1. Prove that the trace is cyclic. That is, for three operators A,B, and C, the following relation holds $Tr\{ABC\} = Tr\{CAB\} = Tr\{BCA\}$

proof. The crux of our argument relies on clever insertion of the *completeness relation* to "shuffle" around the operators, so to speak.

$Tr\{ABC\} = \sum_i \langle i| ABC |i\rangle = \sum_i \langle i| AB(\sum_j |\phi_j\rangle \langle\phi_j|)C |i\rangle$

$Tr\{ABC\} = \sum_{i,j} \langle i| AB |\phi_j\rangle \langle\phi_j| C |i\rangle = \sum_{i,j} \langle\phi_j| C |i\rangle\langle i| AB |\phi_j\rangle$

$Tr\{ABC\} = \sum_j \langle\phi_j| CAB |\phi_j\rangle = Tr\{CAB\}$

The rest of the proof follows in similar fashion.

2. Suppose the ensemble has a degenerate probability distribution, say $p_X(0) = 1$ and $p_X(x) = 0$. What is the density operator of this degenerate ensemble?

Recall how we defined the density operator ρ of an ensemble:

$$\sum_{x\in\chi} P_X(x) |\psi_x\rangle \langle\psi_x| \tag{1}$$

Then, for the ensemble described above,

$$\rho = (1) |\psi_0\rangle \langle\psi_0| + \sum_{x\in\chi, x\neq 0} P_X(x) |\psi_x\rangle \langle\psi_x| = |\psi_0\rangle \langle\psi_0| \tag{2}$$

The density operator of any *pure state* is analogous to a degenerate probability distribution.

3. Prove the following equality:

$$Tr\{A\} = \langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS}, \tag{3}$$

where A is a square operator acting on a Hilbert space $\mathcal{H}_S$, I_R is the identity operator acting on a Hilbert space $\mathcal{H}_R$ isomorphic to $\mathcal{H}_S$ and $|\Gamma\rangle_{RS}$ is the unnormalized maximally entangled vector from 3.233. This gives an alternate formula for the trace of a square operator A.

chug and plug homie, chug and plug.

4. Prove that $Tr\{f(G^\dagger G\} = Tr\{f(GG^\dagger\}$

Note that the operator $G^\dagger G$ is hermitian, for any operator G. Hence, $G^\dagger G$ commutes with $GG^\dagger$, which means that they are **simultaneously diagonalizable**. Spectrally decomposing the respective operators into sums of their eigenprojections, applying the function $f()$ to the eigenvalues, and then summing yeilds the same value for both quantities of interest. Hence, $Tr\{f(G^\dagger G\} = Tr\{f(GG^\dagger\}$, as desired.

5. Show that the ensembles in exercise **4.1.5** have the same density operator

Just compute it homie.

6. Show that the coefficients λ_x are probabilities using the fact that $Tr\{\rho\} = 1$ and $\rho \geq 0$

proof. We start from the fact that the density operator ρ is hermitian, and we can write it as follows: $\rho = \sum_{x=0}^{d-1} \lambda_x |\phi_x\rangle \langle\phi_x|$. Applying the trace operator to ρ, we get that $\sum_{x=0}^{d-1} \lambda_x(\sum_i \langle i|\phi_x\rangle \langle\phi_x|i\rangle) = \sum_{x=0}^{d-1} \lambda_x = 1$. Also, since the density operator is positive semidefinite, we have that $\sum_{x=0}^{d-1} \lambda_x \langle i|\phi_x\rangle \langle\phi_x|i\rangle \geq 0$, which immediately implies that all the λ_x's must be positive real numbers.

Since the collection of λ_x's are positive reals that sum to 1, we have shown that they are probabilities, as desired.

7. Show that π is the density operator of an ensemble that chooses $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$ with equal probability.

The density operator of the described ensemble ρ is as follows:

$$\rho = \frac{1}{4}(|0\rangle \langle 0| + |1\rangle \langle 1| + |+\rangle \langle +| + |-\rangle \langle -|) \tag{4}$$

$$\rho = \frac{1}{4}(2I) = \frac{1}{2}I = \pi \tag{5}$$

since all the $|i\rangle$'s live in a two-dimensional hilbert space $\mathcal{H}$.

8. Show that the set of density operators acting on a given Hilbert space is a convex set.

proof. To show that the set $D(\mathcal{H})$ is convex, it suffices to show that for any $\lambda \in [0,1]$ and arbitrary density operators ρ, σ, $\lambda\rho + (1-\lambda)\sigma$ is also a density operator.

That is, the burden of proof is to show that $\lambda\rho + (1-\lambda)\sigma$ has unit trace, is hermitian, and is positive semi-definite.

9. Prove that the purity of a density operator ρ is equal to 1 if and only if ρ is a pure state, such that it can be written as $\rho = |\psi\rangle\langle\psi|$ for some unit vector ψ

proof. We prove the substantially more difficult direction first.

Let us suppose that the purity of a density operator ρ is 1 (and we want to show that ρ is then pure).

$$Tr\{\rho^2\} = \sum_i \langle i| \rho^2 |i\rangle = \sum_i \langle i| (\sum_{x=0}^{d-1} \lambda_x |\phi_x\rangle\langle\phi_x|)^2 |i\rangle \tag{6}$$

$$Tr\{\rho^2\} = \sum_i \sum_{x=0}^{d-1} \lambda_x^2 \langle i|\phi_x\rangle\langle\phi_x|i\rangle = \sum_{x=0}^{d-1} \lambda_x^2 \langle\phi_x|\phi_x\rangle \tag{7}$$

$$Tr\{\rho^2\} = \sum_{x=0}^{d-1} \lambda_x^2 = 1. \tag{8}$$

but we also know the following via the "cannonical representation" (Wilde) of the density operator

$$\sum_{x=0}^{d-1} \lambda_x = 1 \tag{9}$$

Note that all the λ_x's are bounded above 0 and below 1 ($0 \leq \lambda_x \leq 1$). The only possible way equation (8) and (9) both hold is if only one $\lambda_k = 1$ and the other λ_x's excluding it are all 0. One can reason through this argument geometrically (squaring a number that is *geq* 0 and *leq* 1 will decrease when squared, unless it is equal to 1).
Hence, since only one of the λ_x's is equal to 1, the density operator ρ is a pure state, written as $\rho = |\phi_k\rangle\langle\phi_k|$ for the k where $\lambda_k = 1$.

The other direction is left as an exercise to the reader, including the author himself.

10. Show that the matrix in **4.45** has unit trace, is Hermitian, and is non-negative for all

$\mathbf{r}$ such that $\|r\|_2 \leq 1$. It thus corresponds to any valid density matrix.

Note, we are dealing with the following matrix M: $\frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$, where $r_x = sin(\theta)cos(\varphi)$, $r_y = sin(\theta)sin(\varphi)$, $r_z = cos(\theta)$ where the angles are the cannonical bloch sphere parametrization angles. It is trivial to show that the matrix has unit trace and that it is hermitian.

Denote an arbitrary vector in the two-dimensional hilbert space $\mathcal{H}$ by $|\psi\rangle$. Its matrix representation is simply $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Computing $\langle\psi| M |\psi\rangle$ we get $\frac{1}{2}((1+r_z)\alpha\alpha^* + \alpha\beta(r_x - ir_y) + \alpha\beta(r_x + ir_y) + \beta\beta^*(1 - r_z))$ which is ≥ 0 given the constraints. Hence, the matrix M corresponds to any valid density matrix (on the bloch sphere).

11. Show that we can compute the Bloch sphere coordinates r_x, r_y, and r_x with the respective formulas $Tr\{X\rho\}$, $Tr\{Y\rho\}$, and $Tr\{Z\rho\}$ using the representation in (**4.45**) and the result of Exercise **3.3.6**

One can simply go through the computations and the desired results pop out without much difficulty. The author is not sure how to make use of Exercise **3.3.6** to make the problem easier, however.

12. Show that the eigenvalues of a general qubit density operator with density matrix representation in (**4.45**) are as follows: $\frac{1}{2}(1 \pm \|r\|_2)$

We compute the eigenvalues by solving for the λ's that give us $det(\lambda I - M) = 0$ (we will account for the factor of $\frac{1}{2}$ that comes along with M at the very end). This leaves us with having to solve for the roots of the following polynomial: $\lambda^2 - 2\lambda + (1 - \|r\|_2^2)$. Applying the quadratic formula gives us $\lambda_{\pm} = 1 \pm \|r\|_2$. Adding the factor $\frac{1}{2}$ of M at the very end gives us the desired $\frac{1}{2}(1 \pm \|r\|_2)$.

13. Show that a mixture of pure states $|\psi_j\rangle$ each with Bloch vector $\mathbf{r}_j$ and probability $p(j)$ gives a density matrix with the Bloch vector $\mathbf{r}$ where $\mathbf{r} = \sum_j p(j)\mathbf{r}_j$

Let us first recall what it means (in context) for a density matrix to have a *Bloch-vector* **r**. If ψ denotes a pure state, its density operator, or matrix, is simply $\rho_\psi = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ as given in **4.44** (Wilde). We then say that ψ has a bloch-vector $r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$

Analogously, for a mixture of pure states ρ_{ψ_j} its density operator can be written as follows:

$$\sum_{j\in\chi} P_J(j) |\psi_j\rangle \langle\psi_j| = \frac{1}{2}\sum_{j\in\chi} P_J(j)(I + r_{x,j}X + r_{y,j}Y + r_{z,j}Z) \tag{10}$$

where $r_{k,j}$ denotes the $k-th$ component of the bloch-vector of the $j-th$ pure state in the mixture, kind of confusing at first – I know... Hence, the overall mixture has a bloch-vector $\mathbf{r}$ where $\mathbf{r} = \sum_j p_J(j)\mathbf{r}_j$

14. Show that $Tr\{\Pi(A \cup B)\rho\} = Pr\{\text{A}\} + Pr\{\text{B}\}$ whenever the projectors $\Pi(A)$ and $\Pi(B)$ satisfy $\Pi(A)\Pi(B) = 0$ and the density operator ρ is diagonal in the same basis as $\Pi(A)$ and $\Pi(B)$.

For context, go to **4.1.4** in Wilde.
proof. Since the projectors $\Pi(A)$ and $\Pi(B)$ commute to the *zero-operator*, we get immediately that they are simultaneously diagonalizable and that the quantum phase states sets (informal language I know) are mutually exclusive [a]. Hence, $Tr\{\Pi(A \cup B)\rho\} = Tr\{(\Pi(A) + \Pi(B))\rho\}$. Distributing the trace, we get that $Tr\{(\Pi(A) + \Pi(B))\rho\} = Tr\{\Pi(A)\rho\} + Tr\{\Pi(B)\rho\}$.

$$Tr\{\Pi(A)\rho\} = Tr\{\sum_{x\in A} |x\rangle \langle x| \sum_{x'\in\chi} p_X(x') |x'\rangle \langle x'|\} \tag{11}$$

The only non-vanishing terms occur when $x' \in A$ [b]

$$Tr\{\Pi(A)\rho\} = Tr\{\sum_{x\in A} p_X(x) |x\rangle \langle x|\} = \sum_{x\in A} p_X(x) Tr\{|x\rangle \langle x|\} \tag{12}$$

$$Tr\{\Pi(A)\rho\} = Pr(A) \tag{13}$$

Tracing out the exact same sets as above will give us that

$$Tr\{\Pi(B)\rho\} = Pr(B), \tag{14}$$

[a] there is no overlap in the dyads that make up the projections
[b] This is due due to the fact that ρ is diagonal in the same basis as $\Pi(A)$ and $\Pi(B)$.

15. Show that $Tr\{\Pi(A)\Pi(B)\rho\} = Pr\{A \cap B\}$ whenever the density operator ρ is diagonal

in the same basis as $\Pi(A)$ and $\Pi(B)$.

For context, go to **4.1.4** in Wilde.
proof.

16. **Union Bound** Prove a union bound for commuting projectors Π_1 and Π_2 where $0 \leq \Pi_1, \Pi_2 \leq I$ and for an arbitrary density operator ρ (not necessarily diagonal in the same basis as Π_1 and Π_2: $Tr\{(I - \Pi_1\Pi_2)\rho\} \leq Tr\{(I - \Pi_1)\rho\} + Tr\{(I - \Pi_2)\rho\}$

proof.

It helps to build intuition for the **union bound**.

Let us consider a *one-dimensional quantum harmonic oscillator*. It is well known that the energy levels to this system are discretized as $E_n = \hbar\omega(n + \frac{1}{2})$ These discrete energies induce corresponding eigenstates in the hilbert space. Let $\Pi_1 = |1\rangle\langle 1| + |2\rangle\langle 2| + \ldots$ and $\Pi_2 = |0\rangle\langle 0| + |1\rangle\langle 1|$, which have the following physical interpretations: Π_1 represents that the quantum particle does not carry an energy that is outside the interval $E > 1$. Similarly, the Π_2 operator has the physical interpretation that corresponds to the quantum particle not carrying an energy value that is outside the interval $E < 2$. These projections commute; that is, it is easy to check that $\Pi_1\Pi_2 = \Pi_2\Pi_1$. Since the Π_1 and Π_2 commute, we can define negation, conjunction, and disjunction (NOT, AND, OR) operations that follow the laws set forth in traditional [a]logic. Negation is rather simple. The projections are in direct correspondence (the projections are isomorphic) to physical, observable properties of the quantum system as a matter of fact. $\neg(\Pi_1) = I - \Pi_1$, $\neg(\Pi_2) = I - \Pi_2$, which correspond the interpretations that the particle carries an energy that is not in the $E > 1$ interval and that the energy is not in the $E < 2$ interval.

Put more simply, and perhaps overally heuristically, $\neg(\Pi_1)$ implies $E \leq 1$, $\neg(\Pi_2)$ implies $E \geq 2$. Now, what about the following proposition: $\neg(\Pi_1\Pi_2)$. $\neg(\Pi_1\Pi_2) = \neg\Pi_1$ OR $\neg\Pi_2$.

The **union bound** can be translated into the following english phrase (in context, of course): $Tr\{(I - \Pi_1\Pi_2)\rho\} \leq Tr\{(I - \Pi_1)\rho\} + Tr\{(I - \Pi_2)\rho\} \cong$ *"The probability that the particle has energy that is less than or equal to 1 or has energy greater than or equal to 2 is bounded above by the probability that the particle has energy less than or equal to 1 PLUS the probability that the particle has energy greater than or equal to 2"*. The union bound, in context, makes complete sense actually.

[a]This is precisely why we cannot define a measurement that captures up to some arbitrary precision the position and momentum of a quantum particle – the laws of traditional logic break down and it becomes impossible to physically define such a measurement. Another example would be considering $\frac{1}{2}$ of spin values of fermions along different directional axes. The respective operators in the examples above don't commute with each other; they are deemed *incompatible* according to **R. Griffiths** (1995 CQT, Ch 4.)

17. Consider the following five *"Chrysler" states*

$$|e_k\rangle \equiv cos(2\pi k/5)\,|0\rangle + sin(2\pi k/5)\,|1\rangle \tag{15}$$

where $k \in \{0, ..., 4\}$. These states are the "Chrysler" states because they form a pentagon on the $XZ - plane$ of the Bloch sphere. Show that the following set of operators forms a valid $POVM$: $\{\frac{2}{5}|e_k\rangle\langle e_k|\}$

Good exercise for practicing computing and checking definitions. We will leave this to the author/reader as an exercise.

18. Suppose we have an ensemble $\{p_X(x), \rho(x)\}$ of density operators and a $POVM$ with elements $\{\Lambda_x\}$ that should identify the states ρ_x with high probability, i.e., we would like $Tr\{\Lambda_x \rho_x\}$ to be as high as possible. The expected success probability of the $POVM$ is then
$$\sum_x p_X(x) Tr\{\Lambda_x \rho_x\} \tag{16}$$
Suppose that there exists some operator τ such that
$$\tau \geq p_X(x)\rho_x, \tag{17}$$
where the condition $\tau \geq p_X(x)\rho_x$ is the same as $\tau - p_X(x)\rho_x \geq 0$. Show that $Tr\{\tau\}$ is an upper bound on the expected success probability of the $POVM$. After doing so, consider the case of encoding n bits into a $d - dimensional$ subspace. By choosing states uniformly at random, show that the expected success probability is bounded above by $d2^{-n}$, better written as $\frac{d}{2^n}$. Thus, it is not possible to store more than n classical bits in n qubits and have a perfect success probability of retrieval.

We want to prove the following inequality:
$$\sum_x p_X(x) Tr\{\Lambda_x \rho_x\} \leq Tr\{\tau\} \tag{18}$$
proof. We generate the first bound as follows:
$$\sum_x p_X(x) Tr\{\Lambda_x \rho_x\} \leq \sum_x p_X(x) Tr\{\Lambda_x \frac{\tau}{p_X(x)}\} \tag{19}$$
which follows via rearrangement of the terms (13[a]). Then, equation (15) becomes
$$\sum_x p_X(x) Tr\{\Lambda_x \rho_x\} \leq \sum_x Tr\{\Lambda_x \tau\} = Tr\{\sum_x \Lambda_x \tau\} \tag{20}$$
Using the fact that any $POVM$ resolves the identity (under the summation), we get that
$$\sum_x p_X(x) Tr\{\Lambda_x \rho_x\} \leq Tr\{\sum_x \Lambda_x \tau\} = Tr\{I\tau\} = Tr\{\tau\} \tag{21}$$
QED. Now let us consider the case of encoding n bits into a $d - dimensional$ subspace. This is described by the follows ensemble: $\varepsilon \equiv \{2^{-n}, \rho_i\}_{i \in \{0,1\}^n}$. The *expected success probability* in this case is ... need to create a τ based on the situation to prove the upper bound – serves as a useful general strategy not going to lie.

[a] should include a formal argument as to how the operator version of $\geq$ translates into being able to make such a claim

19. Show that the purity $P(\rho_A)$ is equal to the following expression:

$$P(\rho_a) = Tr\{(\rho_A \otimes \rho_{A'})F_{AA'}\}; \tag{22}$$

where system A' has a Hilbert space structure isomorphic to that of system A and $F_{AA'}$ is the swap operator that has the following action on kets in A and A':

$$\forall x, y F_{AA'} \left|x\right\rangle_A \left|y\right\rangle_{A'} = \left|y\right\rangle_A \left|x\right\rangle_{A'} \tag{23}$$

proof. Since $\mathcal{H}_{\mathcal{A}} \cong \mathcal{H}_{\mathcal{A}'}$, they have the same dimension which we will assume to be d (we are in the finite dimensional Hilbert space setting). We will continue to employ the trusty "cannonical representation" of a density operator: $\rho_A = \sum_i \lambda_i |i\rangle \langle i|_A$, $\rho_{A'} = \sum_j \lambda_j |j\rangle \langle j|_{A'}$. The just defined basis sets $\{|i\rangle_A\}$, $\{|j\rangle_{A'}\}$ denote orthonormal eigenbases for $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{A}'}$, respectively. It is convenient to also define the following basis for the space $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}'}$: $\{|i\rangle_A \otimes |j\rangle_{A'}\}$.

$$Tr\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = Tr\{F_{AA'}(\rho_A \otimes \rho_{A'})\} \tag{24}$$

$$F_{AA'}(\rho_A \otimes \rho_{A'}) = F_{AA'}(\sum_i \sum_j \lambda_i \lambda_j (|i\rangle \langle i|_A \otimes |j\rangle \langle j|_{A'})) \tag{25}$$

$$F_{AA'}(\sum_i \sum_j \lambda_i \lambda_j (|i\rangle \langle i|_A \otimes |j\rangle \langle j|_{A'})) = F_{AA'}(\sum_i \sum_j \lambda_i \lambda_j ((|i\rangle_A \otimes |j\rangle_{A'})(\langle i|_A \otimes \langle j|_{A'})) \tag{26}$$

which follows from the handy way [a] to re-express tensor products. Applying the swap function, we get that (22) equals the following:

$$(\sum_{i,j} \lambda_i \lambda_j ((|j\rangle_A \otimes |i\rangle_{A'})(\langle i|_A \otimes \langle j|_{A'})) \tag{27}$$

We now take the trace of (23) as follows[b]:

$$\sum_{i',j'} \langle (\langle i'|_A \otimes \langle j'|_{A'} | \sum_{i,j} \lambda_i \lambda_j ((|j\rangle_A \otimes |i\rangle_{A'})(\langle i|_A \otimes \langle j|_{A'})) \sum_{i',j'} ||i'\rangle_A \otimes |j'\rangle_{A'}\rangle \tag{28}$$

Adjoining the summations and sliding the bra's and ket's to the middle we get

$$\sum_{i',j'} \sum_{i,j} \lambda_i \lambda_j \langle i'|j\rangle_A \langle j'|i\rangle_{A'} \langle i|i'\rangle_A \langle j|j'\rangle_{A'} \tag{29}$$

The only sums that evaluate to anything that is non-zero, due to our choices of orthonormal bases for $\mathcal{H}_{\mathcal{A}}$; $\mathcal{H}'_{\mathcal{A}}$; and $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}'_{\mathcal{A}}$; is when $i' = j$, $j' = i$, $i = i'$, $j = j'$. Then, (25) reduces to

$$\sum_i \lambda_i^2 = P(\rho_A) \tag{30}$$

[a] $|(\psi_x\rangle \otimes |\phi_y\rangle)(\langle \psi_x \otimes \langle \phi_y||) = |\psi_x\rangle \langle \psi_x| \otimes |\phi_y\rangle \langle \phi_y|$

[b] we will be using the same tensor product O.N basis defined above, but just re-indexing as i' and j' to avoid abuse of notation

20. By ignoring Bob's system, we can determine Alice's local density operator. Show that

$$\mathbb{E}_{X,Y,Z}\{|\psi_{X,Z}\rangle \langle \psi_{X,Z}|\} = \sum_z p_Z(z)\rho_z \tag{31}$$

so that the above expression is the density operator for Alice. It similarly follows that the local density operator for Bob is

$$\mathbb{E}_{X,Y,Z}\{|\phi_{Y,Z}\rangle\langle\phi_{Y,Z}|\} = \sum_z p_Z(z)\sigma_z \tag{32}$$

For context, please view section ***4.3.2 Seperable States*** *in Wilde.*

$$\mathbb{E}_{X,Y,Z}\{|\psi_{X,Z}\rangle\langle\psi_{X,Z}|\} = \sum_z P_Z(z)\sum_x P_{X|Z}(x|z)\,|\psi_{x,z}\langle\psi_{x,z}|\rangle \tag{33}$$

by definition of expected value. Notice that since X and Y are independent when conditioned on Z (as assumed in **4.3.2**) the probabilities conditioned on Y are all disregarded. (22) terms into (23) simply via definition of Alice's local density operator in context:

$$\sum_z P_Z(z)\sum_x P_{X|Z}(x|z)\,|\psi_{x,z}\rangle\langle\psi_{x,z}| = \sum_z P_Z(z)\rho_z \tag{34}$$

We proceed in exactly the same fashion to derive Bob's local density operator.

$$\mathbb{E}_{X,Y,Z}\{|\phi_{Y,Z}\rangle\langle\phi_{Y,Z}|\} = \sum_z P_Z(z)\sum_y P_{Y|Z}(y|z)\,|\phi_{y,z}\rangle\langle\phi_{y,z}| \tag{35}$$

$$\sum_z P_Z(z)\sum_y P_{Y|Z}(y|z)\,|\phi_{y,z}\rangle\langle\phi_{y,z}| = \sum_z P_Z(z)\sigma_z \tag{36}$$

21. Show that we can always write a state of the form in **4.112** as a convex combination of pure product states:

$$\sum_w p_W(w)\,|\phi_w\rangle\langle\phi_w|\otimes|\psi_w\rangle\langle\psi_w| \tag{37}$$

by manipulating the general form in **4.112**.

22. Show that the set of separable states acting on a given tensor product Hilbert space is a convex set. That is, if $\lambda\in[0,1]$ and ρ_{AB} and σ_{AB} are separable states, then $\lambda\rho_{AB}+(1-\lambda)\sigma_{AB}$ is a separable state.

23. Show that the projection operators corresponding to a measurement of the observable $Z_A\otimes Z_B$ are as follows:

$$\Pi_{even} \equiv \frac{1}{2}(I_A\otimes I_B + Z_A\otimes Z_B) = |00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}, \tag{38}$$

$$\Pi_{odd} \equiv \frac{1}{2}(I_A \otimes I_B - Z_A \otimes Z_B) = |01\rangle\langle 01|_{AB} + |10\rangle\langle 10|_{AB}, \tag{39}$$

To show that the aforementioned projection operators correspond to a measurement of the observable $Z_A \otimes Z_B$ it suffices to show that $\Pi^\dagger_{even}\Pi_{even} + \Pi^\dagger_{odd}\Pi_{odd}$ resolves the identity.

$$\Pi^\dagger_{even}\Pi_{even} = (|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB})(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}) \tag{40}$$

the cross terms, so to speak, cancel out due to the orthonormal nature of the computational basis and tensor product rules giving us

$$\Pi^\dagger_{even}\Pi_{even} = (|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}) \tag{41}$$

Performing symmetric calculations for the *odd* projections,

$$\Pi^\dagger_{odd}\Pi_{odd} = (|01\rangle\langle 01|_{AB} + |10\rangle\langle 10|_{AB}) \tag{42}$$

which gives us

$$\Pi^\dagger_{even}\Pi_{even} + \Pi^\dagger_{odd}\Pi_{odd} = I_{AB} \tag{43}$$

which is the resolution of the identity in context, as desired.

24. Show that a parity measurement (defined in the previous exercise) of the state $|\Phi^+\rangle_{AB}$ returns an even parity result with probability one, and a parity measurement of the state $\pi_A \otimes \pi_B$ returns even or odd parity with equal probability. Thus, despite the fact that these states have the same local description, their global behavior is very different. Show that the same is true for the phase parity measurement given by,

$$\Pi^X_{even} \equiv \frac{1}{2}(I_A \otimes I_B + X_X \otimes X_B) = \tag{44}$$

$$\Pi^X_{odd} \equiv \frac{1}{2}(I_A \otimes I_B - X_A \otimes X_B) = \tag{45}$$

To show that a parity measurement of the state $|\Phi^+\rangle_{AB}$ returns an even parity result with probability one, we compute $\langle \Phi^+_{AB} | \Pi_{even} | \Phi^+_{AB} \rangle$ which does in fact return 1. Next, we compute [a] $\langle \pi_A \otimes \pi_B | \Pi_{even} | \pi_A \otimes \pi_B \rangle$ and $\langle \pi_A \otimes \pi_B | \Pi_{odd} | \pi_A \otimes \pi_B \rangle$ to confirm whether both computations do return $\frac{1}{2}$. Plugging and chugging does give *even* with probability 1/2. Now, we don't even have to compute the *odd*, but the justification is important. The primary idea, once again, extends from the extensive commentary I gave in **problem 16**. The projection operators literally have no "overlap" (check the literal dyads for yourself) – they commute with each other and the commutations equal the 0 operator, and they resolve the identity without the help of any other projections. This means that the projections correspond, isomorphically, to mutually exclusive (physical properties) events that make up the quantum phase space.

The author leaves the second half of the exercise for the author and the reader as an exercise.

[a] recall that $\pi_A = \frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A)$, check **Definition 4.1.4 (Maximally Mixed State)** in Wilde for more details

25. Show that the maximally correlated state $\bar{\Phi}_{AB}$, where

$$\bar{\Phi}_{AB} = \frac{1}{2}(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}), \tag{46}$$

gives results for local measurements that are the same as those for the maximally entangled state $|\Phi^+\rangle_{AB}$. Show that the above parity measurements can distinguish these states.

26. (**Local Density Operator**) Let ρ_{AB} be a density operator acting on a bipartite Hilbert space. Prove that $\rho_A = Tr_B\{\rho_{AB}\}$ is a density operator, meaning that it is positive semi-definite and has trace equal to one.

proof. We first show that *tracing out B* leaves us with a valid density operator that is positive semi-definite. We will show that it has unit trace directly afterwards. Let $\psi \in \mathcal{H}_{\mathcal{A}}$ be arbitrary.

$$\langle\psi|_A \, [\sum_l (I_A \otimes \langle l|_B)\rho_{AB}(I_A \otimes |l\rangle_B)] \, |\psi\rangle_A \tag{47}$$

Bringing the $|\psi\rangle$'s inside the summation,

$$\sum_l \langle\psi|_A \, (I_A \otimes \langle l|_B)\rho_{AB}(I_A \otimes |l\rangle_B) \, |\psi\rangle_A \tag{48}$$

Referring directly to equations **(4.142)** and **(4.143)** in Wilde [a], we get

$$\sum_l (\langle\psi|_A \otimes \langle l|_B)\rho_{AB}(|\psi\rangle_A \otimes |l\rangle_B) \geq 0 \tag{49}$$

Since ρ_{AB} is positive semidefinite, (45) and therefore (43) is bounded above ($\geq$) 0; we have shown positive semi-definiteness. Now, let us take $\{|k\rangle_A\}$ to be an orthonormal basis for $\mathcal{H}_{\mathcal{A}}$. We now compute the trace of ρ_A directly as follows:

$$\sum_k \langle k|_A \, [\sum_l (I_A \otimes \langle l|_B)\rho_{AB}(I_A \otimes |l\rangle_B)] \, |k\rangle_A \tag{50}$$

which we can quickly turn into the following:

$$\sum_k \sum_l \langle k|_A \, (I_A \otimes \langle l|_B)\rho_{AB}(I_A \otimes |l\rangle_B) \, |k\rangle_A \tag{51}$$

employing (**4.142**) and (**4.143**) again,

$$\sum_{k,l} (\langle k|_A \otimes \langle l|_B)\rho_{AB}(|k\rangle_A \otimes |l\rangle_B) \tag{52}$$

(48) is simply the expression for the trace of the bipartite density operator, which is 1. Hence,

$$\sum_{k,l} (\langle k|_A \otimes \langle l|_B)\rho_{AB}(|k\rangle_A \otimes |l\rangle_B) = 1 = Tr\{\rho_A\} \tag{53}$$

as desired.

[a] reviewing how to derive the expression for the partial trace will essentially solve this problem

27. Show that the two notions of the partial trace operation are consistent. That is, show

that

$$Tr_B\{|x_1\rangle\langle x_2|_A \otimes |y_1\rangle\langle y_2|_B\} = \sum_i \langle i|_B |x_1\rangle\langle x_2|_A \otimes |y_1\rangle\langle y_2|_B |i\rangle_B = |x_1\rangle\langle x_2|_A \langle y_2|y_1\rangle \tag{54}$$

proof.

$$Tr_B\{|x_1\rangle\langle x_2|_A \otimes |y_1\rangle\langle y_2|_B\} = \sum_i (I_A \otimes \langle i|_B) |x_1\rangle\langle x_2|_A \otimes |y_1\rangle\langle y_2|_B (I_A \otimes |i\rangle_B) \tag{55}$$

by definition. Collecting the right-most [a] terms,

$$\sum_i (I_A \otimes \langle i|_B) |x_1\rangle\langle x_2|_A \otimes |y_1\rangle\langle y_2|i\rangle_B \tag{56}$$

Collecting terms once more,

$$\sum_i |x_1\rangle\langle x_2|_A \otimes \langle i|y_1\rangle_B \langle y_2|i\rangle_B = |x_1\rangle\langle x_2|_A \sum_i \langle i|y_1\rangle_B \langle y_2|i\rangle_B \tag{57}$$

Finally, (53) gives us that

$$|x_1\rangle\langle x_2|_A \sum_i \langle i|y_1\rangle_B \langle y_2|i\rangle_B = |x_1\rangle\langle x_2|_A \langle y_2|y_1\rangle \tag{58}$$

[a] working out some toy cases in $\mathbb{C}^2$ terms will convince you of the following implicit computations

Link to the template used