CS 113: Mathematical Structures for Computer Science

Dr. Francis Parisi

Pace University

Fall 2018

Mathematical Induction

Chapter 4

Mathematical Induction

- The Principle of Mathematical Induction
- Additional Examples of Induction Proofs
- Sequences
- The Strong Principle of Mathematical Induction

Mathematical Induction

We have studied several proof techniques for verifying the truth of quantified statements like

$$\forall x \in S,\ R(x)$$

namely

- Direct proof
- Proof by contrapositive
- Proof by contradiction
- Now we add another method of proof to our arsenal ... mathematical induction

Mathematical Induction – The Principle of Mathematical Induction

- If we wish to prove that a quantified statement like

$$\forall n \in \mathbb{N},\ P(n)$$

  for some open sentence $P(n)$, $P(n)$ must be true for each positive integer n
- To show this is true for every n would be impossible
- Thus we use induction for our method of proof

The statement $\forall n \in \mathbb{N},\ P(n)$ is true if

1. $P(1)$ is true
2. $\forall k \in \mathbb{N},\ P(k) \Rightarrow P(k+1)$ is true

Mathematical Induction – The Principle of Mathematical Induction

- Step (1) above is the **basis step**
- Step (2) is the **induction step**

Simply put the steps for proof by mathematical induction are as follows:

1. Establish the base case, $n = 1$
2. Assume it holds for some integer k
3. Show it is true for $k + 1$

Mathematical Induction – The Principle of Mathematical Induction

Example

Show that for every positive integer n

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Proof

For $n = 1$, $\frac{1}{1\cdot 2} = \frac{1}{1+1}$ *which establishes the base step.*

Assume the inductive step

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We need to show

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Mathematical Induction – The Principle of Mathematical Induction

Proof

$$
\begin{aligned}
\tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \cdots + \tfrac{1}{(k+1)(k+2)} &= \left[\tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3} + \cdots + \tfrac{1}{k(k+1)}\right] + \tfrac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}
\end{aligned}
$$

Mathematical Induction – Additional Examples of Induction Proofs

- So far we have looked at problems concerning sums of numbers over the domain $\mathbb{N}$
- We now look at a generalized version of induction that goes by the same name!

The Principle of Mathematical Induction (again).
For a fixed integer m, let

$$S = \{i \in \mathbb{Z} : i \geq m\}.$$

The statement

$$\forall n \in S,\ P(n) : \text{ i.e., for every integer } n \geq m,\ P(n)$$

is true if

1. $P(m)$ is true
2. $\forall k \in S,\ P(k) \Rightarrow P(k+1)$ is true

Mathematical Induction – Sequences

- A **sequence** is a listing of the elements of some set, A
- A sequence can be either finite or infinite
- The elements of a sequence typically belong to $\mathbb{Z}$ or $\mathbb{R}$
- We denote an infinite sequence by $a_1, a_2, a_3, \ldots$ or by $\{a_n\}$
- a_1 is the first term, a_2, the second, ... a_n the nth

Definition

A **geometric sequence** is a sequence in which the ratio of every two consecutive terms a_n and a_{n+1} is some constant r

Definition

An **arithmetic sequence** is a sequence in which the difference of every two consecutive terms a_n and a_{n+1} is some constant k

Mathematical Induction – Sequences

Example

Consider the seqence $a_0, a_1, a_2, \ldots,$ where

$$a_0 = -\frac{1}{3},\ a_1 = \frac{2}{5},\ a_2 = -\frac{4}{7},\ \text{ and } a_3 = \frac{8}{9}.$$

Given the first four terms, find a_n.
What we have...

- Alternating signs
- Numerators are $1, 2, 4, 8, \ldots$
- The denominators are $3, 5, 7, 9, \ldots$

From inspection we see $(-1)^{n+1}$ gives the right sign, 2^n gives the right numerators, and $2n+3$ gives the denominators. Therefore,

$$a_n = (-1)^{n+1}\frac{2^n}{2n+3}$$

Mathematical Induction – Sequences

While many sequences can be expressed in a closed form – evaluated with a given expression – some sequences are **recursively defined**

Definition

A sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to be **recursively defined** if:

1. For some fixed positive integer t, the terms $a_1, a_2, \ldots, a_t$ are given
2. For each integer $n > t$, a_n is defined in terms of one or more of $a_1, a_2, \ldots, a_{n-1}$

Mathematical Induction – Sequences

Example

$n!$ is defined as

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

with $0! = 1$.

An intuitive was to define this recursively is

$$f_n = \begin{cases} 1 & \text{if } n = 0 \\ nf_{n-1} & \text{if } n \in \mathbb{Z}. \end{cases}$$

Mathematical Induction – Sequences

Fibonacci Numbers create one of the most widely studied recursively defined sequences.

Example

The **Fibonacci sequence** $F_1, F_2, F_3, \ldots$ is defined recursively by

$$F_n = \begin{cases} 1 & \text{if } n = 1, 2 \\ F_{n-2} + F_{n-1} & \text{if } n \geq 3. \end{cases}$$

The numbers $F_1, F_2, F_3, \ldots$ are Fibonacci numbers and they show up in areas of mathematics, natural sciences, and even finance!

Mathematical Induction – The Strong Principle of Mathematical Induction

The statement

$$\forall n \in \mathbb{N},\ P(n): \text{ For every positive integer } n, P(n)$$

is true if

1. $P(1)$ is true
2. the statement $\forall k \in \mathbb{Z},\ P(1) \wedge P(2) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)$ is true

Mathematical Induction – The Strong Principle of Mathematical Induction

For a fixed integer m, let

$$S = \{i \in \mathbb{Z} : i \geq m\}.$$

The statement

$$\forall n \in S,\ P(n) : \text{ For every integer } n \geq m,\ P(n)$$

is true if

1. $P(m)$ is true
2. the statement
   $\forall k \in \mathbb{Z},\ P(m) \land P(m+1) \land \cdots \land P(k) \Rightarrow P(k+1)$ is true

Mathematical Induction – Key Results

- **The Principle of Mathematical Induction**
  The statement

  $$\forall n \in \mathbb{N},\ P(n): \text{ For every positive integer } n, P(n)$$

  is true if $P(1)$ is true and $\forall k \in \mathbb{N},\ P(k) \Rightarrow P(k+1)$ is true.

- **The Principle of Mathematical Induction**
  For a fixed integer m, let $S = \{i \in \mathbb{Z} : i \geq m\}$. The statement

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Mathematical Induction – Key Results

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