

Diagrama de Bode

Sinais & Sistemas I
Prof. Fernando Passold
Nov/2023

Tópicos previstos

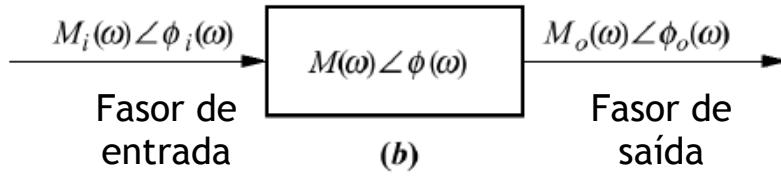
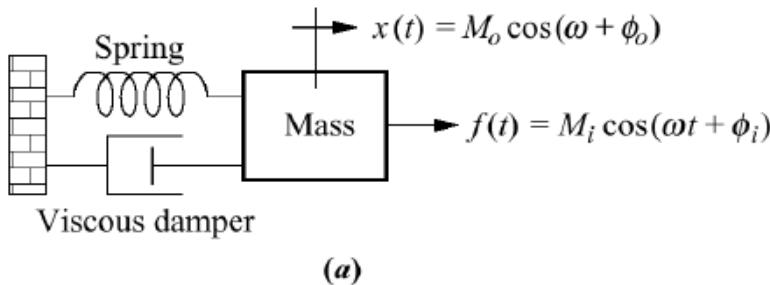
- ▶ Intro: o que é Resposta em Frequência;
 - ▶ Como graficar resposta em frequência;
 - ▶ Usando Matlab para obter diagramas de Bode;
 - ▶ Como usar resposta em frequência para analisar resposta transitória de um sistema;
- ▶

Introdução

- ▶ Estabilidade e projeto da resposta transitória mediante ajuste de ganho:
 - ▶ Métodos baseados em resposta em frequência, diferentes do método baseado em RL, podem ser realizados sem a obrigatoriedade de uma ferramenta computacional usando aproximações assintóticas.
- ▶ O projeto da resposta transitória mediante compensação em cascata:
 - ▶ Métodos baseados em resposta em frequência não são tão intuitivos como os baseados em RL.
- ▶ Projeto dos erros de estado estacionário mediante compensação en cascata:
 - ▶ Métodos baseados em resposta em frequência facilitam o projeto de compensadores derivativos de forma a acelerar a resposta do sistema ao mesmo tempo respeitando requerimentos de erros de regime permanente.

Resposta em frequência: Definição...

- Ondas sinusoidais podem ser representadas como números complexos chamados fasores

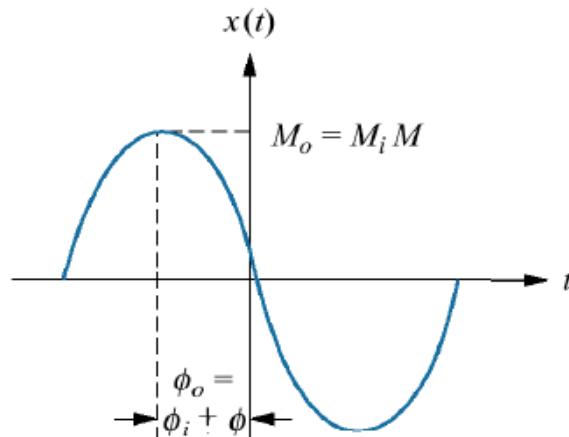
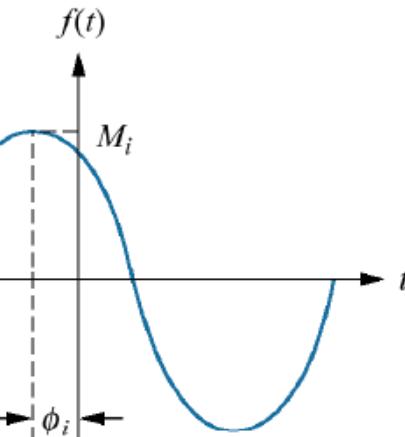


$f(t)$ = entrada de força, sinusoidal neste caso:

Definição:

$$M(w) = \frac{M_o(w)}{M_i(w)}$$

$$\phi(w) = \phi_o(w) - \phi_i(w)$$



Definições Matemáticas...

Relações de Euler:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

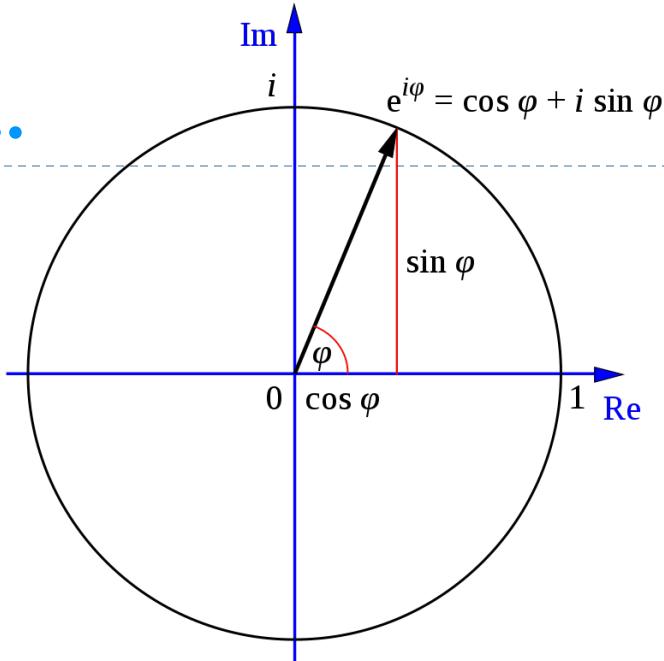
$$z = x + jy = |z| [\cos(\varphi) + j \sin(\varphi)] = r e^{j\varphi}$$

x = Parte real de z ;

y = Parte Imaginária de z ;

$$r = |z| = \sqrt{x^2 + y^2} = \text{magnitude de } z;$$

$$\varphi = \text{atan2}(y, x)$$



- ▶ Definição: $M(w)\angle\phi(w)$
- ▶ Magnitude: $M(w) = M_o(w)/M_i(w)$
- ▶ Fase: $\phi(w) = \phi_o(w) - \phi_i(w)$

- ▶ Formatos de expressão:

$$r(t) = A \cos(wt) + B \sin(wt) = \sqrt{A^2 + B^2} \cos\left[wt - \tan^{-1}(B/A)\right]$$

1. Forma Polar: $M_i\angle\phi_i$
2. Forma Retangular: $A - jB$
3. Equação de Euler: $M_i e^{j\phi_i}$



Expressões analíticas p/ resposta em frequêcia

Seja o sistema mostrado na fig. ao lado.

Este sistema é excitado por uma entrada sinusoidal:

$$r(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$r(t) = \sqrt{A^2 + B^2} \cos [\omega t - \tan^{-1}(B/A)]$$

$$r(t) = M_i \angle \phi_i = M_i e^{j\phi_i}, \text{ onde: } M_i = \sqrt{A^2 + B^2} \text{ e } \phi_i = -\tan^{-1}(B/A)$$

A resposta forçada do sistema $G(s)$ para esta entrada resulta em:

$$C(s) = \frac{As + B\omega}{(s^2 + \omega^2)} \cdot G(s)$$

Expandindo usando frações parciais, teremos:

$$C(s) = \frac{As + B\omega}{(s + j\omega)(s - j\omega)} = \frac{K_1}{s + j\omega} + \frac{K_2}{s - j\omega} + \text{Termos frações parciais de } G(s)$$

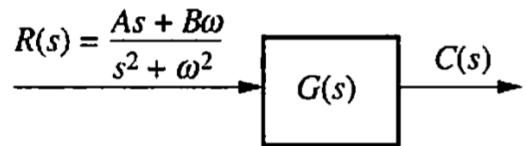
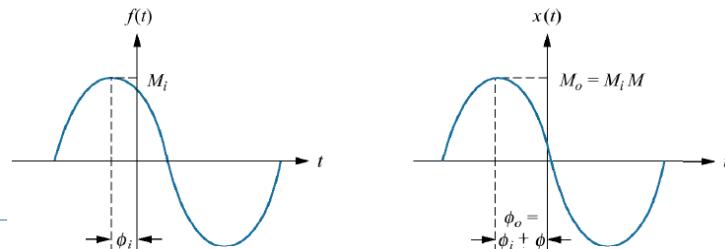
$$K_1 = \left. \frac{As + B\omega}{s - j\omega} G(s) \right|_{s \rightarrow -j\omega} = \frac{1}{2}(A + jB)G(-j\omega) = \frac{1}{2}M_i e^{-j\phi_i} M_G e^{-j\phi_G}$$

$$K_1 = \frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}$$

$$K_2 = \left. \frac{As + B\omega}{s + j\omega} G(s) \right|_{s \rightarrow -j\omega} = \frac{1}{2}(A - jB)G(j\omega) = \frac{1}{2}M_i e^{j\phi_i} M_G e^{j\phi_G}$$

$$K_2 = \frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)} = K_1^*$$

onde: $M_G = |G(j\omega)|$ e $\phi_G = \angle G(j\omega)$.



Então $C(s)$ resulta:

$$C(s) = \frac{\frac{M_i M_G}{2} e^{-j(\phi_i + \phi_G)}}{s + j\omega} + \frac{\frac{M_i M_G}{2} e^{j(\phi_i + \phi_G)}}{s - j\omega}$$

Cuja transformada inversa de Laplace rende:

$$c(t) = M_i M_G \left(\frac{e^{-j(\omega t + \phi_i + \phi_g)} + e^{j(\omega t + \phi_i + \phi_g)}}{2} \right)$$

$$c(t) = M_i M_G \cos(\omega t + \phi_i + \phi_G)$$

$$c(t) = M_o \angle \phi_o = (M_i \angle \phi_i) \cdot (M_G \angle \phi_G)$$

Note que:

$M_G \angle \phi_G$ = resposta em frequêcia.

Ou seja:

$$G(j\omega) = G(s) \Big|_{s \rightarrow j\omega}$$

Exemplo 1: $G(s) = \frac{1}{s+2}$

Como: $s = j\omega$, para obter: $G(j\omega) = G(s)|_{s \rightarrow j\omega}$

$$G(j\omega) = \frac{1}{j\omega + 2} = \frac{1}{2 + j\omega}$$

$$G(j\omega) = \frac{1}{j\omega + 2} = \frac{1}{2 + j\omega} \cdot \frac{2 - j\omega}{2 - j\omega} = \frac{2 - j\omega}{4 + \omega^2}$$

Magnitude: $|G(j\omega)| = \frac{1}{|2 + j\omega|} = \frac{1}{\sqrt{2^2 + \omega^2}} = \frac{1}{\sqrt{\omega^2 + 4}}$

Magnitude(dB): $|G(j\omega)| \Big|_{dB} = 20 \log \left(\frac{1}{\sqrt{\omega^2 + 4}} \right)$

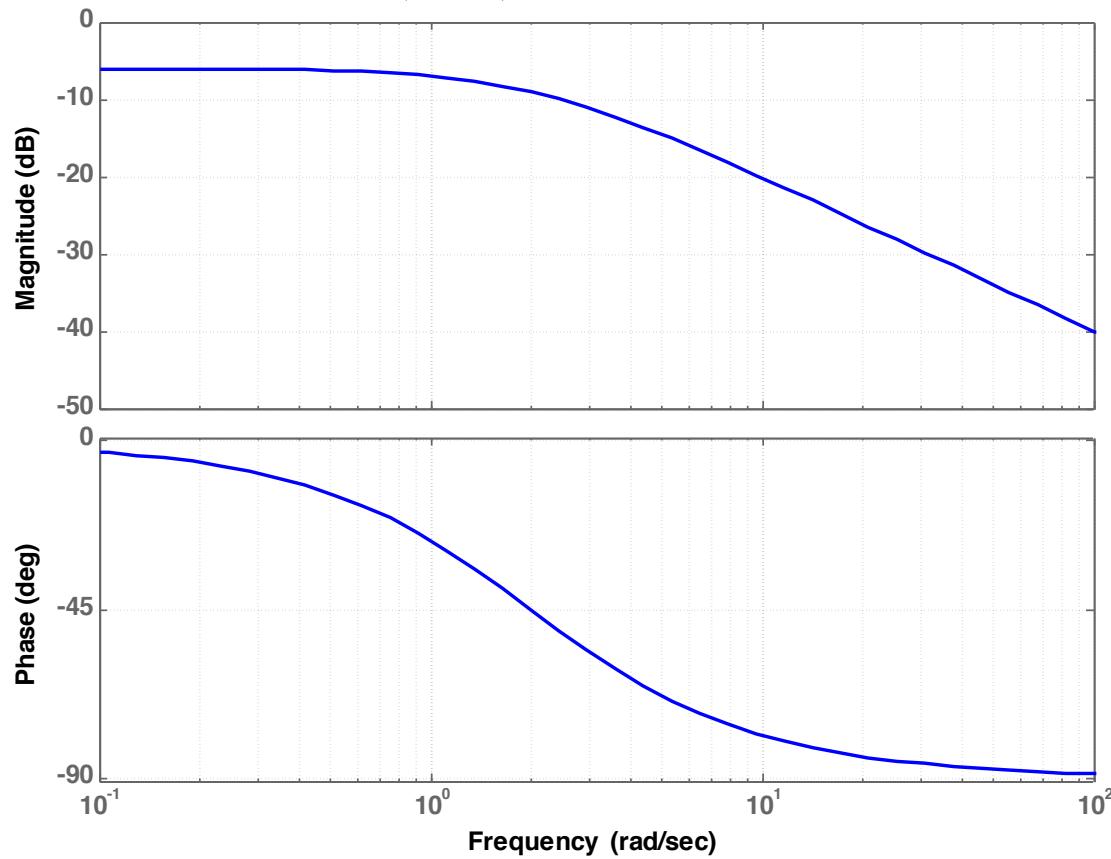
Fase: $\angle G(j\omega) = \frac{1}{\tan^{-1}(\omega/2)} = -\tan^{-1} \left(\frac{\omega}{2} \right)$

Exemplo 1:

$$G(s) = \frac{1}{s + 2}$$

Grafico de Magnitude: $= 20 \log(1/\sqrt{\omega^2 + 4})$

Grafico de Fase: $= -\tan^{-1}(\omega/2)$



```
>> clear all  
>> numg=1;  
>> deng=[1 2];  
>> g=tf(numg,deng);  
>> zpk(g)
```

Zero/pole/gain:
1

(s+2)

```
>> bode(g), grid
```

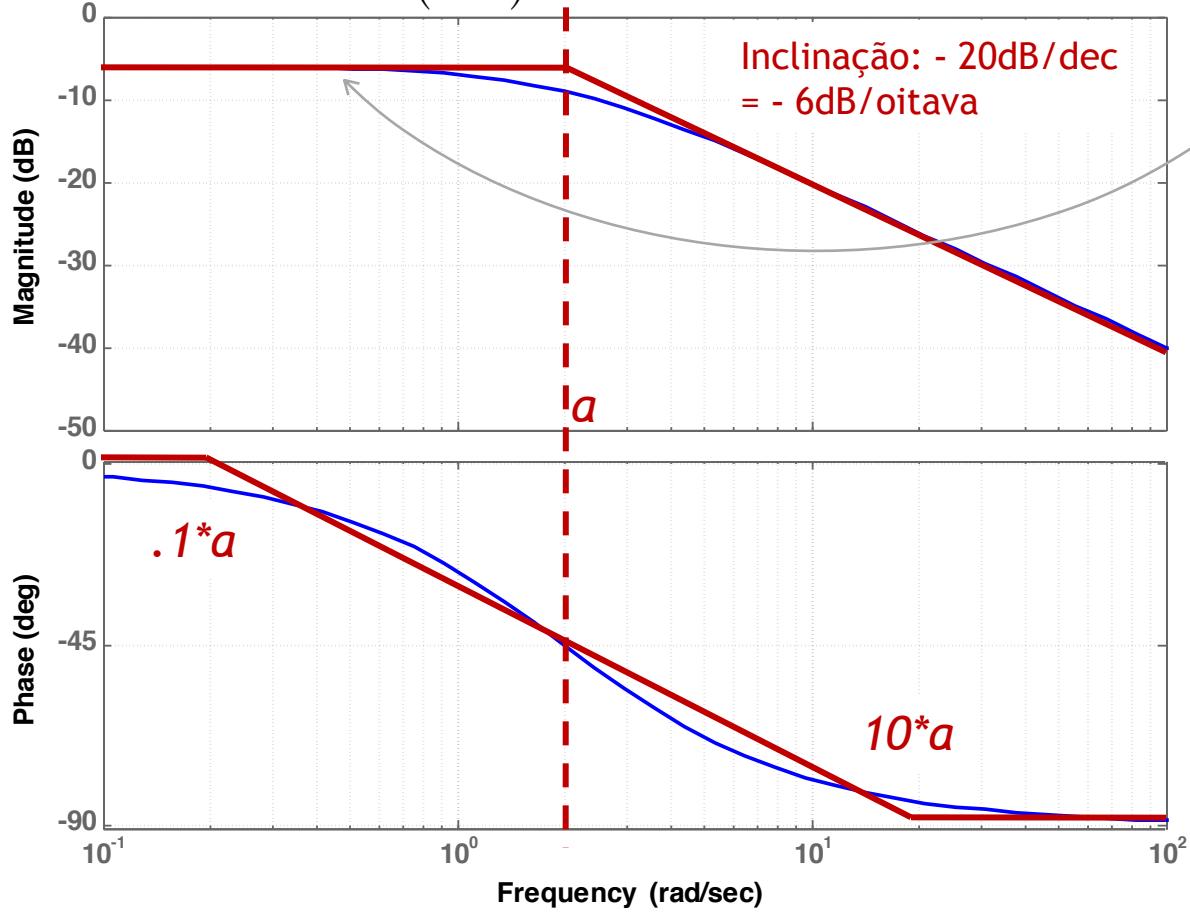
Ref: URL: <http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/BodeHow.html>

Assintóticamente:

$$\text{Exemplo 1: } G(s) = \frac{1}{s+2}$$

Grafico de Magnitude: $= 20 \log(1/\sqrt{\omega^2 + 4})$ ou $= 20 \log\left(1/\sqrt{\omega^2 + a^2}\right)$

Grafico de Fase: $= -\tan^{-1}(\omega/2)$ **Bode Diagram**



$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a}+1\right)}$$

Para baixas freq. ($j\omega < a$; suponha $w=a/10$):

$$|G(j\omega)| = 20 \log\left(\frac{1}{\sqrt{\left(\frac{a}{10}\right)^2 + a^2}}\right) = 20 \log\left(\frac{1}{\sqrt{\frac{101a^2}{100}}}\right)$$

$$|G(j\omega)| \cong 20 \log\left(\frac{1}{a}\right) = -20 \log(a)$$

$$|G(j\omega)| = -20 \log(2) = -6,0205999133 \text{ dB}$$

Lembrando que:

$$\log_a b = x \quad \therefore \quad a^x = b$$

$$\log(1) = 0$$

$$\log(2) = 0,3010299957$$

$$\log(10) = 1$$

$$\log(100) = 2 \quad \therefore \quad 10^2 = 100$$

$$\log(1/10) = -\log(10) = -1$$

$$\log(a \cdot b) = \log(a) + \log(b)$$

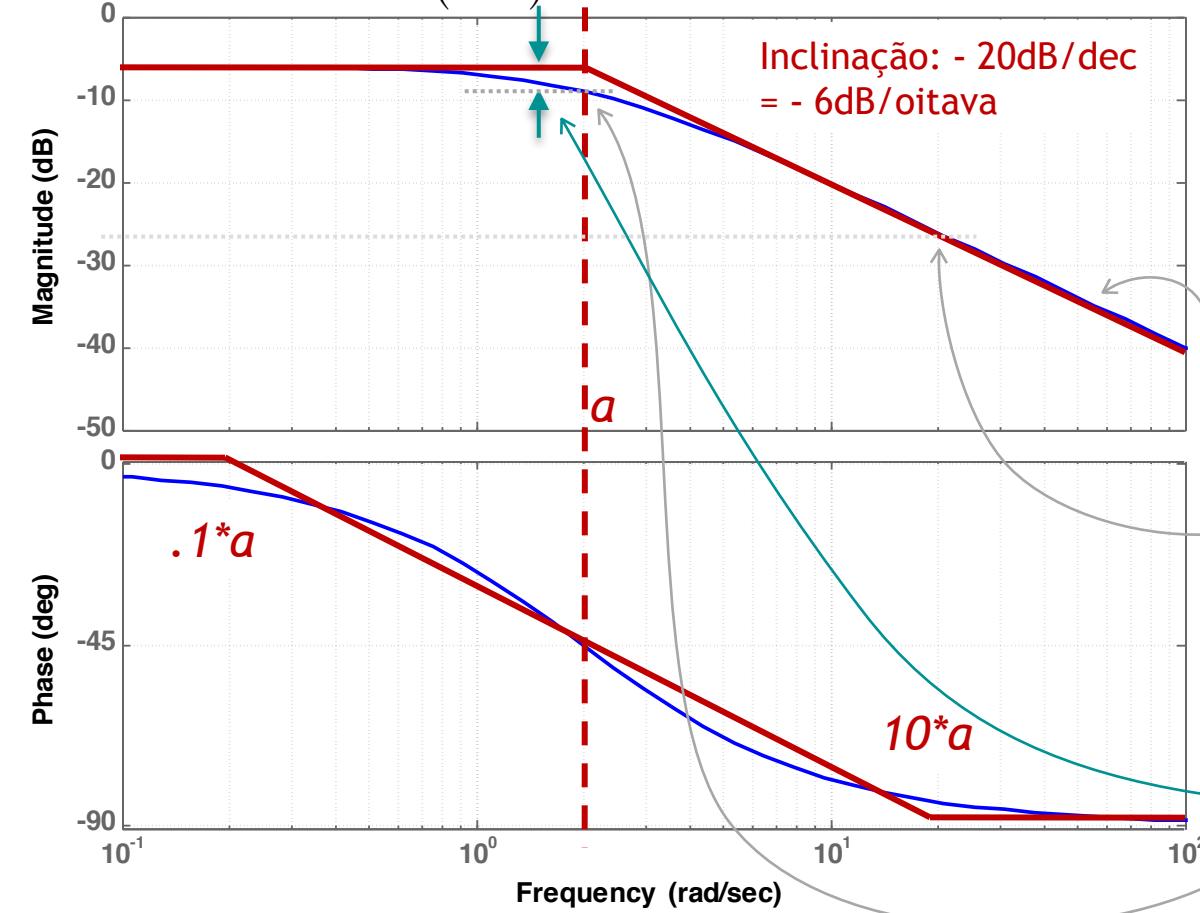
$$\log(a/b) = \log(a) - \log(b)$$

Assintóticamente:

$$\text{Exemplo 1: } G(s) = \frac{1}{s + 2}$$

Grafico de Magnitude: $= 20 \log(1/\sqrt{\omega^2 + 4})$ ou $= 20 \log\left(1/\sqrt{\omega^2 + a^2}\right)$

Grafico de Fase: $= -\tan^{-1}(\omega/2)$ **Bode Diagram**



$$G(s) = \frac{1}{s + a} = \frac{1}{a\left(\frac{s}{a} + 1\right)}$$

Para baixas freq. ($j\omega < a$; suponha $w=a/10$):

$$|G(j\omega)| = 20 \log\left(\frac{1}{\sqrt{\left(\frac{a}{10}\right)^2 + a^2}}\right) = 20 \log\left(\frac{1}{\sqrt{\frac{101a^2}{100}}}\right)$$

$$|G(j\omega)| \cong 20 \log\left(\frac{1}{a}\right) = -20 \log(a)$$

$$|G(j\omega)| = -20 \log(2) = -6,0205999133 \text{ dB}$$

Para altas freq. ($j\omega > a$; suponha $w=10a$):

$$|G(j\omega)| = 20 \log\left(\frac{1}{\sqrt{(10a)^2 + a^2}}\right) = 20 \log\left(\frac{1}{\sqrt{101a^2}}\right)$$

$$|G(j\omega)| \cong 20 \log\left(\frac{1}{\sqrt{100a}}\right) \cong 20 \underbrace{\log(1)}_0 - 20 \log(10a)$$

$$|G(j\omega)| \cong -20 \log(\omega)$$

$$|G(j\omega)| \Big|_{\omega=10a} \cong -20 \log(2 \cdot 10) = -26,0205999133 \text{ dB}$$

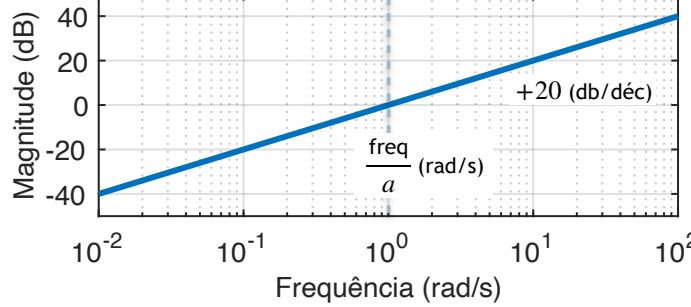
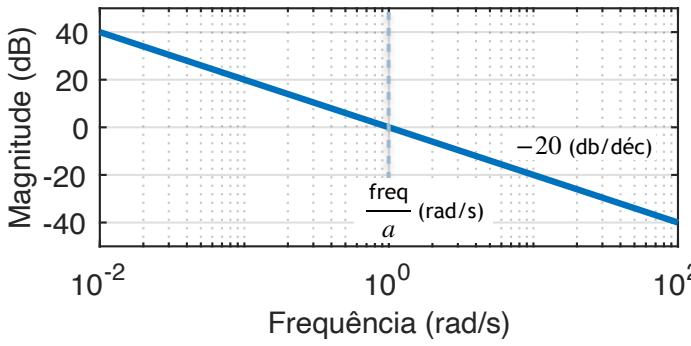
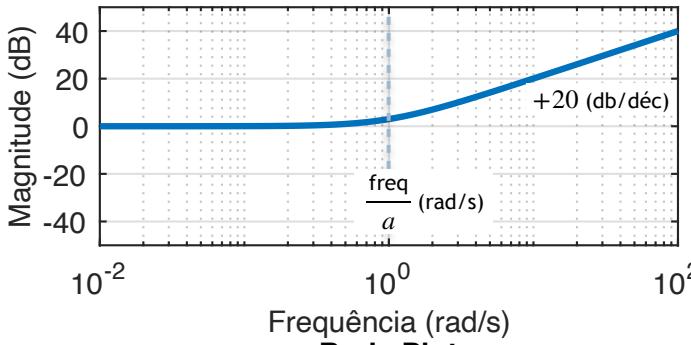
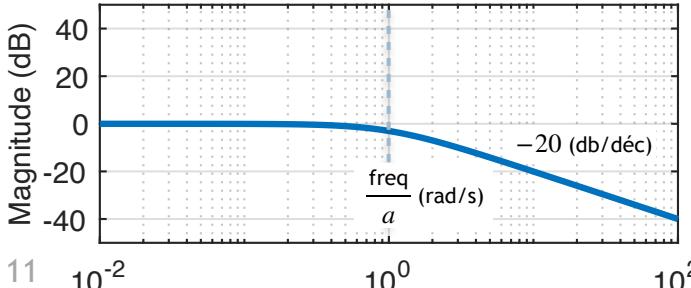
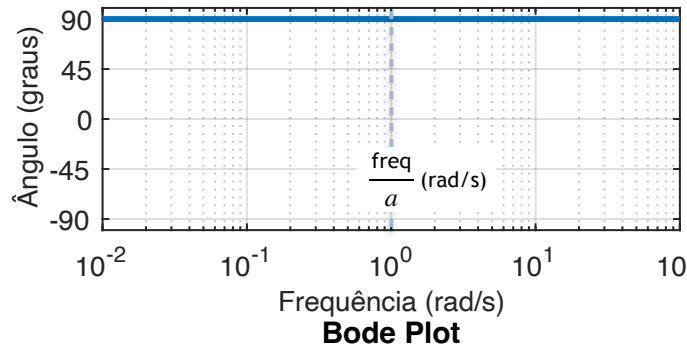
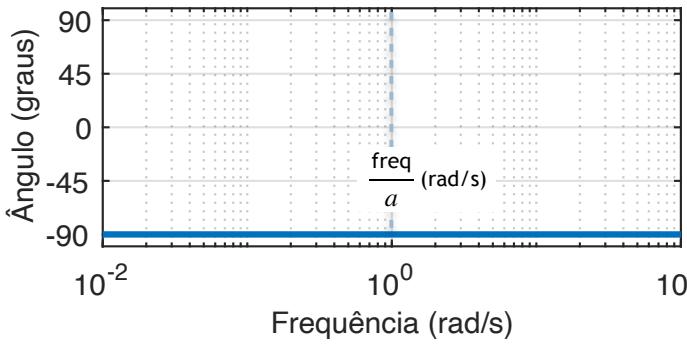
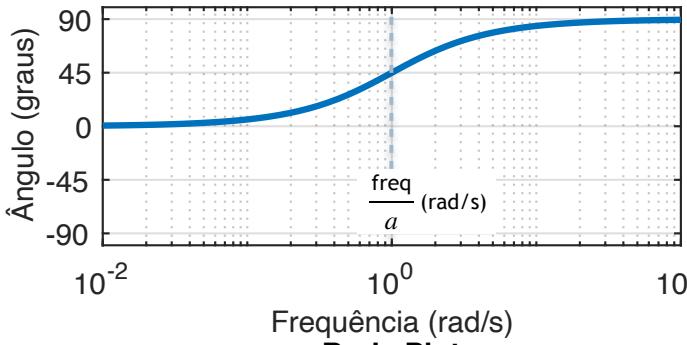
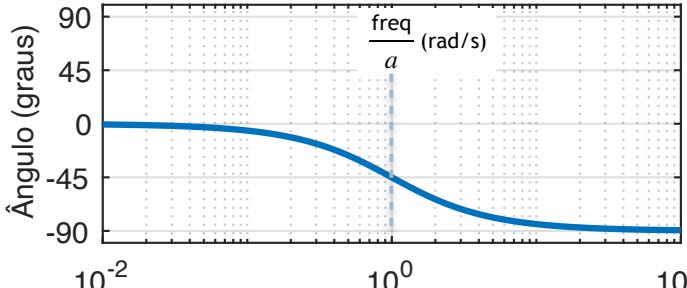
E para $\omega = a$, teremos:

$$|G(j\omega)| = 20 \log\left(\frac{1}{\sqrt{a^2 + a^2}}\right)$$

$$|G(j\omega)| = -20 \log(\sqrt{2a^2}) = \underbrace{-20 \log(\sqrt{2})}_{-3,0103 \text{ dB}} - 20 \log a$$

$$\underbrace{-20 \log(\sqrt{2})}_{-3,0103 \text{ dB}} - 20 \log 2 = -9,0309 \text{ dB}$$

Ref: URL: <http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/Bode/BodeHow.html>

Bode Plot**Bode Plot****Bode Plot****Bode Plot****Bode Plot****Bode Plot****Bode Plot****Bode Plot**

a) $G(s) = s$
(Derivador Puro);

b) $G(s) = \frac{1}{s}$
(Integrador Puro);

c) $G(s) = (s + a)$

d) $G(s) = \frac{1}{(s + a)}$

bode_lado_lado.m

```
% Mostrar diagrama de Bode: Magnitude x Fase, lado a lado
% Entrada: variável G = tf(.)
% Fernando Passold, em 10.06.2022

W=[0.01,100];
[MAG,PHASE,Wb] = bode(G,W);

% Diagrama de Magnitude
subplot(121);
% bodemag(G,W); % até poderia ser usado
MAG=squeeze(MAG); % reduce dimensions
mag = 20*log10(MAG);
semilogx(Wb,mag,'LineWidth',3)
grid on
axis([0.01 100 -50 50])
title ('Bode Plot')
xlabel('Frequência (rad/s)');
ylabel('Magnitude (dB)');
yticks([-40 -20 0 20 40]);

% Diagrama de Fase
subplot(122);
% Detalhe: PHASE = 1 x 1 x 41 !!!
PHASE=squeeze(PHASE); % reduce dimensions
semilogx(Wb,PHASE,'LineWidth',3)
axis([0.01 100 -100 100])
grid on
title ('Bode Plot')
xlabel('Frequência (rad/s)');
ylabel('Ângulo (graus)');
yticks([-90 -45 0 45 90]);
```



Revisão de traçados de Diagramas de Bode...

- Diagrama de Bode para:

- Se $s = jw$: (Derivador Puro)

$$G(jw) = (jw + a) = a \left(j \frac{w}{a} + 1 \right)$$

- Para baixas frequências ($w < a$):

$$G(jw) \approx a$$

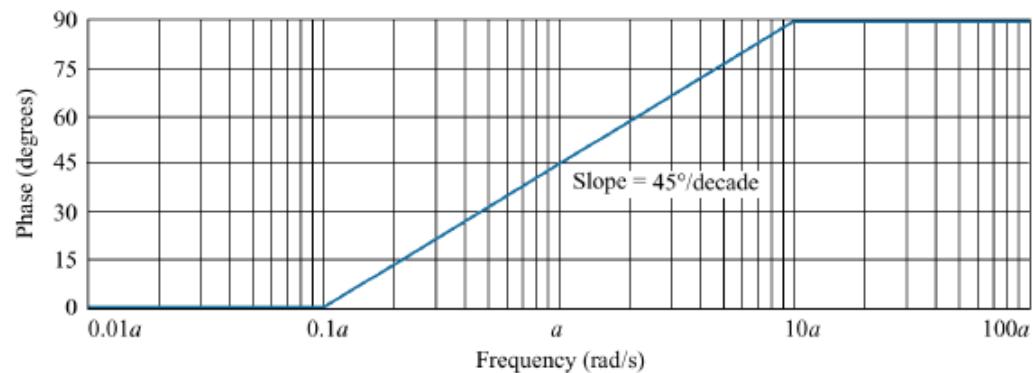
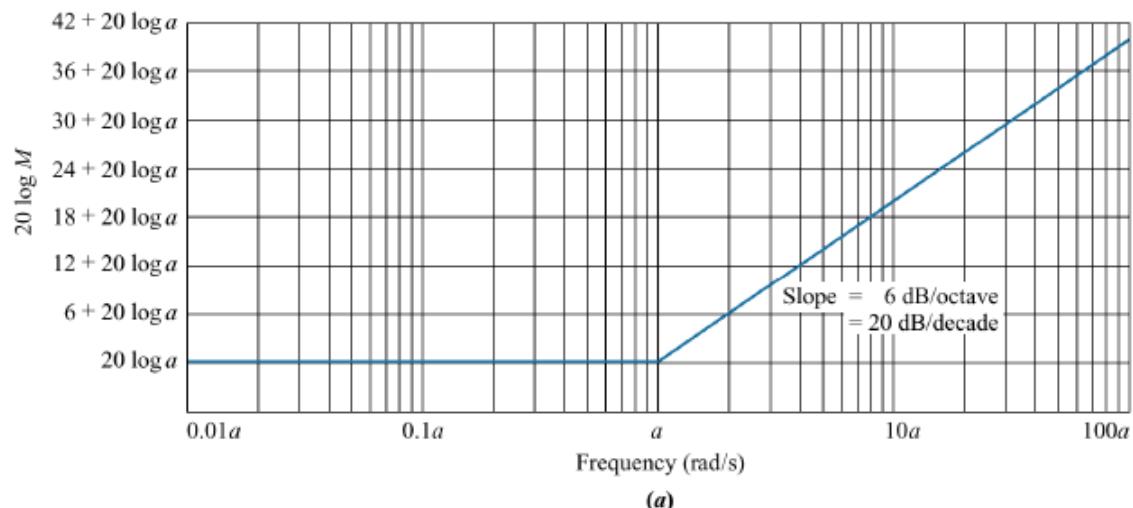
$$20\log(M) = 20\log(a)$$

- Para frequências elevadas ($w > a$)

$$G(jw) \approx a$$

$$20\log(M) = 20\log(a)$$

$$G(s) = (s + a)$$



Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

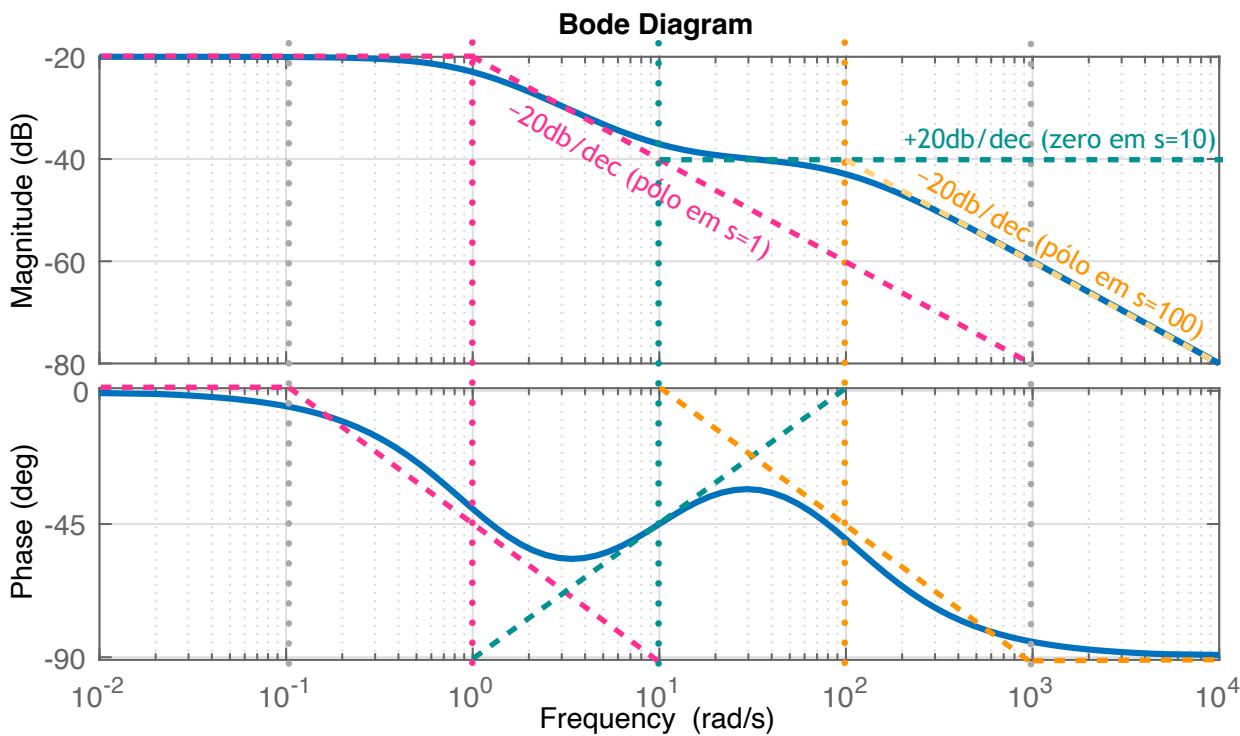
$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} \underset{\text{degrau}}{\overset{\text{Degrau}}{|}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

2. Lembrar que:

cada pólo decrementa ganho de: -20 db/déc;

cada zero incrementa ganho de: +20 db/déc.



```
>> G=tf([1 10],poly([-1 -100]))
G =
  s + 10
  -----
  s^2 + 101 s + 100
>> zpk(G)
  (s+10)
  -----
  (s+100) (s+1)
>> bode(G)
>> grid
```

Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

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$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} \left|_{\substack{\text{Degrau} \\ \text{dB}}} \right. = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

Valores em $\omega = 0,1 \text{ rad/s}$
(1 década abaixo do pólo em $s = -1$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

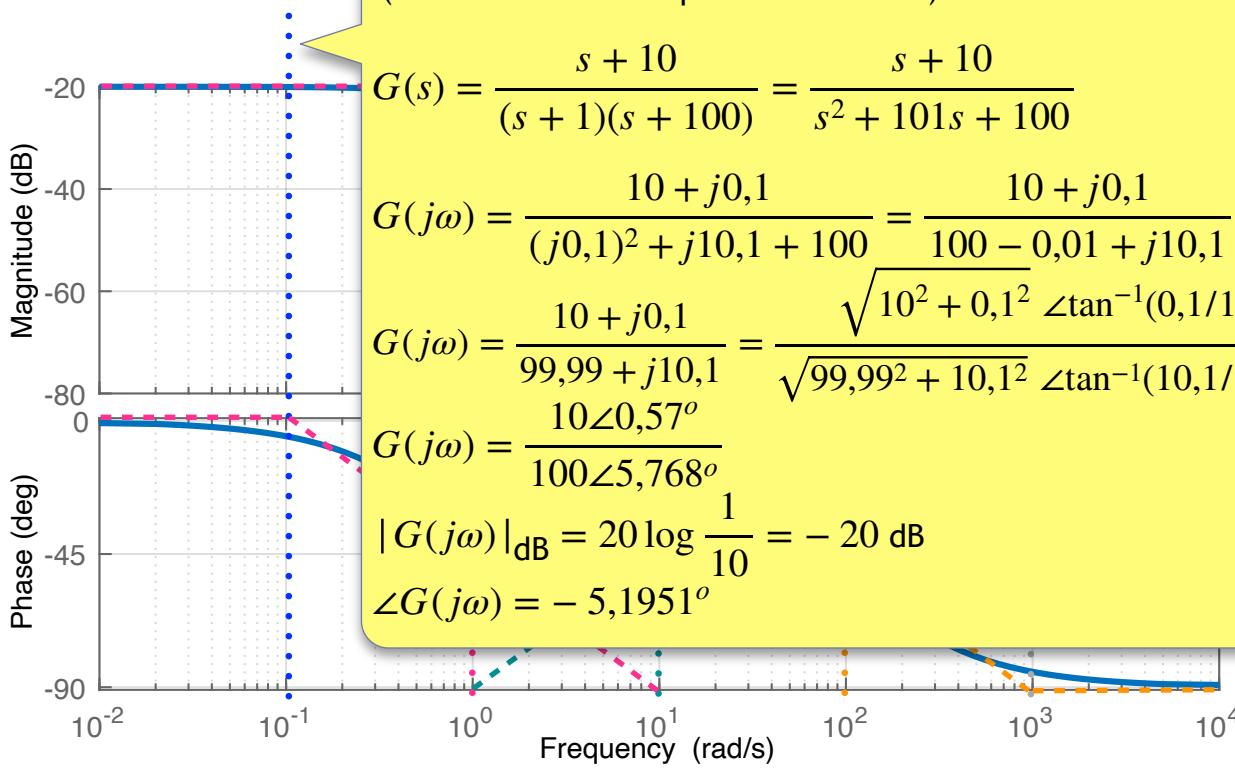
$$G(j\omega) = \frac{10 + j0,1}{(j0,1)^2 + j10,1 + 100} = \frac{10 + j0,1}{100 - 0,01 + j10,1}$$

$$G(j\omega) = \frac{10 + j0,1}{99,99 + j10,1} = \frac{\sqrt{10^2 + 0,1^2} \angle \tan^{-1}(0,1/10)}{\sqrt{99,99^2 + 10,1^2} \angle \tan^{-1}(10,1/99,99)}$$

$$G(j\omega) = \frac{10 \angle 0,57^\circ}{100 \angle 5,768^\circ}$$

$$|G(j\omega)|_{\text{dB}} = 20 \log \frac{1}{10} = -20 \text{ dB}$$

$$\angle G(j\omega) = -5,1951^\circ$$



de: -20 db/déc;
de: +20 db/déc.

```
>> G=tf([1 10],poly([-1 -100]))
G =
      s + 10
      -----
      s^2 + 101 s + 100
>> zpk(G)
      (s+10)
      -----
      (s+100) (s+1)
>> bode(G)
>> grid
```

Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} \left|_{\substack{\text{Degrau} \\ \text{dB}}} \right. = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

Valores em $\omega = 1$ rad/s (no pólo em $s = -1$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j1}{(j)^2 + j101 + 100} = \frac{10 + j1}{100 - 1 + j101}$$

$$G(j\omega) = \frac{10 + j1}{99 + j101} = \frac{\sqrt{10^2 + 1^2} \angle \tan^{-1}(1/10)}{\sqrt{99^2 + 101^2} \angle \tan^{-1}(101/99)}$$

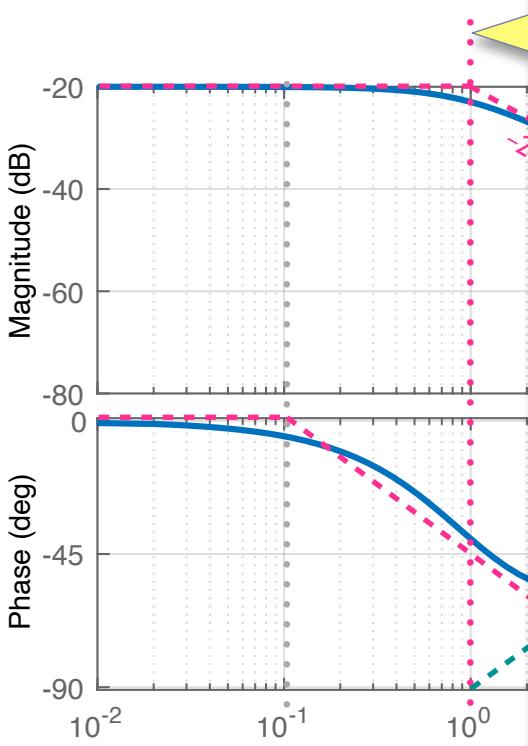
$$G(j\omega) = \frac{\sqrt{101} \angle 5.7106^\circ}{\sqrt{20002} \angle 45.573^\circ}$$

$$\sqrt{100^2 + 100^2} = \sqrt{20000}$$

$$G(j\omega) = \frac{10,05 \angle 5,7106^\circ}{141,43 \angle 45,573^\circ} = 0,07106 \angle -39,862^\circ$$

$$|G(j\omega)|_{\text{dB}} = 20 \log(0,07106) = -22,968 \text{ dB}$$

$$\angle G(j\omega) = -39,862^\circ$$



déc;
déc.

poly([-1 -100]))

100

Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

Valores em $\omega = 10 \text{ rad/s}$ (no pólo em $s = -10$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j10}{(j10)^2 + j1010 + 100} = \frac{10 + j10}{100 - 100 + j1010}$$

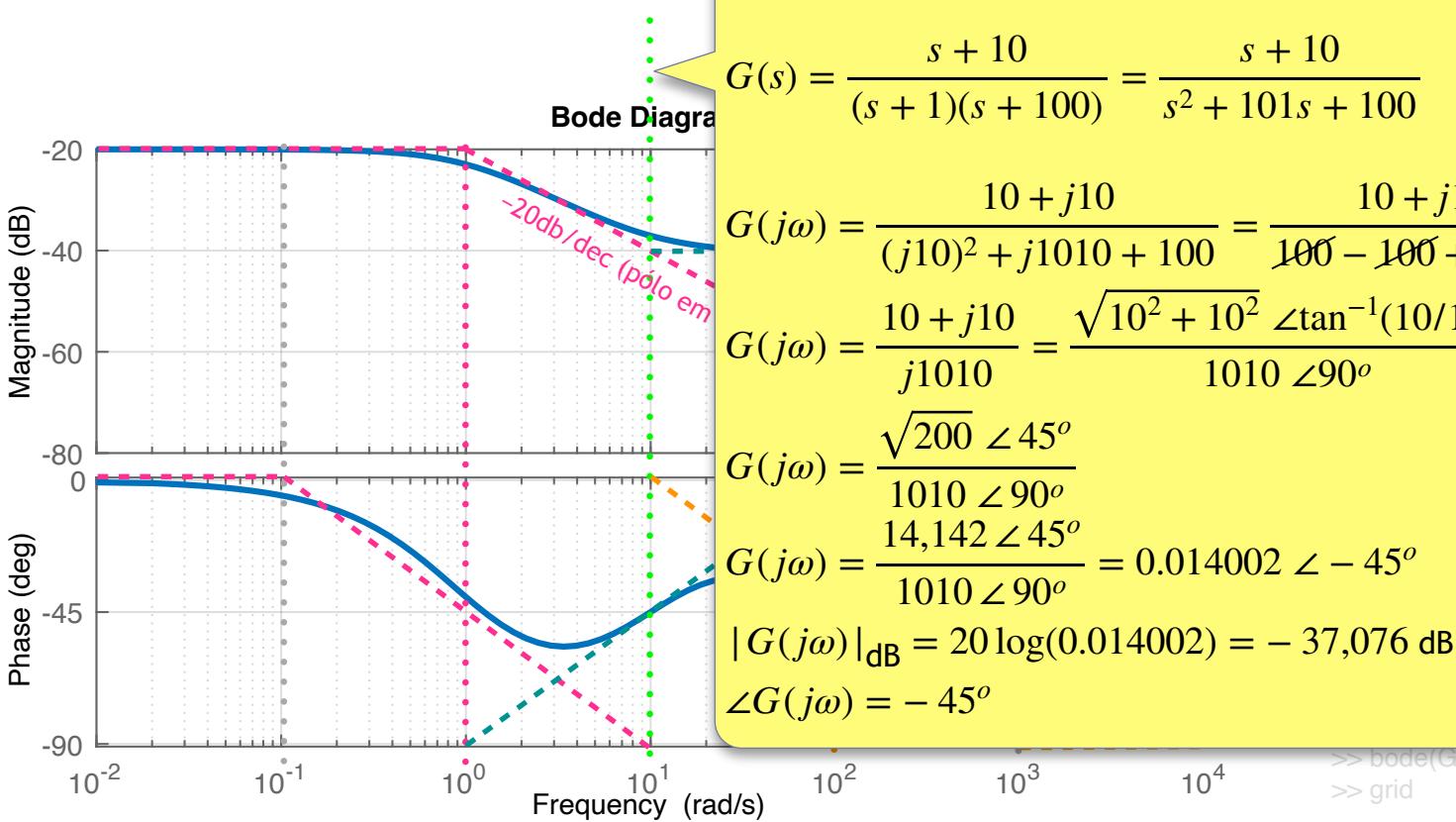
$$G(j\omega) = \frac{10 + j10}{j1010} = \frac{\sqrt{10^2 + 10^2} \angle \tan^{-1}(10/10)}{1010 \angle 90^\circ}$$

$$G(j\omega) = \frac{\sqrt{200} \angle 45^\circ}{1010 \angle 90^\circ}$$

$$G(j\omega) = \frac{14,142 \angle 45^\circ}{1010 \angle 90^\circ} = 0,014002 \angle -45^\circ$$

$$|G(j\omega)|_{\text{dB}} = 20 \log(0,014002) = -37,076 \text{ dB}$$

$$\angle G(j\omega) = -45^\circ$$



>> bode(G)
>> grid

Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

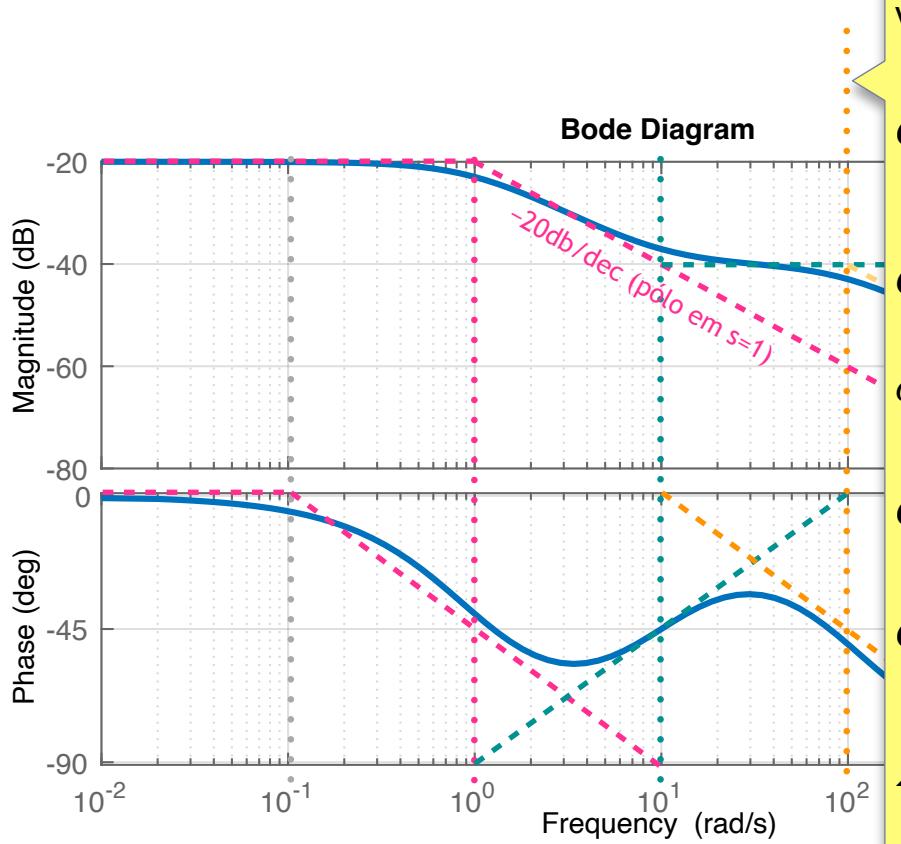
1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \underbrace{\left[\frac{s(s+10)}{(s+1)(s+100)} \right]}_{\text{Degrau}} = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} = -20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$



Valores em $\omega = 100$ rad/s (no pólo em $s = -100$):

$$G(s) = \frac{s+10}{(s+1)(s+100)} = \frac{s+10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j100}{(j100)^2 + j10100 + 100} = \frac{10 + j100}{100 - 10000 + j10100}$$

$$G(j\omega) = \frac{10 + j100}{-9900 + j10100} = \frac{\sqrt{10^2 + 100^2} \angle \tan^{-1}(100/10)}{\sqrt{(-9900)^2 + 10100^2} \angle \tan^{-1}(10100/-9900)}$$

$$G(j\omega) = \frac{\sqrt{10100} \angle 84,289^\circ}{\sqrt{200020000} \angle 134,43^\circ}$$

$$G(j\omega) = \frac{100,5 \angle 84,289^\circ}{14143 \angle 134,43^\circ} = 0,007106 \angle -50,138^\circ$$

$$|G(j\omega)|_{\text{dB}} = 20 \log(0,007106) = -42,968 \text{ dB}$$

$$\angle G(j\omega) = -50,138^\circ$$

Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Valores em $\omega = 1000$ rad/s

Nos:

(1 década acima do pólo em $s = -100$):

$$G(s) = \frac{s + 10}{(s + 1)(s + 100)} = \frac{s + 10}{s^2 + 101s + 100}$$

$$G(j\omega) = \frac{10 + j1000}{(j1000)^2 + j10100 + 100} = \frac{10 + j1000}{100 - 1000000 + j10100}$$

$$G(j\omega) = \frac{10 + j1000}{-999900 + j10100}$$

$$G(j\omega) = \frac{\sqrt{1000100}}{\sqrt{9,999 \times 10^{11}}} \angle 89,427^\circ$$

$$G(j\omega) = \frac{1000 \angle 89,427^\circ}{999950 \angle 179,42^\circ} = \underbrace{0.0010001}_{\cong 1/1000} \angle -89,994^\circ$$

e

$$|G(j\omega)|_{\text{dB}} = 20 \log(0.0010001) = \underbrace{-59,999}_{\cong -60} \text{ dB}$$

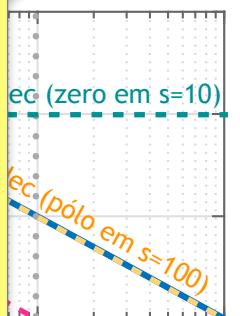
$$\angle G(j\omega) \cong -90^\circ$$

$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s + 10)}{(s + 1)(s + 100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\overset{\text{Degrau}}{\underset{\text{dB}}{|DC|}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

rar que:

pólo decrementa ganho de: -20 db/déc;
zero incrementa ganho de: +20 db/déc.



```
>> G=tf([1 10],poly([-1 -100]))
G =
      s + 10
-----
      s^2 + 101 s + 100
>> zpk(G)
      (s+10)
-----
      (s+100) (s+1)
>> bode(G)
>> grid
```

Exemplo 2)

$$G(s) = K \cdot \frac{(s + 10)}{(s + 1)(s + 100)}$$

1. Cálculo do ganho "DC" (ganho de $G(s)$ em regime permanente):

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} s \cdot G(s)$$

Neste caso: Ganho DC quando aplicado degrau

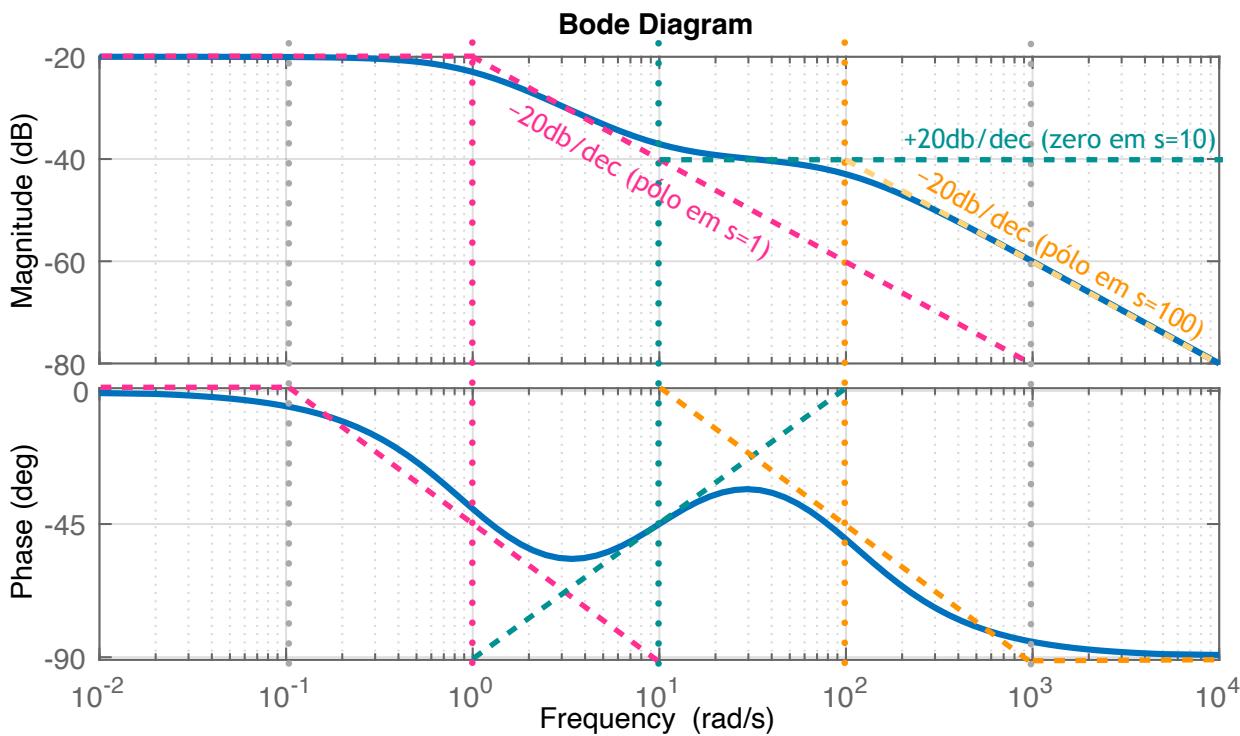
$$= K \cdot \lim_{s \rightarrow 0} \left(\frac{1}{s} \right) \cdot \left[\frac{s(s+10)}{(s+1)(s+100)} \right] = K \cdot \frac{10}{(1)(100)} = \frac{1}{10} = 0,1$$

$$\text{Ganho DC} \underset{\text{dB}}{\overset{\text{Degrau}}{|}} = 20 \log(K \cdot 1/10) = -20 + 20 \log(K) \text{ dB}$$

2. Lembrar que:

cada pólo decrementa ganho de: -20 db/déc;

cada zero incrementa ganho de: +20 db/déc.



```
>> G=tf([1 10],poly([-1 -100]))
G =
      s + 10
-----
      s^2 + 101 s + 100

>> zpk(G)
      (s+10)
-----
      (s+100) (s+1)

>> bode(G)
>> grid
```

Outros exemplos:

$$\triangleright G_1(s) = \frac{s}{(s+10)}$$

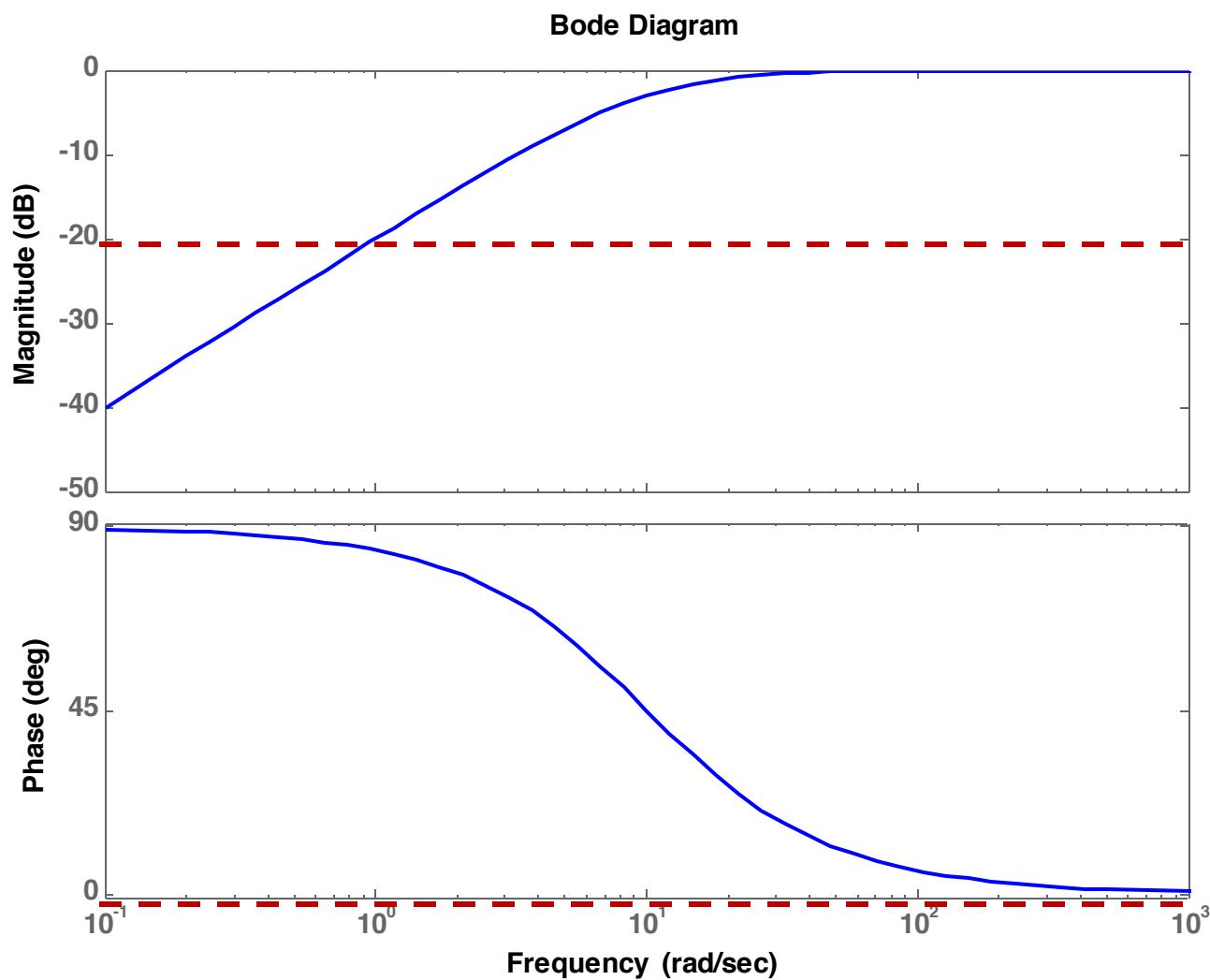
$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10} + 1\right)}$$



Linha de base,
ganho = 0,1
(-20 dB)

$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



Outros exemplos:

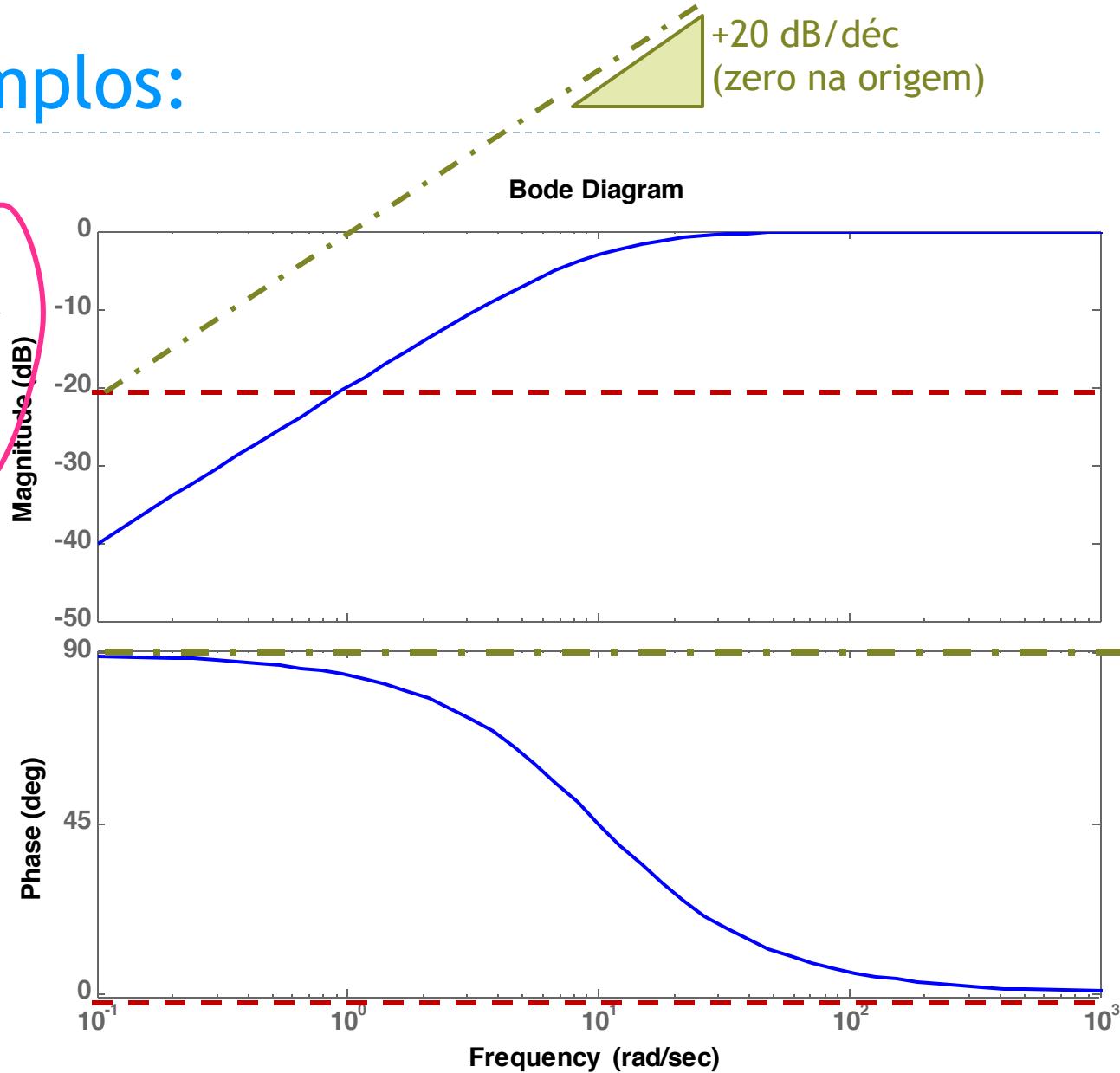
$$\triangleright G_1(s) = \frac{s}{(s+10)}$$

$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10} + 1\right)}$$

Zero na origem

$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



Outros exemplos:

► $G_1(s) = \frac{s}{(s + 10)}$

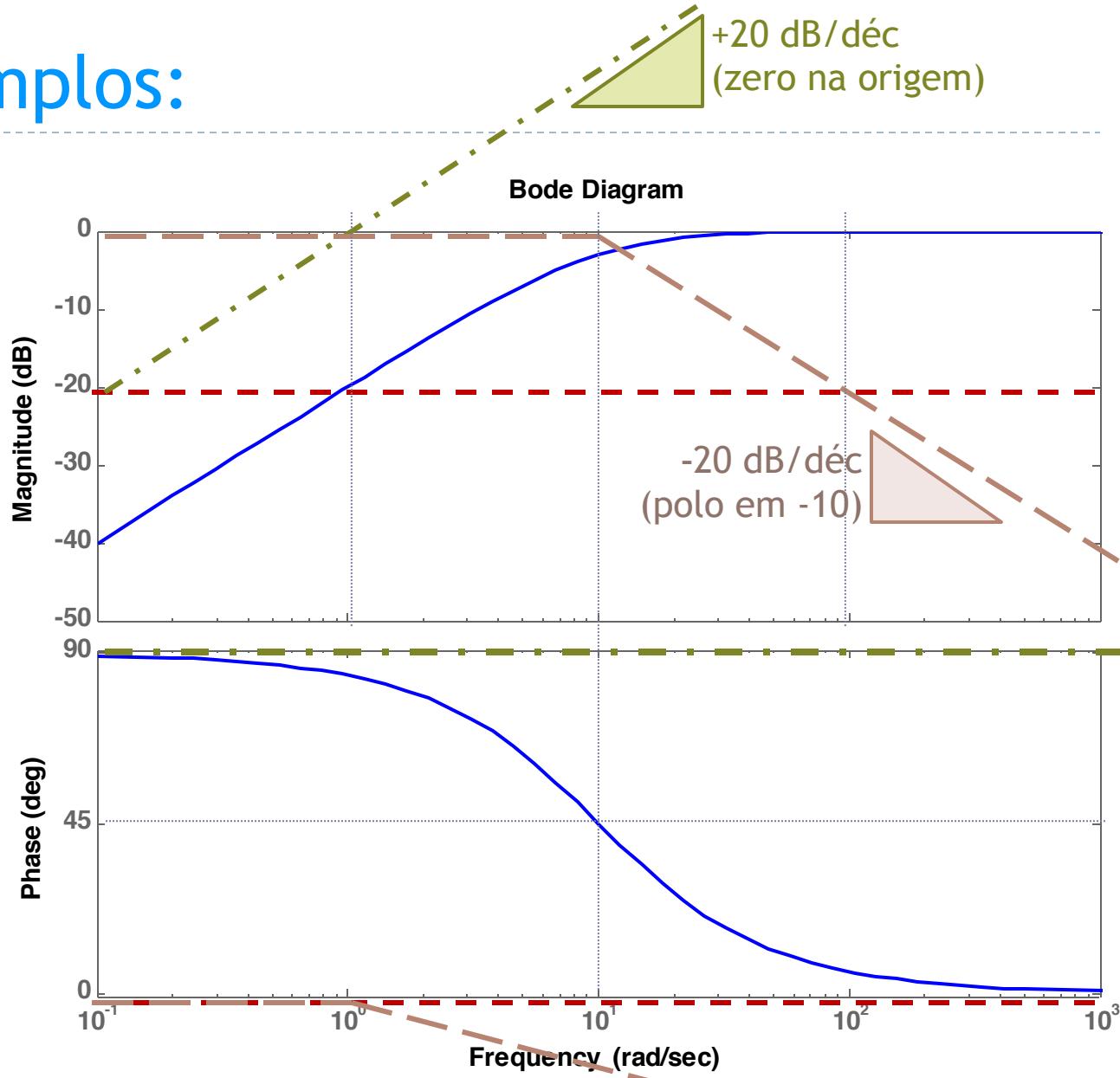
$$G_1(s) = \frac{1}{10} \cdot \frac{s}{\left(\frac{s}{10} + 1\right)}$$

Polo real em $s = -10$



$$\log(10) = 1$$

$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$



Outros exemplos:

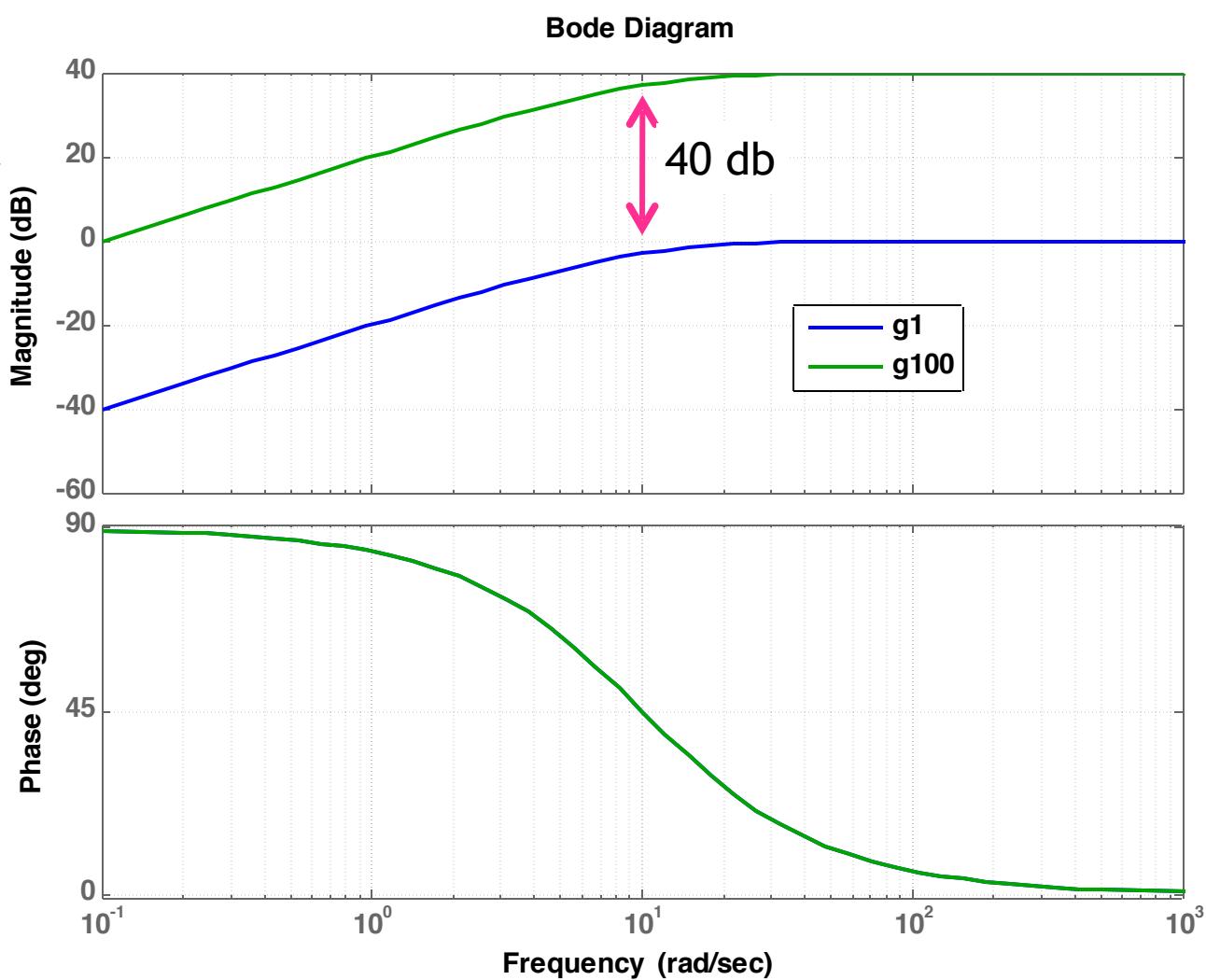
► $G_1(s) = \frac{s}{(s+10)}$

$$G_{100}(s) = 100 \frac{s}{(s+10)}$$

$$G_{100}(s) = 10 \frac{s}{s + \frac{s}{10}}$$

Diferença:
Ganho de magnitude:

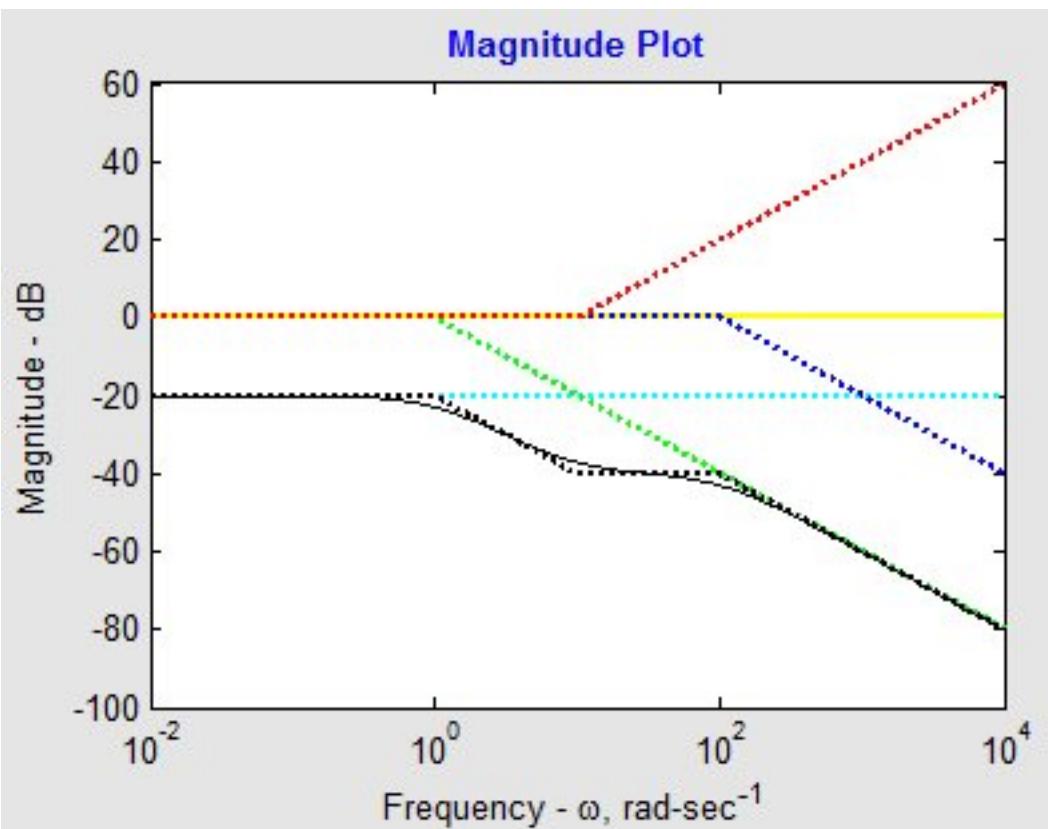
$$\log(100) = 2$$



Outros exemplos:

$$\begin{aligned}\triangleright G_2(s) &= \frac{s+10}{(s+1)(s+100)} \\ &= \frac{10\left(\frac{s}{10} + 1\right)}{(s+1) \cdot 100\left(\frac{s}{100} + 1\right)} \\ &= \frac{10}{100} \cdot \frac{\left(\frac{s}{10} + 1\right)}{(s+1)\left(\frac{s}{100} + 1\right)}\end{aligned}$$

- Exact Bode Plot
- ... Asymptotic Plot
- Zero Value (for reference only)
- ... Constant = 0.1 (-20 dB)
- ... Real Pole at -1e+002
- ... Real Pole at -1
- ... Real Zero at -10

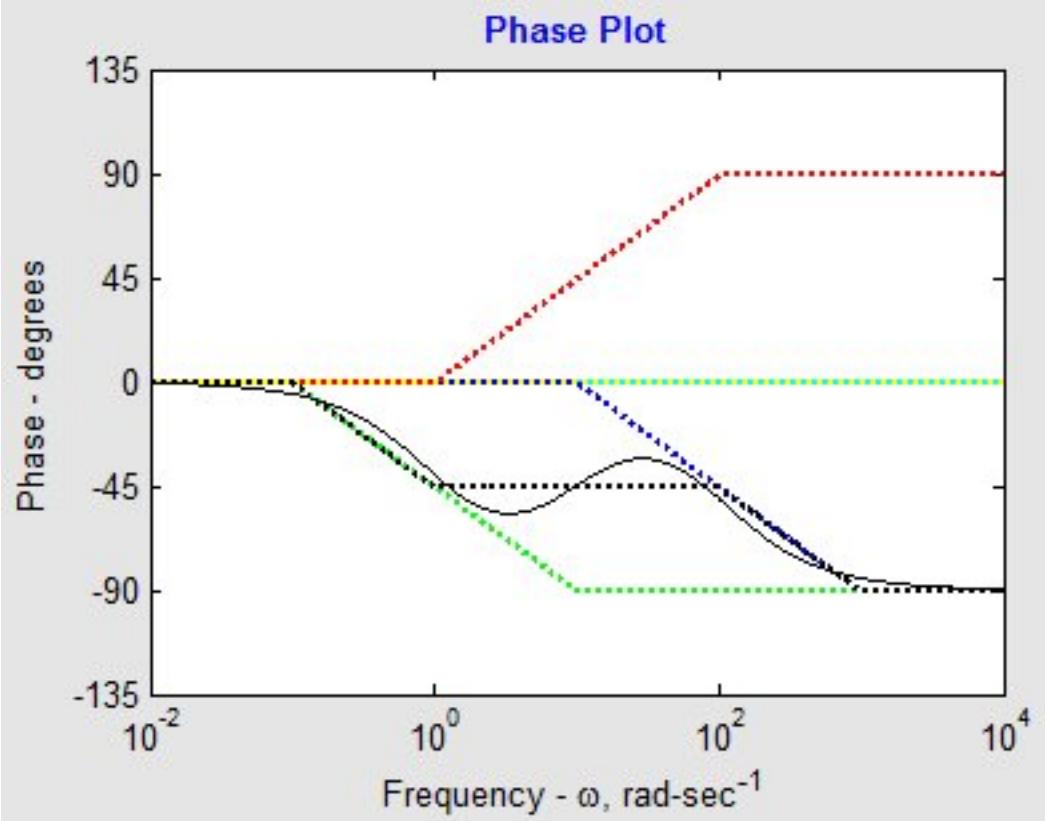


Outros exemplos:

►
$$G_2(s) = \frac{s+10}{(s+1)(s+100)}$$

$$= \frac{10\left(\frac{s}{10} + 1\right)}{(s+1) \cdot 100\left(\frac{s}{100} + 1\right)}$$
$$= \frac{10}{100} \cdot \frac{\left(\frac{s}{10} + 1\right)}{(s+1)\left(\frac{s}{100} + 1\right)}$$

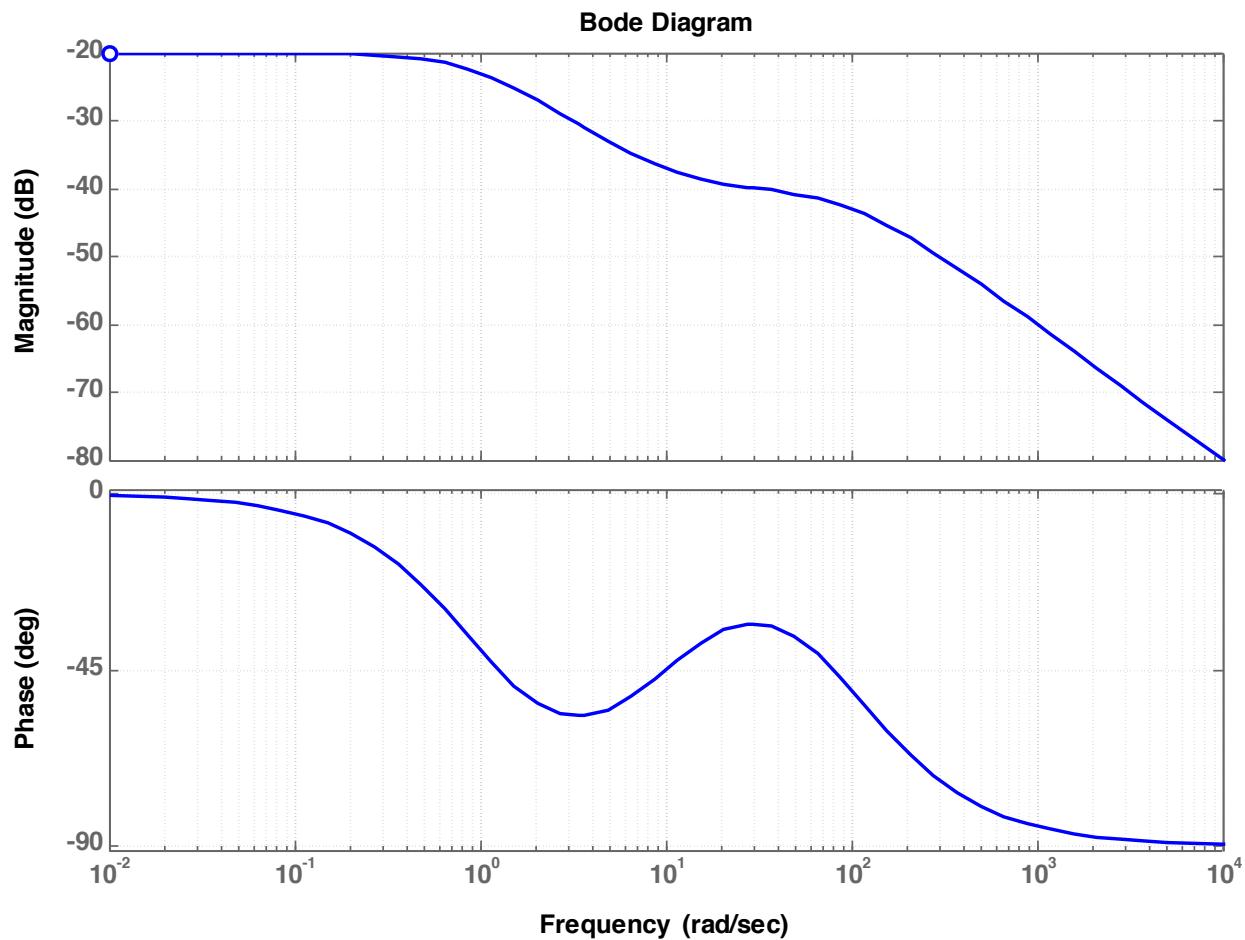
- Exact Bode Plot
- ... Asymptotic Plot
- Zero Value (for reference only)
- ... Constant = 0.1 (-20 dB)
- ... Real Pole at -1e+002
- ... Real Pole at -1
- ... Real Zero at -10



Outros exemplos:

►
$$G_2(s) = \frac{s + 10}{(s + 1)(s + 100)}$$

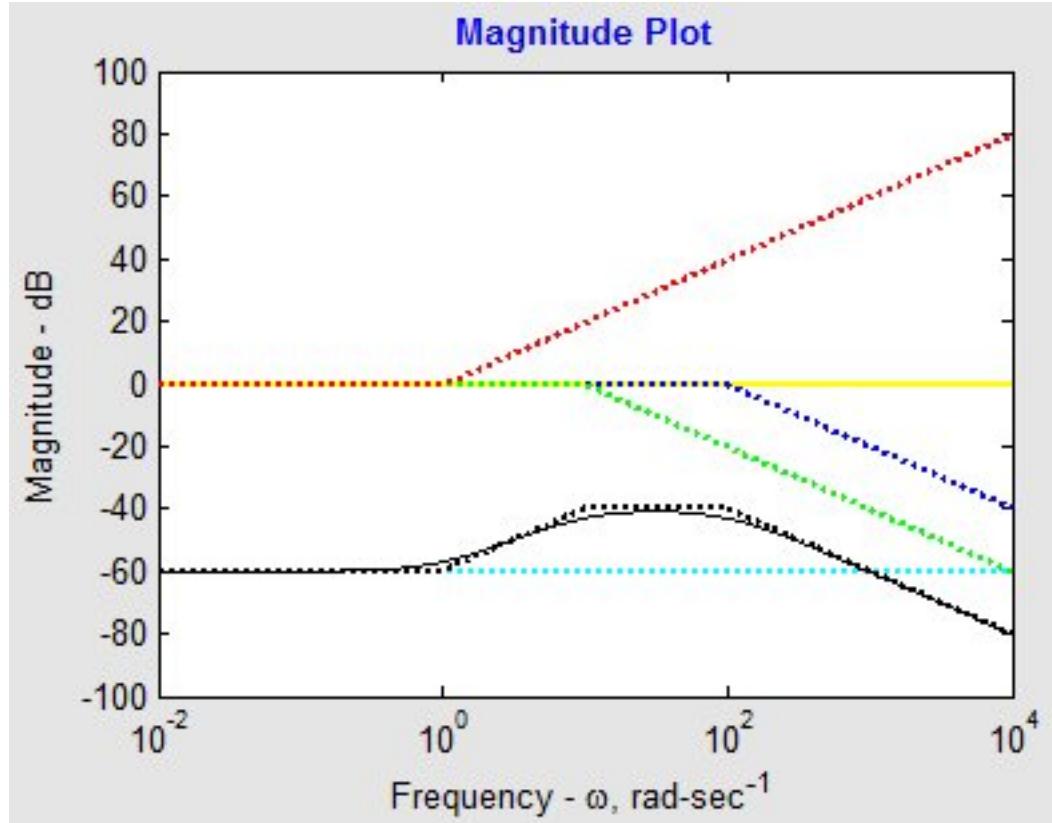
$$= \frac{10\left(\frac{s}{10} + 1\right)}{(s + 1) \cdot 100\left(\frac{s}{100} + 1\right)}$$
$$= \frac{10}{100} \cdot \frac{\left(\frac{s}{10} + 1\right)}{(s + 1)\left(\frac{s}{100} + 1\right)}$$



Outros Exemplos:

$$\begin{aligned}\triangleright G_3(s) &= \frac{s+1}{(s+10)(s+100)} \\ &= \frac{(s+1)}{10\left(\frac{s}{10}+1\right)\cdot 100\left(\frac{s}{100}+1\right)} \\ &= \frac{(s+1)}{1000\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}\end{aligned}$$

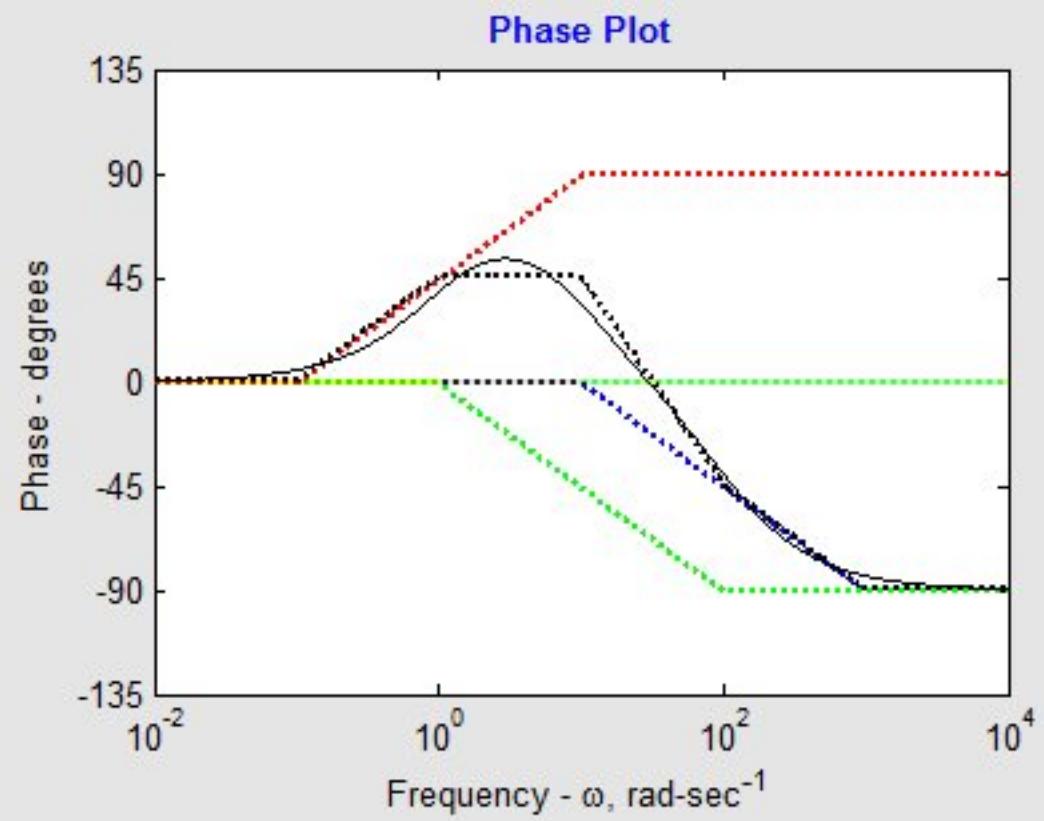
- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 0.001 (-60 dB)
- Real Pole at -1e+002
- Real Pole at -10
- Real Zero at -1



Outros Exemplos:

$$\begin{aligned}\triangleright G_3(s) &= \frac{s+1}{(s+10)(s+100)} \\ &= \frac{(s+1)}{10\left(\frac{s}{10}+1\right)\cdot 100\left(\frac{s}{100}+1\right)} \\ &= \frac{(s+1)}{1000\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}\end{aligned}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 0.001 (-60 dB)
- Real Pole at -1e+002
- Real Pole at -10
- Real Zero at -1



Outros Exemplos:

$$\triangleright G_4(s) = \frac{s+100}{(s+1)(s+10)}$$

$$= \frac{100\left(\frac{s}{100} + 1\right)}{(s+1) \cdot 10\left(\frac{s}{10} + 1\right)}$$

$$= 10 \frac{\left(\frac{s}{100} + 1\right)}{(s+1) \cdot \left(\frac{s}{10} + 1\right)}$$

— Exact Bode Plot

..... Asymptotic Plot

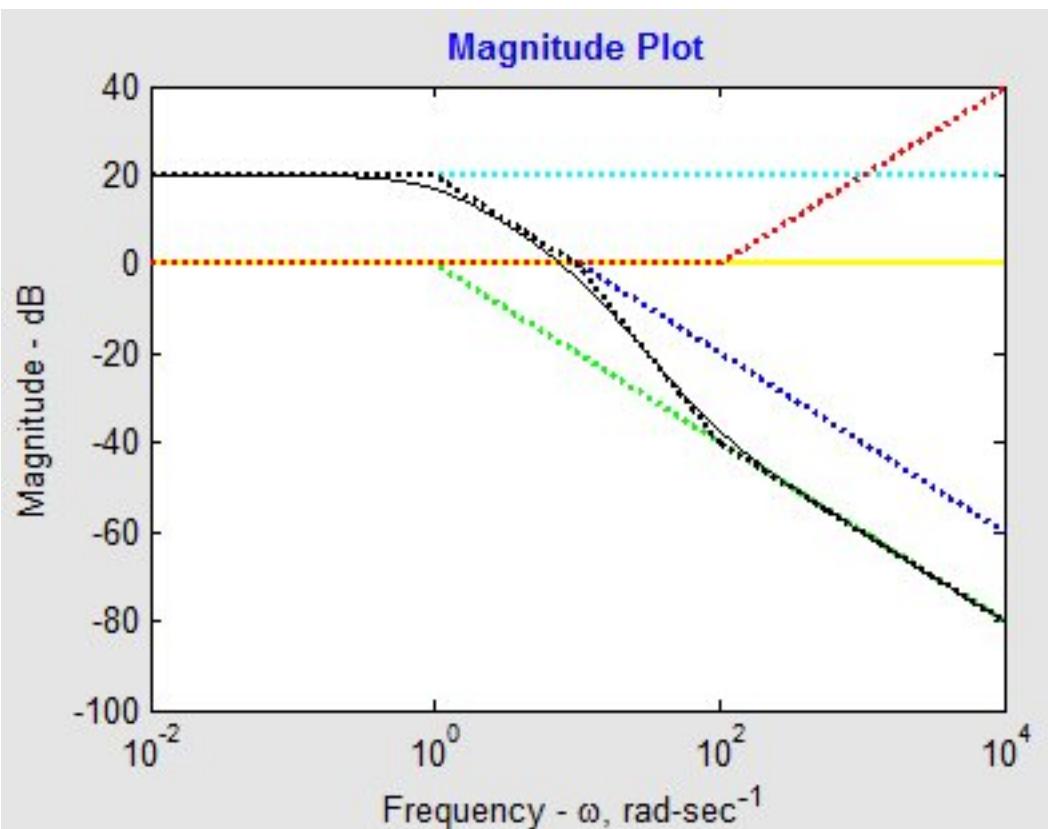
— Zero Value (for reference only)

..... Constant = 10 (20 dB)

..... Real Pole at -10

..... Real Pole at -1

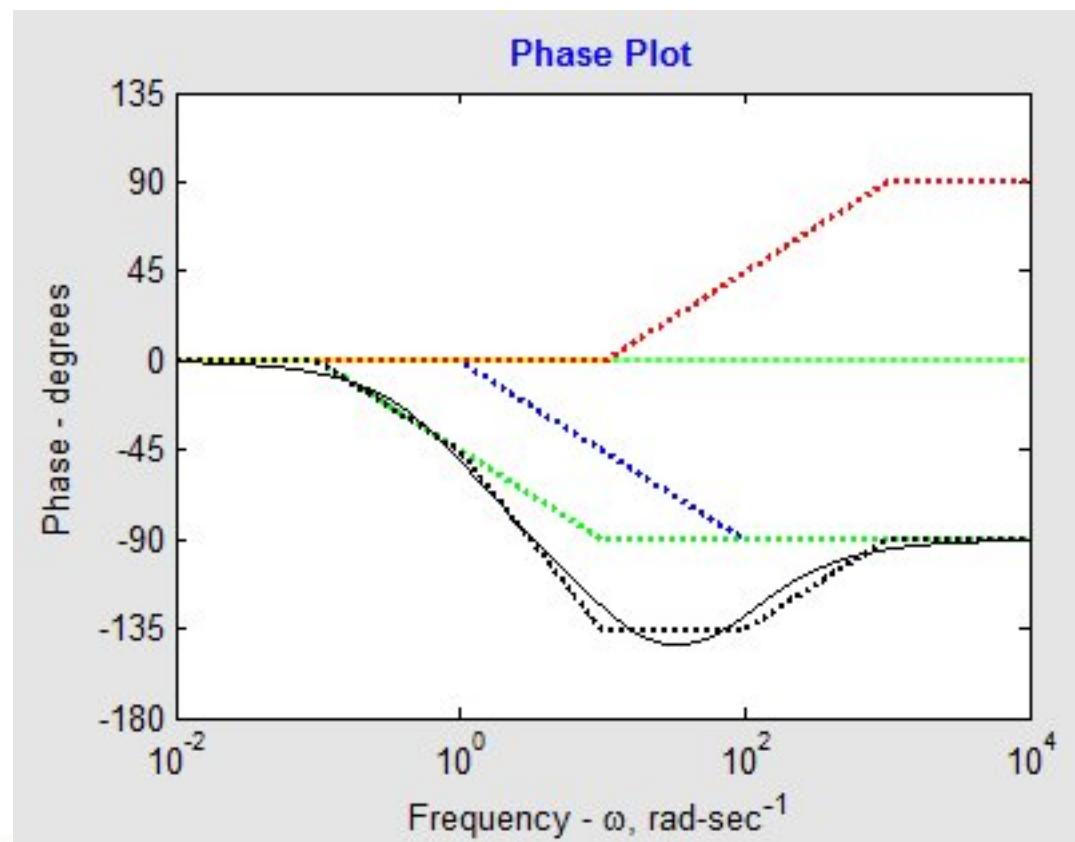
..... Real Zero at -1e+002



Outros Exemplos:

$$\begin{aligned} \triangleright G_4(s) &= \frac{s+100}{(s+1)(s+10)} \\ &= \frac{100\left(\frac{s}{100} + 1\right)}{(s+1) \cdot 10\left(\frac{s}{10} + 1\right)} \\ &= 10 \frac{\left(\frac{s}{100} + 1\right)}{(s+1) \cdot \left(\frac{s}{10} + 1\right)} \end{aligned}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 10 (20 dB)
- Real Pole at -10
- Real Pole at -1
- Real Zero at -1e+002



Outros Exemplos:

$$\triangleright G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot 2\left(\frac{s}{2} + 1\right)}$$

$$= \frac{3}{2} \cdot \frac{\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot \left(\frac{s}{2} + 1\right)}$$

— Exact Bode Plot

..... Asymptotic Plot

— Zero Value (for reference only)

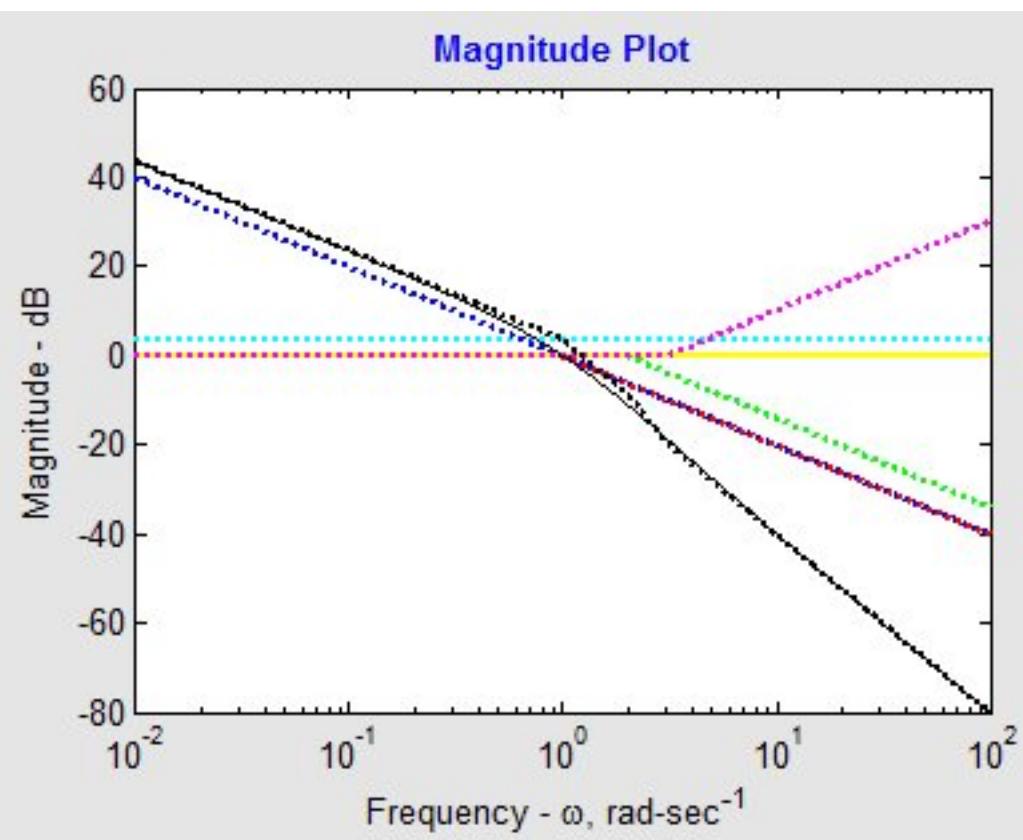
— Constant = 1.5 (3.5 dB)

..... Pole at origin

..... Real Pole at -2

..... Real Pole at -1

..... Real Zero at -3



$$20 \times \log(3/2) = 20 \times 0,1761 = 3,5218$$

Outros Exemplos:

► $G_5(s) = \frac{s+3}{s(s+1)(s+2)}$

$$= \frac{3\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot 2\left(\frac{s}{2} + 1\right)}$$

$$= \frac{3}{2} \cdot \frac{\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot \left(\frac{s}{2} + 1\right)}$$

— Exact Bode Plot

..... Asymptotic Plot

— Zero Value (for reference only)

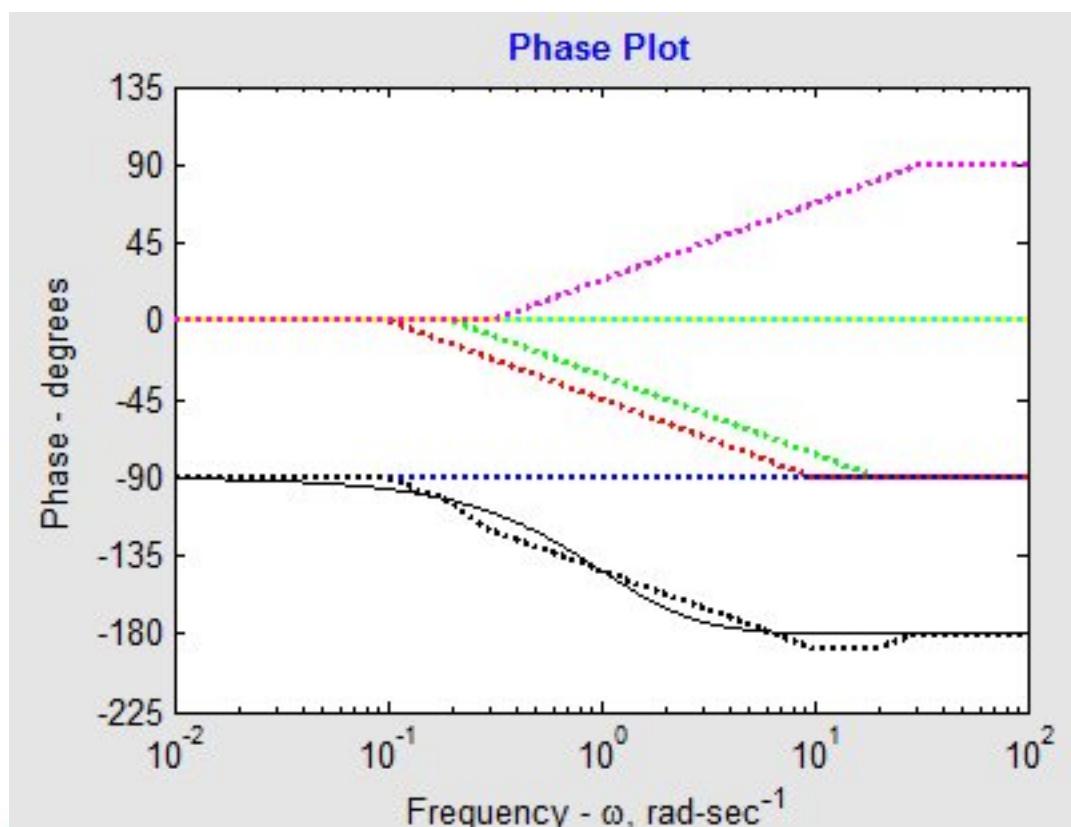
— Constant = 1.5 (3.5 dB)

..... Pole at origin

..... Real Pole at -2

..... Real Pole at -1

..... Real Zero at -3

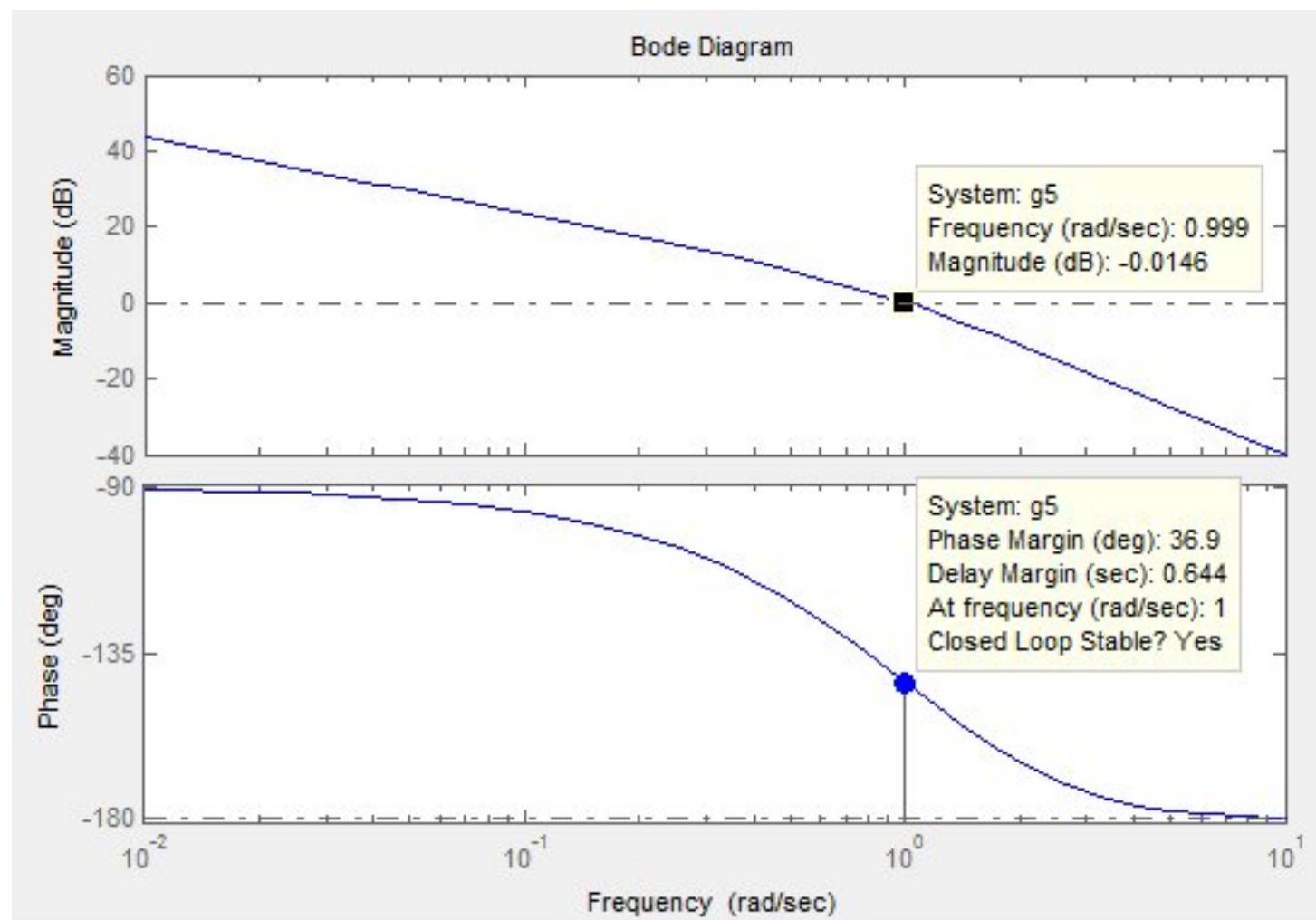


Outros Exemplos:

$$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot 2\left(\frac{s}{2} + 1\right)}$$

$$= \frac{3}{2} \cdot \frac{\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot \left(\frac{s}{2} + 1\right)}$$



Sistemas com polos complexos

▶ Seja:

$$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2 \left(\frac{s^2}{w_n^2} + 2\zeta \frac{s}{w_n} + 1 \right)$$

▶ Em baixas frequências:

$$G(s) \approx w_n^2 = w_n^2 \angle 0^\circ$$

$$20 \log M = 20 \log |G(jw)| = 20 \log w_n^2$$

▶ Em altas frequências:

$$G(s) \approx s^2$$

$$20 \log M = 20 \log |G(jw)| = 20 \log w^2 = 40 \log w$$

$$G(jw) \approx -w^2 = w^2 \angle 180^\circ$$

▶ Detalhes:

▶ w_n : frequência de corte (quebra).

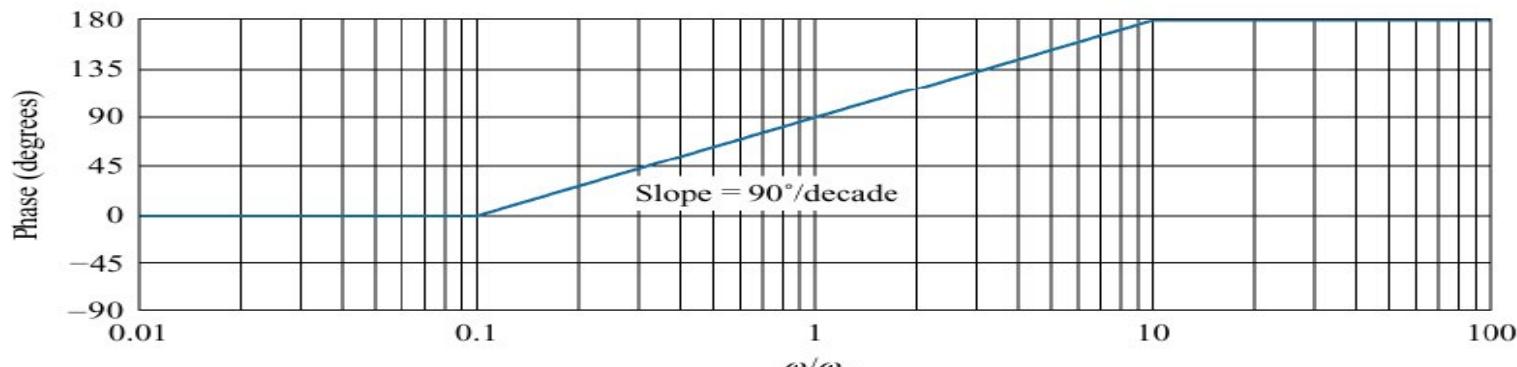
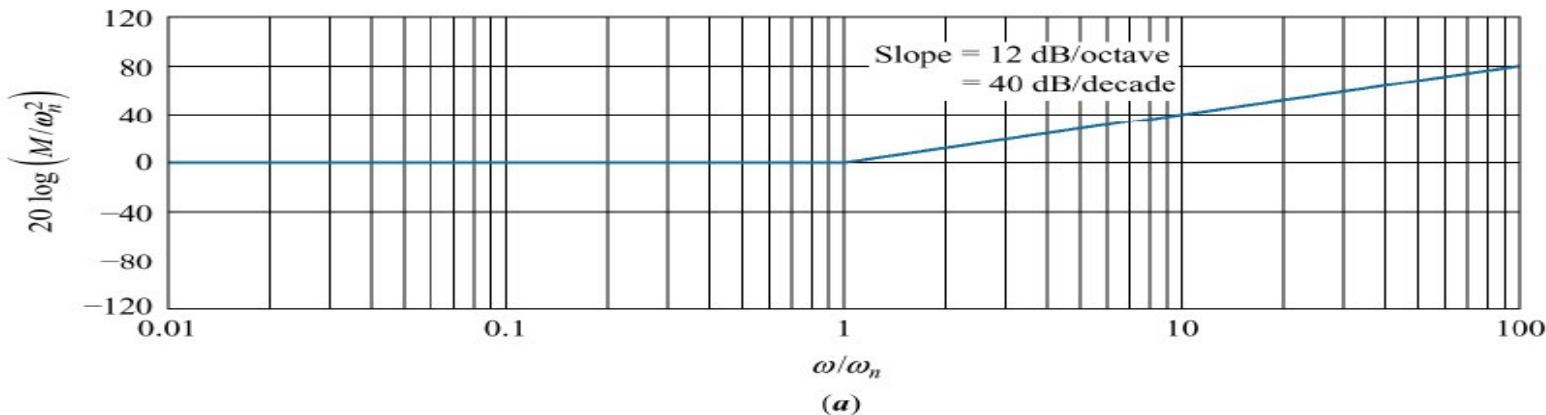
▶ Fase em w_n :

$$G(jw) = s^2 + 2\zeta w_n s + w_n^2 \Big|_{s \rightarrow jw} = (w_n^2 - w^2) + j2\zeta w_n w$$

em w_n o resultado é: $j2\zeta w_n^2$ assim a fase na frequência natural é de $+90^\circ$

Sistemas com polos complexos

- ▶ Seja: $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\zeta \frac{s}{\omega_n} + 1 \right)$
- ▶ Em baixas frequências: $G(s) \approx \omega_n^2 = \omega_n^2 \angle 0^\circ \quad 20 \log M = 20 \log |G(j\omega)| = 20 \log \omega_n^2$
- ▶ Em altas frequências: $G(s) \approx s^2 \quad 20 \log M = 20 \log |G(j\omega)| = 20 \log \omega^2 = 40 \log \omega$
 $G(j\omega) \approx -\omega^2 = \omega^2 \angle 180^\circ$



Sistemas com polos complexos

Correções em diagrama assintótico..

▶ Seja:

$$G(s) = s^2 + 2\zeta w_n s + w_n^2 = w_n^2 \left(\frac{s^2}{w_n^2} + 2\zeta \frac{s}{w_n} + 1 \right)$$

- Um polinômio de 1^a-ordem resulta numa diferença não superior a 3,01 dB na magnitude e 5,71° em relação à fase (no ponto do polo).
- Um polinômio de 2^a-ordem pode implicar maior disparidade, depende do valor de ζ (na localização dos polos complexos):

$$G(jw) = s^2 + 2\zeta w_n s + w_n^2 \Big|_{s \rightarrow jw} = (w_n^2 - w^2) + j2\zeta w_n w$$

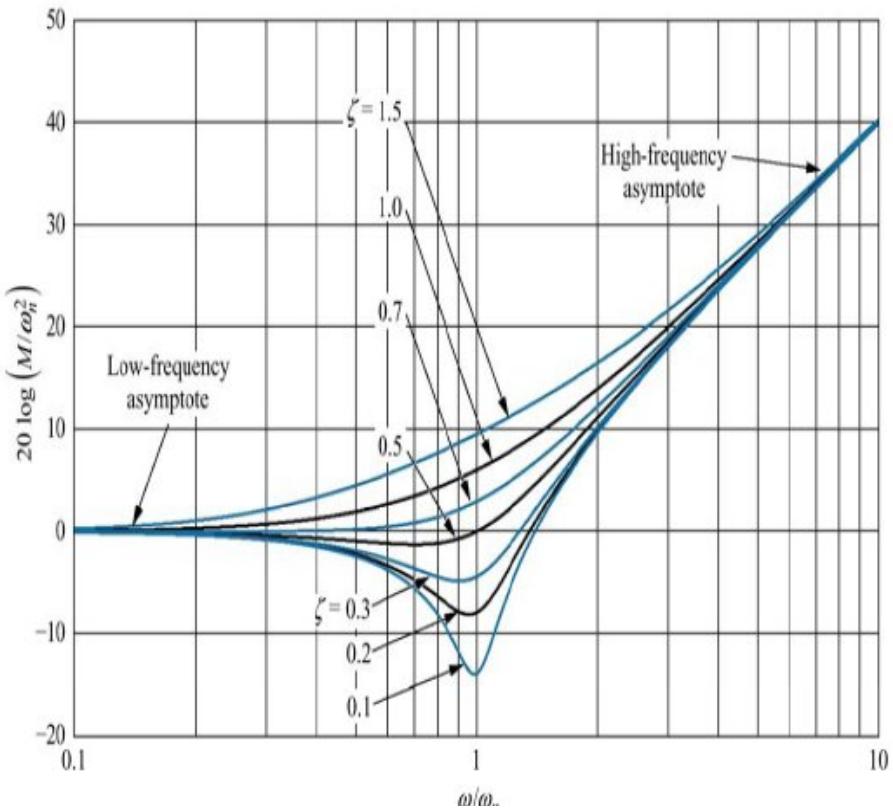
$$M = \sqrt{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}$$

$$\text{Fase} = \tan^{-1} \left(\frac{2\zeta w_n w}{w_n^2 - w^2} \right)$$

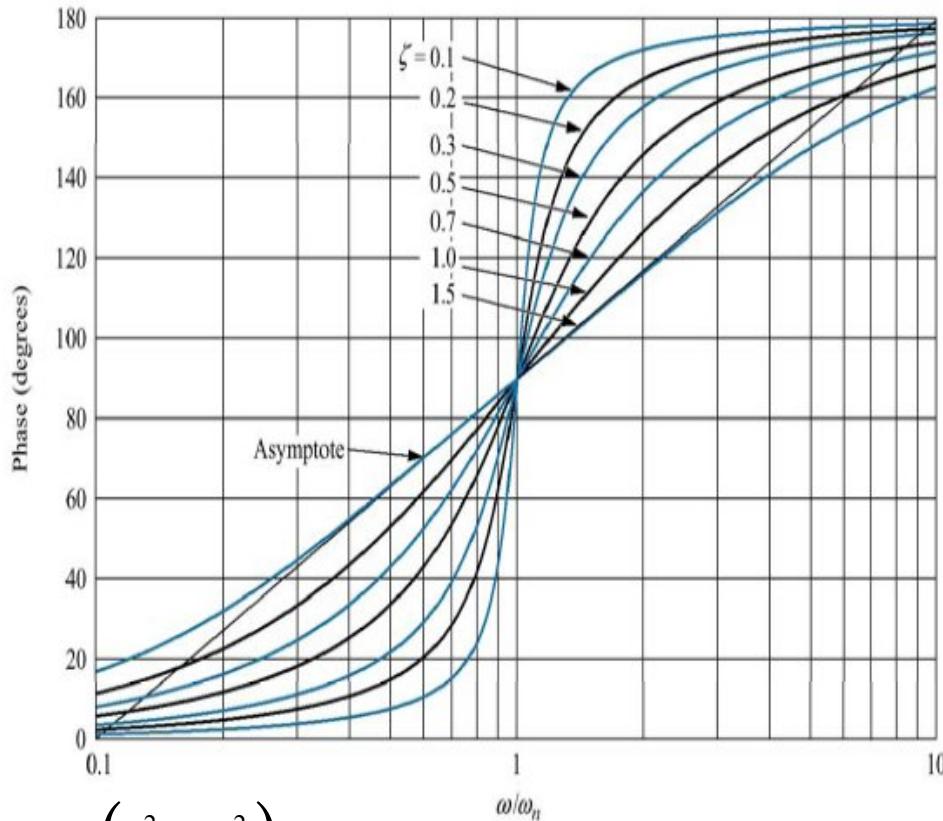
Sistemas com polos complexos

Correções em diagrama assintótico..

- Seja: $G(s) = s^2 + 2\xi\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\xi \frac{s}{\omega_n} + 1 \right)$
- $$M = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}$$
- $$\text{Fase} = \tan^{-1} \left(\frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2} \right)$$



$$G(j\omega) = s^2 + 2\xi\omega_n s + \omega_n^2 \Big|_{s \rightarrow j\omega} = (\omega_n^2 - \omega^2) + j2\xi\omega_n\omega$$

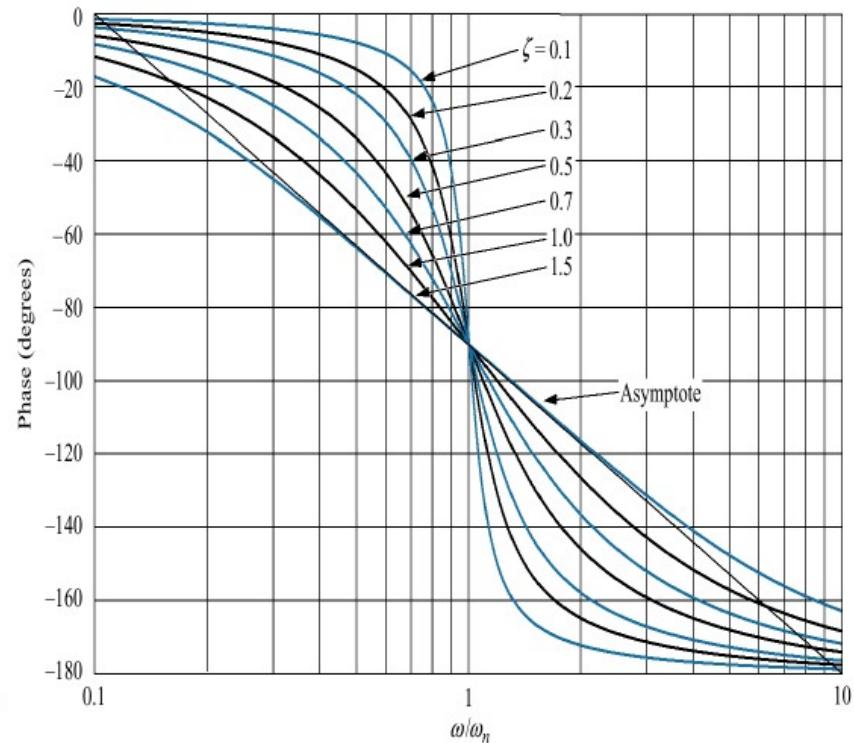
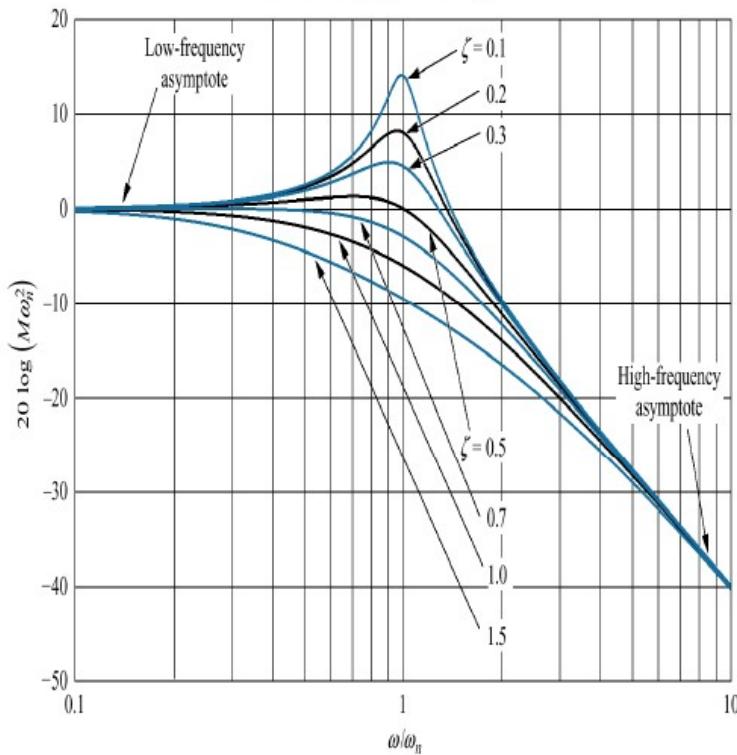


Sistemas com polos complexos

Correções em diagrama assintótico..

$$G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$

- The slope is -40dB/decade .
- The normalized magnitude at the scaled natural frequency is $-20\log 2\zeta\omega_n^2$



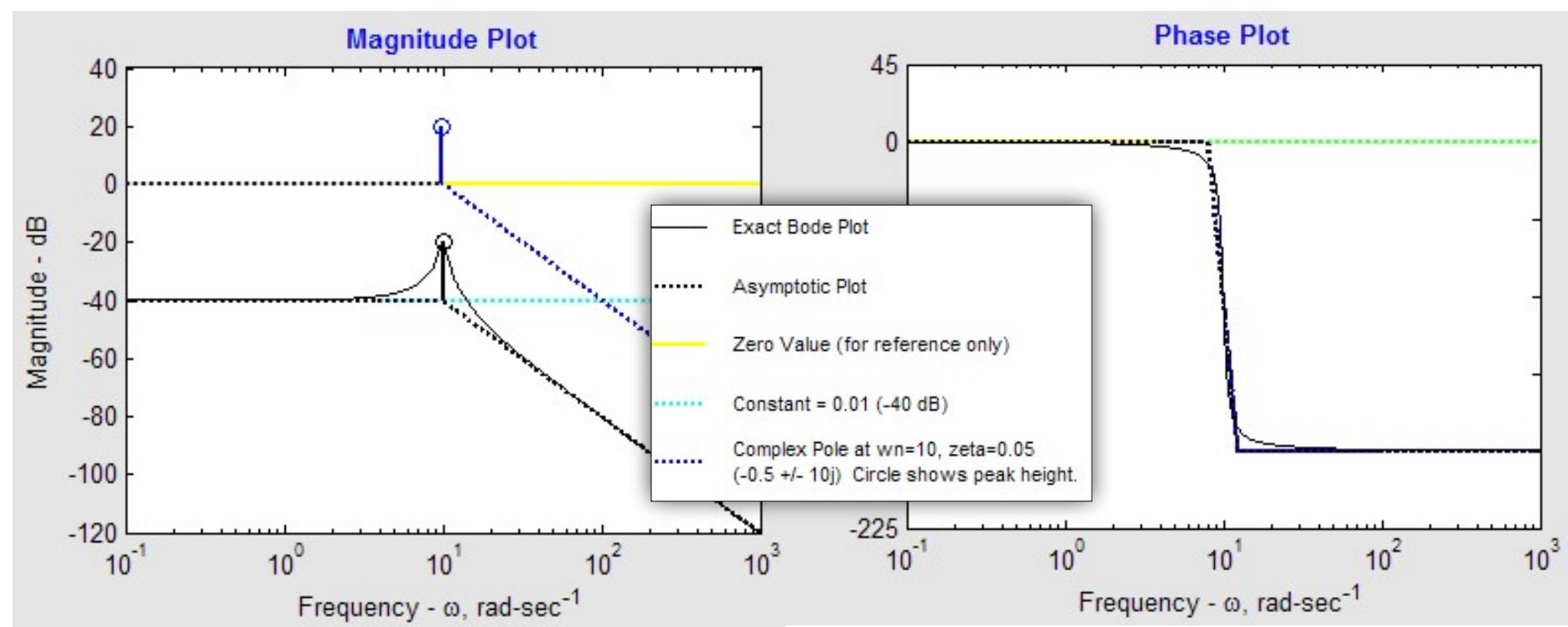
Exemplo polos complexos

$$G(s) = \frac{1}{s^2 + s + 100}$$

$$G(s) = \frac{1}{(s + 0,5 + j10)(s + 0,5 - j10)}$$

$$G(s) = s^2 + 2\xi\omega_n s + \omega_n^2 = \omega_n^2 \left(\frac{s^2}{\omega_n^2} + 2\xi \frac{s}{\omega_n} + 1 \right)$$

$$\omega_n = \sqrt{100} = 10 \quad \xi = 1/2\omega_n = 1/20 = 0,05$$



$$20 \cdot \log_{10}(1/100) = -40 \text{ dB}$$

```

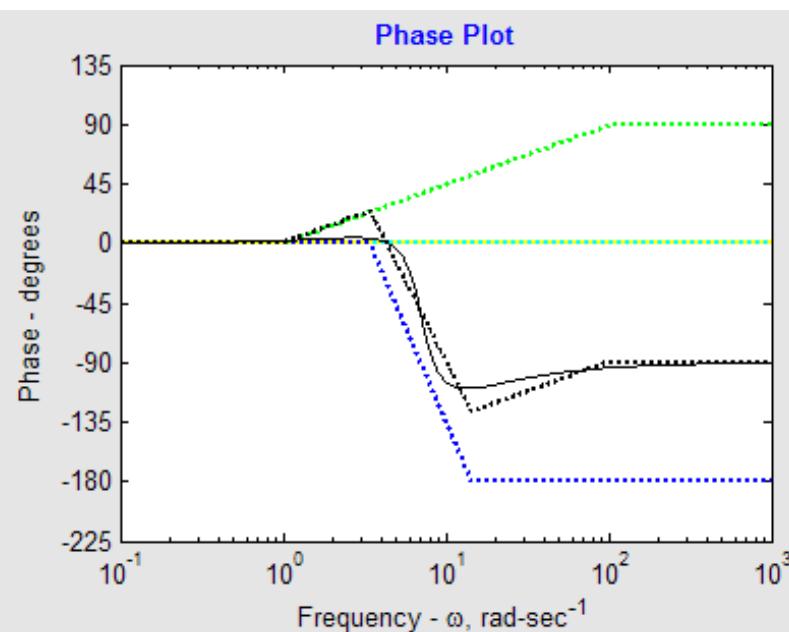
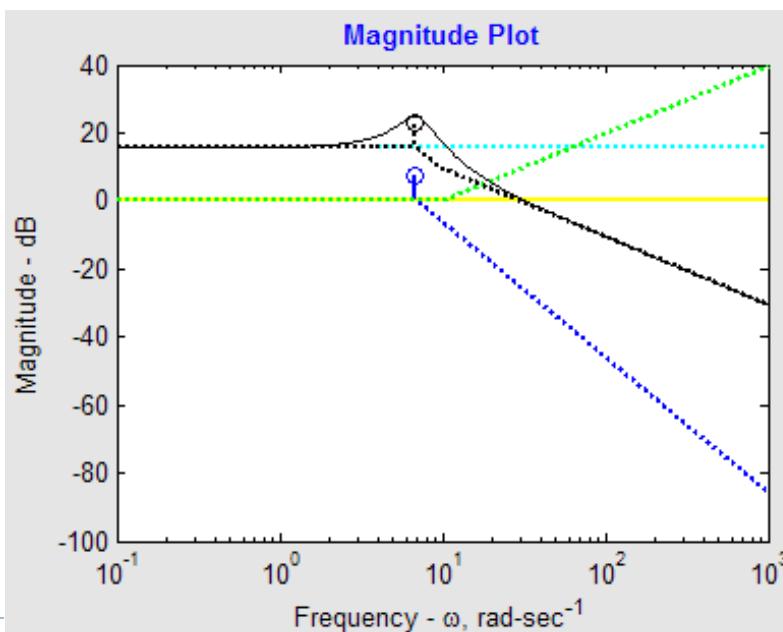
>> roots([1 3 50])
ans =
-1.5000 + 6.9101i
-1.5000 - 6.9101i
>>

```

Problemas sugeridos:

$$H_1(s) = 30 \frac{s+10}{s^2 + 3s + 50}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 6 (16 dB)
- Complex Pole at $\omega_n=7.1$, $\zeta=0.21$
(-1.5 ± 6.9j) Circle shows peak height.
- Real Zero at -10



$$H(s) = 30 \frac{s+10}{s^2 + 3s + 50} = 30 \frac{10}{50} \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1} = 6 \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1}$$

- valor constante = 6,
- um zero em $s=-10$,
- e um par de polos complexos conjugados em: raízes de: $s^2+3s+50=0$;
- Polos complexos em $s=-1.5 \pm j6.9$ (onde $j=\sqrt{-1}$).

Uma maneira mais comum (e útil para nossos propósitos) de expressar isso é usar a notação padrão para um polinômio de segunda ordem:

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1 \quad \omega_0 = \sqrt{50} = 7.07, \quad \zeta = \frac{3\sqrt{50}}{2 \cdot 50} = 0.21$$

Problemas sugeridos:

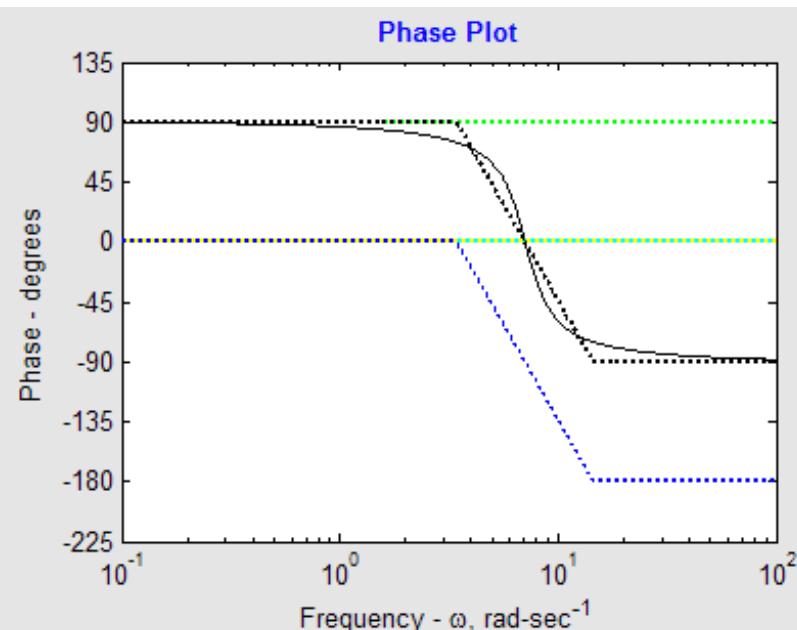
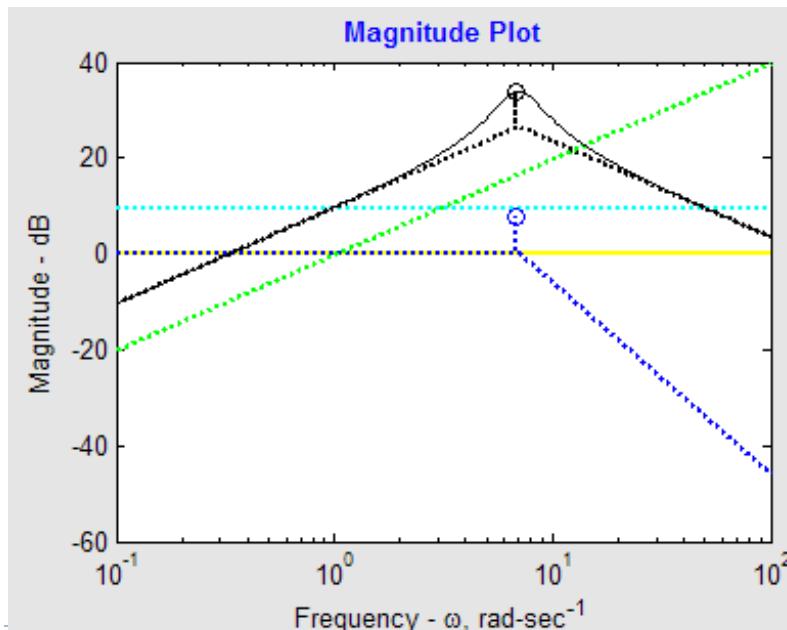
$$H_2(s) = 30 \frac{5s}{s^2 + 3s + 50}$$

- Exact Bode Plot
- Asymptotic Plot
- Zero Value (for reference only)
- Constant = 3 (9.5 dB)
- Complex Pole at $\omega_n=7.1$, $\zeta=0.21$
(-1.5 +/- 6.9j) Circle shows peak height.
- Zero at origin

$$H(s) = 30 \frac{s+10}{s^2 + 3s + 50} = 30 \frac{10}{50} \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1} = 6 \frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1}$$

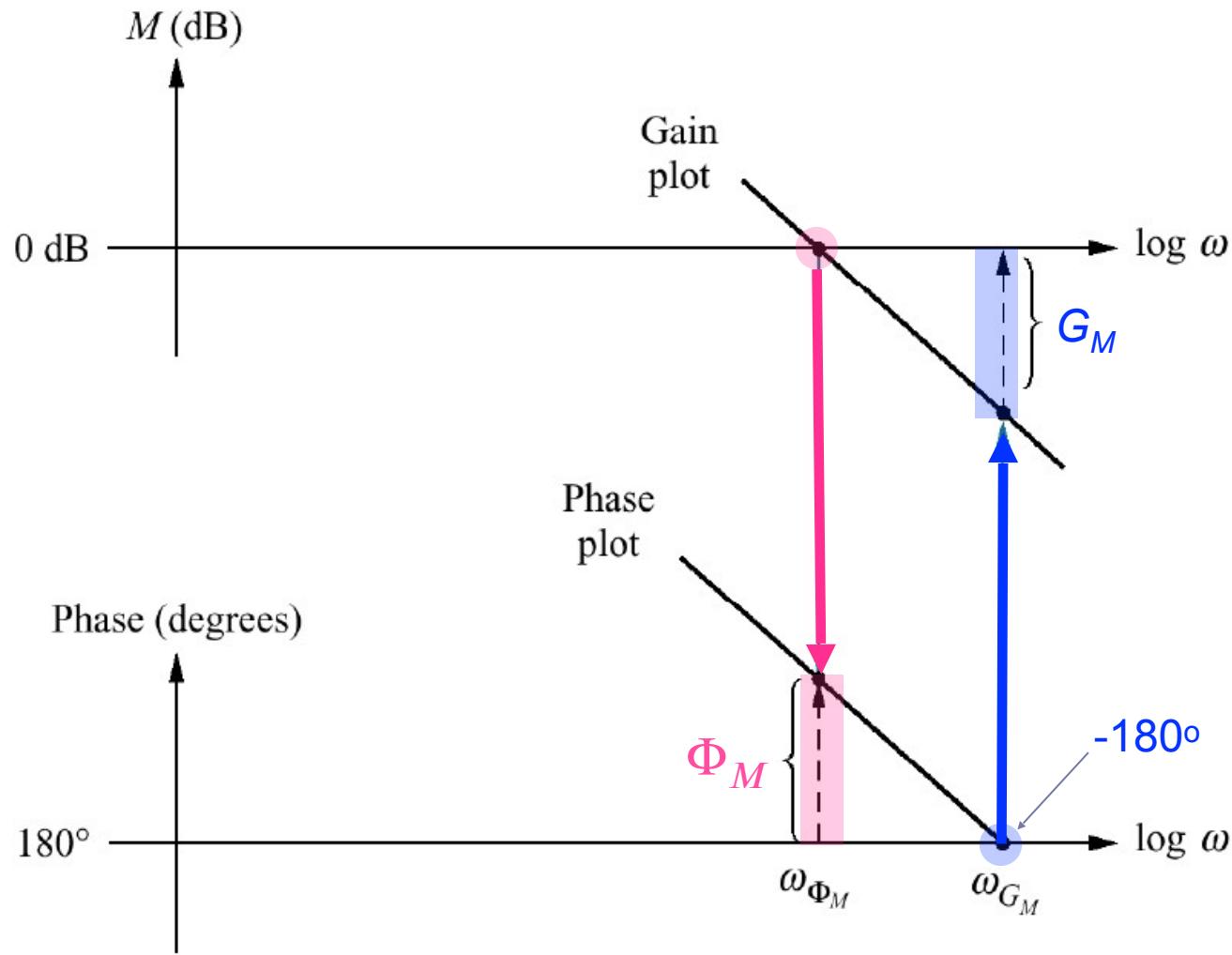
- valor constante = 6,
- um zero e $s=-10$,
- e par de polos complexos conjugados em: raízes de: $s^2+3s+50=0$, ou:

$$\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1 \quad \omega_0 = \sqrt{50} = 7.07, \quad \zeta = \frac{3\sqrt{50}}{2 \cdot 50} = 0.21$$



Estabilidade, Margem de Ganho, G_M e Margem de Fase, Φ_M através do Diagrama de Bode...

Aplicação em Controle Automático



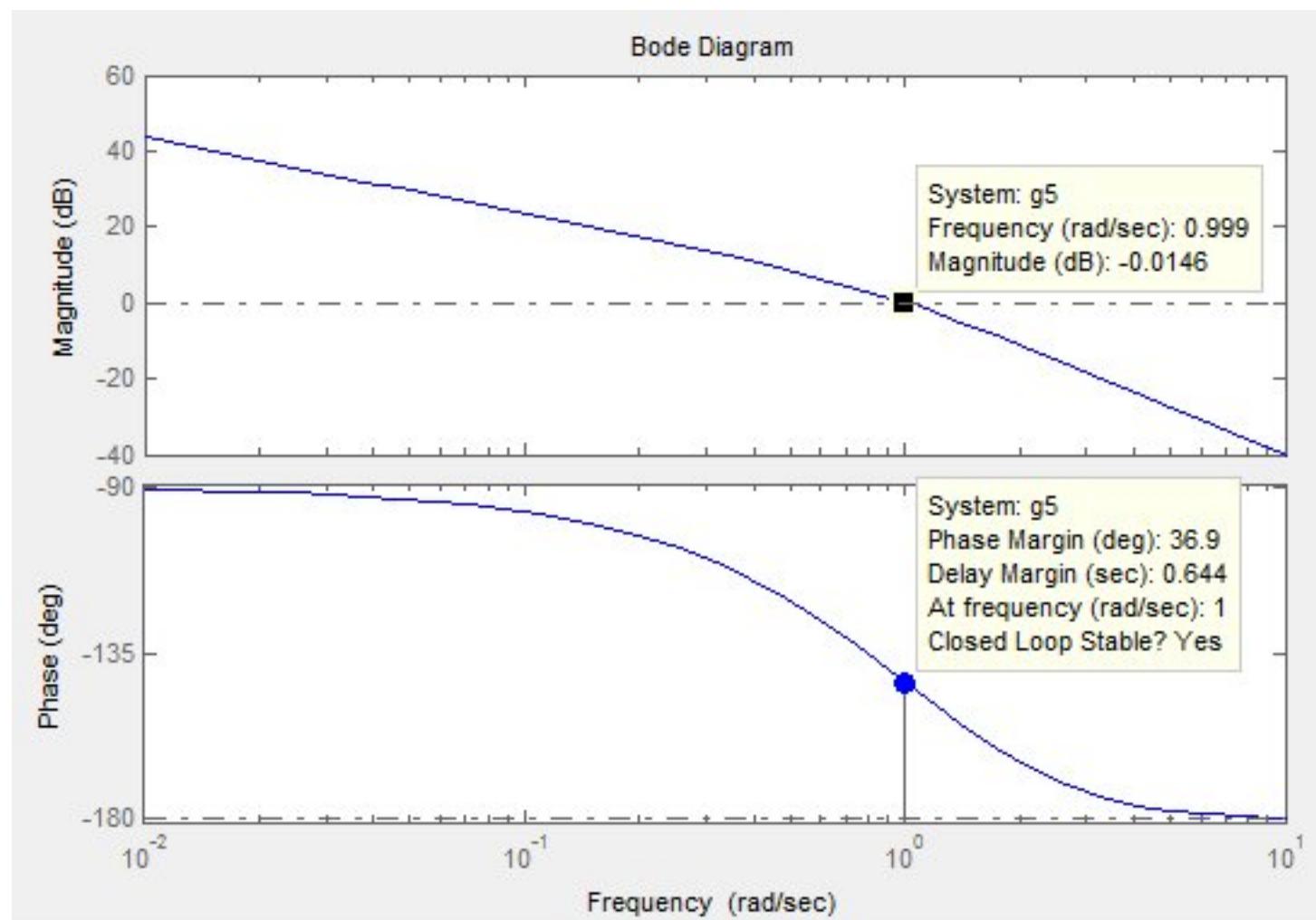
Outros Exemplos:



$$G_5(s) = \frac{s+3}{s(s+1)(s+2)}$$

$$= \frac{3\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot 2\left(\frac{s}{2} + 1\right)}$$

$$= \frac{3}{2} \cdot \frac{\left(\frac{s}{3} + 1\right)}{s(s+1) \cdot \left(\frac{s}{2} + 1\right)}$$



Resumo:

Caso de pólo simples real...

$$G(s) = 1 + \frac{s}{\omega_p} = \frac{1}{1 + j \frac{\omega}{\omega_p}}$$

Onde ω_p = freq. de corte (-3 dB neste ponto);

Magnitude:

$$|G(j\omega)| = \left| \frac{1}{1 + j \frac{\omega}{\omega_p}} \right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}}$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10} \left[\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_p}\right)^2}} \right]$$

Quando $\omega \ll \omega_p$, teremos: $\omega/\omega_p \rightarrow 0$ e:

$$\sqrt{1 + (\omega/\omega_p)^2} \approx 1 \therefore |G(j\omega)|_{dB} \approx -20 \log_{10}(1/1) = 0$$

Quando $\omega \gg \omega_p$, acontece: $\omega/\omega_p \rightarrow \infty$ e:

$$\sqrt{1 + (\omega/\omega_p)^2} \approx \sqrt{(\omega/\omega_p)^2} \approx \omega/\omega_p$$

$$|G(j\omega)|_{dB} \approx -20 \log_{10}(\omega/\omega_p)$$

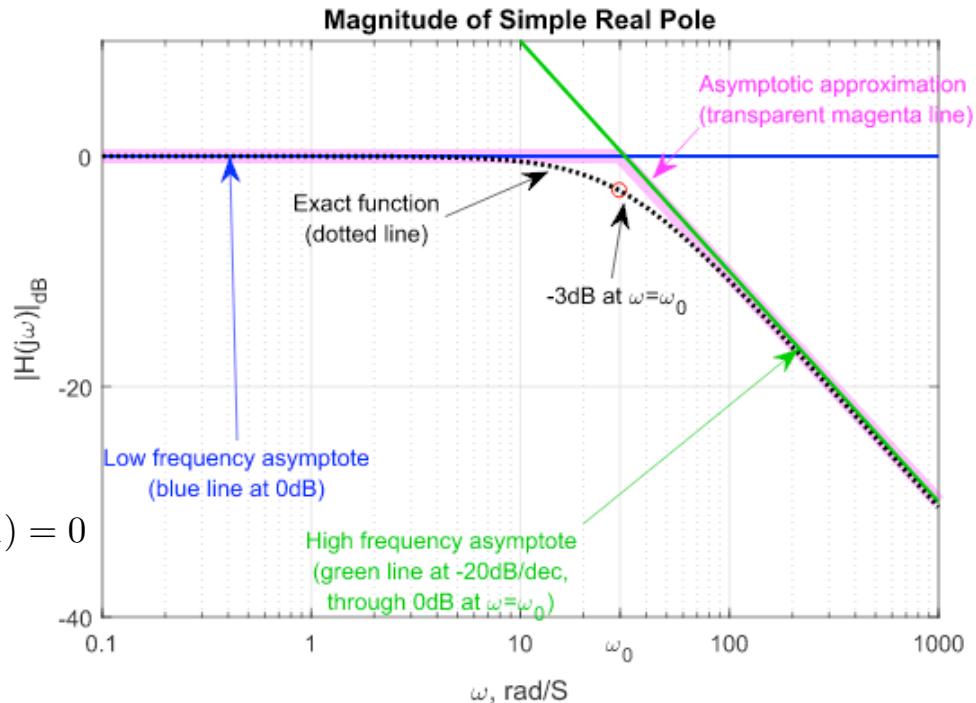
Se: $\omega = 10\omega_p \therefore |G(j\omega)|_{dB} = -20 \log_{10}(10) = -20$ dB

$$|G(j\omega)|_{dB} = -20 \log_{10}(10) = -20 \text{ dB}$$

Quando $\omega = \omega_p$:

$$|G(j\omega_p)|_{dB} = 20 \log_{10} \left(\frac{1}{\sqrt{1 + (\omega_p/\omega_p)^2}} \right)$$

$$|G(j\omega_p)|_{dB} = 20 \log_{10} (1/\sqrt{2}) \approx -3 \text{ dB}$$



Resumo: Caso de pólo simples real...

$$G(s) = 1 + \frac{s}{\omega_p} = \frac{1}{1 + j \frac{\omega}{\omega_p}}$$

Onde ω_p = freq. de corte (-45° neste ponto);

Fase:

$$\angle G(j\omega) = \angle \left[1 / \left(1 + j \frac{\omega}{\omega_p} \right) \right]$$

$$\angle G(j\omega) = -\angle \left(1 + j \frac{\omega}{\omega_p} \right) = -\tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

Quando $\omega \ll \omega_p$, teremos: $\omega/\omega_p \rightarrow 0$ e:

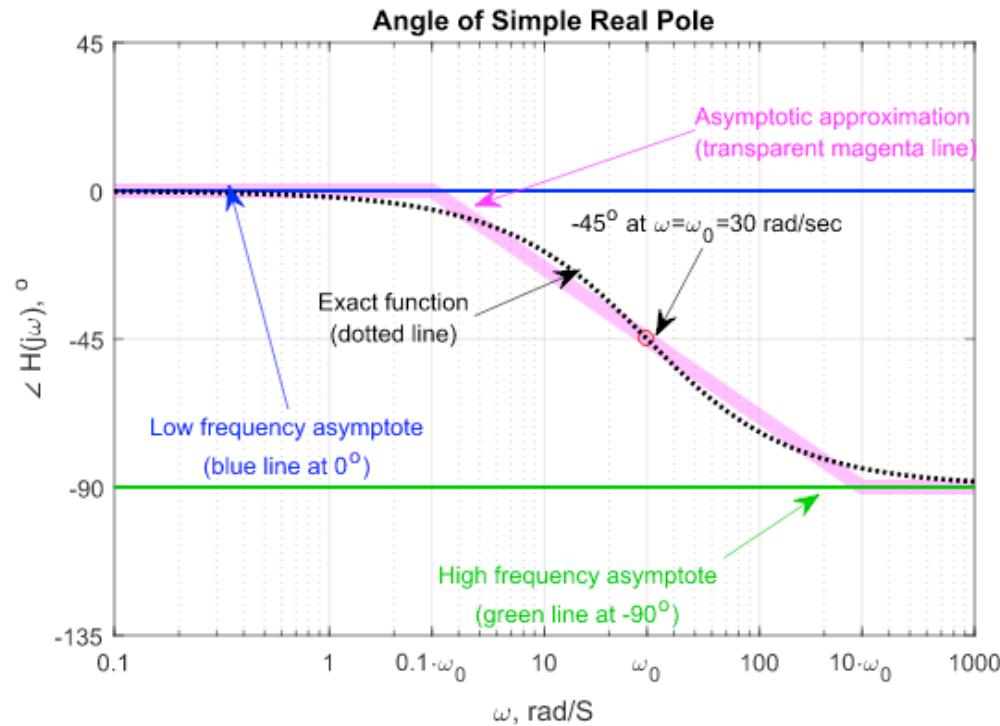
$$\angle G(j\omega) \approx -\tan^{-1}(0) = 0^\circ = 0 \text{ rad}$$

Quando $\omega > > \omega_p$, acontece: $\omega/\omega_p \rightarrow \infty$ e:

$$\angle G(j\omega) \approx -\tan^{-1}(\infty) = -90^\circ = -\frac{\pi}{2} \text{ rad}$$

Quando $\omega = \omega_p$:

$$\angle G(j\omega) \approx -\tan^{-1}(1) = -45^\circ = -\frac{\pi}{4} \text{ rad}$$



Ref.: <https://lpsa.swarthmore.edu/Bode/BodeHow.html> (acessado Jun/2022)

Resumo: Caso de zero simples real...

$$G(s) = 1 + \frac{s}{\omega_z} = 1 + j \frac{\omega}{\omega_z}$$

Onde ω_z = freq. de corte (+3 dB neste ponto);

Magnitude:

$$|G(j\omega)| = \left| 1 + j \frac{\omega}{\omega_z} \right|$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10} \left[\sqrt{1 + (\omega/\omega_z)^2} \right]$$

Quando $\omega \ll \omega_z$, teremos: $\omega/\omega_z \rightarrow 0$ e:

$$\sqrt{1 + (\omega/\omega_p)^2} \approx 1 \therefore |G(j\omega)|_{dB} \approx 20 \log_{10}(1) = 0$$

Quando $\omega \gg \omega_z$, acontece: $\omega/\omega_z \rightarrow \infty$ e:

$$\sqrt{1 + (\omega/\omega_z)^2} \approx \sqrt{(\omega/\omega_z)^2} \approx \omega/\omega_z$$

$$|G(j\omega)|_{dB} \approx 20 \log_{10}(\omega/\omega_z)$$

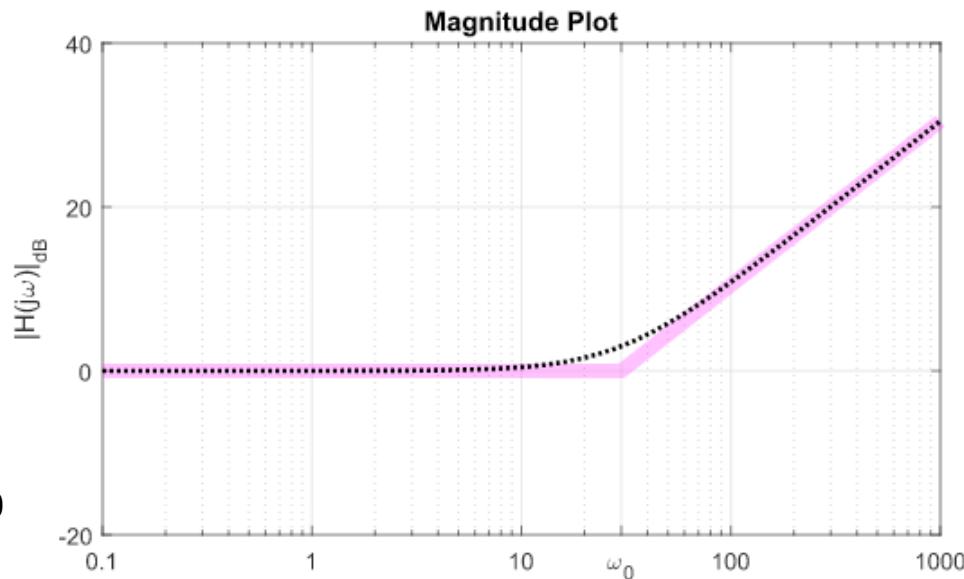
Se: $\omega = 10 \omega_z \therefore |G(j\omega)|_{dB} = 20 \log_{10}(10)$

$$|G(j\omega)|_{dB} = 20 \log_{10}(10) = 20 \text{ dB}$$

Quando $\omega = \omega_z$:

$$|G(j\omega_z)|_{dB} = 20 \log_{10} \left(\sqrt{1 + (\omega_z/\omega_z)^2} \right)$$

$$|G(j\omega_p)|_{dB} = 20 \log_{10} \left(\sqrt{2} \right) \approx 3 \text{ dB}$$



Resumo: Caso de zero simples real...

$$G(s) = 1 + \frac{s}{\omega_z} = 1 + j \frac{\omega}{\omega_z}$$

Onde ω_z = freq. de corte ($+45^\circ$ neste ponto);

Fase:

$$\angle G(j\omega) = \angle (1 + j\omega/\omega_p)$$

$$\angle G(j\omega) = \angle \left(1 + j \frac{\omega}{\omega_p}\right) = \tan^{-1} \left(\frac{\omega}{\omega_p}\right)$$

Quando $\omega < < \omega_z$, teremos: $\omega/\omega_z \rightarrow 0$ e:

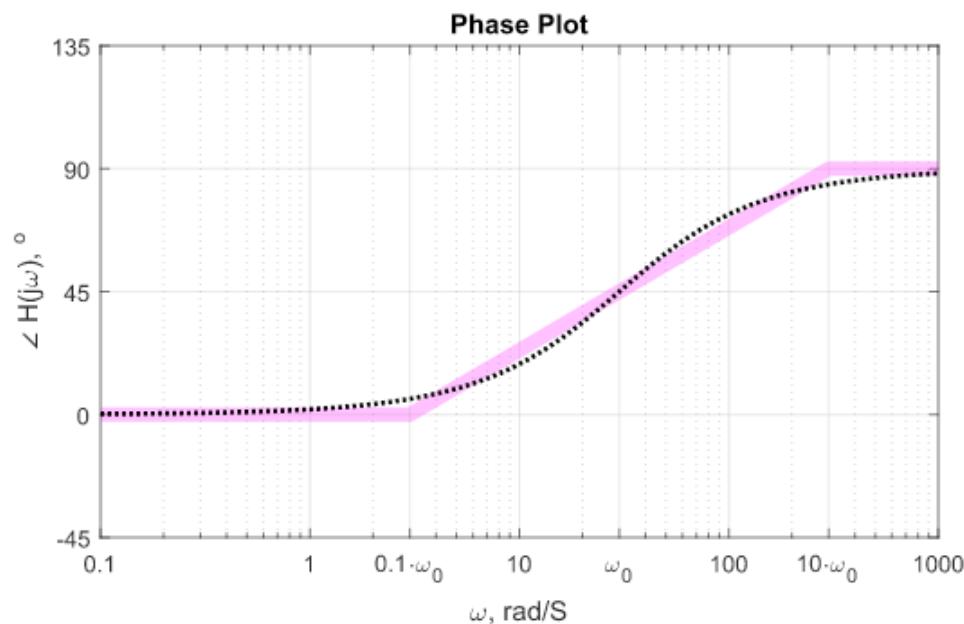
$$\angle G(j\omega) \approx \tan^{-1}(0) = 0^\circ = 0 \text{ rad}$$

Quando $\omega > > \omega_z$, acontece: $\omega/\omega_z \rightarrow \infty$ e:

$$\angle G(j\omega) \approx \tan^{-1}(\infty) = 90^\circ = \frac{\pi}{2} \text{ rad}$$

Quando $\omega = \omega_z$:

$$\angle G(j\omega) \approx \tan^{-1}(1) = 45^\circ = \frac{\pi}{4} \text{ rad}$$



Ref.: <https://lpsa.swarthmore.edu/Bode/BodeHow.html> (acessado Jun/2022)

Resumo: Pólo na origem

$$G(s) = \frac{1}{s} \quad \therefore \quad G(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega}$$

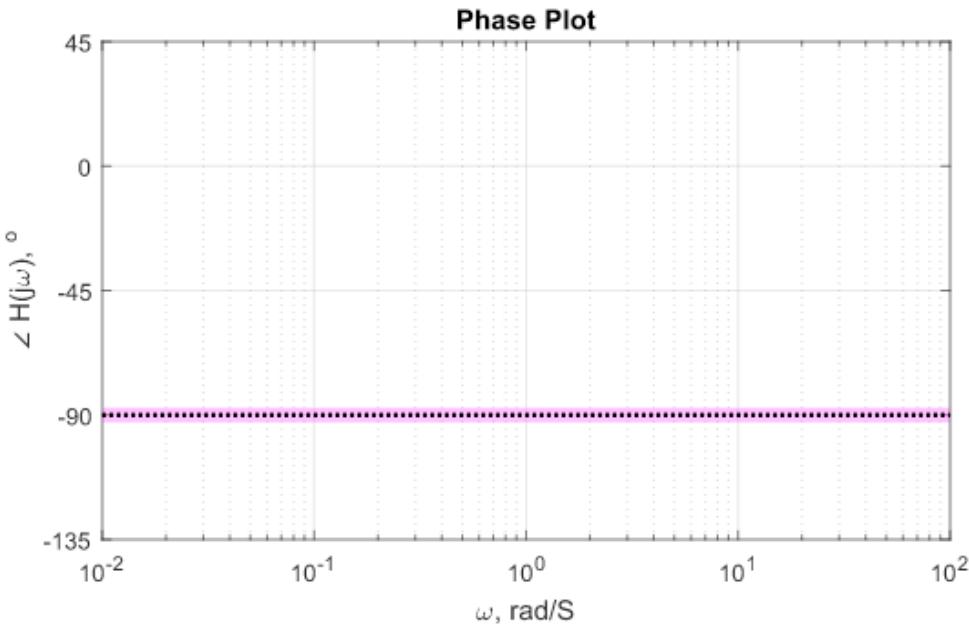
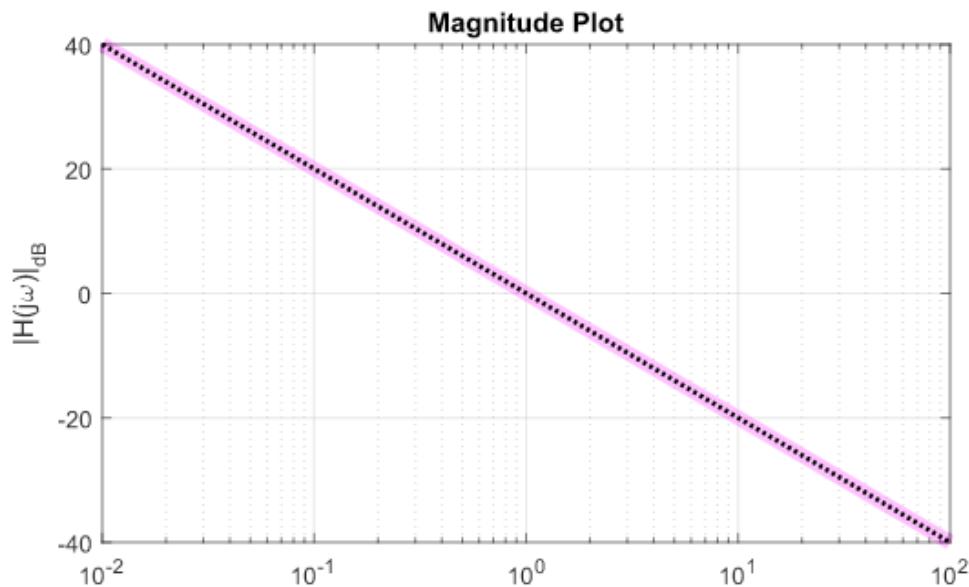
Magnitude:

$$|G(j\omega)| = \left| -\frac{j}{\omega} \right| = \frac{1}{\omega}$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10} \left(\frac{1}{\omega} \right) = -20 \log_{10}(\omega)$$

Fase:

$$\angle G(j\omega) = \angle \left(-\frac{j}{\omega} \right) = -90^\circ$$



Resumo: Zero na origem

$$G(s) = s \quad \therefore \quad G(j\omega) = j\omega$$

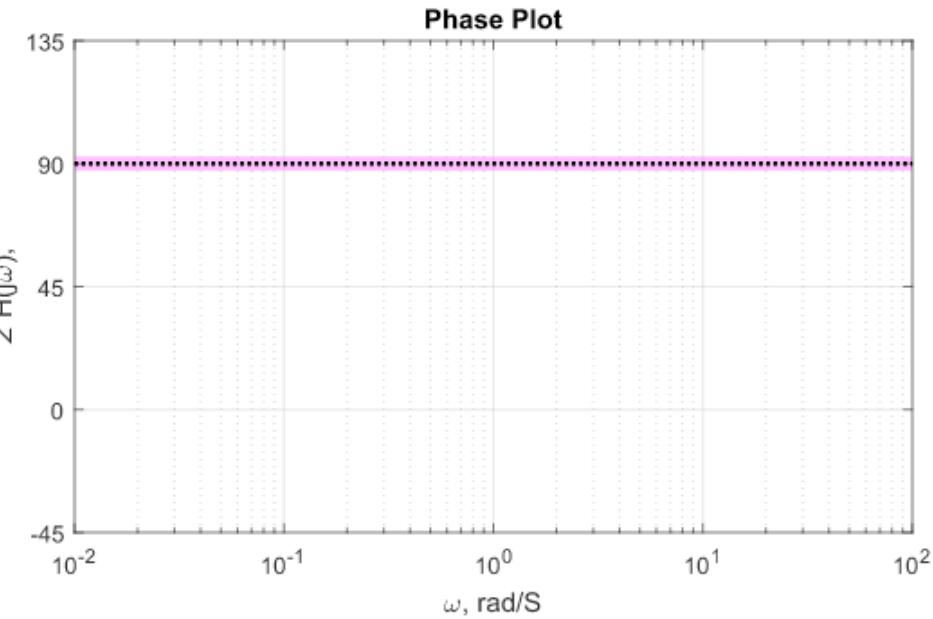
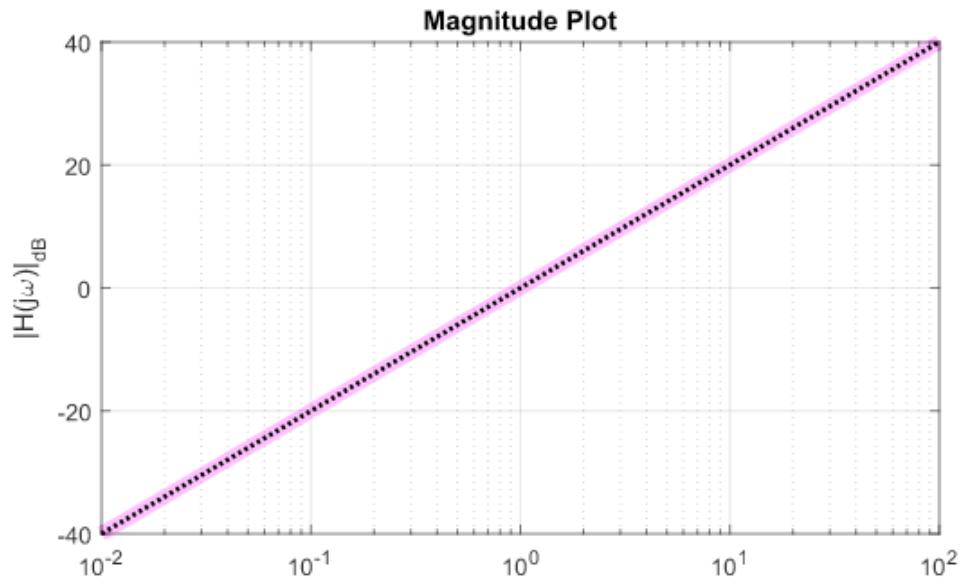
Magnitude:

$$|G(j\omega)| = |j\omega| = \omega$$

$$|G(j\omega)|_{dB} = 20 \cdot \log_{10}(\omega) = 20 \log_{10}(\omega)$$

Fase:

$$\angle G(j\omega) = \angle (j\omega) = 90^\circ$$



Resumo: Pólos complexos

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

Magnitude:

$$|G(j\omega)| = \left| \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right| = \left| \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right) + 1} \right|$$

$$|G(j\omega)| = \left| \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]} \right| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10} \left\{ \sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \right\} \quad (1)$$

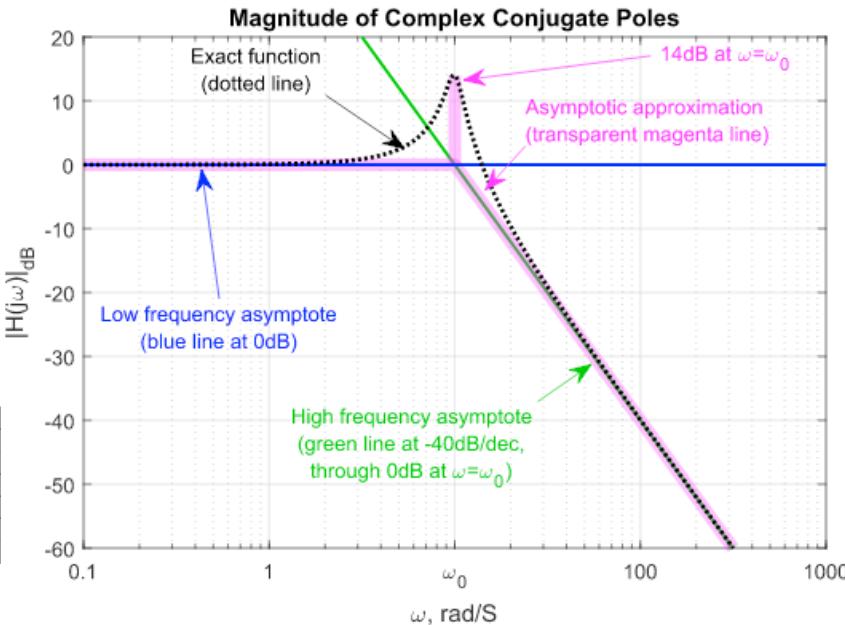
Caso de $\omega \ll \omega_n$: pode-se aproximar (1) para:

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10}(1) = 0$$

Caso de $\omega \gg \omega_n$: pode-se aproximar (1) para:

$$|G(j\omega)|_{dB} = -20 \cdot \log_{10} \left[\left(\frac{\omega}{\omega_n}\right)^2 \right] = -40 \log_{10} \left(\frac{\omega}{\omega_n} \right)$$

Isto implica que para década acima de ω_n , magnitude = -40 (dB/déc).



Caso de $\omega \approx \omega_n$: ocorre um pico próximo de ω_n :

$$\text{Ocorre em } \omega_\tau: \omega_\tau = \omega_n \sqrt{1 - 2\zeta^2}$$

Pico ocorre para: $0 < \zeta < 1/2$, com valor de:

$$|G(j\omega_\tau)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Ver próximo slide para maiores detalhes.

Resumo

Pólos complexos ($0 < \zeta < 1$)

Seja:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + s2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

Um pico de ganho pode ser identificado em $|G(j\omega)|$, fazendo:

$$\frac{d}{d\omega} (|G(j\omega)|) \Big|_{\omega \rightarrow 0}$$

Este pico ocorre em: $\omega_\tau = \omega_n \sqrt{1 - 2\zeta^2}$,

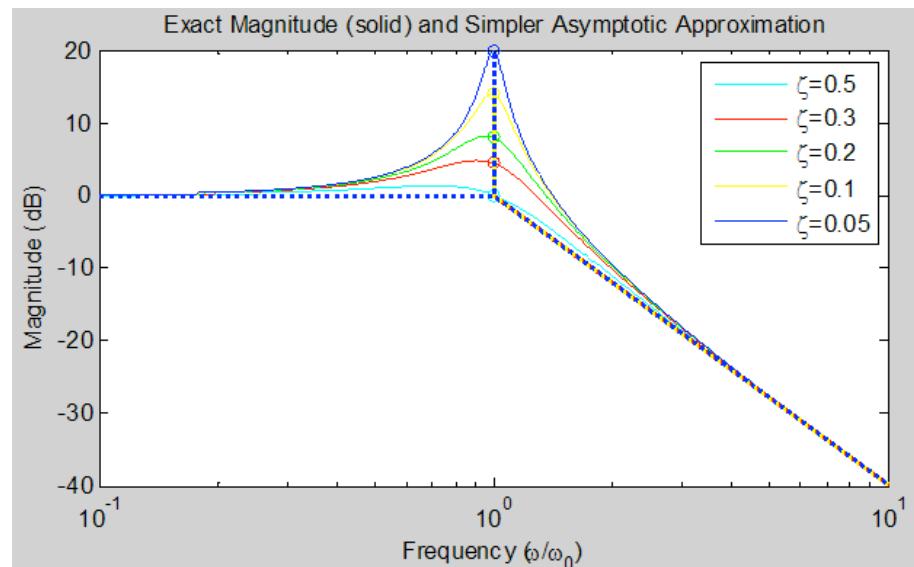
Com magnitude de:

$$|G(j\omega)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \approx \frac{1}{2\zeta}$$

$$|G(j\omega)| = -20 \log_{10} \left(2\zeta\sqrt{1 - \zeta^2} \right)$$

O pico apenas existe para: $0 < \zeta < 0,707$ ($= 1/\sqrt{2}$)

ζ	Peak frequency			Peak Height		
	Exact	Approx	% diff	Exact	Approx	% diff
	$\omega_0 \sqrt{1 - 2\zeta^2}$	ω_0		$\frac{1}{2\zeta\sqrt{1 - \zeta^2}}$	$\frac{1}{2\zeta}$	
0.5	$0.71\omega_0$	ω_0	29%	1.15	1.00	15%
0.4	$0.83\omega_0$	ω_0	17%	1.37	1.25	9.1%
0.3	$0.91\omega_0$	ω_0	9.5%	1.75	1.67	4.8%
0.2	$0.96\omega_0$	ω_0	4.1%	2.55	2.50	2.1%
0.1	$0.99\omega_0$	ω_0	1.0%	5.02	5.00	0.5%
0.05	$1.00\omega_0$	ω_0	0.3%	10.0	10.0	0.1%



Ref.: <https://lpsa.swarthmore.edu/Bode/underdamped/underdampedApprox.html#Simpler> (acessado em Jun/2022)

Resumo: Pólos complexos

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

Fase:

$$\angle G(j\omega) = \angle \left[\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} \right]$$

$$\angle G(j\omega) = -\angle \left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1 \right]$$

$$\angle G(j\omega) = -\angle \left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) \right]$$

$$\angle G(j\omega) = -\tan^{-1} \left[\left(2\zeta \frac{\omega}{\omega_n} \right) \Bigg/ \left(1 - \left(\frac{\omega}{\omega_n} \right)^2 \right) \right] \quad (2)$$

Caso de $\omega < < \omega_n$: pode-se aproximar (2) para:

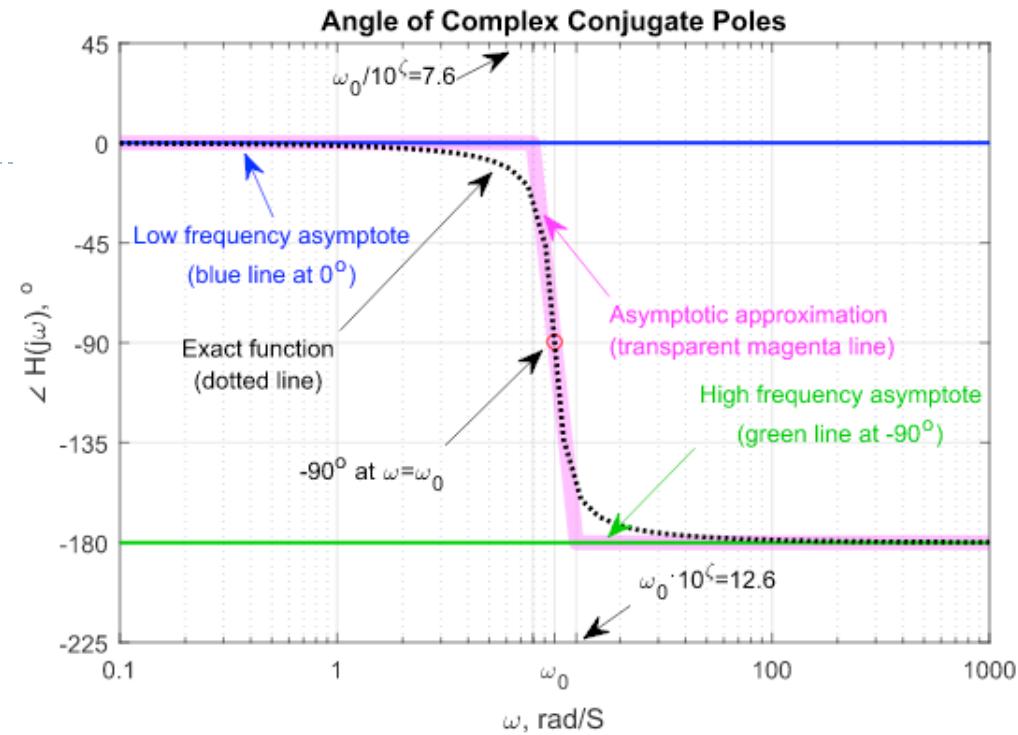
$$\angle G(j\omega) \approx -\tan^{-1} \left(\frac{2\zeta\omega}{\omega_n} \right) \approx -\tan^{-1}(0) = 0^\circ = 0 \text{ rad}$$

Caso de $\omega > > \omega_n$: pode-se aproximar (2) para:

$$\angle G(j\omega) \approx -180^\circ = -\pi \text{ rad}$$

Caso de $\omega = \omega_n$:

$$\angle G(j\omega) \approx -90^\circ = -\pi/2 \text{ rad}$$



Resumo

Pólos complexos ($0 < \zeta < 1$)

Seja:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$
$$\angle G(j\omega) = -\tan^{-1} \left[\left(2\zeta \frac{\omega}{\omega_n} \right) \Bigg/ \left(1 - \left(\frac{\omega}{\omega_n} \right)^2 \right) \right] \quad (2)$$

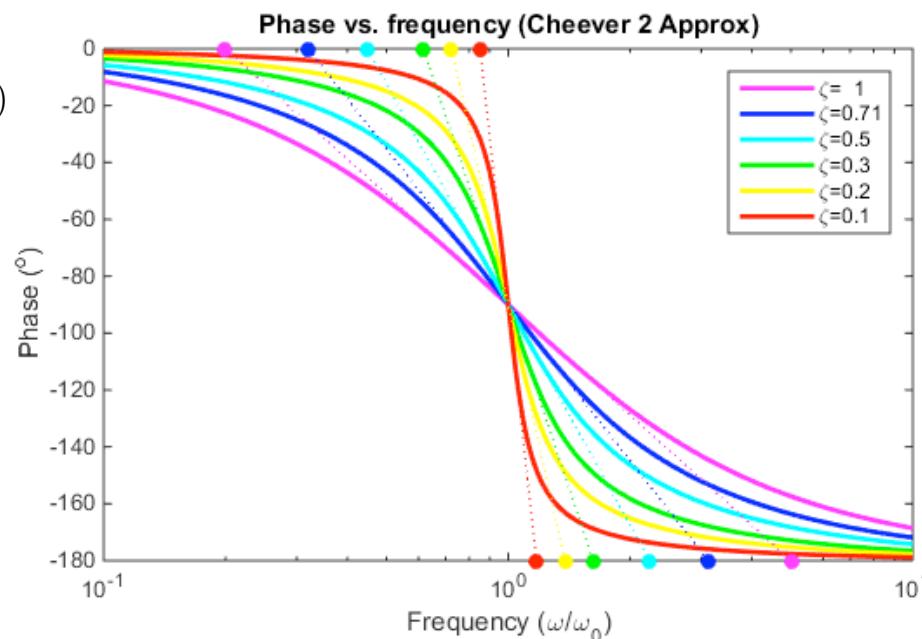
A declividade de $\angle G(j\omega)$, pode ser obtida, fazendo:

$$\frac{d}{d\omega} (\angle G(j\omega)) \Bigg|_{\omega \rightarrow 0}$$

O que permite identificar os pontos:

$$\omega_{low} = \omega_n e^{-\frac{\pi}{2}\zeta} \approx \omega_n \cdot 0,2^\zeta$$

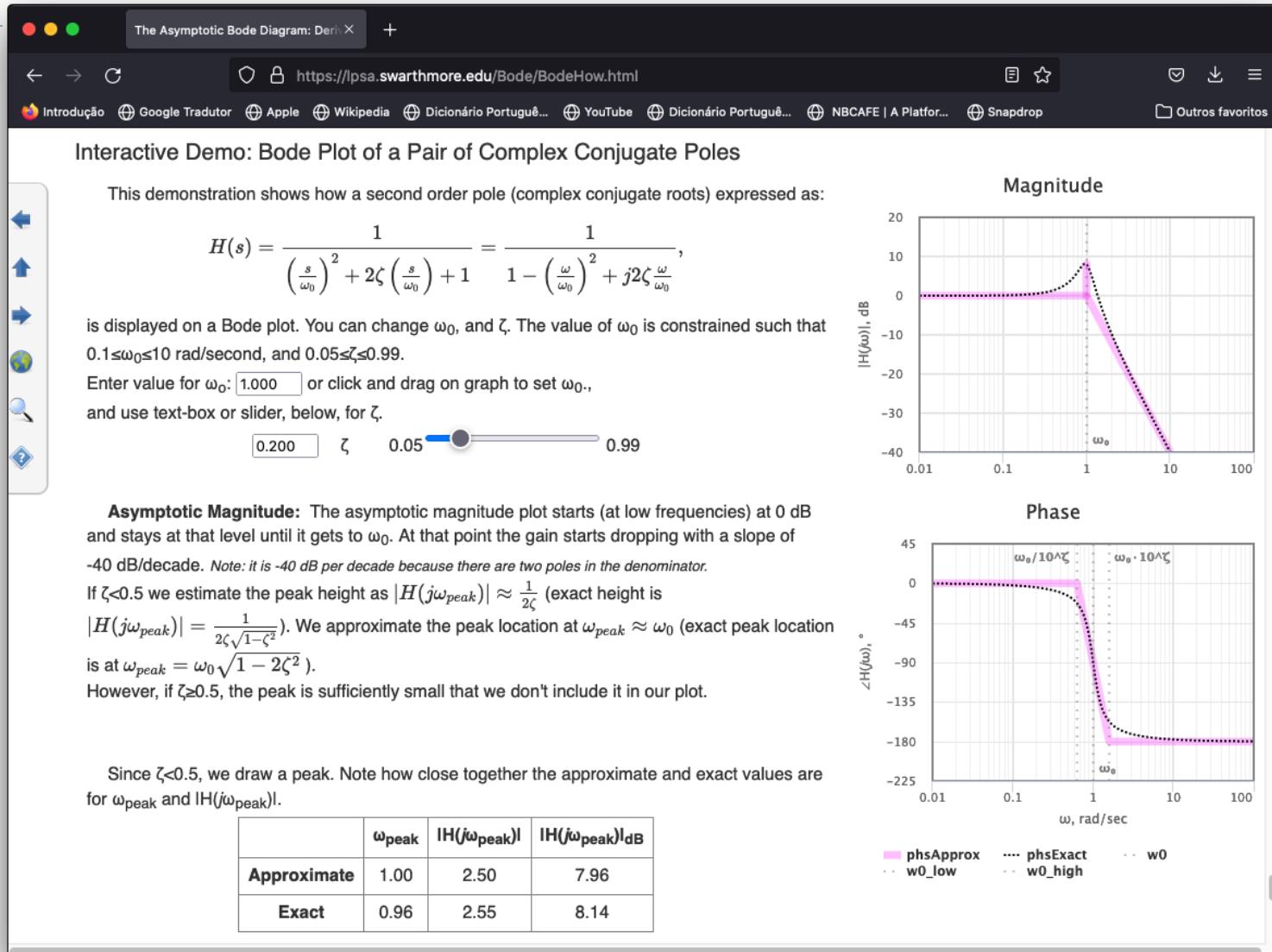
$$\omega_{high} = \frac{\omega_n}{e^{-\frac{\pi}{2}\zeta}} \approx \frac{\omega_n}{0,2^\zeta} = \omega \cdot 5^\zeta$$



Ref.: <https://lpsa.swarthmore.edu/Bode/underdamped/underdampedApprox.html#Simpler> (acessado em Jun/2022)

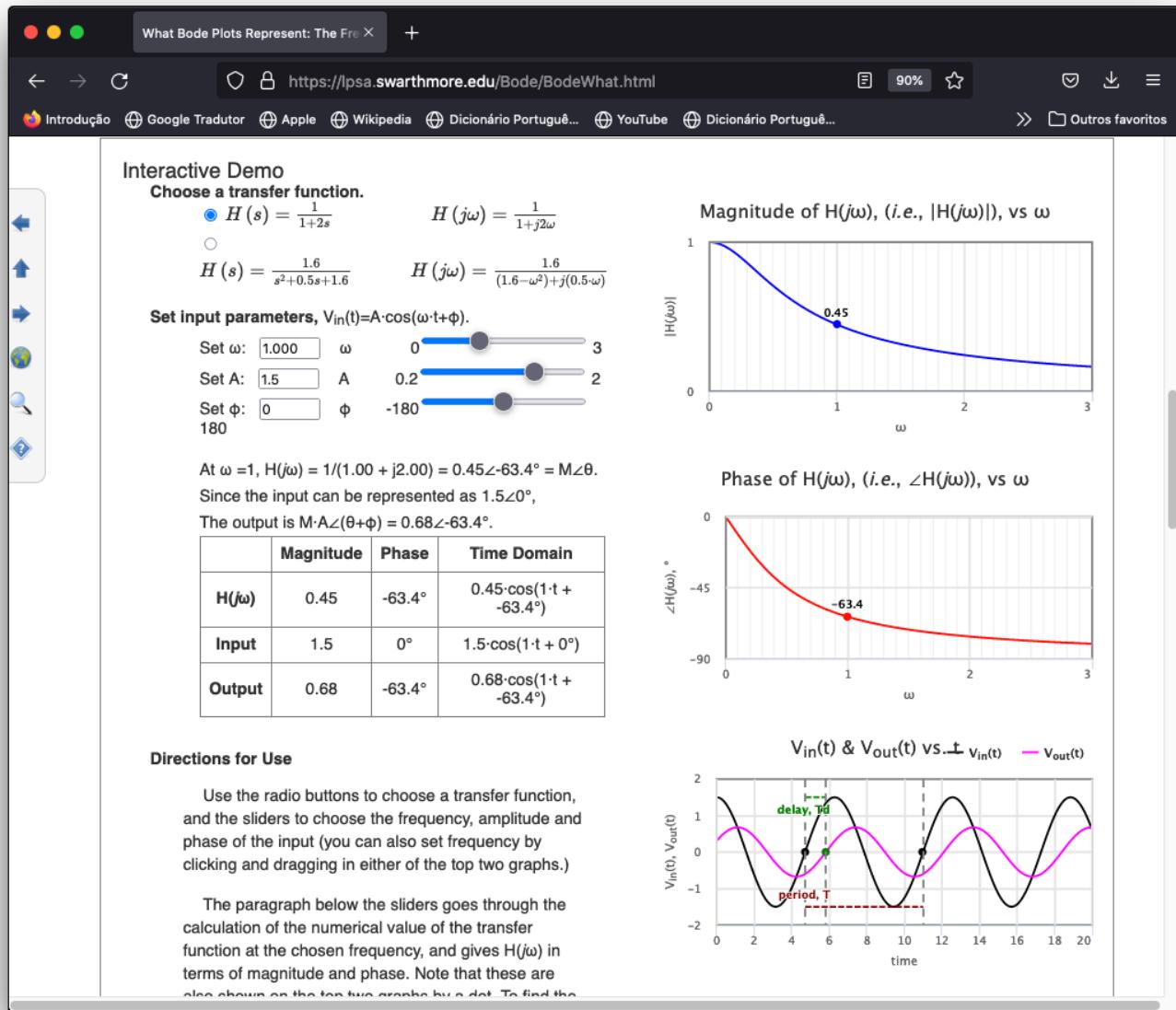
Resumo: Esboços Diagramas de Bode

► <https://lpsa.swarthmore.edu/Bode/BodeHow.html>



Resposta frequencial:: Aplicação

► <https://lpsa.swarthmore.edu/Bode/BodeWhat.html>



Resposta frequencial

Aplicação

- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \mu\text{F}$
- Solução:

$$H(s) = \frac{1}{1 + sRC} = \frac{1}{1 + 2 \times 10^6 \cdot 1 \times 10^{-6} \cdot s} = \frac{1}{1 + 2s}$$

$$H(j\omega) = \frac{1}{1 + j2\omega} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

$$|H(j\omega)| = \left| \frac{1}{1 + j\frac{\omega}{\omega_p}} \right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}}$$

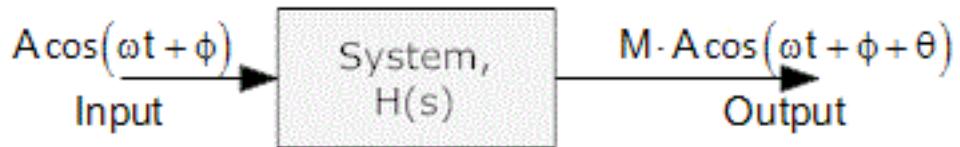
$$\angle H(j\omega) = -\angle \left(1 + j\frac{\omega}{\omega_p} \right) = -\tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

$$\omega_p = 1/2 \text{ (rad/s)}$$

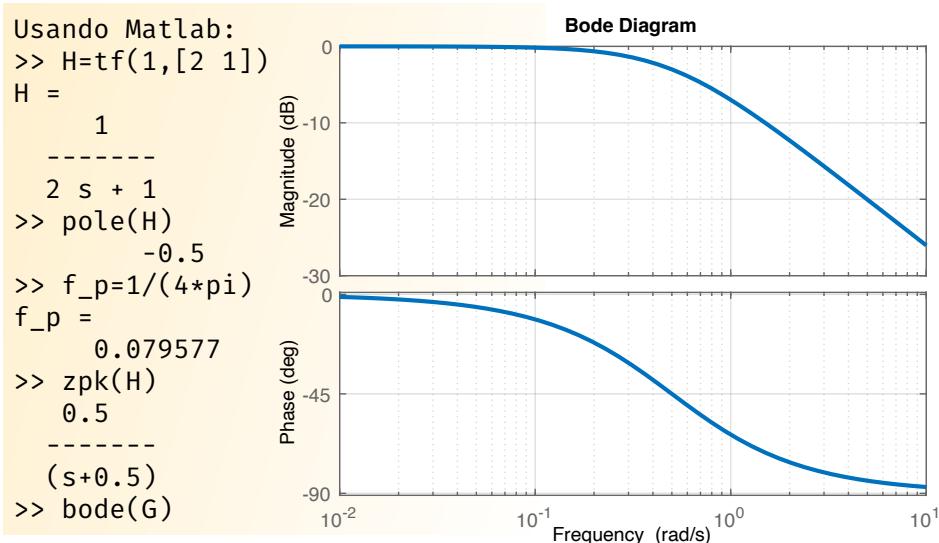
$$\therefore f_p = \frac{\omega}{2\pi} = \frac{1}{4\pi} = 0,079577 \text{ Hz}$$

$$V_{out}(j\omega) = V_{in}(j\omega) \cdot H(j\omega)$$

$$|V_{out}(j\omega)| =$$



Circuit	Transfer Function
	$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + sRC}$



Resposta frequencial

Aplicação

- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \mu\text{F}$
- Solução:

$$H(s) = \frac{1}{1 + sRC} = \frac{1}{1 + 2 \times 10^6 \cdot 1 \times 10^{-6} \cdot s} = \frac{1}{1 + 2s}$$

$$H(j\omega) = \frac{1}{1 + j2\omega} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

$$|H(j\omega)| = \left| \frac{1}{1 + j\frac{\omega}{\omega_p}} \right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}}$$

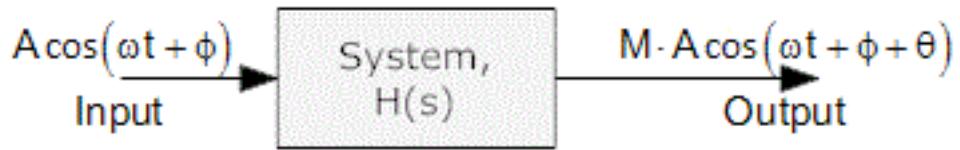
$$\angle H(j\omega) = -\angle \left(1 + j\frac{\omega}{\omega_p} \right) = -\tan^{-1} \left(\frac{\omega}{\omega_p} \right)$$

$$\omega_p = 1/2 \text{ (rad/s)}$$

$$\therefore f_p = \frac{\omega}{2\pi} = \frac{1}{4\pi} = 0,079577 \text{ Hz}$$

$$V_{out}(j\omega) = V_{in}(j\omega) \cdot H(j\omega)$$

$$|V_{out}(j\omega)| =$$



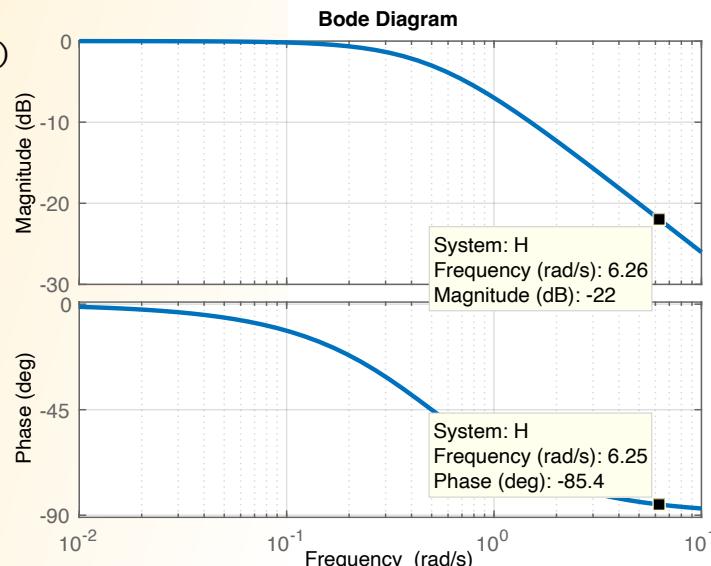
Circuit	Transfer Function
	$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + sRC}$

Usando Matlab:

```

>> H=tf(1,[2 1])
H =
  1
  -----
  2 s + 1
>> pole(H)
  -0.5
>> f_p=1/(4*pi)
f_p =
  0.079577
>> zpk(H)
  0.5
  -----
  (s+0.5)
>> bode(G)
>> f=2*pi
f =
  6.2832

```



Resposta frequencial

Aplicação

- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \mu\text{F}$
- Solução:

$$H(s) = \frac{1}{1 + sRC} = \frac{1}{1 + 2 \times 10^6 \cdot 1 \times 10^{-6} \cdot s} = \frac{1}{1 + 2s}$$

$$H(j\omega) = \frac{1}{1 + j2\omega} = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

Ganho em $\omega = 1 \text{ Hz} = 2\pi \text{ rad/s}$:

$$|G(j\omega)|_{dB} = -22 \text{ dB}$$

$$-22 = 20 \log_{10}(|G(j\omega)|)$$

$$|G(j\omega)| = 10^{-\frac{22}{20}} = 0,079433$$

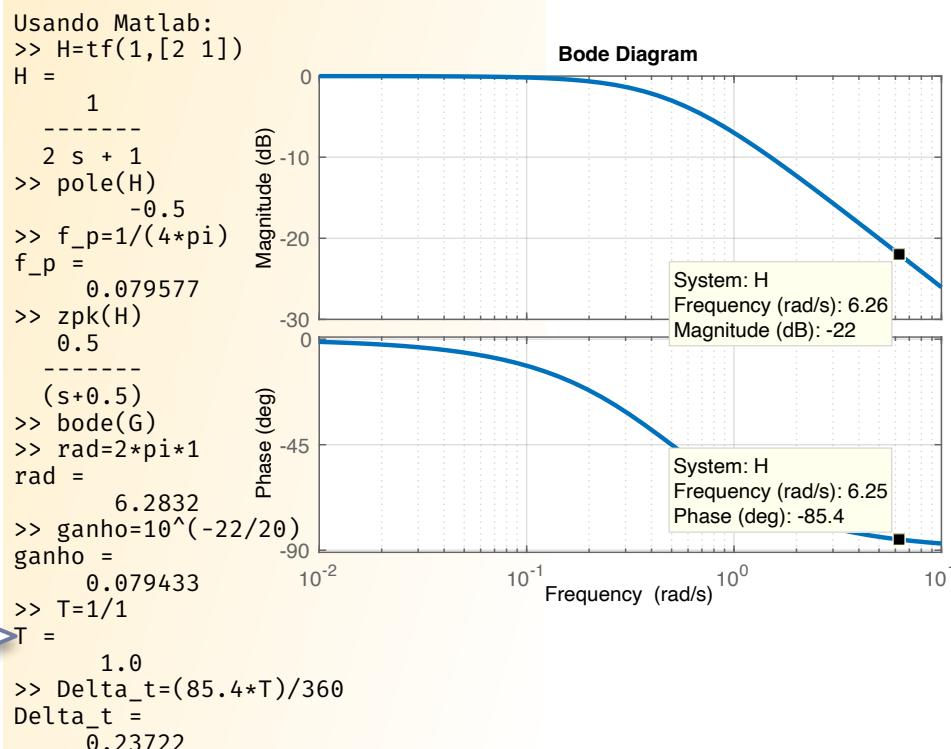
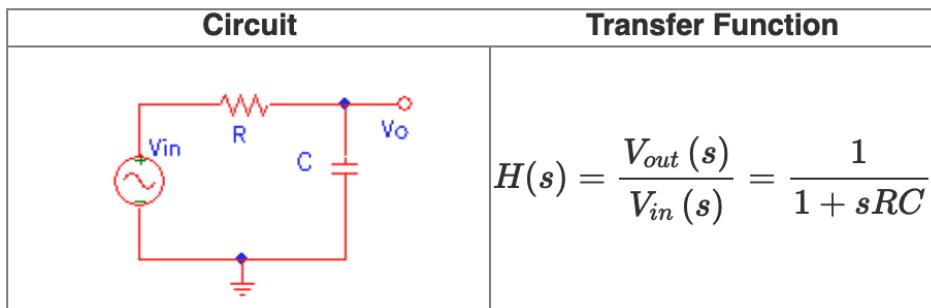
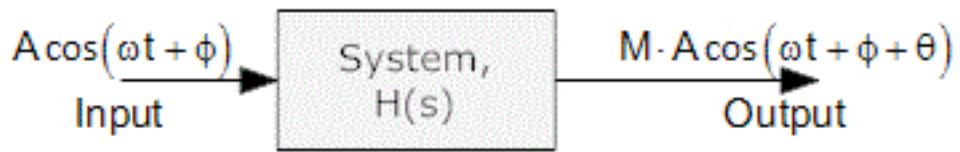
$$\angle G(j\omega) = -85,4^\circ = -85,4^\circ \cdot \frac{\pi}{180^\circ} = -1.4905 \text{ rad}$$

$$T \rightarrow 360^\circ (2\pi)$$

$$\Delta t \leftarrow 85,4^\circ$$

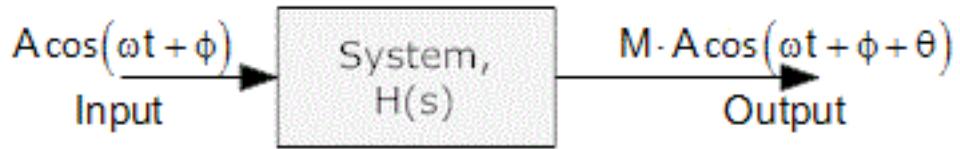
$$\Delta t = \frac{85,4^\circ \cdot T}{360^\circ} = 0,037755 \text{ segundos}$$

Considerando $T = 0,15915$ segundos.



Resposta frequencial

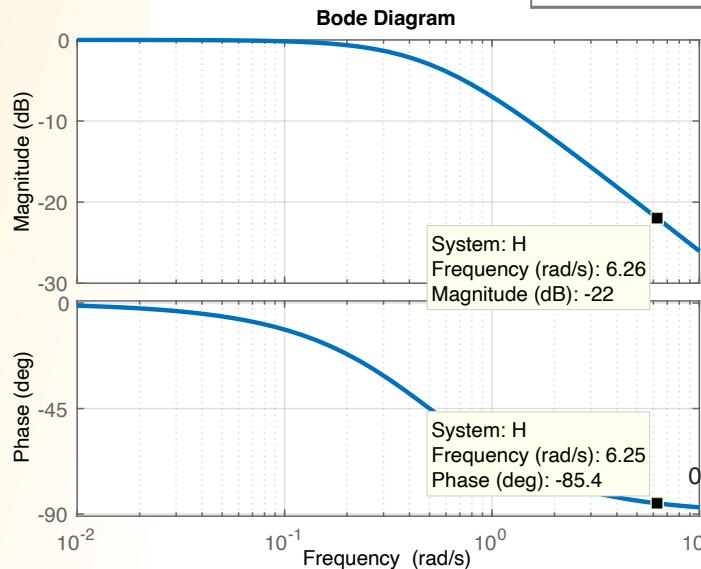
Aplicação



- Problema: mostre o sinal de saída para o sistema ao lado quando se aplica ao mesmo um cosseno de 2 Vpp à frequência de 1 Hz.
- Considerar: $R = 2 \text{ M}\Omega$ e $C = 1 \mu\text{F}$
- Solução:

Usando Matlab:

```
>> H=tf(1,[2 1])
H =
  1
  -
  2 s + 1
>> pole(H)
  -0.5
>> f_p=1/(4*pi)
f_p =
  0.079577
>> zpk(H)
  0.5
  -
  (s+0.5)
>> bode(G)
>> rad=2*pi*1
rad =
  6.2832
>> ganho=10^(-22/20)
ganho =
  0.079433
>> T=1/1
T =
  1.0
>> Delta_t=(85.4*T)/360
Delta_t =
  0.23722
```



Circuit	Transfer Function
	$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + sRC}$

```
>> rad=(85.4*pi)/180
rad =
  1.4905
>> ezplot(@(t)1*cos(2*pi*1*t),[0 2])
% onda entrada
>> hold on
>> ezplot(@(t)ganho*cos(2*pi*1*t-
rad),[0 2]) // onda saída
>> axis([0 2 -1 1])
>> grid
>> legend('V_{in}', 'V_{out}')
```

