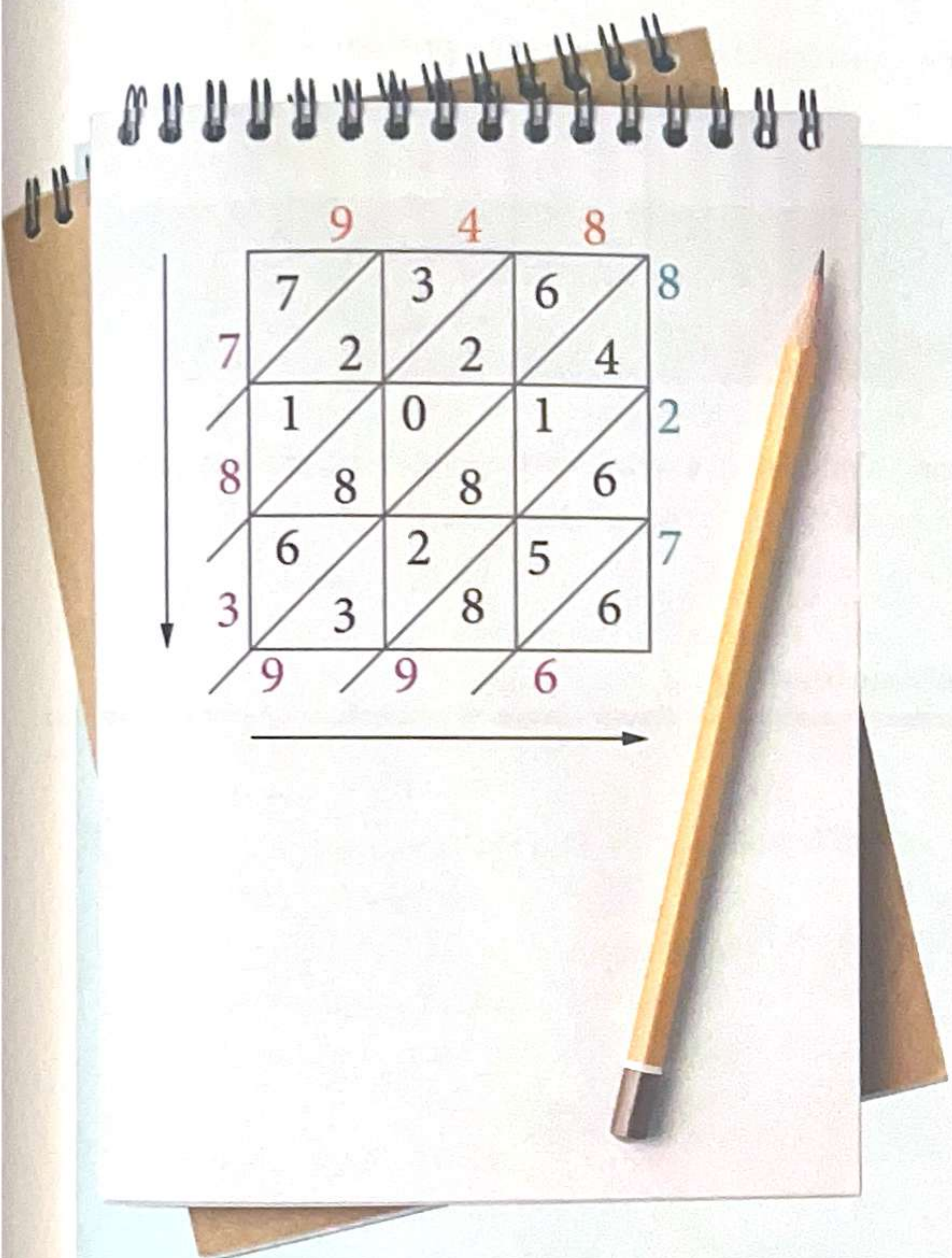
CHAPTER 4

Expansion and Factorisation using Special Algebraic Identities



The use of symbols in algebra provides a system of **notation** for mathematicians to communicate ideas clearly and efficiently. More importantly, the ability to work with algebraic expressions can open up new opportunities for mathematicians to make connections between numbers and geometry.

For example, in the 10th century, people used lattice multiplication to work out the product of two whole numbers. How can we show that this method will work for any two whole numbers?

Another interesting example involves the square of numbers ending with '5'. For instance, $35 \times 35 = 1225$; $65 \times 65 = 4225$; and so on. Can you find a shortcut to find the square of any number ending with '5'? Why does it work?

It turns out that we can represent all these situations using geometric figures and demonstrate the generality of these ideas using algebra. In this chapter, we will explore some useful algebraic identities that can be derived from these connections.

Learning Outcomes

What will we learn in this chapter?

- What the three special algebraic identities are
- How to apply the three special algebraic identities to expand and factorise algebraic expressions
- Why the three special algebraic identities have useful applications in mathematics

Introductory Problem

Without using a calculator, find the value of $2022^2 - 2021^2$. Is there a shorter method?

In this chapter, we will learn three special algebraic identities which can help us solve such problems.

4.1 Expansion using special algebraic identities

In this section, we will learn how to use three special algebraic identities to expand certain algebraic expressions.

Investigation — First special algebraic identity

1. We have learnt that x^2 means $x \times x$.
   What does $(a + b)^2$ mean?

2. Expand $(a + b)^2$ using the Distributive Law: $(a + b)^2 = (a + b)(a + b)$

   $=$

   $=$

3. Expand $(a + b)^2$ using a multiplication frame:

$\times$	a	$+b$
a		
$+b$		

$\therefore (a + b)^2 =$

4. (i) Do you get the same answer for Questions 2 and 3?
   (ii) Is $(a + b)^2 = a^2 + b^2$? Why or why not?
   Hint: Consider your answer to Questions 2 and 3.

5. Fig. 4.1 shows a square $PQRS$ formed by two smaller squares and two rectangles, whose dimensions are given in the figure.

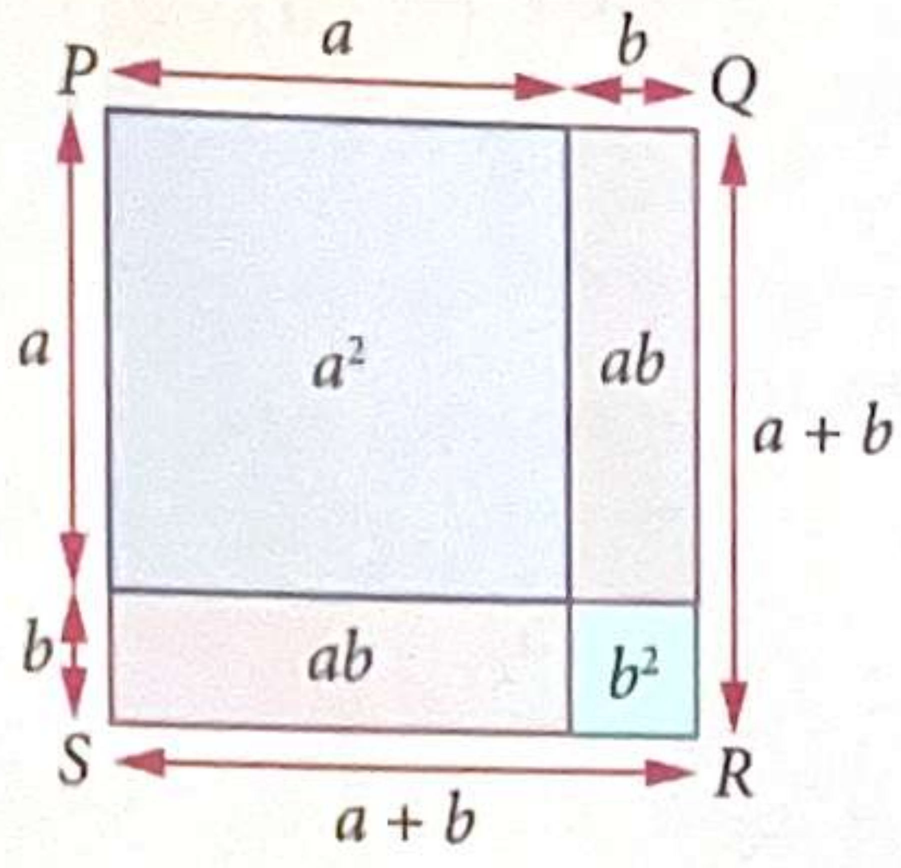


Fig. 4.1

(i) What is the area of the square $PQRS$ in terms of its length?

(ii) What is the area of the square $PQRS$ in terms of the total area of the two smaller squares and two rectangles?

(iii) Are the two expressions in parts **(i)** and **(ii)** equal? Explain.

(iv) From part **(iii)**, $(a + b)^2 =$ ______

(v) Is your answer in part **(iv)** the same as your answer in Questions 2 and 3?

(vi) Using Fig. 4.1, explain why $(a + b)^2 \neq a^2 + b^2$.

Hint: What is the area of the square $PQRS$?

6. In Book 1, we learnt that numbers such as 7^2 and 16^2 are called perfect squares.
In algebra, the expression a^2 is also called a **perfect square**.

(i) By referring to Fig. 4.1, explain why a^2 is called a perfect square.

(ii) Is b^2 a perfect square? Explain.

(iii) Is $(a + b)^2$ a perfect square? Explain.

7. In Book 1, we learnt that an **identity** is an equation that is true for all values of the variable, e.g. $3(x + 2) = 3x + 6$.
Is $(a + b)^2 = a^2 + 2ab + b^2$ an identity? Explain.

From the above Investigation, we have discovered the first special algebraic identity:

First special algebraic identity (or first perfect square identity)

$$(a + b)^2 = a^2 + 2ab + b^2$$

Big Idea

Equivalence

In Book 1, we learnt that two expressions are equivalent if the value of both expressions is the same for any value we substitute into the same variables in the expressions. In an *identity*, the expressions on both sides are *equivalent*. So, substituting the same values of each a and b in $(a + b)^2$ and in $a^2 + 2ab + b^2$ will always give the same result.

Worked Example 1

Expanding algebraic expressions of the form $(a + b)^2$

Expand each of the following expressions.

(a) $(x + 4)^2$ **(b)** $\left(3y+\frac{1}{3}\right)^2$ **(c)** $(4a + 3b)^2$

Solution

(a) $(x + 4)^2 = x^2 + 2(x)(4) + 4^2$ apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = x$ and $b = 4$

$= x^2 + 8x + 16$

(b) $\left(3y+\frac{1}{3}\right)^2$

$= (3y)^2 + 2(3y)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2$ apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 3y$ and $b = \frac{1}{3}$

$= 9y^2 + 2y + \frac{1}{9}$

Attention

(b) $(3y)^2 = 3y \times 3y$
$= 9y^2$
$(3y)^2 \neq 3y^2$

(c) $(4a + 3b)^2 = \underbrace{(4a)^2}_{\text{square 1st term}} + \underbrace{2(4a)(3b)}_{2 \times \text{1st term} \times \text{2nd term}} + \underbrace{(3b)^2}_{\text{square 2nd term}}$ 1st term = $4a$; 2nd term = $3b$

$= 16a^2 + 24ab + 9b^2$

Practise Now 1

Similar and Further Questions
Exercise 4A
Questions 1(a)–(f), 7(a), (b)

Expand each of the following expressions.

(a) $(x + 6)^2$ **(b)** $(4y + 3)^2$ **(c)** $(7 + 3a)^2$

(d) $\left(\frac{1}{2}x+8\right)^2$ **(e)** $(2x + 3y)^2$ **(f)** $(5a + 2b)^2$

Investigation

Second special algebraic identity

1. Expand $(a - b)^2$ using the Distributive Law: $(a - b)^2 = (a - b)(a - b)$

$=$

$=$

2. Expand $(a - b)^2$ using a multiplication frame:

$\times$	a	$-b$
a		
$-b$		

$\therefore (a - b)^2 =$

3. Replace b with $-b$ in the first special algebraic identity:

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 =$ ______

$=$ ______

Attention

$(-b)^2 = (-b) \times (-b)$
$= b^2$
$\neq -b^2$

4. (i) Do you get the same answer for Questions 1, 2 and 3?
   (ii) Is $(a - b)^2 = a^2 - b^2$? Why or why not?
   Hint: Consider your answer to Questions 1, 2 and 3.

5. Fig. 4.2 shows a square $PTUV$ of length a, with two rectangles covering parts of the square.

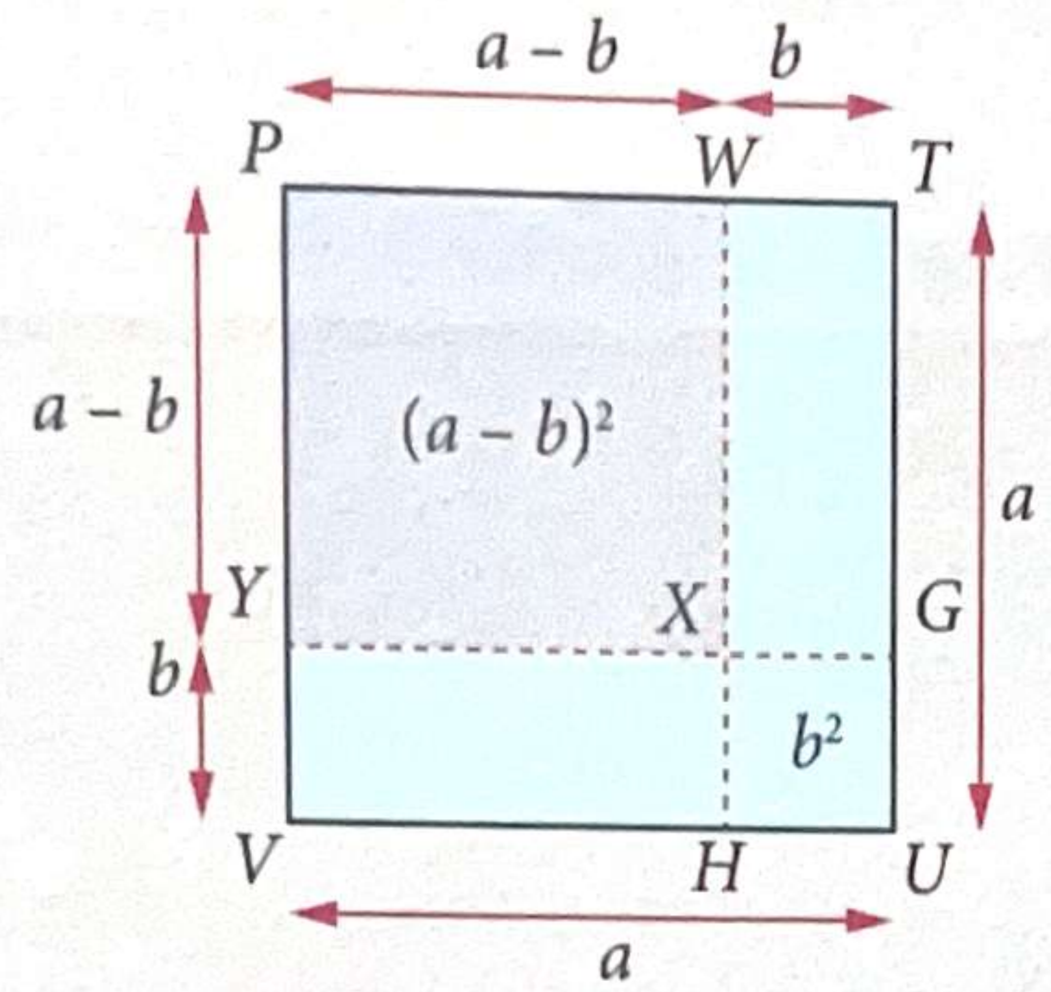


Fig. 4.2

Attention

Does Fig. 4.2 help you see why $(a - b)^2 = a^2 - 2ab + b^2$? Discuss with your classmates.

(i) What is the length of the square $PWXY$?
(ii) Do you think $(a - b)^2$ is a **perfect square**? Explain.
(iii) Using Fig. 4.2, explain why $(a - b)^2 \neq a^2 - b^2$.
Hint: What is the area of each of the squares $PTUV$, $XGUH$ and $PWXY$?

6. Is $(a - b)^2 = a^2 - 2ab + b^2$ an **identity**? Explain.

From the above Investigation, we have discovered the second special algebraic identity:

Second special algebraic identity (or second perfect square identity)

$$(a - b)^2 = a^2 - 2ab + b^2$$

Worked Example 2

Expanding algebraic expressions of the form $(a - b)^2$

Expand each of the following expressions.

(a) $(x - 3)^2$ (b) $\left(\frac{7}{2} - 4y\right)^2$ (c) $(5a - 2b)^2$

Solution

(a) $(x - 3)^2 = x^2 - 2(x)(3) + 3^2$ apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = x$ and $b = 3$
$= x^2 - 6x + 9$

(b) $\left(\frac{7}{2} - 4y\right)^2 = \left(\frac{7}{2}\right)^2 - 2\left(\frac{7}{2}\right)(4y) + (4y)^2$ apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = \frac{7}{2}$ and $b = 4y$
$= \frac{49}{4} - 28y + 16y^2$

Attention

(b) $(4y)^2 = 4y \times 4y$
$= 16y^2$
$(4y)^2 \neq 4y^2$

square 1st term ↓ | 2 × 1st term × 2nd term ↓ | square 2nd term ↓

(c) $(5a - 2b)^2 = (5a)^2 \; - \; 2(5a)(2b) \; + \; (2b)^2$ 1st term = $5a$; 2nd term = $2b$

$= 25a^2 - 20ab + 4b^2$

Practise Now 2

Similar and Further Questions
Exercise 4A
Questions 2(a)–(f), 8(a), (b)

Expand each of the following expressions.

(a) $(x - 4)^2$ **(b)** $(5y - 3)^2$ **(c)** $(8 - 2a)^2$

(d) $\left(\frac{2}{3}x - 6\right)^2$ **(e)** $(b - 3a)^2$ **(f)** $(3a - 4b)^2$

Investigation — Third special algebraic identity

1. Expand $(a + b)(a - b)$ using the Distributive Law: $(a + b)(a - b) =$ ______

$=$ ______

2. Expand $(a + b)(a - b)$ using a multiplication frame:

$\times$	a	$-b$
a		
$+b$		

$\therefore (a + b)(a - b) =$ ______

3. Do you get the same answer for Questions 1 and 2?

4. Fig. 4.3**(a)** shows a polygon $PTGXHV$ formed by cutting the small square $XGUH$ (with length b) from the big square $PTUV$ (with length a).

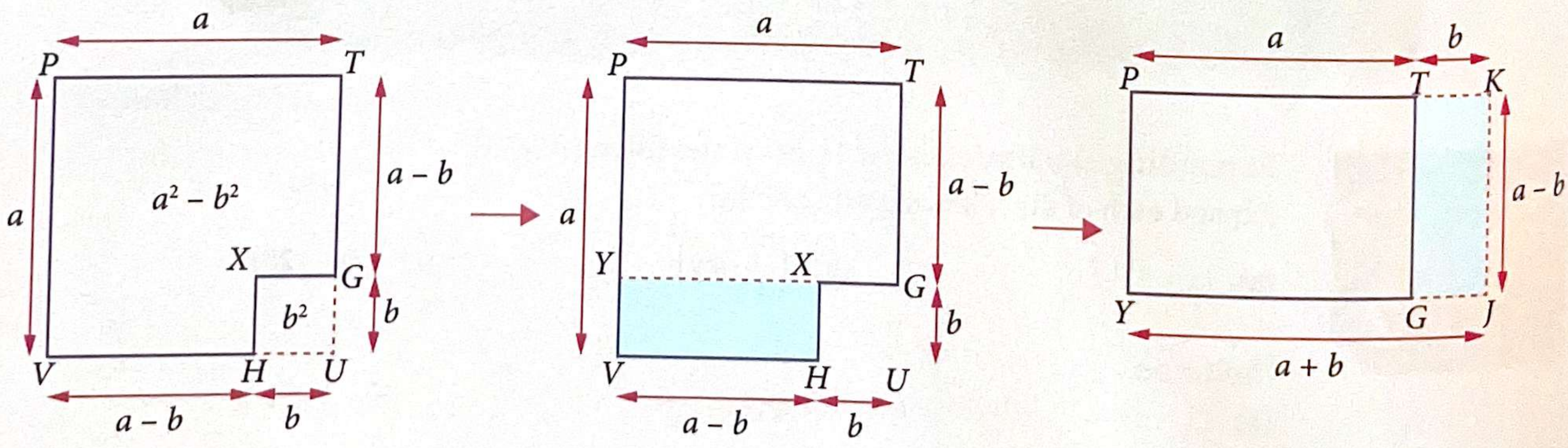


Fig. 4.3

(i) What is the area of the polygon $PTGXHV$ in Fig. 4.3**(a)**?

(ii) Do you think $a^2 - b^2$ is a perfect square or a **difference of two squares**? Explain.

5. Fig. 4.3(**b**) shows the same polygon $PTGXHV$ with the dotted line YX parallel to VH. What are the dimensions of the rectangle $YXHV$?

6. Remove the rectangle $YXHV$ from the polygon $PTGXHV$ in Fig. 4.3(**b**) and place it as shown in Fig. 4.3(**c**).
   (i) Is the area of the rectangle $PKJY$ equal to the area of the polygon $PTGXHV$? Why?
   (ii) What are the dimensions of the rectangle $PKJY$?
   (iii) Find the area of the rectangle $PKJY$ in terms of its length and breadth.
   (iv) By relating the area of rectangle $PKJY$ in parts **(i)** and **(iii)**, write down the third special algebraic **identity**:

   $(a + b)(a - b) =$ ______

From the above Investigation, we have discovered the third special algebraic identity:

Third special algebraic identity (or difference of two squares identity)

$$(a + b)(a - b) = a^2 - b^2$$

Worked Example 3

Expanding algebraic expressions of the form $(a + b)(a - b)$

Expand each of the following expressions.

(a) $(x + 5)(x - 5)$ **(b)** $\left(4a-\frac{7}{2}b\right)\left(4a+\frac{7}{2}b\right)$

Solution

(a) $(x + 5)(x - 5) = x^2 - 5^2$ apply $(a + b)(a - b) = a^2 - b^2$, where $a = x$ and $b = 5$

$= x^2 - 25$

(b) $\left(4a-\frac{7}{2}b\right)\left(4a+\frac{7}{2}b\right) = \underbrace{(4a)^2}_{\text{square 1st term}} - \underbrace{\left(\frac{7}{2}b\right)^2}_{\text{square 2nd term}}$ 1st term = $4a$; 2nd term = $\frac{7}{2}b$

$= 16a^2 - \frac{49}{4}b^2$

Practise Now 3

Similar and Further Questions
Exercise 4A
Questions 3(a)–(d), 9(a)–(d), 10(a), (b), 13

Expand each of the following expressions.

(a) $(x + 3)(x - 3)$ **(b)** $(5y - 4)(5y + 4)$

(c) $(-3 + 2a)(-3 - 2a)$ **(d)** $\left(\frac{1}{4}x+8\right)\left(8-\frac{1}{4}x\right)$

(e) $(2x + 7y)(2x - 7y)$ **(f)** $(6b - a)(a + 6b)$

Problem-solving Tip

(d) Is it easier to re-order the terms in $\left(8-\frac{1}{4}x\right)$ or $\left(\frac{1}{4}x+8\right)$?

4.2 Factorisation using special algebraic identities

In Section 4.1, we have learnt how to *expand* certain algebraic expressions using the three special algebraic identities. Since factorisation is the reverse of expansion, we can use these same identities to *factorise* certain algebraic expressions.

$$(a+b)^2 = a^2 + 2ab + b^2$$ (expand → ; ← factorise)

$$(a-b)^2 = a^2 - 2ab + b^2$$ (expand → ; ← factorise)

$$(a+b)(a-b) = a^2 - b^2$$ (expand → ; ← factorise)

Attention

$(a+b)^2 \neq a^2 + b^2$
$(a-b)^2 \neq a^2 - b^2$

Worked Example 7

Factorising algebraic expressions of the form $a^2 + 2ab + b^2$

Factorise each of the following expressions completely.

(a) $x^2 + 6x + 9$ (b) $16y^2 + 20y + \frac{25}{4}$ (c) $4a^2 + 12ab + 9b^2$

Solution

(a) $x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$ express as $a^2 + 2ab + b^2$, where $a = x$ and $b = 3$

$= (x+3)^2$ apply $a^2 + 2ab + b^2 = (a+b)^2$

Attention

(a) x^2 and 9 are *perfect squares*. Can we can write $6x$ as $2 \times x \times \sqrt{9}$? If yes, we can make use of $a^2 + 2ab + b^2 = (a+b)^2$ to factorise the expression.

(b) $16y^2 + 20y + \frac{25}{4} = (4y)^2 + 2(4y)\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2$ express as $a^2 + 2ab + b^2$, where $a = 4y$ and $b = \frac{5}{2}$

$= \left(4y + \frac{5}{2}\right)^2$ apply $a^2 + 2ab + b^2 = (a+b)^2$

(c) $4a^2 + 12ab + 9b^2 = \underbrace{(2a)^2}_{\text{square 1st term}} + \underbrace{2(2a)(3b)}_{2 \times \text{1st term} \times \text{2nd term}} + \underbrace{(3b)^2}_{\text{square 2nd term}}$ 1st term $= 2a$; 2nd term $= 3b$

$= (\underbrace{2a}_{\text{1st term}} + \underbrace{3b}_{\text{2nd term}})^2$

Practise Now 7

Similar and Further Questions
Exercise 4B
Questions 1(a)–(e), 5(a)–(f), 6

If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to do so, state N.A. (for not applicable).

(a) $x^2 + 10x + 25$ (b) $x^2 + 18x + 36$ (c) $9y^2 + 24y + 16$

(d) $36a^2 + 8a + \frac{4}{9}$ (e) $25a^2 + 40ab + 16b^2$ (f) $16x^2 + 28xy + 49y^2$

Worked Example 8

Factorising algebraic expressions of the form $a^2 - 2ab + b^2$

Factorise each of the following expressions completely.

(a) $2x^2 - 16x + 32$ (b) $36 - 8y + \frac{4y^2}{9}$ (c) $81a^2 - 36ab + 4b^2$

Attention

(a) x^2 and 16 are *perfect squares*. Can we write $8x$ as $2 \times x \times \sqrt{16}$? If yes, we can make use of $a^2 - 2ab + b^2 = (a - b)^2$ to factorise the expression.

Solution

(a) $2x^2 - 16x + 32 = 2(x^2 - 8x + 16)$ HCF of 2, 16 and 32 = 2

$= 2[x^2 - 2(x)(4) + 4^2]$ express as $a^2 - 2ab + b^2$, where $a = x$ and $b = 4$

$= 2(x - 4)^2$ apply $a^2 - 2ab + b^2 = (a - b)^2$

(b) $36 - 8y + \frac{4y^2}{9} = 6^2 - 2(6)\left(\frac{2y}{3}\right) + \left(\frac{2y}{3}\right)^2$ express as $a^2 - 2ab + b^2$, where $a = 6$ and $b = \frac{2y}{3}$

$= \left(6 - \frac{2y}{3}\right)^2$ apply $a^2 - 2ab + b^2 = (a - b)^2$

(c) $81a^2 - 36ab + 4b^2 = \underbrace{(9a)^2}_{\text{square 1st term}} - \underbrace{2(9a)(2b)}_{2 \times \text{1st term} \times \text{2nd term}} + \underbrace{(2b)^2}_{\text{square 2nd term}}$ 1st term = $9a$; 2nd term = $2b$

$= (\underbrace{9a}_{\text{1st term}} - \underbrace{2b}_{\text{2nd term}})^2$

Practise Now 8

Similar and Further Questions
Exercise 4B
Questions 2(a)-(d), 7(a)-(f)

If possible, factorise each of the following expressions completely using an algebraic identity. If it is not possible to use an algebraic identity to factorise, state N.A. (for not applicable).

(a) $8x^2 - 56x + 98$ (b) $\frac{4}{3}t^2 - 4t + 3$ (c) $1 - \frac{2}{3}q + \frac{1}{9}q^2$

(d) $\frac{16}{25} - \frac{24}{5}n + 9n^2$ (e) $25x^2 - 10xy + y^2$ (f) $49h^2 - 42hk + 36k^2$

Worked Example 11

Solving problem using special algebraic identity

(i) Factorise $x^2 - 9y^2$.

(ii) Given that x and y are positive integers, solve the equation $x^2 - 9y^2 = 13$.

***Solution**

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: Understand the problem

For part (i), what does the term 'factorise' imply?

For part (ii), what can we understand from the term 'positive integers'? Based on the word 'solve', what do we have to find?

Stage 2: Think of a plan

For part (i), what are some methods of factorisation that we have learnt? Is there any special algebraic identity that we can use?

For part (ii), have we encountered such an equation before?

We have learnt how to solve a linear equation in one variable as well as two simultaneous linear equations. However, the given equation is a quadratic equation in two variables, which we have not learnt how to solve.

How can we use the answer from part (i)?

Stage 3: Carry out the plan

(i) $x^2 - 9y^2 = x^2 - (3y)^2$

$= (x + 3y)(x - 3y)$ apply $a^2 - b^2 = (a + b)(a - b)$, where $a = x$ and $b = 3y$

(ii) $x^2 - 9y^2 = 13$

$(x + 3y)(x - 3y) = 13$

Go back to Stage 2: Think of a plan

Is there a way to convert this equation into two simultaneous linear equations? What have we not used? Is there anything special about 13?

Since x and y are positive integers, then $(x + 3y)$ and $(x - 3y)$ are also integers.

13 is a prime number, so it has exactly two integer factors: 1 and 13. As $(x + 3y)(x - 3y) = 13$, then the larger factor $(x + 3y)$ must be 13 and the smaller factor $(x - 3y)$ must be 1.

Stage 3: Carry out the plan

(ii) $x^2 - 9y^2 = 13$

$(x + 3y)(x - 3y) = 13$

Since 13 is a prime number, it has exactly two factors: 1 and 13.

Since x and y are positive integers, $x - 3y$ is smaller than $x + 3y$.

$\therefore x - 3y = 1$ — (1)

$x + 3y = 13$ — (2)

(1) + (2): $2x = 14$

$x = 7$

Subst. into (2): $7 + 3y = 13$

$3y = 6$

$y = 2$

$\therefore x = 7$ and $y = 2$

Attention

If $(x + 3y)(x - 3y) = 6$, which is not a prime number, can we conclude that $x + 3y = 6$ and $x - 3y = 1$? Or is it possible for $x + 3y = 3$ and $x - 3y = 2$?