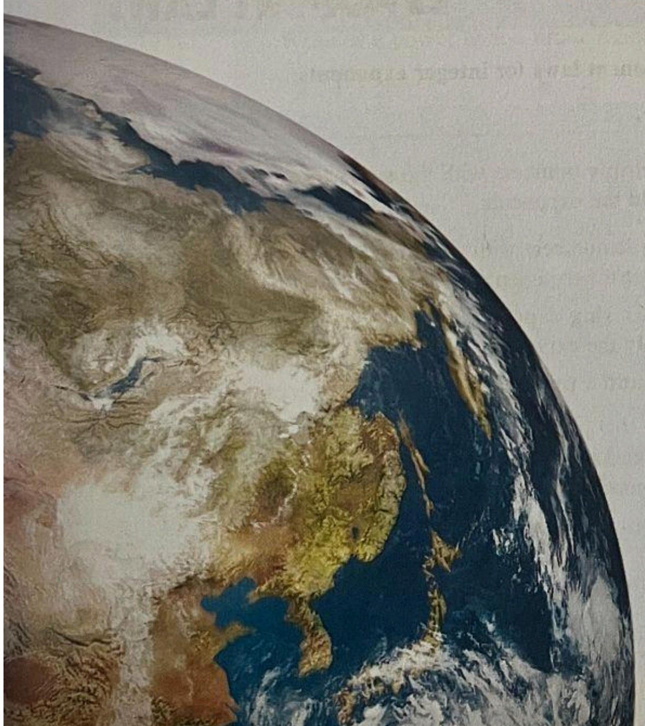
Chapter 1

Exponents

Contents:



OPENING PROBLEM

The **parsec**, the **light-year**, and the **astronomical unit** are all units of length used to measure large distances in space. The length of each unit in metres is shown alongside.

1 parsec $\approx 3.1 \times 10^{16}$ m
1 light-year $\approx 9.5 \times 10^{15}$ m
1 astronomical unit $\approx 1.5 \times 10^{11}$ m

Things to think about:

a Can you write the length of each unit in metres as an ordinary number?
b What advantages are there to writing the lengths in the form given above?
c How much farther is a parsec than a light-year?
d How many astronomical units are there in one parsec?

Exponent notation can be used to represent repeated multiplications of the same number.

For example, we represent $5 \times 5 \times 5$ as 5^3, which reads "5 to the power 3". We say that 5 is the **base**, and 3 is the **exponent**, **power**, or **index**.

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \dots \times a}_{n \text{ factors}}$$

A EXPONENT LAWS

In previous years we have seen the following **exponent laws** for **integer exponents**:

If m and n are integers, then:

- $a^m \times a^n = a^{m+n}$ — To **multiply** numbers with the **same base**, keep the base and **add** the exponents.
- $\dfrac{a^m}{a^n} = a^{m-n}$ provided $a \neq 0$ — To **divide** numbers with the **same base**, keep the base and **subtract** the exponents.
- $(a^m)^n = a^{mn}$ — When **raising** a **power** to a **power**, keep the base and **multiply** the exponents.
- $(ab)^n = a^n b^n$ — To **expand** a **product** to a power, raise each factor to the power.
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ provided $b \neq 0$ — To **expand** a **quotient** to a power, raise both the numerator and denominator to the power.
- $a^0 = 1$ provided $a \neq 0$ — Any non-zero number raised to the power of zero is 1.
- $a^{-n} = \dfrac{1}{a^n}$ — a^n and a^{-n} are **reciprocals** of one another. In particular, $a^{-1} = \dfrac{1}{a}$.

Example 1 **Self Tutor**

Simplify using the exponent laws:

a $7^8 \times 7^4$ **b** $\frac{x^{11}}{x^k}$ **c** $(5^a)^3$

a $7^8 \times 7^4$
$= 7^{8+4}$
$= 7^{12}$

b $\frac{x^{11}}{x^k}$
$= x^{11-k}$

c $(5^a)^3$
$= 5^{a\times 3}$
$= 5^{3a}$

EXERCISE 1A

1 Simplify using the exponent laws:

a $3^2 \times 3^5$ **b** $x^6 \times x^3$ **c** $x^5 \times x^n$ **d** $t^3 \times t^4 \times t^5$

e $\frac{7^9}{7^5}$ **f** $\frac{x^7}{x^3}$ **g** $\frac{t^6}{t^x}$ **h** $t^{3m} \div t$

i $(5^3)^2$ **j** $(t^4)^3$ **k** $(y^3)^m$ **l** $(a^{3m})^4$

2 Simplify using the exponent laws:

a $(x^4)^t \times x^5$ **b** $\frac{k^2 \times k^m}{k^n}$ **c** $\frac{c^5}{(c^x)^4}$ **d** $\frac{(m^2)^t}{m^3 \times m^s}$

Example 2 **Self Tutor**

Write as a power with a prime number base:

a 4×2^p **b** $\frac{3^x}{9^y}$ **c** 25^{x-1}

a 4×2^p
$= 2^2 \times 2^p$
$= 2^{2+p}$

b $\frac{3^x}{9^y}$
$= \frac{3^x}{(3^2)^y}$
$= \frac{3^x}{3^{2y}}$
$= 3^{x-2y}$

c 25^{x-1}
$= (5^2)^{x-1}$
$= 5^{2(x-1)}$
$= 5^{2x-2}$

3 Write as a power with a prime number base:

a 121 **b** 32 **c** 81 **d** 4^3

e 25^2 **f** $7^t \times 49$ **g** $3^a \div 9$ **h** $8^p \div 4$

i $\frac{7^n}{7^{n-2}}$ **j** $\frac{9}{3^x}$ **k** $(25^t)^2$ **l** $16^{k-3} \times 2^{-k}$

m $\frac{4^a}{2^b}$ **n** $\frac{8^x}{16^y}$ **o** $\frac{125^{x+1}}{5^{x-1}}$ **p** $\frac{27^{a+2}}{3^a \times 9^a}$

8 Write as a fraction in lowest terms:

a 6^{-1} b 4^{-1} c 13^{-1} d 4^{-2}
e 5^{-3} f 7^{-2} g 10^{-3} h 3^{-4}
i $5^0 - 5^{-1}$ j $3^0 + 3^1 - 3^{-1}$ k $2^{-3} + 2^{-4}$ l $12^{-1} - 12^{-2}$

9 Write as a power with a negative exponent:

a $\dfrac{5^3}{5^5}$ b $\dfrac{2^6}{2^{10}}$ c $\dfrac{x^4}{x^9}$ d $\dfrac{y^3}{(y^5)^2}$

10 Write as a fraction in lowest terms:

a $\left(\frac{3}{8}\right)^{-1}$ b $\left(\frac{2}{3}\right)^{-1}$ c $\left(\frac{1}{5}\right)^{-1}$ d $\left(\frac{1}{3}\right)^{-2}$
e $\left(\frac{2}{3}\right)^{-3}$ f $\left(\frac{4}{5}\right)^{-2}$ g $\left(1\frac{1}{2}\right)^{-3}$ h $\left(3\frac{1}{2}\right)^{-2}$

11 Write without brackets or negative exponents:

a $(5x)^{-1}$ b $5x^{-1}$ c $7a^{-1}$ d $(7a)^{-1}$
e $\left(\dfrac{1}{t}\right)^{-2}$ f $\left(\dfrac{3x}{y}\right)^{-1}$ g $(5t)^{-2}$ h $(5t^{-2})^{-1}$
i xy^{-1} j $(xy)^{-1}$ k xy^{-3} l $(xy)^{-3}$
m $(3pq)^{-1}$ n $3(pq)^{-1}$ o $3pq^{-1}$ p $\dfrac{(xy)^3}{y^{-2}}$
q $(5x^{-2}y^3)^3$ r $\left(\dfrac{c}{2d^3}\right)^{-2}$ s $\left(\dfrac{3r^{-3}}{t}\right)^{-2}$ t $\left(\dfrac{2p}{5q^{-2}}\right)^{-3}$

12 Use the exponent laws to show that, for positive a and b and integer n:

a $\dfrac{1}{a^{-n}} = a^n$ b $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$

13 The units for speed *kilometres per hour* or km/h can be written in exponent form as $\text{km}\,\text{h}^{-1}$. Write these units in exponent form:

a m/s b cubic metres/hour
c square centimetres per second d cubic centimetres per minute
e grams per second f kilogram metres per second
g metres per second per second.

14 Find the smaller of 2^{125} and 3^{75} without using a calculator. **Hint:** $2^{125} = (2^5)^{25}$

15 Order these numbers from smallest to largest: 2^{90}, 3^{60}, 5^{36}, 10^{24}.

B RATIONAL EXPONENTS

A **rational** number is a number which can be written in the form $\dfrac{m}{n}$ where m and n are integers, $n \neq 0$.

Notice that the integers themselves are rational numbers, since for example $5 = \dfrac{5}{1}$.

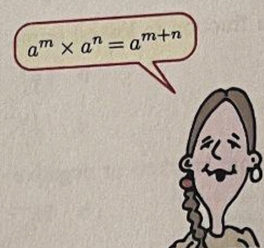
For any positive base a, we choose to define a raised to a rational exponent so that the exponent laws still hold. This helps give meaning to values such as $3^{\frac{1}{2}}$ and $5^{\frac{2}{3}}$.

RATIONAL EXPONENTS OF THE FORM $\frac{1}{n}$

INVESTIGATION 1

This Investigation will help you discover the meaning of $a^{\frac{1}{n}}$ where n is an integer.

What to do:



1 Notice that $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$
and that $\sqrt{5} \times \sqrt{5} = 5$ also.

a Copy and complete:

i $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{2}} = 3^1 = 3$
$\sqrt{3} \times \sqrt{3} = 3$

ii $13^{\frac{1}{2}} \times 13^{\frac{1}{2}} = \ldots\ldots = \ldots\ldots = \ldots\ldots$
$\sqrt{13} \times \sqrt{13} = \ldots\ldots$

iii $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = \ldots\ldots = \ldots\ldots = \ldots\ldots$
$\sqrt{a} \times \sqrt{a} = \ldots\ldots$

b Copy and complete: $a^{\frac{1}{2}} = \ldots\ldots$

2 Notice that $7^{\frac{1}{3}} \times 7^{\frac{1}{3}} \times 7^{\frac{1}{3}} = 7^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 7^1 = 7$
and that $\sqrt[3]{7} \times \sqrt[3]{7} \times \sqrt[3]{7} = 7$ also.

a Copy and complete:

i $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = \ldots\ldots = \ldots\ldots = \ldots\ldots$
$\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = \ldots\ldots$

ii $27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}} = \ldots\ldots = \ldots\ldots = \ldots\ldots$
$\sqrt[3]{27} \times \sqrt[3]{27} \times \sqrt[3]{27} = \ldots\ldots$

iii $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = \ldots\ldots = \ldots\ldots = \ldots\ldots$
$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = \ldots\ldots$

b Copy and complete: $a^{\frac{1}{3}} = \ldots\ldots$

3 Suggest a rule for the general case: $a^{\frac{1}{n}} = \ldots\ldots$

From the **Investigation**, you should have concluded that $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ and $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$.

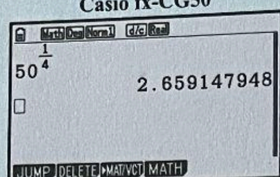
In general, if n is a positive integer, then $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ is called the **nth root of a.**

Example 7 🔊 **Self Tutor**

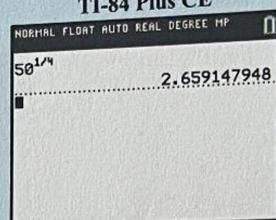
Use your calculator to evaluate $\sqrt[4]{50}$ correct to 2 decimal places.

$\sqrt[4]{50} = 50^{\frac{1}{4}}$

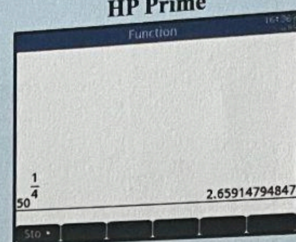
Casio fx-CG50



TI-84 Plus CE



HP Prime



$\sqrt[4]{50} \approx 2.66$

5 Use your calculator to evaluate correct to 2 decimal places:

a $\sqrt[4]{20}$ **b** $\sqrt[5]{300}$ **c** $\dfrac{1}{\sqrt[4]{80}}$ **d** $\dfrac{1}{\sqrt[6]{12}}$

RATIONAL EXPONENTS OF THE FORM $\frac{m}{n}$

INVESTIGATION 2

This Investigation will help you discover the meaning of $a^{\frac{m}{n}}$ where m, n are integers, $n > 0$.

What to do:

1 **a** Copy and complete by applying the rule $(a^m)^n = a^{mn}$:

i $(8^2)^{\frac{1}{3}} = 8^{\cdots}$ **ii** $\left(8^{\frac{1}{3}}\right)^2 = 8^{\cdots}$

b Copy and complete by applying the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$:

i $(8^2)^{\frac{1}{3}} = \sqrt[\cdots]{8^2}$ **ii** $\left(8^{\frac{1}{3}}\right)^2 = (\sqrt[\cdots]{8})^2$

c Hence copy and complete: $8^{\frac{2}{3}} = \sqrt[\cdots]{8^{\cdots}} = (\sqrt[\cdots]{8})^{\cdots}$

2 Copy and complete:

a $a^{\frac{m}{n}} = (a^{\cdots})^{\frac{1}{\cdots}}$
$= \sqrt[\cdots]{a^{\cdots}}$

b $a^{\frac{m}{n}} = \left(a^{\frac{1}{\cdots}}\right)^{\cdots}$
$= (\sqrt[\cdots]{a})^{\cdots}$

From the **Investigation**, you should have discovered that if m, n are integers, $n > 0$, then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

When dealing with exponents in this form, it is often easiest to write the base number as a prime raised to a power. We simplify the result using the exponent laws.

6 Use your calculator to evaluate, rounded to 3 significant figures where necessary:

a $25^{\frac{3}{2}}$ **b** $27^{\frac{2}{3}}$ **c** $8^{\frac{7}{3}}$ **d** $9^{\frac{2}{5}}$

e $10^{\frac{3}{7}}$ **f** $15^{\frac{5}{3}}$ **g** $16^{\frac{3}{11}}$ **h** $4^{-\frac{5}{2}}$

i $27^{-\frac{5}{3}}$ **j** $15^{-\frac{2}{5}}$ **k** $53^{-\frac{3}{7}}$ **l** $3^{-\frac{7}{5}}$

7 Without using your calculator, evaluate $\dfrac{\sqrt[3]{9} \times \sqrt[4]{27}}{\sqrt[12]{243}}$.

GAME

This game can be played by either 1 or 2 players.

PRINTABLE BOARD

$\frac{1}{4}$	5	3	$\frac{1}{3}$	2	4	$\frac{1}{8}$	2	$\frac{4}{5}$
2	9	4	$\frac{2}{5}$	$\frac{2}{3}$	81	3	$-\frac{1}{2}$	81
32	-3	$\frac{1}{27}$	125	$\frac{1}{16}$	64	$\frac{1}{2}$	243	5
4	3	$\frac{1}{8}$	-2	3	32	7	$\frac{3}{4}$	2
5	0	2	$\frac{1}{2}$	$-\frac{2}{3}$	1	6	2	36
49	25	$\frac{1}{2}$	-3	0	-2	64	7	$\frac{1}{5}$
25	$\frac{1}{81}$	-4	27	2	-3	5	4	27
-1	6	$-\frac{1}{3}$	16	-2	$\frac{1}{16}$	3	125	$\frac{1}{2}$
$\frac{1}{25}$	16	343	$\frac{1}{2}$	5	3	1	9	$\frac{1}{5}$

What to do:

In this game, three squares on the board are selected at a time to create a statement of the form $a^b = c$. Once used, each square is crossed out and cannot be used again.

One player variant: Try to use all 81 squares in 27 selections.

Two player variant: Take alternate turns. The last player who is able to make a valid selection is the winner.

C STANDARD FORM (SCIENTIFIC NOTATION)

We can use **standard form** or **scientific notation** to write very large and very small numbers in a way that is easier to understand.

> **Standard form** involves writing any positive number as *a number* between 1 inclusive and 10, multiplied by an *integer power of* 10.
>
> The result has the form $a \times 10^k$ where $1 \leqslant a < 10$ and k is an integer.

For example:

- $68\,000\,000$ is written in standard form as 6.8×10^7
- $0.000\,471$ is written in standard form as 4.71×10^{-4}.

When a number is written in standard form:
- its significant figures are the same and in the same order
- the decimal point is immediately after the first significant figure
- 10^k is the *place value* of the first significant figure in the original number.
  - If the original number is greater than or equal to 10, then k is positive.
  - If the original number is between 1 inclusive and 10, then $k = 0$.
  - If the original number is less than 1, then k is negative.

Example 9 Self Tutor

Write in scientific notation:

a 23 600 000 **b** 0.000 023 6

a $23\,600\,000$
$= 2.36 \times 10^7$

b $0.000\,023\,6$
$= 2.36 \div 10^5$
$= 2.36 \times 10^{-5}$

EXERCISE 1C

1 Write in scientific notation:

a 230 **b** 53 900 **c** 0.0361 **d** 0.006 80
e 3.26 **f** 0.5821 **g** 361 000 000 **h** 0.000 001 674

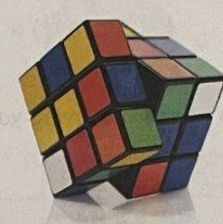
2 Write each quantity in standard form:

a There are approximately 4 million red blood cells in a drop of blood.

b The thickness of a coin is about 0.0008 m.

c A Rubik's Cube has approximately 43 252 000 000 000 000 000 possible arrangements.

d The probability of winning the major prize in a lottery is approximately 0.000 000 003 422 .



Example 10 Self Tutor

Write as an ordinary number:

a 2.57×10^4 **b** 7.853×10^{-3}

a 2.57×10^4
$= 2.5700 \times 10\,000$
$= 25\,700$

b 7.853×10^{-3}
$= 0007.853 \div 10^3$
$= 0.007\,853$

3 Write as an ordinary number:

a 2.3×10^3 **b** 2.3×10^{-2} **c** 5.64×10^5 **d** 7.931×10^{-4}
e 9.97×10^0 **f** 6.04×10^7 **g** 4.215×10^{-1} **h** 3.621×10^{-8}