CS 113: Mathematical Structures for Computer Science

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Boolean Algebra

Background

- The circuits in computers and other electronic devices have inputs and outputs
- Each input is either a 0 or a 1, and produces outputs that are also 0s and 1s
- Circuits can be constructed using any basic element that has two different states
- These include switches that can be in either the on or the off position
- . . . and optical devices that can be either lit or unlit

Boolean Algebra

Background

- Boolean algebra provides the set of operations for working with the set $\{0, 1\}$
- It is helpful to compare the operations of Boolean algebra to the rules of logic
- In Boolean algebra $0 \equiv \textbf{False}$, and $1 \equiv \textbf{True}$
- The most common operations are complementation (complement of an element), Boolean sum, and Boolean product
- The complement of an element, denoted with a bar, is defined by $\overline{0} = 1$ and $\overline{1} = 0$
- Boolean sum, denoted by $+$ or by OR, has the following values:
  - $1 + 1 = 1, 1 + 0 = 1, 0 + 1 = 1, 0 + 0 = 0$
  - The logical equivalent is disjunction $\mathbf{T} \vee \mathbf{T} \equiv \mathbf{T}$, $\mathbf{T} \vee \mathbf{F} \equiv \mathbf{T}$, $\mathbf{F} \vee \mathbf{T} \equiv \mathbf{T}$, $\mathbf{F} \vee \mathbf{F} \equiv \mathbf{F}$,

Boolean Algebra

Background

- The Boolean product, denoted by $\cdot$ or by AND, has the following values:
  - $1 \cdot 1 = 1, 1 \cdot 0 = 0, 0 \cdot 1 = 0, 0 \cdot 0 = 0$
  - The logical equivalent is conjunction $\mathbf{T} \wedge \mathbf{T} \equiv \mathbf{T}$, $\mathbf{T} \wedge \mathbf{F} \equiv \mathbf{F}$, $\mathbf{F} \wedge \mathbf{T} \equiv \mathbf{F}$, $\mathbf{F} \wedge \mathbf{F} \equiv \mathbf{F}$,
  - We can exclude the symbol $\cdot$ if there is no danger of confusion
- Rules of precedence for Boolean operators are: first, all complements are computed, second all Boolean products, third all Boolean sums, unless we use parentheses to explicitly override the order of precedence

Boolean Algebra

Example

Find the value of $1 \cdot 0 + \overline{(0 + 1)}$

$$
\begin{aligned}
1 \cdot 0 + \overline{(0 + 1)} &= 1 \cdot 0 + \overline{1} \\
&= 1 \cdot 0 + 0 \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

Boolean Algebra

Example

Translate $1 \cdot 0 + \overline{(0 + 1)}$ into a logical equivalence

$$(\mathbf{T} \wedge \mathbf{F}) \vee \sim (\mathbf{F} \vee \mathbf{T})$$

$$(\mathbf{T} \wedge \mathbf{F}) \vee \sim (\mathbf{F} \vee \mathbf{T}) \equiv \mathbf{F}$$

Boolean Algebra

Definition

A Boolean algebra is a set S with two or more elements, binary operations addition $+$ and multiplication $\cdot$ and these properties, often denoted $(S, +, \cdot)$:

- **Commutative Laws**
  $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in S$
- **Associative Laws**
  $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$
- **Distributive Laws**
  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $a + (b \cdot c) = (a + b) \cdot (a + c)$ for all $a, b, c \in S$
- **Existence of Zero and Unity**
  There exist distinct elements $0, 1 \in S$ such that $(a + 0) = a$ and $a \cdot 1 = a$ for each $a \in S$
- **Existence of Complements**
  For every $a \in S$, there exists $\overline{a} \in S$ such that $a + \overline{a} = 1$ and $a\overline{a} = 0$

Boolean Algebra

Other properties that Boolean algebras satisfy

- **Absorption Laws**
  $a \cdot (a + b) = a$ and $a + (a \cdot b) = a$ for all $a, b \in S$
  Logical equivalent: $A \wedge (A \vee B) \equiv A$, and $A \vee (A \wedge B) \equiv A$
- **Idempotent Laws**
  $a + a = a$ and $a \cdot a = a$ for every $a \in S$
  Logical equivalent: $A \vee A \equiv A$, and $A \wedge A \equiv A$
- **Domination Laws**
  $a + 1 = 1$ and $a \cdot 0 = 0$ for every $a \in S$
  Logical equivalent: $A \vee T \equiv T$ and $A \wedge F \equiv F$

Boolean Algebra

- Each property of a Boolean algebra has a dual property
- **The Duality Principle for Boolean Algebras** For each theorem concerning a Boolean algebra S there is a dual theorem concerning S obtained by interchanging $+$ and $\cdot$ and by interchanging 0 and 1
- Each of DeMorgan's Laws can be derived from the other using the duality principle

$$\overline{a+b} = \overline{a} \cdot \overline{b} \text{ and } \overline{a \cdot b} = \overline{a} + \overline{b}$$

Boolean Algebra

Taking a closer look at the simplest Boolean algebra with only two elements

Let's take another look at the disjunction, conjunction, and negation of statements

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	$\sim P$
T	F
F	T

Boolean Algebra

If we interpret $\vee$ and $\wedge$ as binary operations on the set $\{T, F\}$

$\vee$	T	F
T	T	T
F	T	F

$\vee$	T	F
T	T	F
F	F	F

This gives us a Boolean algebra if we let $1 = T$ and $0 = F$
Let $B = \{0, 1\}$ and define addition, multiplication and complementation as follows:

$+$	1	0
1	1	1
0	1	0

$\cdot$	1	0
1	1	0
0	0	0

$$\overline{1} = 0$$
$$\overline{0} = 1$$

Boolean Algebra – Boolean Functions

The Basics

- Let $B = \{0, 1\}$
- x is a **Boolean variable** if it takes a value from B, i.e., 0 or 1
- A **Boolean function of degree** n is a function $f : B^n \to B$, where $B^n = B \times B \times \cdots \times B$ is the Cartesian product of n sets
- We can represent a Boolean function by a **Boolean expression** from Boolean variables and operations

Boolean Algebra – Boolean Functions

Example

For Boolean variables x, y, and z, function f defined by $f(x, y, z) = x \cdot \overline{y} + z$ is a Boolean function. Since $f : B^3 \rightarrow B$ f has degree 3. We can display the values of f in a table, similar to the truth tables we learned in logic.

x	y	z	$\overline{y}$	$x \cdot \overline{y}$	$f(x, y, z) = x \cdot \overline{y} + z$
1	1	1	0	0	1
1	1	0	0	0	0
1	0	1	1	1	1
1	0	0	1	1	1
0	1	1	0	0	1
0	1	0	0	0	0
0	0	1	1	0	1
0	0	0	1	0	0

Boolean Algebra – Boolean Functions

Example

For Boolean variables x, y, and z, determine the values of function f of degree 3 defined by $f(x, y, z) = (\bar{x} + \bar{y}) \cdot z$.

Boolean Algebra – Boolean Functions

Example

Find the Boolean expression that represents the function f from the values given in the table.

x	y	z	f
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

This is called the **sum-of-products expansion** or **disjunctive normal form** (**DNF**) of the Boolean function f

Boolean Algebra – Boolean Functions

Example

Find the DNF of the Boolean function f of degree 3 defined by

$$f(x, y, z,) = (x + \overline{y})z.$$

Solution: Another way to find the DNF without using a table.

$$\begin{aligned} f(x, y, z) &= (x + \overline{y})z = xz + \overline{y}z \\ &= x \cdot 1 \cdot z + 1 \cdot \overline{y}z \\ &= x(y + \overline{y})z + (x + \overline{x})\overline{y}z \\ &= xyz + x\overline{y}z + x\overline{y}z + \overline{x}\,\overline{y}z \\ &= xyz + x\overline{y}z + \overline{x}\,\overline{y}z \end{aligned}$$

Boolean Algebra – Boolean Functions

- Another common form of a Boolean expression is the **product-of-sums expansion** or **conjunctive normal form** (**CNF**)
- Recall that to find the DNF we looked for all triples (x, y, z) for which $f(x, y, z) = 1$
- To find the CNF we look for all triples (x, y, z) for which $f(x, y, z) = 0$

Boolean Algebra – Boolean Functions

Example

Find the CNF that represents the function f from the values given in the table.

x	y	z	f
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

Solution $f(x,y,z) = (\overline{x} + \overline{y} + \overline{z})(\overline{x} + y + z)(x + y + z)$
(*there's a typo in the text*)

Boolean Algebra – Combinatorial Circuits

Definition

A **combinatorial circuit** converts a combination of inputs each 0 or 1, into a uniquely defined output, which is also wither 0 or 1

- Every combinatorial circuit can be designed using the rules of Boolean algebra
- The basic elements of these circuits are **gates**
- An **OR gate** converts inputs x and y into the output $x + y$
- An **AND gate** converts inputs x and y into the output xy
- An **inverter** or **NOT gate** converts an input x into the output $\overline{x}$

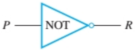
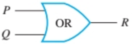
Boolean Algebra – Combinatorial Circuits

(a) Inverter

(b) OR gate

(c) AND gate

Boolean Algebra – Combinatorial Circuits

<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td><table><tr><th>Input</th><th>Output</th></tr><tr><th>P</th><th>R</th></tr><tr><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td></tr></table></td></tr>
<tr><td>AND</td><td>
</td><td><table><tr><th colspan="2">Input</th><th>Output</th></tr><tr><th>P</th><th>Q</th><th>R</th></tr><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table></td></tr>
<tr><td>OR</td><td>
</td><td><table><tr><th colspan="2">Input</th><th>Output</th></tr><tr><th>P</th><th>Q</th><th>R</th></tr><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table></td></tr>
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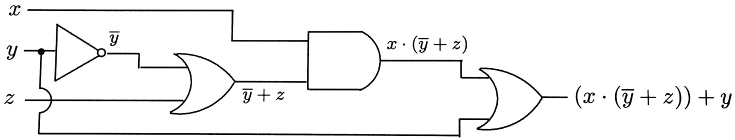
Boolean Algebra – Combinatorial Circuits

Example

Find a combinatorial circuit that produces the output
$x \cdot (\overline{y} + z) + y$



Boolean Algebra – Key Results

- If $(S, +, \cdot)$ is a Boolean algebra, then

  Idempotent Laws $a + a = a$ and $a \cdot a = a$ for every $a \in S$
  Domination Laws $a + 1 = 1$ and $a \cdot 0 = 0$ for every $a \in S$
  Absorption Laws
  $a \cdot (a + b) = a \text{ and } a + (a \cdot b) = a \text{ for all } a, b \in S$
- In a Boolean algebra every element has a unique complement
- Let S be a Boolean algebra then for every $a \in S$, $\overline{\overline{a}} = a$
- Every Boolean algebra S satisfies De Morgan's Laws for all $a, b \in S$:

$$\overline{a + b} = \overline{a} \cdot \overline{b} \text{ and } \overline{a \cdot b} = \overline{a} + \overline{b}$$

Boolean Algebra – Key Results

- **The Duality Principle for Boolean Algebras** For each theorem concerning a Boolean algebra S there is a dual theorem concerning S obtained by interchanging $+$ and $\cdot$ and by interchanging 0 and 1
- Let S be a Boolean algebra and let $a, b \in S$, then $a \cdot \overline{b} + a \cdot b = a$
- For elements a and b in a Boolean algebra, $a \cdot b = a$ if and only if $a \cdot \overline{b} = 0$
- For elements a and b in a Boolean algebra, $a + b = a$ if and only if $a + \overline{b} = 1$