

Department of Mechanical Engineering
 Massachusetts Institute of Technology
2.160 Identification, Estimation, and Learning

Mid-Term Examination

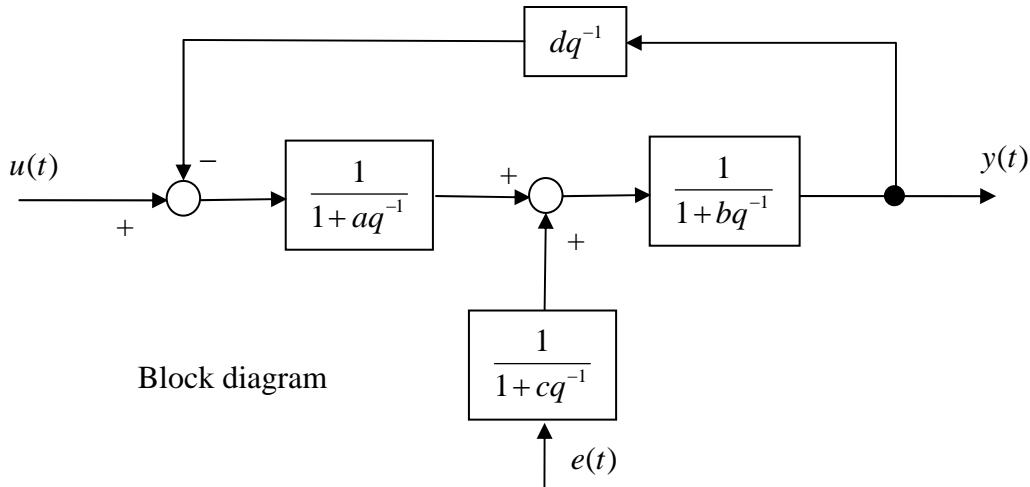
April 3, 2006

1:00 – 3:00 pm

Close book. Two sheets of notes are allowed.
 Show how you arrived at your answer.

Problem 1 (30 points)

Consider a linear stable system, as shown below. Variable $u(t)$ is input, $y(t)$ is output, and $e(t)$ is uncorrelated white noise with zero mean values. Answer the following questions.



- Obtain the one-step-ahead predictor of the output: $\hat{y}(t | t-1)$.
- Prove that the prediction error $y(t) - \hat{y}(t | t-1)$ is equivalent to the uncorrelated noise $e(t)$, if the parameters involved in the system are exactly known.

Problem 2 (40 points)

Consider a second order system with the following transfer function having a pair of complex conjugate poles:

$$G(s) = \frac{1}{s^2 + 2bs + b^2 + c^2}$$

where both parameters are positive real numbers, $b > 0, c > 0$. In an attempt to represent this system with a reduced-order Finite Impulse Response model, the continuous Laguerre series expansion method is used. Answer the following questions.

- Obtain the poles of the above transfer function, p_1 and p_2 , and find the locations of both poles in the z -plane when p_1 and p_2 are transformed to the z -domain by

$$z = \frac{s+a}{s-a}$$

where parameter $a > 0$ gives the pole of Laguerre series expansion, $-a$. Plot the transformed poles, z_1 and z_2 , on the z -plane by determining the magnitude and phase angle of each pole: $|z_i|, \angle z_i$.

b). We want to place the transformed poles close to the origin of the z -plane. Show that the distance between the transformed poles and the origin becomes minimal when the Laguerre pole is set to

$$a = \sqrt{b^2 + c^2}.$$

Problem 3 (30 points)

Consider the following state equation and measurement equation for a scalar dynamical system having uncorrelated noises with zero-mean Gaussian distributions:

$$x_{t+1} = a x_t + w_t \quad w_t \sim N(0, Q)$$

$$y_t = x_t + v_t \quad v_t \sim N(0, R)$$

$$E[w_t v_t] = 0$$

where parameter a is unknown but is assumed to be constant. We want to estimate both state x_t and the unknown parameter a by using Kalman filter. To this end the state space is augmented to the one defined as

$$X_t = \begin{pmatrix} x_t \\ a_t \end{pmatrix}$$

Note that the unknown parameter is now treated as another state variable to estimate. Consider the state equation of the augmented state vector X_t :

$$\begin{pmatrix} x_{t+1} \\ a_{t+1} \end{pmatrix} = \begin{pmatrix} a_t \cdot x_t \\ a_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(t)$$

The idea is to estimate X_t by designing a Kalman filter whose augmented state \hat{X}_t can converge to the true augmented state, given an initial estimate, say $\hat{X}_0 = (0, 0)^T$.

Answer the following questions.

- a). Is the augmented state equation linear or nonlinear? If nonlinear, obtain a linearized state equation using $x_t = \bar{x}_t + \Delta x_t$ and $a_t = \bar{a}_t + \Delta a_t$, where $\Delta \bullet$ is a deviation from reference/average $\bar{\bullet}$.
- b). Obtain Kalman filter for this augmented system. Write out equations for state estimate update, the Kalman gain, error covariance update, and state and covariance propagation. (Writing merely the general formula of KF is not good enough. Provide a specific formula needed for estimating a and x , recursively. In your answer, indicate clearly which variable is a real measurement, an average value, or an estimate.)