Notes on dMRI Inverse Problem

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1 Function representation

We start with the subspace of spherical harmonics

$$\mathcal{H}_N = \text{span}\left\{Y_n^m\left(\mathbf{\Omega}\right) \,:\, |m| \le n,\, 0 \le n \le N\right\}, \tag{1}$$

where N is a given natural number and Y_n^m is a spherical harmonic of degree n and order m. Introducing the reproducing kernel $K\left(\mu\right) = \sum_{n=0}^{N} \frac{2n+1}{4\pi} P_n\left(\mu\right)$, we have the identity

$$f\left(\mathbf{\Omega}\right) = \int_{\mathbb{S}^2} K\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) f\left(\mathbf{\Omega}'\right) d\mathbf{\Omega}', \quad \forall f \in \mathcal{H}_N, \tag{2}$$

which can be verified using the Addition Theorem

$$\frac{2n+1}{4\pi} P_n\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) = \sum_{|m|\le n} \bar{Y}_n^m\left(\mathbf{\Omega}'\right) Y_n^m\left(\mathbf{\Omega}\right).$$

Suppose now that there exists an exact quadrature for $\mathcal{H}_{2N}$ with nodes and weights $\left\{\mathbf{\Omega}_i, w_i\right\}_{i=1}^{M}$. Discretizing Eq (2), we obtain

$$f\left(\mathbf{\Omega}\right) = \sum_{i=1}^{M} f\left(\mathbf{\Omega}_i\right) w_i K\left(\mathbf{\Omega}\cdot\mathbf{\Omega}_i\right). \tag{3}$$

Because of the inherent symmetry in the dMRI problem, we want to be able to project onto the even harmonics in $\mathcal{H}_N$. To this end, let the even part of f be defined as

$$f_e\left(\mathbf{\Omega}\right) = \frac{1}{2}\left[f\left(\mathbf{\Omega}\right) + f\left(-\mathbf{\Omega}\right)\right]$$

and the odd part as

$$f_o\left(\mathbf{\Omega}\right) = \frac{1}{2}\left[f\left(\mathbf{\Omega}\right) - f\left(-\mathbf{\Omega}\right)\right].$$

Then

$$f\left(\mathbf{\Omega}\right) = f_e\left(\mathbf{\Omega}\right) + f_o\left(\mathbf{\Omega}\right).$$

Using Eq. (2), we can define the even and odd projectors via even and odd reproducing kernels

$$\begin{aligned} f(\mathbf{\Omega}) &= \int_{\mathbb{S}^2} K\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) f\left(\mathbf{\Omega}'\right) d\mathbf{\Omega}' \\ &= \int_{\mathbb{S}^2} K_e\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) f\left(\mathbf{\Omega}'\right) d\mathbf{\Omega}' + \int_{\mathbb{S}^2} K_o\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) f\left(\mathbf{\Omega}'\right) d\mathbf{\Omega}' \\ &= f_e(\mathbf{\Omega}) + f_o(\mathbf{\Omega}), \end{aligned}$$

where

$$K_e(\mu) \equiv \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{2(2n)+1}{4\pi} P_{2n}(\mu)$$

and

$$K_o(\mu) \equiv \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{2(2n+1)+1}{4\pi} P_{2n+1}(\mu).$$

Thus, assuming f is an even function and using Eq. (3) we have

$$f(\mathbf{\Omega}) = f_e(\mathbf{\Omega}) = \sum_{i=1}^{M} f(\mathbf{\Omega}_i)\, w_i K_e(\mathbf{\Omega}\cdot\mathbf{\Omega}_i). \tag{4}$$

We'll use representation Eq. (4) going forward. Note that representation Eq. (4) does not require the quadrature points to be invariant under inversion.

We also need to calculate the spherical Laplacian and Funk-Radon transform of K_e. Recall that the spherical harmonics are eigenfunctions of the spherical Laplacian, i.e.

$$\triangle Y_n^m(\mathbf{\Omega}) = -n(n+1)\, Y_n^m(\mathbf{\Omega}).$$

Thus

$$\triangle K_e\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) = -\sum_{n=0}^{\lfloor N/2 \rfloor} (2n)(2n+1)\left[\frac{2(2n)+1}{4\pi}\right] P_{2n}\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right).$$

The Funk-Radon transform is defined by

$$(\mathcal{G}f)(\mathbf{\Omega}) = \int_{\mathbb{S}^2} \delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right) f\left(\mathbf{\Omega}'\right) d\mathbf{\Omega}',$$

where δ is the Dirac delta on the sphere. We now calculate the Funk-Radon transform of $\triangle K_e(\mathbf{\Omega}\cdot\mathbf{\Omega}_i)$:

$$
\begin{aligned}
(\mathcal{G}\triangle K_e(\cdot\mathbf{\Omega}_i))(\mathbf{\Omega}) &= \int_{\mathbb{S}^2}\delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right)\triangle K_e\left(\mathbf{\Omega}'\cdot\mathbf{\Omega}_i\right)d\mathbf{\Omega}' \\
&= -\int_{\mathbb{S}^2}\delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right)\sum_{n=0}^{\lfloor N/2\rfloor}(2n)(2n+1)\left[\frac{2(2n)+1}{4\pi}\right]P_{2n}\left(\mathbf{\Omega}'\cdot\mathbf{\Omega}_i\right)d\mathbf{\Omega}' \\
&= -\int_{\mathbb{S}^2}\delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right)\sum_{n=0}^{\lfloor N/2\rfloor}(2n)(2n+1)\sum_{|m|\le 2n}\bar{Y}_{2n}^{m}\left(\mathbf{\Omega}'\right)Y_{2n}^{m}(\mathbf{\Omega}_i)\,d\mathbf{\Omega}' \\
&= -\sum_{n=0}^{\lfloor N/2\rfloor}(2n)(2n+1)\sum_{|m|\le 2n}Y_{2n}^{m}(\mathbf{\Omega}_i)\int_{\mathbb{S}^2}\delta\left(\mathbf{\Omega}\cdot\mathbf{\Omega}'\right)\bar{Y}_{2n}^{m}\left(\mathbf{\Omega}'\right)d\mathbf{\Omega}' \\
&= -\sum_{n=0}^{\lfloor N/2\rfloor}(2n)(2n+1)\sum_{|m|\le 2n}Y_{2n}^{m}(\mathbf{\Omega}_i)\left[2\pi\int_{-1}^{1}P_{2n}(\mu)\,\delta(\mu)\,d\mu\right]\bar{Y}_{2n}^{m}(\mathbf{\Omega}) \\
&= -\sum_{n=0}^{\lfloor N/2\rfloor}(2n)(2n+1)\left[2\pi P_{2n}(0)\right]\sum_{|m|\le 2n}Y_{2n}^{m}(\mathbf{\Omega}_i)\,\bar{Y}_{2n}^{m}(\mathbf{\Omega}) \\
&= -\sum_{n=0}^{\lfloor N/2\rfloor}(2n)(2n+1)\left[2\pi P_{2n}(0)\right]\left[\frac{2(2n)+1}{4\pi}\right]P_{2n}(\mathbf{\Omega}\cdot\mathbf{\Omega}_i)
\end{aligned}
$$

Note that the first non-zero term is for $n = 1$. We now define a new kernel based on the inverse spherical Laplacian and inverse Funk-Radon transform, namely

$$
H(\mu) \equiv -\sum_{n=1}^{\lfloor N/2\rfloor}\frac{2(2n)+1}{8\pi^2 P_{2n}(0)(2n)(2n+1)}P_{2n}(\mu).
$$

That is

$$
\mathcal{G}\triangle H(\mathbf{\Omega}\cdot\mathbf{\Omega}_i) = K_e(\mathbf{\Omega}\cdot\mathbf{\Omega}_i). \tag{5}
$$

We will represent the measured dMRI signal in terms of linear combinations of the functions $H_i(\mathbf{\Omega}) = H(\mathbf{\Omega}\cdot\mathbf{\Omega}_i)$.

2 Formulation of Aganj, Lenglet et al.

Here we use the formulation developed in [1] along with the representation Eq. (4). The ODF derived in is given by

$$
ODF(\mathbf{\Omega}) = \frac{1}{4\pi} + \frac{1}{16\pi^2}\mathcal{G}\left\{\triangle\ln\left[-\ln\left(\tilde{\mathrm{E}}\right)\right]\right\}(\mathbf{\Omega}), \tag{6}
$$

where $\tilde{E}$ is the measured dMRI signal. To reconstruct $ODF(\mathbf{\Omega})$, we assume

$$
\ln\left[-\ln\tilde{E}(\mathbf{\Omega})\right] \approx \sum_{i=1}^{M}\psi_i\,H_i(\mathbf{\Omega})
$$

and assume we have measured data at the points $\boldsymbol{\omega}_j,\ j = 1, 2, 3, \cdots M'$. Then we have the linear system of equations

$$\ln\left[-\ln\tilde{E}\left(\boldsymbol{\omega}_j\right)\right] \approx \sum_{i=1}^{M} \psi_i\, H_i\left(\boldsymbol{\omega}_j\right),\ j = 1, 2, 3, \cdots M'$$

to solve. We solve this (usually under-determined) system of equations for ψ_i by using an L_1-penalized least-squares approach. We remark that some care must be taken when evaluating the iterated logarithms when $\tilde{E}$ is near zero or unity. Once the coefficients ψ_i are found, we use Eqs. (5) and (6) to obtain

$$\begin{aligned} ODF\left(\boldsymbol{\Omega}\right) &\approx \frac{1}{4\pi} + \frac{1}{16\pi^2}\mathcal{G}\left\{\triangle\sum_{\mathrm{i}=1}^{\mathrm{M}} \psi_{\mathrm{i}}\,\mathrm{H}_{\mathrm{i}}\right\}\left(\boldsymbol{\Omega}\right) \\ &\approx \frac{1}{4\pi} + \frac{1}{16\pi^2}\sum_{i=1}^{M} \psi_i\mathcal{G}\left\{\triangle\mathrm{H}_{\mathrm{i}}\right\}\left(\boldsymbol{\Omega}\right) \\ ODF\left(\boldsymbol{\Omega}\right) &\approx \frac{1}{4\pi} + \frac{1}{16\pi^2}\sum_{i=1}^{M} \psi_i\mathrm{K}_{\mathrm{e}}\left(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_i\right) \end{aligned} \tag{7}$$

Since the functions $\mathrm{K}_{\mathrm{e}}\left(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_i\right)$ are localized, at least localized compared to spherical harmonics, the "dominant" coefficients ψ_i indicate the directions in which the ODF is peaked.

References

[1] Aganj Iman, Lenglet Christophe, Sapiro Guillermo, Yacoub Essa, Ugurbil Kamil, and Harel Noam. Reconstruction of the orientation distribution function in single- and multiple-shell q-ball imaging within constant solid angle. *Magnetic Resonance In Medicine*, 64:554–566, 2010.