CS 113: Mathematical Structures for Computer Science

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Sets

Chapter 2

Sets

- Sets and Subsets
- Set Operations and Their Properties
- Cartesian Products of Sets
- Partitions

Sets

- The set is a fundamental concept in mathematics
- A set S is a collection of objects $S = \{x, y, z\}$
- We denote an element x of S as $x \in S$
- $\ldots$ and $w \notin S$ because w is not an element of the set S

Definition

Two sets A and B are **equal**, denoted $A = B$ if they have exactly the same elements. Otherwise they are not equal, $A \neq B$

If $S = \{x, y, z\}$ and $T = \{x, y, z, w\}$ then $S \neq T$

Sets

- A set with no elements is the **null** or **empty** set denoted $\emptyset$ or $\{\}$
- The **cardinality** of a set A is the number of elements in the set denoted $|A|$
- The elements of a set may be other sets $S = \{x, y, \{a, b, c\}\}$; there are three elements in S namely x, y, and the set $\{a, b, c\}$

Sets – Cardinality of Sets

Example

If A is the set of letters in the English language and C is the set of playing cards in a standard deck the $|A| = 26$ and $|C| = 52$.

Example

The set $A = \{1, \{1, 3\}, \emptyset, a\}$ has four elements, two of which are sets, $\{1, 3\}$, and $\emptyset$. Therefore $|A| = 4$.

Remark

In the first example $|A| = 26$ and $|C| = 52$, so the cardinality of A is less than that of C and we write $|A| < |C|$.

Sets – Fundamental Sets of Numbers

$\mathbb{N}$	The set of natural numbers	positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	The set of integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\frac{a}{b}$ where a, b are integers
$\mathbb{R}$	The set of real numbers	$\{-\infty, \ldots, \infty\}$

Sets – Setbuilder Notation

The notation for defining or describing a set is

$$A = \{x \in S : P(x)\}$$

This is read as: *The set A is the set of all x in S for which $P(x)$ holds.* Alternate forms include

$$A = \{x \in S | P(x)\} \text{ and } A = \{x : P(x)\}$$

the latter used when the set S is understood

Sets – Setbuilder Notation

Example

The set

$$A = \{n \in \mathbb{Z} : n^2 \leq 4\}$$

is the set of all integers n such that $n^2 \leq 4$. This is true for $-2 \leq n \leq 2$. Therefore,

$$A = \{n \in \mathbb{Z} : n^2 \leq 4\} = \{n \in \mathbb{Z} | n^2 \leq 4\} = \{-2, -1, 0, 1, 2\}$$

Sets – Setbuilder Notation

Example

List the elements of each set

$$A = \{x \in \mathbb{R} : x^2 - x - 6 = 0\}$$

$$B = \{x \in \mathbb{R} : x^2 + 1 = 0\}$$

$A = \{-2, 3\}$

$B = \{\} = \emptyset$

Sets – Subsets

Definition

A set A is a **subset** of a set B written $A \subseteq B$ if every element of A also belongs to B.

- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
- $A \subseteq A$
- For every set A the empty set $\emptyset \subseteq A$

Definition

A set A is a **proper subset** of a set B written $A \subset B$ if $A \subseteq B$ but $A \neq B$.

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Sets – Subsets

Example

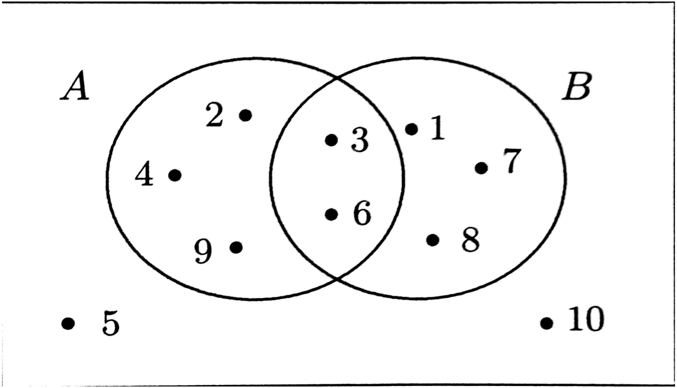
For the sets $A = \{a, b, c\}$ and $C = \{a, b, c, d, e\}$, find all sets B such that $A \subset B \subset C$

$B = \{a, b, c, d\}$ and $B = \{a, b, c, e\}$

Sets – Venn Diagrams

- Venn Diagrams are useful for pictorially representing a set or sets
- Often we draw a rectangle to represent the universal set U
- Within the rectangle we draw circles or ellipses to represent each set

Sets – Venn Diagrams



From the Venn diagram we see $A = \{2, 3, 4, 6, 9\}$,
$B = \{1, 3, 6, 7, 8\}$, $U = \{1, 2, \ldots, 10\}$

Sets – Power Sets

Definition

The set of all subsets of a set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Example

Let $A = \{a, b, c\}$, and $B = \{a, \{b, d\}, \{\emptyset\}\}$. Find $\mathcal{P}(A)$ and $\mathcal{P}(B)$.

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

$\mathcal{P}(B) =$
$\{\emptyset, \{a\}, \{\{b, d\}\}, \{\{\emptyset\}\}, \{a, \{b, d\}\}, \{a, \{\emptyset\}\}, \{\{b, d\}, \{\emptyset\}\}, B\}$

Sets – Intersections and Unions

Definition

Let A and B be two sets. The **intersection** $A \cap B$ of A and B is the set of elements belonging to both A and B. Thus

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Definition

Let A and B be two sets. The **union** $A \cup B$ of A and B is the set of elements belonging to at least one of A and B. Thus

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Sets – Intersections and Unions

Example

For the sets $C = \{1, 2, 4, 5\}$ and $D = \{1, 3, 5\}$,

$$C \cap D = \{1, 5\} \text{ and } C \cup D = \{1, 2, 3, 4, 5\}.$$

In general for $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ the intersection of these sets is

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots A_n$$

and the union of these sets is

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots A_n$$

Sets – Intersections and Unions

Theorem

For every three sets A, B, and C

(a) ***Commutative Laws***

$$A \cap B = B \cap A \text{ and } A \cup B = B \cup A$$

(b) ***Associative Laws***

$$(A \cap B) \cap C = A \cap (B \cap C) \text{ and } (A \cup B) \cup C = A \cup (B \cup C)$$

(c) ***Distributive Laws***

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Sets – Intersections and Unions

Definition

Two sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \emptyset$. A collection of sets is said to be **pairwise disjoint** if every two distinct sets in the collection are disjoint.

- The set of even integers and the set of odd integers
- The set of rational numbers and the set of irrational numbers
- The set of white swans and the set of black swans

Sets – Difference and Symmetric Difference

Definition

The **difference** $A - B$ of two sets A and B is defined as

$$A - B = \{x : x \in A \text{ and } x \notin B.$$

Definition

The **symmetric difference** $A \oplus B$ between two sets A and B is defined as

$$A \oplus B = (A - B) \cup (B - A).$$

Sets – Complement of a Set

Definition

For set A (which is a subset of the universal set U) the **complement** $\overline{A}$ of A is the set of elements in the universal set not belonging to A. That is,

$$\overline{A} = \{x \in U : x \notin A\}.$$

Example

Let $\mathbb{Z}$ be th universal set and E be the set of even integers. Then the complement $\overline{E}$ of E is the set O of odd integers. Moreover, $E \cup \overline{E} = E \cup O = \mathbb{Z}$ and $E \cap \overline{E} = E \cap O = \emptyset$

Sets – Complement of a Set

Theorem

De Morgan's Laws: *For two sets A and B*

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \text{ and } \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Proof

$$\begin{aligned}
x \in \overline{A \cup B} &\equiv \sim(x \in A \cup B) \equiv \sim(x \in A \text{ or } x \in B) \\
&\equiv \sim((x \in A) \vee (x \in B)) \equiv (x \notin A) \wedge (x \notin B) \\
&\equiv (x \in \overline{A}) \wedge (x \in \overline{B}) \equiv x \in \overline{A} \cap \overline{B}. \\
x \in \overline{A \cap B} &\equiv \sim(x \in A \cap B) \equiv \sim(x \in A \text{ and } x \in B) \\
&\equiv \sim((x \in A) \wedge (x \in B)) \equiv (x \notin A) \vee (x \notin B) \\
&\equiv (x \in \overline{A}) \vee (x \in \overline{B}) \equiv x \in \overline{A} \cup \overline{B}.
\end{aligned}$$

Thus $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *and* $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Sets – Cartesian Products of Sets

Definition

For sets A and B the **cartesian product** $A \times B$ of A and B is the set of all orders pairs whose first coordinate belongs to A and whose second belongs to B.

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Sets – Partitions

- At times it may be useful to divide a nonempty set A into subsets
- Each element in a subset belongs to one and only one subset
- In such a case we say the set A is **partitioned** into subsets

Definition

A **partition** of a nonempty set A is a collection of nonempty subsets of A such that every element of A belongs to exactly one of these subsets.

Sets – Partitions

Example

Divide the set of integers $\mathbb{Z}$ into the set E of even integers, and the set O of odd integers. For any element $x \in E$, $x \notin O$ - the partition of $\mathbb{Z}$ into E and O is disjoint.

Example

Given the set $A = \{1, 2, \ldots, 10\}$ and the subsets $S_1 = \{1, 7, 8\}$, $S_2 = \{2, 4, 9\}$,, $S_3 = \{3\}$,, $S_4 = \{5, 6, 10\}$,, the set $\mathcal{P} = \{S_1, S_2, S_3, S_4\}$ is a partition of A.

Sets – Key Results

- If A is a set with $|A| = n$, where n is a nonnegative integer, then $|P(A)| = 2^n$
- For any three sets $A,\ B,$ and C, the following laws hold:
  Commutative Laws
  $A \cap B = B \cap A$ and $A \cup B = B \cup A$
  Associative Laws
  $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$
  Distributive Laws
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  De Morgan's Laws
  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$