CS 113: Mathematical Structures for Computer Science

Dr. Francis Parisi

Pace University

Fall 2018

Graphs

Chapter 12

Graphs

- Fundamental Concepts of Graph Theory
- Connected Graphs
- Eulerian Graphs
- Hamiltonian Graphs
- Weighted Graphs

Graphs

- Graphs and trees show up in many areas
- Trees help visual multistep processes and help in making decisions
- A PERT diagram shows the precedence of tasks in executing a project
- Applications of graphs and trees include
  1. artificial intelligence
  2. chemistry
  3. scheduling problems
  4. transportation systems

Graphs – Fundamental Concepts of Graph Theory

- A **graph** consists of a finite nonempty set V of objects called **vertices** (singular is **vertex**), and a set E of 2-element subsets of V
- Each element of E is an **edge**
- *Vertices may be called **nodes** or **points**, and edges may be called **links** or **lines***
- V is the **vertex set** of G and E is the **edge set** of G

Graphs – Fundamental Concepts of Graph Theory

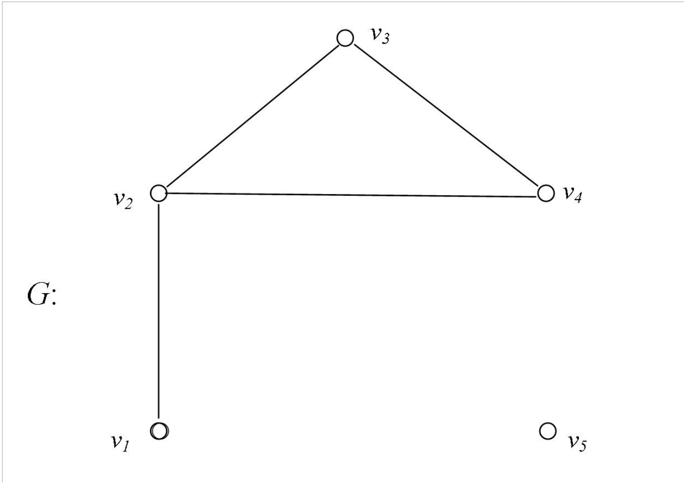
- A graph G with vertex set V and edge set E is denoted $G = (V, E)$
- The Vertex Set of G is often denoted by $V(G)$
- The Edge Set of G is often denoted by $E(G)$
- The edge of set G may be denoted $\{x, y\}$ or by xy or yx
- If $xy \in E$ then x and y are called **adjacent** vertices; otherwise they are **nonadjacent** vertices

Graphs – Fundamental Concepts of Graph Theory

- If x and y are adjacent vertices, then the edge xy **joins** x and y
- Any vertex adjacent to a vertex x is a **neighbor** of x and the set of neighbors is the **neighborhood** of x, denoted by $N(x)$
- The edge xy is **incident** to x and y
- If xy and yz are distinct edges of G, then xy and yz are **adjacent** edges
- wx and yz are nonadjacent edges if and only if $\{w,x\} \cap \{y,z\} = \emptyset$

Graphs – Fundamental Concepts of Graph Theory

A graph G with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$



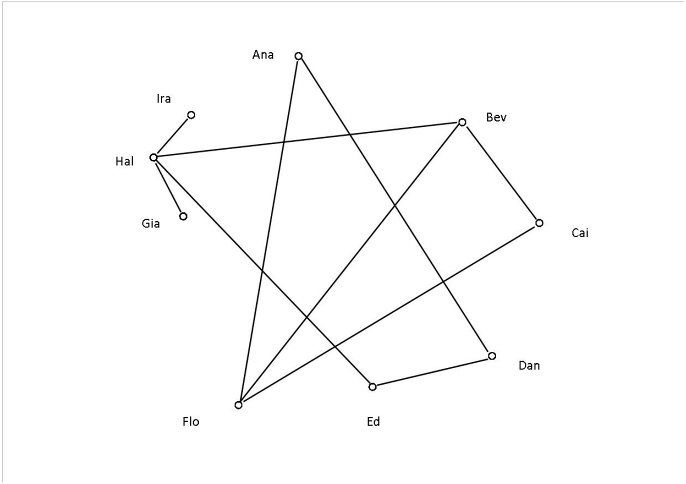
Graphs – Fundamental Concepts of Graph Theory

Nine employees are assigned to a project. The project manager wants to maximize productivity and create teams of three of employees who have worked together before. The data appear in the table below.

Name	Past Partners
Ana	Dan, Flo
Bev	Cai, Flo, Hal
Cai	Bev, Flo
Dan	Ana, Ed
Ed	Dan, Hal
Flo	Cai, Bev, Ana
Gia	Hal
Hal	Gia, Ed, Bev, Ira
Ira	Hal

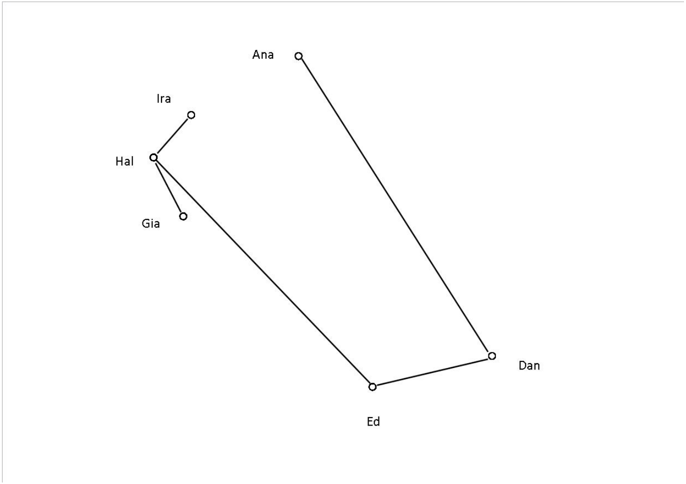
Graphs – Fundamental Concepts of Graph Theory

Using a graph to visualize the same problem



Graphs – Fundamental Concepts of Graph Theory

After we selected Bev, Cai and Flo, the remaining teams are clear



Graphs – Matrix Representation

Matrix Fundamentals

Definition

A **matrix** is a rectangular array of numbers.

- A **matrix** $\mathbf{A}$ having m rows and n columns is an $m \times n$ matrix
- m and n are the **size** of $\mathbf{A}$
- $\mathbf{A}$ is a **square matrix** if $m = n$
- The element in row i and column j of $\mathbf{A}$ is the (i, j)**-entry** of $\mathbf{A}$
- The (i, j)**-entry** of $\mathbf{A}$ is denoted a_{ij}
- The matrix $\mathbf{A}$ is written as $\mathbf{A} = [a_{ij}]$

Graphs – Matrix Representation

Matrix Fundamentals

The following is a 2×3 matrix ...

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 7 \\ 8 & 4 & 0 \end{bmatrix}$$

A matrix is a **zero-one matrix** if every entry is either 0 or 1

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Graphs – Matrix Representation

Operations on Matrices

Two matrices may be added or multiplied, but to be added they must be of the same size, *i.e.*, the same number of rows and columns

Definition

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two $m \times n$ matrices. The sum of $\mathbf{A}$ and $\mathbf{B}$ is the $m \times n$ matrix

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}].$$

Operations on Matrices

Example (Matrix Addition)

$$\begin{bmatrix} -2 & 1 & 0 \\ 3 & -4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & -4 & 3 \\ -1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} (-2)+5 & 1+(-4) & 0+3 \\ 3+(-1) & (-4)+2 & 2+6 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 3 \\ 2 & -2 & 8 \end{bmatrix}$$

A **row matrix** has a single row and a **column matrix** has a single column

Graphs – Matrix Representation

Definition

Let $\mathbf{A}$ be a row matrix and $\mathbf{B}$ be a column matrix, each with n entries. Hence $\mathbf{A}$ is a $1 \times n$ matrix and $\mathbf{B}$ is an $n \times 1$ matrix. The **inner product** or **dot product** $\mathbf{A} \cdot \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is obtained by adding the products of the corresponding elements of $\mathbf{A}$ and $\mathbf{B}$. That is, if

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

then

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Graphs – Matrix Representation

The product $\mathbf{AB}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$

Definition

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times p$ matrix, where $\mathbf{A} = [a_{ij}]$ with $1 \leq i \leq m$ and $1 \leq j \leq n$, and $\mathbf{B} = [b_{jk}]$, with $1 \leq j \leq n$ and $1 \leq k \leq p$. Then the **product** $\mathbf{C} = \mathbf{AB}$ of $\mathbf{A}$ and $\mathbf{B}$ is the $m \times p$ matrix, where $\mathbf{C} = [c_{ik}]$ where $1 \leq i \leq m$ and $1 \leq k \leq p$ such that

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

To calculate the (i,k)-entry c_{ik} of $\mathbf{C}$, we compute the inner product of row i of $\mathbf{A}$ (which has n columns) and column k of $\mathbf{B}$ (which has n rows).

Graphs – Matrix Representation

Definition

For an $n \times n$ square matrix $\mathbf{A}$, the **powers** of $\mathbf{A}$ are defined by

$$\mathbf{A}^1 = \mathbf{A} \text{ and } \mathbf{A}^k = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{k \text{ factors}} \text{ for each integer } k \geq 2.$$

For example, if $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, then

$$\mathbf{A}^1 = \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \ \mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, \text{ and}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A} = \mathbf{A}\mathbf{A}^2 = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}.$$

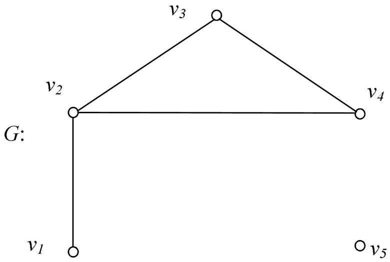
Graphs – Matrix Representation

- Using the information about matrices we can describe a matrix representation of a graph
- Suppose G is a graph of order n, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$
- The **adjaceny matrix** of G is the $n \times n$ zero-one matrix $\mathbf{A} = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{if } v_i v_j \notin E(G). \end{cases}$$

Graphs – Matrix Representation

Graph G and its adjacency matrix



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Graphs – Degree of a Vertex

- The **degree** of a vertex $x$is the number of neighbors of x, denoted by $\deg_G(x), \deg(x),$ or $\deg x$
- The degree is also the number of edges incident to x, the number of vertices adjacent to x and $|N(x)|$

For graph G on the previous slide,

$$\deg v_1 = 1, \deg v_2 = 3, \deg v_3 = \deg v_4 = 2, \deg v_5 = 0$$

- A vertex of degree 0 is an **isolated vertex**
- A vertex of degree 1 is an **end-vertex**, a **leaf** or a **pendant vertex**: v_5 is an isolated vertex, v_1 is an end-vertex
- The **maximum degree** of a graph G is the maximum of its degrees denoted by $\Delta(G)$
- The **minimum degree** of G is denoted by $\delta(G)$

Graphs – Degree of a Vertex

In general we have

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1$$

... and

Theorem (The First Theorem of Graph Theory)

If G is a graph of order n and size m with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$\sum_{i=1}^{n} \deg v_i = 2m$$

Graphs – Degree of a Vertex

Example

A graph G of size 29 has three vertices of each of the degrees 3, 5, and 6. The remaining vertices of G have degree 4. What is the order of G?

Solution: Suppose G has x vertices of degree 4. From the First Theorem of Graph Theory the sum of the degrees of the vertices of G is

$$3 \cdot 3 + 3 \cdot 5 + 3 \cdot 6 + 4x = 2 \cdot 29.$$

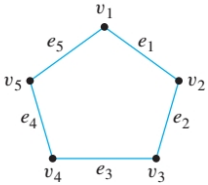
So $x = 4$, and the order of G is $9 + x = 13$.

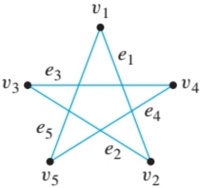
Graphs – Degree of a Vertex

- A vertex in a graph is **even** if its degree is even and **odd** if its degree is odd
- Every graph contains and even number of odd vertices
- A graph is **regular** if every vertex of G has the same degree
- ...and is r-regular if this degree is r

Graphs – Degree of a Vertex

Regular graphs of order 5

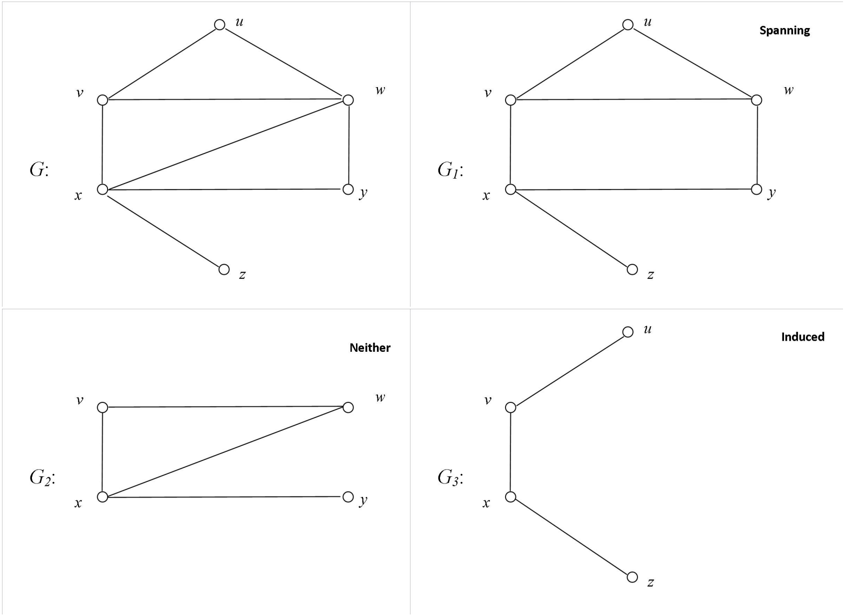




Graphs – Degree of a Vertex

- A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
- If $H \neq G$ then H is a proper subgraph of G
- A subgraph H of a graph G is an **induced subgraph** of G if whenever two vertices of H are adjacent in G, they are adjacent in H
- A subgraph H of a graph G is a **spanning subgraph** of G if $V(H) = V(G)$

Graphs – Degree of a Vertex



Graphs – Degree of a Vertex

- A graph is **complete** if every two distinct vertices are adjacent
- A complete graph of order n is denoted by K_n
- The graph K_n is $(n-1)$-regular and its size is $\binom{n}{2}$
- The **complement** of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ such that two distinct vertices u and v of G are adjacent in $\overline{G}$ if and only if they are not adjacent in G
- A graph G of order n is a **path** if the vertices of G can be labeled so that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i < n\}$
- A graph G of order $n \geq 3$ is a **cycle** if the vertices of G can be laabeled so that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i < n\} \cup \{v_1 v_n\}$
- The **length** of a path or cycle is the number of edges it contains

Graphs – Isomorphic Graphs

What does it mean for two graphs to be the same?

- Two graphs G and H are considered the same if they have *the same structure* – the term for this is *isomorphic*
- Informally, two graphs G and H are isomorphic if either can be drawn so that it looks like the other

Definition

A graph G is **isomorphic** to a graph H, written $G \cong H$ if there is a bijective function $\phi : V(G) \rightarrow V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The graphs G and H are then called **isomorphic graphs** and the function ϕ an **isomorphism.**

Graphs – Isomorphic Graphs

Theorem

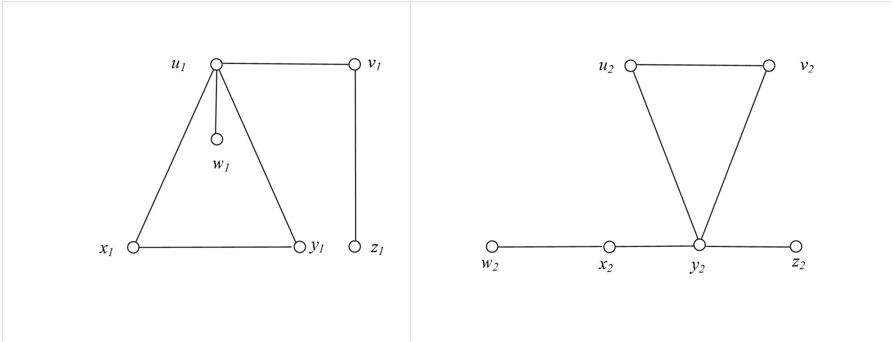
If G and H are isomorphic graphs, then

(a) *G and H have the same order,*

(b) *G and H have the same size and*

(c) *the degrees of the vertices of G are the same as the degrees of the vertices of H*

Graphs – Isomorphic Graphs

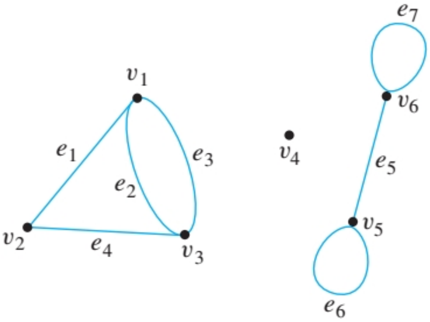


$\phi(u_1) = y_2, \phi(v_1) = x_2, \phi(w_1) = z_2, \phi(x_1) = u_2, \phi(y_1) = v_2, \phi(z_1) = w_2$

Graphs

Exercise

Consider the following graph:



Write the vertex set and the edge set, and give a table showing the endpoints for each edge.

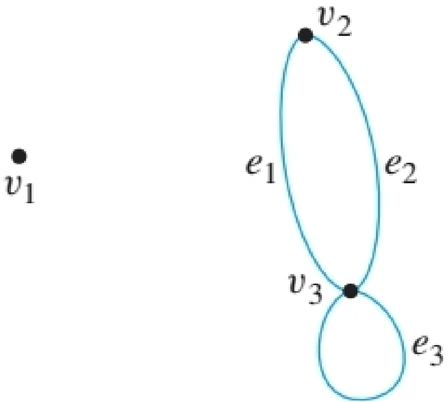
Exercise

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2, and e_3 is a loop at v_1.

Graphs

Exercise

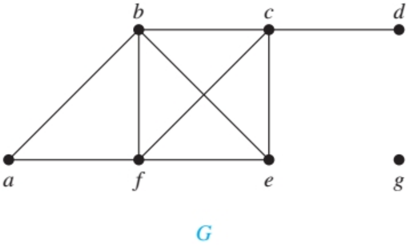
Find the degree of each vertex of the graph G shown below. Then find the total degree of G. Finally find $\delta(G)$ and $\Delta(G)$.

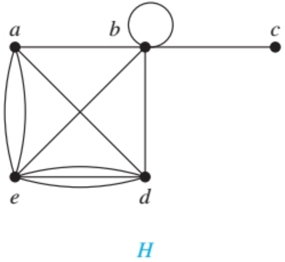


Graphs

Exercise

What are the degrees and what are the neighborhoods of the vertices in the graphs G and H?





Exercise

Draw a graph with vertices a, b, c, d with adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Graphs – Connected Graphs

Consider starting at a vertex u in a graph G then proceeding to a neighbor v of u then to a neighbor of v, and so on, concluding at some vertex w — This is a *walk* from u to w in G

Definition

A **walk** in a graph G is a sequence $u_0, u_1, \ldots, u_k$ of vertices of G such that u_i and u_{i+1} are adjacent for $1 \leq i \leq k-1$. We define the sequence by

$$W = (u_0, u_1, \ldots, u_k).$$

The W is a $u_0 - u_k$ walk, where u_0 is the **initial vertex** of W and u_k is the **terminal vertex** of W.

Graphs – Connected Graphs

Definition

A **trail** is a walk in which no edge is encountered more than once.

Definition

A **path** is a walk in which no vertex is repeated.

Definition

A **circuit** is a closed trail of length 3 or more.

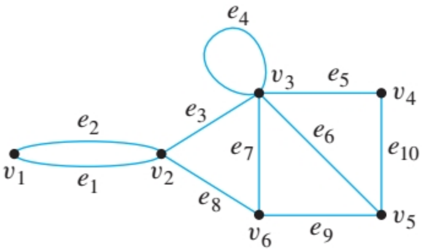
Definition

A **cycle** is a circuit in which no vertex is repeated except that the initial vertex is the terminal vertex. A cycle of length k is called a k-**cycle** and a 3-cycle is also referred to as a triangle.

Exercise

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

(a) $v_1e_1v_2e_3v_3e_4v_3e_5v_4$ (b) $e_1e_3e_5e_5e_6$ (c) $v_2v_3v_4v_5v_3v_6v_2$
(d) $v_2v_3v_4v_5v_6v_2$ (e) $v_1e_1v_2e_1v_1$ (f) v_1



Graphs – Connected Graphs

Definition

A graph G is **connected** if for every two vertices u and v, G contains a $u - v$ path. If G contains a $u - v$ path, then we say that u and v are **connected** in G. That is, G is a connected graph if every two vertices of G are connected.

Definition

A subgraph G' of a graph G is a **component** of G if G' is connected but G' is not a proper subgraph of any connected subgraph of G.

Graphs – Connected Graphs

Often a connected graph may contain several $u - v$ paths for two vertices u and v

Definition

Let u and v be two vertices in a connected graph G. The **distance** $d(u, v)$ from u to v is the length of a shortest $u - v$ path in G.

Definition

A graph G is a **bipartite graph** if it is possible to partition its vertex set into two subsets U and V, called **partite sets** so the every edge of G joins a vertex of U and a vertex of W,

Graphs – Connected Graphs

Theorem

If G is a disconnected graph then $\overline{G}$ is connected.

Theorem

If G is a graph of order n such that

$$\deg v \geq \frac{n-1}{2}$$

for every vertex v of G, then G is connected.

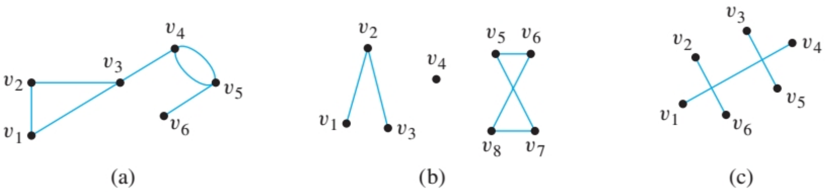
Definition

An edge e in a graph G is called a **bridge** of G if $G - e$ has more components than G. A vertex v of a nontrivial graph G is a **cut-vertex** of G if $G - v$ has more components than G.

Graphs

Exercise

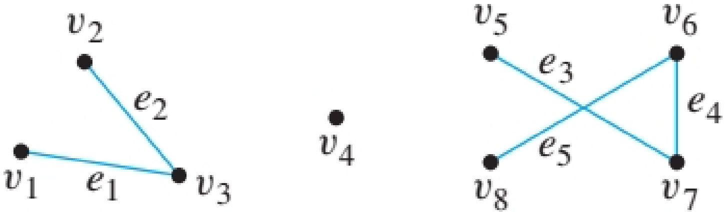
Which of the following graphs are connected?



Graphs

Exercise

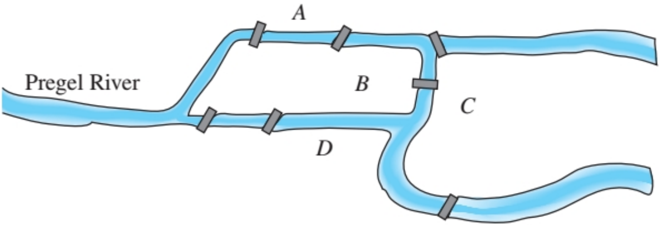
Find all connected components of the following graph G.



Graphs – Eulerian Graphs

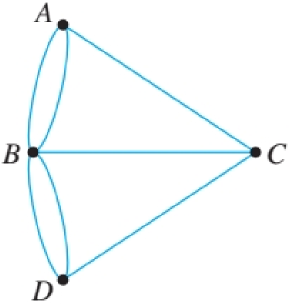
Leonhard Euler and the **Königsberg Bridge Problem**

Is it possible to stroll around Königsberg and cross each of the seven bridges only once?



Graphs – Eulerian Graphs

The **Königsberg Bridge Problem** as a graph



Is it possible to find a route through the graph that starts and ends at some vertex, one of A, B, C, or D, and traverses each edge exactly once?

Graphs – Eulerian Graphs

The **Königsberg Bridge Problem** as a graph

- Let's start at vertex A; each time we go through vertex B, C, or D, we use two edges
- If a route exists that uses all the edges of the graph and starts and ends at A, then the total number of arrivals and departures from each vertex B, C, and D must be a multiple of 2, and the degrees of the vertices B, C, and D must be even
- But $\deg(B) = 5, \deg(C) = 3$, and $\deg(D) = 3$, so there is no route that solves the puzzle by starting and ending at A
- The same reasoning shows that there are no routes that solve the puzzle by starting and ending at B, C, or D either
- Therefore, it's not possible to travel all around the city crossing each bridge exactly once

Graphs – Eulerian Graphs

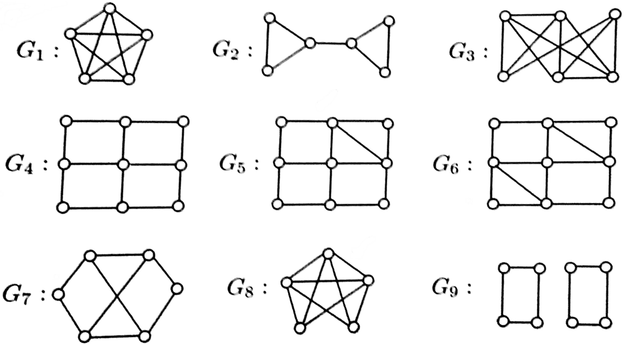
Definition

Let G be a connected graph or connected multigraph. An open trail that contains every edge of G is called an **Eulerian trail**, while a circuit that contains every edge of G is an **Eulerian circuit**. A graph or multigraph containing an Eulerian circuit is itself **Eulerian**.

Graphs – Eulerian Graphs

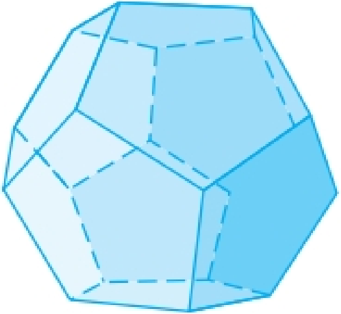
Exercise

For the following graphs, $G_1 - G_9$, determine which contain an Eulerian circuit, Eulerian trail or neither.



Graphs – Hamiltonian Graphs

- A *dodecahedron* is a geometric solid with 20 vertices, 30 edges, and 12 faces



- Is is possible to travel along the edges of a dodecahedron passing through every vertex only once and returning to the starting point?

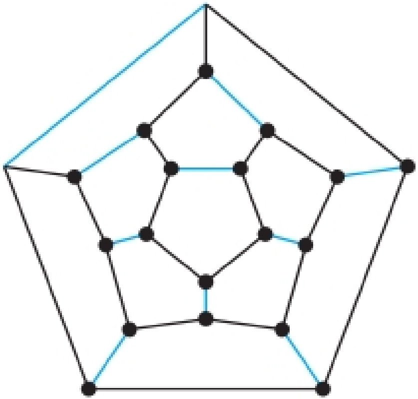
Graphs – Hamiltonian Graphs

Definition

A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle** of G and a graph containing a Hamiltonian cycle is called a **Hamiltonian graph**

Graphs – Hamiltonian Graphs

A Hamiltonian cycle in the graph of a dodecahedron



Graphs – Weighted Graphs

- Graphs can be used to represent a variety of situations involving networks
- Rather than having an interest in the distance between cities for example, we may be interested in the minimum distance or minimum cost
- For this type of analysis we would assign a number to each edge called its **weight**

Definition

A **weighted graph** is a graph G in which every edge e of G is assigned a real number weight (usually positive) denoted by $w(e)$ and called the weight of e.

Graphs – Weighted Graphs

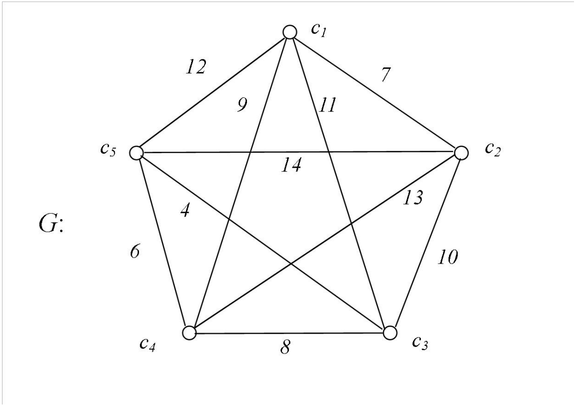
Definition

Let u and v be two vertices in a weighted graph$G$. The **length** of a $u - v$ path P in G is the sum of the weight of the edges of P. The the **distance** $d(u, v)$ between u and v is the minimum length of a $u - v$ path P in G.

Graphs – Weighted Graphs

The Traveling Salesman Problem

Suppose that a salesman must travel to each city exactly once, starting and ending in the same city. Which route will minimize the total distance that must be traveled?



Graphs – Weighted Graphs

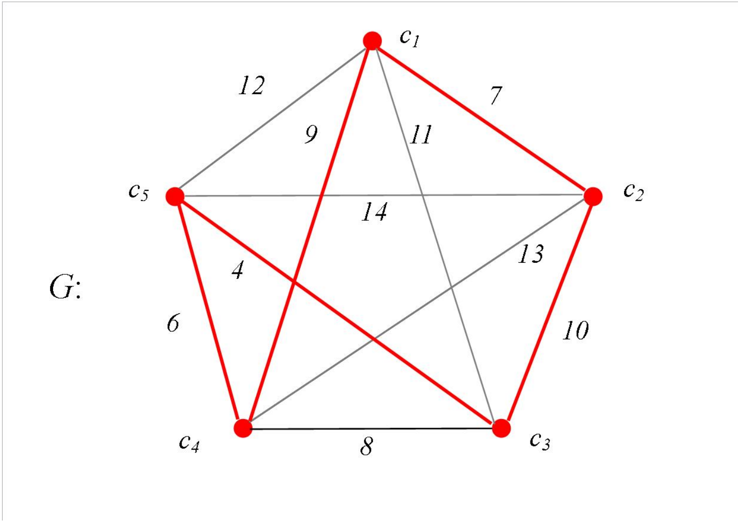
There are $(n-1)!/2$ Hamiltonian cycles in G. Since the order of G is 5, there are

$$\frac{(5-1)!}{2} = 12$$

Hamiltonian cycles in G.

If we enumerate all 12 cycles we would find that $s_2 = (c_1, c_2, c_3, c_5, c_4, c_1)$ has the minimum distance of 36 miles.

Graphs – Weighted Graphs



Graphs – Key Results

- **The First Theory of Graph Theory:** If a G is a graph or multigraph of size m, then $\sum_{v \in V(G)} \deg v = 2m$.
- Every graph contains an even number of odd vertices.
- If G and H are isomorphic graphs, then
  (a) G and H have the same order,
  (b) G and H have the same size and
  (c) the degrees of the vertices of G are the same as the degrees of the vertices of H.
- Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix $\mathbf{A}$. For each positive integer k, the number of different $v_i - v_j$ walks of length k in G is the (i, j)-entry in the matrix $\mathbf{A}^k$.
- If a graph G contains a $u - v$ walk for vertices u and v in G, then G also contains a $u - v$ path.

Graphs – Key Results

- A nontrivial graph G is bipartite if and only if G does not contain an odd cycle.
- If G is a connected graph, then $\overline{G}$ is connected.
- If G is a graph of order n such that $\deg v \geq \frac{n-1}{2}$ for every vertex v of G, then G is connected.
- An edge e of a graph G is a bridge of G if and only if e lies on no cycle of G.
- A vertex v of a connected graph G is a cut-vertex of G if and only if there exist vertices u and w distinct from v such that v lies on every $u - w$ path of G.
- Let G be a nontrivial connected graph. Then G is Eulerian if and only of every vertex of G is even.

Graphs – Key Results

- Let G be a connected graph. Then G has an Eulerian trail if and only if G has exactly two odd vertices. Furthermore, any Eulerian trail of G begins at one of the odd vertices and ends at the other.
- Let G be a connected multigraph with one or more edges. Then
  (1) G is Eulerian if and only if every vertex of G is even and
  (2) G has an Eulerian trail if and only if G has exactly two odd vertices.
- If G is a Hamiltonian graph, then $k(G - S) \leq |S|$ for every nonempty proper subset set S of $V(G)$.
- If there exists a nonempty proper subset S of the vertex set of a graph G such that $k(G - S) > |S|$, then G is not Hamiltonian.
- If G be a graph of order $n \geq 3$ such that $\deg v \geq \frac{n}{2}$ for each vertex v of a graph G, then G is Hamiltonian.

Graphs – Key Results

- **Dijkstra's Algorithm:** an algorithm that determines the distances from a fixed vertex v_i in a connected weighted graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ to all other vertices of G and provides a shortest $v_1 - v_i$ path for each i $(2 \leq i \leq n)$.