1. [1]Show that state $|\psi\rangle_{RA}$ as defined in (**5.3**), is a purification of the density operator ρ_A, with a spectral decomposition as given in (**5.2**)

proof. We proceed to compute $Tr_R\{|\psi\rangle\langle\psi|_{RA}\}$ as follows:

$$Tr_R\{|\psi\rangle\langle\psi|_{RA}\} = Tr_R\{(\sum_x \sqrt{p_X(x)}\,|x\rangle_R\,|x\rangle_A)(\sum_x \sqrt{p_X(x)}\,|x\rangle_R\,|x\rangle_A)^\dagger\} \quad (1)$$

Evaluating the 'daggered' term,

$$Tr_R\{|\psi\rangle\langle\psi|_{RA}\} = Tr_R\{(\sum_x p_X(x)\,|xx\rangle_{RA}\,\langle xx|_{RA})\} \quad (2)$$

Now, before we take the partial trace of the bracketed expression, refer to the alternate notion of the partial trace operator as described in exercise (**4.3.9**) in Wilde. Re-expressing the tensor product[a],

$$Tr_R\{\sum_x p_X(x)\,|x\rangle_R\,\langle x|_R \otimes |x\rangle_A\,\langle x|_A\} = \sum_x p_X(x)\,\langle x|x\rangle_R\,|x\rangle\,\langle x|_A \quad (3)$$

which leaves us with the desired expression:

$$\sum_x p_X(x)\,\langle x|x\rangle_R\,|x\rangle\,\langle x|_A = \rho_A \quad (4)$$

[a]this overall procedure described in **4.3.9** saves a lot of time and effort – add it to the toolbox.

2. (**Cannonical Purification**) Let ρ_A be a density operator and let $\sqrt{\rho_A}$ be its unique positive semidefinite square root. We define the cannonical purification of ρ_A as follows:

$$(I_R \otimes \sqrt{\rho_A})\,|\Gamma\rangle_{RA} \quad (5)$$

where $|\Gamma\rangle_{RA}$ is the maximally unnormalized vector from (**3.233**).[2] Show that (**5.4**) is a purification of ρ_A

[1]https://arxiv.org/pdf/1106.1445 (the copy of Wilde we are using)
[2]Recall exercise (**3.7.12**) in Wilde, *the transpose trick or ricochet* – didn't use it here but helpful to know

proof. Spectrally decomposing $\rho_A = \sum_x p_X(x) |x\rangle \langle x|_A$, where $\{|x\rangle_A\}$ forms an O.N eigenbasis over $\mathcal{H}_\mathcal{A}$. Then, $\sqrt{\rho_A} = \sum_x \sqrt{p_X(x)} |x\rangle \langle x|_A$. Take the orthonormal basis over $\mathcal{H}_\mathcal{R}$, denoting it by $\{|x\rangle_R\}$. Then the unnormalized maximally entangled vector [a] $|\Gamma\rangle_{RA} = \sum_x |x\rangle_R |x\rangle_A$. [b] Now, all we have left to do is compute $(I_R \otimes \sqrt{\rho_A}(|\Gamma\rangle_{RA}))(I_R \otimes \sqrt{\rho_A})^\dagger |\Gamma\rangle_{RA}$ and trace over $\mathcal{H}_\mathcal{R}$.

$$(I_R \otimes \sqrt{\rho_A}) |\Gamma\rangle_{RA} = (I_R \otimes \sum_x \sqrt{p_X(x)} |x\rangle_A \langle x|_A) \sum_{x'} |x'\rangle_R |x'\rangle_A \tag{6}$$

$$(I_R \otimes \sum_x \sqrt{p_X(x)} |x\rangle_A \langle x|_A) \sum_{x'} |x'\rangle_R |x'\rangle_A = \sum_x |x\rangle_R \otimes \sqrt{p_X(x)} |x\rangle_A \tag{7}$$

since the only non-vanishing terms are when $x' = x$. $(I_R \otimes \sqrt{\rho_A}(|\Gamma\rangle_{RA}))(I_R \otimes \sqrt{\rho_A})^\dagger |\Gamma\rangle_{RA}$ and trace over $\mathcal{H}_\mathcal{R}$ is then

$$\sum_x |x\rangle \langle x|_R \otimes p_X(x) |x\rangle \langle x|_A \tag{8}$$

Tracing out R gives us the following:

$$\sum_x \langle x|x\rangle_R \, p_X(x) |x\rangle \langle x|_A = \sum_x p_X(x) |x\rangle \langle x|_A = \rho_A, \tag{9}$$

as desired.

[a] https://physics.stackexchange.com/questions/267293/does-the-entanglement-depend-on-the-basis

[b] https://quantumcomputing.stackexchange.com/questions/9242/maximally-entangled-state-definition-and-orthonormal-basis-of-maximally-entangl

Link to the template used