1. Prove that the triangle inequality holds for square operators $M, N \in \mathcal{L}(\mathcal{H})$

proof. Let $M, N \in \mathcal{L}(\mathcal{H})$ be arbitrary square operators. Then, the **variational characterization** property guarantees us the following inequality: [a]

$$\|N + M\|_1 = \max_U |Tr\{(N + M)U\}| \tag{1}$$

Under the assumption we are working with the maximal unitary U, we induce the next inequality:

$$|Tr\{(N + M)U\}| = |Tr\{NU\} + Tr\{MU\}| \leq Tr\{NU\}| + |Tr\{MU\}| \tag{2}$$

Using [b] Cauchy-Schwarz

$$|Tr\{NU\}| + |Tr\{MU\}| \leq \|N\|_1 + \|M\|_1 \tag{3}$$

Therefore,

$$\|N + M\|_1 \leq \|N\|_1 + \|M\|_1, \tag{4}$$

QED.

[a] Property 9.1.6 in Wilde
[b] refer to proof of Property 9.1.6

2. Show that the trace distance obeys a telescoping property:

$$\|\rho_1 \otimes \rho_2 - \sigma_1 \otimes \sigma_2\|_1 \leq \|\rho_1 - \sigma_1\|_1 + \|\rho_2 - \sigma_2\|_1 \tag{5}$$

for any density operators $\rho_1, \rho_2, \sigma_1, \sigma_2$.

proof.

3. Show that the trace distance is invariant with respect to an isometric quantum channel, in the following sense:

$$\|\rho_1 - \sigma_1\|_1 = \|U\rho_1 U^\dagger - U\sigma_1 U^\dagger\|_1 \tag{6}$$

where U is an isometry. The physical implication of the equality is that an isometric quantum channel applied to both states does not increase or decrease the distinguishability of the two states.

proof. Let us take U to be an arbitrary isometry applied to both ρ and σ. Then,

$$\|U\rho_1U^\dagger - U\sigma_1U^\dagger\|_1 = Tr\{\sqrt{(U\rho_1U^\dagger - U\sigma_1U^\dagger)^\dagger(U\rho_1U^\dagger - U\sigma_1U^\dagger)}\} \tag{7}$$

because $(U\rho_1U^\dagger - U\sigma_1U^\dagger)$ is self-adjoint, its square root is well-defined

$$Tr\{\sqrt{(U\rho_1U^\dagger - U\sigma_1U^\dagger)^\dagger(U\rho_1U^\dagger - U\sigma_1U^\dagger)}\} = Tr\{(U\rho_1U^\dagger - U\sigma_1U^\dagger)\} \tag{8}$$

we split the trace via linearity and then use the cyclic property of the trace and the definition of an isometry to claim the following:

$$Tr\{(U\rho_1U^\dagger - U\sigma_1U^\dagger)\} = Tr\{\rho_1\} - Tr\{\sigma_1\} \tag{9}$$

then, we arrive at our desired equality as shown below:

$$Tr\{\rho_1\} = Tr\{\sqrt{\rho_1^\dagger\rho_1}\} = \|\rho_1\|_1 \tag{10}$$

$$Tr\{\sigma_1\} = Tr\{\sqrt{\sigma_1^\dagger\sigma_1}\} = \|\sigma_1\|_1 \tag{11}$$

where the above two equations make clever use of the definition of a density operator. Hence,

$$\|\rho_1 - \sigma_1\|_1 = \|U\rho_1U^\dagger - U\sigma_1U^\dagger\|_1, \tag{12}$$

as desired.

4. Show that the trace norm of any Hermitian operator ω is given by the following optimization:

$$\|\omega\|_1 = \max_{-I\leq\Lambda\leq I} Tr\{\Lambda\omega\} \tag{13}$$

proof. Let $-I \leq \Lambda \leq I$ be arbitrary. Then, $Tr\{\Lambda\omega\} \leq |Tr\{\Lambda\omega\}|$. Applying Cauchy-Schwarz,

$$|Tr\{\Lambda\omega\}| \leq \sqrt{Tr\{\Lambda\Lambda^\dagger\}}\sqrt{Tr\{\omega\omega^\dagger\}} \tag{14}$$

Recall that since Λ is bounded between $-I \leq \Lambda \leq I$,

$$\sqrt{Tr\{\Lambda\Lambda^\dagger\}}\sqrt{Tr\{\omega\omega^\dagger\}} \leq \sqrt{Tr\{\omega\omega^\dagger\}} = Tr\{\omega\} = \|\omega\|_1 \tag{15}$$

where the last two equalities arise from the fact that ω is hermitian. Therefore, when $-I \leq \Lambda \leq I$ and ω is hermitian,

$$\|\omega\|_1 = \max_{-I\leq\Lambda\leq I} Tr\{\Lambda\omega\} \tag{16}$$

where $\Lambda = I$ is maxima for the optimization.

5. Suppose that the prior probabilities in the above hypothesis testing scenario are not uniform but are rather equal to p_0 and p_1. Show that the success probability is instead

given by

$$p_{succ} = \frac{1}{2}(1 + \|p_0\rho_0 - p_1\rho_1\|_1) \tag{17}$$

proof. For context, please read section (**9.1.4**) in Wilde. Suppose that Bob prepares one of two quantum states, ρ_0 and ρ_1 with *a priori* probabilities p_0 and p_1, respectively. That is, if X denotes the Bernoulli random variable assigned to the prior probabilities, $p_X(0) = p_0$ and $p_X(1) = p_1$. Alice can perform a binary POVM with elements $\Lambda \equiv \{\Lambda_0, \Lambda_1\}$ to distinguish the two states. That is, Alice guesses the state in question is ρ_0 if she receives outcome 0 and vice versa. Let Y denote the Bernoulli random variable assigned to the classical outcomes of her measurement.

The success probability $p_{succ}(\Lambda)$ for this hypothesis testing scenario is as follows:

$$p_{succ}(\Lambda) = p_{Y|X}(0|0)p_0 + p_{Y|X}(1|1)p_1 \tag{18}$$

$$p_{succ}(\Lambda) = Tr\{\Lambda_0\rho_0\}p_0 + Tr\{\Lambda_1\rho_1\}(1 - p_0) \tag{19}$$

Using the completeness relation of the POVM,

6. Prove that (**9.57**) holds for arbitrary Hermitian operators ρ and σ by exploiting the result of Exercuse (**9.1.6.**)

proof. Let ρ and σ be arbitrary hermitian operators and let $\Pi \in \mathcal{L}(\mathcal{H})$ such that $0 \leq \Pi \leq I$ be arbitrary as well. [a] Since ρ and σ are hermitian, so is their difference. Hence,

$$\|\rho - \sigma\|_1 = \max_{-I \leq \Lambda \leq I} Tr\{\Lambda(\rho - \sigma)\} \tag{20}$$

Given the constraint on Π (since we cannot guarantee Π and Λ are equivalent), the following inequality holds:

$$\|\rho - \sigma\|_1 \leq Tr\{\Pi(\rho - \sigma)\} = Tr\{\Pi\rho\} - Tr\{\Pi\sigma)\} \tag{21}$$

Re-arranging the above equation gives us that

$$Tr\{\Pi\rho\} \leq Tr\{\Pi\sigma\} - \|\rho - \sigma\|_1, \tag{22}$$

as desired.

[a] Suppose we have two quantum states, ρ, $\sigma \in \mathcal{D}(\mathcal{H})$ and an operator $\Pi \in \mathcal{L}(\mathcal{H})$ such that $0 \leq \Pi \leq I$. Then, $Tr\{\Pi\rho\} \leq Tr\{\Pi\sigma\} - \|\rho - \sigma\|_1$ is Corollary 9.1.1 which contains (9.57)

Link to the template used