

Massachusetts Institute of Technology  
Department of Mechanical Engineering

2.160 Identification, Estimation, and Learning  
Spring 2006

***Problem Set No. 3***

Out: March 1, 2006 Due: March 8, 2006

For the first two problems below, hand calculation is recommended.

**Problem 1**

Consider a simple scalar random process governed by the following state transition equation:

$$x_{t+1} = x_t + w_t$$

where  $x_t$  is state, and  $w_t$  is a zero-mean process noise with

$$E[w_t w_s] = \begin{cases} Q > 0 & \forall t = s \\ 0 & \forall t \neq s \end{cases} .$$

The output of the process is observed with a single sensor having the following properties:

$$\begin{aligned} y_t &= x_t + v_t \\ E[v_t v_s] &= \begin{cases} R > 0 & \forall t = s \\ 0 & \forall t \neq s \end{cases} \\ E[v_t w_s] &= 0 \quad \forall t, \forall s \end{aligned}$$

We want to build a Kalman filter and examine its characteristics with respect to the state estimation error covariance and the sensor and process noise properties.

- (a). Write out all the recursive equations of Kalman filter for this scalar process. Simplify the equations as much as you can.
- (b). Given  $P_0 = 100$ ,  $Q = 1$ , and  $R = 1$ , plot the values of a posteriori and a priori error covariances,  $P_0, P_{1|0}, P_1, P_{2|1}, P_2, P_{3|2}, P_3, \dots$  against time. Also plot the Kalman gain  $K_1, K_2, K_3, \dots$ . Repeat it for different values of  $Q$  and  $R$ , and discuss how the process noise variance and the sensor noise variance are used in the Kalman filter for optimal state estimation.

## Problem 2

A vector discrete-time random sequence  $x_t$  is given by

$$x_{t+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_t + w_t$$

$w_t \sim N(0, 1)$  and uncorrelated (White)

The observation equation is given by

$$y_t = (1 \ 0) x_t + v_t$$

$v_t \sim N(0, 2 + (-1)^t)$  and uncorrelated (White)

$$E[v_t w_s] = 0 \quad \forall t, \forall s$$

Calculate the values of  $P_t$ ,  $P_{t+1|t}$ ,  $K_t$  for  $t = 1, \dots$  with

$$P_0 = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}.$$

## Problem 3

A heat pump is an important component involved in various energy systems ranging from air conditioners and refrigerators to fuel cells. Figure 1 depicts its principle, the refrigeration cycle. Refrigerant is first compressed at the compressor. The resultant high-pressure, high-temperature refrigerant is led to a heat exchanger, called a condenser, where the refrigerant becomes liquid while loosing heat. The liquid refrigerant is then delivered to a place that we wish to cool down. At the inlet, the high-pressure, liquid refrigerant is discharged through an expansion valve, which brings the refrigerant temperature to low. Then the low-temperature refrigerant is led to another heat exchanger, called an evaporator, where the refrigerant absorbs heat and changes its phase back to vapor. As this cycle is continued, heat is removed from the evaporator side to the condenser side.

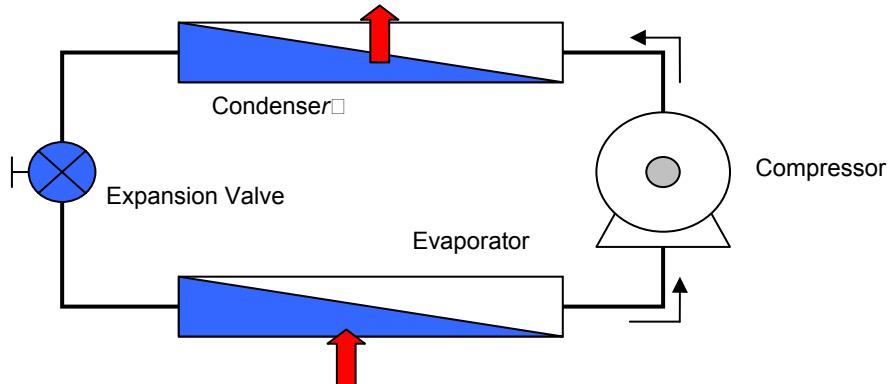
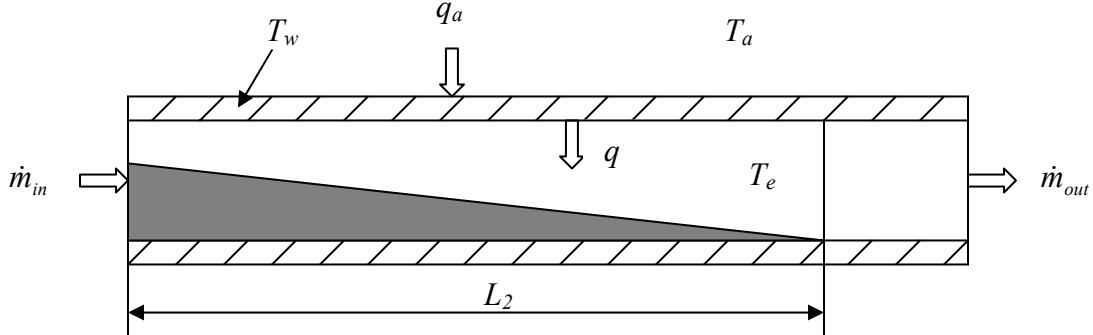


Figure 1 Refrigeration cycle

In both condenser and evaporator, refrigerant is changing its phase between liquid and gas. One side is all liquid and the other side is all gas. In between the refrigerant is in two-phase where liquid and gas co-exist and the temperature is uniform throughout the phase transition. One of the critical state variables involved in this refrigeration cycle is the length of the two-phase section,  $L_2$ , as shown in the figure.



**Figure 2 Evaporator:**  $T_e$  is the evaporating temperature,  $L_2$  is the length of the two-phase section.  $T_w$  is the wall temperature of the tube.  $T_a$  is the room air temperature.  $\dot{m}_{in}$  and  $\dot{m}_{out}$  are the inlet and outlet refrigerant mass flow rates, respectively.  $q$  is the heat transfer rate from the tube wall to the two-phase refrigerant.  $q_a$  is the heat transfer rate from the room to the tube wall.

If we know the two-phase length, we can easily obtain the total amount of heat transferred through the heat exchanger, and the control of the heat pump can significantly be improved. Unfortunately, however, no sensor is commercially available for directly measuring the two-phase length. The problem can be solved with use of Kalman filter estimating the two-phase length based on the process model and available low-cost sensors, such as thermocouples.

The real heat exchanger dynamics is nonlinear. A low-order, linearized evaporator model is given by

$$\begin{aligned} \begin{pmatrix} \partial \dot{T}_e \\ \partial \dot{T}_w \\ \partial \dot{L}_2 \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \partial T_e \\ \partial T_w \\ \partial L_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \\ b_2 \end{pmatrix} u + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} w \\ y &= (1 \quad 0 \quad 0) \begin{pmatrix} \partial T_e \\ \partial T_w \\ \partial L_2 \end{pmatrix} + v \end{aligned}$$

The discrete version of the above model is

$$\begin{aligned} X_{t+1} &= A_t X_t + B_t U_t + G_t w_t \\ y_t &= H_t \cdot x_t + v_t \end{aligned}$$

Using the data posted on the web, implement the Kalman filter of this system and reconstruct the two-phase length  $L_2$ .