Numerik 2 - Übung 1

Florian Lüthi, i10b

February 28, 2012

Aufgabe 1

a) $v(u) = \alpha u + \beta$

b) $b = \begin{pmatrix} 0.95 \\ 2.14 \\ 2.98 \end{pmatrix}$, $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$

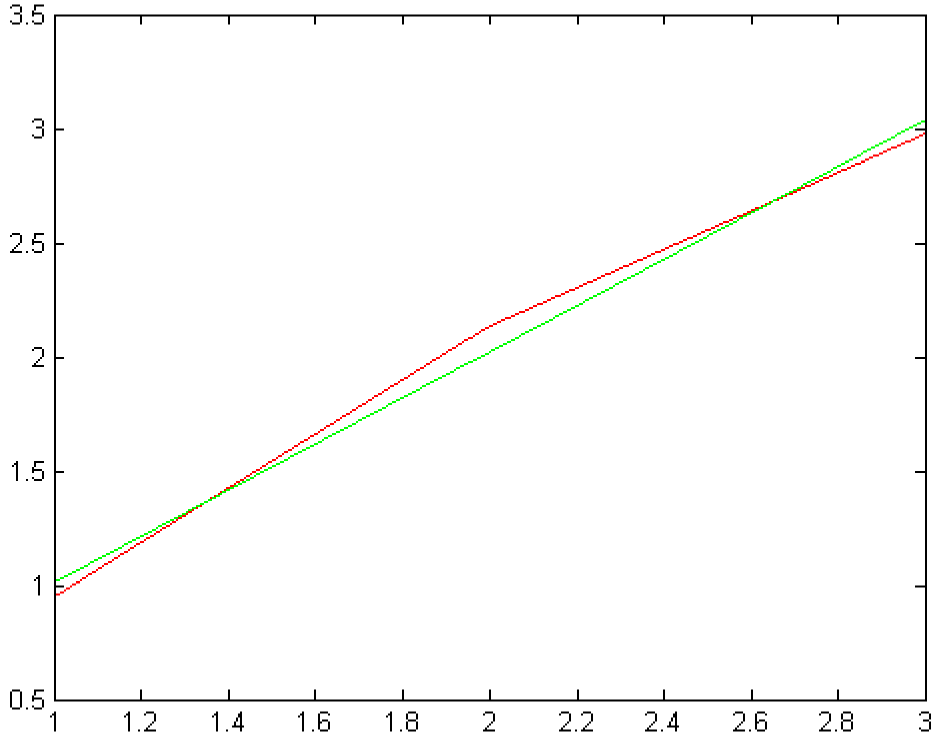
c) durch Normalgleichung:

$$\begin{aligned}
A^T A &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \\
A^T b &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0.95 \\ 2.14 \\ 2.98 \end{pmatrix} = \begin{pmatrix} 14.17 \\ 6.07 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} &= \begin{pmatrix} 14.17 \\ 6.07 \end{pmatrix} \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1.0150 \\ -0.0067 \end{pmatrix} \\
\Rightarrow v(u) &= 1.015u - 0.0067
\end{aligned}$$

oder durch analytische Minimierung der Abweichung:

$$
\begin{aligned}
F(\alpha,\beta) &= (-0.95+\alpha+\beta)^2+(-2.14+2\alpha+\beta)^2+(-2.98+3\alpha+\beta)^2 \\
\Rightarrow \frac{\partial F}{\partial \alpha} &= -1.9+2\alpha+2\beta-8.56+8\alpha+4\beta-17.88+18\alpha+6\beta \overset{!}{=} 0 \\
\frac{\partial F}{\partial \beta} &= -1.9+2\alpha+2\beta-4.28+4\alpha+2\beta-5.96+6\alpha+2\beta \overset{!}{=} 0 \\
\Rightarrow \beta &= \frac{-28\alpha+28.34}{12} \\
\Rightarrow \alpha &= \frac{-6\beta+12.14}{12} = \frac{14\alpha-2.03}{12} \\
\Rightarrow 12\alpha &= 14\alpha-2.03 \Rightarrow \alpha = 1.015 \\
\Rightarrow \beta &= \frac{-28\cdot 1.015+28.34}{12} = -0.0067 \\
\Rightarrow v(u) &= 1.015u-0.0067
\end{aligned}
$$

d) Bitteschön. Rot sind die verbundenen Messpunkte, grün die Ausgleichsgerade.



Aufgabe 2

a)

$$F(\alpha,\beta) = (a_{11}\alpha + a_{12}\beta - b_1)^2 + (a_{21}\alpha + a_{22}\beta - b_2)^2 + (a_{31}\alpha + a_{32}\beta - b_3)^2$$

b)

$$\begin{aligned}
\frac{\partial F}{\partial \alpha} &= 2a_{11}(a_{11}\alpha + a_{12}\beta - b_1) + 2a_{21}(a_{21}\alpha + a_{22}\beta - b_2) + 2a_{31}(a_{31}\alpha + a_{32}\beta - b_3) \\
\frac{\partial F}{\partial \beta} &= 2a_{12}(a_{11}\alpha + a_{12}\beta - b_1) + 2a_{22}(a_{21}\alpha + a_{22}\beta - b_2) + 2a_{32}(a_{31}\alpha + a_{32}\beta - b_3) \\
\Rightarrow \operatorname{grad}(F) &= \begin{pmatrix} 2a_{11}(a_{11}\alpha + a_{12}\beta - b_1) + 2a_{21}(a_{21}\alpha + a_{22}\beta - b_2) + 2a_{31}(a_{31}\alpha + a_{32}\beta - b_3) \\ 2a_{12}(a_{11}\alpha + a_{12}\beta - b_1) + 2a_{22}(a_{21}\alpha + a_{22}\beta - b_2) + 2a_{32}(a_{31}\alpha + a_{32}\beta - b_3) \end{pmatrix}
\end{aligned}$$

c) Gesucht: $A^TAx = A^Tb$

$$\begin{pmatrix} 2a_{11}(a_{11}\alpha + a_{12}\beta - b_1) \\ 2a_{12}(a_{11}\alpha + a_{12}\beta - b_1) \end{pmatrix} + \begin{pmatrix} 2a_{21}(a_{21}\alpha + a_{22}\beta - b_2) \\ 2a_{22}(a_{21}\alpha + a_{22}\beta - b_2) \end{pmatrix} + \begin{pmatrix} 2a_{31}(a_{31}\alpha + a_{32}\beta - b_3) \\ 2a_{32}(a_{31}\alpha + a_{32}\beta - b_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}(a_{11}\alpha + a_{12}\beta - b_1) + 2\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}(a_{21}\alpha + a_{22}\beta - b_2) + 2\begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix}(a_{31}\alpha + a_{32}\beta - b_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}(a_{11}\alpha + a_{12}\beta - b_1) + \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}(a_{21}\alpha + a_{22}\beta - b_2) + \begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix}(a_{31}\alpha + a_{32}\beta - b_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} a_{11}\alpha + a_{12}\beta - b_1 \\ a_{21}\alpha + a_{22}\beta - b_2 \\ a_{31}\alpha + a_{32}\beta - b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} a_{11}\alpha + a_{12}\beta \\ a_{21}\alpha + a_{22}\beta \\ a_{31}\alpha + a_{32}\beta \end{pmatrix} - \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} a_{11}\alpha + a_{12}\beta \\ a_{21}\alpha + a_{22}\beta \\ a_{31}\alpha + a_{32}\beta \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix}}_{A^T} \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix}}_{A^T} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{b}$$

qed.

Aufgabe 3

a) Ansatz: $v(u) = \alpha u + \beta$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ \vdots & \vdots \end{pmatrix}, b = \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \\ \vdots \end{pmatrix}, x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 385 & 55 \\ 55 & 11 \end{pmatrix}, A^T b = \begin{pmatrix} 1554 \\ 126 \end{pmatrix}$$

$$\alpha^* = 8.4, \beta^* = -30.5455$$
$$\Rightarrow v(u) = 8.4u - 30.5455$$

b) Ansatz: $v(u) = \alpha u^2 + \beta u + \gamma$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ \vdots & \vdots & \vdots \end{pmatrix}, b = \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \\ \vdots \end{pmatrix}, x = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 25333 & 3025 & 385 \\ 3025 & 385 & 55 \\ 385 & 55 & 11 \end{pmatrix}, A^T b = \begin{pmatrix} 15782 \\ 1554 \\ 126 \end{pmatrix}$$

$$\alpha^* = 2.4848, \beta^* = -16.4485, \gamma^* = 6.7273$$
$$\Rightarrow v(u) = 2.4848u^2 - 16.4485u + 6.7273$$

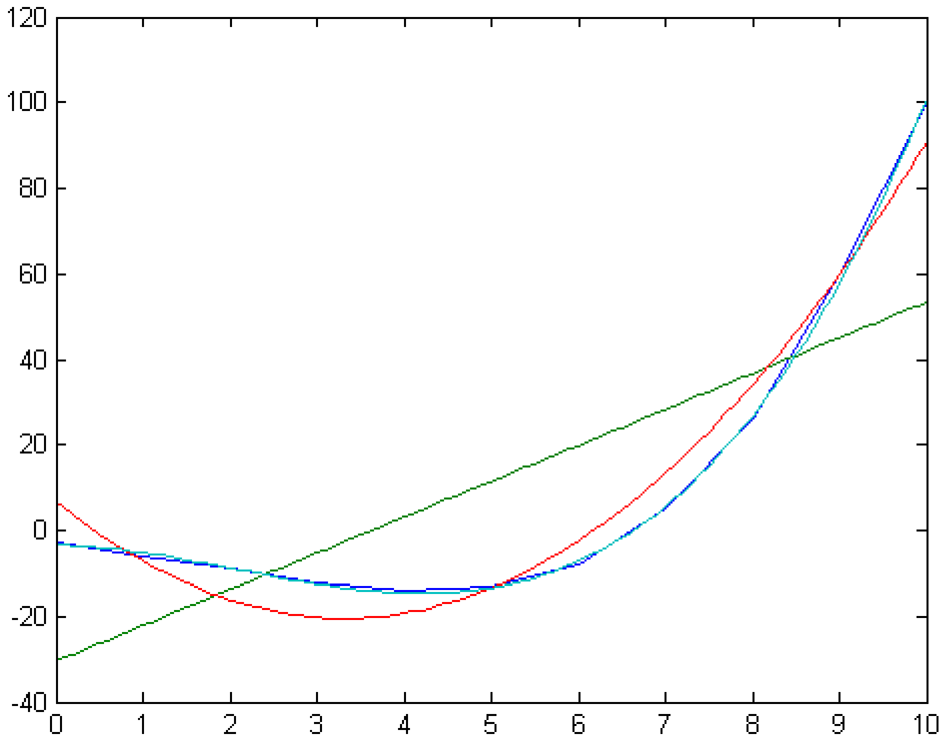
c) Ansatz: $v(u) = \alpha u^3 + \beta u^2 + \gamma u + \delta$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, b = \begin{pmatrix} -3 \\ -6 \\ -9 \\ -12 \\ \vdots \end{pmatrix}, x = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1978405 & 220825 & 25333 & 3025 \\ 220825 & 25333 & 3025 & 385 \\ 25333 & 3025 & 385 & 55 \\ 3025 & 385 & 55 & 11 \end{pmatrix}, A^Tb = \begin{pmatrix} 154116 \\ 15782 \\ 1554 \\ 126 \end{pmatrix}$$

$$\alpha^* = 0.2815, \beta^* = -1.7372, \gamma^* = -0.3485, \delta^* = -3.4056$$
$$\Rightarrow v(u) = 0.2815u^3 - 1.7372u^2 - 0.3485u - 3.4056$$

d) Die dunkelblaue Kurve zeigt die Messwerte, die grüne Kurve den linearen Ansatz, die rote den quadratischen und die hellblaue den kubischen:



e) Dass der lineare Ansatz nicht passen würde, war vorauszusehen. Dass hingegen der kubische die Messreihe ziemlich genau abbildet ist schon erstaunlich.