CS 113: Mathematical Structures for Computer Science

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Counting

Chapter 8

Counting

- Multiplication and Addition Principles
- Inclusion-Exclusion Principle
- Pigeonhole Principle
- Permutations
- Combinations

Counting

- Combinatorics deals with the configuration and arrangement of objects
- The primary focus is on topics like
  1. Existence: Is it possible to have such an arrangement or configuration?
  2. Enumeration: How many such configurations are there?
  3. Optimization: Is there a specific arrangement that is somehow more desirable?
- Our particular focus will be on **enumerative combinatorics** or simply counting

Counting – The Multiplication Principle

Definition

The Multiplication Principle A procedure consists of a sequence of two tasks. To perform this procedure, one performs the first task followed by performing the second task. If there are n_1 ways to perform the first task and n_2 ways to perform the second task after the first task has been performed, then there are $n_1 n_2$ ways to perform the procedure.

Counting – The (General) Multiplication Principle

Definition

The (General) Multiplication Principle Performing a certain procedure consists of performing a sequence of $m \geq 2$ tasks $T_1, T_2, \ldots, T_m$. If there are n_i ways of performing task T_i after any preceding tasks have been performed for $i = 1, 2, \ldots m$, then the total number of ways of performing the procedure is $n_1 n_2 \cdots n_m$.

Counting – The (General) Multiplication

Example

How many different license plates can be made consisting of two letters followed by four digits?

There are 26 letters in the English alphabet and 10 digits. Since the first two slots are letters and the last four are digits we have by the multiplication rule that we can make

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^4 = 6,760,000$$

different license plates.

Counting – The (General) Multiplication Principle

Let's consider another example...

Example

Suppose a quiz has 10 true-false questions. How many sequences of possible answers are there?

Since each question is binary (true-false) there are

$$2^{10} = 1024$$

sequences of possible answers.

We can generalize this result into the following theorem

Counting – The (General) Multiplication Principle

Theorem

If A and B are two finite nonempty sets with $|A| = m$ and $|B| = n$, then the number of different functions from A to B is $|B|^{|A|} = n^m$.

Proof

Let $A = \{a_1, a_2, \ldots, a_m\}$. Any function $f : A \to B$ has the appearance

$$f = \{(a_1, \_\_), (a_2, \_\_), \ldots, (a_m, \_\_)\},$$

where each blank (the image of an element of A) is to be filled in with an element of the codomain B. Since there are n choices for each image, it then follows by the Multiplication Principle that the number of such functions is the product of m factors n, or n^m.

Counting – The (General) Multiplication Principle

A result related to the previous theorem as applied to one-to-one functions is as follows

Theorem

If A and B are two sets with $|A| = m$ and $|B| = n$, where $m \leq n$, then the number of different one-to-one functions from A to B is $\dfrac{n!}{(n-m)!}$.

Counting – The Addition Principle

- When we have a procedure that can be done with either of two tasks but the two tasks that cannot be performed at the same time, the procedure is done when either of the two is done
- The number of ways the procedure can be done is found using the Addition Principle

Definition

The Addition Principle A procedure consists of two tasks that cannot be performed simultaneously. To perform this procedure, either of the two tasks is performed. If the first task can be performed in n_1 ways and the second can be performed in n_2 ways, then the number of ways of performing this procedure is $n_1 + n_2$.

Counting – The Addition Principle

Definition

The (General) Addition Principle Performing a certain procedure consists of performing one of $m \geq 2$ tasks $T_1, T_2, \ldots, T_m$, no two of which can be performed at the same time. If task T_i can be performed in n_i ways for $1 \leq i \leq m$, then the number of ways of performing this procedure is $n_1 + n_2 + \cdots + n_m$.

Let's work on some exercises ...

Counting

Exercise

Determine the number of one-to-one functions from A to B where $|A| = 4,\ |B| = 7$.

Counting

Exercise

Students go to a movie and then go for desert. The movie choices are: An animated movie, a comedy, and an adventure movie. The desert choices are: Hot fudge sundae, strawberry shortcake, apple pie, and chocolate milkshake. How many possible movie-desert choices are there?

Counting

Exercise

Let A, and B be two sets and $|A| = 5,\ |B| = 6.$

1. How many functions $f : A \to B$ are there?
2. How many functions $g : B \to A$ are there?
3. How many different one-to-one functions from A to B are there?
4. How many different one-to-one functions from B to A are there?

Counting

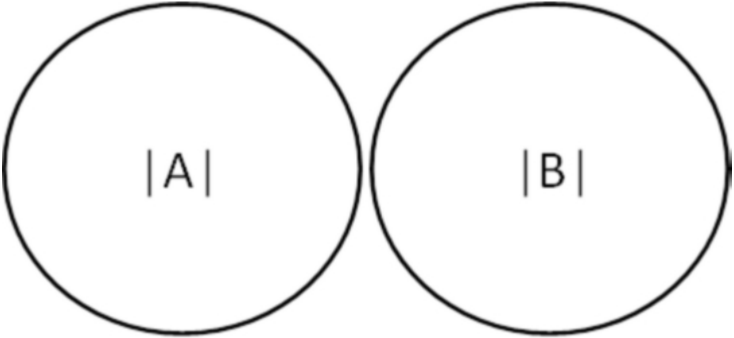
Exercise

How many four-digit numbers can be formed from the digits 1,2,3,4,5,6 if

1. digits can be repeated?
2. digits can't be repeated?

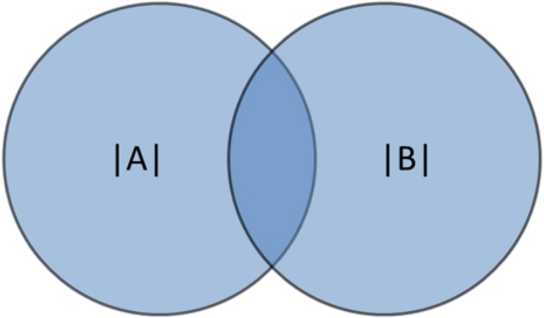
Counting – The Principle of Inclusion-Exclusion

- Recall that for disjoint sets the number of elements in the union of the two sets is the sum of the elements
  $|A \cup B| = |A| + |B|$



- When the two sets are not disjoint we need to consider carefully what we are counting

Counting – The Principle of Inclusion-Exclusion



- The shaded region in the middle double counts
- Therefore $|A \cup B| = |A| + |B| - |A \cap B|$

Counting – The Principle of Inclusion-Exclusion

The Principle of Inclusion-Exclusion A procedure consists of two tasks. To perform the procedure, one performs either of the two tasks. If the first task can be performed in n_1 ways and the second task can be performed in n_2 ways and the two tasks can be performed simultaneously in n_{12} ways, then the total number of ways of performing the procedure is $n_1 + n_2 - n_{12}$.

The Principle of Inclusion-Exclusion (for two sets) For every two finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In particular, if A and B are disjoint, then $|A \cup B| = |A| + |B|$.

Counting – The Principle of Inclusion-Exclusion

Example

How many 8-bit sequences begin with 110 *or* end with 1100?

We have either of the two sequences

$$1\ 1\ 0\ \_\ \_\ \_\ \_\ \_ \text{ or } \_\ \_\ \_\ \_\ 1\ 1\ 0\ 0$$

The number of sequences beginning with 110 is $2 \cdot 2 \cdot 2 = 2^5 = 32$, and the number of 8-bit sequences that end with 1100 is $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$.
There two sequences that start with 110 *and* end with 1100, that is $1\ 1\ 0\ \_\ 1\ 1\ 0\ 0$.

So we have by the Inclusion-Exclusion Principle
$32 + 16 - 2 = 46$ 8-bit sequences with the desired property.

The Principle of Inclusion-Exclusion

Exercise

In a discrete math course 17 students are computer science majors, 11 are math majors, and 5 are dual majors. How many students are either computer science majors or math majors.

The Principle of Inclusion-Exclusion

Exercise

There are 36 students who have enrolled in at least one of the following courses, Discrete Math, Algebra, and Calculus, this semester at Pace. Of these student we know:

- 23 students enrolled in Discrete Math
- 19 students enrolled in Algebra
- 18 students enrolled in Calculus
- 7 students enrolled in Discrete Math and Algebra
- 9 students enrolled in Discrete Math and Calculus
- 11 students enrolled in Algebra and Calculus

1. How many students enrolled in all three courses?
2. How many students enrolled in exactly one course?
3. How many students enrolled in exactly two courses?

Counting – The Pigeonhole Principle

- Consider an Olympic athlete who won four medals.
  ... at least two medals must be the same type
- In a group of five Pace students
  ... at least three must be female or at least three must be male
- In a set of 12 integers
  ... at least six must be even or at least seven must be odd
- In general, for $n > 1$ integers at least $\lceil n/2 \rceil$ of these are even or at least $\lceil n/2 \rceil$ are odd

  Recall that $\lceil x \rceil$ is the ceiling of x which is the smallest integer greater than or equal to x.

Counting – The Pigeonhole Principle

The Pigeonhole Principle If a set S with n elements is divided into k pairwise disjoint subsets $S_1, S_2, \ldots, S_k$, then at least one of these subsets has $\lceil n/k \rceil$ elements.

First note that $0 \leq \lceil x \rceil - x < 1$ so

$$x \leq \lceil x \rceil < x + 1 \tag{7.1}$$

for every $x \in \mathbb{R}$. We will use this in our proof of the Pigeonhole Principle.

Counting – The Pigeonhole Principle

Proof.

Assume to the contrary that none of the sets $S_1, S_2, \ldots, S_k$ has at least $\lceil n/k \rceil$ elements.

Since $\lceil n/k \rceil$ is an integer, every one of the sets $S_1, S_2, \ldots, S_k$ has at most $\lceil n/k \rceil - 1$ elements.

By (7.1) $\lceil n/k \rceil - 1 < n/k$, and each of the sets $S_1, S_2, \ldots, S_k$ has less than $\lceil n/k \rceil$ elements.

Since $S_1, S_2, \ldots, S_k$ are pairwise disjoint and $S = S_1 \cup S_2 \cup \ldots \cup S_k$, it follows that

$$n = |S| = |S_1 \cup S_2 \cup \ldots \cup S_k| = |S_1| + |S_2| + \ldots + |S_k| < k(n/k) = n,$$

so $n < n$ which is impossible. $\square$

Counting – The Pigeonhole Principle

What if we want to find the minimum number of elements N of a set S such that if we divide S into k subsets, then at least one of these subsets has at least r elements?

- From the Pigeonhole Principle we know $\lceil N/k \rceil = r$.
- And $\lceil x \rceil < x + 1$ for every real x, so

$$r = \left\lceil \frac{N}{k} \right\rceil < \frac{N}{k} + 1$$

  and $N > k(r-1)$.
- Since N and $k(r-1)$ are integers we have $N \geq k(r-1) + 1$, and $N = k(r-1) + 1$.

Let r and k be positive integers. If N is the minimum cardinality of a set S such that however we divide S into k subsets, at least one of these subsets must contain at least r elements, then

$$N = k(r-1) + 1.$$

The Pigeonhole Principle

Exercise

There are 28 students in a Discrete Math course at Pace.

1. The professor must assign each student a letter grade of A, B, C, D, or F. What is the largest number of students who **must** be assigned the same grade?
2. The ages of the students range from 16 to 23. What is the largest number of students who **must** be of the same age?
3. In order to register for the course each student must have passed one of three prerequisite courses. What is the maximum number of students who **must** have passed the course?

The Pigeonhole Principle

Exercise

How many people are needed to be sure that at least two of them have a birthday during the same month?

How many people are needed to be sure that at least three of them have a birthday during the same month?

Permutations

Definition

A **permutation** of a nonempty set S is an arrangement or ordered list of the elements of S

Example

Take the set $S = \{1, 2, 3, 4\}$.
There are four possible numbers for the first position, three for he second, two for the third and one for the final number
The total number of permutations is then $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$.

Counting – Permutations and Combinations

Definition

An ordered list of r elements of an n-element set S is called an r-**permutation** of the elements of S. The number of r-permutations of an n-element set is denoted by $P(n,r)$.

Theorem

The number of r-permutations of an n-element set (where $1 \leq r \leq n$) is

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+2)(n-r+1) = \frac{n!}{(n-r)!}.$$

For permutations of a set order matters. How does our analysis change when order does *not* matter?

Counting – Permutations and Combinations

Exercise

Seven students, 4 females and 3 males, present solutions to seven different problems.

(a) How many ways can they present?

(b) How many ways can they present if they alternate males, females?

(c) What if male students must present consecutively, and female students must present consecutively?

Combinations

Example

Consider the set $A = \{1, 2, 3, 4\}$. How many subsets contain exactly 2 elements?

The two element subsets are

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$$

six in all.

Counting – Permutations and Combinations

Definition

Let S be an n-element set. An r-element subset of S, where $0 \leq r \leq n$, is called an r-**combination** of S. Therefore, the number of r-combinations of an n-element set is $C(n, r)$. An r-combination is also referred to as an unordered set of r-elements or as an r-**selection**.

Counting – Permutations and Combinations

Theorem

For integers r and n with $0 \leq r \leq n$, the number of r-element subsets of an n-element set (also called r-combinations or r-selections of an n-element set) is

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Counting – Permutations and Combinations

Example

A farmer buys 3 cows, 2 pigs, and 4 hens from a man who has 6 cows, 5 pigs, and 8 hens. Find the number m of choices that the farmer has.

The farmer can choose the cows in $C(6,3)$ ways, the pigs in $C(5,2)$ ways, and the hens in $C(8,4)$ ways. Thus the number m of choices is

$$m = \binom{6}{3}\binom{5}{2}\binom{8}{4} = 20 \cdot 10 \cdot 70 = 14,000$$

Counting – Permutations and Combinations

Properties of $\binom{n}{r}$

Suppose we have $n = 3$ then

- $\binom{3}{0} = 1$ represents the number of 0-element sets
- $\binom{3}{1} = 3$ represents the number of 1-element sets
- $\binom{3}{2} = 3$ represents the number of 2-element sets
- $\binom{3}{3} = 1$ represents the number of 3-element sets
- $$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8$$

  represents the number of all subsets of a 3-element set
- As we saw in Chapter 2 and again 5, the number of subsets of an n-element set is 2^n
- Also note that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$

Counting – Permutations and Combinations

Theorem (8.45)

For each integer $n \geq 0$,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Theorem (8.46)

For every two integers r *and* n *with* $0 \leq r \leq n$,

$$C(n,r) = C(n,n-r) \text{ or } \binom{n}{r} = \binom{n}{n-r}.$$

Permutations and Combinations

Exercise

A committee needs to meet 3 weekdays during a month that has 20 weekdays. How many different choices are there for the meetings?

Permutations and Combinations

Exercise

Six freshmen, five sophmores, and four juniors volunteer on a four person committee. How many ways can the committee be formed if:

(a) Any student may serve?

(b) The committee must have at least one freshman, at least one sophomore, and at least one junior?

(c) At least one freshman must serve?

Counting – Key Results

- **The Multiplication Principle** A procedure consists of a sequence of two tasks. To perform this procedure, one performs the first task followed by performing the second task. If there are n_1 ways to perform the first task and n_2 ways to perform the second task after the first task has been performed, then there are $n_1 n_2$ ways to perform the procedure.
- **The (General) Multiplication Principle** Performing a certain procedure consists of performing a sequence of $m \geq 2$ tasks $T_1, T_2, \ldots, T_m$. If there are n_i ways of performing task $T_i$after any preceding tasks have been performed for $i = 1, 2, \ldots m$, then the total number of ways of performing the procedure is $n_1 n_2 \cdots n_m$.

Counting – Key Results

- **The Addition Principle** A procedure consists of two tasks that cannot be performed simultaneously. To perform this procedure, either of the two tasks is performed. If the first task can be performed in n_1 ways and the second can be performed in n_2 ways, then the number of ways of performing this procedure is $n_1 + n_2$.
- **The (General) Addition Principle** Performing a certain procedure consists of performing one of $m \geq 2$ tasks $T_1, T_2, \ldots, T_m$, no two of which can be performed at the same time. If task T_i can be performed in n_i ways for $1 \leq i \leq m$, then the number of ways of performing this procedure is $n_1 + n_2 + \cdots + n_m$.

Counting – Key Results

- **The Principle of Inclusion-Exclusion** A procedure consists of two tasks. To perform the procedure, one performs either of the two tasks. If the first task can be performed in n_1 ways and the second task can be performed in n_2 ways and the two tasks can be performed simultaneously in n_{12} ways, then the total number of ways of performing the procedure is $n_1 + n_2 - n_{12}$.
- **The Principle of Inclusion-Exclusion (for $n \geq 2$ sets)** Let $A_1, A_2, \ldots, A_n$ be $n \geq 2$ finite sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| + \sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A$$
$$- \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Counting – Key Results

- **The Pigeonhole Principle** If a set S with n elements is divided into k pairwise disjoint subsets $S_1, S_2, \ldots, S_k$, then at least one of these subsets has $\lceil n/k \rceil$ elements.
- Let r and k be positive integers. If N is the minimum cardinality of a set S such that however we divide S into k subsets, at least one of these subsets must contain at least r elements, then $N = k(r-1)+1$.
- **The (General) Pigeonhole Principle** A set S with n elements is divided into k pairwise disjoint subsets $S_1, S_2, \ldots, S_k$, where $|S_i| \geq n_i$ for a positive integer n_i for $i = 1, 2, \ldots, k$. Then each subset of S with at least $1 + \sum_{i=1}^{k}(n_i - 1)$ elements contains at least n_i elements of S_i for some integer i with $1 \leq i \leq k$

Counting – Key Results

- $P(n, r) = \dfrac{n!}{(n-r)!}$
- $C(n, r) = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$
- $2^n = \dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \cdots + \dbinom{n}{n-1} + \dbinom{n}{n}$
- $\dbinom{n}{r} = \dbinom{n}{n-r}$ for $0 \leq r \leq n$

Advanced Counting

Chapter 9

Advanced Counting

- The Pascal Triangle
- The Hockey Stick Theorem
- The Binomial Theorem

Advanced Counting – The Pascal Triangle

- The Pascal Triangle has some interesting properties
- Each element in the triangle is an integer and can be expressed as $\binom{n}{r}$
- In rectangular form the first row and first column is filled with 1's
- Each element is the sum of the number to its left and above

Advanced Counting – The Pascal Triangle

1	1	1	1	1	$\cdots$
1	2	3	4	5	$\cdots$
1	3	6	10	15	$\cdots$
1	4	10	20	35	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Advanced Counting – The Pascal Triangle

$$
\begin{array}{|cccccc}
\hline
1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & \cdots \\
1 & 3 & \color{red}{\mathbf{6}} & 10 & 15 & \cdots \\
1 & 4 & 10 & 20 & 35 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Advanced Counting – The Pascal Triangle

row 0				1			
row 1			1		1		
row 2		1		2		1	
row 3	1		3		3		1

- The elements of row n are the numbers of r-combinations of an n-element set
- Note that each element in the triangle is the sum of the two elements above it
- We can re-write the elements as $\binom{n}{r}$

Advanced Counting – The Pascal Triangle

row 0				$\binom{0}{0}$				
row 1			$\binom{1}{0}$		$\binom{1}{1}$			
row 2		$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$		
row 3	$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$	

- We can see from the triangle that, for example, $\binom{3}{1} = \binom{2}{0} + \binom{2}{1}$
- This result is generalized in the following Theorem

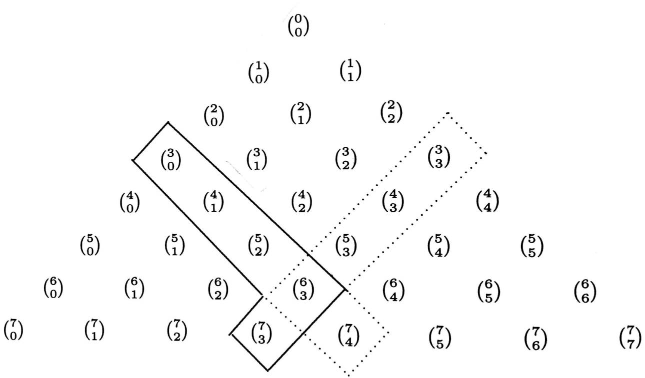
Advanced Counting – The Pascal Triangle

Theorem

For integers r and n with $1 \leq r \leq n-1$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Advanced Counting – The Hockey Stick Theorem



- By Theorem 8.46 we have $\binom{n}{n-r} = \binom{n}{r}$
- Therefore, with $n = 7$ and $r = 3$, $\binom{7}{4} = \binom{7}{3}$
- $\binom{7}{4} = \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3}$ (The Hockey Stick Theorem)

Advanced Counting – The Hockey Stick Theorem

Theorem

For every two integers r *and* n *with* $0 \leq r \leq n$

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$$

Advanced Counting – The Binomial Theorem

- The Pascal Triangle provides another interesting property that led to the Binomial Theorem as stated by Pascal

$$
\begin{gathered}
(x+y)^0 = 1 \\
(x+y)^1 = 1x + 1y \\
(x+y)^2 = 1x^2 + 2xy + 1y^2 \\
(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4
\end{gathered}
$$

- The coefficients of the nth power of the binomial $(x+y)^n$ are the numbers in the nth row of the Pascal Triangle
- A (perhaps) useful application of this is the squaring of numbers that end in 5

Advanced Counting – The Binomial Theorem

- Consider the integer $n = 75$
- We can write 75 as $70 + 5$ or $10 \cdot 7 + 5$
- Therefore $n = 10a + 5$ where $a = 7$ in this case, and $n^2 = 100a(a+1) + 25$
- So $75^2 = 100 \cdot 7 \cdot 8 + 25 = 5,625$

Advanced Counting – The Binomial Theorem

Theorem (The Binomial Theorem)

$$
\begin{aligned}
(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n \\
&= \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r
\end{aligned}
$$

- Recall when we considered the number of 4-digit numbers without restrictions we found that to be 10^4 since each digit can be $0, 1, \ldots, 9$

  We generalize this as

Theorem (Theorem 9.10)

The number of r-permutations with repetition of an n-element set is n^r.

- Let's consider permutations with repetitions, but the repeated elements are distinguishable

Advanced Counting

- Take the word STREETS which has 2 S's, 2 T's, 2 E's, and 1 R
- If we make the letters distinguishable we have $S_1, S_2, T_1, T_2, E_1, E_2,$ and R.
- Say the number of distinct permutations of STREETS is N
- If we distinguish the S's we then have $2!N$ permutations
- Additionally, distinguishing the T's gives $2!2!N$; adding the E's we get $2!2!2!N$
- Finally, with each letter distinct we have $7!$ so $2!2!2!1N = 7!$ or

$$N = \frac{7!}{2!2!2!1}$$

This example is generalized in the following Theorem

Theorem (Theorem 9.12)

Suppose that there are n objects, exactly k of which are distinguishable. If there are n_1 objects of type 1, n_2 objects of type 2 and so on, up to n_k objects of type k, where $n_1 + n_2 + \cdots + n_k = n$, then the number of different permutations of these n objects is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

- Recall from Chapter 2 that a set is a collection of objects and repetition doesn't matter
- The sets $\{1, 2, 3, 3\}$ and $\{1, 2, 2, 3\}$ are equal
- If repetition of elements is allowed so that $\{1, 2, 3, 3\}$ and $\{1, 2, 2, 3\}$ are distinct sets we call these **multisets**

Advanced Counting

Example

A lady enters Ben's Bagels to buy 4 bagels. The restaurant currently has 5 kinds of bagels: plain, cherry, blueberry, crunchy and coconut. How many different assortments are possible?

Solution

We can think of dividing the bagels into the 5 kinds by 4 separating lines:

$$\text{plain} \;\Big|\; \text{cherry} \;\Big|\; \text{blueberry} \;\Big|\; \text{crunchy} \;\Big|\; \text{coconut}$$

Let the symbol $*$ indicate a bagel to be purchased. For example,

$$\Big| \; * \; \Big| \quad \Big| \; ** \; \Big| \; *$$

Solution (Cont'd)

So there are 4 positions for the bagels and 4 positions for the lines

The number of ways to choose the 4 bagels becomes

$$\binom{8}{4} = \frac{8!}{4!\,4!} = 70$$

Advanced Counting

If we let A be the set of bagels, t is the number of different kinds of bagels (5), and s the number of bagels the lady buys (4), then the number of ways is summarized in the following theorem:

Theorem

Let A be a set containing t different kinds of elements, where there are at least s elements of each kind. The number of different selections of s elements from A is

$$\binom{s+t-1}{s}$$

Advanced Counting – Key Results

- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \leq r \leq n$
- $\binom{n}{r}\frac{n-r}{r+1} = \binom{n}{r+1}$ for $0 \leq r \leq n$
- **The Hockey Stick Theorem**

  $$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$$

  for $0 \leq r \leq n$
- **The Binomial Theorem**

  $$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$