Aspects of Differential Geometry in HoTT

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$$\text{Smooth Manifolds} \hookrightarrow \text{Formal Smooth Sets} = \mathrm{Sh}(\{\mathbb{R}^n \times \mathbb{D} | n \in \mathbb{N}\})$$

$$\text{Schemes} \hookrightarrow \text{Zariski-sheaves} = \mathrm{Sh}(\mathrm{Rings}^{\mathrm{op}})$$

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Where "Sh" means the topos of set- or ∞-groupoid-valued sheaves on:

1. $\mathbb{R}^n \times \mathbb{D}$ with smooth open good covers (ignoring the $\mathbb{D}$s).
2. Commutative, unital rings with jointly surjective inclusions of Zariski-open affine subsets.

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Let us see, what this functor does to a sheaf S, representing a k-Scheme:

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But $(k[X]/(X^2))_{\mathrm{red}}$ is just k, so the tangent vectors at k-points of $\Im(S)$ are just the k-points:

$$\Im(S)(\mathrm{Spec}(k[X]/(X^2))) = S(\mathrm{Spec}(k[X]/(X^2))_{\mathrm{red}}) = S(\mathrm{Spec}(k))$$

So: $\Im$ removes all differential geometric information!

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For all nilpotent $\mathbb{R}$-algebra V, formally extend the left category by

$$\mathbb{R}^n \times \mathbb{D}_V = \mathrm{Spec}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V))$$

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and call this category FC. For any k, we can restrict the order:

$$\mathrm{FC}_k := \{\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V) | n \in \mathbb{N}, V^{k+1} = 0\}^{\mathrm{op}} \subseteq \mathbb{R}\mathrm{-algebras}^{\mathrm{op}}$$

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Now, define $\mathfrak{I}\colon \mathrm{Sh}(\mathrm{FC}) \to \mathrm{Sh}(\mathrm{FC})$ by

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and, respectively $\mathfrak{I}_k\colon \mathrm{Sh}(\mathrm{FC}_k) \to \mathrm{Sh}(\mathrm{FC}_k)$ by the same equation

$$\mathfrak{I}_k(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}_V) := \mathcal{F}(\mathbb{R}^n)$$

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$$\begin{tikzcd}
T_x M \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & 1 \arrow[d, "_ \mapsto \iota_M(x)"] \\
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The tangent bundle is also given as a pullback:

$$\begin{tikzcd} TM \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & M \arrow[d, "\iota_M"] \\ M \arrow[r, "\iota_M"'] & \mathfrak{I}_1 M \end{tikzcd}$$

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3. For any type A, $\Im A$ is coreduced.
4. For any $B : \Im A \to \mathcal{U}$, such that $\prod_{a : \Im A} B(a)$ is coreduced, a section $s : \prod_{a : \Im A} B(a)$ is defined by $s_0 : \prod_{a : A} B(\iota_A(a))$.

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5. Coreduced types have coreduced identity types.

Internal geometric notions

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Definition

For any point $x\colon A$, $\mathbb{D}_x$ is defined by

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The *formal disk bundle over* A, $T_\infty A$ is defined by the pullback

$$\begin{tikzcd}
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$$\prod_{x : X} \mu(e, x) = x \text{ and } \prod_{x : X} \mu(x, e) = x.$$

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$$\prod_{x : X} \mu(e, x) = x \text{ and } \prod_{x : X} \mu(x, e) = x.$$
4. Proof that for any $a : X$ the right-translation $x \mapsto \mu(x, a)$ is an equivalence, i.e. there is a term of type
$$\prod_{a : X} (x \mapsto \mu(x, a)) \text{ is an equivalence.}$$

The triviality theorem

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Theorem
Let V be a left invertible H-space and $\mathbb{D}$ the formal disk at the unit in V, then:

$$
\begin{tikzcd}
T_\infty V \arrow[rr, "\simeq"] \arrow[dr, "{(\pi_1, \pi_2)}"'] & & \mathbb{D} \times V \arrow[dl, "{(d,g) \mapsto (\mu(\pi_1(d), g), g)}"] \\
& V \times V &
\end{tikzcd}
$$

Differential structure preserving morphisms

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A map $f\colon A \to B$ is called *formally étale* if the naturality square

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & \Im A \arrow[d, "\Im f"] \\ B \arrow[r, "\iota_B"'] & \Im B \end{tikzcd}$$

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Remark

For smooth manifolds formally étale maps correspond to local diffeomorphisms.

For noetherian schemes, they correspond to étale maps.

Structured spaces

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Definition

Let V be a left invertible H-space. A type M is called a V-Manifold, if there is a span of formally étale maps

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Theorem (needs Univalence)
Any V-Manifold has a locally trivial formal disk bundle witnessed by a classifying map

$$\chi_M \colon M \to \mathrm{BAut}(\mathbb{D}_V)$$

Cartan Geometry

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If we have a delooping BG of a group G with a map $\varphi\colon \mathrm{BG} \to \mathrm{BAut}(\mathbb{D}_V)$, we can ask if there is a lift:

$$\begin{tikzcd} & \mathrm{BG} \arrow[d, "\varphi"] \\ M \arrow[ur, dashed] \arrow[r, "\chi_M"'] & \mathrm{BAut}(\mathbb{D}_V) \end{tikzcd}$$

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For example, such a lift for $G = O(n)$ together with another condition is a Pseudo-Riemannian structure on M.