25Su ORMC Compiled Notes

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Contents

1 How to read this

However you want to! This is just for you to see some of my written solutions as well as some supplementary material to the packets we went over throughout the summer. The sections with a $*$ I marked to either signal that the material is a bit challenging (so definitely rewarding) but also supplementary. ** is very difficult materiall, but nonetheless very interesting and related to some of our problems and discussions. This document is also bound to (hopefully not riddled with but) have errors and typos. Feel free to let me know via email if you have any questions or corrections; I'll also upload it on this github repo here so that I don't have to email this to you. It was a real pleasure to have you guys as students this summer!

2 Peano Axioms and Induction

2.1 What is Induction?

Rather than to try and immediately define *what* induction is, I think its better for us to first view it is a *tool*; we will gain familiarity with the tool by studying what problems we can solve using it. After this, it'll be easier to attempt to rigorously define what exactly *induction* is.

Say I was being mean and I assigned you with the following problem: *Evaluate the following sums:*

$$\begin{aligned} 1 &=? \\ 1+2 &=? \\ 1+2+3 &=? \\ 1+2+3+4 &=? \\ 1+2+3+4+5 &=? \\ &\vdots \\ 1+2+3+4+5+6+\cdots+9+10 &=? \end{aligned}$$

If you **computed each line after discarding your answer to the previous sum**, you'd be in a world of pain. **Your answer to a previous line can be used to compute, with relative ease, the answer to the next sum**. More explicitly, if you locked in and computed

$$1+2+3+4+5=15$$

the next sum $\sum_{i=1}^{6} i = 15 + 6 = \text{previous answer} + 6$. We will see this is one defining property of this so called "induction". If you stare at the sum for long enough, or you compute more sums, you will notice a pattern.

$$1+2+3+4+5=\frac{5(5+1)}{2}$$

Check this this holds for the other sums. It is then natural to conjecture

$$1 + 2 + 3 + \cdots + n \underbrace{=}_{?} \frac{n(n+1)}{2}$$

where n is some natural number $\mathbb{N}$. Our first exploration into induction will be motivated by proving our conjecture above.

The idea of the proof hinges on an observation we made previously: *computing the sums is made easier if we possess knowledge of the answer to the previous sum!* If our proposed formula for the sum of the first n natural numbers were to be true, it MUST BE TRUE for the FIRST case. Namely, it must hold for $n = 1$; otherwise how could we possibly even claim that it holds for all natural numbers? Let's check if our formula correctly predicts the sum for the first 1 natural numbers. Yes, this seems pointless, but this is crucial.

$$1 = \frac{1(1+1)}{2}$$

Ok, so our formula, in computer science terms, passes the first and easiest test case. Suppose now that for some arbitrary natural number n, we know that our formula works. That is to say, we already have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

If we can, using this assumption, prove that

$$1 + 2 + \cdots + n + (n+1) \underbrace{=}_{?} \frac{(n+1)(n+2)}{2}$$

then we are done. Why? Well because we chose n arbitrarily. This is something to dwell on. n, for example could have been $n = 2$. This means that the truth of our formula for the case $n = 2$ implies the truth of our formula for the case $n = 3$. Since we've verified that our assumption also works for $n = 1$, we have established an ascending chain of implications:

$$p(1) \implies p(2) \implies p(3) \implies \cdots \implies p(n) \implies p(n+1) \implies \cdots$$

where $p(\cdot)$ refers to the truth of our statement for a certain value of n and the $\implies$ arrow means "implies". This allows us to verify that our formula works for all $n \in \mathbb{N}$ without having to spend the rest of our lifetimes computing and checking our formula works for each case!

> **Try it out:** Look back at our proof of the formula, and in accordance to the packet, identify what the **base case, inductive hypothesis, and inductive step** were in our proof.

2.2 Selected Solutions

(**Problem 2.6**) *Use mathematical induction to prove the following*:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

Proof. It is easy to check that the base case holds (the first domino falls). Thus, let us inductively assume that the above sum holds for some n. The goal now is to establish the validity of the following:

$$\sum_{i=1}^{n+1} i^2 \underbrace{=}_{?} \frac{(n+1)(n+2)(2n+3)}{6} \tag{2}$$

The following series of computations yields the desired conclusion:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \tag{3}$$

$$= \frac{(n+1)\Big(n(2n+1) + 6(n+1)\Big)}{6} = \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(2n+3)(n+2)}{6} \tag{4}$$

and we are done. □

(Problem 2.7) *Experiment with sums of cubes of the form* $1^3 + 2^3 + 3^3 + \ldots n^3$ *for various natural numbers* n*. Try to find a formula similar to the ones we derived previously. Then, use mathematical induction to prove it.*

Proof. (*Sketch*) My approach is always to tinker a little bit, play around with the sums and find a pattern. Let's start off small, but let's skip looking at 1^3 because I mean that's not too interesting.

$$1^3 + 2^3 = 9; \quad 1^3 + 2^3 + 3^3 = 36; \quad 1^3 + 2^3 + 3^3 + 4^3 = 100; \tag{5}$$

You see the pattern right? We formally [1] deduce that

$$\boxed{1^3 + 2^3 + 3^3 + \ldots n^3 \underbrace{=}_{?} (1+2+3+\cdots+n)^2 = (\frac{n(n+1)}{2})^2 = \frac{n^2(n+1)^2}{4}} \tag{6}$$

The proof via induction is left as an exercise for the reader. □

[1] We have not proved this yet! This is what we will prove using induction

2.2.1 A Generalization*?

Let me present to you another solution to the previous problem. This idea was communicated to the author by our very own Professor Greene during a class on finite group theory.

Proof. (**Alternative solution to Problem 2.7**) Look at

$$(N+1)^3 - N^3 = 3N^2 + 3N + 1 \tag{7}$$

and sum the LHS (left hand side) from $N = 1$ to $N = n$. Notice that we have at our disposal $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Like always, however, let's start out small and experiment a little. take $N = 3$. Then, we have the following equations:

$$(1+1)^3 - 1^3 = 3(1)^2 + 3(1) + 1 \tag{8}$$

$$(2+1)^3 - 2^3 = 3(2)^2 + 3(2) + 1 \tag{9}$$

$$(3+1)^3 - 3^3 = 3(3)^2 + 3(3) + 1 \tag{10}$$

The key is to notice to add the equations as desirable cancellation occurs [2].

$$(3+1)^3 - 1^3 = 3(\sum_{N=1}^{3} N^2) + 3(\sum_{N=1}^{3} N) + \sum_{N=1}^{3} 1 \tag{11}$$

Generalizing this process for N gives us the following when we let $N = n$:

$$(n+1)^3 - 1^3 = 3(\sum_{N=1}^{n} N^2) + 3(\sum_{N=1}^{n} N) + \sum_{N=1}^{n} 1 \tag{12}$$

Solving for $\sum_{N=1}^{n} N^2$,

$$\sum_{N=1}^{n} 3N^2 = (n+1)^3 - 1^3 - 3\sum_{N=1}^{n} N - \sum_{N=1}^{n} 1 \tag{13}$$

Solving gives us that

$$3\sum_{N=1}^{n} N^2 = (n+1)^3 - 1 - 3(\frac{n(n+1)}{2}) - n \tag{14}$$

$$6\sum_{N=1}^{n} N^2 = 2(n+1)^3 - 3n^2 - 5n - 2 = n(n+1)(2n+1) \tag{15}$$

$$\sum_{N=1}^{n} N^2 = \frac{n(n+1)(2n+1)}{6} \tag{16}$$

$\square$

[2] If you want to sound cool this is referred to as *telescoping*

Can you do the process in the above problem for higher powers (inductively on the power)? How does it work for 3rd powers, i.e. $\sum_1^n N^3 =?$

Yes, this process can be done inductively. To determine $\sum_{n=1}^n N^3$ we proceed as follows: First, it is fruitful to examine the following quantity:

$$(N+1)^4 = N^4 + 4N^3 + 6N^2 + 4N + 1 \tag{17}$$

which can be computed directly or quickly via the binomial theorem (and knowledge of pascal's triangle). Then, notice that the following development is similar to the method in the above problem:

$$(N+1)^4 - N^4 = N^3 + 6N^2 + 4N + 1 \tag{18}$$

Summing both sides from $N = 1$ to $N = n$ gives us that

$$(n+1)^4 - 1^4 = 4\sum_{N=1}^{n} N^3 + 6\sum_{N=1}^{n} N^2 + 4\sum_{N=1}^{n} N + \sum_{N=1}^{n} 1 \tag{19}$$

and we can solve for $\sum_1^n N^3$ as follows:

$$4\sum_{N=1}^{n} N^3 = (n+1)^4 - 1 - 6\left(\frac{n(n+1)(2n+1)}{6}\right) - 4\left(\frac{n(n+1)}{2}\right) - n \tag{20}$$

$$4\sum_{N=1}^{n} N^3 = n^4 + 2n^3 + n^2 = n^2(n+1)^2 \tag{21}$$

$$\sum_{N=1}^{n} N^3 = \frac{n^2(n+1)^2}{4} \tag{22}$$

The reason this method is so nice is because we can use this method to solve **Problem 2.8** in the induction packet.

(**Problem 2.8**) *The following remarkable formula holds for the sum of the fourth powers of the first n natural numbers.*

$$\sum_{i=1}^{4} i^4 = 1^4 + 2^4 + \cdots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \tag{23}$$

Prove (23).

Many of you have already solved this problem using induction. Try it with our newly introduced method!

2.3 Peano's Axioms

2.3.1 Quick comment on Axioms

As our lead instructor emphasized in class, **P1,P2,P3,P4,P5** we are taking for granted. These are things we can assume to be entirely true, mainly because they are so obvious and fundamental that it seems almost pointless to try and prove them. That being said, other intuitive facts like

$$m + n = n + m$$

we have no yet proven and this fact is not immediate from the axioms we have assumed. What were are building here is a *theory* of the natural numbers. Generally speaking, our goal is to see how far we can extend these axioms.

2.3.2 Selected Solutions

There are bunch of corollaries and lemmas thrown around in the packet, so this section is probably best viewed with the original packet right next to you. This one is a particularly annoying and tricky problem, mainly because of our rather "constrained" addition operator. Try and parse my following solution carefully.

> *Use the claim reason chart to prove that* $2 + 2 = 4$

Proof.

$$2 + 2 = S(1) + S(1) \ : \text{definition of the symbol 2} \tag{24}$$
$$S(1) + S(1) = S(S(1) + 1) \ : \text{P6} \tag{25}$$
$$S(S(1) + 1) = S(2 + 1) \ : \text{definition of the symbol 2} \tag{26}$$
$$S(2 + 1) = S(S(3)) \ : \text{Theorem 3.1} \tag{27}$$
$$S(S(2)) = S(S(S(S(0)))) \ : \text{definition of S(2)} \tag{28}$$
$$S(S(S(S(0))) = 4 \ : \text{definition of 4} \tag{29}$$

□

Recall now **Theorem 3.2:** $(l + m) + n = l + (m + n)$.

> *Finish the proof of Theorem 3.2 by providing reasons for the claims that complete the inductive step.*

Proof.

$$(l+m)+S(n) = S((l+m)+n) \;:\; \text{P6} \tag{30}$$
$$(l+m)+n = l+(m+n) \;:\; \text{inductive hypothesis} \tag{31}$$
$$S((l+m)+n) = S(l+(m+n)) \;:\; \text{IH + def of S} \tag{32}$$
$$S(l+(m+n)) = l+S(m+n) \;:\; \text{P6} \tag{33}$$
$$l+\underbrace{S(m+n)}_{P6} = l+(m+S(n)) \;:\; \text{P6} \tag{34}$$
$$\ldots \tag{35}$$

□

Lemma 1: $0+n=n$. *Prove Lemma 1 by induction on n.*

Proof. **Base Case:** $0+0=0$ follows from **proposition 5**, so we have established our base case. Posit then that $0+n=n$. We wish to show that $0+S(n)\underbrace{=}_{?}S(n)$. Well, notice that

$$0+S(n) = S(0+n) \tag{36}$$

(why?) and using our inductive hypothesis yields that

$$S(0+n) = S(n) \tag{37}$$

and we are done as desired. □

Prove the following: **Lemma 2:** $1+n=n+1$.

Proof. We will argue via induction on n. **Base case:** $1+0=0+1$ follows directly from *corollary 2*. Suppose then that $1+n=n+1$. We wish to show that $1+S(n)=S(n)+1$.

$$1+S(n) = S(1+n) \;:\; \text{P6} \tag{38}$$
$$1+n = n+1 \;:\; \text{inductive hypothesis} \tag{39}$$
$$S(1+n) = S(n+1) \;:\; \text{inductive hypothesis + def of S} \tag{40}$$
$$n+1 = (n+1)+0 \;:\; \text{P5} \tag{41}$$
$$S(n+1) = S((n+1)+0) \;:\; \text{above + def of S} \tag{42}$$
$$S((n+1)+0) = (n+1)+S(0) \;:\; \text{P6} \tag{43}$$
$$n+1 = S(n) \;:\; \text{Theorem 3.1} \tag{44}$$
$$S(0) = 1 \;:\; \text{definition of symbol 1} \tag{45}$$
$$(n+1)+S(0) = S(n)+1 \;:\; \text{last two reasons combined} \tag{46}$$
$$\ldots \tag{47}$$

□

Prove: **Theorem 3.3: For any two non-negative integers m and n, $m+n = n+m$.**

Proof. We will argue via induction on n.

$$m + S(n) = S(m+n) \ : \text{P6} \tag{48}$$
$$m + n = n + m \ : \text{inductive hypothesis} \tag{49}$$
$$S(m+n) = S(n+m) \ : \text{substitute above} \tag{50}$$
$$S(n+m) = (n+m)+1 \ : \text{Theorem 3.1} \tag{51}$$
$$(n+m)+1 = n+(m+1) \ : \text{Theorem 3.2} \tag{52}$$
$$m+1 = 1+m \ : \text{Lemma 2} \tag{53}$$
$$n+(m+1) = (n+m)+1 \ : \text{Theorem 3.2} \tag{54}$$
$$n+1 = S(n) \ : \text{Theorem 3.1} \tag{55}$$
$$(n+1)+m = S(n)+m \ : „ \tag{56}$$
$$m+S(n) = S(n)+m \ : \text{All of the above} \tag{57}$$

□

3 The Josephus Problem

3.1 Selected Solutions

(**Problem 3**) Show that $J(n)$ cannot be an even number.

Proof. Let $n \in \mathbb{N}_{\geq 2}$ be arbitrary. Situate then the n fighters from left to right (or in a circle, it doesn't matter), assigning to each individual a number (that denotes position) from $1, 2, \ldots n$ respectively. Notice that a *stage* necessarily eliminates every fighter situated on an even square, given that fighter 1 starts with the sword. Thus, it is not possible for $J(n)$ to be even. □

(**Problem 4**) *Show that if $n = 2^m$, $m \in \mathbb{N}$, then $J(n) = 1$.*

Proof. Proceed via the arrangement; we assume that $n = 2^m$ for some $m \in \mathbb{N}$. After one stage, there remain 2^{m-1} fighters whose positions may be relabeled based on position via:

$$1 \mapsto 1; \quad n \mapsto \left\lfloor \frac{n}{2} \right\rfloor \tag{58}$$

and recursive repetition of the above procedure (*precisely m times*) leaves fighter 1 alive always. □

(**Problem 5**) *On a separate paper sheet, continue the above table for* $n = 4, 5, 6, \ldots, 20$. Observe the pattern and try to guess the $J(n)$ formula for any $n \in \mathbb{N}$.

Hint: This problem is innately explorative; it think its probably best only for me to leave some hints, also most of yall have probably solved this. Anyways, there are a couple ways to spot the solution, but the big theme is **parity**. Namely, you want to separate your experimentation and computations and see what happens when n is an even number and compare it to when n is odd. This, you will notice quickly, will give you a lot of information to work with. This sort of "splitting" is also very thematic. Another general heuristic to keep in mind when solving these sort of problems is the **invariance principle**. This is sort of like *Occam's Razor*; this is not a method of proof, but a technique that will help you solve a problem. Namely, when something is being repeated, so for us this rather unfortunate game, you want to look for something that **never changes**. For those of your interested in math competitions, I highly recommend the beginning chapters in **Arthur Engel**'s book: *Problem Solving Strategies* (Engel 1998).

(**Problem 7**) For $n = 2, 3, \ldots, 10$ write n next to $J(n)$ using binary numbers. Find the pattern and formulate the algorithm for getting $J(n)$ out of n in the binary notations

Proof. There are many many solutions that work for this. First, given an arbitrary n, notice that the following is always going to hold:

$$2^m \leq n < 2^{m+1} \tag{59}$$

for some $m \in \mathbb{N}$. Then, the binary expansion of n is going to have m digits composed of either 1's or 0's. This can be checked directly. We propose the following algorithm (where $a_i \in \{0, 1\}$):

$$\boxed{n = a_1 a_2 \ldots a_m \implies J(n) = 1 + 2^1 a_m + 2^2 a_{m-1} + \ldots 2^{m-1} a_2}$$

Note we are not asked for a formal proof of our observations in this problem; that being said I'd like to ask: **Can you think of why we even considered looking at the binary representations of n?** □

(**Problem 8**) Compute $J(41)$

Proof. Let us use the algorithm proposed above. Then, writing 41 in binary yields

$$41 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

After conversion into binary we get that

$$41 \cong 101001_2 \underbrace{\implies}_{\text{algorithm}} J(41) = 1 + 2^1 a_5 + 2^4 a_2 = 19$$

□

4 Permutations, Combinations, and Did Dream Cheat?

The packets themselves for these topics are rather thorough in their explanations, so there isn't really much else I can expand upon. Ill come back to this later, maybe.

5 Measurement Errors and Differential Calculus

5.1 Some helpful preliminaries*

Our goal is to understand what the *limit of a sequence is* as well as learn about something called the triangle inequality, which will be essential for some of the proofs to the problems in the packet for this and next session.

A *sequence*, roughly speaking, is some countably infinite list of numbers, which we can denote by $\{a_n\}_{n=1}^{\infty}$. That is, $a_n \in \mathbb{R}$ and each $n \in \mathbb{N}$. If I told you to find the limit of the following sequence

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$$

you would tell me that the sequence *approaches* or "equals" $0 \in \mathbb{R}$. This is correct, but we can't be sure until we've defined what a *limit* of a sequence is.

Definition: (Limit of a Sequence) We say that a sequence $\{a_n\}_{n=1}^{\infty}$ has a limit $L \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists some $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$, it follows that

$$|L - a_n| < \varepsilon$$

We say that such a sequence is **convergent**.

There's already a couple things to unpack here. First, take a look at the following picture: (1). A sequence of points on the real line converges to a limit if **for any** $\varepsilon > 0$, I can find a number large enough N, which crucially depends on ε, such that for all terms in the sequence $\{a_n\}_{n=1}^{\infty}$ with the subscript greater than or equal to N, it lies within some ε- interval of L. Intuitively, if we have a convergent sequence in $\mathbb{R}$, it means I can get however close I want to the limit point L if I look "far enough" along the sequence $\{a_n\}_{n=1}^{\infty}$. Of course, this notion of convergence doesn't just hold in $\mathbb{R}$! As a matter of fact, there are some nice ways to represent convergence in $\mathbb{R}^2, \mathbb{R}^3$ as well. I've included some pictures for you to think about here: (2) and (3). Here are a couple mini exercises you can try out.

Try it out: Using the ε-N definition of the limit of a sequence, prove that $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{1}{n}$ converges to 0. **You may use for free the following statement**: Let $x \in \mathbb{R}$. Then, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

Remark 5.1. The "free" statement is called the *Archimedean Principle*! To use it in the above problem, think reciprocals!

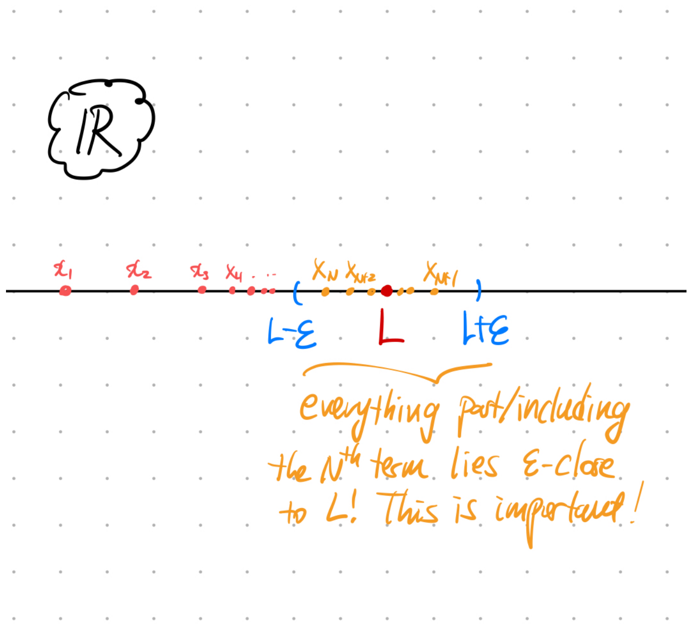

Figure 1: Convergence of a sequence in $\mathbb{R}$

Try it out: Show that the sequence $(-1, 1, -1, 1, \cdots)$, a sequence that alternates between -1 and 1 does not converge to any real number. *Hint*: If the sequence were to converge to a real number, what are the only possibilities? Using this pick a $\varepsilon > 0$ such that the sequence cannot satisfy the ε-N definition.

We should also talk about the most goated inequalities in all of analysis.

(**The Triangle Inequality**) Let $x, y \in \mathbb{R}$. Then,

$$|x + y| \leq |x| + |y|$$

Geometrically, you can interpret this as saying that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides!

5.2 Selected Solutions

(**Problem 7**) Prove the following formula, called the sum/difference rule:

$$\Delta(x \pm y) = \Delta x + \Delta y$$

Proof. Let us denote $\sup S := \Delta(x + y)$ to avoid confusion. Recall that we define

$$\Delta(x) = \sup\{|\overline{x} - x| : |\overline{x} - x| \leq \Delta(x)|\}$$

Recall that $\Delta(x)$ induces an interval; that is,

$$x - \Delta(x) \leq \overline{x} \leq x + \Delta(x)$$

which is a closed interval in $\mathbb{R}$. Thus, it follows that the sup in our inquired quantity is met at the endpoint of the above closed interval. We will argue that $\Delta(x) + \Delta(y)$ is indeed an upper bound for $|\overline{x} + \overline{y} - (x + y)|$ and then show that it is the least such one.

$$|\overline{x} + \overline{y} - x + y| \leq |\overline{x} - x| + |\overline{y} - y| \leq \Delta(x) + \Delta(y)$$

So $\Delta(x) + \Delta(y)$ is an upper bound, i.e., $\sup S \leq \Delta(x) + \Delta(y)$. Pick the worst possible admissible pairs, namely $\overline{x} = x + \Delta(x)$ and $\overline{y} = y + \Delta(y)$. Then,

$$|\overline{x} - x + \overline{y} - y| = \Delta(x) + \Delta(y) \implies \Delta(x) + \Delta(y) \in S$$

Hence, $\sup S \geq \Delta(x) + \Delta(y)$. For the $-$ version, pick the opposite pair for $\overline{y} = y - \Delta(y)$. $\square$

(**Problem 11, 12**) Prove that for $q \neq 1$, $1 + q + \cdots + q^n = \frac{1-q^{n+1}}{1-q}$. After, prove that for $|q| < 1$, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

Proof. Note the identity:

$$(1 - q)(1 + q + q^2 + \cdots + q^n) = 1 - q^{n+1}$$

due to the telescoping induced by the $-q$ in the left parentheses of the LHS. Now, let $|q| < 1$.

$$\sum_{n=0}^{\infty} q^n = \lim_{k\to\infty} \sum_{n=0}^{k} q^k = \lim_{k\to\infty} \frac{1 - q^{k+1}}{1 - q} = \frac{1}{1 - q}$$

since $q^{k+1} \to 0$ as $k \to \infty$. Note that if have that q is sufficiently small,

$$\frac{1}{1 - q} \approx 1 + q$$

$\square$

(**Problem 17**) Use

$$\Delta(x^n) = nx^{n-1}\Delta(x)$$

to prove that

$$\Delta(\sqrt[n]{x}) = \frac{1}{n} x^{\frac{1-n}{n}} \Delta(x)$$

Proof. As much as we'd like to just wave our hands and formally compute the desired result, we can't because our power rules works only on natural numbered exponents. Set $y^n = x \iff y = x^{1/n}$ and implicitly differentiate. That is,

$$\Delta(x) = \Delta(y^n) = ny^{n-1}\Delta(y)$$

Now, what we want is $\Delta(y)$, so we can solve for it now by rearranging what we have above:

$$\frac{1}{n}y^{n-1}\Delta(x) = \Delta(y) \longrightarrow \Delta(y) = \boxed{\frac{1}{n}x^{\frac{1-n}{n}}\Delta(x)}$$

$\square$

(**Problem 19**) Assume that the length p of the pole at Eratosthenes used was two meters. Recall that Alexandria and Syene are nearly on the same meridian and that their latitudes $31°12'56.3''$ and $23°26'12.7''$ N respectively. Find s, the length of the shade case by the Alexandria.

Proof. It suffices to find the angle α that induces the sunlight length s. $\alpha \approx$ AOS which we compute as follows:

$$\phi_{\text{alexandria}} = 31 + \frac{12}{60} + \frac{56.3}{3600} \approx 31.215639°$$
$$\phi_{\text{Syene}} = 23 + \frac{26}{60} + \frac{12.7}{3600} \approx 23.436861°$$

Then we arrive that the fact that

$$\text{AOS} \approx 31.215639° - 23.436861° \approx 7.77878°$$

Now, it'll help to move over into the radian setting, which yields $7.7788 \times \frac{\pi}{180} = 0.1358$ rad. Since α is sufficiently small, we can use that

$$\alpha \approx \tan(\alpha) \approx \frac{s}{p} \implies s \approx p\alpha = 2 \times 0.1358 \approx 0.27m$$

$\square$

Problem 23 Prove that $\delta(xy) = \delta(x) + \delta(y)$

Proof. Just like in the derivation of the discretized product rule, we assume that $x > 0$ and $y > 0$. Otherwise, we run into sign issues because of how we've defined *relative error*.

$$\begin{aligned}\delta(xy) = \frac{\Delta(xy)}{|xy|} \times 100\% &= \frac{x\Delta(y) + y\Delta(x)}{|x||y|} \times 100\% \\ &= (x\Delta(y)/|x||y| + y\Delta(x)/|x|||y|) \times 100\% \\ &= \delta(x) + \delta(y)\end{aligned}$$

$\square$

5.3 Continuous Functions and Measurements*

This idea of significant digits and measurement can be formulated in a more general, functional language.

> I be an open interval in $\mathbb{R}$. We say that a function $f : I \to \mathbb{R}$ is continuous at $x \in I$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
>
> $$|x - \overline{x}| < \delta \implies |f(x) - f(\overline{x})| < \varepsilon$$
>
> where $\overline{x} \in I$. We say that f is *continuous* on I if f is continuous for all $x \in I$.

In fact, a continuous function can be thought of as **taking a measurement**! I never thought of it this way until having read (Gleason 2009) sometime last fall. Let's suppose that you are asked to in chemistry class to measure some quantity to measure the concentration of solution such that your measurement error is less than some specified value, say ε. We can model concentration using a function of two variables: $f(s, s')$ where s denotes the amount of solute and s' denotes the amount of solvent. Of course, this ε is going to have to be small, or your lab grade is cooked. But, the good news is that you know that you can calibrate your measurement device (for us in the packet this is our ruler) so that your error, say δ, in the variables s, s' to guarantee that your final measurement error is within the specified ε-bound! This means that $f(s, s')$ is a continuous function (of two variables but ignore that fact)! Convince yourself of this by looking at our definition (you can ignore the second variable s' when comparing definitions, it's the idea that matters). Also, go back to the earlier section in our packets and think about the relation between significant digits, ε, and how "fine" our ruler should be.

6 Euler's Number

6.1 Metric Spaces*

Ok, so remember our class on *Peano's Axioms* and using induction to rigorously justify operations on $\mathbb{N}$? Well today's session, we are going to be dealing with properties of real numbers, which we denote by $\mathbb{R}$, and working to examine the *transcendental number*, e.

We've been dealing with, albeit implicitly, the notion of *distances* numerous times throughout this summer; take last week's packet on Eratosthenes for example. You and I alike have an intuitive grasp of how *distance* works; if we have some point $\alpha = (x_1, y_1) \in \mathbb{R}^2$, and $\beta = (x_2, y_2)$, we say that the (euclidean) distance between these two points is

$$|x - y| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and this notion generalizes well to higher dimensions. It's natural to ask if we can construct some ambient space, give it some axioms, and arrive upon familiar (like $\mathbb{R}$) or new structures that we can further study. Do there exist more types of *distances*? This leads us to the notion of a **metric space** which we will define and axiomatize right now.

Definition: (Metric Space) Let M be nonempty set [a] and we refer to elements of M often as *points*. Let $d_M : M \times M \to \mathbb{R}_0^+$ be some instantiated function. We require that d_M behave as follows:

1. $d_M(x,x) = 0$, $(\forall x \in M)$ and if $x \neq y$, $d_M(x,y) > 0$
2. $d_M(x,y) = d_M(y,x)$, $(\forall x, y \in M)$
3. $d_M(x,y) \leq d_M(x,z) + d_M(z,y)$

[a](I don't want to study sets with nothing in em)

If all of the above are satisfied, we say that (M, d_M) is a **metric space**. This is all just a fancy shmancy way of saying *hey we are working over a set with a notion of distance between its elements.*

Remark: You should recognize the 3. condition from the definition above as the *triangle inequality*. This was crucial in our class involving measurement error and discretized calculus. It is not a joke when I say this is probably the most useful inequality in all of analysis.

Try it out: *Write out in plain old english, as informal as you'd like, what the conditions* $1, 2, 3$ *say. Do they make sense, are they natural? If so, why? If not, tell me about it.*

We are going to move away from this abstraction very quickly, don't worry. If you need an anchor think of $(\mathbb{R}, |\cdot|)$. This is the metric space we have been working over for the past couple weeks. It is very very important to realize that a metric space is a pair, a tag team. $(M, d) \neq (M, \tilde{d})$, the behaviors of sets endowed with different metric are very different and obviously different sets with the same metric can also behave very differently as well. Here's something fun to think about.

The Discrete Metric: Let us set $M = \mathbb{N}$, the naturals. Define $d_{\text{disc}} : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ as

$$\begin{cases} d_{\text{disc}}(x,y) = 1 & \text{if } x \neq y \\ d_{\text{disc}}(x,y) = 0 & \text{only if x = y} \end{cases}$$

Convince yourself that this is indeed a valid metric on $\mathbb{N}$. Here is an interesting example I got from reading **Pugh** (Pugh 2015). Can you assign a mental picture to what $(\mathbb{N}, d_{\text{disc}})$ looks like? First try with smaller sets of the naturals, like $(\{0,1\}, d_{\text{disc}})$ or $(\{0,1,2\}, d_{\text{disc}})$ [3] Try and draw pictures for the latter *smaller* metric spaces! This stuff is interesting to think about and know that the discrete metric is a very useful metric to consider when coming up with counterexamples. If you have taken calculus and are familiar with the notion of a continuous function, attempt the *try it out* below! If not, forge ahead.

[3]Try for four points as well. Do you notice anything?

Try it out: Let $C([0,1])$ denote the space of all continuous functions $f : [0,1] \to \mathbb{R}$. We can endow this $C([0,1])$ with a metric

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

and this in fact makes $C([0,1])$ into a legitimate metric space. You don't have to prove this, but I think its very useful to develop the following mental map. Also, if the sup scares you can treat it like a max, but I encourage you to get used to dealing with sup's. *Try and think of these functions as points. Close your eyes and in your mind pick a point and call it $f \in C([0,1])$. Can you tell me what it means for another point, in our case some function $\overline{f}$, to be say distance 1 away from our chosen f? What does it mean for $d_\infty(f,g) < \varepsilon$ for small ε? Can you come up with a way to interpret this statement graphically? Try and draw a picture.*

6.2 Reviewing Limits

Given that the packet is lim heavy, we ought to go over what exactly $\lim_{n\to\infty} x_n$ means and how to determine whether or not it exists. Before that, let's look over quantifiers. The symbol $\forall :=$ for all and $\exists :=$ there exists.

Definition: Take a sequence $(x_n)_n$ in $\mathbb{R}$. We say that $\lim_{n\to\infty} x_n = L \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a natural $N = N(\varepsilon) \in \mathbb{N}$ [a] such that for all $n \geq N(\varepsilon)$ we have that

$$|x_n - L| < \varepsilon$$

Note that we are dealing with convergence with respect to the standard metric $|\cdot|$ on $\mathbb{R}$! If no such L exists, we simply say the sequence diverges.

[a] I write $N(\varepsilon)$ because the choice of N depends on ε

Try it out: *Adapt the above limit definition and define convergence of a sequence in an arbitrary metric space (M, d_M).*

You can visualize this on the real line as follows: If $(x_n)_n \to L$ [4], for any $\varepsilon > 0$ of your choosing there exists a large enough number $N(\varepsilon)$ such that all terms in your sequence with index greater than or equal to $N(\varepsilon)$ lie within the open interval $(L - \varepsilon, L + \varepsilon)$. Look at the pictures (1), (2), and (3).

Now some of you may be asking yourselves the following question: *If I have a sequence of points in which the points get arbitrarily close to each other, does the sequence converge?* Well, intuitively this sounds obvious and in $\mathbb{R}$ this statement turns out to be true. This is because $\mathbb{R}$ is **complete** as indicated in your packet! In general, however, this is not

[4] This is shorthand for $\lim_{n\to\infty} x_n = L$

always true. But for our purposes things work out as expected. Before that, let's define mathematically what it means for a sequence $(x_n)_n$ to exhibit the aforementioned behavior (where the points get arbitrarily close to each other).

Definition: Let (x_n) be a sequence in $\mathbb{R}$ such that for any $\varepsilon > 0$, there exists an $N(\varepsilon) = N \in \mathbb{N}$ such that for all $m, n \geq N$, we have that

$$|x_n - x_m| < \varepsilon$$

Such a sequence is called **Cauchy** and we take it as a fact that all Cauchy sequences in $\mathbb{R}$ converge to some point in $\mathbb{R}$.

Try it out: *Let (M, d_M) be a metric space. We say that a metric space is* ***complete*** *if every Cauchy sequence in M converges to a point in M. Adapt the above definition of a Cauchy sequence in $\mathbb{R}$ and recast it in the metric space setting.*

Remark: The completeness of $\mathbb{R}$ is very useful because gives us a way to tell whether or not a sequence in $\mathbb{R}$ converges, without having to compute or guess its limit. This, however, is at times a drawback when we are interested in what exactly the limit point is, but there are ways around this.

Misc: Some of you may be asking, if $\mathbb{R}$ is complete what about $\mathbb{Q}$? We don't have time to actually deal with this (at least in class) but I'll tell you upfront that $\mathbb{Q}$ is not complete. Can you come up with an example? Think $\sqrt{2}$, big hint right there. Anyways, the reason you probably thought of that question is because intuitively $\mathbb{Q}$ is embedded in $\mathbb{R}$ and its quite frequent; we can get arbitrarily close to any real number using only rationals. This is precisely how $\mathbb{R}$ is actually constructed mathematically. The 'numbers' in $\mathbb{R}$ are actually equivalence classes of various sequences of $\mathbb{Q}$ that converge to the same thing, these fill in the gaps of $\mathbb{Q}$. If the prospect of understanding why $\mathbb{Q}$ is not complete and how $\mathbb{R}$ is the *completion* of $\mathbb{Q}$ we can talk about at the start of next session.

Lastly, I think its worth discussing the solution to **Lemma 2(a)** in the Euler packet – think of this as some kind of a proof skeleton.

Lemma 2(a): *Suppose* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$. *Then,* $\lim_{n\to\infty} a_n + b_n = A + B$

Before giving the proof, let me outline what my thought process is when approaching such a problem. My goal is to show that for any $\varepsilon > 0$, I can find or produce an $N(\varepsilon) = N \in \mathbb{N}$, depending on ε, for which I have

$$|a_n + b_n - (A + B)| < \varepsilon \ \ \forall n \geq N(\varepsilon)$$

Since $\lim_{n\to\infty} a_n = A$, I know I can find a sufficiently large $N^a \in \mathbb{N}$ such that

$$|a_n - A| < \varepsilon/2$$

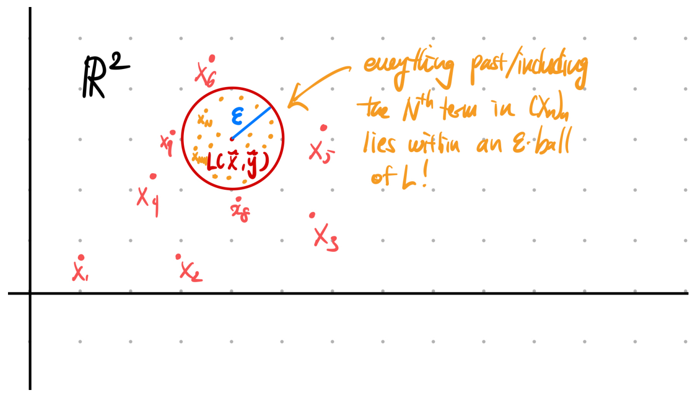

Figure 2: Convergence of a sequence in $\mathbb{R}^2$

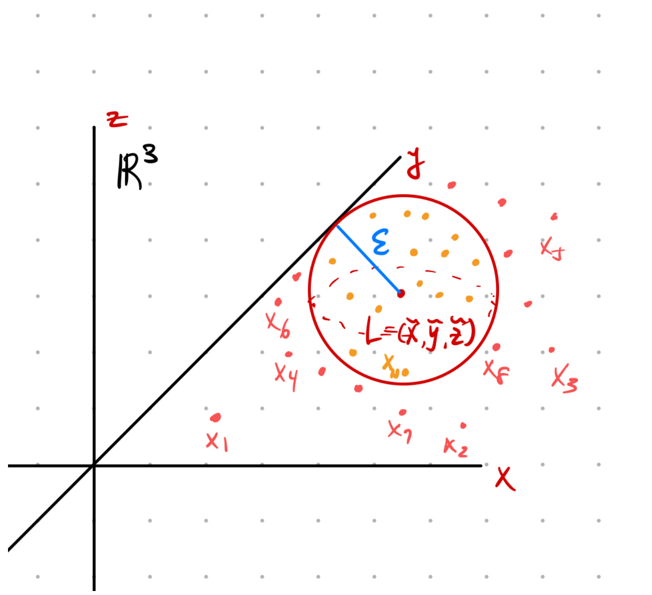

Figure 3: Convergence of a sequence in $\mathbb{R}^3$

for all $n \geq N^a$. I can similarly find an N^b sufficiently large so that my b_n's get $\varepsilon/2$-close to B. I also know that $|x - y| \leq |x| + |y|$ by the triangle inequality. I'll combine all these observations to write a formal proof.

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = A$, there exists an $N^a \in \mathbb{N}$ sufficiently large such that for all $n \geq N^a$, we have

$$|a_n - A| < \varepsilon/2$$

Similarly, since $\lim_{n\to\infty} b_n = B$, there exists an $N^b \in \mathbb{N}$ sufficiently large such that for all $n \geq N^b$, it follows that

$$|b_n - B| < \varepsilon/2$$

Set

$$N := \max(N^a, N^b)$$

We claim that for all $n \geq N$, $|a_n + b_n - (A + B)| < \varepsilon$. Indeed, if we take $n \geq N$, we have

$$\begin{aligned} |a_n + b_n - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

and thus we have shown that $\lim_{n\to\infty} a_n + b_n = A + B$ ☐

6.3 Juggling Infinities

Let's fix X, Y to denote arbitrary sets. I'm going to hurl a bunch of definitions at you, but bear [5] with me :)

Definition We say that a function $f : X \to Y$ is **one-to-one** if

$$f(x_1) = f(x_2) \underbrace{\implies}_{\text{implies}} x_1 = x_2$$

for $x_1, x_2 \in X$. This pretty much just means that f is a function that assigns a unique output to each input. Such functions are also called **injective**.

Definition: We say that a function $f : X \to Y$ is **onto** if $\forall y \in Y, \exists x \in X$ such that

$$f(x) = y$$

This means that for every element in Y, I can find some $x \in X$ for which $f(x) = y$. Everything in the space we are mapping into is hit, so to speak. Such functions are also called **surjective**.

[5] yes, this was a UCLA pun I'm sorry

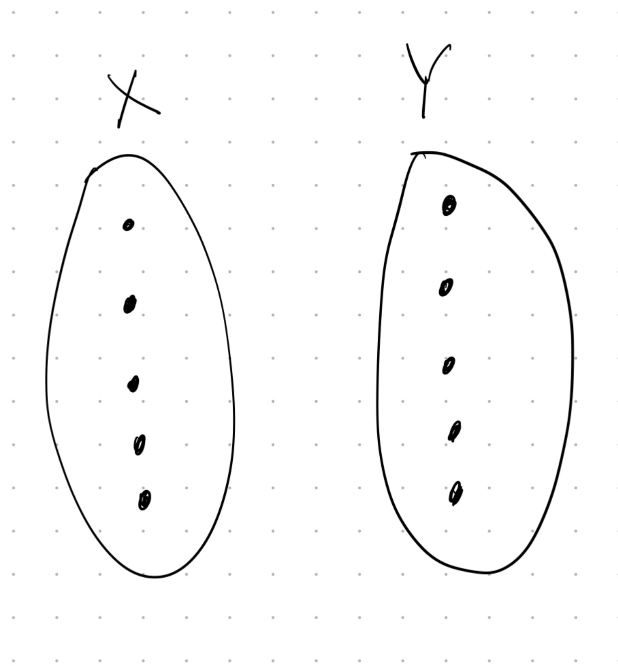

Figure 4: Try drawing arrows between the dots!

The following *try it out* motivates why we care about any of this context of the packet.

Try it out: *Let A, B be sets with the same number of elements. Let's assume $|A| = |B| = n$, for some $n \in \mathbb{N}$. Note the $||$'s here denote the number of elements of the set. If $f : X \to Y$ is one-to-one is it onto? Conversely if f is onto is it necessarily one-to-one?* Hint: Draw a picture!

Here is a picture (above) to help you get started: (4).

Definition: If $f : X \to Y$ is both one-to-one and onto, we say that it is **bijective**.

Do you see why the previous *try it out* mattered? It tells us that we can use **bijective** maps to compare the sizes of sets! Well, this is only somewhat accurate. When we deal with infinities, this notion of size becomes a bit more coarse-grained and fuzy, but the intuition still carries over.

Definition: We say that a set X is **countable** if there exists a bijective mapping $f : X \to \mathbb{N}$. This means that X, though infinite, is denumerable. We can list (the list is infinite though) out the elements of this set. We say that a set Y is **uncountable** if it is infinite (so not finite) but not countable. That is, there does not exist a bijection between $\mathbb{N}$ and Y.

We are going to work through a very famous example and it'll help you when doing problem 2 on the packet.

Cantor's Diagonal Argument: Let S denote the set of all infinite lists containing only 0's and 1's. More explicitly,

$$S := \{(a_1, a_2, \dots) : a_i \in \{0,1\} \ \forall i \in \mathbb{N}\}$$

I claim that S is **uncountable**.

Proof. We will show that S is uncountable by showing that it cannot be countable. Assume on the contrary that S is countable. Then, it is denumerable, so denoting by $x_k := (a_1^k, a_2^k, \cdots)$ for all $k \in \mathbb{N}$, we can list them as follows:

$$\begin{aligned} x_1 &= (a_1^1, a_2^1, a_3^1, a_4^1, \cdots) \\ x_2 &= (a_1^2, a_2^2, a_3^2, a_4^2, \cdots) \\ x_3 &= (a_1^3, a_2^3, a_3^3, a_4^3, \cdots) \\ x_4 &= (a_1^4, a_2^4, a_3^4, a_4^4, \cdots) \\ &\vdots \end{aligned}$$

If we are able to produce an x consisting only of zeroes and ones that is not in this infinite list we are done. We build such an $x = (b_1, b_2, \cdots)$ by taking its first element to be the opposite of a_1^1, so if $a_1^1 = 1$ we take $b_1 = 0$ and $b_1 = 1$ if $a_1^1 = 0$. Proceed to do this for all inductively to generate a sequence that whose kth term differs from a_k^k, the diagonal in the schematic above. This produces an element x, an infinite list of 0's and 1's that not in the list $\{x_k\}_{k\in\mathbb{N}}$. Thus, S cannot be countable and this finishes the proof. □

6.4 Schröeder-Bernstein**

under construction

6.5 Selected Solutions

Problem 9: *Prove the following: A monotonically increasing sequence of real numbers bounded from above has a limit.*

Proof. We prove this directly. Set

$$L = \sup\{x_n : n \in \mathbb{N}\}$$

and we claim that $(x_n)_n \nearrow L$. Fix any $\varepsilon > 0$. Then, by definition of sup there exists $x_N \in (x_n)$ such that

$$x_N > L - \varepsilon$$

By monotonicity, this holds for all $n \geq N$. Thus,

$$|x_n - L| < \varepsilon$$

and this finishes the proof. $\square$

Problem 11: *Prove that*

$$(1+\frac{1}{n})^n < 3 - \frac{1}{n}$$

for $n = 3, 4, \ldots$

Proof.

$$\begin{aligned}(1+\frac{1}{n})^n &= \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{n})^k \\ < 2 + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} &= 2 + \sum_{k=2}^{n} \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \\ < 2 + \sum_{k=2}^{n} \frac{1}{k!} < 2 + \sum_{k=2}^{n} \frac{1}{k(k-1)} &\underbrace{=}_{\text{telescope}} 2 + 1 - \frac{1}{n}\end{aligned}$$

$\square$

Problem 13: Use Bernoulli's inequality to prove that the sequence e_n is monotone increasing.

Proof.

$$(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^n \iff \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^{n+1}} > (1+\frac{1}{n})^{-1} = 1 - \frac{1}{n+1}$$

Noting then that

$$\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} = 1 - \frac{1}{(n+1)^2}$$

Then apply Bernoulli's to

$$(1-\frac{1}{(n+1)^2})^{n+1} > 1 - \frac{n+1}{(n+1)^2} = 1 - \frac{1}{n+1}$$

and this finishes the proof. $\square$

6.6 Interchanging Limits*

One of my table-group students asked about swapping limits, and more generally to what extent we can extend limiting operations to our traditional arithmetic. This is a super deep topic, so I think it's best to talk only about why we can't always swap limits and also how we got to be careful when we compose various limiting operations in general.

Let's look at the sequence of $\{a_{m,n}\}$ defined by

$$a_{m,n} = \frac{m}{m+n}$$

where we can just assume for simplicities sake that m, n are just natural numbers. Notice that

$$\lim_{m\to\infty} (\lim_{n\to\infty} \frac{m}{m+n}) = \lim_{m\to\infty} (0) = 0$$

because

$$\lim_{n\to\infty} \frac{m}{m+n} = 0$$

Remember we are treating m as a fixed number when evaluating $n \to \infty$. If you compute the swapped limit,

$$\lim_{n\to\infty} (\lim_{m\to\infty} \frac{m}{m+n})$$

you should notice you get something different!

For those of you guys who have seen or taken calculus, here's a fun and also very popular example. If you haven't take calculus, you can understand this funny $\int_0^1 f(x)\, dx$ as

$$\int_0^1 f(x)\, dx = \text{"area under the graph from [0,1]"}$$

This is what is known as the (definite) *integral* of $f(x)$ from $[0, 1]$. For example, you can look at (6). If the function was a triangle whose base lies on the x-axis, the integral of such a function is just the area of the triangle. Similar logic holds for various types of "shapes".

Now, let's consider the following functions note $x \in [0, 1]$:

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \le 1 \end{cases}$$

Note that as n grows really large, the graph of the function, namely the triangle, get skinnier and taller, but a majority of the graph is just the 0 function. I'm going to handwave a bit, but visually you should convince yourself that as n grows really really large, the graph pretty much becomes "identical" (in the pointwise sense) to $f(x) = 0$ on $[0, 1]$. Remember, as n grows this things is just 0 for a long time and then some really thin peaked triangle whose area is 1. However, no matter what n we choose, also note that we have defined $f_n(x)$ so that

$$\int_0^1 f_n(x)\, dx = 1$$

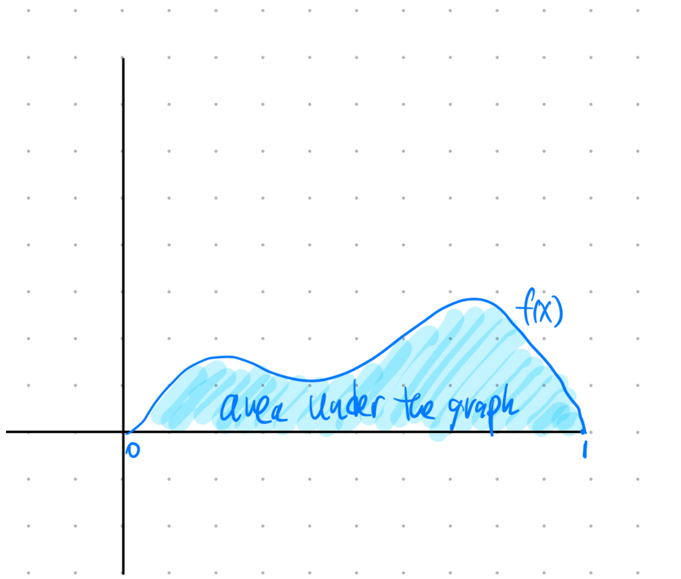

Figure 5: $\int_0^1 f(x)\,dx$ is the shaded area

Thus,

$$\boxed{1 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0}$$

References

Engel, Arthur (1998). *Problem-Solving Strategies*. 1st ed. Problem Books in Mathematics. Springer-Verlag, New York, NY, pp. X, 403. ISBN: 978-0-387-98219-9 (softcover), 978-0-387-22641-5 (eBook). DOI: 10.1007/b97682.

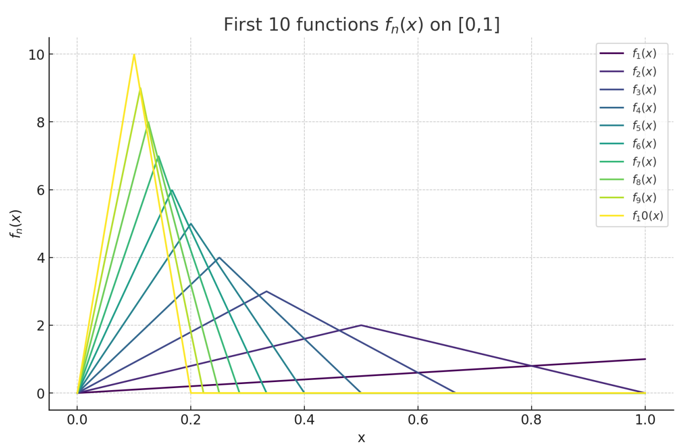

Figure 6: Triangles approaching the origin: courtesy of GPT

Gleason, Jonathan James (Aug. 2009). “The C-Algebraic Formalism of Quantum Mechanics”. Unpublished manuscript.

Pugh, Charles C. (2015). *Real Mathematical Analysis*. 2nd. Undergraduate Texts in Mathematics. Springer, Cham. ISBN: 978-3-319-17771-7 (eBook), 978-3-319-17770-0 (Hardcover). DOI: 10.1007/978-3-319-17771-7.