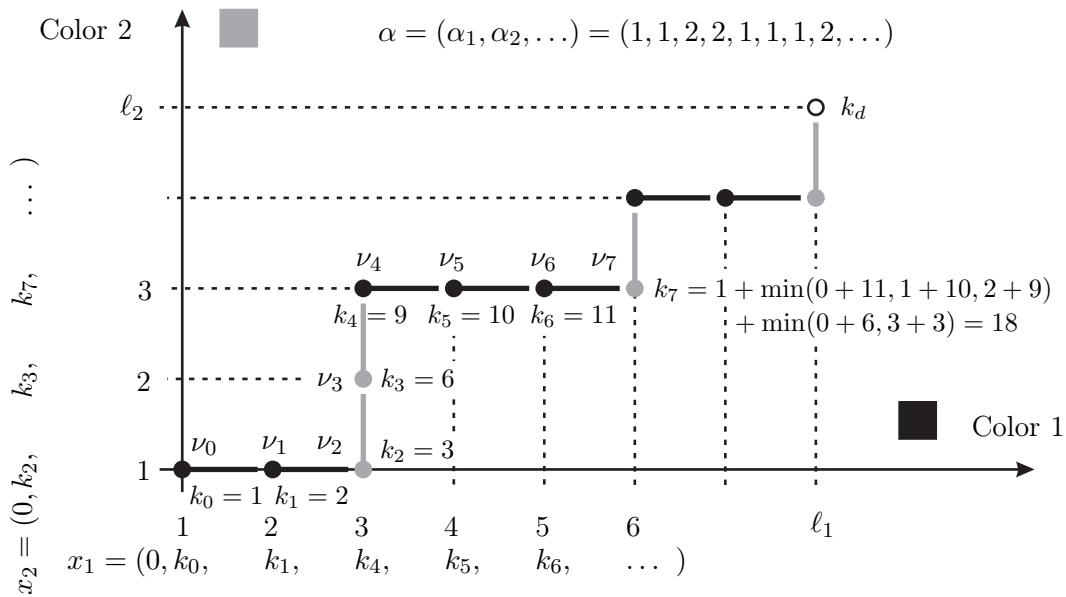
$$k_i := 1 + \sum_{s \in [r]} \min_{\substack{j_1, j_2 \geq 0: \\ j_1 + j_2 = \nu_{i,s} - 1}} (x_{s,j_1} + x_{s,j_2}) \quad (5.3c)$$

and

$$x_{s,\nu_{i,s}} := k_i \quad \text{if } \alpha_{i+1} = s , \quad (5.3d)$$

where the values $\nu_{i,s}$ are defined in (5.3a) for the given sequence α . (One can check that after step i of the recursion exactly the values k_0, \dots, k_i and, for each $s \in [r]$, the values $x_{s,0}, \dots, x_{s,\nu_{i+1,s}-1}$ have been computed.) An example illustrating these definitions is given in Figure 5.1.

Note that we can think of the sequence $(k_i)_{i \geq 0}$ as being computed along the walk corresponding to α , and for each $s \in [r]$ the entries of the sequence x_s are obtained by selecting those values $(k_i)_{i \geq 0}$ where the walk takes a step in direction s (see Figure 5.1). As we shall see, for any $s \in [r]$ and any $j \geq 0$ the number $x_{s,j}$ equals the number of vertices in the smallest component (=tree)



containing a path on j vertices in color s if Painter plays according to the strategy corresponding to the sequence α (see Lemma 5.12 below).

LEMMA 5.9 (Monotonicity along the recursion). *For any $\alpha = (\alpha_i)_{i \geq 1}$, $\alpha_i \in [r]$, the sequence $(k_i)_{i \geq 0}$ and in particular each of the sequences x_1, \dots, x_r defined in (5.3) is strictly increasing.*

$$d = d(\ell_1, \dots, \ell_r) := \sum_{s \in [r]} (\ell_s - 1) \quad (5.4a)$$
$$k(\alpha) := k_d \quad . \quad (5.4b)$$

The following proposition is the main result of this section and characterizes the parameter $k^*(P_{\ell_1}, \dots, P_{\ell_r})$ from the $(P_{\ell_1}, \dots, P_{\ell_r})$ -avoidance game in terms of the recursion defined above.

$$k^*(P_{\ell_1}, \dots, P_{\ell_r}) = \max_{\alpha \in W(\ell_1, \dots, \ell_r)} k(\alpha) \ , \quad (5.5)$$

5.3.2. Proof of Proposition 5.10. We begin by proving that the right hand side of (5.5) is an upper bound on $k^*(P_{\ell_1}, \dots, P_{\ell_r})$. We do so by describing a Builder strategy that closely resembles the structure of the recursion (5.3).

PROOF OF PROPOSITION 5.10 (UPPER BOUND). We describe a winning strategy for Builder in the $(P_{\ell_1}, \dots, P_{\ell_r})$ -avoidance game with tree size restriction

$$k := \max_{\alpha \in W(\ell_1, \dots, \ell_r)} k(\alpha) . \quad (5.6)$$

We may and will assume w.l.o.g. that Painter plays consistently in the sense of Section 5.2, implying that if Builder has enforced a copy of some tree on the board, then he can enforce as many additional copies of this tree as he needs. Moreover, we will ignore such repeated steps when counting the number of steps it takes until Builder has enforced a copy of some tree on the board. Intuitively, Builder's strategy follows the recursion defined in (5.3) for a sequence $\alpha = (\alpha_i)_{i \geq 1}$, $\alpha_i \in [r]$, that is extracted step by step from Painter's coloring decisions during the game.

Specifically, Builder maintains in each color $s \in [r]$ a list $T_s = (T_{s,0}, T_{s,1}, \dots, T_{s,\nu_s-1})$, where $T_{s,0}$ is the null graph ($v(T_{s,0}) = 0$) and $T_{s,j}$, $1 \leq j \leq \nu_s - 1$, is a tree containing a monochromatic P_j in color s for which Builder has already enforced a copy on the board. Initially, we have $T_s = (T_{s,0})$ for all $s \in [r]$. In each step, Builder does the following: Given the lists $T_s = (T_{s,0}, \dots, T_{s,\nu_s-1})$, $s \in [r]$, he adds a new vertex v to the board and, for each color $s \in [r]$, connects it to copies of two trees from the list T_s for which the sum $v(T_{s,j_1}) + v(T_{s,j_2})$, $j_1 + j_2 = \nu_s - 1$, is minimized, in such a way that if Painter assigns color s to v , a path on $j_1 + j_2 + 1 = \nu_s$ many vertices in color s is created (if one of the contributing graphs is the null graph, then no corresponding edge is added). Let $\sigma \in [r]$ denote the color Painter assigns to v , thus creating a tree that contains a copy of P_{ν_σ} in color σ . If $\nu_\sigma < \ell_\sigma$, then Builder adds this tree to the end of the list T_σ , which therefore grows by one element. Otherwise the game ends with a monochromatic P_{ℓ_σ} in color σ . Let $d' + 1$ denote the number of steps until the game ends (we consider these steps indexed from 0 to d'), and $\alpha' \in [r]^{\{1, \dots, d'\}}$ the sequence of all coloring decisions of Painter except the last one during Builder's strategy. (Thus Painter's decision in step i , $0 \leq i \leq d' - 1$, is given by α'_{i+1} , in line with (5.3d).) As each time Painter uses some color $s \in [r]$ the length of the list T_s grows by exactly one, the sequence α' has at most $\ell_s - 1$ entries equal to s .

It follows easily by induction that this Builder strategy satisfies the following property: For each $0 \leq i \leq d'$ the lists $T_s = (T_{s,0}, \dots, T_{s,\nu_{i,s}-1})$, $s \in [r]$, satisfy

$$(v(T_{s,0}), v(T_{s,1}), \dots, v(T_{s,\nu_{i,s}-1})) = (x_{s,0}, \dots, x_{s,\nu_{i,s}-1}) ,$$

and the tree constructed in step i has k_i many vertices, where $\nu_{i,s}$, k_i and the sequences x_1, \dots, x_s are defined in (5.3) for the given α' .

From this property it follows with Lemma 5.9 that the largest tree Builder constructs is the one in the last step of the game, and that it has $k_{d'}$ many vertices. Letting α denote any sequence from the set $W(\ell_1, \dots, \ell_r)$ with prefix α' , and k_d (with d as in (5.4a)) the value defined in (5.3) for this α , we obtain with Lemma 5.9 that

$$k_{d'} \leq k_d \stackrel{(5.4b)}{=} k(\alpha) \stackrel{(5.6)}{\leq} k ,$$

showing that Builder adhered to the given tree size restriction. \square

For proving the lower bound in Proposition 5.10 we will need the following observation. (If the reader is deterred by the technical-looking statement, we recommend looking at the very elementary proof first.)

LEMMA 5.11 (Choosing a color). *Let $\ell_1, \dots, \ell_r \geq 1$ be integers and $\alpha \in W(\ell_1, \dots, \ell_r)$. Then for any integers $\lambda_1, \dots, \lambda_r$ with $1 \leq \lambda_s \leq \ell_s$, $s \in [r]$, and $\lambda_s < \ell_s$ for at least one $s \in [r]$, the following holds: There is a unique integer $0 \leq i \leq d - 1$ such that for $\sigma := \alpha_{i+1}$ we have*

$$\nu_{i,\sigma} = \lambda_\sigma, \quad (5.7a)$$

$$\nu_{i,s} \leq \lambda_s, \quad s \in [r] \setminus \{\sigma\}, \quad (5.7b)$$

where $\nu_{i,s}$, $s \in [r]$, is defined in (5.3a) for the given α and $d = d(\ell_1, \dots, \ell_r)$ is defined in (5.4a). Moreover, we then have $\lambda_\sigma < \ell_\sigma$.

PROOF. Geometrically, the box $B := [1, \lambda_1] \times \dots \times [1, \lambda_r]$ is contained in the larger box $[1, \ell_1] \times \dots \times [1, \ell_r]$. As the walk corresponding to the sequence α starts at $(1, \dots, 1)$ and ends at (ℓ_1, \dots, ℓ_r) , there is a unique first step where it leaves the box B . It is easy to see that the starting point ν_i of this step (which lies on the boundary of B) is the unique integer i that satisfies the conditions of the lemma. \square

Consider now the following Painter strategy for the $(P_{\ell_1}, \dots, P_{\ell_r})$ -avoidance game, which is defined for an arbitrary fixed $\alpha \in W(\ell_1, \dots, \ell_r)$, and which we denote by $\text{AVOIDPATHS}_\alpha(P_{\ell_1}, \dots, P_{\ell_r})$. For each new vertex v , Painter determines for each color $s \in [r]$ the number of vertices λ'_s of the longest monochromatic path in color s that would be completed if that color were assigned to v , and defines $\lambda_s := \min(\lambda'_s, \ell_s)$. If $(\lambda_1, \dots, \lambda_r) = (\ell_1, \dots, \ell_r)$, then she assigns an arbitrary color to v (and the game ends). Otherwise one of the values λ_s is strictly smaller than ℓ_s . Painter then chooses an $0 \leq i \leq d - 1$ such that for $\sigma := \alpha_{i+1}$ the relations (5.7) hold (such a choice is possible by Lemma 5.11), and assigns color σ to v . (As we have $\lambda_\sigma < \ell_\sigma$ in this case, this does not create a monochromatic P_{ℓ_σ} in color σ . Moreover, using color σ does not increase the length of any monochromatic path in a color different from σ , implying that the game does not end in this step.)

For the rest of this chapter we usually refer to a sequence $\alpha \in W(\ell_1, \dots, \ell_r)$ as a *strategy sequence*, having the above interpretation in mind. Note that the greedy strategy analyzed in Lemma 5.6 is exactly $\text{AVOIDPATHS}_\alpha(P_{\ell_1}, \dots, P_{\ell_r})$ for the strategy sequence $\alpha = (r)^{\ell_r-1} \circ (r-1)^{\ell_{r-1}-1} \circ \dots \circ (1)^{\ell_1-1}$. Here and throughout we use \circ to denote concatenation of sequences, and integer exponents to indicate repetitions.

The next lemma states the strategy invariant that we already briefly mentioned when we introduced the recursion (5.3).

LEMMA 5.12 (Strategy invariant). *Playing according to the strategy $\text{AVOIDPATHS}_\alpha(P_{\ell_1}, \dots, P_{\ell_r})$ ensures that the following invariant holds throughout (except possibly in the last step when the game ends): For each $s \in [r]$ and each $0 \leq t \leq \ell_s - 1$, each monochromatic P_t in color s on the board is contained in a component (=tree) with at least $x_{s,t}$ vertices, where $x_{s,t}$ is defined in (5.3) for the given α .*

As we shall see, the above invariant is also maintained in the last step when the game ends, but for technical reasons we do not prove this here.

PROOF. To show that this invariant holds, we argue by induction over the number of steps of the game: Initially, no graph is present on the board, and the statement is trivially true (with $t = 0$ and $x_{s,0} = 0$). For the induction step consider a fixed step where the game does not end, and let λ_s , $s \in [r]$, be as defined in Painter's strategy. Furthermore, let i denote the index guaranteed by Lemma 5.11 for these values λ_s , and let $\sigma = \alpha_{i+1}$ denote the color Painter assigns to the

new vertex v in this step. Clearly, the invariant is maintained for all colors $s \in [r] \setminus \{\sigma\}$, and it remains to show that it holds for σ . By Lemma 5.9 we have

$$x_{\sigma,0} < x_{\sigma,1} < \cdots < x_{\sigma,\ell_\sigma-1} ,$$

implying that it suffices to consider a *longest* monochromatic path in color σ that is completed by Painter's decision to assign color σ to v . Let Q_σ denote such a path, and set $t := v(Q_\sigma)$ (as the game does not end in the current step we have $t \leq \ell_\sigma - 1$). By definition of Painter's strategy, we have

$$\lambda_\sigma = t , \tag{5.8}$$

and for each $s \in [r] \setminus \{\sigma\}$, assigning color s to v would have completed some (not necessarily maximal) path Q_s in color s on λ_s vertices. Note that the paths Q_1, \dots, Q_r only share the vertex v , and that v divides each of these paths into two paths $Q_{s,1}$ and $Q_{s,2}$ which for $j_{s,1} := v(Q_{s,1})$ and $j_{s,2} := v(Q_{s,2})$ satisfy

$$j_{s,1} + j_{s,2} = \lambda_s - 1 \stackrel{(5.7)}{\geq} \nu_{i,s} - 1 . \tag{5.9}$$

Furthermore, observe that the $2r$ paths $Q_{s,1}$ and $Q_{s,2}$, $s \in [r]$, were contained in $2r$ distinct components (=trees) $T_{s,1}$ and $T_{s,2}$ before being joined by the vertex v in the current step. (If $Q_{s,1}$ or $Q_{s,2}$ has no vertices, then we also let $T_{s,1}$ or $T_{s,2}$ be the null graph, i.e., the graph with empty vertex set.) By induction we have

$$\begin{aligned} v(T_{s,1}) &\geq x_{s,j_{s,1}} , \\ v(T_{s,2}) &\geq x_{s,j_{s,2}} . \end{aligned} \tag{5.10}$$

Combining our previous observations, we obtain that the vertex v is contained in a tree T satisfying

$$v(T) = 1 + \sum_{s \in [r]} (v(T_{s,1}) + v(T_{s,2})) \stackrel{(5.10)}{\geq} 1 + \sum_{s \in [r]} (x_{s,j_{s,1}} + x_{s,j_{s,2}}) \stackrel{(5.3c),(5.9)}{\geq} k_i , \tag{5.11}$$

where we also used Lemma 5.9 in the last step. Combining (5.3d), (5.7a) and (5.8) shows that the right hand side of (5.11) equals $x_{\sigma,t}$, proving that the claimed invariant holds. \square

We are now in a position to prove the lower bound in Proposition 5.10. The argument is very similar to the inductive argument in the previous proof, but due to some subtleties we have to treat the step in which the game ends separately.

PROOF OF PROPOSITION 5.10 (LOWER BOUND). We will argue that the Painter strategy $\text{AVOID-PATHS}_\alpha(P_{\ell_1}, \dots, P_{\ell_r})$ is a winning strategy in the $(P_{\ell_1}, \dots, P_{\ell_r})$ -avoidance game with tree size restriction $k(\alpha) - 1$, where $k(\alpha)$ is defined in (5.3) and (5.4). Optimizing over the choice of $\alpha \in W(\ell_1, \dots, \ell_r)$, we thus obtain a winning strategy for Painter in the game with tree size restriction $\max_{\alpha \in W(\ell_1, \dots, \ell_r)} k(\alpha) - 1$, as required.

Let $\alpha \in W(\ell_1, \dots, \ell_r)$ be fixed and suppose Painter plays according to the strategy $\text{AVOID-PATHS}_\alpha(P_{\ell_1}, \dots, P_{\ell_r})$. Suppose for the sake of contradiction that Painter loses with a monochromatic path P_{ℓ_s} in some color $s \in [r]$. By the definition of Painter's strategy, this means that in the last step of the game assigning *any* of the colors $s \in [r]$ to the last vertex v would complete a path P_{ℓ_s} in color s . This implies that in each color $s \in [r]$ the vertex v joins two (not necessarily maximal) paths $Q_{s,1}$ and $Q_{s,2}$ in color s which for $j_{s,1} := v(Q_{s,1})$ and $j_{s,2} := v(Q_{s,2})$ satisfy

$$j_{s,1} + j_{s,2} = \ell_s - 1 . \tag{5.12}$$

Denoting for every $s \in [r]$ by $T_{s,1}$ and $T_{s,2}$ the components (=trees) that were joined by v and that contain $Q_{s,1}$ and $Q_{s,2}$, respectively, we obtain from Lemma 5.12 that the vertex v is contained in a tree T satisfying

$$v(T) = 1 + \sum_{s \in [r]} (v(T_{s,1}) + v(T_{s,2})) \geq 1 + \sum_{s \in [r]} (x_{s,j_{s,1}} + x_{s,j_{s,2}}) \stackrel{(5.3c), (5.4b), (5.12)}{\geq} k(\alpha) ,$$

where in the last step we also used that for d defined in (5.4a) we have $\nu_d = (\ell_1, \dots, \ell_r)$. This yields the desired contradiction and completes the proof. \square

5.3.3. Exact values of $k^*(P_\ell, 2)$ for small values of ℓ . The values in Table 1 were found by implementing the recursion in (5.3) and (5.4) and using Proposition 5.10. The computationally most expensive part in this approach is the maximization in (5.5), as e.g. for the (symmetric) P_ℓ -avoidance game with $r = 2$ colors it requires maximizing over all strategy sequences from $W(\ell, \ell)$, of which there are $\binom{2(\ell-1)}{\ell-1} = 4^{(1+o(1))\ell}$ many. However, by using an appropriate branch-and-bound technique, the set of strategy sequences to be considered in the maximization can be reduced substantially. A program that implements this and further optimizations to compute $k^*(P_{\ell_1}, P_{\ell_2})$ is available from the authors' website [Müt].

We conclude this section by giving an example of a Painter strategy for the (symmetric) P_ℓ -avoidance game with $r = 2$ colors that outperforms the greedy strategy. For $\ell = 28$, there are four strategy sequences from the set $W(28, 28)$ achieving the optimal performance $k^*(P_{28}, 2) = 28^2 + 7 = 791$ (cf. Table 1). They are given by $\alpha = (1)^6 \circ (2, 2) \circ (1)^7 \circ (2) \circ (1)^{14} \circ (2)^{24}$, $\alpha' = (1, 1, 2, 1, 2, 2) \circ (1)^{24} \circ (2)^{24}$, and the sequences $\bar{\alpha}$ and $\bar{\alpha}'$ that are obtained from α and α' by interchanging the 1 and 2 entries, exploiting the obvious symmetry.

5.4. Analyzing the recursion

In this section we prove Theorems 5.2–5.5 by analyzing the recursion defined in (5.3) and (5.4) and using Proposition 5.10. We focus on the asymmetric path-avoidance game in most of the upcoming arguments, and derive our results for the symmetric game at the very end. For the rest of this chapter we restrict our attention to the case of $r = 2$ colors.

5.4.1. Asymptotic behavior. A crucial ingredient in our analysis of the recursion in (5.3) and (5.4) is the study of its asymptotic behavior along a walk as described in Section 5.3.1 which after some initial turns moves towards infinity only along one coordinate direction (think e.g. of infinitely extending the walk in Figure 5.1 in coordinate direction 1). The following completely self-contained lemma is the basis for this approach.

For any sequence $(x_\nu)_{\nu \geq 0}$ we define the corresponding sequence of first differences as $\Delta(x) := (x_{\nu+1} - x_\nu)_{\nu \geq 0}$.

LEMMA 5.13 (Recursion becomes periodic). *Let x_0, \dots, x_t and β be arbitrary integers, and recursively define*

$$x_\nu := \beta + \min_{\substack{j_1, j_2 \geq 0: \\ j_1 + j_2 = \nu - 1}} (x_{j_1} + x_{j_2}) , \quad \nu \geq t + 1 . \quad (5.13)$$

Furthermore, let p be an integer from the set $\arg \min_{0 \leq j \leq t} \frac{x_j + \beta}{j+1}$. Then the sequence $\Delta(x) = (x_{\nu+1} - x_\nu)_{\nu \geq 0}$ becomes periodic with period length $p + 1$, and for all $\nu \geq t + 1$ we have

$$x_\nu - x_{\nu-(p+1)} \leq x_p + \beta \quad (5.14)$$

with equality for all large enough ν . Moreover, for all $k \geq 1$ we have

$$x_{p+k(p+1)} - x_p \geq k(x_p + \beta) . \quad (5.15)$$

Note that Lemma 5.13 quantifies the asymptotic behavior of the recursion (5.13): In the long run, the values will change by $x_p + \beta$ every $p + 1$ steps, i.e., for $\nu \rightarrow \infty$ we have

$$x_\nu = (\delta + o(1)) \cdot \nu ,$$

where

$$\delta = \delta(x_0, \dots, x_t, \beta) := \min_{0 \leq j \leq t} \frac{x_j + \beta}{j + 1} . \quad (5.16)$$

PROOF. For any two integers $a \geq 1$ and b , applying the transformation

$$y_\nu = a(x_\nu + \beta) - b(\nu + 1) \quad (5.17)$$

to (5.13) yields an integer sequence $(y_\nu)_{\nu \geq 0}$ that satisfies the recursion

$$y_\nu = \min_{\substack{j_1, j_2 \geq 0: \\ j_1 + j_2 = \nu - 1}} (y_{j_1} + y_{j_2}) , \quad \nu \geq t + 1 . \quad (5.18)$$

Furthermore, by (5.17) the first differences of the sequences $(x_\nu)_{\nu \geq 0}$ and $(y_\nu)_{\nu \geq 0}$ are related via

$$\Delta(y) = a\Delta(x) - b . \quad (5.19)$$

Applying the transformation (5.17) with

$$a := p + 1 \quad \text{and} \quad b := x_p + \beta , \quad (5.20)$$

we obtain with the definition of p in the lemma that $y_p = 0$ and $y_\nu \geq 0$ for all $0 \leq \nu \leq t$. By these initial conditions and by (5.18), *all* elements of the sequence $(y_\nu)_{\nu \geq 0}$ are non-negative. Furthermore, using that $y_p = 0$ it follows from (5.18) that

$$y_\nu \leq y_{\nu-(p+1)} \quad \text{for all } \nu \geq t + 1 . \quad (5.21)$$

Combining this with the non-negativity of the sequence $(y_\nu)_{\nu \geq 0}$, we obtain that for each residue class modulo $p + 1$ the corresponding subsequence of $(y_\nu)_{\nu \geq 0}$ becomes constant, and that consequently the sequence itself becomes periodic with period length $p + 1$. It follows that the sequence $\Delta(y)$ and by (5.19) also the sequence $\Delta(x)$ become periodic with period length $p + 1$.

Note that for all $\nu \geq t + 1$ we have

$$x_\nu - x_{\nu-(p+1)} \stackrel{(5.19)}{=} \frac{y_\nu - y_{\nu-(p+1)} + b(p+1)}{a} \stackrel{(5.21)}{\leq} \frac{b(p+1)}{a} \stackrel{(5.20)}{=} x_p + \beta ,$$

with equality for all large enough ν , proving (5.14).

Similarly, using the non-negativity of the sequence $(y_\nu)_{\nu \geq 0}$ and $y_p = 0$ we obtain for all $k \geq 1$

$$x_{p+k(p+1)} - x_p \stackrel{(5.19)}{=} \frac{y_{p+k(p+1)} - y_p + bk(p+1)}{a} \geq \frac{bk(p+1)}{a} \stackrel{(5.20)}{=} k(x_p + \beta) ,$$

proving (5.15). □

5.4.2. Explicit version of Theorem 5.3. Using Lemma 5.13 we will show that asymptotically optimal Painter strategies for the asymmetric (P_ℓ, P_c) -avoidance game (i.e., strategies achieving the lower bound stated in Theorem 5.3) can be constructed as follows. Intuitively, we distinguish two phases of the corresponding walks: a short initial ‘preparation’ phase and a long ‘payoff’ phase, which is just a straight segment of the walk extending into coordinate direction 1. The goal of the preparation phase is *not* to directly optimize the resulting recursion values during this phase, but to optimize the constant δ as defined in (5.16) that arises when applying Lemma 5.13 to the payoff phase.

These ideas lead to the following definition of the constant $\delta(c)$ appearing in Theorem 5.3. For any strategy sequence $\alpha \in W(\ell, c)$ we define

$$\beta(\alpha) := 1 + \min_{\substack{j_1, j_2 \geq 0: \\ j_1 + j_2 = c-1}} (x_{2, j_1} + x_{2, j_2}) , \quad (5.22a)$$

$$\delta(\alpha) := \min_{0 \leq j \leq \ell-1} \frac{x_{1, j} + \beta(\alpha)}{j+1} , \quad (5.22b)$$

where $x_1 = (x_{1,0}, \dots, x_{1,\ell-1})$ and $x_2 = (x_{2,0}, \dots, x_{2,c-1})$ are defined via the recursion (5.3). Using those definitions we set

$$\delta(1) := 1 , \quad (5.22c)$$

and for any $c \geq 2$,

$$\delta(c) := \sup_{\substack{\ell \geq 1 \\ \alpha \in W(\ell, c): \alpha_{\ell+c-2}=2}} \delta(\alpha) , \quad (5.22d)$$

where the condition $\alpha_{\ell+c-2} = 2$ expresses that the last step of the walk corresponding to α is towards the second coordinate. (We will see in Lemma 5.15 below that $\delta(c)$ is indeed a well-defined finite value.)

THEOREM 5.14 (Explicit version of Theorem 5.3). *For any $c \geq 1$ and any $\ell \geq 1$ we have*

$$k^*(P_\ell, P_c) \leq \delta(c) \cdot \ell$$

and

$$k^*(P_\ell, P_c) \geq (\delta(c) - o(1)) \cdot \ell ,$$

where $\delta(c)$ is defined in (5.22), and $o(1)$ stands for a non-negative function of c and ℓ that tends to 0 for c fixed and $\ell \rightarrow \infty$.

We prove Theorem 5.14 (and thus Theorem 5.3) in the next section.

5.4.3. Proof of Theorem 5.14. We will prove the following three lemmas by induction. Note that Lemma 5.17 is exactly the upper bound part of Theorem 5.14, and that moreover Lemma 5.15 yields the upper bound part of Theorem 5.5 (we have $\log_2(3) = 1.584\dots < 1.59$).

LEMMA 5.15 (Upper bound for $\delta(c)$). *For any $c \geq 1$ we have $\delta(c) \leq c^{\log_2(3)}$.*

LEMMA 5.16 (Monotonicity of $\delta(c)$). *For all $1 \leq \hat{c} \leq c$ we have $\delta(\hat{c}) \leq \delta(c)$.*

LEMMA 5.17 (Upper bound for $k^*(P_\ell, P_c)$ via $\delta(c)$). *For any $c \geq 1$ and any $\ell \geq 1$ we have $k^*(P_\ell, P_c) \leq \delta(c) \cdot \ell$.*

PROOF OF LEMMA 5.15, 5.16 AND 5.17. We argue by induction on c . For $c = 1$ all claims are trivially satisfied. For the induction step let $c \geq 2$.

Induction step for Lemma 5.15. For any fixed $\ell \geq 1$ consider an arbitrary fixed strategy sequence $\alpha \in W(\ell, c)$ with $\alpha_{\ell+c-2} = 2$. Note that α can be uniquely written in the form

$$\alpha = (1)^{\ell_1-1} \circ (2) \circ (1)^{\ell_2-\ell_1} \circ (2) \circ \dots \circ (1)^{\ell_{c-1}-\ell_{c-2}} \circ (2) , \quad (5.23)$$

where $1 \leq \ell_1 \leq \ell_2 \leq \dots \leq \ell_{c-2} \leq \ell_{c-1} = \ell$.

In the following we derive upper bounds for the entries of the sequences x_1 and x_2 defined in (5.3) for this α . For any $1 \leq j \leq c-1$ we define $\alpha^{(j)}$ as the maximal prefix of α containing exactly $j-1$ entries equal to 2. By (5.23) we have $\alpha^{(j)} \in W(\ell_j, j)$. Moreover, by the definitions in (5.3) and (5.4) and by Proposition 5.10 we have

$$x_{2,j} = k_{(\ell_j-1)+(j-1)} = k(\alpha^{(j)}) \stackrel{(5.5)}{\leq} k^*(P_{\ell_j}, P_j) , \quad 1 \leq j \leq c-1 . \quad (5.24)$$

By induction and Lemma 5.17 we hence obtain from (5.24) that

$$x_{2,j} \leq \delta(j) \cdot \ell_j , \quad 1 \leq j \leq c-1 . \quad (5.25)$$

Using that by (5.3) and (5.23) the integers x_{1,ℓ_j-1} and $x_{2,j}$ correspond to sequence elements k_i and $k_{i'}$ as defined in (5.3c) with $i < i'$, we obtain with Lemma 5.9 that

$$x_{1,\ell_j-1} \leq x_{2,j} - 1 \stackrel{(5.25)}{\leq} \delta(j) \cdot \ell_j - 1 , \quad 1 \leq j \leq c-1 . \quad (5.26)$$

By setting $j_1 = \lfloor (c-1)/2 \rfloor$ and $j_2 = \lceil (c-1)/2 \rceil$ in (5.22a) we obtain

$$\beta(\alpha) \stackrel{(5.22a)}{\leq} 1 + x_{2,\lfloor (c-1)/2 \rfloor} + x_{2,\lceil (c-1)/2 \rceil} \leq 1 + 2x_{2,\lceil (c-1)/2 \rceil} \stackrel{(5.25)}{\leq} 1 + 2\delta(\lceil \frac{c-1}{2} \rceil) \cdot \ell_{\lceil (c-1)/2 \rceil} , \quad (5.27)$$

where we again used Lemma 5.9 in the second estimate. Similarly, setting $j = \ell_{\lceil (c-1)/2 \rceil} - 1$ in (5.22b) yields

$$\delta(\alpha) \stackrel{(5.22b)}{\leq} \frac{x_{1,\ell_{\lceil (c-1)/2 \rceil}-1} + \beta(\alpha)}{\ell_{\lceil (c-1)/2 \rceil}} \stackrel{(5.26),(5.27)}{\leq} 3 \cdot \delta(\lceil \frac{c-1}{2} \rceil) . \quad (5.28)$$

As the bound in (5.28) holds for all $\ell \geq 1$ and all strategy sequences $\alpha \in W(\ell, c)$ with $\alpha_{\ell+c-2} = 2$ simultaneously, we obtain with the definition in (5.22d) that

$$\delta(c) \leq 3 \cdot \delta(\lceil \frac{c-1}{2} \rceil) \leq 3 \cdot \lceil \frac{c-1}{2} \rceil^{\log_2(3)} \leq 3 \cdot \left(\frac{c}{2}\right)^{\log_2(3)} = c^{\log_2(3)} ,$$

where the second estimate is the induction hypothesis. This completes the proof of Lemma 5.15.

Induction step for Lemma 5.16. By induction we have $\delta(1) \leq \dots \leq \delta(c-1)$, so it suffices to show that $\delta(c-1) \leq \delta(c)$ (by Lemma 5.15 we already know that $\delta(c)$ is a well-defined finite value). For $c = 2$, note that the strategy sequence $\alpha = (2) \in W(1, 2)$ yields $\beta(\alpha) = 2$ and $\delta(\alpha) = 2$ and consequently guarantees a lower bound of $\delta(2) \geq 2$, implying in particular that $\delta(1) \leq \delta(2)$ (recall (5.22c)). For $c \geq 3$ we argue as follows: For each strategy sequence $\alpha^- \in W(\ell, c-1)$ with $\alpha_{\ell+(c-1)-2}^- = 2$ consider the extended sequence $\alpha := \alpha^- \circ (2) \in W(\ell, c)$. By Lemma 5.9 and (5.22a) we have $\beta(\alpha^-) < \beta(\alpha)$, which by (5.22b) implies that $\delta(\alpha^-) < \delta(\alpha)$. Using (5.22d) this shows that $\delta(c-1) \leq \delta(c)$, completing the proof of Lemma 5.16.

Induction step for Lemma 5.17. For the reader's convenience, Figure 5.2 illustrates the notations used in this proof.

Let $\ell \geq 1$ and $\alpha \in W(\ell, c)$ be fixed. We show that for $k(\alpha)$ as defined in (5.3) and (5.4) we have $k(\alpha) \leq \delta(c) \cdot \ell$, from which the claim follows by Proposition 5.10. For the proof it is convenient to extend the sequences $x_1 = (x_{1,0}, \dots, x_{1,\ell-1})$ and $x_2 = (x_{2,0}, \dots, x_{2,c-1})$ defined in (5.3) for the given α by setting

$$x_{1,\ell} := k_d \stackrel{(5.4b)}{=} k(\alpha) \quad (5.29)$$

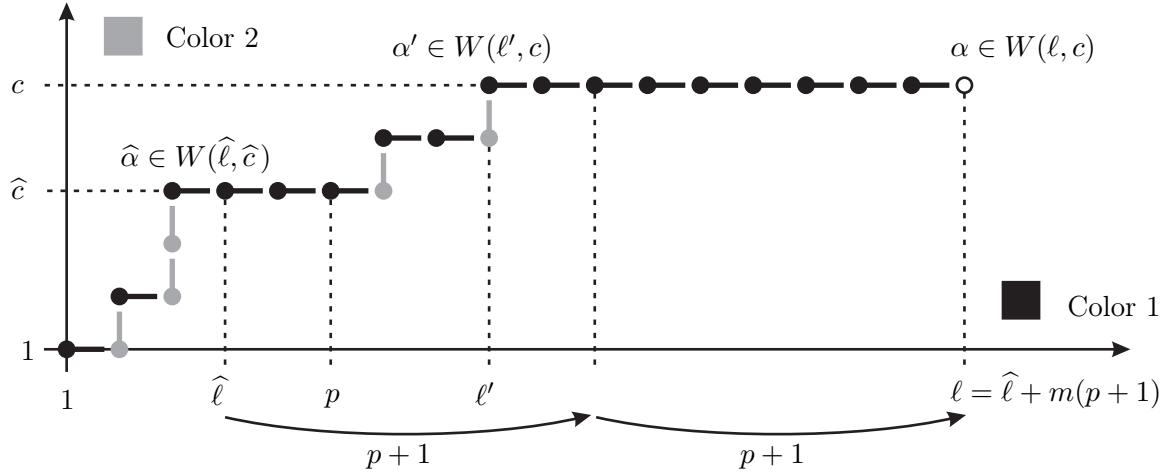


FIGURE 5.2. Notations used in the proof of Lemma 5.17.

with $d = d(\ell, c)$ defined in (5.4a). Let ℓ' be such that $\alpha = \alpha' \circ (1)^{\ell-\ell'}$ with $\alpha' \in W(\ell', c)$ and $\alpha'_{\ell'+c-2} = 2$. Fixing some integer

$$p \in \arg \min_{0 \leq j \leq \ell'-1} \frac{x_{1,j} + \beta(\alpha')}{j+1}, \quad (5.30)$$

where $\beta(\alpha')$ is defined in (5.22a), we have

$$\frac{x_{1,p} + \beta(\alpha')}{p+1} \stackrel{(5.22b), (5.30)}{=} \delta(\alpha') \stackrel{(5.22d)}{\leq} \delta(c). \quad (5.31)$$

Let $\hat{\ell} \leq \ell' - 1$ be the largest integer such that

$$\ell = \hat{\ell} + m(p+1) \quad (5.32)$$

for some integer m .

By (5.3), (5.29) and the definition of $\beta(\alpha')$ in (5.22a) we have

$$x_{1,\nu} = \beta(\alpha') + \min_{\substack{j_1, j_2 \geq 0: \\ j_1 + j_2 = \nu - 1}} (x_{1,j_1} + x_{1,j_2}), \quad \ell' \leq \nu \leq \ell.$$

We may hence apply Lemma 5.13, and using (5.30) we obtain that

$$x_{1,\ell} \stackrel{(5.32)}{=} x_{1,\hat{\ell}+m(p+1)} \stackrel{(5.14)}{\leq} x_{1,\hat{\ell}} + m(x_{1,p} + \beta(\alpha')). \quad (5.33)$$

If $\hat{\ell} \geq 1$, we let \hat{c} denote the maximal value of \bar{c} for which $W(\hat{\ell}, \bar{c})$ contains a prefix of α' , and we let $\hat{\alpha} \in W(\hat{\ell}, \hat{c})$ denote the corresponding prefix (see Figure 5.2). Clearly we have $\hat{c} < c$ and

$$k(\hat{\alpha}) \stackrel{(5.3), (5.4)}{=} x_{1,\hat{\ell}}. \quad (5.34)$$

As $\hat{c} < c$ we may apply the induction hypothesis and obtain together with Proposition 5.10 that

$$k(\hat{\alpha}) \stackrel{(5.5)}{\leq} k^*(P_{\hat{\ell}}, P_{\hat{c}}) \leq \delta(\hat{c}) \cdot \hat{\ell},$$

which combined with (5.34) and Lemma 5.16 yields

$$x_{1,\hat{\ell}} \leq \delta(c) \cdot \hat{\ell}. \quad (5.35)$$

If $\hat{\ell} = 0$, then (5.35) holds trivially (both sides of this inequality are equal to zero).

Combining our previous observations we obtain

$$k(\alpha) \stackrel{(5.29),(5.33)}{\leq} x_{1,\widehat{\ell}} + m(x_{1,p} + \beta(\alpha')) \stackrel{(5.31),(5.35)}{\leq} \delta(c) \cdot (\widehat{\ell} + m(p+1)) \stackrel{(5.32)}{=} \delta(c) \cdot \ell ,$$

completing the proof of Lemma 5.17. \square

It remains to prove the lower bound in Theorem 5.14.

PROOF OF THEOREM 5.14 (LOWER BOUND). As in the proof of Lemma 5.17 it is also convenient here to extend the definition in (5.3d) for any $\alpha \in W(\ell, c)$ by setting

$$x_{1,\ell} := k_d \stackrel{(5.4b)}{=} k(\alpha) \tag{5.36}$$

with $d = d(\ell, c)$ defined in (5.4a). Note that the claim holds trivially if $c = 1$, so we consider a fixed $c \geq 2$ in the following. By the definition in (5.22d) there are families $(\ell_t)_{t \geq 0}$ and $(\alpha^{(t)})_{t \geq 0}$, where $\alpha^{(t)} \in W(\ell_t, c)$ with $\alpha_{\ell_t+c-2}^{(t)} = 2$, satisfying

$$\lim_{t \rightarrow \infty} \delta(\alpha^{(t)}) = \delta(c) . \tag{5.37}$$

Fix any such strategy sequence $\alpha^{(t)}$, and for every $\ell \geq \ell_t$ consider the extended sequence $\widehat{\alpha}^{(t)} := \alpha^{(t)} \circ (1)^{\ell-\ell_t} \in W(\ell, c)$. Using (5.22a) we obtain that for any such extended sequence $\widehat{\alpha}^{(t)}$, the sequence $x_1 = (x_{1,0}, \dots, x_{1,\ell})$ defined in (5.3) and (5.36) satisfies

$$x_{1,\nu} = \beta(\alpha^{(t)}) + \min_{\substack{j_1, j_2 \geq 0: \\ j_1 + j_2 = \nu - 1}} (x_{1,j_1} + x_{1,j_2}) , \quad \ell_t \leq \nu \leq \ell .$$

Moreover, by (5.36) we have $k(\widehat{\alpha}^{(t)}) = x_{1,\ell}$. By the first part of Lemma 5.13 and the definition in (5.22b) we thus have

$$k(\widehat{\alpha}^{(t)}) = (\delta(\alpha^{(t)}) + o(1)) \cdot \ell$$

for c and t fixed and $\ell \rightarrow \infty$ (recall that (5.14) holds with equality for all large enough indices). Combining this with (5.37) and applying Proposition 5.10 yields that

$$k^*(P_c, P_\ell) \geq (\delta(c) + o(1)) \cdot \ell$$

for c fixed and $\ell \rightarrow \infty$. Moreover, the upper bound given by Lemma 5.17 shows that the term $o(1)$ must be non-positive. \square

5.4.4. Proof of Theorem 5.4. In this section we derive the exact values of $\delta(c)$ stated in Theorem 5.4 by carrying out explicitly the optimization over lattice walks that appears in the definition (5.22d). Note that for small values of c , the walk corresponding to a strategy sequence $\alpha \in W(\ell, c)$ has only few turning points. We will derive upper bounds for the entries of the sequences x_1 and x_2 computed by the recursion (5.3) as a function of the 1-coordinates of these turning points. To do so we will only consider a few carefully selected terms when evaluating the minimization in (5.3c). The upper bound on $\delta(c)$ we obtain this way is the minimum over a small number of functions of 1-coordinates of turning points, and turns out to be irrational. We will derive asymptotically matching lower bounds by describing families of strategy sequences for which the ratios between the 1-coordinates of successive turning points approximate the optimal (irrational!) ratios.

We only give the proof for $c = 4$ here, and omit the (similar but more complicated) proofs for $c \in \{5, 6\}$ (those proofs can be found in the preprint version of [MS11] on arXiv).

PROOF OF $\delta(4) = \frac{1}{2}(\sqrt{13} + 5)$. Consider a strategy sequence $\alpha \in W(\ell, 4)$ with $\alpha_{\ell+4-2} = 2$, and note that α can be uniquely written in the form

$$\alpha = (1)^{\ell_1-1} \circ (2) \circ (1)^{\ell_2-\ell_1} \circ (2) \circ (1)^{\ell-\ell_2} \circ (2) ,$$

where $1 \leq \ell_1 \leq \ell_2 \leq \ell$.

Let $p \geq 1$ and $0 \leq m < \ell_1$ be the unique integers satisfying

$$\ell_2 = p\ell_1 + m . \quad (5.38)$$

The recursion (5.3) yields by straightforward calculations that

$$\begin{aligned} x_{1,j} &= j , \quad 0 \leq j \leq \ell_1 - 1 , \quad (*) \\ x_{2,1} &= \ell_1 , \\ x_{1,2\ell_1-1} &\leq x_{1,\ell_1-1} + x_{1,\ell_1-1} + x_{2,1} + 1 = 3\ell_1 - 1 , \\ x_{1,3\ell_1-1} &\leq x_{1,2\ell_1-1} + x_{1,\ell_1-1} + x_{2,1} + 1 \leq 5\ell_1 - 1 , \\ &\dots \\ x_{1,p\ell_1-1} &\leq x_{1,(p-1)\ell_1-1} + x_{1,\ell_1-1} + x_{2,1} + 1 \leq 2p\ell_1 - \ell_1 - 1 , \quad (**) \\ x_{1,\ell_2-1} &\leq x_{1,p\ell_1-1} + x_{1,m-1} + x_{2,1} + 1 \leq 2p\ell_1 + m - 1 \stackrel{(5.38)}{=} 2\ell_2 - m - 1 , \quad (***) \\ x_{2,2} &\leq x_{1,p\ell_1-1} + x_{1,m} + x_{2,1} + 1 \leq 2p\ell_1 + m \stackrel{(5.38)}{=} 2\ell_2 - m , \end{aligned} \quad (5.39)$$

where a priori the inequality marked with (**) holds only if $p \geq 2$ (as $x_{1,(p-1)\ell_1-1}$ is undefined otherwise). For $p = 1$ the resulting inequality reads $x_{1,\ell_1-1} \leq \ell_1 - 1$, which is true nevertheless, as a comparison with (*) shows. Similarly, a priori the inequality marked with (***) holds only if $m \geq 1$ (as $x_{1,m-1}$ is undefined otherwise). For $m = 0$ the resulting inequality reads $x_{1,\ell_2-1} \leq 2p\ell_1 - 1 = 2\ell_2 - 1$, which is true nevertheless, as a comparison with (**) shows.

Defining

$$\mu := m/\ell_1 \quad (5.40)$$

we thus obtain

$$\begin{aligned} \delta(\alpha) &\stackrel{(5.22a),(5.22b)}{\leq} \min \left\{ \frac{x_{1,p\ell_1-1} + x_{2,1} + x_{2,2} + 1}{p\ell_1}, \frac{x_{1,\ell_2-1} + x_{2,1} + x_{2,2} + 1}{\ell_2} \right\} \\ &\stackrel{(5.39),(5.40)}{\leq} \min \left\{ 4 + \frac{\mu}{p}, 4 + \frac{1-2\mu}{p+\mu} \right\} . \end{aligned} \quad (5.41)$$

In order to determine the best bound resulting from this analysis, we need to find the maximum of the function on the right side of (5.41). Relaxing this problem to a maximization problem with the integer variable $p \in \{1, 2, \dots\}$, and the *real-valued* variable $0 \leq \mu < 1$ (cf. the definition in (5.40) and recall that $m \in \mathbb{N}_0$ is chosen such that $m < \ell_1$), it is easy to see that the right hand side of (5.41) attains its maximum for

$$p = 1 \quad \text{and} \quad \mu = \frac{1}{2}(\sqrt{13} - 3) , \quad (5.42)$$

corresponding to a ratio $\ell_2/\ell_1 = p + \mu = \frac{1}{2}(\sqrt{13} - 1)$ and yielding

$$\delta(\alpha) \stackrel{(5.41),(5.42)}{\leq} \frac{1}{2}(\sqrt{13} + 5) . \quad (5.43)$$

As the bound in (5.43) holds for all $\ell \geq 1$ and all $\alpha \in W(\ell, 4)$ with $\alpha_{\ell+4-2} = 2$ simultaneously, we obtain with the definition in (5.22d) that

$$\delta(4) \leq \frac{1}{2}(\sqrt{13} + 5) .$$

To show that this upper bound is tight, by (5.22d) it suffices to specify a family of strategy sequences $(\alpha^{(t)})_{t \geq 0}$, where $\alpha^{(t)} \in W(\ell_t, 4)$ for some $\ell_t \geq 1$ and $\alpha_{\ell_t+4-2}^{(t)} = 2$, with $\lim_{t \rightarrow \infty} \delta(\alpha^{(t)}) = \frac{1}{2}(\sqrt{13} + 5)$. Define

$$\alpha^{(t)} := (1)^{\ell_{1,t}-1} \circ (2) \circ (1)^{\ell_{2,t}-\ell_{1,t}} \circ (2, 2) \in W(\ell_{2,t}, 4)$$

for all $t \geq 0$, where $\ell_{1,t} := 10^t$ and $\ell_{2,t} := \lfloor \frac{1}{2}(\sqrt{13} - 1) \cdot 10^t \rfloor$, i.e., we have

$$\lim_{t \rightarrow \infty} \frac{\ell_{2,t}}{\ell_{1,t}} = \frac{1}{2}(\sqrt{13} - 1) . \quad (5.44)$$

For each such strategy sequence $\alpha^{(t)}$ we obtain from (5.3), using that $\ell_{2,t} < 2\ell_{1,t}$,

$$\begin{aligned} x_{1,j} &= j , \quad 0 \leq j \leq \ell_{1,t} - 1 , \\ x_{2,1} &= \ell_{1,t} , \\ x_{1,j} &= \ell_{1,t} + j , \quad \ell_{1,t} \leq j \leq \ell_{2,t} - 1 , \\ x_{2,2} &= \ell_{1,t} + \ell_{2,t} , \\ x_{2,3} &= 2\ell_{1,t} + \ell_{2,t} , \end{aligned} \quad (5.45)$$

implying that

$$\beta(\alpha^{(t)}) \stackrel{(5.22a), (5.45)}{=} 2\ell_{1,t} + \ell_{2,t} + 1 \quad (5.46)$$

and

$$\delta(\alpha^{(t)}) \stackrel{(5.22b), (5.45), (5.46)}{=} \min \left\{ 3 + \frac{\ell_{2,t}}{\ell_{1,t}}, 2 + 3 \left(\frac{\ell_{2,t}}{\ell_{1,t}} \right)^{-1} \right\} . \quad (5.47)$$

Using (5.44) it follows from (5.47) that $\delta(\alpha^{(t)}) \rightarrow \frac{1}{2}(\sqrt{13} + 5)$ as $t \rightarrow \infty$. \square

5.4.5. Lower bound for $\delta(c)$ via ‘bootstrapping’. In the previous section, we determined asymptotically optimal strategy sequences for the asymmetric (P_ℓ, P_4) -avoidance game. We now show how these can be ‘bootstrapped’ to derive good strategy sequences for the asymmetric (P_ℓ, P_c) -avoidance game with any fixed c of form $c = 4^t$.

Specifically, we construct these strategy sequences by nesting scaled copies of nearly-optimal strategy sequences for the (P_ℓ, P_4) -avoidance game into each other. The lattice walks corresponding to these strategy sequences thus have a self-similar structure (see Figure 5.3). As we will see, the equations arising in the analysis of this construction are, up to some error terms, exactly the same as in the proof that $\delta(4) = \frac{1}{2}(\sqrt{13} + 5)$ in the previous section.

LEMMA 5.18 (Lower bound for $\delta(c)$ via ‘bootstrapping’). *For any integer $t \geq 0$, the function $\delta(c)$ defined in (5.22) satisfies*

$$\delta(4^t) \geq (\delta(4))^t \stackrel{Thm. 5.4}{=} \left(\frac{1}{2}(\sqrt{13} + 5) \right)^t .$$

Note that together with the monotonicity guaranteed by Lemma 5.16, Lemma 5.18 shows that $\delta(c) = \Omega(c^{\log_4(\delta(4))}) = \Omega(c^{1.052\dots})$ as a function of c , proving the bound claimed in Theorem 5.5.

REMARK 5.19. Similar lower bound statements can be proven by bootstrapping asymptotically optimal strategy sequences for the (P_ℓ, P_5) - or the (P_ℓ, P_6) -avoidance game. The resulting exponent of c is marginally better for the case (P_ℓ, P_5) but worse for the case (P_ℓ, P_6) : Using the values stated in Theorem 5.4 we obtain $\log_4(\delta(4)) = \log_4(\frac{1}{2}(\sqrt{13} + 5)) = 1.052\dots$, $\log_5(\delta(5)) = \log_5(\frac{1}{2}(\sqrt{24} + 6)) = 1.053\dots$, and $\log_6(\delta(6)) = \log_6(\frac{1}{2}(\sqrt{37} + 7)) = 1.048\dots$

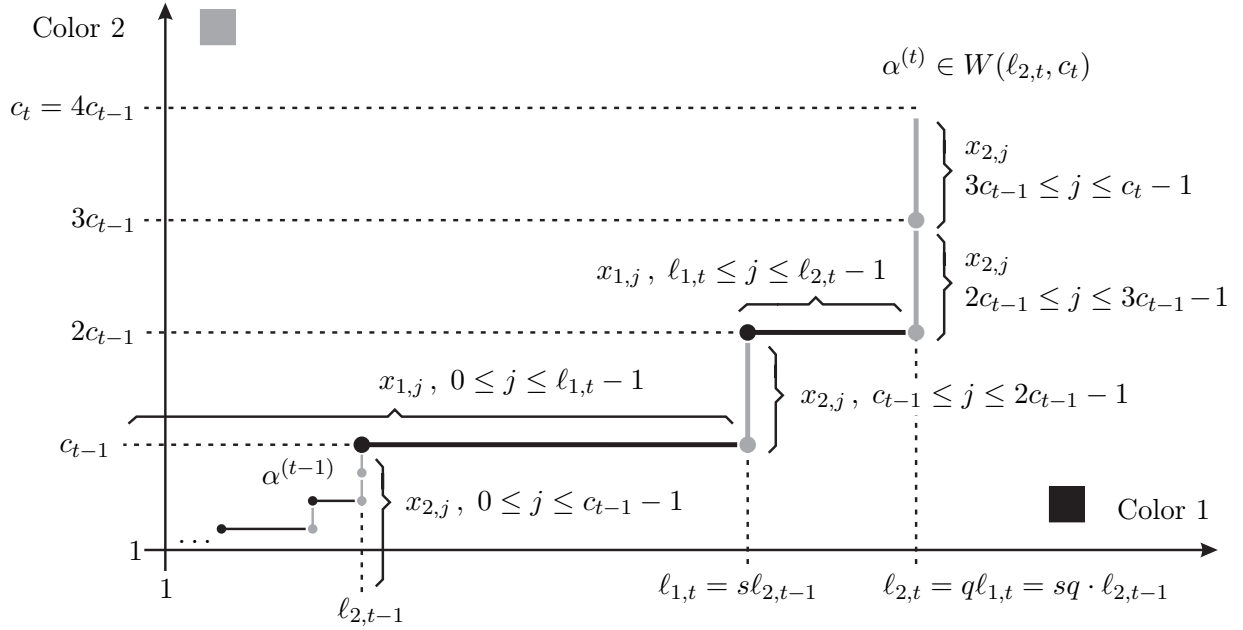


FIGURE 5.3. Notations used in the proof of Lemma 5.18.

PROOF OF LEMMA 5.18. In the following we specify a family of finite strategy sequences $(\alpha^{(t)})_{t \geq 0}$ such that $\alpha^{(t)}$ is a prefix of $\alpha^{(t+1)}$ for all $t \geq 0$. This defines an infinite sequence α , and we denote by x_1 and x_2 the sequences defined in (5.3) for this α .

Let q be a rational number with

$$1.3 \leq q < \frac{1}{2}(\sqrt{13} - 1) = 1.302 \dots,$$

and let $s \geq 100$ be an integer such that $sq \in \mathbb{N}$. We will consider these parameters fixed throughout the proof; at the very end we will take the limit $q \rightarrow \frac{1}{2}(\sqrt{13} - 1)$ and $s \rightarrow \infty$.

We define

$$\ell_{1,0} := 1, \quad \ell_{2,0} := 1, \quad c_0 := 1, \quad \alpha^{(0)} := () \in W(\ell_{2,0}, c_0), \quad (5.48a)$$

and for every $t \geq 1$,

$$\ell_{1,t} := s\ell_{2,t-1}, \quad \ell_{2,t} := q\ell_{1,t} = sq \cdot \ell_{2,t-1}, \quad c_t := 4c_{t-1} = 4^t \quad (5.48b)$$

and

$$\alpha^{(t)} := \alpha^{(t-1)} \circ (1)^{\ell_{1,t} - \ell_{2,t-1}} \circ (2)^{c_{t-1}} \circ (1)^{\ell_{2,t} - \ell_{1,t}} \circ (2)^{2c_{t-1}} \in W(\ell_{2,t}, c_t). \quad (5.48c)$$

For the reader's convenience those definitions are illustrated in Figure 5.3.

By the definition in (5.22b) we clearly have

$$\delta(\alpha^{(0)}) = 1. \quad (5.49)$$

We proceed by deriving lower bounds for $\delta(\alpha^{(t)})$, $t \geq 1$. For t fixed, let

$$p \in \arg \min_{0 \leq j \leq \ell_{2,t-1} - 1} \frac{x_{1,j} + \beta(\alpha^{(t-1)})}{j + 1}, \quad (5.50)$$

where $\beta(\alpha^{(t-1)})$ is defined in (5.22a). By Lemma 5.13 we have

$$\frac{x_{1,p+k(p+1)} - x_{1,p}}{k(p+1)} \stackrel{(5.15)}{\geq} \frac{x_{1,p} + \beta(\alpha^{(t-1)})}{p+1} \stackrel{(5.22b), (5.50)}{=} \delta(\alpha^{(t-1)})$$

for all $k \geq 1$ with $p + k(p + 1) \leq \ell_{1,t} - 1$. As the entries of the sequence x_1 are non-negative and increasing (recall Lemma 5.9) this implies

$$x_{1,j} \geq \delta(\alpha^{(t-1)}) \cdot (j - 2p) \geq \delta(\alpha^{(t-1)}) \cdot (j - 2\ell_{2,t-1} + 2) , \quad 0 \leq j \leq \ell_{1,t} - 1 , \quad (5.51a)$$

where we used that $p \leq \ell_{2,t-1} - 1$ in the second estimate. Using (5.51a) and the trivial bound

$$x_{2,j} \geq 0 , \quad 0 \leq j \leq c_{t-1} - 1 , \quad (5.51b)$$

we obtain from (5.3), using that $\ell_{2,t} < 2\ell_{1,t}$,

$$\begin{aligned} x_{2,j} &\geq \delta(\alpha^{(t-1)}) \cdot (\ell_{1,t} - 4\ell_{2,t-1} + 3) , \quad c_{t-1} \leq j \leq 2c_{t-1} - 1 , \\ x_{1,j} &\geq \delta(\alpha^{(t-1)}) \cdot (\ell_{1,t} + j - 8\ell_{2,t-1} + 6) , \quad \ell_{1,t} \leq j \leq \ell_{2,t} - 1 , \\ x_{2,j} &\geq \delta(\alpha^{(t-1)}) \cdot (\ell_{1,t} + \ell_{2,t} - 8\ell_{2,t-1} + 6) , \quad 2c_{t-1} \leq j \leq 3c_{t-1} - 1 , \\ x_{2,j} &\geq \delta(\alpha^{(t-1)}) \cdot (2\ell_{1,t} + \ell_{2,t} - 12\ell_{2,t-1} + 9) , \quad 3c_{t-1} \leq j \leq 4c_{t-1} - 1 , \end{aligned} \quad (5.51c)$$

where we ignored the summand $+1$ arising from (5.3c) in all these lower bounds.

It follows that

$$\beta(\alpha^{(t)}) \stackrel{(5.22a),(5.51)}{\geq} \delta(\alpha^{(t-1)}) (2\ell_{1,t} + \ell_{2,t} - 12\ell_{2,t-1}) \quad (5.52)$$

(where we ignored the summand $+1$ from (5.22a) and the summand $+9\delta(\alpha^{(t-1)})$ from (5.51)). It is readily checked that for the lower bounds given in (5.51) and (5.52), the minimum in (5.22b) is attained either for $j = \ell_{1,t} - 1$ or for $j = \ell_{2,t} - 1$, yielding

$$\begin{aligned} \delta(\alpha^{(t)}) &\stackrel{(5.22b),(5.51),(5.52)}{\geq} \delta(\alpha^{(t-1)}) \cdot \min \left\{ \frac{3\ell_{1,t} + \ell_{2,t} - 14\ell_{2,t-1} + 1}{\ell_{1,t}}, \frac{3\ell_{1,t} + 2\ell_{2,t} - 20\ell_{2,t-1} + 5}{\ell_{2,t}} \right\} \\ &\stackrel{(5.48b)}{\geq} \underbrace{\delta(\alpha^{(t-1)}) \cdot \min \left\{ 3 + q - \frac{14}{s}, 2 + 3q^{-1} - \frac{20}{qs} \right\}}_{=: f(q,s)} = \delta(\alpha^{(t-1)}) \cdot f(q, s) . \end{aligned} \quad (5.53)$$

(Note the similarities between (5.45), (5.46), (5.47) and (5.51), (5.52), (5.53), respectively.)

Combining (5.49) and (5.53) and using the definition in (5.22d) we thus have

$$\delta(c_t) \geq \delta(\alpha^{(t)}) \geq (f(q, s))^t \quad (5.54)$$

for all $t \geq 0$. Observing that for $q \rightarrow \frac{1}{2}(\sqrt{13} - 1)$ and $s \rightarrow \infty$ we have

$$f(q, s) \rightarrow \frac{1}{2}(\sqrt{13} + 5) = \delta(4)$$

(recall Theorem 5.4), we obtain that for any $t \geq 0$ we have

$$\delta(4^t) = \delta(c_t) \geq (\delta(4))^t .$$

□

5.4.6. Lower bound for $k^*(P_\ell, 2)$. An extension of the construction in the previous section finally yields the lower bound on $k^*(P_\ell, 2)$ claimed in Theorem 5.2.

PROOF OF THEOREM 5.2 (LOWER BOUND). We will reuse most of the analysis from the previous proof for the fixed parameters

$$q := 1.3 , \quad s := 320 . \quad (5.55)$$

Note that for q and s as in (5.55) and any $t \geq 0$, the analysis of the strategy sequence $\alpha^{(t)} \in W(\ell_{2,t}, c_t)$ defined in (5.48) yields that

$$\delta(\alpha^{(t)}) \stackrel{(5.54)}{\geq} (f(q, s))^t \geq \left(\frac{17}{4}\right)^t. \quad (5.56)$$

For $t \geq 0$, we now set

$$\widehat{\ell}_t := 10\ell_{2,t} \stackrel{(5.48b)}{=} 10 \cdot (sq)^t \stackrel{(5.55)}{=} 10 \cdot 416^t \quad (5.57)$$

and extend $\alpha^{(t)} \in W(\ell_{2,t}, c_t)$ to a strategy sequence $\widehat{\alpha}^{(t)} \in W(\widehat{\ell}_t, \widehat{\ell}_t)$ by defining

$$\widehat{\alpha}^{(t)} := \alpha^{(t)} \circ (1)^{\widehat{\ell}_t - \ell_{2,t}} \circ (2)^{\widehat{\ell}_t - c_t}.$$

Similarly to the previous proof, we obtain that the sequences x_1 and x_2 defined for $\widehat{\alpha}^{(t)}$ in (5.3) satisfy

$$x_{1,j} \geq \delta(\alpha^{(t)}) \cdot (j - 2\ell_{2,t} + 2), \quad 0 \leq j \leq \widehat{\ell}_t - 1,$$

and

$$x_{2,j} \geq \delta(\alpha^{(t)}) \cdot (k-1)(\widehat{\ell}_t - 4\ell_{2,t} + 3), \quad (k-1)c_t \leq j \leq kc_t - 1,$$

for every integer $1 \leq k \leq \widehat{\ell}_t/c_t$. Putting everything together, we obtain

$$\begin{aligned} k(\widehat{\alpha}^{(t)}) &\geq \delta(\alpha^{(t)}) \cdot \frac{\widehat{\ell}_t}{c_t} \cdot (\widehat{\ell}_t - 4\ell_{2,t} + 3) \\ &\geq \left(1 - 4\frac{\ell_{2,t}}{\widehat{\ell}_t}\right) \cdot \frac{\delta(\alpha^{(t)})}{c_t} \cdot \widehat{\ell}_t^2 \\ &\stackrel{(5.48b), (5.56), (5.57)}{\geq} 0.6 \cdot \left(\frac{17}{16}\right)^t \cdot \widehat{\ell}_t^2 \\ &\stackrel{(5.57)}{\geq} 0.6 \cdot \left(\frac{\widehat{\ell}_t}{10}\right)^{\log_{416}(17/16)} \cdot \widehat{\ell}_t^2 \geq 0.5 \cdot \widehat{\ell}_t^{2.01}. \end{aligned} \quad (5.58)$$

Applying Proposition 5.10 we obtain from (5.58) that $k^*(P_\ell, 2) = k^*(P_\ell, P_\ell) = \Omega(\ell^{2.01})$, proving the claimed lower bound. \square

REMARK 5.20. Observe that when performing the analysis of the previous proof with the scaling factor s as a variable (and q near $\frac{1}{2}(\sqrt{13} - 1)$ fixed), we obtain an improvement of

$$\log_{sq}(f(s, q)/4)$$

over 2 in the exponent of $\widehat{\ell}_t$ in (5.58). The choice of $s := 320$ in (5.55) roughly maximizes this gain.

5.4.7. Putting everything together. In this last section we complete the proofs of Theorem 5.5 and Theorem 5.2 by collecting our findings from the previous sections.

PROOF OF THEOREM 5.5. As already mentioned, the upper bound follows by combining Theorem 5.14 with Lemma 5.15, observing that $\log_2(3) = 1.584\dots < 1.59$. The lower bound follows by combining Theorem 5.14 with Lemma 5.18, using the monotonicity guaranteed by Lemma 5.16 and observing that $\log_4(\delta(4)) = \log_4(\frac{1}{2}(\sqrt{13} + 5)) = 1.052\dots > 1.05$. \square

PROOF OF THEOREM 5.2. As already mentioned, the upper bound follows immediately by combining Theorem 5.3 with the upper bound on $\delta(c)$ stated in Theorem 5.5, using the non-negativity of the $o(1)$ term in Theorem 5.3. The proof of the lower bound was given in Section 5.4.6. \square

Bibliography

- [AF99] D. Achlioptas and E. Friedgut, *A sharp threshold for k -colorability*, Random Structures Algorithms **14** (1999), no. 1, 63–70.
- [AK97] N. Alon and M. Krivelevich, *The concentration of the chromatic number of random graphs*, Combinatorica **17** (1997), no. 3, 303–313.
- [AKS99] N. Alon, M. Krivelevich, and B. Sudakov, *List coloring of random and pseudo-random graphs*, Combinatorica **19** (1999), no. 4, 453–472.
- [BB10] J. Balogh and J. Butterfield, *Online Ramsey games for triangles in random graphs*, Discrete Math. **310** (2010), no. 24, 3653–3657.
- [BCG04] E. Berlekamp, J. Conway, and R. Guy, *Winning ways for your mathematical plays. Vols. 1–4*, second ed., A K Peters Ltd., Natick, MA, 2001–2004.
- [Bec83a] J. Beck, *On size Ramsey number of paths, trees, and circuits. I*, J. Graph Theory **7** (1983), no. 1, 115–129.
- [Bec83b] ———, *There is no fast method for finding monochromatic complete subgraphs*, J. Combin. Theory Ser. B **34** (1983), no. 1, 58–64.
- [Bec93] ———, *Achievement games and the probabilistic method*, Combinatorics, Paul Erdős is eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, pp. 51–78.
- [Bec94] ———, *Deterministic graph games and a probabilistic intuition*, Combin. Probab. Comput. **3** (1994), no. 1, 13–26.
- [Bec96] ———, *Foundations of positional games*, Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995), vol. 9, 1996, pp. 15–47.
- [Bec08] ———, *Combinatorial games*, Encyclopedia of Mathematics and its Applications, vol. 114, Cambridge University Press, Cambridge, 2008.
- [BEL76] S. Burr, P. Erdős, and L. Lovasz, *On graphs of Ramsey type*, Ars Combinatoria **1** (1976), no. 1, 167–190.
- [BF01] T. Bohman and A. Frieze, *Avoiding a giant component*, Random Structures Algorithms **19** (2001), no. 1, 75–85.
- [BFS08] T. Bohman, A. Frieze, and B. Sudakov, *The game chromatic number of random graphs*, Random Structures Algorithms **32** (2008), no. 2, 223–235.
- [BGK⁺11] J. Butterfield, T. Grauman, W. Kinnersley, K. Milans, C. Stocker, and D. West, *On-line Ramsey theory for bounded degree graphs*, Electron. J. Combin. **18** (2011), no. 1, Research Paper 136, 17 pp. (electronic).
- [BMS11⁺] M. Belfrage, T. Mütze, and R. Spöhel, *Probabilistic one-player Ramsey games via deterministic two-player games*, Submitted.
- [Bol81] B. Bollobás, *Threshold functions for small subgraphs*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 2, 197–206.
- [Bol88] ———, *The chromatic number of random graphs*, Combinatorica **8** (1988), no. 1, 49–55.
- [Bol98] ———, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998.
- [Bol01] ———, *Random graphs*, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.
- [Bou02] C. Bouton, *Nim, a game with a complete mathematical theory*, Ann. of Math. (2) **3** (1901/02), no. 1–4, 35–39.
- [BT87] B. Bollobás and A. Thomason, *Threshold functions*, Combinatorica **7** (1987), no. 1, 35–38.
- [BT95] ———, *Generalized chromatic numbers of random graphs*, Proceedings of the Sixth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science, “Random Graphs ’93” (Poznań, 1993), vol. 6, 1995, pp. 353–356.

- [CE78] V. Chvátal and P. Erdős, *Biased positional games*, Ann. Discrete Math. **2** (1978), 221–229, Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976).
- [CG83] F. Chung and C. Grinstead, *A survey of bounds for classical Ramsey numbers*, J. Graph Theory **7** (1983), no. 1, 25–37.
- [Con10] D. Conlon, *On-line Ramsey numbers*, SIAM J. Discrete Math. **23** (2009/10), no. 4, 1954–1963.
- [Con09] ———, *A new upper bound for diagonal Ramsey numbers*, Ann. of Math. (2) **170** (2009), no. 2, 941–960.
- [COPS08] A. Coja-Oghlan, K. Panagiotou, and A. Steger, *On the chromatic number of random graphs*, J. Combin. Theory Ser. B **98** (2008), no. 5, 980–993.
- [DFR10] A. Dudek, P. Frankl, and V. Rödl, *Some recent results on Ramsey-type numbers*, Discrete Appl. Math. **In Press, Corrected Proof** (2010), –.
- [DR08] A. Dudek and V. Rödl, *On the Folkman number $f(2, 3, 4)$* , Experiment. Math. **17** (2008), no. 1, 63–67.
- [DR10] ———, *An almost quadratic bound on vertex Folkman numbers*, J. Combin. Theory Ser. B **100** (2010), no. 2, 132–140.
- [EFRS78] P. Erdős, R. Faudree, C. Rousseau, and R. Schelp, *The size Ramsey number*, Period. Math. Hungar. **9** (1978), no. 1-2, 145–161.
- [EH67] P. Erdős and A. Hajnal, *Research problems 2–3*, J. Combin. Theory **2** (1967), 104–105.
- [ER59] P. Erdős and A. Rényi, *On random graphs. I*, Publ. Math. Debrecen **6** (1959), 290–297.
- [ER60] ———, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **5** (1960), 17–61.
- [ER61a] ———, *On the evolution of random graphs*, Bull. Inst. Internat. Statist. **38** (1961), 343–347.
- [ER61b] ———, *On the strength of connectedness of a random graph*, Acta Math. Acad. Sci. Hungar. **12** (1961), 261–267.
- [ER63] ———, *Asymmetric graphs*, Acta Math. Acad. Sci. Hungar. **14** (1963), 295–315.
- [ER64] ———, *On random matrices*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1964), 455–461 (1964).
- [ER66] ———, *On the existence of a factor of degree one of a connected random graph*, Acta Math. Acad. Sci. Hungar. **17** (1966), 359–368.
- [ER68] ———, *On random matrices. II*, Studia Sci. Math. Hungar. **3** (1968), 459–464.
- [Erd47] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294.
- [ES35] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470.
- [FK98] U. Feige and J. Kilian, *Zero knowledge and the chromatic number*, J. Comput. System Sci. **57** (1998), no. 2, 187–199, Complexity 96—The Eleventh Annual IEEE Conference on Computational Complexity (Philadelphia, PA).
- [FK00] E. Friedgut and M. Krivelevich, *Sharp thresholds for certain Ramsey properties of random graphs*, Random Structures Algorithms **17** (2000), no. 1, 1–19.
- [FKR⁺03] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, and P. Tetali, *Ramsey games against a one-armed bandit*, Combin. Probab. Comput. **12** (2003), no. 5-6, 515–545, Special issue on Ramsey theory.
- [FL07] J. Fox and K. Lin, *The minimum degree of Ramsey-minimal graphs*, J. Graph Theory **54** (2007), no. 2, 167–177.
- [Fol70] J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring.*, SIAM J. Appl. Math. **18** (1970), 19–24.
- [FR86] P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without K_4* , Graphs Combin. **2** (1986), no. 2, 135–144.
- [Fri05] E. Friedgut, *Hunting for sharp thresholds*, Random Structures Algorithms **26** (2005), no. 1-2, 37–51.
- [FRRT06] E. Friedgut, V. Rödl, A. Ruciński, and P. Tetali, *A sharp threshold for random graphs with a monochromatic triangle in every edge coloring*, Mem. Amer. Math. Soc. **179** (2006), no. 845, vi+66.
- [FS02] R. Faudree and R. Schelp, *A survey of results on the size Ramsey number*, Paul Erdős and his mathematics, II (Budapest, 1999), Bolyai Soc. Math. Stud., vol. 11, János Bolyai Math. Soc., Budapest, 2002, pp. 291–309.
- [GHK04] J. Grytczuk, M. Hałuszczak, and H. Kierstead, *On-line Ramsey theory*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 60, 10 pp. (electronic), Paper number later changed by the publisher from 60 to 57.
- [GKP08] J. Grytczuk, H. Kierstead, and P. Prałat, *On-line Ramsey numbers for paths and stars*, Discrete Math. Theor. Comput. Sci. **10** (2008), no. 3, 63–74.

- [GM75] G. Grimmett and C. McDiarmid, *On colouring random graphs*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 313–324.
- [GR87] R. Graham and V. Rödl, *Numbers in Ramsey theory*, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 111–153.
- [GRS90] R. Graham, B. Rothschild, and J. Spencer, *Ramsey theory*, second ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 1990.
- [Gru39] P. Grundy, *Mathematics and games*, Eureka **2** (1939), 6–8, (reprinted 1964, Eureka **27**, 9–11).
- [HKL95] P. Haxell, Y. Kohayakawa, and T. Łuczak, *The induced size-Ramsey number of cycles*, Combin. Probab. Comput. **4** (1995), no. 3, 217–239.
- [Häl11] M. Hälg, *Avoiding forests in vertex-coloring games*, Master thesis, ETH Zurich, Switzerland, 2011.
- [JŁR00] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.
- [JT95] T. Jensen and B. Toft, *Graph coloring problems*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 1995.
- [Kar72] R. Karp, *Reducibility among combinatorial problems*, Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972), Plenum, New York, 1972, pp. 85–103.
- [KK09] H. Kierstead and G. Konjevod, *Coloring number and on-line Ramsey theory for graphs and hypergraphs*, Combinatorica **29** (2009), no. 1, 49–64.
- [KLS09] M. Krivelevich, P.-S. Loh, and B. Sudakov, *Avoiding small subgraphs in Achlioptas processes*, Random Structures Algorithms **34** (2009), no. 1, 165–195.
- [KM10] R. Kang and C. McDiarmid, *The t -improper chromatic number of random graphs*, Combin. Probab. Comput. **19** (2010), no. 1, 87–98.
- [KP09] M. Krivelevich and B. Patkós, *Equitable coloring of random graphs*, Random Structures Algorithms **35** (2009), no. 1, 83–99.
- [KR94] A. Kurek and A. Ruciński, *Globally sparse vertex-Ramsey graphs*, J. Graph Theory **18** (1994), no. 1, 73–81.
- [KR97] M. Karoński and A. Ruciński, *The origins of the theory of random graphs*, The mathematics of Paul Erdős, I (R. Graham and J. Nešetřil, eds.), Springer, Berlin, 1997, pp. 311–336.
- [KR05] A. Kurek and A. Ruciński, *Two variants of the size Ramsey number*, Discuss. Math. Graph Theory **25** (2005), no. 1-2, 141–149.
- [ŁRV92] T. Łuczak, A. Ruciński, and B. Voigt, *Ramsey properties of random graphs*, J. Combin. Theory Ser. B **56** (1992), no. 1, 55–68.
- [LS08] P.-S. Loh and B. Sudakov, *On the strong chromatic number of random graphs*, Combin. Probab. Comput. **17** (2008), no. 2, 271–286.
- [Łuc91a] T. Łuczak, *The chromatic number of random graphs*, Combinatorica **11** (1991), no. 1, 45–54.
- [Łuc91b] ———, *A note on the sharp concentration of the chromatic number of random graphs*, Combinatorica **11** (1991), no. 3, 295–297.
- [Mat87] D. Matula, *Expose-and-merge exploration and the chromatic number of a random graph*, Combinatorica **7** (1987), no. 3, 275–284.
- [McD90] C. McDiarmid, *On the chromatic number of random graphs*, Random Structures Algorithms **1** (1990), no. 4, 435–442.
- [MMS07] M. Marciniszyn, D. Mitsche, and M. Stojaković, *Online balanced graph avoidance games*, European J. Combin. **28** (2007), no. 8, 2248–2263.
- [MRS11⁺] T. Mütze, T. Rast, and R. Spöhel, *Coloring random graphs online without creating monochromatic subgraphs*, Submitted. An extended abstract appeared in the proceedings of SODA ’11.
- [MS10] M. Marciniszyn and R. Spöhel, *Online vertex-coloring games in random graphs*, Combinatorica **30** (2010), no. 1, 105–123, An extended abstract appeared in the proceedings of SODA ’07.
- [MS11] T. Mütze and R. Spöhel, *On the path-avoidance vertex-coloring game*, Electron. J. Combin. **18** (2011), no. 1, Research Paper 163, 33 pp. (electronic), An extended abstract appeared in the proceedings of EuroComb ’11.
- [MSS09a] M. Marciniszyn, R. Spöhel, and A. Steger, *Online Ramsey games in random graphs*, Combin. Probab. Comput. **18** (2009), no. Special Issue 1-2, 271–300.
- [MSS09b] ———, *Upper bounds for online Ramsey games in random graphs*, Combin. Probab. Comput. **18** (2009), no. Special Issue 1-2, 259–270.

- [MST11] T. Mütze, R. Spöhel, and H. Thomas, *Small subgraphs in random graphs and the power of multiple choices*, J. Combin. Theory Ser. B **101** (2011), no. 4, 237 – 268.
- [Müt] T. Mütze, currently <http://www.as.inf.ethz.ch/muetze>.
- [Nas51] J. Nash, *Non-cooperative games*, Ann. of Math. (2) **54** (1951), 286–295.
- [Neš95] J. Nešetřil, *Ramsey theory*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 1331–1403.
- [NR76] J. Nešetřil and V. Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, J. Combin. Theory Ser. B **20** (1976), no. 3, 243–249.
- [PST09] A. Prakash, R. Spöhel, and H. Thomas, *Balanced online Ramsey games in random graphs*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 11, 22 pp. (electronic).
- [Rad94] S. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin. (1994), Dynamic survey, latest revision 2009.
- [Ram30] F. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **s2-30** (1930), no. 1, 264–286.
- [RR93] V. Rödl and A. Ruciński, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, pp. 317–346.
- [RR94] ———, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures Algorithms **5** (1994), no. 2, 253–270.
- [RR95] ———, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), no. 4, 917–942.
- [Rut86] V. Rutenburg, *Complexity of generalized graph coloring*, Mathematical Foundations of Computer Science 1986 (J. Gruska, B. Rován, and J. Wiedermann, eds.), Lecture Notes in Computer Science, vol. 233, Springer Berlin / Heidelberg, 1986, 10.1007/BFb0016284, pp. 573–581.
- [Sch92] E. Scheinerman, *Generalized chromatic numbers of random graphs*, SIAM J. Discrete Math. **5** (1992), no. 1, 74–80.
- [SG11] R. Spöhel and L. Gugelmann, *On balanced coloring games in random graphs*, Proceedings of the 6th European Conference on Combinatorics, Graph Theory and Applications (EuroComb ’11), 2011.
- [Spe75] J. Spencer, *Ramsey’s theorem—a new lower bound*, J. Combin. Theory Ser. A **18** (1975), 108–115.
- [Spr36] R. Sprague, *Über mathematische Kampfspiele*, Tohoku Math. J. **41** (1936), 438–444.
- [SZZ10] T. Szabó, P. Zumstein, and S. Zürcher, *On the minimum degree of minimal Ramsey graphs*, J. Graph Theory **64** (2010), no. 2, 150–164.
- [Tho88] A. Thomason, *An upper bound for some Ramsey numbers*, J. Graph Theory **12** (1988), no. 4, 509–517.
- [Tho01] C. Thomassen, *Chromatic graph theory*, Challenges for the 21st century (Singapore, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 183–195.
- [vN28] J. v. Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), no. 1, 295–320.
- [vNM44] J. v. Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, 1944.
- [Vyg11] J. Vygen, *Splitting trees at vertices*, Discrete Math. **311** (2011), no. 1, 67–69.
- [Wac11] C. Wachter, *Computing the online vertex-ramsey density of forests*, Bachelor thesis, ETH Zurich, Switzerland, 2011.