CS 113: Mathematical Structures for Computer Science

Dr. Francis Parisi

Pace University

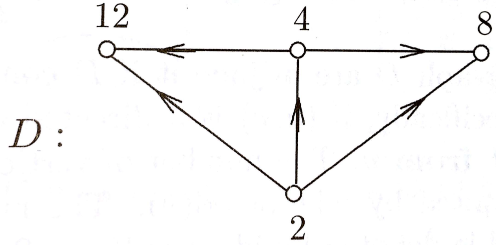
Fall 2018

Directed Graphs

Chapter 15

Directed Graphs – Digraph Theory

- A directed graph or digraph is a common modeling tool when the ordering of the vertices of a graph are important
- Suppose for vertex set $V = \{2, 4, 8, 12\}$ there is a directed line segment from vertex i to vertex j, and $i \neq j$, if $i | j$.



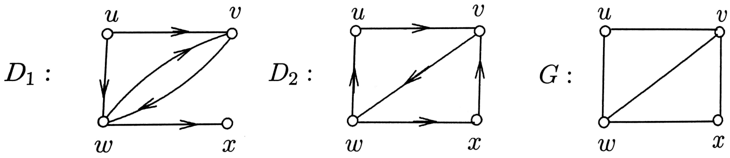
Directed Graphs – Digraph Theory

Definition

A **digraph** or **directed graph** D is a finite nonempty set of $V(D)$ of **vertices** and set $E(D)$ of ordered pairs of distinct vertices of D, each ordered pair of which is called a **directed edge** or an **arc**.

- A directed edge (u, v) is drawn as a directed line segment (arrow)
- For a directed edge $e = (u, v)$ u is the **initial vertex**
- v is the **terminal vertex**
- In a graph (u, v) and (v, u) represent the same edge
- In a digraph (u, v) and (v, u) represent different directed edges
- If for each pair u, v at most one of (u, v) and (v, u) is a directed edge, then the graph is an **oriented graph**

Directed Graphs – Digraph Theory



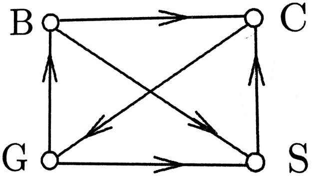
- D_1 is not oriented, but D_2 is oriented
- D_2 is and orientation of graph G

Directed Graphs – Digraph Theory

Example

Soccer teams from Brazil, China, Germany, and Spain play one another. Brazil beats China and Spain, Germany beats Brazil and Spain, Spain beats China, and China defeats Germany. How do we model this as a digraph?

$D:$



Directed Graphs – Digraph Theory

- Two vertices u and v in a digraph D are **adjacent** if there is at least one directed edge (u, v) or (v, u)
- If (u, v) is a directed edge we say u is **adjacent to** v and v is **adjacent from** u
- The number of vertices *to which* a vertex v is adjacent is the **outdegree** of v denoted by od v or od(v)
- The number of vertices *from which* a vertex v is adjacent is the **indegree** of v denoted by $\mathrm{id}\, v$ or $\mathrm{id}(v)$

Directed Graphs – Digraph Theory

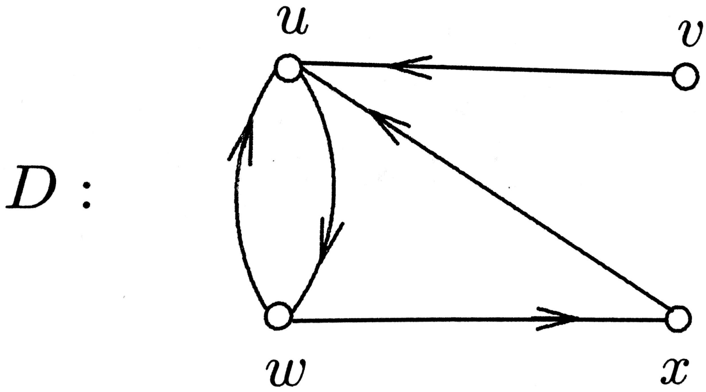
Theorem (The First Theorem of Digraph Theory)

If D is a digraph of size m, then

$$\sum_{v \in V(D)} \operatorname{od} v = \sum_{v \in V(D)} \operatorname{id} v = m.$$

This is illustrated in the following figure

Directed Graphs – Digraph Theory



Directed Graphs – Digraph Theory

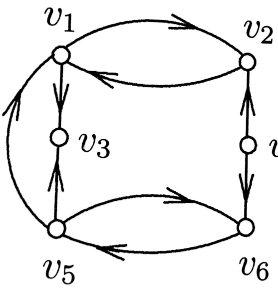
Definition

The **adjacency matrix** $\mathbf{A} = [a_{ij}]$ of a digraph D with $V(D) = \{v_1, v_2, \ldots, v_n\}$ is defined as expected just like for a graph,

$$a_{ij} = \begin{cases} 1 \text{ if } (v_i, v_j) \in E(G) \\ 0 \text{ otherwise.} \end{cases}$$

Directed Graphs – Digraph Theory

$D:$



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Directed Graphs – Finite-State Machines

- **Finite state machines** describe and increasing number of input-output devices – some examples
  - pushing a sequence of buttons on your phone followed by the talk button connects to another person's phone
  - entering information at a web site and clicking "complete order" places an online order
  - placing coins in a vending machine and pushing a button to make a selection for a candy bar
  - and of course a computer is an input-output device
- A finite state machine is also know as a **sequential circuit**

Directed Graphs – Finite-State Machines

Example

Consider a laundromat that has a machine to dispense laundry detergent and fabric softener. It costs $75¢$ to purchase laundry detergent or fabric softener. This can be modeled as a finite state machine.

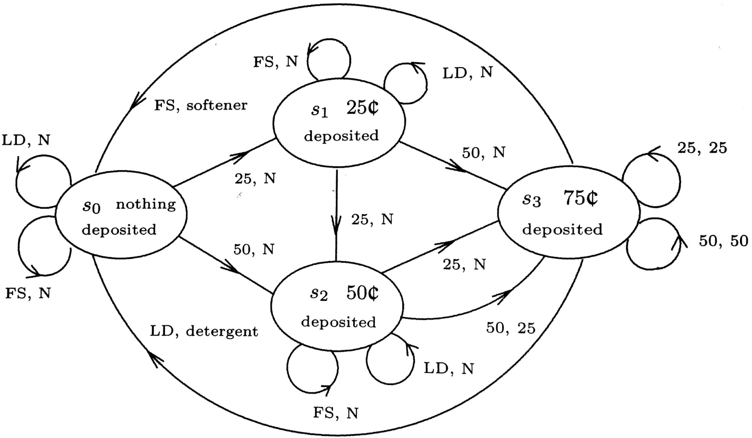
Time	t_0	t_1	t_2	t_3	t_4
State	s_0	s_1	s_2	s_3	s_0
Input	25¢	25¢	25¢	LD	
Output	nothing	nothing	nothing	detergent	

Directed Graphs – Finite-State Machines

A **finite-state machine** consists of:

1. a finite set S of internal states
2. a finite **input set** I
3. a finite **output set** O
4. a **next-state** (or **transition**) **function** $f : S \times I \to S$
5. an **output function** $g : S \times I \to O$

Directed Graphs – Finite-State Machines



Directed Graphs – Finite-State Machines

Finite-State Automata *Finite-State Machines that produce no output*

A **finite-state automaton** (singular for automata) consists of:

1. a finite set S of states
2. a finite set I of input values
3. a **transition function** f that associates a next step with each state-input pair

An example is the network of roads in a town; the inputs are directions (right turn, left turn, go straight) and each new intersection (location) is a state.

Graphs – Key Results

- **The First Theorem of Digraph Theory:** If D is a digraph of size m, then

$$\sum_{v \in V(D)} \operatorname{od} v = \sum_{v \in V(D)} \operatorname{id} v = m.$$

- **Finite-State Machine:** A structure consisting of
  1. a finite set S of internal states
  2. a finite **input set** I
  3. a finite **output set** O
  4. a **next-state** (or **transition**) **function** $f \ : S \times I \to S$
  5. an **output function** $g \ : S \times I \to OS$

- **Finite-State Automaton:** A structure consisting of
  1. a finite set S of states
  2. a finite set I of input values
  3. a **transition function** f that associates a next step with each state-input pair