CS 113: Mathematical Structures for Computer Science

Dr. Francis Parisi

Pace University

Fall 2018

Relations and Functions

Chapter 5

Relations and Functions

- Relations
- Equivalence Relations
- Functions
- Bijective Functions
- Cardinalities of Sets

Relations and Functions – Relations

- Often we have two sets A and B where some of the elements of A are connected to elements of B
- These connections are described by two important concepts: relations and functions
- Which applies depends on the nature of the connection

Relations and Functions – Relations

Recall from Chapter 2 that the Cartesian product of two sets A and B is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$. Symbolically we write

$$A \times B = \{(a,b) : a \in A, b \in B\}.$$

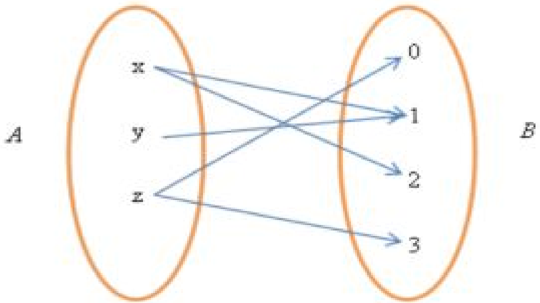
Definition

A **relation** R **from a set** A **to a set** B is a subset of $A \times B$. In addition, R is said to be a relation on $A \times B$. If $(a,b) \in R$, then a is said to be related to b; while if $(a,b) \notin R$, then a is not related to b. If $(a,b) \in R$, we also write $a\ R\ b$; while if $(a,b) \notin R$, we write $a \not R\ b$.

Relations and Functions – Relations

Example

For sets $A = \{x, y, z\}$, and $B = \{0, 1, 2, 3\}$,
$R = \{(x, 1), (x, 2), (y, 1), (z, 0), (z, 3)\}$ is a relation from A to B.



Relations and Functions – Relations

Exercise

Let $A = \left\{\sqrt{2}, e, 3, \pi\right\}$ and $B = \{1, 2, 3, 4\}$. An element $a \in A$ is said to be related to an element $b \in B$ if $|a - b| < 1$. Which elements of A are related to which elements of B?

Relations and Functions – Relations

Exercise

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^{-1}$ denote the set of negative integers. A relation R from $\mathbb{N}$ to $\mathbb{N}^{-1}$ is defined by $a \, R \, b$ if $a + b = \mathbb{N}$. Give examples of pairs of elements that *are* related and some that *are not*.

Relations and Functions – Relations

Definition

A **relation R on a set S** is a relation from S to S. That is, R is a relation on a set S if R is a subset of $S \times S$.

- If a set A has n elements then there are 2^n subsets of A
- There are three key properties of relations defined next

Definition

Let R be a relation defined on a nonempty set S. Then R is:

1. **reflexive** if $a\ R\ a$ for all $a \in S$; that is, if $a \in S$, then $(a, a) \in R$
2. **symmetric** if whenever $a\ R\ b$, then $b\ R\ a$ for all $a, b \in S$; that is if $(a, b) \in R$ then $(b, a) \in R$
3. **transitive** if whenever $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$ for all $a, b, c \in S$; that is if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

Relations and Functions – Relations

- A relation R need not have any one or more of these properties
- A relation R defined on a set S is:
  1′ **not reflexive** if $(x, x) \notin R$ for some $x \in S$
  2′ **not symmetric** if $(x, y) \in R$ but $(y, x) \notin R$ for some pair (x, y) of distinct elements of S
  3′ **not transitive** if $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$ for some $x, y, z \in S$

Relations and Functions – Equivalence Relations

- Relations that have all three properties, reflexive, symmetric and transitive, are equivalence relations
- The best known relation, the equals relation, is an equivalence relation

Definition

A relation R on a non-empty set is an equivalence relation if R is reflexive, symmetric and transitive

For an equivalence relation R defined on a set A there is a subset of A associated with each element of A that is of particular interest.

Definition

Let R be an equivalence relation on a set A. For $a \in A$, the equivalence class $[a]$ is defined by

$$[a] = \{x \in A : x\, R\, a\}$$

Relations and Functions – Equivalence Relations

Exercise

Let $S = \{1, 2, 3, 4, 5, 6\}$. The relation

$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,4),(2,3),(2,6),(3,2),\\ (3,6),(4,1),(6,2),(6,3)\}$$

on S is an equivalence relation. Therefore, there is an an equivalence class associated with each element of S. Find the equivalence classes.

Relations and Functions – Functions

- We saw that in a relation from set A to set B an element of A may relate to some or all of the elements of B
- We will now consider the case when each element of A is related to exactly one element of B

Definition

Let A and B be nonempty sets. A **function** f from A to B is a relation from A to B that associates with each element of A a unique element of B. A function f from A to B is denoted by $f : A \to B$

Relations and Functions – Functions

Definition

Given a function $f : A \to B$ if $b \in B$ is the unique element assigned to $a \in A$ by f then we write $b = f(a)$, and b is the **image** of a under f

Definition

If $f : A \to B$ is a function from a set A to B, then A is called the **domain** of f and B is the **codomain** of f. The **range** $f(A)$ is the set of images of the elements of A, namely,

$$f(A) = \{ f(a) : a \in A \}$$

Relations and Functions – Functions

Definition

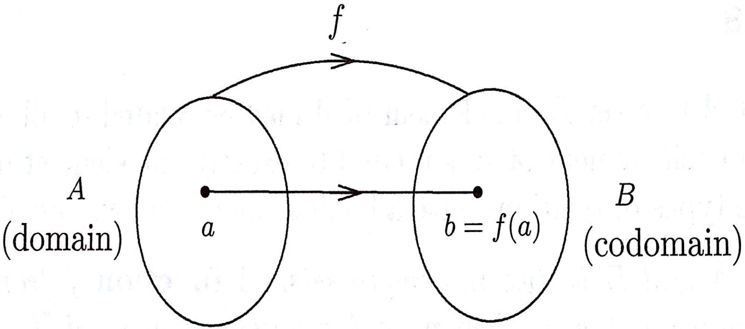
For a function f from a set A to a set B and a subset X of A, the image of X under f is the set

$$f(X) = \{ f(x) : x \in X \}$$

- For every subset X of A, $f(X) \subseteq B$
- If $X = A$ then $f(X) = f(A)$ is the range of f

Relations and Functions – Functions

Visualizing Functions It is often useful to visualize a function via a diagram



Relations and Functions – Functions

Exercise

Let $A = \{a, b, c, d, e\}$ and $B = \{x, y, z\}$ and $f = \{(a, x), (b, x), (c, z), (d, x), (e, z)\}$ be a function from A to B.

(a) Determine the domain, codomain, and range of f.

(b) Determine the image of d.

(c) Is y an image?

(d) Determine $f(X)$ where $x = \{a, c, d\}$.

(e) Give an example of a function g from B to A.

Relations and Functions – Functions

Exercise

Let $f : A \to B$ be a function. Prove that if $A_1, A_2 \subseteq A$, then

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2).$$

Exercise

For every $x \in \mathbb{R}$, let $f(x)$ denote any real number such that (x,y) lies on the circle $x^2 + y^2 = 25$. Is $f(x)$ a function from $\mathbb{R}$ to $\mathbb{R}$?

Relations and Functions – Functions

Common Functions

- **Identity Function** The function $f : A \rightarrow A$ defined by $f(a) = a, \forall a \in A$
- **Absolute Value Function** The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- **Ceiling Function** The function $f : \mathbb{R} \rightarrow \mathbb{Z}$ defined by $f(x) = \lceil x \rceil$

Relations and Functions – Functions

Common Functions

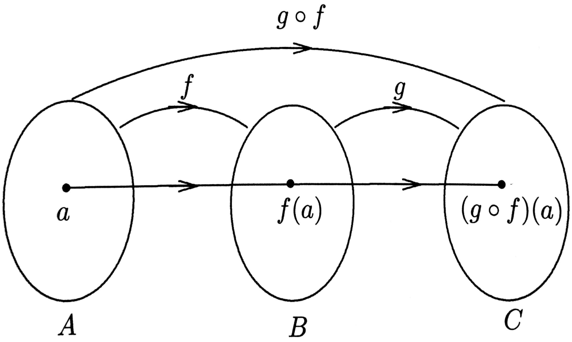
- **Floor Function** The function $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lfloor x \rfloor$
  If $a \in \mathbb{R}^+$, $a \neq 1$, $b \in \mathbb{R}$, and $a^b = c$, then $\log_a c = b$ if and only if $a^b = c$
- **Exponential Function** The function $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = 2^x$
- **Logarithmic Function** The function $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \log_2 x$

Relations and Functions – Functions

Composition of Functions

Definition

Let A, B, and C be sets and suppose that $f : A \to B$ and $g : B \to C$ are two functions. The **composition** $g \circ f$ of f and g is the function from A to C defined by $(g \circ f)(a) = g(f(a))$ for $a \in A$.



Relations and Functions – Functions

Example

Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $C = \{x, y, z\}$ and let $f : A \to B$ and $g : B \to C$ be functions where

$f = \{(1, c), (2, b), (3, a)\}$ and $g = \{(a, y), (b, x), (c, x), (d, z)\}$.

Then

$$(g \circ f)(1) = g(f(1)) = g(c) = x.$$

Relations and Functions – Functions

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$, defined as

$$f(x) = \sin x \text{ and } g(x) = x^2.$$

Determine $(f \circ g)(x)$, and $(g \circ f)(x)$.

Relations and Functions – Bijective Functions

Often functions from a set A to a set B have one or both of the following properties

1. each element of B is the image of **at most** one element of A
2. each element of B is the image of **at least** one element of A

Relations and Functions – Bijective Functions

Property #1 **One-to-One Functions**

Definition

For two nonempty sets A and B, a function $f : A \to B$ is said to be **one-to-one** if every two distinct elements of A have distinct images in B, that is if $a, b \in A$ and $a \neq b$, then $f(a) \neq f(b)$. This function is also known as an **injective function** or an **injection.**

Relations and Functions – Bijective Functions

Property #2 **Onto Functions**

Definition

Let A and B be two nonempty sets. A function $f : A \rightarrow B$ is called **onto** if every element of B is the image of some element of A. This function is also known as an **surjective function** or a **surjection**.

Relations and Functions – Functions

Exercise

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x - 3$ for $x \in \mathbb{R}$. Show $f(x)$ is one-to-one.

Exercise

Show the following are *not* one-to-one.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 1,\ x \in \mathbb{R}$

(b) $g : \mathbb{Z} \to \mathbb{Z},\ g(n) = \lceil n \rceil,\ n \in \mathbb{Z}$

(c) $h : \mathbb{R} \to \mathbb{R},\ h(x) = x^2 - 3x + 1,\ x \in \mathbb{R}$

Exercise

Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2 - 2x + 5$$

dor $x \in \mathbb{R}$ is onto.

Exercise

A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$f(n) = 2n.$$

Determine (and explain) whether f is

(a) one-to-one

(b) onto

Relations and Functions – Bijective Functions

What if a function is both one-to-one and onto?

If a function f from a set A to a set B is both one-to-one and onto then every element of B is the image of at most (one-to-one) one element of A and the image of at least (onto) one element of A. In other words, every element of B is the image of exactly one element of A.

Definition

A function that is one-to-one and onto is called a **bijective function**, a **bijection**, or a **one-to-one correspondence**

Relations and Functions – Bijective Functions

Composition of Bijective Functions

Theorem

Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be two functions.

(a) *If f and g are one-to-one, then so is $g \circ f$*

(b) *If f and g are onto, then so is $g \circ f$*

Corollary

Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be two functions. If f and g are bijective, then so is $g \circ f$.

Relations and Functions – Bijective Functions

Inverse Functions

- Let the nonempty sets A and B be $A = \{a, b, c\}$ and $B = \{x, y, z\}$
- A bijective function $f : A \to B$ pairs off the elements of A with the elements of B
- So for example $f = \{(a, x), (b, y), (c, z)\}$

Definition

The **inverse function** or the **inverse** f^{-1} of f is found by replacing each ordered pair (r, s) with (s, r). The function $f^{-1} : B \to A$ is also bijective.

Theorem

Let A and B be nonempty sets. A function $f : A \to B$ has an inverse function $f^{-1} : B \to A$ if and only if f is bijective. Moreover if f is bijective so is f^{-1}.

Relations and Functions – Cardinalities of Sets

Sets Having the Same Cardinality

- We discussed cardinality, the number of elements in a set, when we covered set theory in chapter 2
- Consider sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$ again – each set has three elements so $|A| = |B|$
- And the bijective function $f = \{(a, x), (b, y), (c, z)\}$ pairs off the elements and motivate a definition for same cardinality

Definition

Two nonempty sets A and B (finite or infinite) are defined to have the same cardinality, written $|A| = |B|$, if there exists a bijective function from A to B

Denumerable Sets

- The definition for same cardinality provides us with a definition for $|A| = |B|$ when A and B are infinite.
- This leads us to an important class of infinite sets

Definition

A set A is called **denumerable** if $|A| = |\mathbb{N}|$.

In other words, set A is denumerable if it has the same number of elements as the set of positive in integers, or natural numbers. Put another way, set A is denumerable if its elements can be put in one-to-one correspondence with the elements of $\mathbb{N}$.

Countable and Uncountable Sets

Definition

A set that is either finite or denumerable is called **countable.** A denumerable set is also called **countably finite.** A set that is not countable is called **uncountable.**

Theorem

Every set that contains an uncountable subset is itself uncountable.

Corollary

The set $\mathbb{R}$ *of real numbers is uncountable.*

Corollary

The set $\mathbb{C}$ *of complex numbers is uncountable.*

Theorem

If A and B are disjoint denumerable sets, then $A \cup B$ is denumerable.

Theorem

The set of irrational numbers is uncountable.

Theorem

Every set has a smaller cardinality than its power set, that is, $|A| < |\mathcal{P}(A)|$ for every set A.

Corollary

There is no set of largest cardinality.

Relations and Functions – Key Results

- Let R be an equivalence relation on a nonempty set A and let a and b be elements of A. Then $[a] = [b]$ if and only if $a \; R \; b$.
- Let R be an equivalence relation defined on a nonempty set A. If $\mathcal{P}$ is the set of distinct equivalence classes of A resulting from R, then $\mathcal{P}$ is a partition of A.
- Let R be an equivalence relation defined on a nonempty set A. If $[a] = [b]$ are equivalence classes of A resulting from R, then either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.
- Let A,B and C be nonempty sets and $f : A \to B$ and $g : B \to C$ two functions. Then the following hold:

  If f and g are one-to-one, then so is $g \circ f$.
  If f and g are onto, then so is $g \circ f$.
  If f and g are bijective, then so is $g \circ f$.

Relations and Functions – Key Results

- A function $f : A \to B$ has an inverse $f^{-1} : B \to A$ if and only if f is bijective; then f^{-1} is also bijective.
- If $f : A \to B$ is a bijective function, then $f^{-1} \circ f$ is the identity function on A and $f \circ f^{-1}$ is the identity function on B.
- The set $\mathbb{Z}$ of integers is denumerable.
- The set $\mathbb{Q}$ of rational numbers is denumerable.
- Every infinite subset of a denumerable set is denumerable
- Every set that contains an uncountable subset is itself uncountable
- The set $\mathbb{R}$ of real numbers is uncountable
- The set $\mathbb{C}$ of complex numbers is uncountable
- If A and B are disjoint denumerable sets, then $A \cup B$ is denumerable
- The set of irrational numbers is uncountable
- For every set A, $|A| < |\mathcal{P}(A)|$
- There is no set of largest cardinality