# CS 113: Mathematical Structures for Computer Science

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# Themes of This Course

**Logic and Proof** Probably the most important goal of a first course in discrete mathematics is to help students develop the ability to think abstractly. This means learning to use logically valid forms of argument and avoid common logical errors, appreciating what it means to reason from definitions, knowing how to use both direct and indirect argument to derive new results from those already known to be true, and being able to work with symbolic representations as if they were concrete objects.

# Themes of This Course

Induction and Recursion An exciting development of recent years has been the increased appreciation for the power and beauty of recursive thinking. To think recursively means to address a problem by assuming that similar problems of a smaller nature have already been solved and figuring out how to put those solutions together to solve the larger problem. Such thinking is widely used in the analysis of algorithms, where recurrence relations that result from recursive thinking often give rise to formulas that are verified by mathematical induction.

# Themes of This Course

**Discrete Structures** Discrete mathematical structures are the abstract structures that describe, categorize, and reveal the underlying relationships among discrete mathematical objects. Those studied in this book are the sets of integers and rational numbers, general sets, Boolean algebras, functions, relations, graphs and trees, formal languages and regular expressions, and finite-state automata.

# Themes of This Course

**Combinatorics and Discrete Probability** Combinatorics is the mathematics of counting and arranging objects, and probability is the study of laws concerning the measurement of random or chance events. Discrete probability focuses on situations involving discrete sets of objects, such as finding the likelihood of obtaining a certain number of heads when an unbiased coin is tossed a certain number of times. Skill in using combinatorics and probability is needed in almost every discipline where mathematics is applied, from economics to biology, to computer science, to chemistry and physics, to business management.

# Logic

*Chapter 1*

# Logic

- Statements (aka Propositions)
- Negation, Conjunction, Disjunction
- Implications
- Biconditionals
- Tautologies and Contradictions

# Logic – Statements

**"Logic is the the science of necessary laws of thought"**,
*Immanuel Kant*

- English sentences are...
  - Declarative
  - Interrogative
  - Imperative
  - Exclamatory

## Example

- The sun is shining. (declarative)
- Close the door. (imperative)
- Run! (exclamatory)
- Are you cold? (interrogative)

# Logic – Statements

Argument form is the central concept of deductive logic and the key to this is the *statement* or *proposition*

### Definition

A **statement** is a declarative sentence that is either true or false but not both

# Logic – Truth Values

- Logic deals with statements and their Truth Values
- Declarative statements have truth values, others do not
- Truth Value of a true statement is **true** denoted by $\mathbf{T}$
- Truth Value of a false statement is **false** denoted by $\mathbf{F}$

# Logic – Open Sentences

## Definition

An **open sentence** is a declarative sentence containing one or more variables, and whose truth depends on the values of the variables.

## Notation

We denote statements or open sentences by symbols like $P, Q, R$ or $P_1, P_2, P_3$

## Example

$$P_1 : |-2| = -2$$

$$P_2 : (-1)^3 = -1$$

$P_1$ is false and $P_2$ is true.

# Logic – Open Sentences

An **open sentence** containing variable $x$ is typically denoted $P(x)$. If $x$ is part of a collection $S$, then $S$ is the *domain* of $x$ and $P(x)$ is the open sentence over the domain $S$.

Example

$$P(x) : 3x - 9 = 0, \quad x \text{ is a real number}$$

$P(x)$ is true for $x = 3$ but false for $x = -3$

# Logic – Open Sentences

### Example

Each of the following is an open sentence, where $n$ represents a positive integer. For which values of $n$ are the following true?

(a) $P_1(n) : |n+1| = 0$

no positive integer $n$ for which $P_1(n)$ is true $(n = -1)$

(b) $P_2(n) : n^2 - 4 \leq 0$

$n = 1, 2 \quad (n^2 \leq 4)$

(c) $P_3(n) : n^2 - n - 6 = 0$

$n = 3$ (factor into $(n-3)(n+2) = 0$, so $n = 3$)

(d) $P_4(n) : n + \frac{1}{n} \leq 2$

$n = 1$ (for $n \geq 2$, $n + \frac{1}{2} > 2$)

# Logic – Negation, Conjunction, Disjunction

- Before we get into logical operators let's consider *Truth Tables*
- Each statement can take on one of two truth values, $T$ or $F$
- When working with multiple statements, say $P$ and $Q$, we can create a truth table that provides the truth values for all possible combinations
- For $n = 2$ statements there are $2^2 = 4$ combinations
- In general for $n$ statements there are $2^n$ combinations

| $P$ | $Q$ |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

# Logic – Negation

## Definition

The **negation** of a statement $P$, is the statement not $P$, or, it is not the case that $P$, denoted $\sim P$.

Negation switches the truth values

| $P$ | $\sim P$ |
|---|---|
| T | F |
| F | T |

## Example

For any real value $x$, the negation of $P(x) : (x-2)^2 > 0$ is $\sim P(x) : (x-2)^2 \leq 0$. $P(x)$ **is** a statement for each specific $x$.

# Logic – Conjunction

## Definition

For two statements $P$ and $Q$, the **conjunction** of $P$ and $Q$ is the statement $P$ and $Q$, denoted $P \wedge Q$.

The truth table for the conjunction $P \wedge Q$ is as follows

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Logic – Conjunction

### Example

$P$: Pace University is located in New York City
$Q$: Albany is the capital of New York State
$P \wedge Q$ is **true**
What if ...
$Q$: Brooklyn is the capital of New York State
$P \wedge Q$ is **false**

# Logic – Disjunction

## Definition

For two statements $P$ and $Q$, the **disjunction** of $P$ and $Q$ is the statement $P$ or $Q$, denoted $P \vee Q$.

The truth table for the disjunction $P \vee Q$ is as follows

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Disjunction is the “inclusive” OR. In English this translates into $P$ or $Q$ or both.

# Logic – Exclusive 'OR'

## Definition

For two statements $P$ and $Q$, the **exclusive or** of $P$ and $Q$ is the statement $P$ or $Q$ but not both, denoted $P \oplus Q$.

The truth table for the exclusive or $P \oplus Q$ is as follows

| $P$ | $Q$ | $P \oplus Q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# Logic – Conjunction, Disjunction, and Exclusive OR

## Example

Consider the two open sentences for integer n

$$P(n) : n^3 + 2n \text{ is even.} \quad Q(n) : n^2 - 4 < 0.$$

The conjunction, disjunction, and exclusive or of $P(n)$ and $Q(n)$ are

$$P(n) \wedge Q(n) : n^3 + 2n \text{ is even } \textbf{and } Q(n) : n^2 - 4 < 0.$$

$$P(n) \vee Q(n) : n^3 + 2n \text{ is even } \textbf{or } Q(n) : n^2 - 4 < 0.$$

$$P(n) \oplus Q(n) : n^3 + 2n \text{ is even } \textbf{or } Q(n) : n^2 - 4 < 0, \textbf{ but not both}.$$

# Logic – Compound Statements

The operations we have discussed so far, $\sim, \wedge, \vee$ and $\oplus$ are examples of logical connectives.

## Definition

A **compound statement** is a statement constructed from one or more given statements and one or more logical connectors. The given statements used to create compound statements are called **component statements**.

- In order to know the truth value of a compound statement we must know the truth values of its component statement(s)
- For example, if $R : P \wedge Q$, then knowing the truth values of $P$ and $Q$ we can derive the truth value of $R$.

# Logic – Compound Statements

## Definition

Two compound statements $R$ and $S$ constructed from the same component statements are called **logically equivalent** if $R$ and $S$ have the same truth value for all combinations of truth values of their component statements. If $R$ and $S$ are logically equivalent, then we write $R \equiv S$; while if $R$ and $S$ are not logically equivalent then we write $R \not\equiv S$.

# Logic – Compound Statements

The logical equivalence of statements such as $P \wedge Q$ and $P \vee Q$ lead to two laws of logic know as the **commutative laws**

### Theorem

*For every two statements* $P$ *and* $Q$*,*

$$P \wedge Q \equiv Q \wedge P \quad \textit{and} \quad P \vee Q \equiv Q \vee P.$$

We can verify the the commutative laws using truth tables.

### Example

Logically there is no difference between *Pace University is in New York, and Brooklyn is part of NYC* and *Brooklyn is part of NYC and Pace University is in New York*

The same holds for open sentences $P(n)$ and $Q(n)$.

# Logic – De Morgan's Laws

Combines negation, disjunction and conjunction to provide two very useful laws

## Theorem

*For every two statements $P$ and $Q$*

(a) $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$

(b) $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

| $P$ | $Q$ | $P \vee Q$ | $\sim (P \vee Q)$ | $\sim P$ | $\sim Q$ | $(\sim P) \wedge (\sim Q)$ |
|---|---|---|---|---|---|---|
| T | T | T | **F** | F | F | **F** |
| T | F | T | **F** | F | T | **F** |
| F | T | T | **F** | T | F | **F** |
| F | F | F | **T** | T | T | **T** |

# Logic – De Morgan's Laws

## Example

For two integers $a$ and $b$ consider

$$P : a \text{ is odd.} \quad Q : b \text{ is odd.}$$

1. State the disjunction of $P \vee Q$
2. State the negation of $\sim (P \vee Q)$ (*it is not the case that ...*)
3. Restate $\sim (P \vee Q)$ using an appropriate De Morgan's Law
4. State the conjunction $P \wedge Q$
5. State the negation $\sim (P \wedge Q)$ (*it is not the case that ...*)
6. Restate $\sim (P \wedge Q)$ using an appropriate De Morgan's Law

# Logic – De Morgan's Laws

- **Double Negation** results in logically equivalent statements.
- For every statement $P$, $P \equiv \sim (\sim P)$

## Example

Let $P$ and $Q$ be statements. Verify

1. $\sim (P \vee (\sim Q)) \equiv (\sim P) \wedge Q$
2. $\sim ((\sim P) \wedge (\sim Q)) \equiv P \vee Q$

# Logic – Associative and Distributive Laws

The Associative law and Distributive law of real numbers have logical equivalents using conjunction and disjunction

### Theorem

*Let $P, Q$ and $R$, be three statements then*
*Associative Laws*

$$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$$

$$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$$

*Distributive Laws*

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

# Logic – Implications

### Definition

For two statements $P$ and $Q$ the implication $P \Rightarrow Q$, is commonly written as **If** $P$ **then** $Q$.

- Implication is also called a conditional
- The statement $P$ is the **hypothesis** and $Q$ is the **conclusion**

# Logic – Implications

Truth table for the implication $P \Rightarrow Q$, for two statements $P$ and $Q$

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

(i) If the hypothesis $P$ is false, then $P \Rightarrow Q$ is true regardless of the truth value of $Q$

(ii) If the conclusion $Q$ is true, then $P \Rightarrow Q$ is true regardless of the truth value of $P$

(iii) $P \Rightarrow Q$ is false only when $P$ is true and $Q$ is false

# Logic – Implications

- Implications appear in language in many different forms
- Be aware of these expressions so you recognize an implication

| $P \Rightarrow Q$ |
|---|
| If $P$ then $Q$ |
| $Q$ if $P$ |
| $P$ **implies** $Q$ |
| $P$ **only if** $Q$ |
| $P$ is **sufficient** for $Q$ |
| $Q$ is **necessary** for $P$ |

# Logic – Converse of an Implication

## Definition

For statements (or open sentences) $P$ and $Q$ the implication $Q \Rightarrow P$ is the **converse** of $P \Rightarrow Q$

## Example

$P$ : You pass this class
$Q$ : You graduate from Pace
$P \Rightarrow Q$ : If you pass this class then you graduate from Pace
$Q \Rightarrow P$ : If you graduate from Pace then you pass this class

# Logic – Contrapositive of an Implication

## Definition

For two statements (or open sentences) $P$ and $Q$,
**contrapositive** of the implication $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$

## Theorem

*For every two statements $P$ and $Q$*

$$P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$$

| $P$ | $Q$ | $P \Rightarrow Q$ | $\sim Q$ | $\sim P$ | $(\sim Q) \Rightarrow (\sim P)$ |
|---|---|---|---|---|---|
| T | T | **T** | F | F | **T** |
| T | F | **F** | T | F | **F** |
| F | T | **T** | F | T | **T** |
| F | F | **T** | T | T | **T** |

# Logic – Contrapositive of an Implication

Two other useful results are stated in the following theorems

**Theorem**

*For every two statements $P$ and $Q$*

$$P \Rightarrow Q \equiv (\sim P) \vee Q$$

**Theorem**

*For every two statements $P$ and $Q$*

$$\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$$

# Logic – Biconditionals

A biconditional is the conjunction of an implication and its converse expressed as “if and only if” or “necessary and sufficient”

## Definition

For two statements $P$ and $Q$ the **biconditional** of $P$ and $Q$ is the conjunction of the implication $P \Rightarrow Q$ and its converse $Q \Rightarrow P$. It is denoted $P \Leftrightarrow Q$ which is the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

In other words the biconditional $P \Leftrightarrow Q$ is $P$ **if and only if** $Q$, or $P$ **is necessary and sufficient** for $Q$. We can verify the logical equivalence of the statements at the end of the definition above with a truth table.

| $P$ | $Q$ | $P \Rightarrow Q$ | $Q \Rightarrow P$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|
| T | T | T | T | **T** |
| T | F | F | T | **F** |
| F | T | T | F | **F** |
| F | F | T | T | **T** |

# Logic – Tautologies and Contradictions

A compound statement is constructed from one or more statements and one or more logical connectives

**Connectives**

| | |
|---|---|
| $\sim$ Negation (not) | $\oplus$ Exclusive or |
| $\wedge$ Conjunction (and) | $\Rightarrow$ Implication (implies) |
| $\vee$ Disjunction (or) | $\Leftrightarrow$ Biconditional (if and only if) |

# Logic – Tautologies and Contradictions

## Definition

A compound statement is a **tautology** if it is true for all possible truth values of its component statements

## Definition

A compound statement is a **contradiction** if it is false for all possible truth values of its component statements

- The compound statement $S$ is a tautology if and only if its negation $\sim S$ is a contradiction
- For every statement $P$ the statement $P \vee \sim P$ is a tautology
- For every statement $P$ the statement $P \wedge \sim P$ is a contradiction
- For any two logically equivalent compound statements $R$ and $S$, the biconditional $R \Leftrightarrow S$ is a tautology

# Logic – Tautologies and Contradictions

Two important rules in logic: **modus ponens** and **modus tollens**

Modus Ponens

$$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$$

Modus Tollens

$$((P \Rightarrow Q) \wedge (\sim Q)) \Rightarrow \sim P$$

*Useful for when we discuss Methods of Proof in Chapter 3*

# Logic – Key Results

The following hold for all statements $P$, $Q$, and $R$

- **Commutative Laws** $P \vee Q \equiv Q \vee P$ and $P \wedge Q \equiv Q \wedge P$
- **De Morgan's Laws** $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$ and $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$
- **Associative Laws** $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ and $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
- **Distributive Laws** $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ and $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
- **Logical Equivalence**
  - $P \equiv \sim (\sim P)$
  - $P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$
  - $P \Rightarrow Q \equiv (\sim P) \vee Q$
  - $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$