

# The reference model approach in feature selection problems

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## Abstract

We carry out a comparison of different methods of feature selection using or not what we call a reference model on top of the procedure, that is a model that describes well our data. We measure the performance and the stability of each method using both the standard approach, that simply relies on data, and the reference model one. We include different procedures in the comparison as a selection through high probability density posterior credibility intervals, a selection made by controlling the Q-value and the local false discovery rate. Finally, we include in the comparison also a real world data example. Results show an increased stability and a better performance of the selection in favour of the reference model approach.

## 1 Introduction

In statistical applications, one of the main steps of the modelling workflow is covariate or feature selection, which can be seen as a special case of model reduction. From a full Bayesian perspective, when the goal of the analysis is to do prediction, no selection should be carried out and integration over all parameters uncertainties, i.e. posterior distributions, is the recommended procedure. In a high dimensional parameter space the choice of the prior plays an even more key role due to different difficulties that can rise, as computation burden and overfitting. In addition to that, sometimes collecting all the features for a future observation is not possible or too expensive. In order to solve or, at least, simplify these problems, different feature selection procedures have been proposed both in the frequentist and the Bayesian setting (e.g. see [Rockova et al., 2012](#); [Vehtari and Ojanen, 2012](#); [Piironen and Vehtari, 2017a](#)). Sometimes the task of the analysis is not prediction, but rather inference on which variables are substantially related to the phenomenon of interest. This typically happens in a *small n large p* scenario, as in many microarray data analysis in medicine or biology. A very large number of genes is usually analysed in two groups of individuals labeled as *positive* or *negative* to some disease. The main goal of the analysis is to spot the subset of explanatory genes in order to perform further experiments (see examples in [Efron, 2012](#)).

Either the target of the selection is the minimal subset of feature with good enough predictive ability, or the whole subset of relevant variables, the selection process is going to be based on assumptions regarding the “true” data generation mechanism. Making assumptions means making models, thus we identify two families of feature selection, or more in general model reduction, approaches depending on how the data generation mechanism is modelled: the data based and the reference model based ones. The former includes all those methods that use the observed empirical distribution as model approximation of the data generation mechanism, as for example happens in the Lasso selection [Tibshirani \(1996\)](#) or in the stepwise backward/forward regression, where the selected submodels balance sparsity with predictive ability on the observed data. The reference model based methods use instead the predictive distribution of a full-encompassing model, i.e. the reference model, as an approximation of the data generation mechanism. Such an idea is widely present in the literature with different names. For example, [Harrell \(2015\)](#) refers to the reference model as a full model that can be thought as the “gold standard” and shows some examples of how it can be used to look for a sparser approximation. He highlights some of the benefits as the possibility to calculate the accuracy with which the submodel approximates the best model and the inheritance of the shrinkage when it is applied to the full model. [Faraggi et al. \(2001\)](#) deal with the necessity of identifying interpretable risk groups in the context of survival data. Neural networks typically perform very well in terms

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of prediction, but it is not very clear how to interpret the significance of each covariate. Faraggi et al. (2001) propose to combine the benefits of the neural networks, which become what we call reference models, with the interpretability of regression trees, in order to provide a framework to make inferences about risk groups. Paul et al. (2008), using the term preconditioning, explore approximating models fitting Lasso (Tibshirani, 1996) or stepwise regression against consistent estimates  $\hat{y}$  of a reference model instead of the observed response quantities  $y$ . Harrell (2015) includes also the possibility of using simple least squares for the approximating models as long as the target is the linear predictor of a full regression model **CHECK**.

The reference model approach has been used for long time also in the Bayesian framework, an example the pioneering work by Lindley (1968). In the case of generalised linear models (GLMs), Tran et al. (2012) provide a procedure to estimate and do variable selection with the goal of prediction. The best approximating model is recovered by optimising the Kullback-Liebler divergence between the predictive distribution of the reference model and the approximating model plus an  $l_1$  penalty on the submodel coefficients. The resulting problem is convex and thus many efficient optimisation algorithm are available. Following Goutis and Robert (1998), Dupuis and Robert (2003) instead project the posterior distribution of the reference model in each candidate model and then select the smallest one with enough explanatory power avoiding any prior elicitation for the submodels. In the special case of candidate models in the family of the GLMs, the optimisation problem correspond to the maximum likelihood estimation problem having the expected model point prediction by the reference model instead of the observed target values. Different extensions have followed Dupuis and Robert (2003), as Nott and Leng (2010) that add an  $l_1$  penalty to select the best approximating model. Piironen et al. (2018) extend the projection algorithm with the clustered projection and choose the best model comparing the predictive performance with the reference model one.

In this paper we try to bring motivations for the use of a reference model. A properly designed model is able to clean part of the noise present in the data, thus to provide an improved and more stable selection process. As an example, we show the improvements in the selection when using the projection predictive approach versus the stepwise regression with a real world dataset. We argue that however the reference model has been used, it can be always seen acting as a filter on the observed data. Following this idea, we devise a simple reference model approach that can be applied on top of any selection procedure. We thus show the reference model benefits, regardless of what specific procedure is applied, using as a benchmark the normal means estimation problem. We apply different state-of-the-art methods comparing in each case the results basing the selection on the data or on the reference model. In case of full Bayesian methods, we report results in terms of shrinkage and accuracy of the posterior estimates, whereas when a full selection is carried out, we compute quantities as false discovery rate, sensitivity and a measure of the stability of the selection. We believe that these results show how the core reason why the reference model based methods perform well is the reference model itself, rather than the specific way of using it. Of course we recognise that all these methods diversify how the subset of variables is actually chosen and which properties it has, thus it remains fundamental to investigate and develop such different procedures. We hope that this work can increase the attractiveness on this family of methods and, thus, it can be of interest to a large number of practitioners. We would like also the reader to note that besides the fact that we consider a Bayesian framework in our discussion, the reference model can and is built also in frequentist settings, as it is done in some of the references previously provided.

The paper is structured as follows. In Section 2 we introduce the idea of the reference model, its benefits with examples, including the projection predictive approach, and how it can be used as a filter on data in a simple way. In Section 3 we use different methods for feature selection comparing the selection results using or not the reference model in the framework of the normal means problem. Finally, in Section 4 we present our conclusions.

## 2 Why the reference model helps

The main assumption under any reference model approach is to be operating in a  $\mathcal{M}$ -complete framework (Bernardo and Smith, 2009; Vehtari and Ojanen, 2012), i.e. to be able to have a model, that is the reference model, which is the best available description of a future observation. There is not a specific definition of

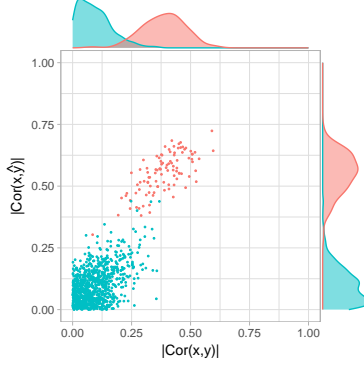


Figure 1: Sample correlation plot of each feature (relevant in red, non-relevant in blue) with the target variable  $y$  and the latent variable  $f$  respectively on the  $x$  and  $y$  axis. Simulation parameters:  $n = 70$ ,  $\rho = 0.3$ ,  $k = 100$ ,  $p = 1000$ .

how a reference model should be; in practice, any full-encompassing model which has a good predictive performance is a candidate reference model. General guidelines on how to devise it are the same about any model in a data analysis context, expect for the fact that no selection is carried out for the reference model.

A good predictive model is able to clean part of the noise present in the data. The noise is the main source of the instability and tends to obscure the relevance of the covariates in relation to the target variable of interest. Indeed, consider the following data generation mechanism for each statistical unit, already used in Piironen et al. (2018):

$$\begin{aligned}
 f &\sim N(0, 1) \\
 Y|f &\sim N(f, 1) \\
 X_j|f &\stackrel{iid}{\sim} N(\sqrt{\rho}f, 1 - \rho) \quad j = 1, \dots, k \\
 X_j|f &\stackrel{iid}{\sim} N(0, 1) \quad j = k + 1, \dots, p
 \end{aligned} \tag{1}$$

$f$  is the latent variable of interest of which  $Y$  is a noisy observation. The first  $k$  covariates are significantly related to the target variable  $Y$  and correlated among themselves. Precisely  $\rho$  is the correlation among any pair of the first  $k$  covariates, whereas  $\sqrt{\rho}$  and  $\sqrt{\rho}/2$  are the level of correlation between any significant covariate and respectively  $f$  and  $Y$ . If we had an infinite amount of observations, the sample correlation would be equal to the true correlation, though, even in this ideal asymptotic regime, the correlation computed would be still a biased indicator of the true relevance of each variable with respect to the latent variable of interest due to the intrinsic noisy nature of  $Y$ . When using a reference model, we are trying to model  $f$  through the covariates  $\{X_j\}_{j=1}^p$  taking in account that what we observe (i.e.  $Y$ ) is corrupted by noise. Thus, if our model is good, we are able to describe  $f$  better than the simply observed variable  $Y$  and to use such information to assess the relevance of the features. Indeed, Figure 1 shows the scatter plot of the absolute value of the sample correlation of each feature with the point prediction by a reference model, in this case see model (12), against the absolute value of the sample correlation of each feature with the noisy observations  $\{y\}_{i=1}^n$ . Looking at the marginal distribution, we observe that using a reference model the two group of features, i.e. relevant and non-relevant, have a minor overlap, and thus can be better distinguished, with respect to when the correlation is computed using directly the observed data. We can think about the correlation as a general indicator of the relevance of a feature.

What we do in Figure 1 is to use the reference model predictive ability as a noise-filter on the observed data. This simple idea is what happens every time a reference model is used, as it is evident for example in the preconditioning by Paul et al. (2008) or in the draw-by-draw projection for GLMs by Dupuis and Robert (2003).

As an another example, let us consider a candidate submodel in a feature selection process and let us refer with  $\pi$  to the parameter distribution of the submodel and with  $q_\pi(\tilde{y})$  to the induced predictive distribution.

We would like to choose  $\pi$  in order that the latter maximises some predictive performance utility, as the expected log-predictive density (elpd) defined as:

$$\text{elpd}[q_\pi] = \int \log q_\pi(\tilde{y}) p_t(\tilde{y}) d\tilde{y} \quad (2)$$

If we refer to the posterior predictive distribution of a reference model with  $p_{\text{ref}}(\tilde{y}|D)$ , where  $D$  stands for the data on which we condition on, we can approximate (2) using  $p_{\text{ref}}(\tilde{y}|D)$  instead of the true, unknown, data generation mechanism  $p_t(\tilde{y})$ . The maximisation of the elpd using the reference model predictive distribution is equivalent to the minimisation of the KL divergence between the reference model predictive distribution and the submodel one:

$$\max_{\pi} \int \log q_\pi(\tilde{y}) p_{\text{ref}}(\tilde{y}|D) d\tilde{y} \leftrightarrow \min_{\pi} \text{KL}[p_{\text{ref}}(\tilde{y}|D) || q_\pi(\tilde{y})] \quad (3)$$

The term on the right-hand side of Equation (6) describes what is referred as the projection of the predictive distribution, which is the general idea behind the projection predictive approach. Again the reference model is acting as a filter, or substitute, on data.

— OR —

What we do in Figure 1 is to use the reference model predictive ability as a noise-filter on the observed data. This simple idea is what happens every time a reference model is used, as it is evident for example in the preconditioning by [Paul et al. \(2008\)](#) or in the draw-by-draw projection for GLMs by [Dupuis and Robert \(2003\)](#). Also the general idea of the projection of the predictive can be seen under this perspective. Indeed, let us call  $p_{\text{ref}}(\tilde{y}|D)$  the posterior predictive distribution for a new observation  $\tilde{y}$  for the reference model and let  $\pi(\theta)$  and  $q_\pi(\tilde{y})$  be respectively the submodel parameter and the induced predictive distributions. The projection of the predictive distribution allows to find the optimal parameter distribution  $\pi^*$  minimising the KL discrepancy among the induced predictive distribution of the submodel and the reference model one:

$$\pi^* \quad \text{s.t.} \quad \text{KL}[p_{\text{ref}}(\tilde{y}|D) || q_{\pi^*}(\tilde{y})] = \min_{\pi} \text{KL}[p_{\text{ref}}(\tilde{y}|D) || q_{\pi}(\tilde{y})] \quad (4)$$

Consider now the predictive performance of such submodel, a common utility in Bayesian context is the expected log-predictive density defined as:

$$\text{elpd}[q_\pi] = \int \log q_\pi(\tilde{y}) p_t(\tilde{y}) d\tilde{y} \quad (5)$$

where  $p_t$  is the distribution of a new observation under the true data generation mechanism. Maximising the elpd of a model is equivalent to minimising the KL-divergence of its predictive distribution from the true data generation mechanism. Therefore the projection corresponds to maximising the elpd using as distribution of the true data generation mechanism the posterior predictive distribution of the reference model, i.e.:

$$\min_{\pi} \text{KL}[p_{\text{ref}}(\tilde{y}|D) || q_{\pi}(\tilde{y})] \leftrightarrow \max_{\pi} \int \log q_\pi(\tilde{y}) p_{\text{ref}}(\tilde{y}|D) d\tilde{y} \quad (6)$$

Equation (6) clearly shows the action of the reference model as a filter on our target observations, substituting the unknown true data generation mechanism, that is usually estimated with the data empirical distribution, with the best known approximation that is available, i.e. the reference model posterior predictive distribution.

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Relying on the idea of the reference model as a noise-filter, a simple, but effective, reference model approach that can be used on top of any feature selection procedure consists in substituting the target observations with a point prediction of the reference model, e.g. posterior predictive mean. Afterwards any method for feature selection can be used. In Section 3 we compare some well-known methods from the literature using both such reference model approach and the original data, observing at the end of the selection higher stability and quality of the selection thanks to the reference model.

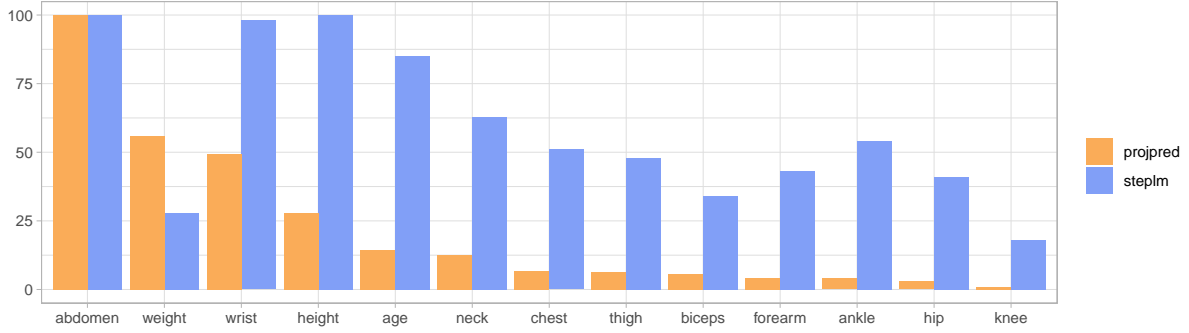


Figure 2: Bootstrap inclusion frequencies after 1000 bootstrap samples.

## 2.1 Benefits of the reference model: an example with the projection predictive approach

The projection predictive approach uses the reference model to project the posterior samples in the submodels. The R-package *projpred* (available at <https://CRAN.R-project.org/package=projpred>) implements the projection in case of submodels in the family of the generalised linear models and additionally provides a framework to choose the optimal submodel size through a predictive utility comparison between the submodels and the reference model. When the goal is prediction, the choice is usually on the smallest submodel that is sufficiently close to the reference model prediction utility score. Further details on the projection predictive approach can be found in Piironen et al. (2018).

We show the benefits of the reference model in the form of the projection approach (*projpred*) with respect to the stepwise backward regression (*steplm*) using the body fat data (Johnson, 1996). The analysis is mainly taken from Vehtari’s R-notebook available at <https://avehtari.github.io/modelselection/bodyfat.html>, adjusting only the number of bootstrap repetitions to 1000 to have a consistent comparison with results from Heinze et al. (2018). The target variable of interest is the amount of body fat, which is measured with a complex and expensive procedure, and the original covariates are 13 anthropometric measurements (e.g. height, weight), some of which are highly correlated among themselves and thus provide some challenges in the model selection, for a total of 251 observations. The goal is to find the simplest model which is able to describe, that is to predict, sufficiently well the amount of fat.

Heinze et al. (2018) report the results using the stepwise backward elimination with a significance level of 0.157 with AIC selection, fixing abdomen and height to be always included in the model. We implement the selection via projection: the reference model includes all the covariates using as a prior the regularised horseshoe, submodels are explored using a forward search and the predictive utility is the expected log-predictive density (elpd) estimated using PSIS-LOO (Vehtari et al., 2017). In order to automatise the procedure, we select the submodel size as the smallest one which has an elpd score higher than the reference model score with probability 0.05. Figure 2 shows the bootstrap inclusions frequencies combining the results from Heinze et al. (2018) for the stepwise backward elimination and our experiments for the projection. The projection predictive approach has only two variables with inclusion frequencies above 50% (abdomen is the only included always), the third most included is ‘wrist’ at 49.5% and the fourth is ‘height’ at 27.8, while the stepwise regression has seven features above 50%, of which abdomen and height are fixed. Such a lower stability of the stepwise regression can be observed also in the bootstrap model selection frequencies reported in Table 1. The first five selected models have a cumulative frequency of almost 70% with *projpred*, whereas with the stepwise regression only of 11.8%. In addition to that we note that the size of the selected model is much smaller when using *projpred*.

Table 2 reports the predictive performances, in terms of root mean square error, of the full model and the correspondent selected model using both the projection predictive approach and the stepwise backward elimination. We also repeat the selection after adding uncorrelated noisy features up to a total of 100

M	projpred	Freq (%)	steplm	Freq (%)
1	abdomen, weight	34.2	abdomen, height, wrist, age, chest, biceps	3.2
2	abdomen, wrist	13.3	abdomen, height, wrist, age, neck, forearm, thigh, hip	2.9
3	abdomen, weight, wrist	7.6	abdomen, height, wrist, age, forearm, chest	1.9
4	abdomen, height	7.4	abdomen, height, wrist, age, neck, forearm, chest	1.9
5	abdomen, height, wrist	6.7	abdomen, height, wrist, age, neck, forearm, chest, thigh, hip	1.9
6	abdomen, age, wrist	2.9	abdomen, height, wrist, age, neck, chest, biceps	1.8
7	abdomen, height, neck	2.2	abdomen, height, wrist, age, neck, thigh, biceps, hip	1.6
8	abdomen, age, height, wrist	1.2	abdomen, height, wrist, age, neck, forearm	1.5
9	abdomen, weight, thigh	1.1	abdomen, height, wrist, age, neck, biceps	1.5
10	abdomen, weight, neck	1.0	abdomen, height, wrist, age, neck, forearm, chest, biceps	1.4

Table 1: Bootstrap model selection frequencies after 1000 bootstrap samples.

	rmse.full	rmse.sel	size.sel	noisy.sel
projpred	4.4	4.5	2	-
steplm	4.4	4.5	7	-
projpred	*4.4	*4.5	2	0
steplm	5.3	5.5	38	32

Table 2: Model predictive performances with original data (first two rows) and adding noisy features (last two rows). Results with an asterisk using 10-fold CV, otherwise 20-fold CV.

covariates and report in such a case both the size of the selected model and the number of selected noisy features. While with the original data we do not observe any gain using projpred in terms of predictive performance (we do observe a smaller size for the selected model, however with a much higher computational cost), when noisy features are added the projection is able to notably improve the predictive performance keeping the same submodel size, whereas the stepwise regression selects a much larger submodel including a large number (32 over 38) of noisy features.

## 2.2 Computational burden

- add more quantitative results: e.g. times for projpred and stepmlm

We can consider the reference model approach as a family of methods, thus providing an answer to the question what the computational burden is, is not possible, because not unique. It is possible to identify a common and inevitable overhead though, that is the cost of fitting the reference model. In the simple reference model approach we have described at the end of Section 2, it is actually the only added cost. In general, a full Bayesian model is advisable to have the best predictive performance, however including all the available features can be often quite computationally demanding. In such a case a first possible speed-up could be to use screening or dimensionality reduction techniques, as for example the supervised principal components (Bair et al., 2006; Piironen and Vehtari, 2017b). However, in some cases this does not always result in an appreciable speed-up, as when there are many predictive and not correlated features and thus many of them have to be included in the reference model anyway. Other alternatives could be fixing some parameters with their maximum marginal likelihood value (typically referred as empirical Bayes) or even use frequentist approach, if predictions are good enough.

The comparison example between projpred and stepmlm shows that such additional computational burden can be worth it. In the case of the projection predictive approach it results in sparser submodels, which can result in saving time and general costs using the selected model for future predictions. 2 shows that there could be even an improvement in the prediction ability, thus justifying more additional computational costs in the selection. More benefits obtainable through the reference model are provided in the experiments of Section 3.

## 3 A comparison in the normal means problem framework

In this Section we show the benefits of the use of a reference model. We consider as reference problem the normal means estimation problem, which consists in estimating the vector of means, usually sparse, of a



normal vector of observations. Let us call  $p$  the dimensionality of the vector of means and let  $\{z_j\}_{j=1}^p$  be the vector of observations, the normal means problem can be written as follows:

$$Z_j | \theta_j, \sigma \stackrel{\text{ind}}{\sim} N(\theta_j, \sigma^2) \quad j = 1, \dots, p \quad (7)$$

Note that it is equivalent to a linear regression where the design matrix is the identity one. This formulation can be retrieved in different ways, a common example is given by microarray data, where a large set of genes are usually tested in two groups of patient labelled as positive or negative to some disease. The objective of the analysis is to spot the whole subset of statistically relevant genes to the disease and one possible way to proceed is to compute the two-samples  $t$ -statistics for each gene and after combining it with the cumulative density function of the standard distribution, the resulting data can be used in the problem formulation (7). For further details see the examples in [Efron \(2008, 2012\)](#).

In our examples we retrieve the normal means problem from the sample correlation using the Fisher  $z$ -transformation approximation ([Hawkins, 1989](#)). Suppose to have a continuous target random variable  $Y$  and a set of  $p$  continuous covariates  $\{X_j\}_{j=1}^p$  and let us call  $\rho_j = \text{Cor}(Y, X_j)$ . Suppose to observe  $n$  statistical units and define  $r_j$  the sample correlation between the observations of the target variables  $\{y_i\}_{i=1}^n$  and the  $j$ -th covariate  $\{x_{ij}\}_{i=1}^n$ . Let finally refer to the function  $\tanh^{-1}(\cdot)$  as  $T_F(\cdot)$ . It holds the following modelling approximation (assuming each pair  $(Y, X_j)$  normally distributed):

$$T_F(r_j) \stackrel{\text{ind}}{\sim} N(T_F(\rho_j), \frac{1}{n-3}) \quad j = 1, \dots, p \quad (8)$$

Therefore rescaling the quantities  $T_F(r_j)$  by  $\sqrt{n-3}$  and referring to the results as variables  $z_j$ , we find again formulation (7) with unit variance:

$$Z_j | \theta_j \stackrel{\text{ind}}{\sim} N(\theta_j, 1) \quad j = 1, \dots, p \quad (9)$$

Note that now the quantity of interest  $\theta_j$  stands for  $\sqrt{n-3} T_F(\rho_j)$ . There are different ways to proceed with the selection from the normal means problem, we consider in our comparison both frequentist and Bayesian state-of-the-art methods. From a full-Bayesian perspective, a Bayesian regression model is fitted to the normal means problem typically using a sparsifying prior as in [Bhattacharya et al. \(2015\)](#), [Bhadra et al. \(2017\)](#) and [Johnstone et al. \(2004\)](#). In case of continuous priors, as the Dirichlet-Laplace (DL) and the horseshoe+ priors, there is not a canonical way to complete the selection; in their experiments [Bhattacharya et al. \(2015\)](#) use k-means clustering over the posterior samples using the DL prior, whereas [Bhadra et al. \(2017\)](#) focus the analysis on the obtained shrinkage and the correct estimation of the sparse signals using the horseshoe+ prior. We include in our experiments the DL and the regularised horseshoe prior ([Piironen and Vehtari, 2017c](#)), which has shown in our opinion better scalability with respect to the horseshoe+, comparing the shrinkage and the correct estimation of the sparse signals using or not the reference model approach. The quantity of interests that we consider are the average SSE (sum of square errors) and the average SESE (sum of expected square errors), defined as follows:

$$\text{SSE} = \sum_{j=1}^p (\hat{\theta}_j - \theta_j^0)^2 \quad (10)$$

$$\text{SESE} = \sum_{j=1}^p \int_{\mathbb{R}} (\theta_j - \theta_j^0)^2 p(\theta_j | z_1, \dots, z_p) d\theta_j = \sum_{j=1}^p \mathbb{E}_{\theta_j} [(\theta_j - \theta_j^0)^2 | z_1, \dots, z_p] \quad (11)$$

where  $\theta_j^0 = \sqrt{n-3} T_F(\rho_j)$  is the true value,  $\hat{\theta}_j$  is the posterior median of each parameter  $\theta_j$  and (11) is a sum of posterior expectations. Note that the quantity SESE takes in account also the uncertainty, thus the shrinkage, of the posterior distributions, whereas the SSE measures only the bias in the point estimates. [Johnstone et al. \(2004\)](#) use a mixture of a delta distribution in zero and a heavy-tailed distribution, when this is a Laplace distribution they show an interesting thresholding property of the median estimator and thus present a complete selection procedure using their prior. We include in our comparison such method using the R-package *EbayesThresh* together with the control of the local false discovery rate ([Efron, 2012, 2008](#)), using the R-package *locfdr*, and a simple selection procedure using inclusion probability at 0.9 for the

HDP posterior credible intervals using the regularised horseshoe prior. All these methods provide a complete selection procedures, we compare the result using or not a reference model on top of the procedure evaluating the average false discovery rate (i.e. ratio of the number of non-relevant selected features over the number of selected features) and the average sensitivity (i.e. ratio of the number of relevant selected features over the total number of relevant features). We also provide a comparison of the stability of the selection using the measure proposed by [Nogueira et al. \(2017\)](#).

### 3.1 Simulated data

First we use simulated data in order to vary the difficulty of the selection problem. We rely on the scheme (1) using different levels of correlation ( $\rho$ ) and number observation ( $n$ ), precisely  $p$  and  $k$  are fixed respectively at 1000 and 100, whereas  $n \in \{50, 70, 100\}$  and  $\rho \in \{0.3, 0.5\}$ . Lower the correlation level and the number of observation are, more challenging the selection is. As mentioned, this data generation mechanism was already proposed by [Piironen et al. \(2018\)](#), who also devise a reference model sufficiently good in predictions. It consists in a linear regression using the first five supervised principal components (SPCs) ([Bair et al., 2006](#); [Piironen and Vehtari, 2017b](#)) as follows:

$$\begin{aligned} Y_i | \beta, \sigma^2, \mathbf{u}_i &\stackrel{ind}{\sim} N(\mathbf{u}_i^T \beta, \sigma^2) & i = 1, \dots, n \\ \beta_j | \tau &\stackrel{iid}{\sim} N(0, \tau^2) & j = 1, \dots, 5 \\ \tau &\sim t_4^+(0, s_{max}^{-2}) \\ \sigma &\sim t_3^+(0, 10) \end{aligned} \tag{12}$$

where  $\mathbf{u}_i$  stays for the  $i$ -th SPC and  $s_{max}$  denotes the sample standard deviation of the largest SPC. All model are implemented in Stan ([Carpenter et al., 2017](#)). The normal means problem formulation is obtained as explained in Section 3 by Fisher-transformation. We label as “ref” the reference approach, which consists in using the reference model on top of the computation of the z-values, and “data” the standard approach based on the observed data only. In Section 3.1.1 we analyse in the context of full-Bayes methods the different shrinkage and signal estimation provided by the use of a reference model or not. Section 3.1.2 shows results for the complete selection procedures.

#### 3.1.1 Shrinkage and signal analysis

In the framework of full-Bayes methods we analyse the effect of the reference model using two continuous hierarchical shrinkage priors: the Dirichlet-Laplace ([Bhattacharya et al., 2015](#)) and the regularised horseshoe prior ([Piironen and Vehtari, 2017c](#)). Although the horseshoe+ prior by [Bhadra et al. \(2017\)](#) is a state-of-the-art prior for sparse signal analysis in the context of the normal means problem, we decided to include the regularised horseshoe due to its faster MCMC inference, since our purpose in this paper is not to study the best prior choice, but rather compare and show the benefits of using a reference model when carrying out variable selection. We used an uniform prior on  $(1/p, 1/2)$  for the Dirichlet hyperparameter of the Dirichlet-Laplace prior, whereas regarding the regularised horseshoe prior we followed the indication provided in [Piironen and Vehtari \(2017c\)](#) setting the prior guess of the effective number of parameters equal to 1 to achieve the highest shrinkage.

As already mentioned, continuous priors do not usually give an automatic selection procedure, in this Section we study the amount of shrinkage and the bias of the posterior median looking at the SESE and the SSE (see definitions (10) and (11)). In the case of the regularised horseshoe prior, we also take into account the posterior mean for the shrinkage factor for each parameter  $\theta_j$ , which for model (9) is defined as:

$$\kappa_j = \frac{1}{1 + \tau^2 \lambda_j^2} \tag{13}$$

with  $\tau$  and  $\lambda_j$  respectively global and local shrinkage parameters, for further details see [Piironen and Vehtari \(2017c\)](#).



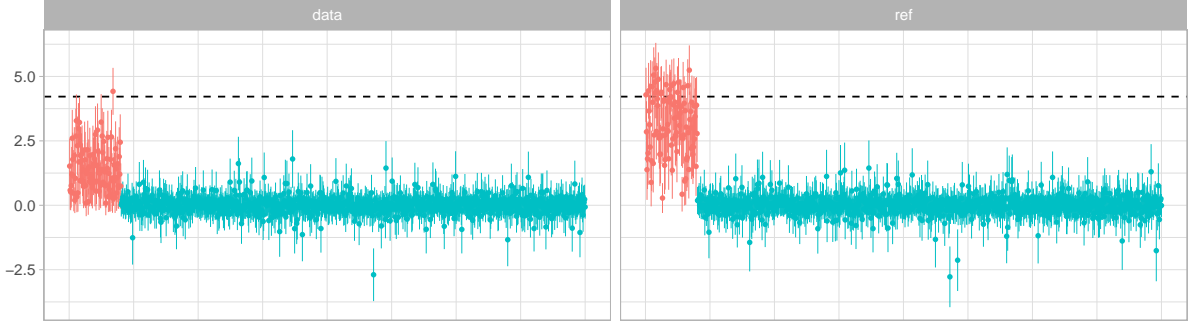


Figure 3: Posterior intervals for the parameters of the normal means problem ( $n = 50$ ,  $\rho = 0.3$ ) using the regularised horseshoe prior. Posterior means and one standard deviation intervals depicted. Respectively in red and in blue relevant and non-relevant features. Horizontal dashed line corresponds to the true value of the mean for the relevant features.

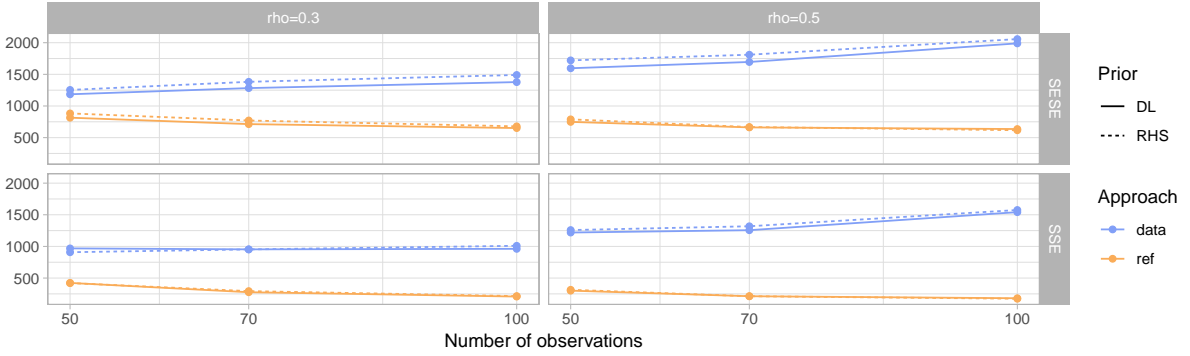


Figure 4: Average SESE and SSE after 100 simulations.

Figure 3 shows an example of posterior distributions for the model (9) using the regularised horseshoe prior. Respectively on the right and on the left the results using or not the reference model approach. It is evident how the reference model helps in tearing apart the signals from the noise, resulting in less biased estimates. This is quantified in Figure 4 where average SESE and SSE are depicted after 100 simulations. We observe that the reference approach (in orange) has a clear lower error estimate for both the SESE and the SSE. As  $n$  and  $\rho$  grow, the amount of error diminish, as expected, when using the reference model, whereas it unexpectedly increases when it is not used (WHY??). We also note that the two priors seem to achieve very similar results, regardless of the approach used.

Figure 5 shows the posterior medians for the shrinkage factors of each parameter, averaged after 100 data simulations. Relevant and non-relevant features are respectively highlighted in red and blue. The ideal shrinkage would have  $\kappa = 0$  when the parameter is a signal, whereas  $\kappa = 1$  when it is noise. Respectively on the y-axis and on the x-axis are depicted the shrinkage factors using the reference model approach or not. If the reference model did not make any difference, we would expect the points to lie on the diagonal. Non-relevant features happen to lie on average on the diagonal for all the simulated scenario, meaning not evident benefits coming from the reference model. Although when considering the shrinkage amount for the relevant ones, we note that the points always lie under the diagonal, meaning that the reference model is able to shrink less the true signals. Such benefit is proportional to the difficulty of the problem, that is it increase with lower correlation and number of observations. Note that the perfect distinction of the two cluster of features (relevant and non-relevant) observable on both the marginal distribution of the shrinkage factors is due to the averaging over the 100 data simulations. In a single data realisation there would be more noise and the cluster would be closer and more overlapping.

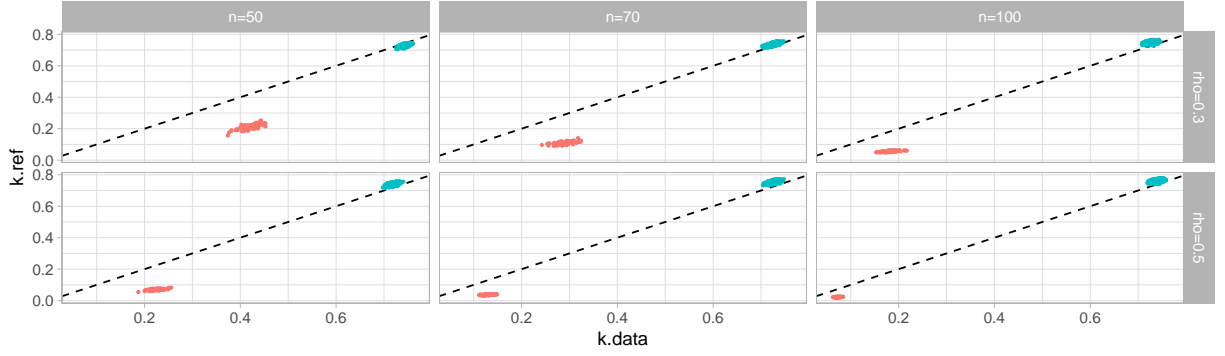


Figure 5: Average posterior median values for the shrinkage factors after 100 simulations. Respectively in red and in blue relevant and non-relevant variables.

### 3.1.2 Complete selection analysis

As complete selection procedures we consider the control of the local false discovery rate (Efron, 2008, 2012), the empirical Bayes median (Johnstone et al., 2004) and the selection by posterior credible intervals. The control of the local false discovery rate is applied through the R-package *locfdr* using splines with 7 degrees of freedom to fit the marginal density and selecting features with local false discovery rate under the value 0.2. Such a value is the common suggested one just for the fact that it corresponds to a Bayes factor large than 36, other values can be considered and in our experience do not affect the results of the comparison. The empirical Bayes median procedure is given by the R-package *EbayesThresh* and consists in fitting a Bayesian model with a prior composed by a mixture of a delta in zero and a heavy-tailed distribution. We use, as suggested, a Laplace distribution resulting in a thresholding property, i.e. there exists a threshold value such that all the data under that threshold have posterior median equal to zero. Therefore the selection is done including those parameters whose posterior median is different from zero. The hyperparameter of the Laplace distribution and the mixing weight of the prior are estimated by marginal maximum likelihood. The selection by posterior credible intervals is done using the regularised horseshoe prior and selecting those features whose highest probability density posterior credible interval does not include zero at the level 0.9.

Figure 6 reports the average sensitivity (on the y-axis) versus the average false discovery rate (x-axis) after 100 data simulations for the different combination of  $n$  and  $\rho$ . For each method (different colours) the square and the circle dots respectively correspond to using or not the reference approach. The best selection performance is on the top left corner, which means lowest false discovery rate and highest sensitivity. It is possible to note that regardless of the method used for the selection, the use of a reference model improves the quality diminishing the false discovery rate (left shifting) and/or augmenting the sensitivity (up shifting). In accordance with what we would expect, higher the number of observation and the correlation are, easier the selection is and thus the benefit of the reference model are less notable, since the unprocessed data already give enough information to identify the significant variables.

Figure 7 shows the estimates of the stability measure proposed by Nogueira et al. (2017) with 0.95 confidence intervals after 100 simulations. Such a measure takes account the variability of the subset of the selected features at each simulation (originally at each bootstrap sample) modelling the selection of each variable as a Bernoulli process. Further details are available in their paper as asymptotic normal distribution, based on which confidence intervals are computed. The reference model helps improving the stability of the selection: again such benefit is notable when the problem is more difficult (small  $n$  and  $\rho$ ). We observe also less uncertainty in the stability estimates for the reference approach (i.e. the width of the 0.95 confidence intervals), which can be still connected to the overall stability of the procedure.

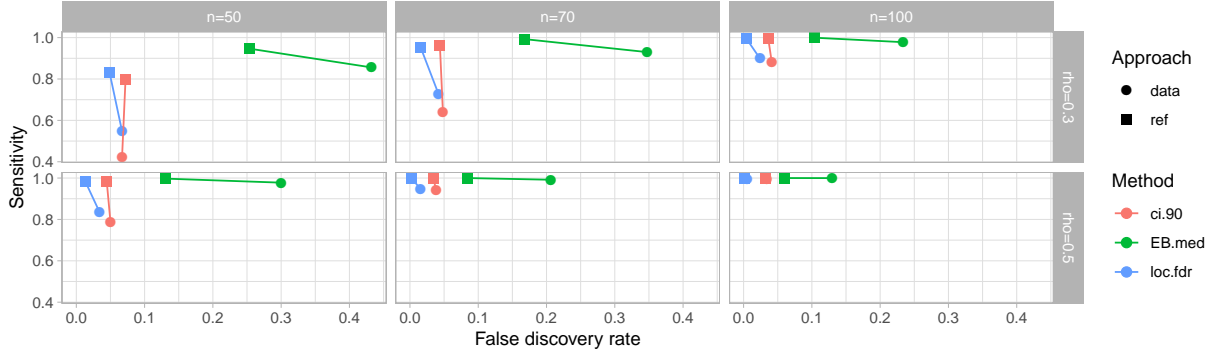


Figure 6: Sensitivity against false discovery rate after 100 data simulations.

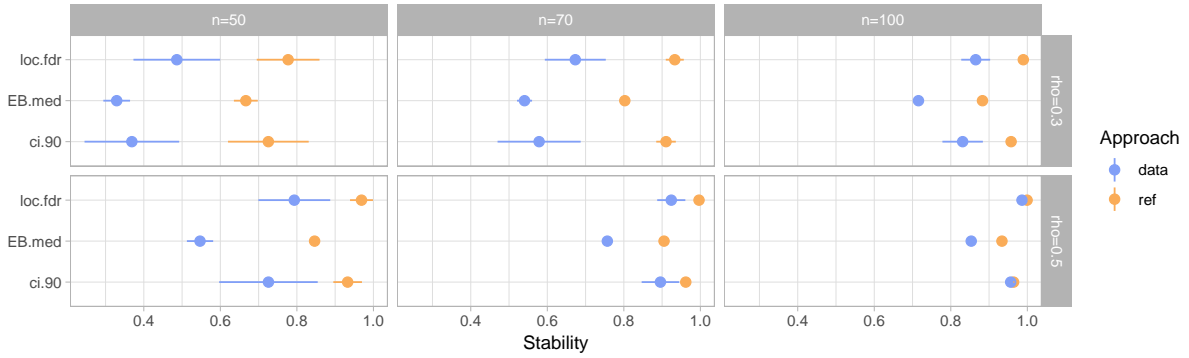


Figure 7: Stability estimates with 0.95 confidence intervals after 100 data simulations.

### 3.2 Real world data: the body fat dataset

We conclude our experiments using the real world dataset body fat, already described in the comparison between the projection predictive approach and the stepwise regression in Section 2.1. Consistently with the previous comparisons, we design the variable selection by normal means problem. We add to the original data noisy uncorrelated covariates up to a total of 1000 features. These covariates are normally distributed and scaled as the original ones. We compute correlations between each variable and the target variable, that is the amount of fat, and transform them by Fisher transformation. The original assumption in order that (8) holds is that the variables are jointly normally distributed. In our experience the normal approximation is still reasonable, but after rescaling by  $\sqrt{n-3}$  we do not fix the variance to be one. The methods we compare are those used in Section 3.1.2, which are the control of the local false discovery rate (loc.fdr), the empirical Bayes median (EB.med) and the selection by posterior credible intervals at level 0.9 (ci.90). The maximum likelihood estimator is used to estimate the variance of the null hypothesis in the local false discovery rate (see Efron, 2012, Chap. 6), whereas the median absolute deviation from zero for the empirical Bayes median method. The selection via posterior credible intervals is done using a regularised horseshoe prior on the parameters for the means, while a standard log-normal prior for the variance. In order to vary the difficulty of the selection, we bootstrap subsamples of different sizes, going from  $n = 50$  up to  $n = 251$ . Each time results are averaged over 100 bootstrap subsamples.

Figure 8 shows the sensitivity against the false discovery rate. Since we do not have a ground truth regarding the original covariates of the data, we believe it is reasonable to consider all of them relevant, at least at some degree. We thus consider as non-relevant all the artificially added ones. When  $n = 70$  we observe an increment in terms of sensitivity, also the false discovery rate slightly increase, but in a lower amount. In all other bootstrapped subsamples, the reference model improves both in terms of sensitivity and false discovery rate. We observe again that the benefits are more evident as the selection is more challenging (i.e.

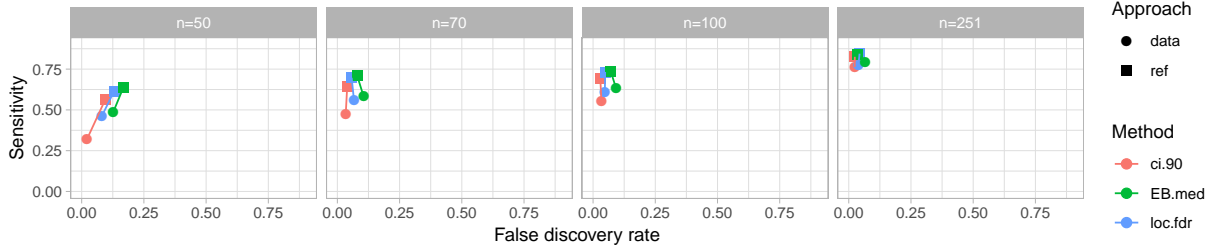


Figure 8: Sensitivity against false discovery rate after 100 data simulations.

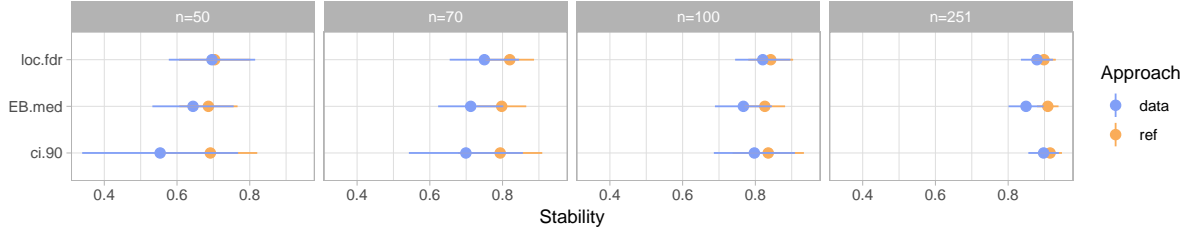


Figure 9: Stability estimates with 0.95 confidence intervals after 100 data simulations.

less number of observations).

Figure 9 shows the stability results, still using the measure provided by [Nogueira et al. \(2017\)](#). The benefit of the reference model are here marginal, small improvements can be observed, but overall not very significant.

## 4 Conclusions and discussion

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