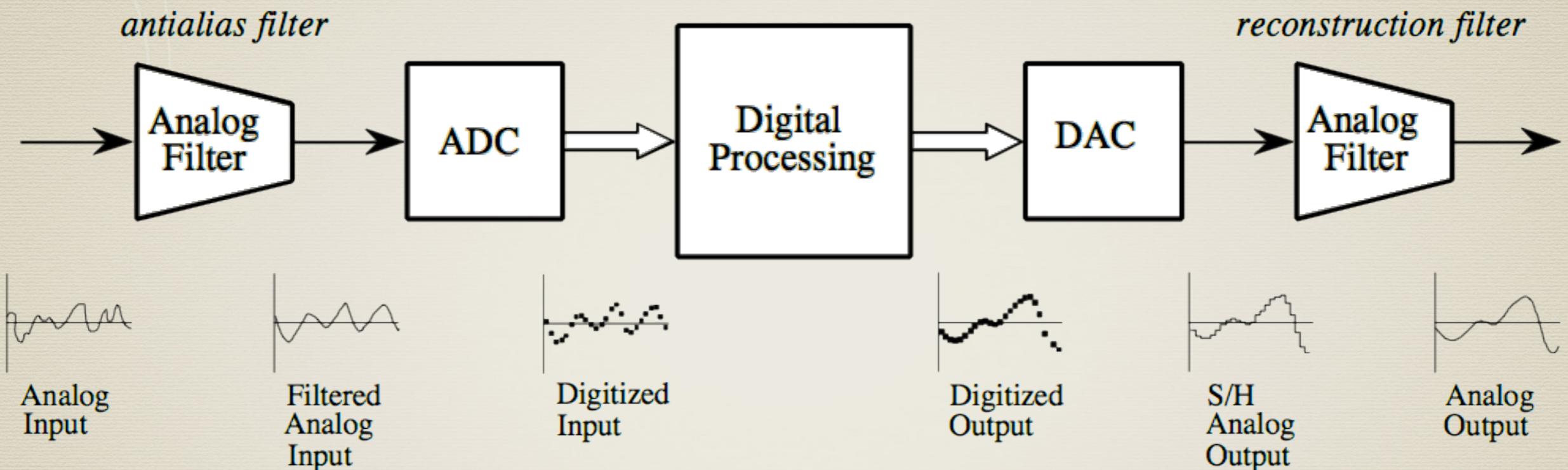


# SISTEMAS AMOSTRADOS

## (Teorema da Amostragem)

Prof. Fernando Passold

# Digitalização de um Sinal



# O Processo de Digitalização

Dados:

Sinal de entrada:  
0 ~ 4.095 Volts;  
A/D de 12-bits:  
saída: 0 ~ 4095

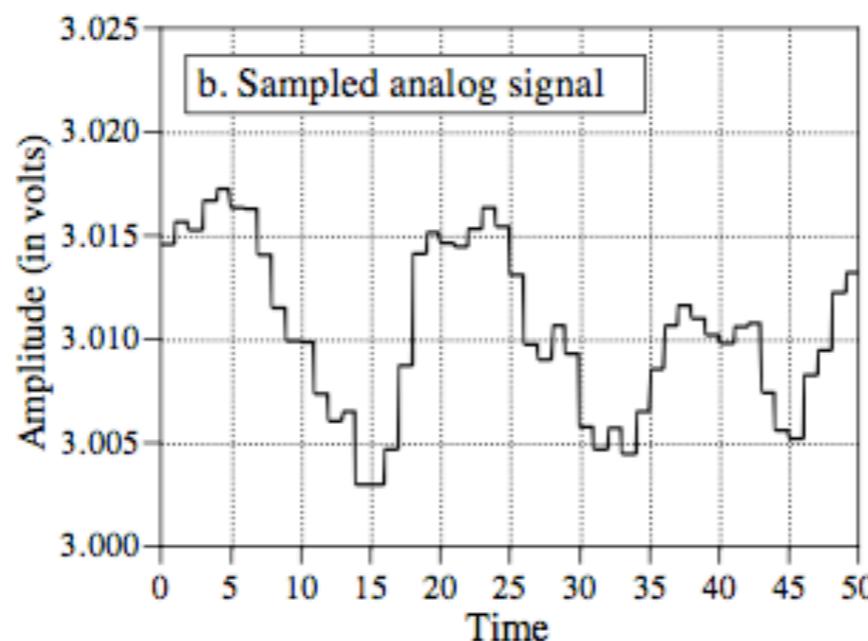
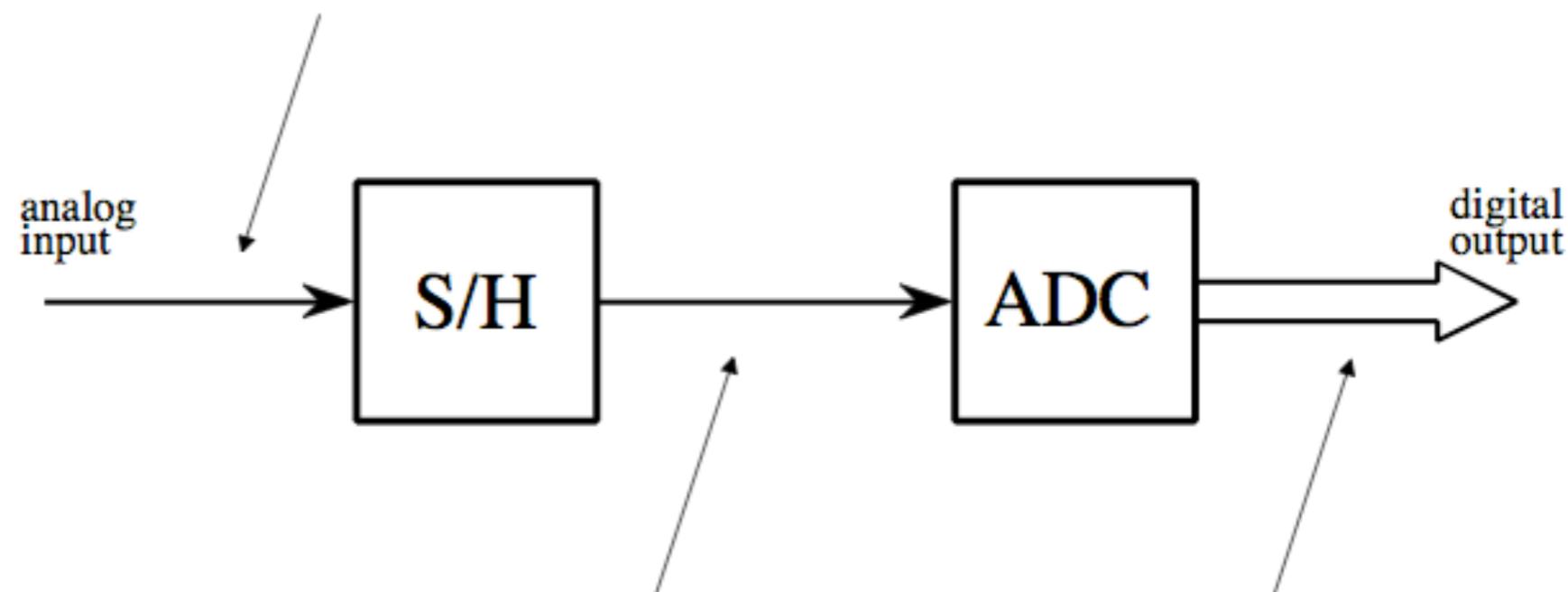
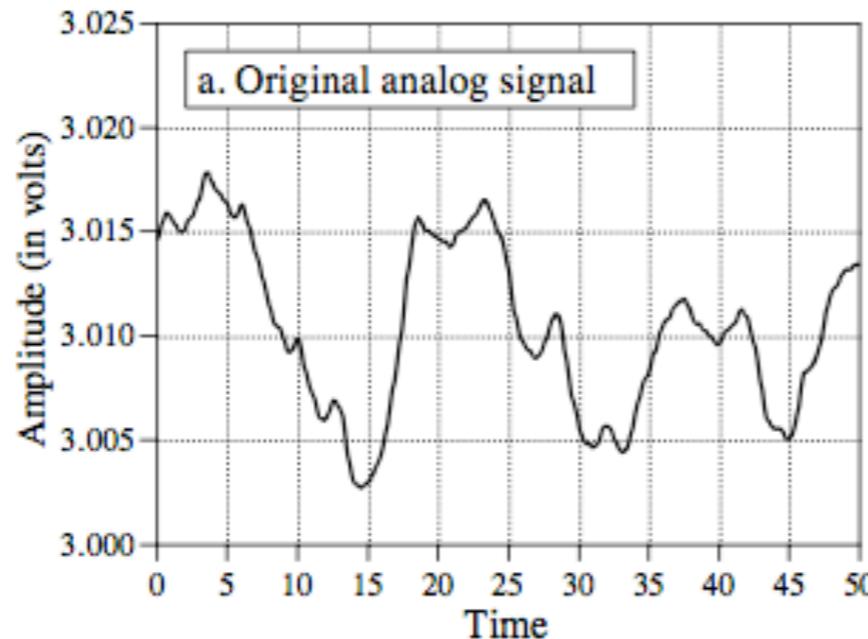
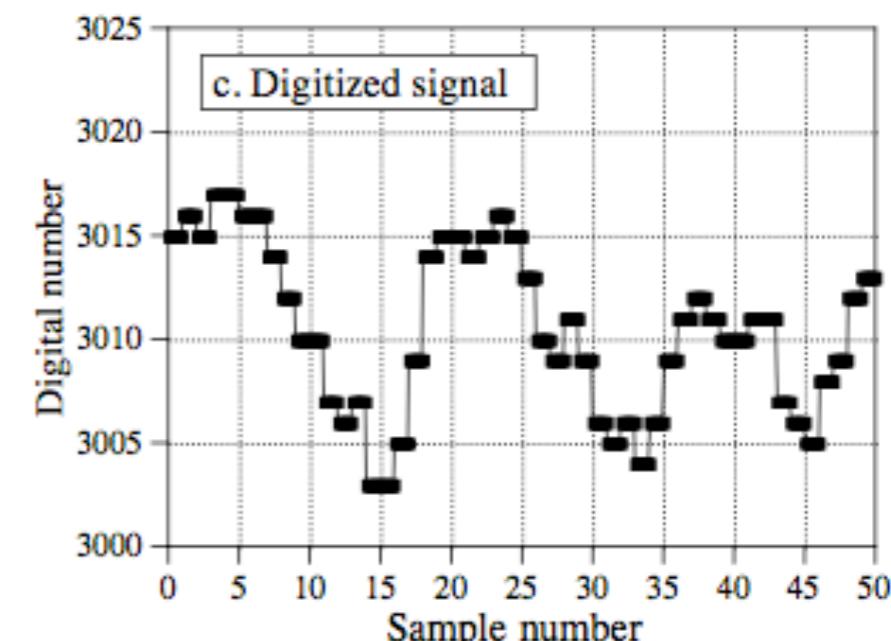


FIGURE 3-1

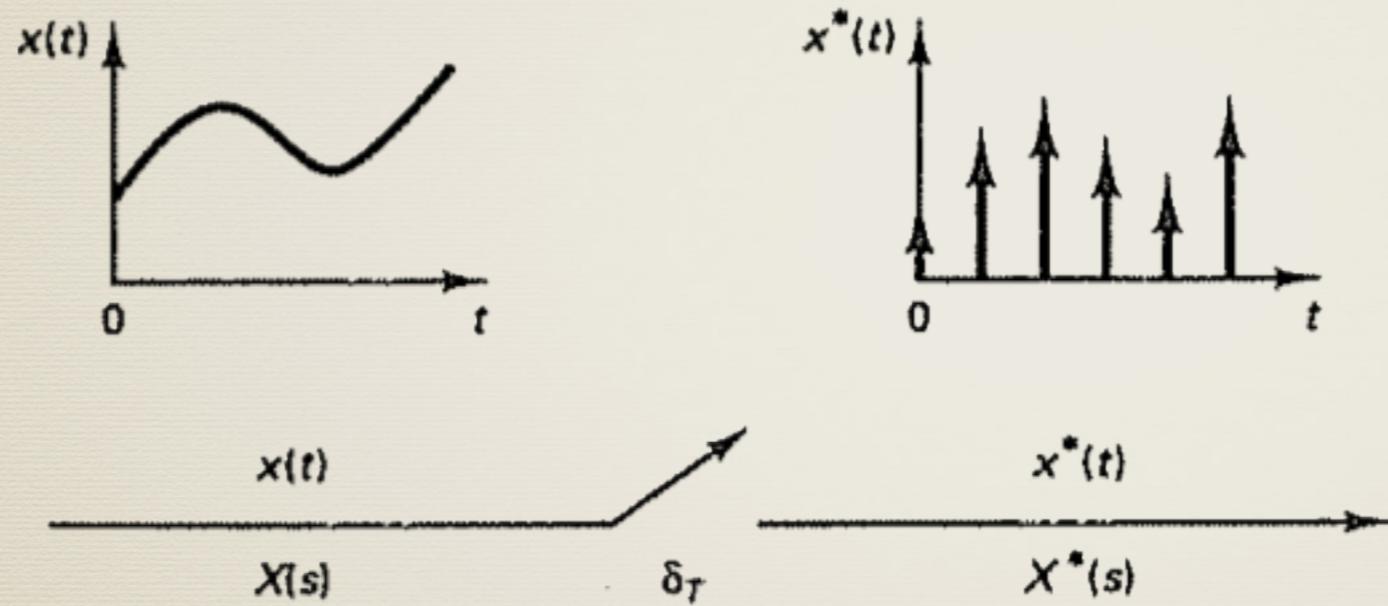
Waveforms illustrating the digitization process. The conversion is broken into two stages to allow the effects of *sampling* to be separated from the effects of *quantization*. The first stage is the sample-and-hold (S/H), where the only information retained is the instantaneous value of the signal when the periodic sampling takes place. In the second stage, the ADC converts the voltage to the nearest integer number. This results in each sample in the digitized signal having an error of up to  $\pm\frac{1}{2}$  LSB, as shown in (d). As a result, quantization can usually be modeled as simply adding noise to the signal.



Ref.: Chap 3. of The  
Scientist and  
Engineer's Guide to  
Digital Signal  
Processing, 2nd. ed.,  
Steven W. Smith,  
California Technical  
Publishing, 1.999  
URL: <http://www.dspguide.com//pdfbook.htm>

# Sinal Amostrado

- \* Suponha que um sistema contínuo no tempo,  $x(t)$  esteja sendo amostrado por um trem de impulsos deslocados no tempo:



Pág. 75, Ogata, Cap. 3: Z-plane Analysis of Discrete-Time Control Systems.  
URL: [http://een.iust.ac.ir/profs/Jahed/digital%20control/e%20book/discrete-time\\_control\\_systems.pdf](http://een.iust.ac.ir/profs/Jahed/digital%20control/e%20book/discrete-time_control_systems.pdf)

**Figure 3-1 Impulse sampler.**

$$(3.1) \quad x^*(t) = \sum_{k=0}^{\infty} x(kT) \delta(t - kT)$$

Trem de impulsos deslocados no tempo (múltiplos do período de amostragem)

$$x^*(t) = x(0)\delta(t) + x(T)\delta(t - T) + \dots + x(kT)\delta(t - kT) + \dots$$

# Sinal Amostrado

- \* O processo de amostragem por trem de pulsos pode ser abordado como um sinal de entrada  $x(t)$  que foi modulado por um trem de impulsos unitários,  $\delta(t)$

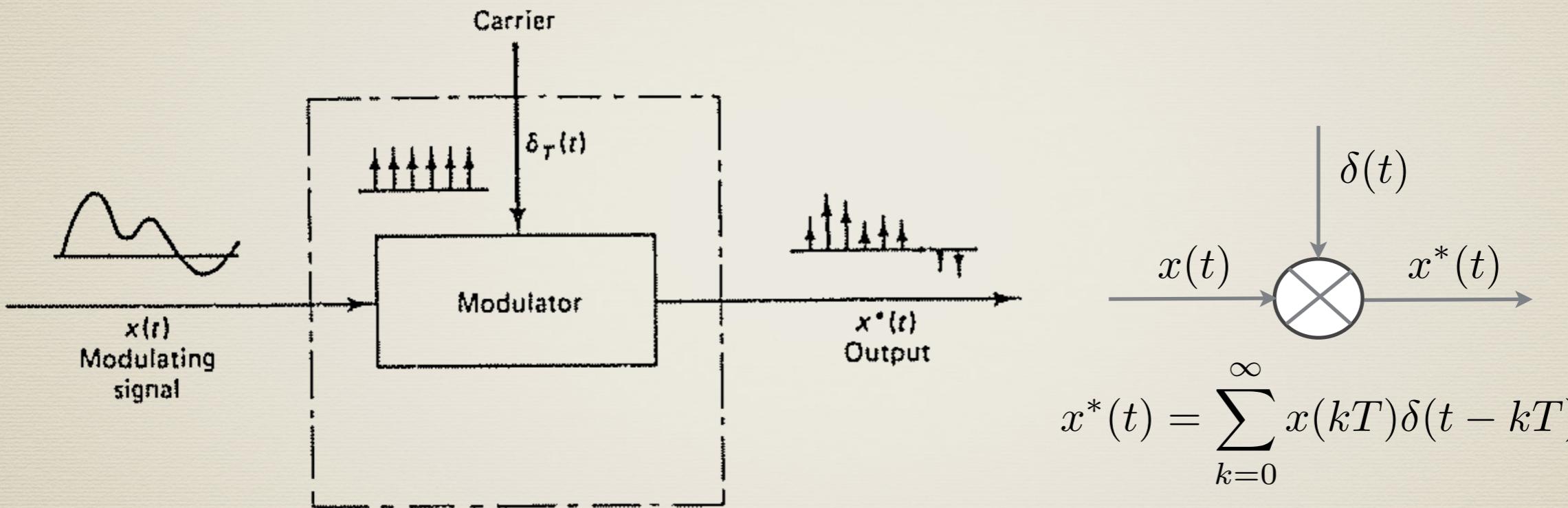


Figure 3–2 Impulse sampler as a modulator.

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t - T)] +$$

- Realizando a transformada de Laplace de (3.1) temos:

$$+ x(2T)\mathcal{L}[\delta(t - 2T)] + \dots$$

$$X^*(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \dots$$

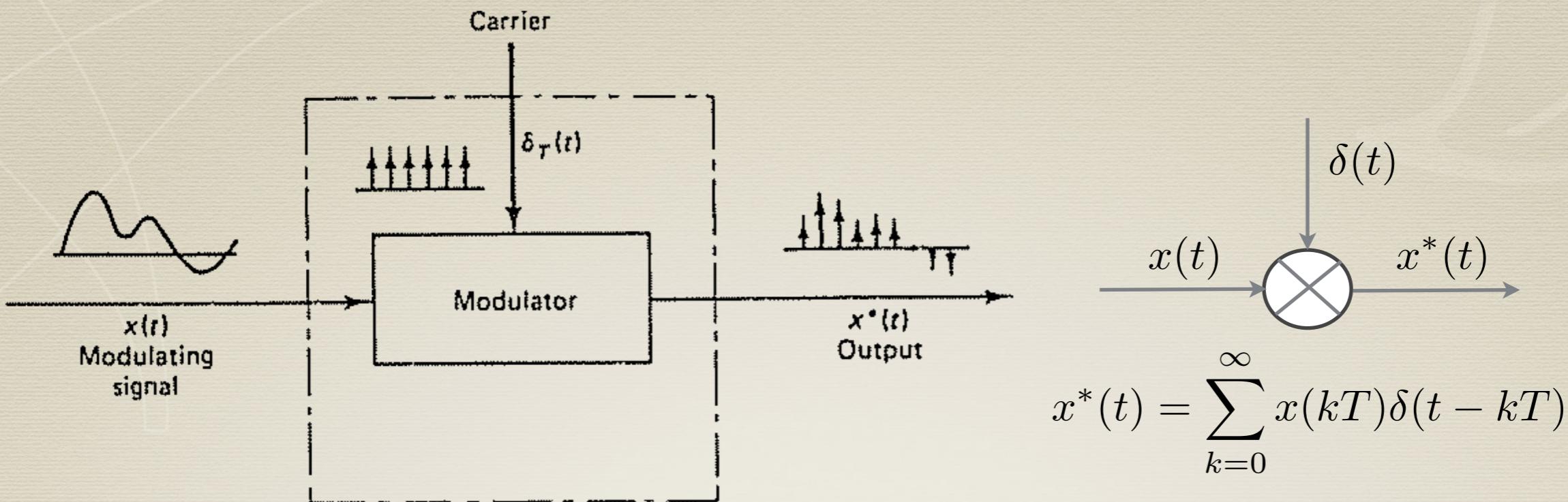


Figure 3-2 Impulse sampler as a modulator.

- Realizando a transformada de Laplace de (3.1) temos:

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t - T)] + x(2T)\mathcal{L}[\delta(t - 2T)] + \dots$$

$$X^*(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \dots$$

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

- Lembrando que:  $\mathcal{L}\{\delta(t - kT)\} = e^{-ks}$

- Pode ser demonstrado que:  $X^{*(s)}|_{s=(1/T) \ln z} = X(z)$

- Realizando a transformada de Laplace de (3.1) temos:

$$X^*(s) = \mathcal{L}[x^*(t)] = x(0)\mathcal{L}[\delta(t)] + x(T)\mathcal{L}[\delta(t - T)] + x(2T)\mathcal{L}[\delta(t - 2T)] + \dots$$

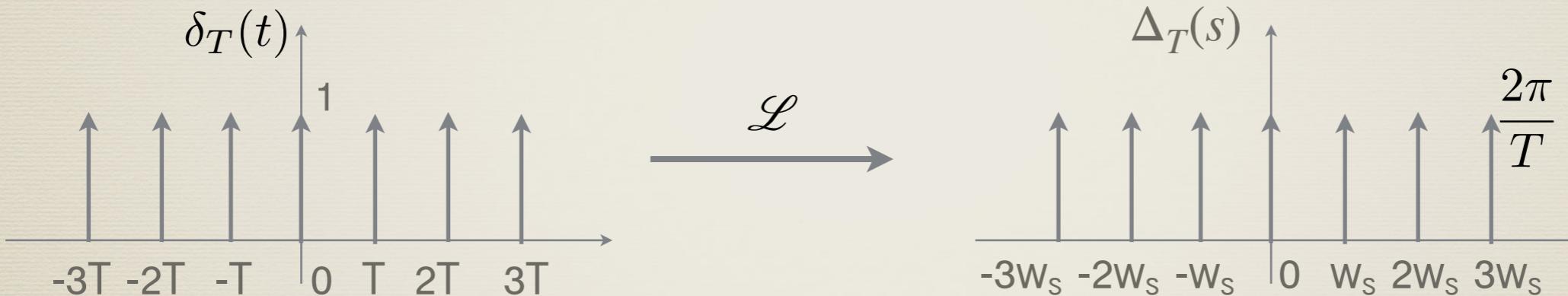
$$X^*(s) = x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \dots$$

$$X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$X^{*(s)}|_{s=(1/T)\ln z} = X(z)$$

- Lembrando que:  $\mathcal{L}\{\delta(t - kT)\} = e^{-Ts}$

- Então no domínio frequêncial, o trem de pulsos é visualizado como:



$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} \cdot e^{jk w_s t}$$

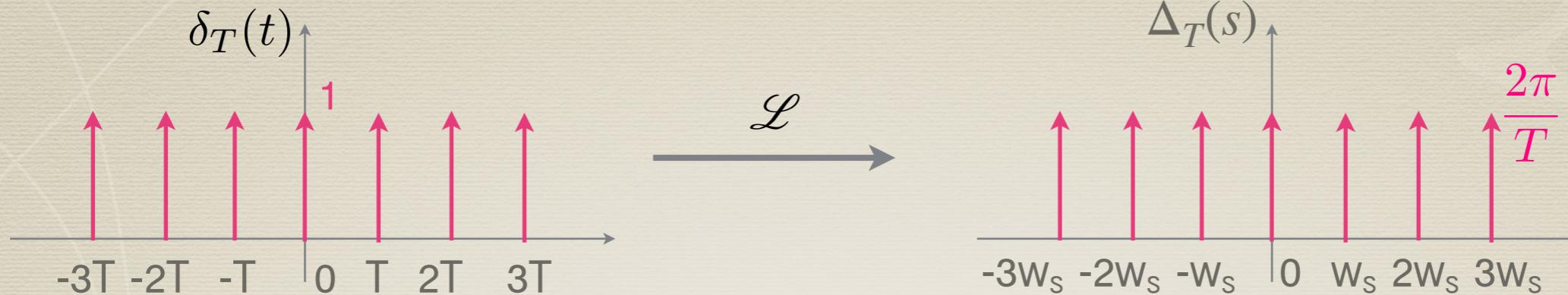
$$T = \frac{1}{f_s}$$

$$w_s = \frac{2\pi}{T}$$

$$\Delta_T(s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - k w_s)$$

- Como:  $x^*(t) = x(t) * \delta_T(t)$  - No Domínio frequêncial teremos uma convolução:

- Então no domínio frequênci, o trem de pulsos é visualizado como:



$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} \cdot e^{jk w_s t}$$

$$T = \frac{1}{f_s}$$

$$w_s = \frac{2\pi}{T}$$

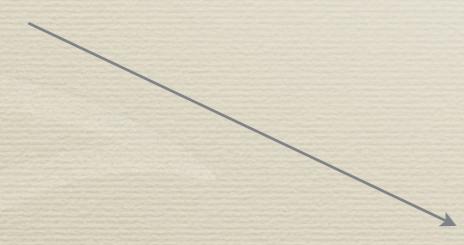
$$\Delta_T(s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - k w_s)$$

- Como:  $x^*(t) = x(t) * \delta_T(t)$  - No Domínio frequênci teremos uma convoluçao:

$$X^*(s) = \frac{1}{2\pi} [X(s) \cdot \Delta(s)]$$

$$\begin{aligned} X^*(s) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \Delta(w - \theta) d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \theta - k w_s) d\theta \\ X^*(s) &= \frac{1}{T} \cdot \sum_{k=-\infty}^{+\infty} X(w - k w_s) \end{aligned}$$

Resultado da amostragem  
de um sinal por um trem  
de impulsos



- Como:  $x^*(t) = x(t) * \delta_T(t)$  - No Domínio frequêncial teremos uma convolução:

$$X^*(s) = \frac{1}{2\pi} [X(s) \cdot \Delta(s)]$$

Resultado da amostragem de um sinal por um trem de impulsos

$$\begin{aligned} X^*(s) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \Delta(w - \theta) d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\theta) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - \theta - k w_s) d\theta \\ &\Rightarrow X^*(s) = \frac{1}{T} \cdot \sum_{k=-\infty}^{+\infty} X(w - k w_s) \end{aligned}$$

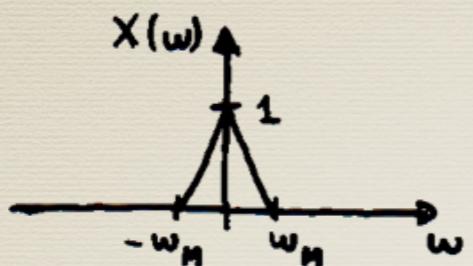
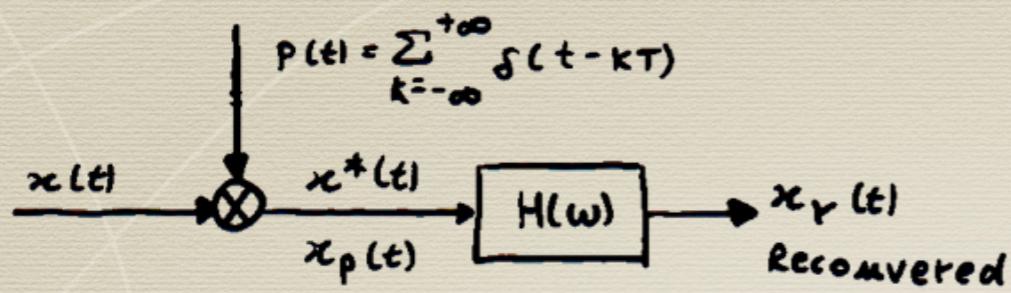
ou:

$$X^*(s) = X^*(s \pm j w_s k), \quad k = 0, 1, 2, \dots$$

Note que  $X(s)$  possui um pólo em  $s=s_I$ , já  $X^*(s)$  possui pólos múltiplos em:

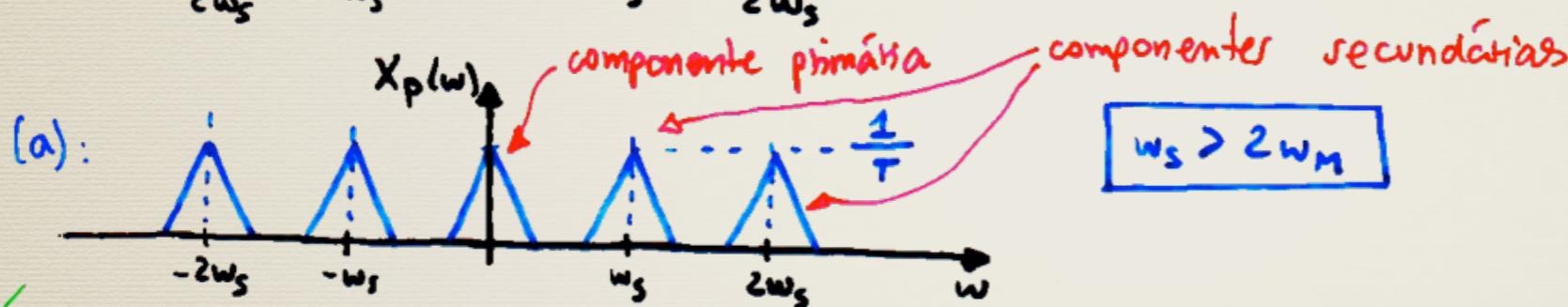
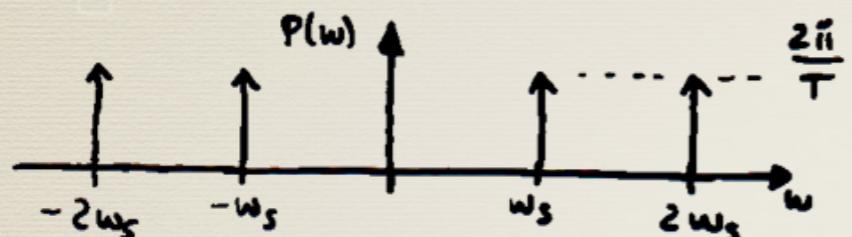
$$s = s_1 \pm j w_s k \quad (k = 0, 1, 2, \dots)$$

Observando o espectro resultante, supondo que  $X(w)$  possui banda limitada.



$$x_p(t) = x(t) * p(t)$$

$$X_p(w) = \frac{1}{2\pi} [X(w) * P(w)]$$



$\curvearrowleft$  p/ recompor o sinal, mas interessa apenas a componente principal

seja um Filtro Passa-Baixas Ideal

$$w_M < w_c < (w_s - w_M)$$

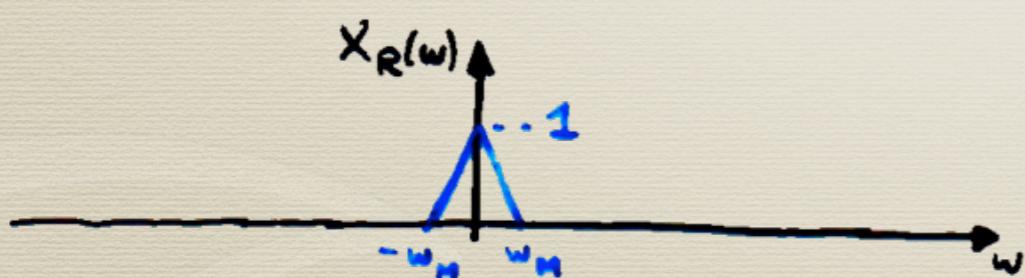
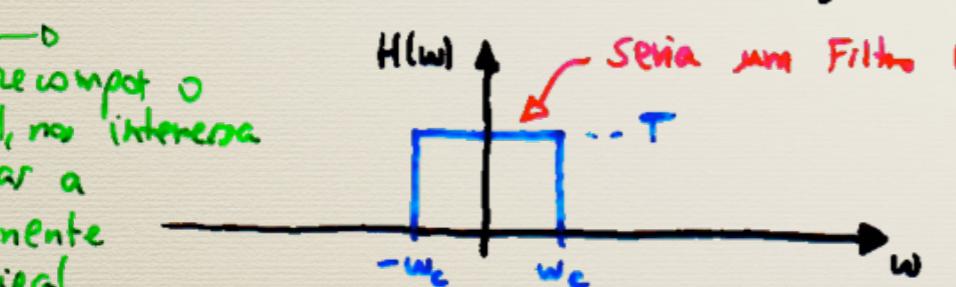


Fig. 2.5 - Amostragem por Impulsos (ideal)

Note:

Para não haver recobrimento de espectros em (a), temos que ter:

$$w_s > 2w_M$$

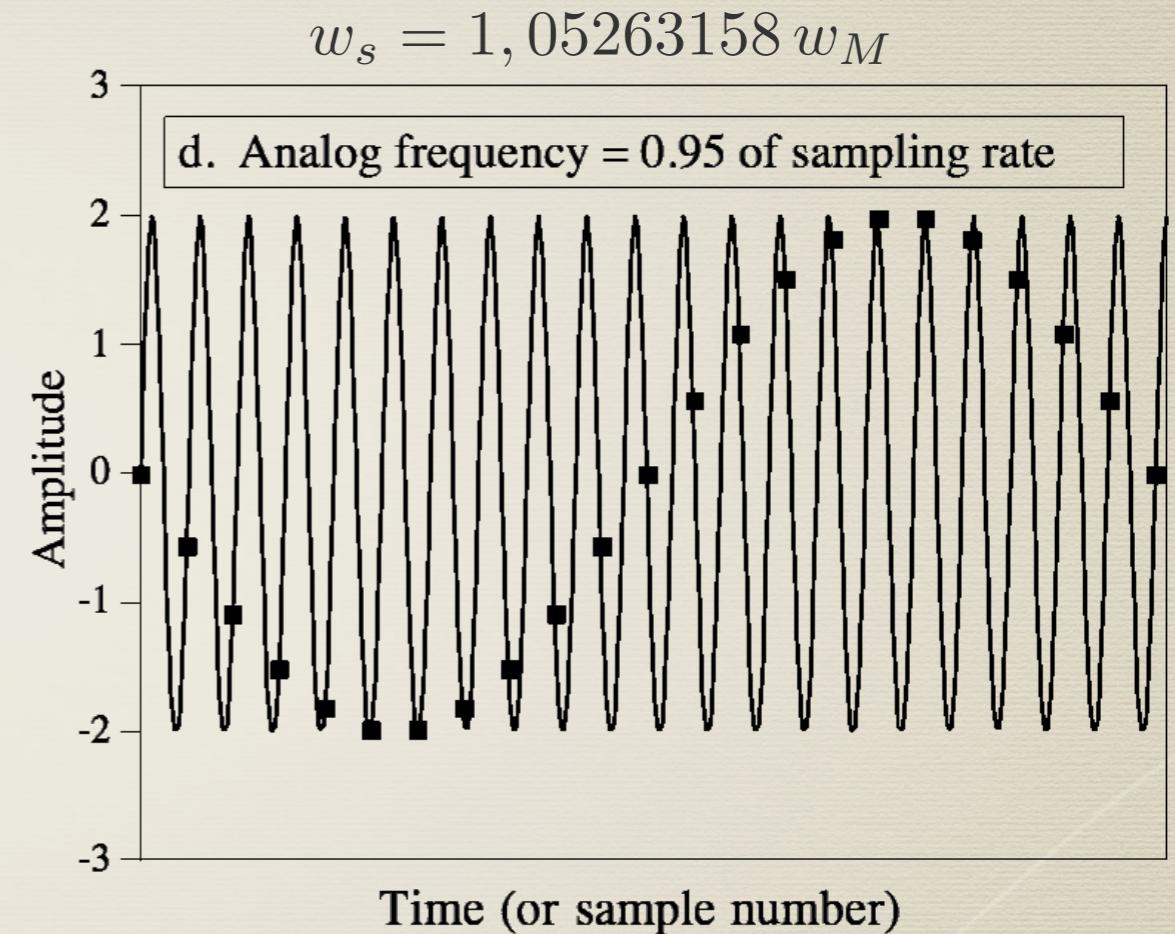
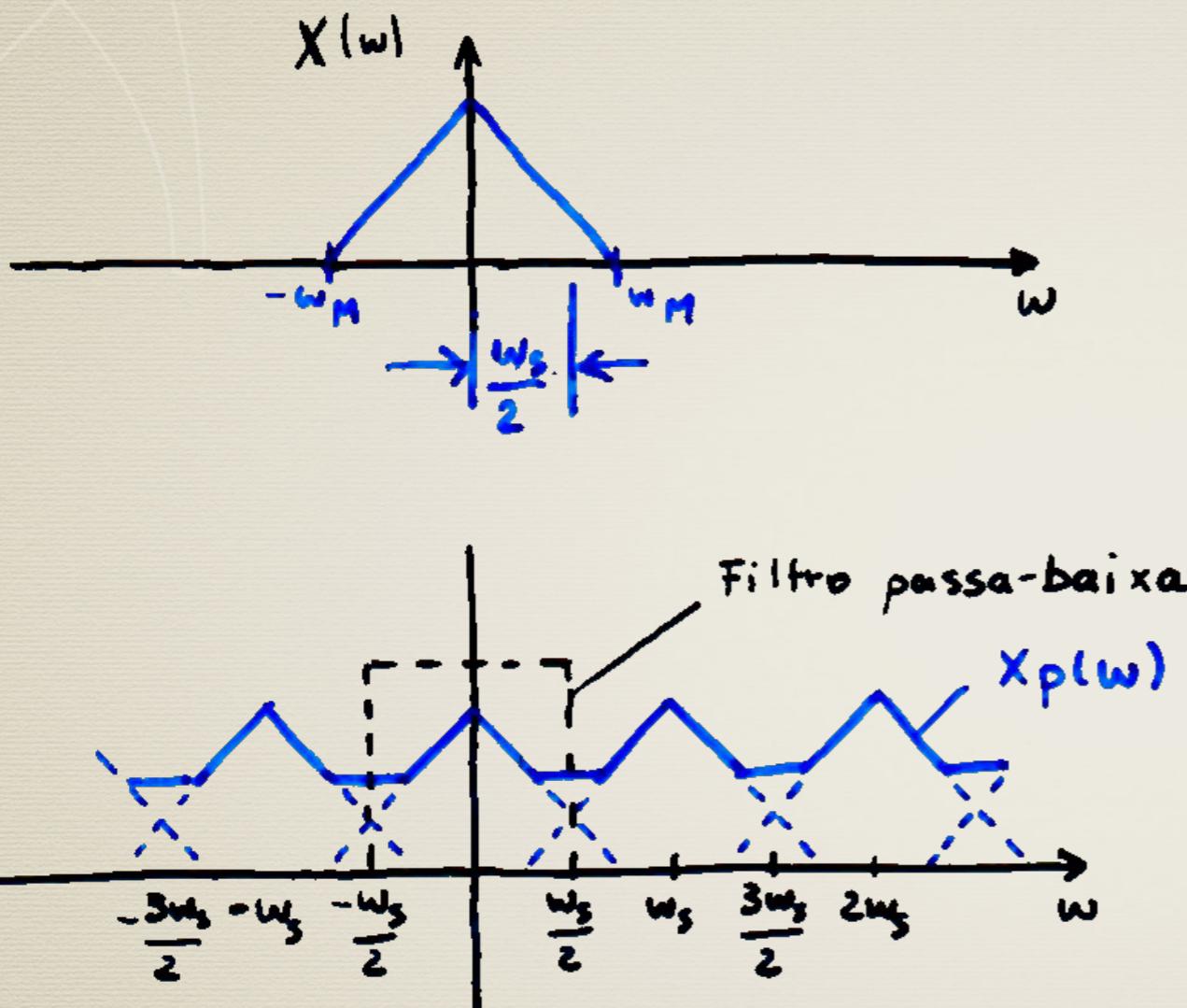
- Teorema da Amostragem (Shanon):

$$w_s = \frac{2\pi}{T} > 2w_M$$

- Freq. (taxa) de Nyquist:

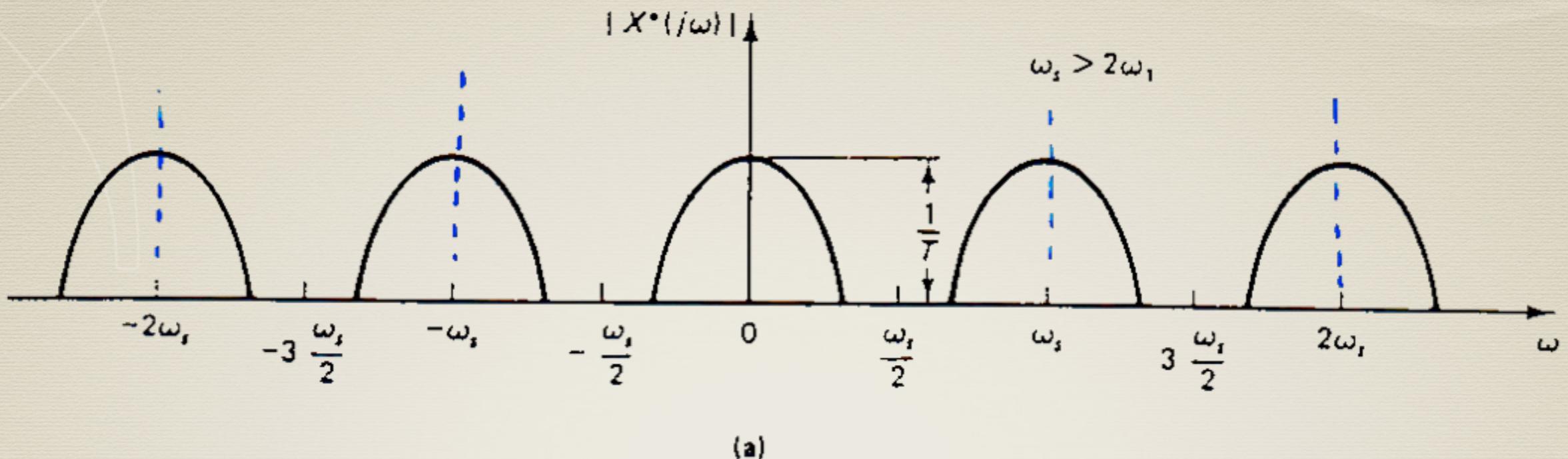
$$w_s = 2w_M$$

# Caso de Sub-amostragem (aliasing)



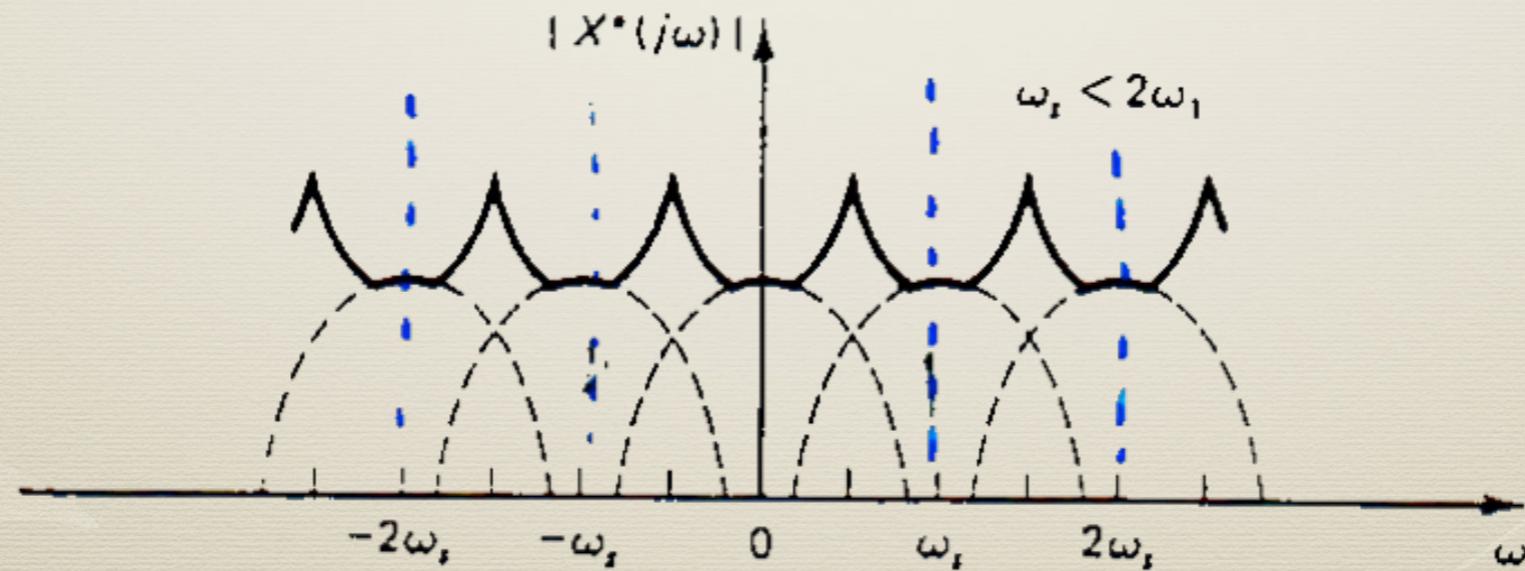
# Outros Casos

$\omega_s > 2\omega_1$ :



(a)

$\omega_s < 2\omega_1$



# Efeitos: “Hidden Oscilation”

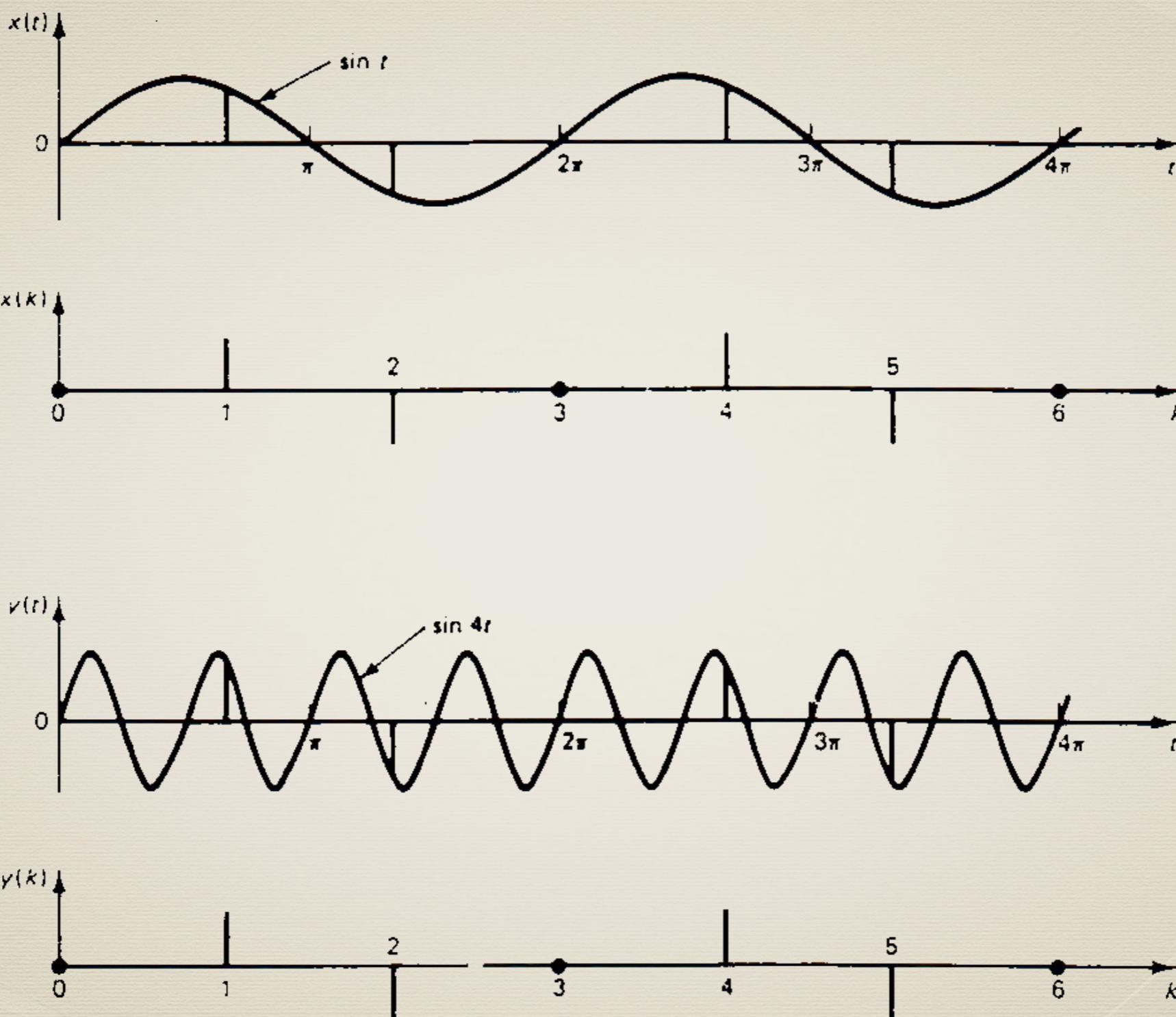
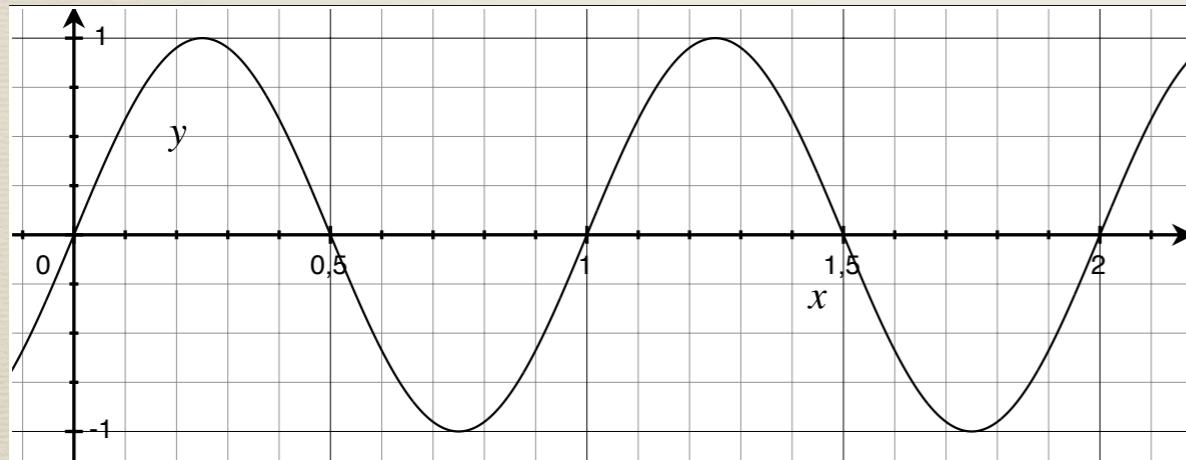


Figure 3–18 Plots of signals  $x(t) = \sin t$  and  $y(t) = \sin 4t$  and their sampled signals.  
 $\omega_S = 3$  rad/s.

# Exercício de Sub-amostragem

- \* Seja uma onda senoidal de 1,0 Vp oscilando à 1 Hz.  
Sua equação seria:  $y(t) = 1 \cdot \sin(2\pi \cdot 1 \cdot t)$   
Lembrando que:  $\omega = 2\pi f$ .
- \* O que acontece se este sinal for amostrado à 2 Hz? Recorde-se que estaríamos respeitando o teorema de Nyquist.

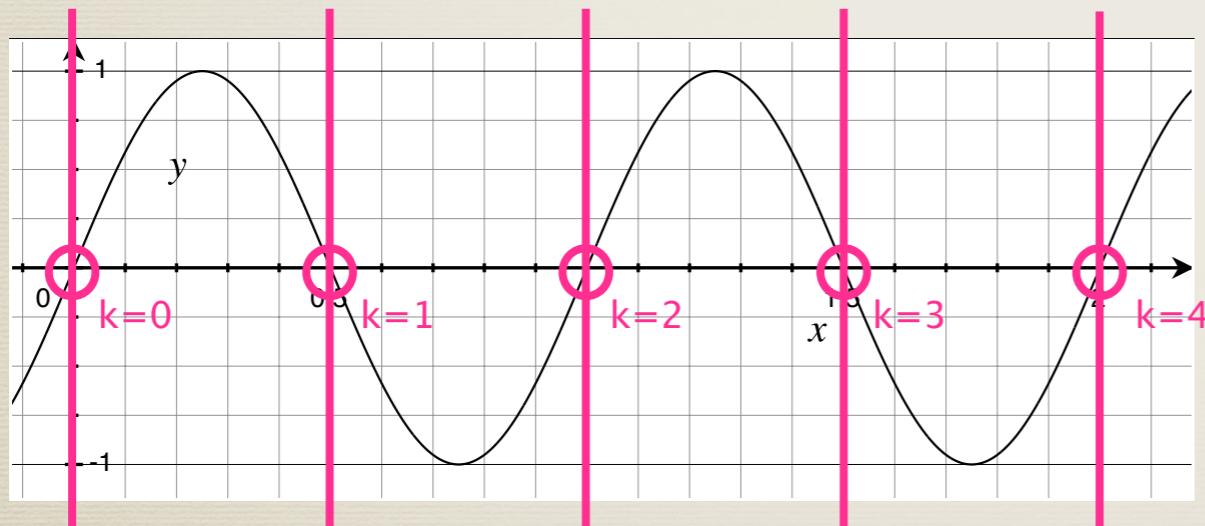


Obs.: Este exercício pode ser retomado ou re-estudado depois de estudada a transformada-Z de uma onda senoidal:

$$\mathcal{Z} \{\sin(\omega t)\} = \frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$$

# Exercício de Sub-amostragem

- \* Seja uma onda senoidal de 1,0 Vp oscilando à 1 Hz.  
Sua equação seria:  $y(t) = 1 \cdot \sin(2\pi \cdot 1 \cdot t)$   
Lembrando que:  $\omega = 2\pi f$ .
- \* O que acontece se este sinal for amostrado à 2 Hz? Recorde-se que estaríamos respeitando o teorema de Nyquist.



Note:

$$y^*(t) = y[kT] = \sin(2\pi \cdot kT)$$
$$T = 1/f_s \therefore T = 1/2 \text{ (} f_s = 2 \text{ Hz)}$$

$$y[kT] = \sin\left(\frac{2\pi k}{2}\right) = \sin(k\pi)$$

k	y[kT]
0	0
1	0
2	0
3	0
4	0
⋮	⋮

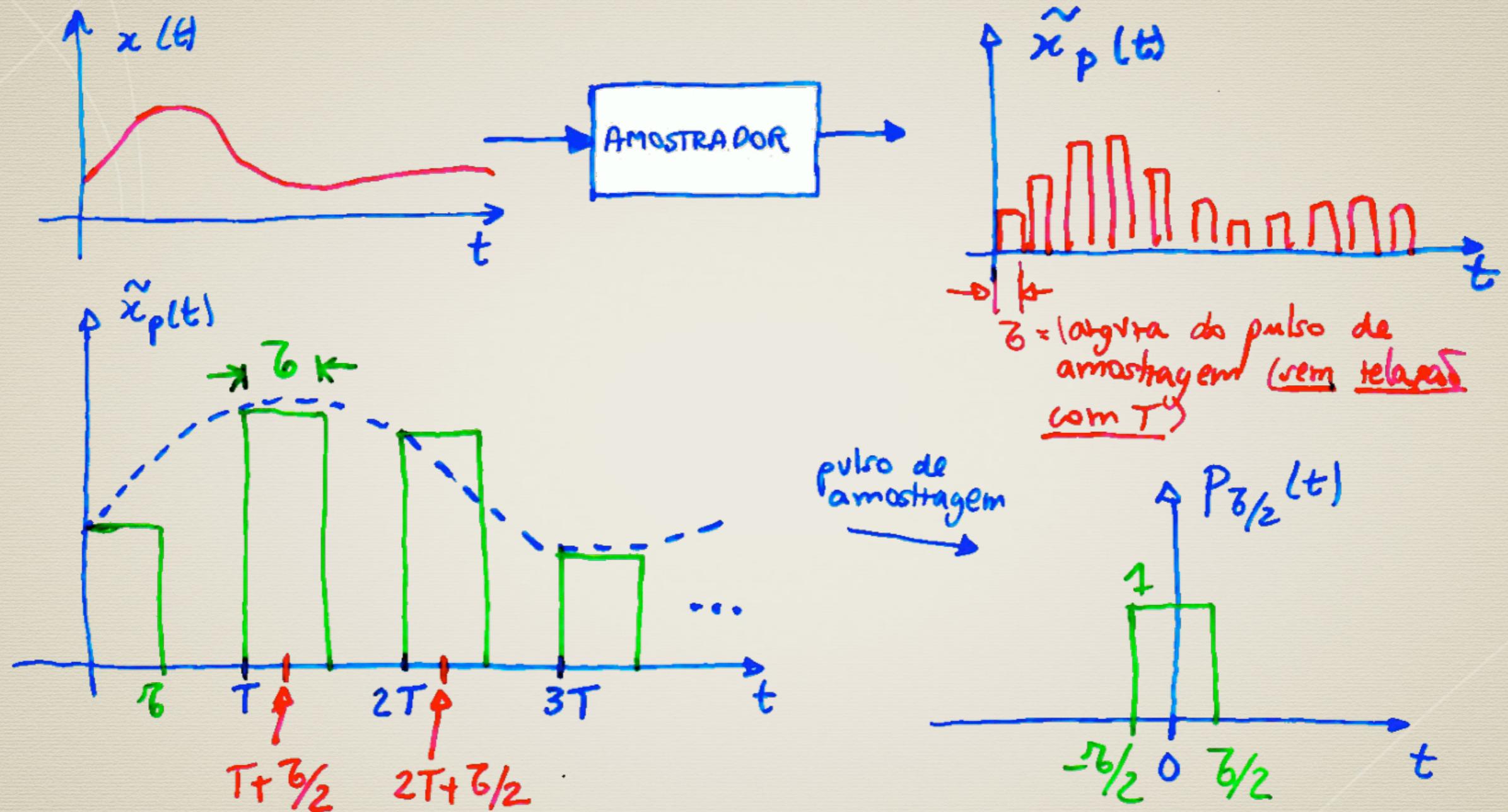
Obs.: Este exercício pode ser retomado ou re-estudado transformada-Z de uma onda senoidal:

$$\mathcal{Z}\{\sin(\omega t)\} = \frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$$

# Exemplo de Subamostragem

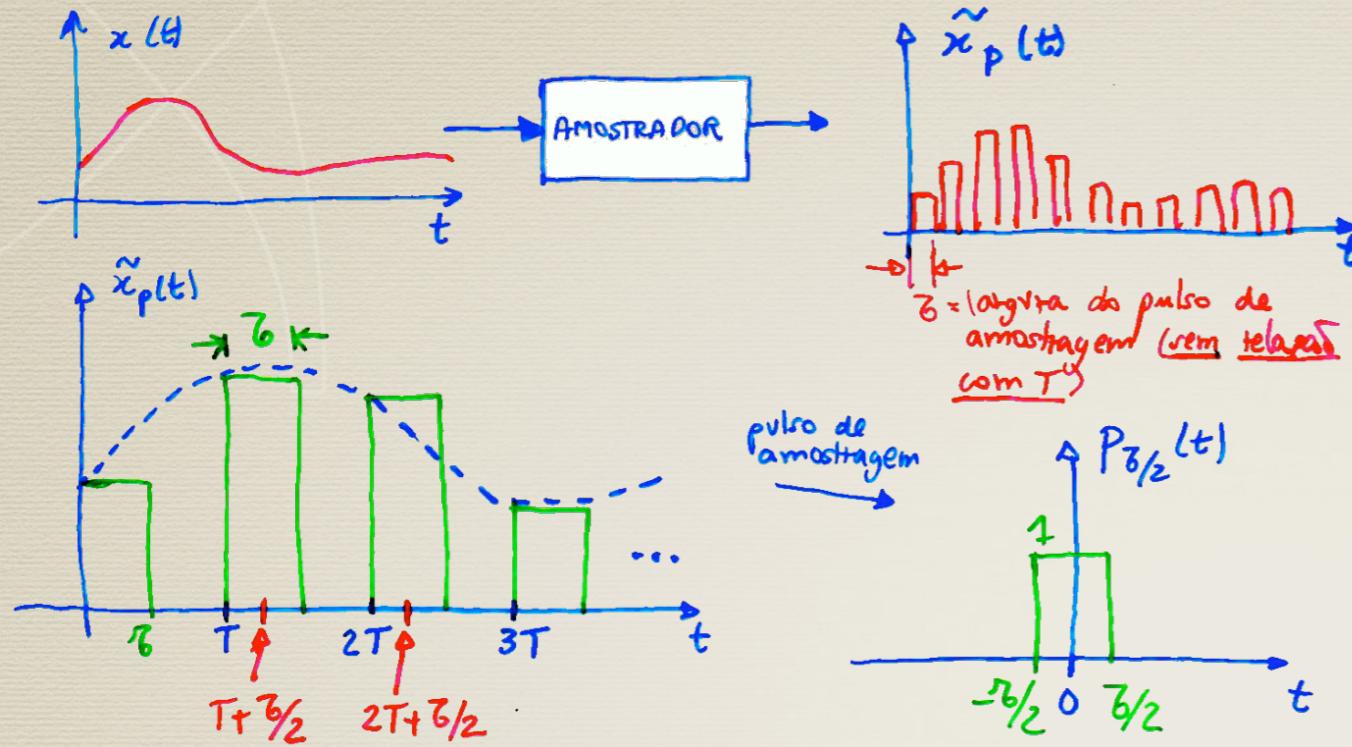
- \* Suponha agora que a onda sinusoidal de 1 Hz seja amostrada à 2 Hz.
- \* O que acontecerá ?

# Amostragem por trem de pulsos



- na prática:
- amostragem realizada por pulsos;
  - filtro passa-baixa ideal não existe (não realizável fisicamente)

# Amostragem por trem de pulsos



Nova modelagem:

$$\tilde{x}_p(t) = \sum_{n=-\infty}^{+\infty} x(kT) \cdot p_{T/2}(t - kT - \frac{T}{2})$$

somatório de pulsos deslocados no tempo:

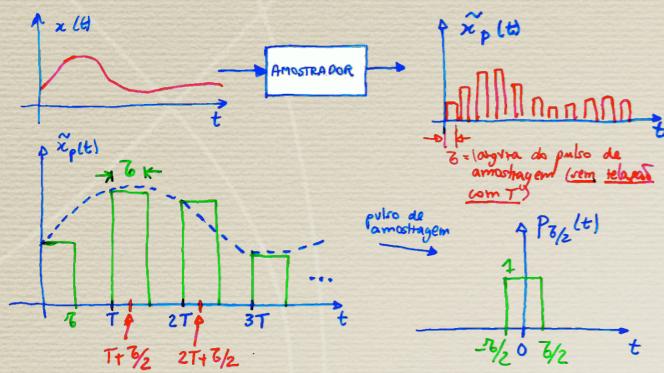
$$\tilde{x}_p(\omega) = \sum_{k=-\infty}^{+\infty} x(kT) \cdot \mathcal{F}\left\{ p_{T/2}(t - kT - \frac{T}{2}) \right\}$$

$$\mathcal{F}\left\{ p_{T/2}(t) \right\} = T \cdot \frac{\sin(\omega T/2)}{\omega T/2}$$

Função sampling

$$\sum_{k=-\infty}^{+\infty} p_{T/2}(t - kT - \frac{T}{2})$$

# Amostragem por trem de pulsos



$$\tilde{x}_p(t) = \sum_{n=-\infty}^{+\infty} x(kT) \cdot p_{\frac{\delta}{2}}(t - kT - \frac{\delta}{2})$$

somatório de pulsos deslocados no tempo:

$$\tilde{X}_p(w) = \sum_{k=-\infty}^{+\infty} x(kT) \cdot \mathcal{F}\left\{ p_{\frac{\delta}{2}}(t - kT - \frac{\delta}{2}) \right\}$$

$$\mathcal{F}\left\{ p_{\frac{\delta}{2}}(t) \right\} = \frac{\pi}{\delta} \cdot \frac{\sin(\omega \frac{\delta}{2})}{\frac{\omega \delta}{2}}$$

Função sampling

$$\mathcal{F}\left\{ p_{\frac{\delta}{2}}(t - kT - \frac{\delta}{2}) \right\} = \frac{\pi}{\delta} \cdot \frac{\sin(\omega \frac{\delta}{2})}{\frac{\omega \delta}{2}} \cdot e^{-j\omega(kT - \frac{\delta}{2})}$$

deslocamento de fase

$$= \frac{\pi}{\delta} \cdot \frac{\sin(\omega \frac{\delta}{2})}{\frac{\omega \delta}{2}} \cdot e^{-j\omega \frac{\delta}{2}} \cdot e^{-j\omega kT}$$

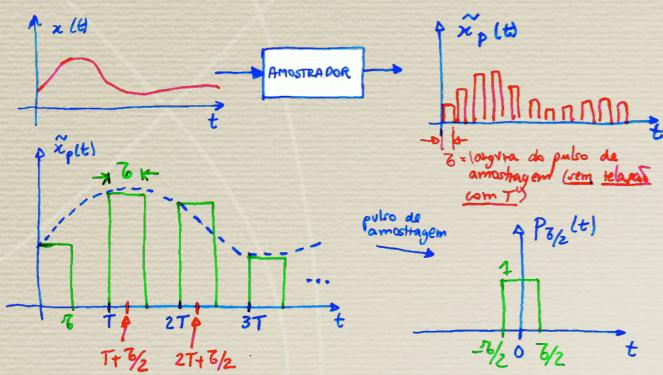
$$\tilde{X}_p(w) = \frac{\pi}{\delta} \cdot \frac{\sin(\omega \frac{\delta}{2})}{\frac{\omega \delta}{2}} \cdot e^{-j\omega \frac{\delta}{2}} \cdot \sum_{k=-\infty}^{+\infty} x(kT) \cdot e^{-j\omega kT}$$

não depende de  $x(t)$ ;  
não depende de  $f_s$ .

$X_p(w)$

Nova modelagem:

# Amostragem por trem de pulsos



$$\begin{aligned} \tilde{x}_p(t) &= \sum_{k=-\infty}^{+\infty} x(kT) \cdot P_{\bar{b}/2}(t - kT) \\ &= \sum_{k=-\infty}^{+\infty} x(kT) \cdot e^{-j\omega_{\bar{b}/2}(kT - \frac{\pi}{2})} \end{aligned}$$

$\underbrace{\sum_{k=-\infty}^{+\infty} x(kT) \cdot}_{\text{não depende de } x(t); \text{ não depende de } f_s.} \underbrace{e^{-j\omega_{\bar{b}/2}(kT - \frac{\pi}{2})}}_{\text{deslocamento de fase}} \cdot \tilde{x}_p(\omega)$

$$\tilde{x}_p(\omega) = \sum_{k=-\infty}^{+\infty} x(kT) \cdot e^{-j\omega_{\bar{b}/2}kT} \cdot X_p(\omega)$$

$\therefore$  distorção em fase e amplitude

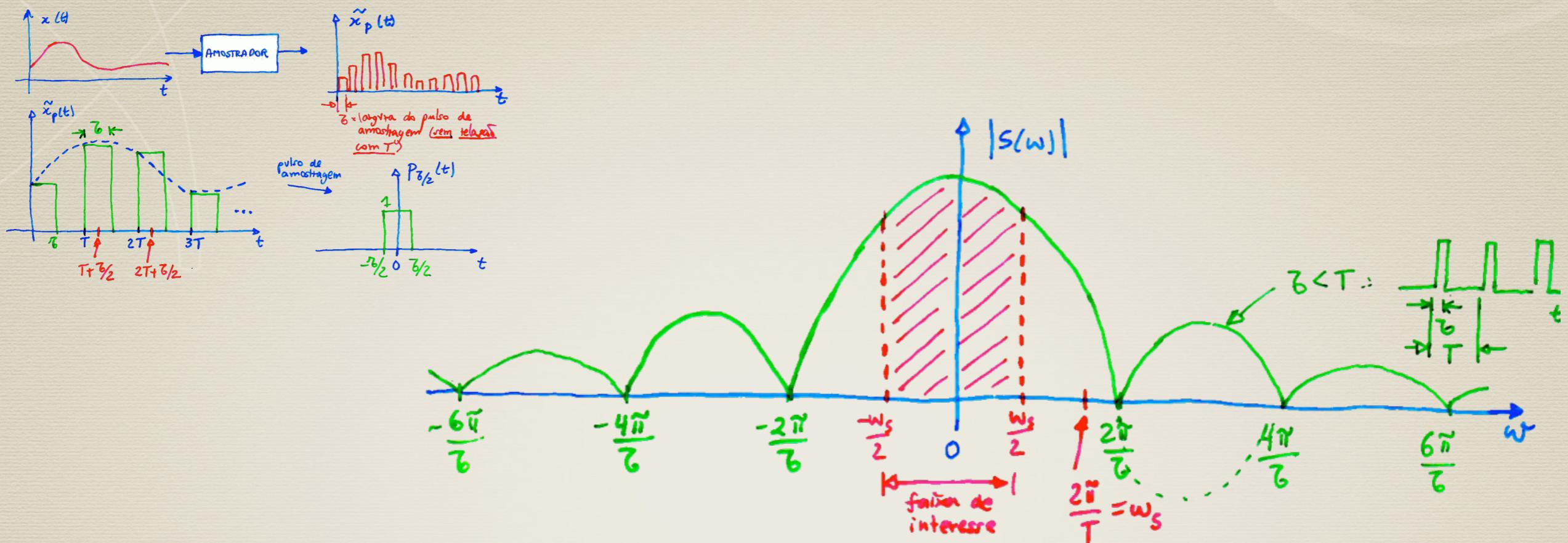
$$e^{-j\omega_{\bar{b}/2}kT} \xrightarrow{\uparrow \text{ devido ao atraso no tempo, de } \frac{\pi}{2}} \frac{\sin(\omega_{\bar{b}/2}kT)}{\omega_{\bar{b}/2}}$$

$$\tilde{x}_p(\omega) = s(\omega) \cdot X_p(\omega)$$

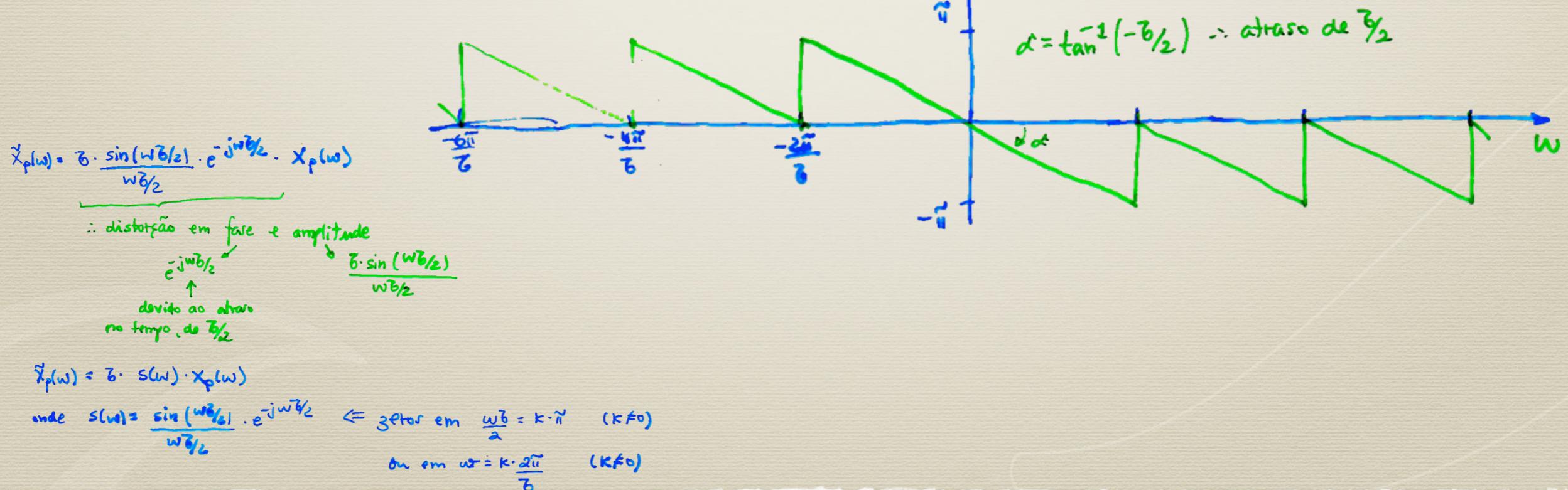
$$\text{onde } s(\omega) = \frac{\sin(\omega_{\bar{b}/2}kT)}{\omega_{\bar{b}/2}} \Leftrightarrow \text{zeros em } \frac{\omega_{\bar{b}}}{2} = k \cdot \frac{\pi}{a} \quad (k \neq 0)$$

$\text{ou em } \omega = k \cdot \frac{2\pi}{\bar{b}} \quad (k \neq 0)$

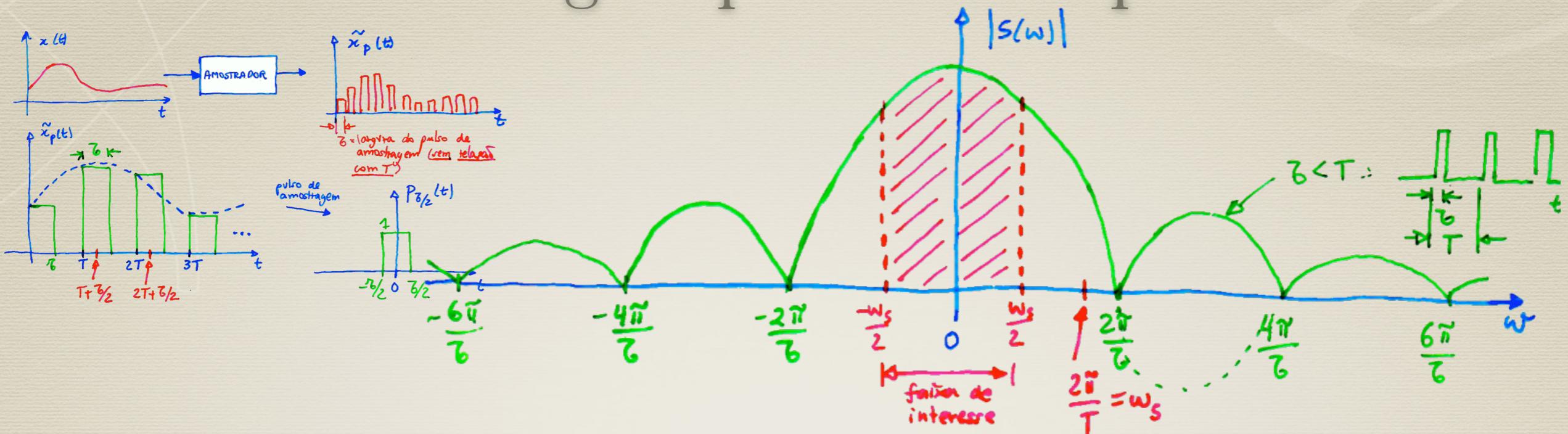
# Amostragem por trem de pulsos



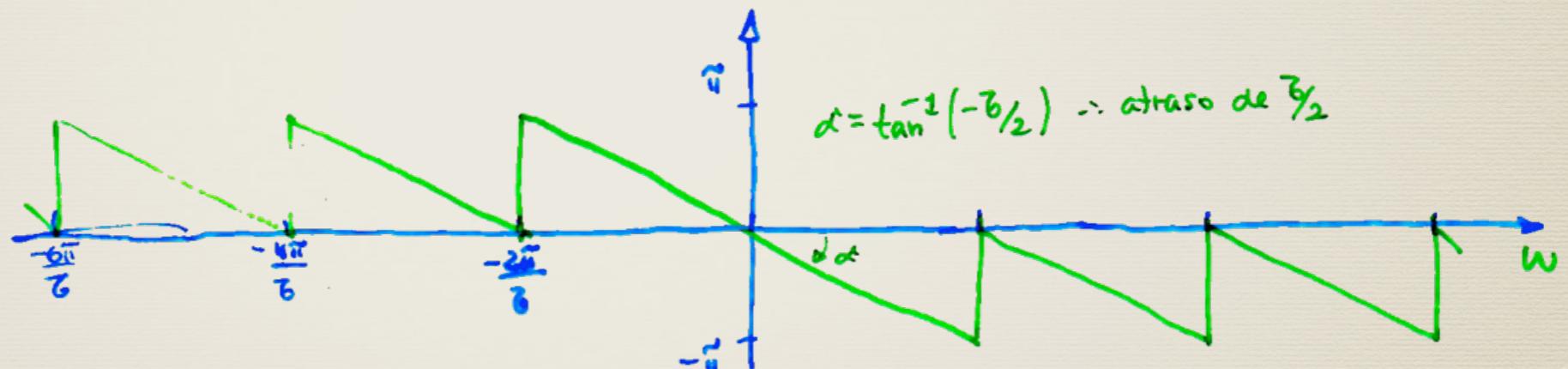
Nova modelagem:



# Amostragem por trem de pulsos



Nova modelagem:



$$\hat{x}_p(w) = b \cdot \frac{\sin(wb/2)}{wb/2} \cdot e^{-jwb/2} \cdot x_p(w)$$

distorção em fase e amplitude

$$e^{-jwb/2}$$

devido ao atraso  
no tempo de  $b/2$

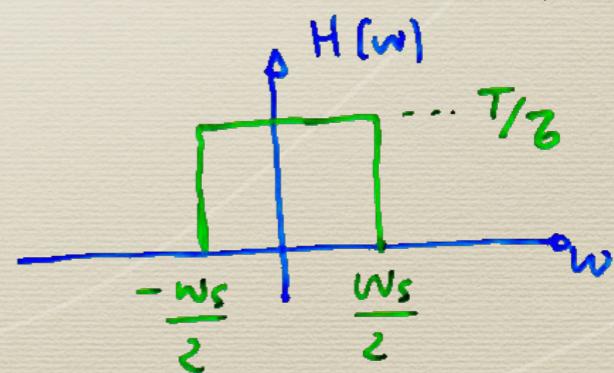
$$\tilde{x}_p(w) = b \cdot s(w) \cdot x_p(w)$$

$$\text{onde } s(w) = \frac{\sin(wb/2)}{wb/2} \cdot e^{-jwb/2} \Leftarrow \text{zeros em } \frac{wb}{2} = k\pi \quad (k \neq 0)$$

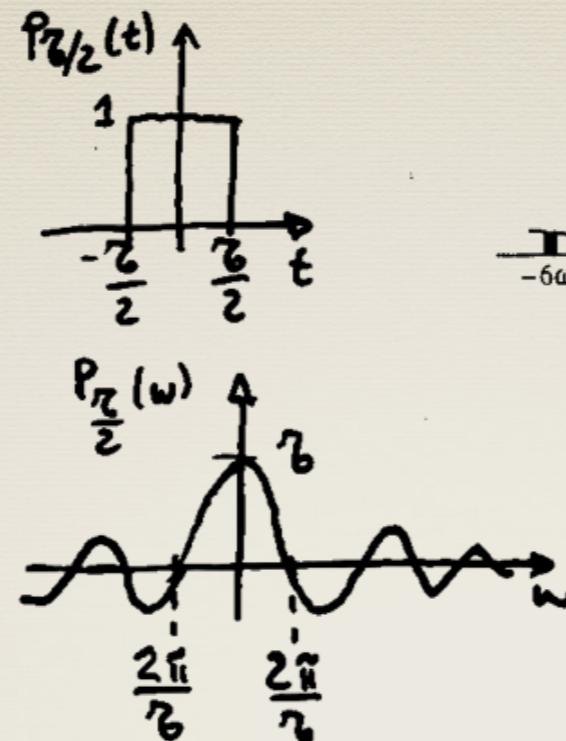
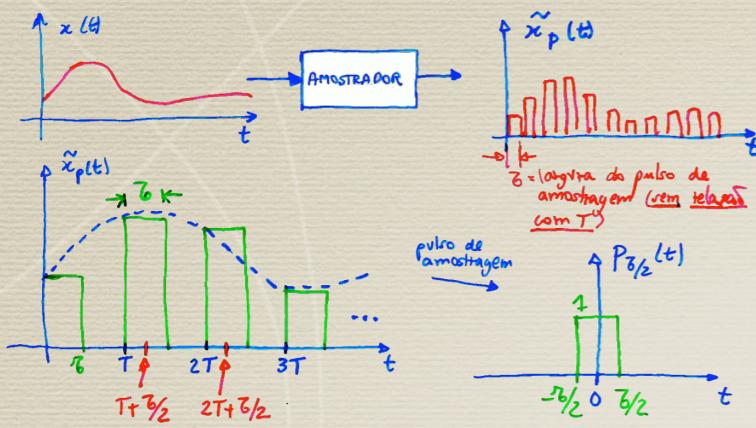
$$\text{ou em } w = k \frac{2\pi}{b} \quad (k \neq 0)$$

Obs.: Posso passar  $\hat{X}_p(w)$  por um filtro parabólico ideal:  $w_c = \frac{w_s}{2}$ .

Sinal reconstituído

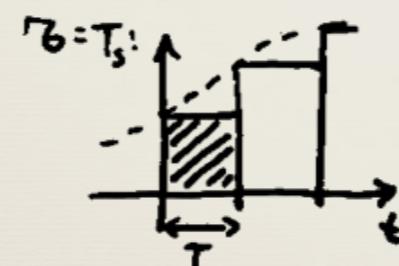


# Amostragem por trem de pulsos

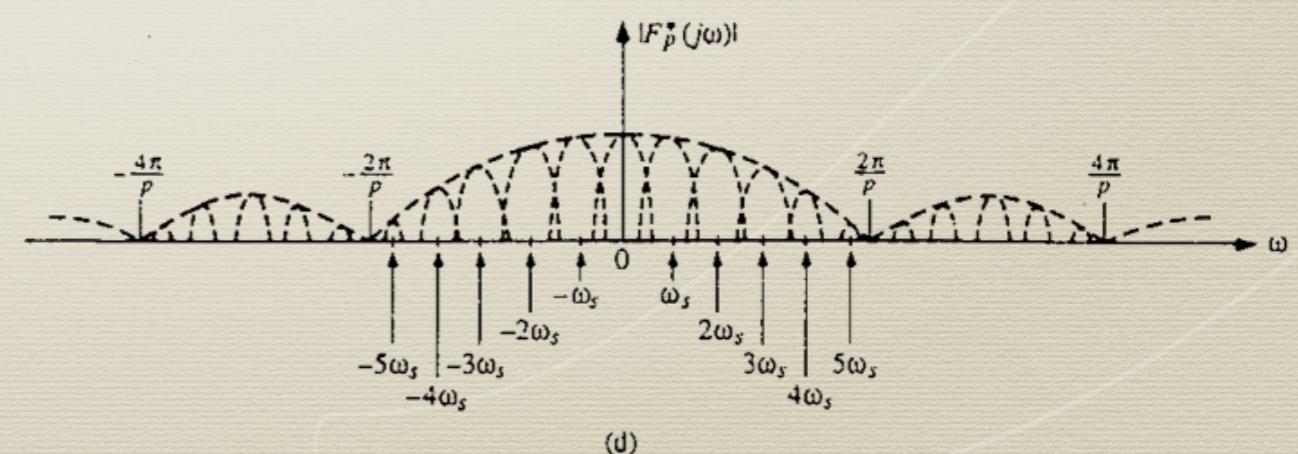
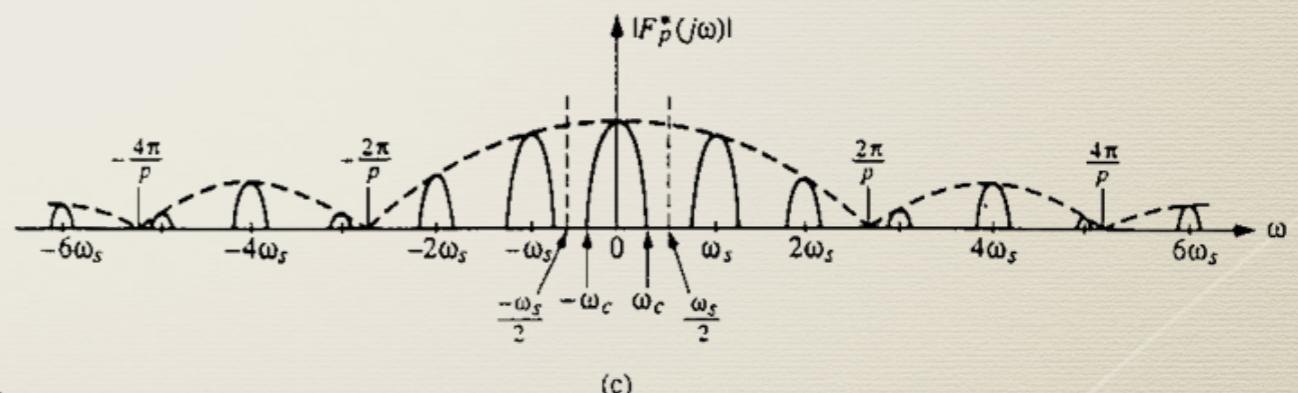
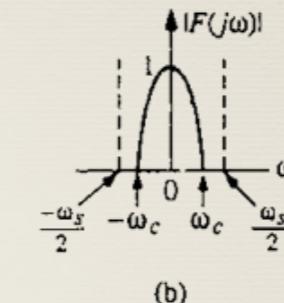
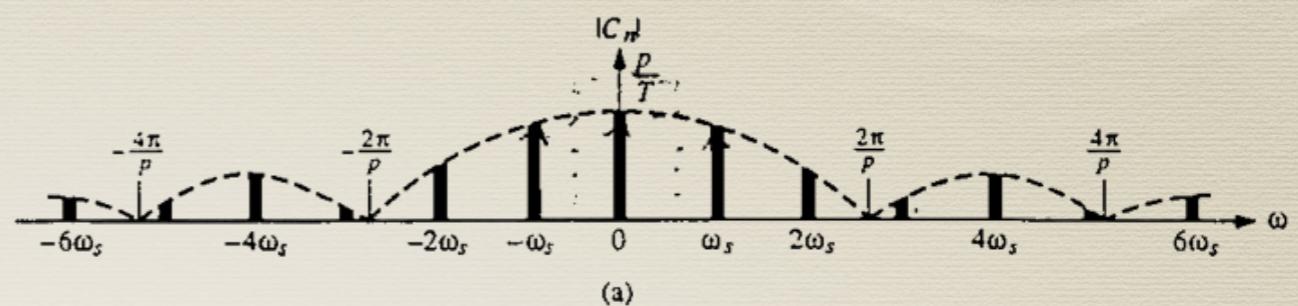
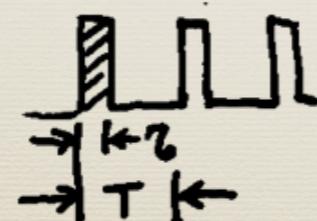


Espectro de amplitudes dos sinais de entrada e saída de um amostrador de largura de pulso finita.

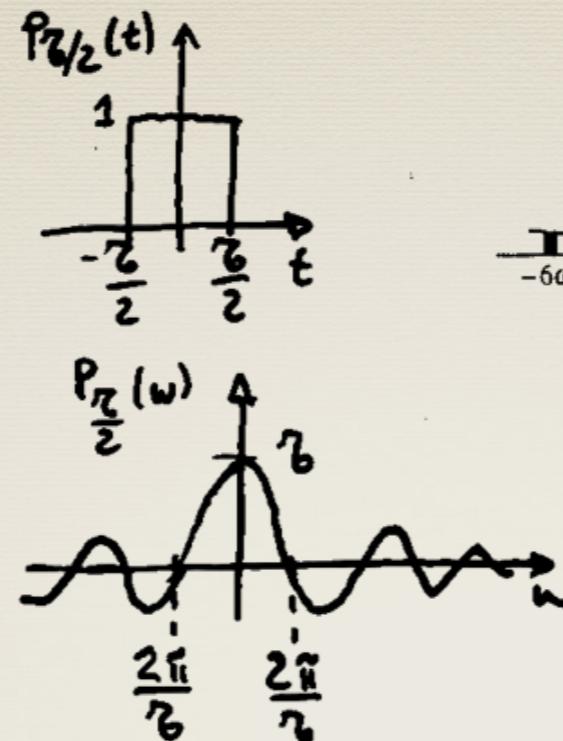
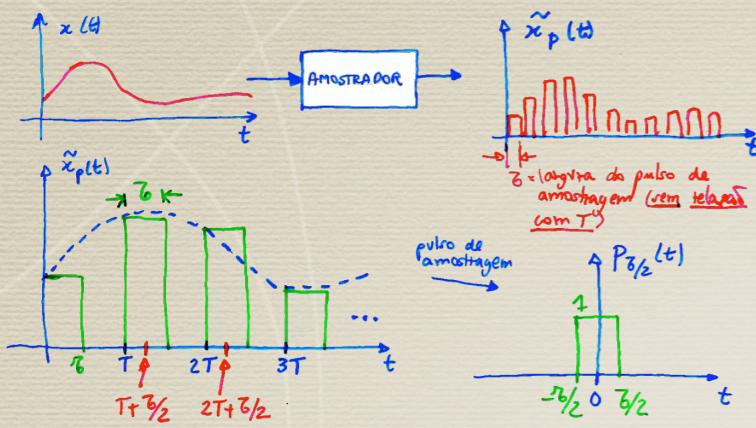
- Espectro de amplitude de um trem unitário de pulsos,  $p(t)$ ;
- Espectro de amplitudes de um sinal contínuo,  $f(t)$ ;
- Espectro de amplitude da saída amostrada ( $\omega_s > 2 \omega_c$ );
- Espectro de amplitude da saída amostrada ( $\omega_s < 2 \omega_c$ ).



$\Delta = T_s \therefore \text{Sample-and-Hold de 100\% (20K)}$

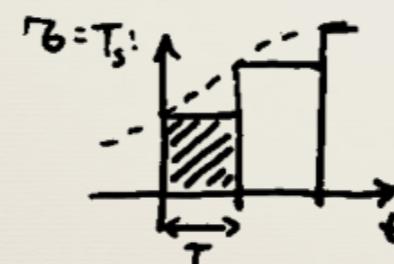


# Amostragem por trem de pulsos

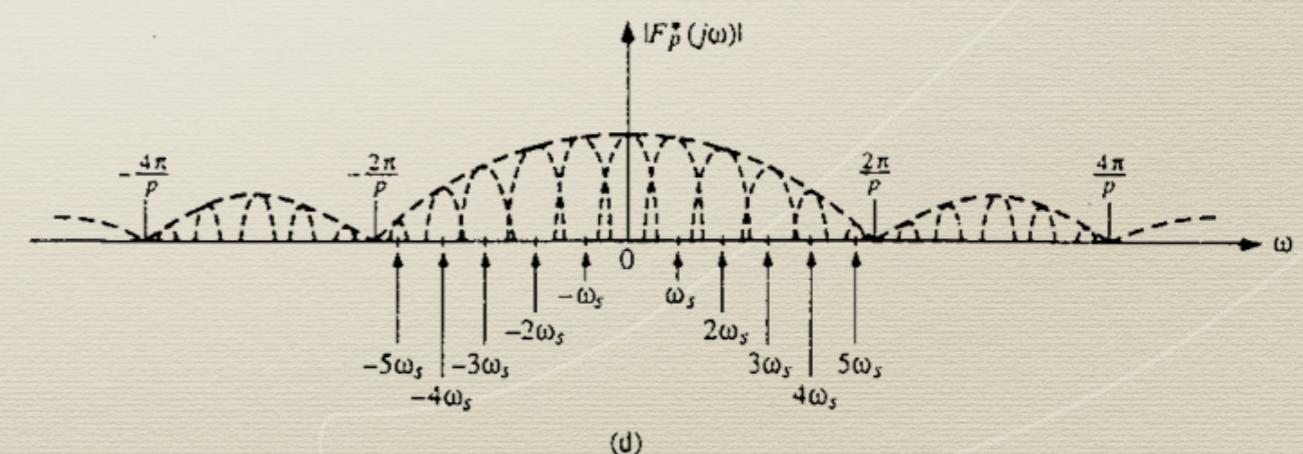
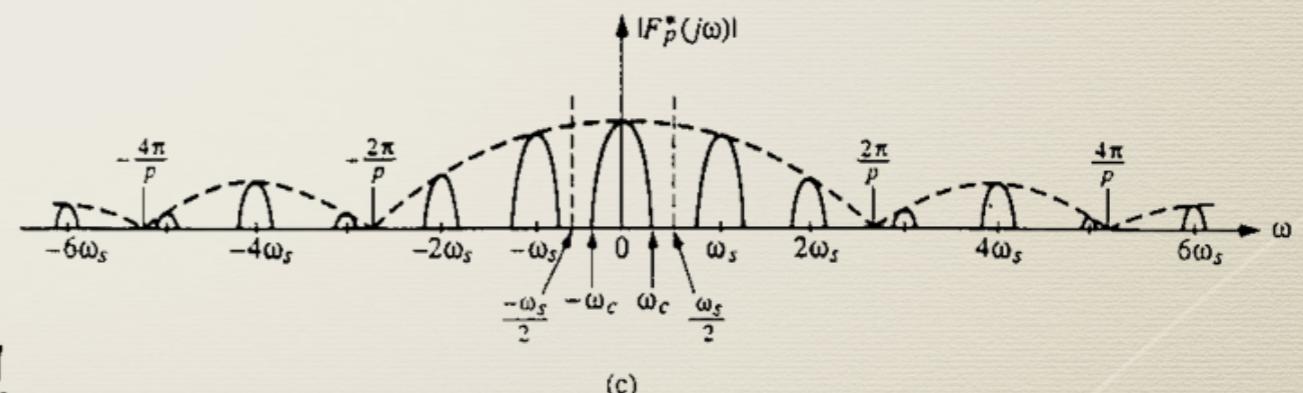
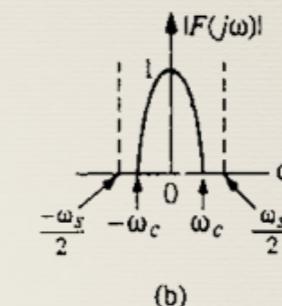
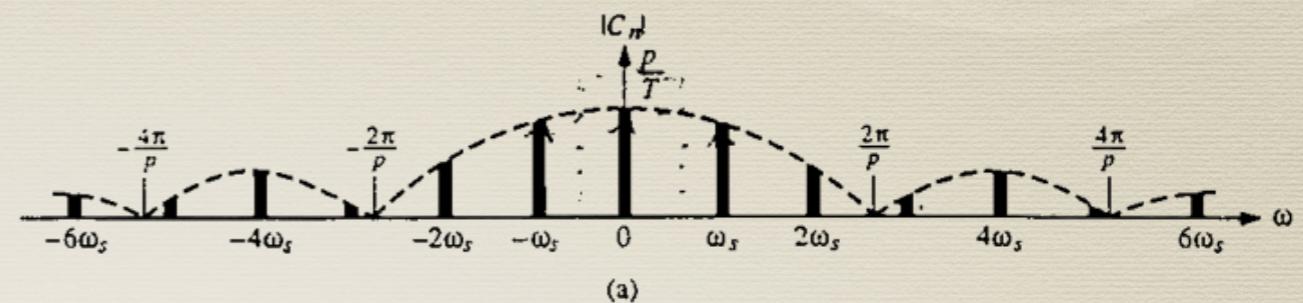
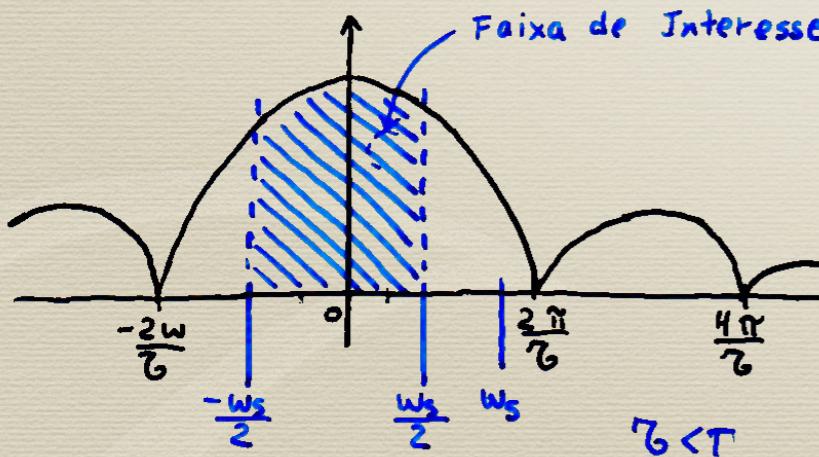


Espectro de amplitudes dos sinais de entrada e saída de um amostrador de largura de pulso finita.

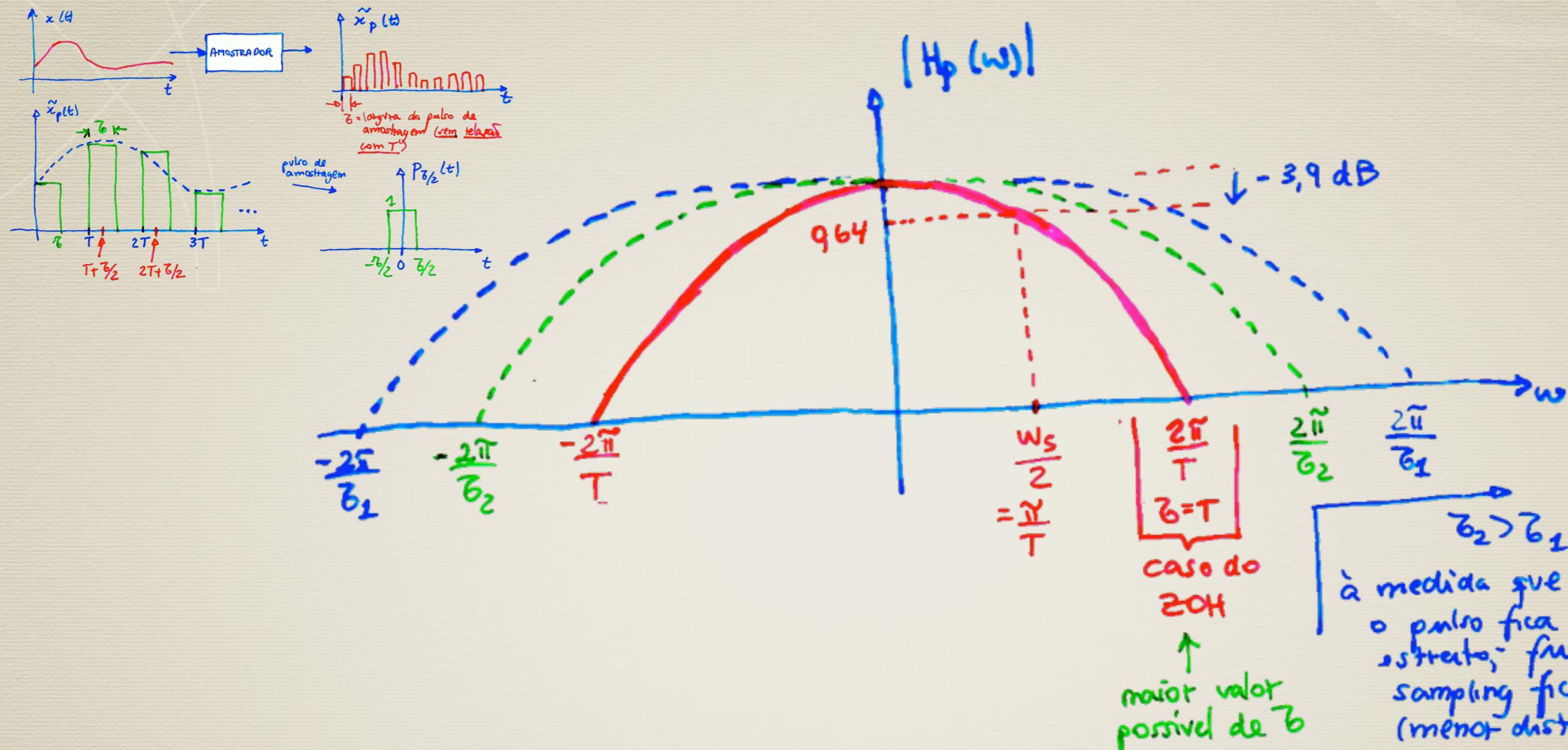
- Espectro de amplitude de um trem unitário de pulsos,  $p(t)$ ;
- Espectro de amplitudes de um sinal contínuo,  $f(t)$ ;
- Espectro de amplitude da saída amostrada ( $\omega_s > 2\omega_c$ );
- Espectro de amplitude da saída amostrada ( $\omega_s < 2\omega_c$ ).



$\Delta = T_s$ :  
T = \Delta : Sample-and-Hold de 100% (20K)



# Distorção na Faixa de Interesse



Quando:  $B = T \therefore$  "sample-and-Hold" de ordem zero (Z.O.H.)  
(100% da ciclo do trabalho)

- embora  $B = T$  gere o maior erro, é o mais usado por ser o mais simples de ser implementado, então:

$$\tilde{x}_p(w) = \frac{B}{T} \cdot S(w) \cdot \sum X(w - k \cdot w_s)$$

→ fator de atenuação do espectro.

# Sample-and-Holder

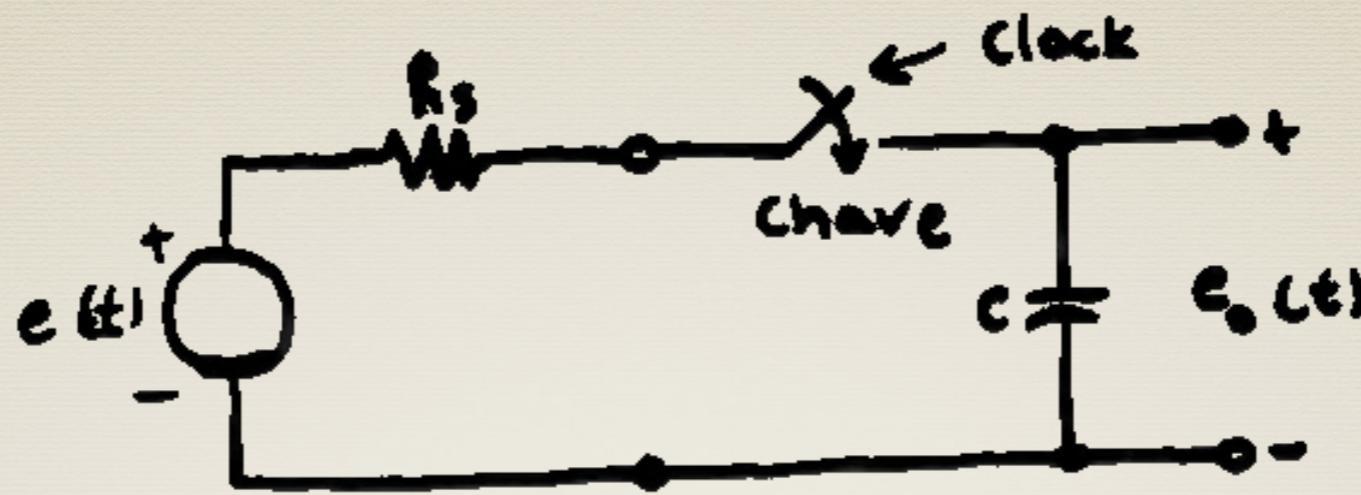


Fig. 2.8 - Circuito simples ilustrando o princípio de “sample-and-hold”.

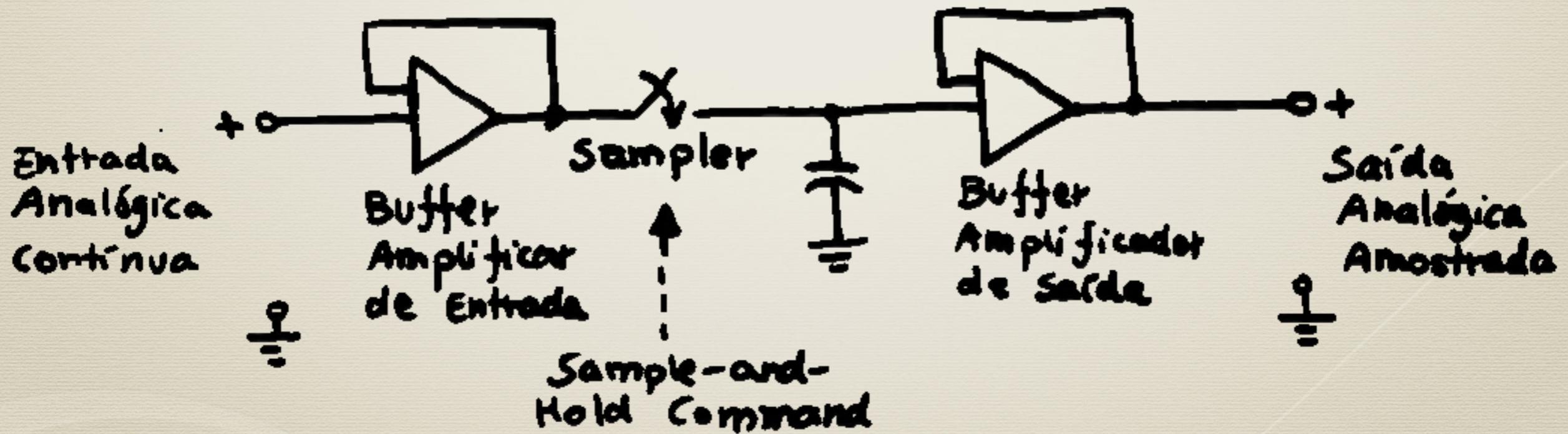


Fig. 2.9 - Dispositivo de “sample-and-hold”.

# Sample-and-Holder

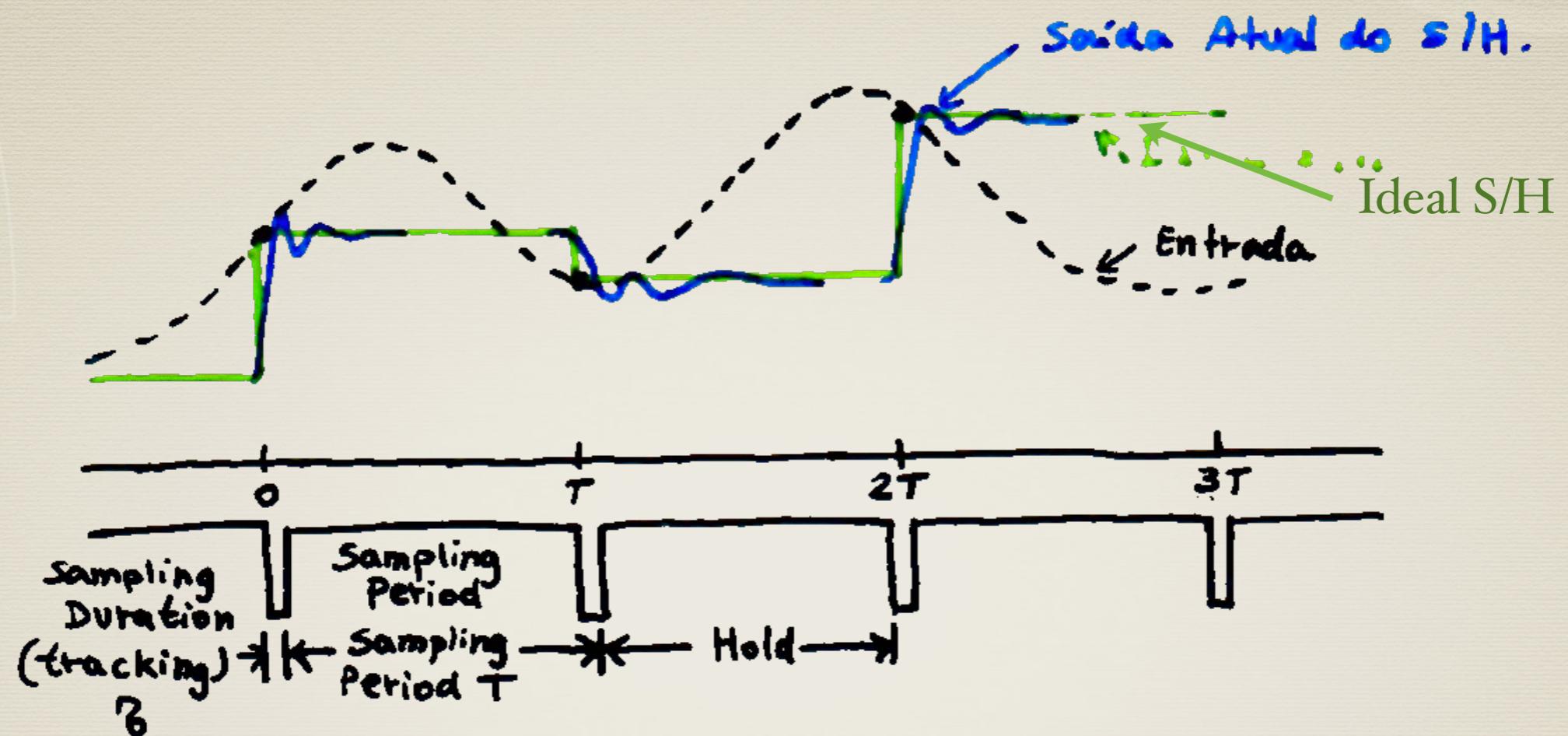


Fig. 2.10 - Entradas e saídas de um dispositivo de S/H com período de amostragem uniforme.

# Resposta em Frequência de um ZOH

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

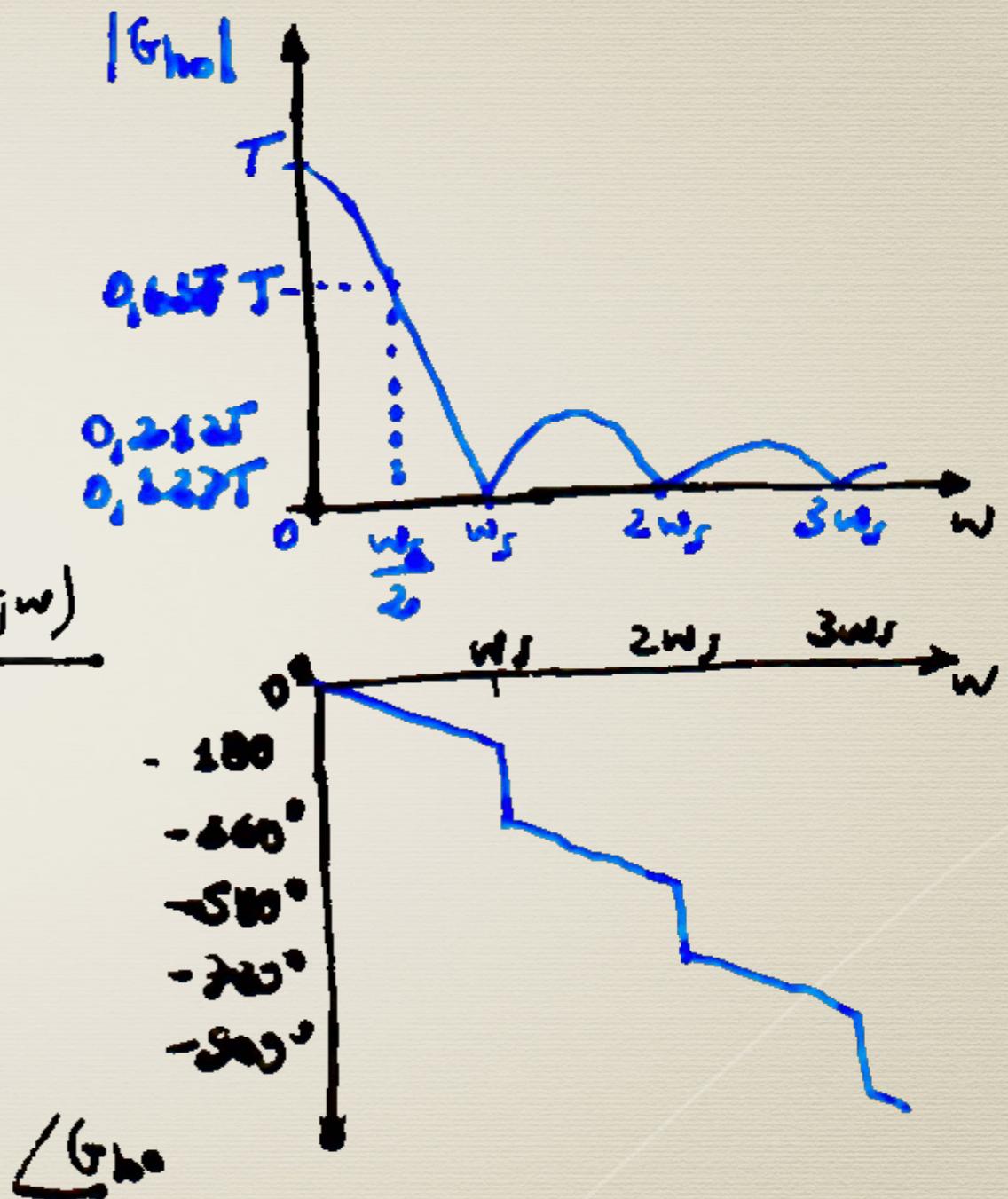
ou

$$\begin{aligned} G_{ho}(j\omega) &= \frac{1 - e^{-Tj\omega}}{j\omega} \\ &= \frac{j\omega}{2e^{-1/2 \cdot Tj\omega}} (e^{j\omega/2 \cdot Tj\omega} - e^{-j\omega/2 \cdot Tj\omega}) \end{aligned}$$

usando relações de Euler:

$$= T \cdot \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-1/2 \cdot Tj\omega}$$

$$|G_{ho}(j\omega)| = T \cdot \frac{\sin(\omega T/2)}{\omega T/2}$$



# Resposta em Frequência de um Filtro Ideal

Note que a transformada de Fourier do filtro ideal se assemelha à da Juntantada de ordem zero:

$$G_1(j\omega) = \begin{cases} 1, & -\frac{1}{2}w_s \leq \omega \leq \frac{1}{2}w_s \\ 0, & \text{outros casos} \end{cases}$$

[Fogata, p. 93]

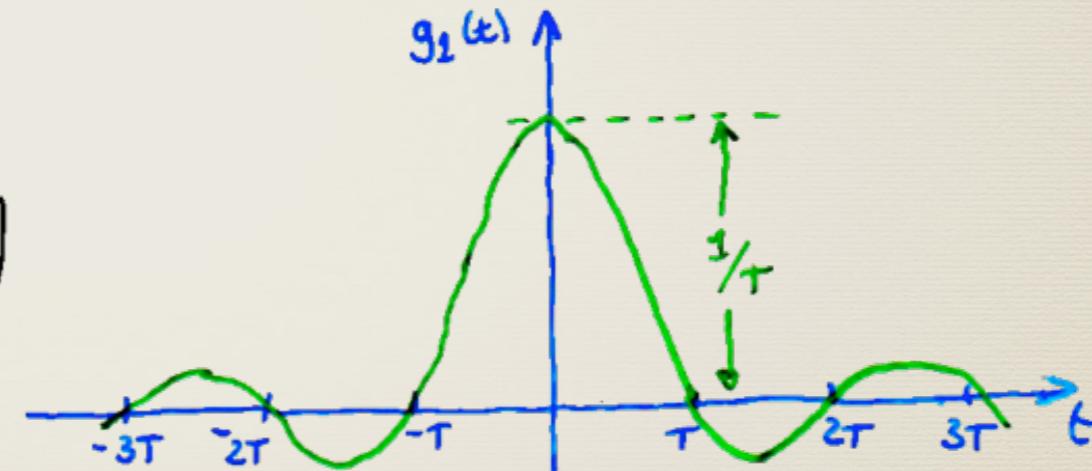
a transformada inversa de Fourier dá:

$$g_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_1(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-w_s/2}^{w_s/2} e^{j\omega t} \cdot d\omega$$

$$= \frac{1}{2\pi \cdot jt} \left( e^{\frac{1}{2}jw_s t} - e^{-\frac{1}{2}jw_s t} \right)$$

$$= \frac{1}{\pi t} \cdot \sin \frac{w_s t}{2}$$

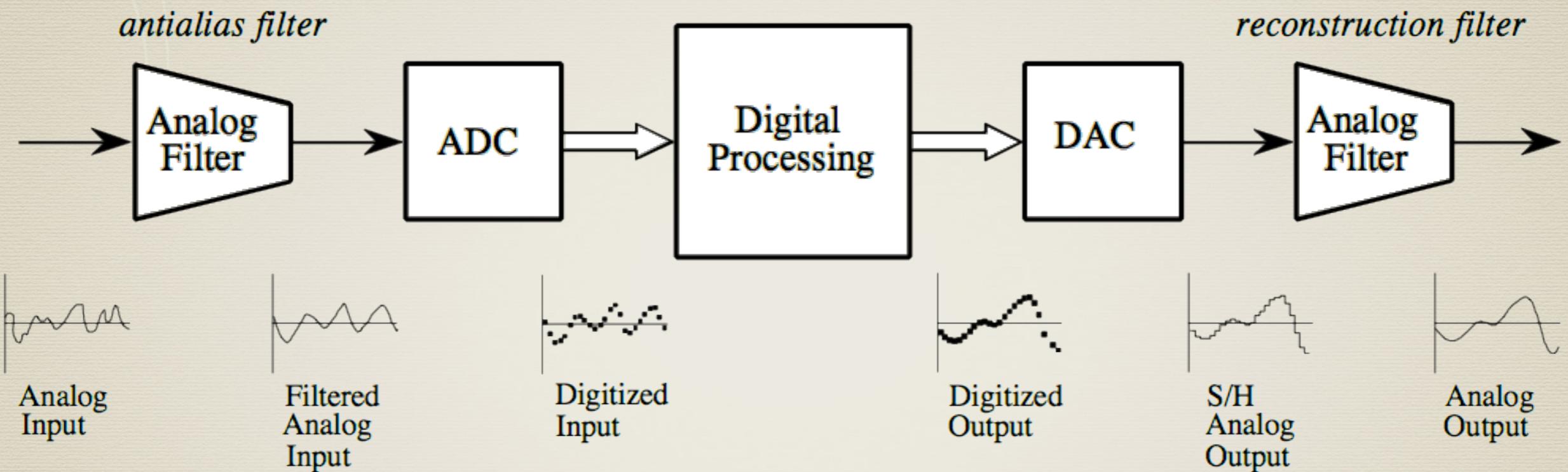


ou

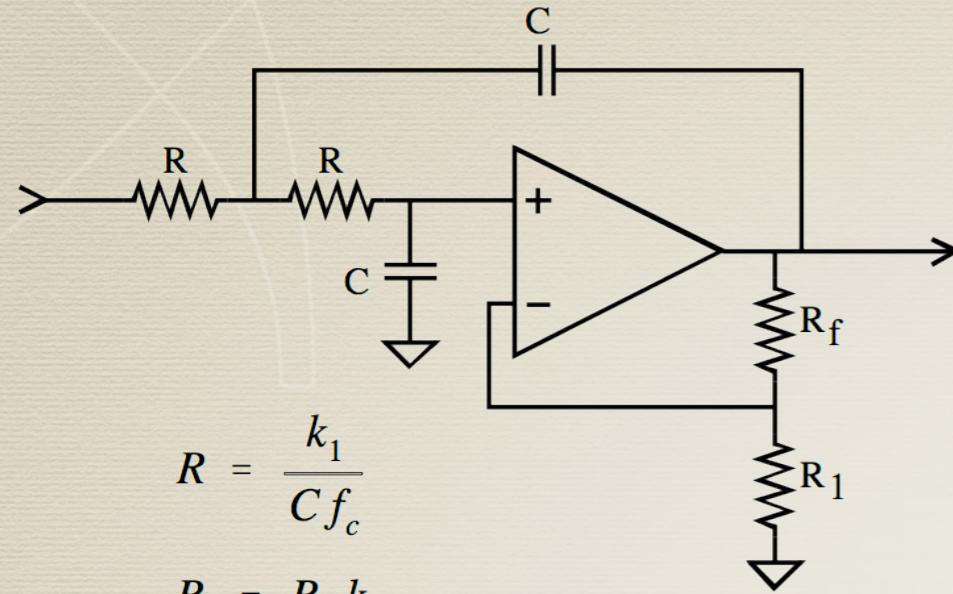
$$g_1(t) = \frac{1}{T} \cdot \frac{\sin(w_s t / 2)}{w_s t / 2} \quad \therefore \text{resposta ao impulso de um Filtro PB ideal.}$$

Note que a resposta se estende de  $t = -\infty$  até  $t = +\infty$ . Isto é, existe (!?) uma resposta para  $t < 0$  para um impulso aplicado em  $t = 0$ . Obviamente isto não é realizável no mundo físico real.

# Filtros Analógicos



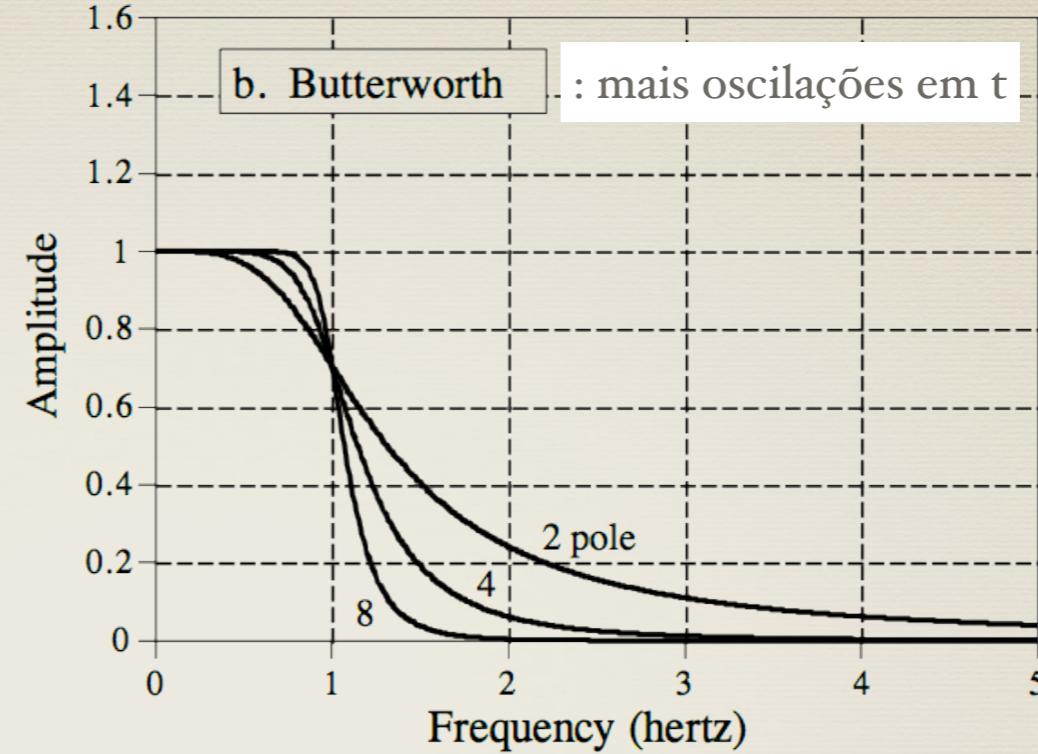
# Filtros Analógicos



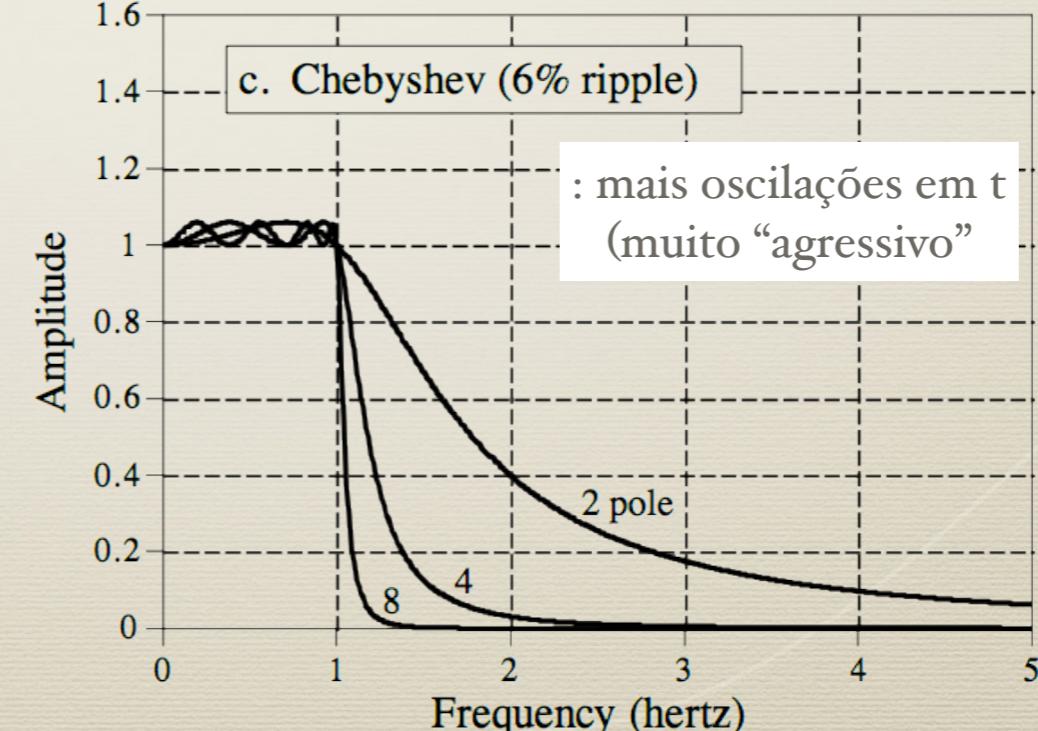
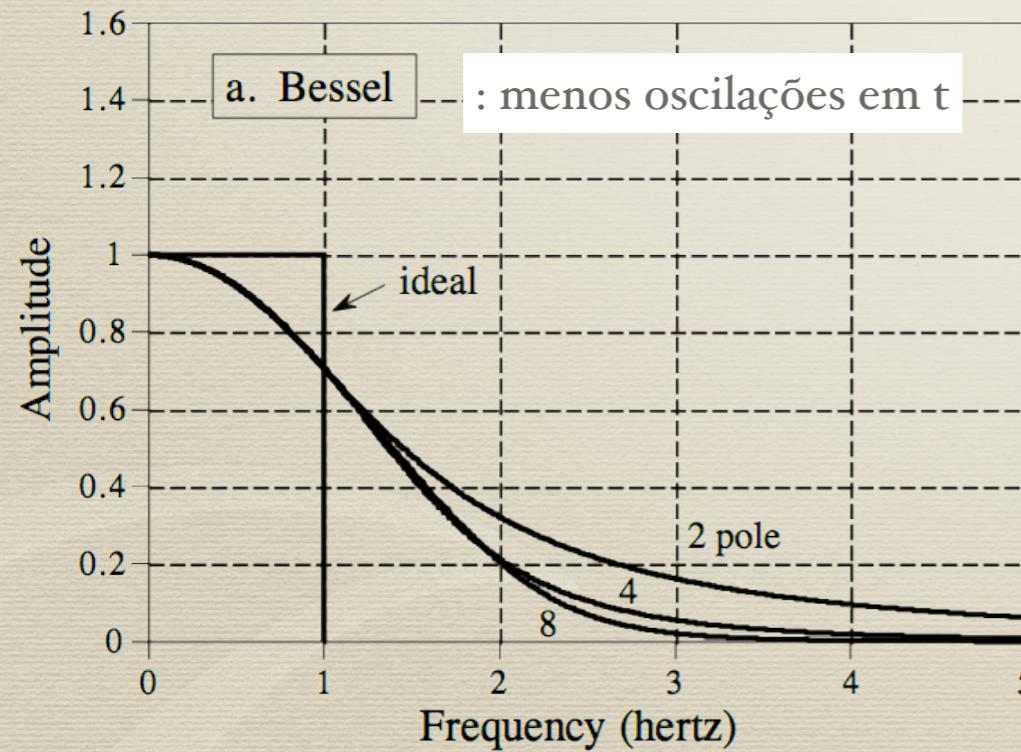
$$R = \frac{k_1}{C f_c}$$

$$R_f = R_1 k_2$$

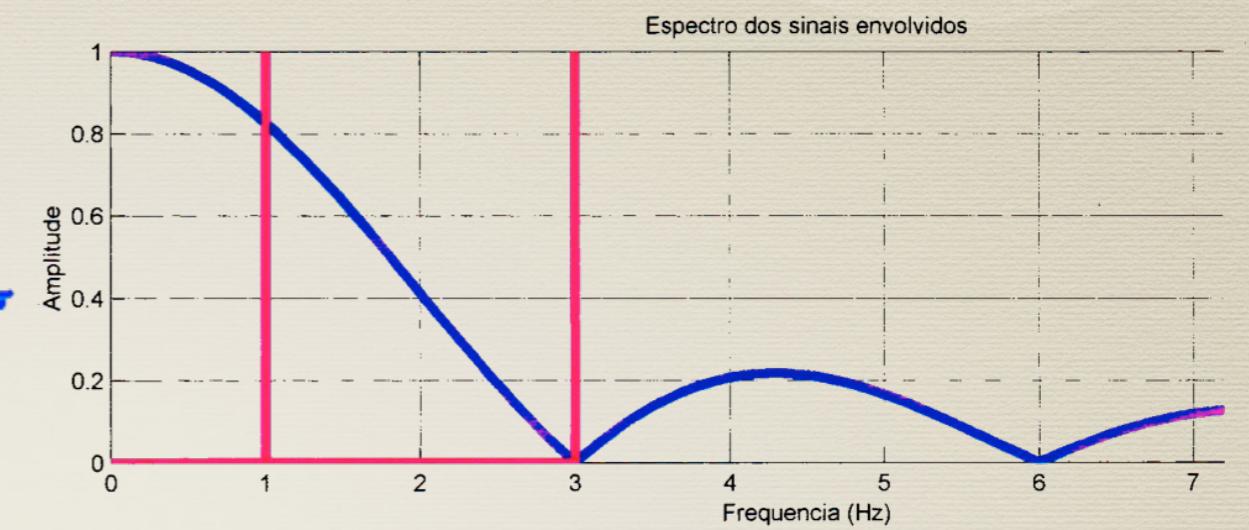
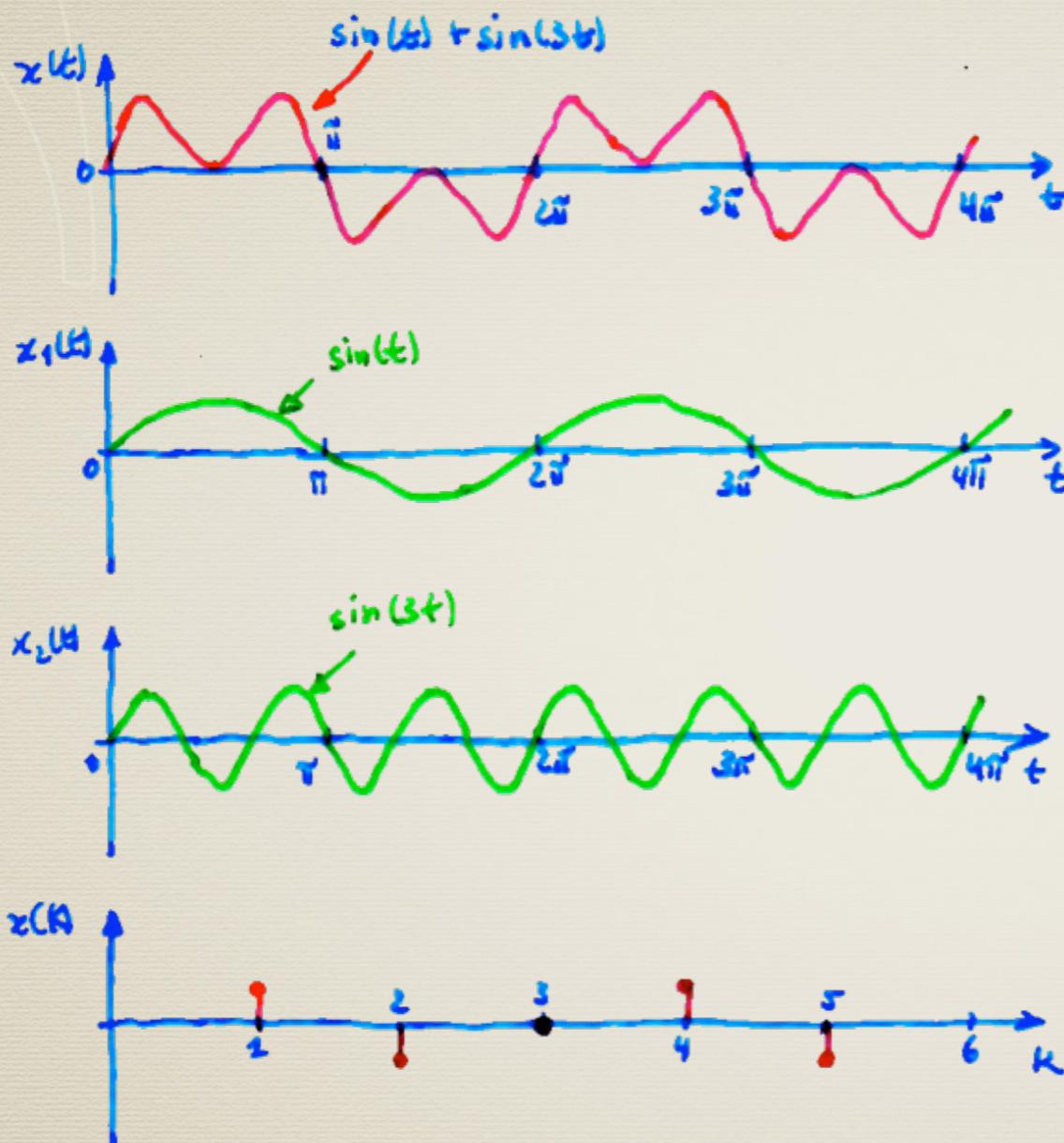
Filtro passa-baixa,  
estrutura Sallen-Key,  
de 2a-ordem.



Obs.: Escalas (freq.) lineares!



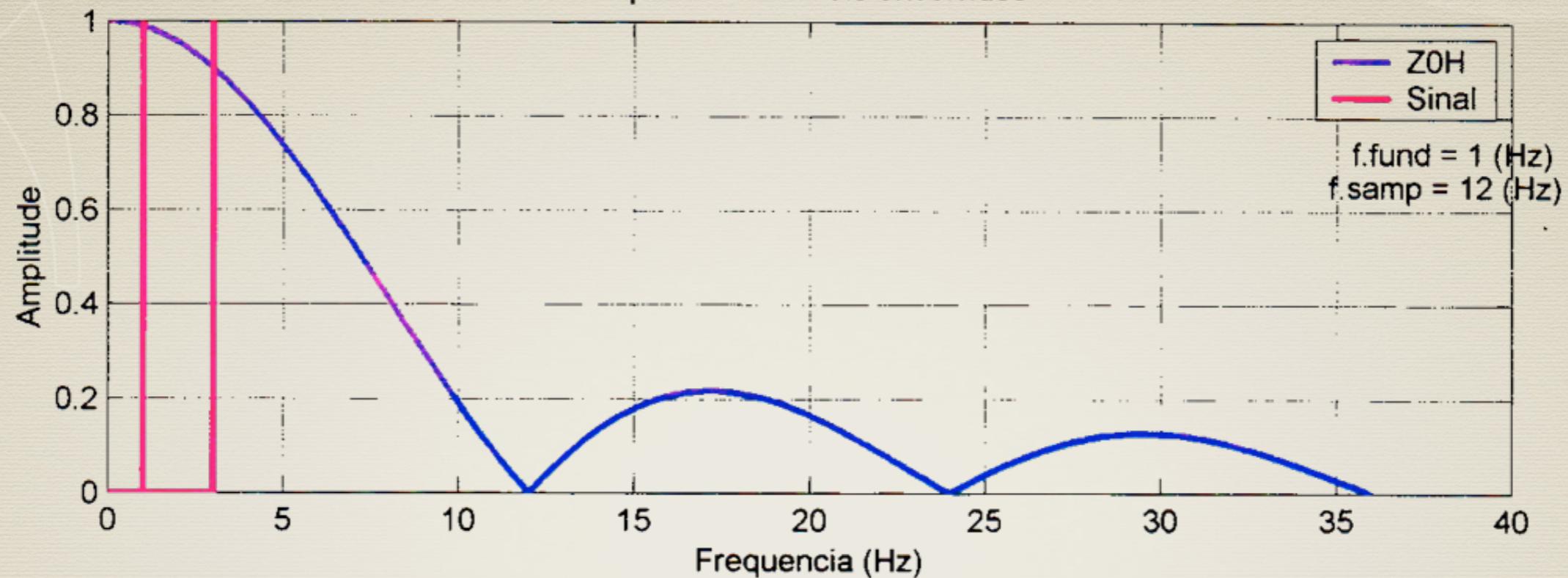
# Simulações:



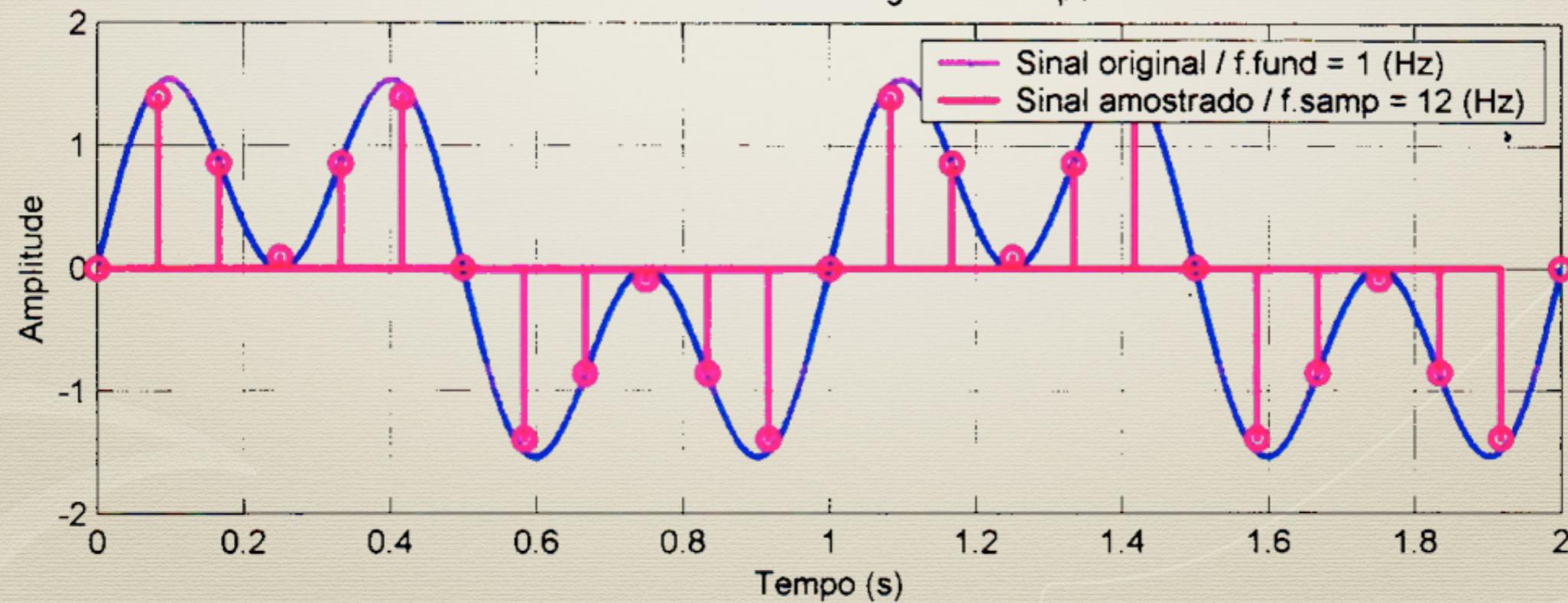
gráficos de  $x(t) = \sin(\omega t) + 3\sin(3\omega t)$ ;  $x_1(t) = \sin(\omega t)$ ;  $x_2(t) = \sin(3\omega t)$ ,  
 sinal Amortiado:  $x(t) = e^{-kt} \sin(\omega t)$ , onde  $\omega = 3 \text{ rad/s}$ .  
 Note que não aparece a frequência:  $\omega = 3 \text{ rad/s}$ .

# Simulações:

Espectro dos sinais envolvidos

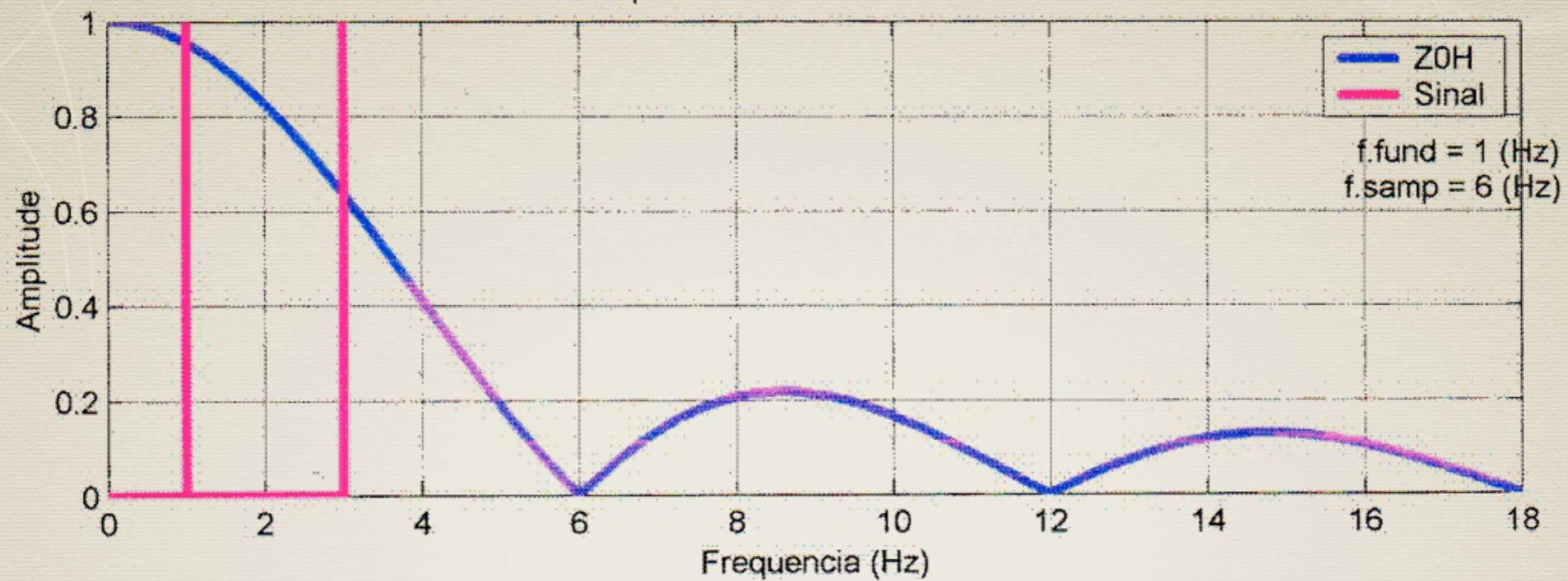


Resultado da amostragem no tempo

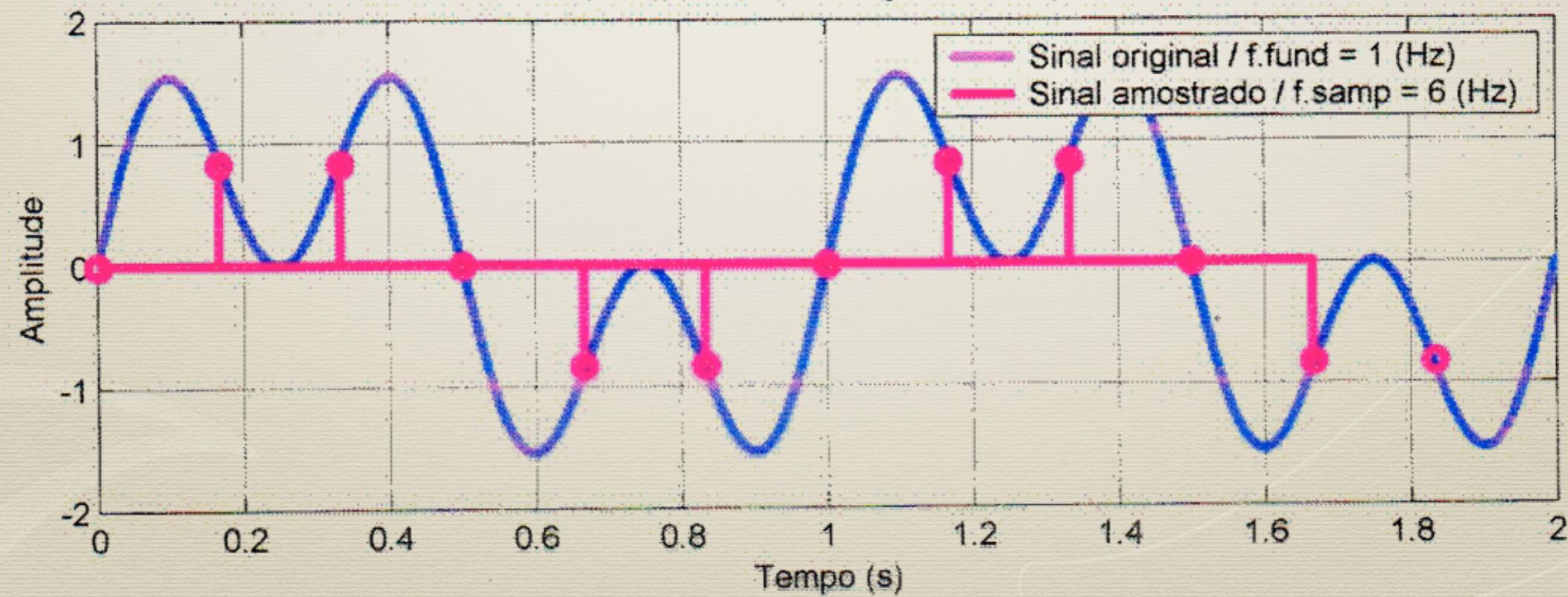


# Simulações:

Espectro dos sinais envolvidos

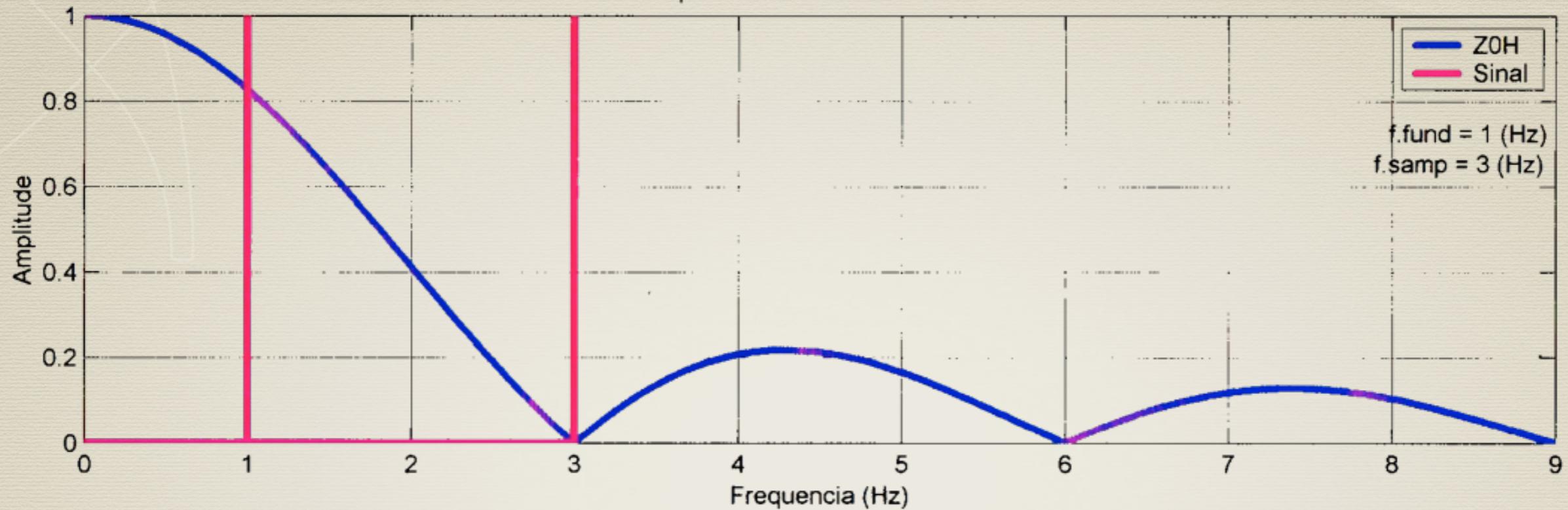


Resultado da amostragem no tempo

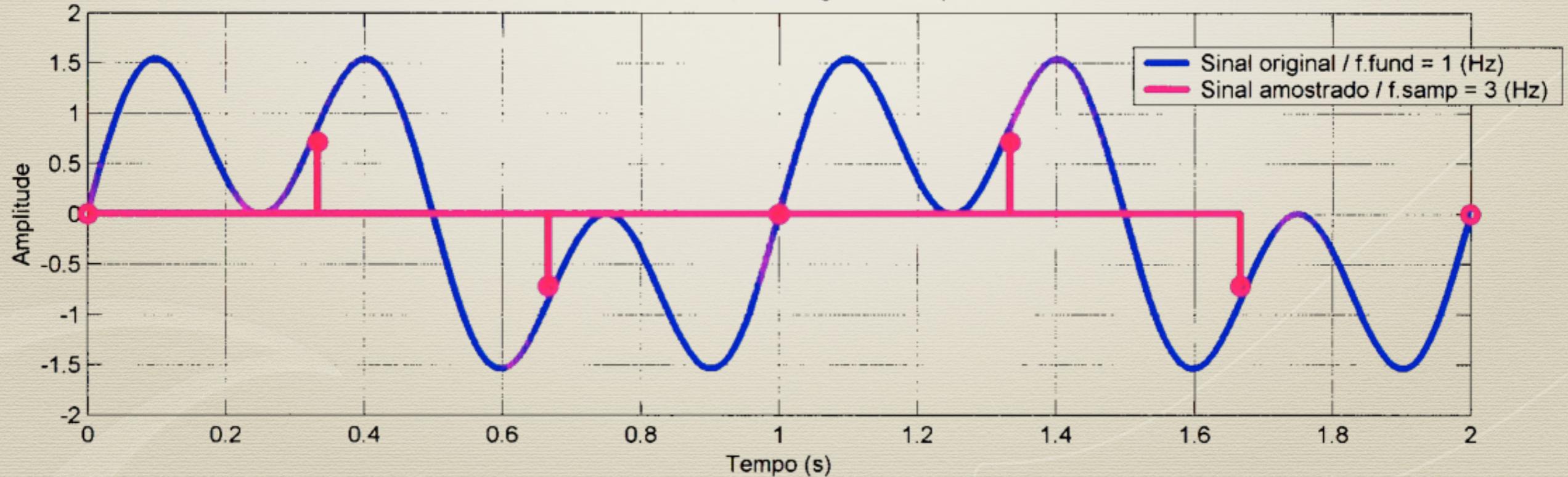


# Simulações:

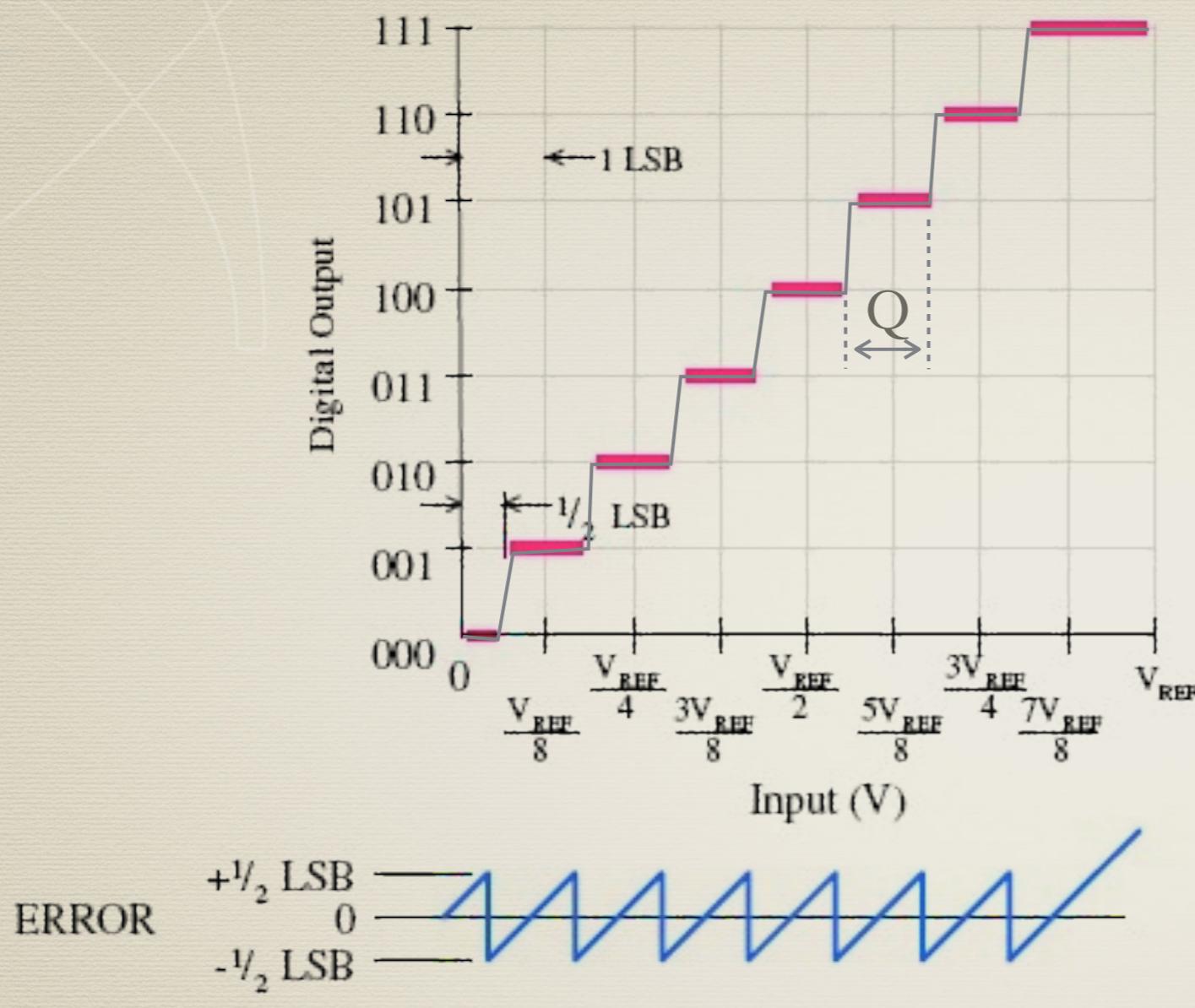
Espectro dos sinais envolvidos



Resultado da amostragem no tempo



# Erro de Quantização



Erro de Quantização,  $\mathcal{Q}$ :

$$Q = \frac{FSR}{2^n}$$

onde:

FSR = Full Scale Range (maior faixa de entrada);

n=No. de bits.

No caso:

$V_{REF}=10$  Volts,

$$Q = \frac{10}{2^3} = 1,25 V$$

Erro na amostragem:

$\pm 0,625 V (\pm 0,195\%)$

Obs.: As maiores fontes de erro num sistema de aquisição digital se concentram no circuito de entrada que condiciona o sinal (filtro), limita sua escala (escala) e no circuito de sample-and-hold. Maiores que os comparados ao erro inerente à quatização do sinal de entrada.