# Aspects of Differential Geometry in HoTT

Felix Wellen

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 ${\sf Smooth\ Manifolds} \buildrel {\sf Smooth\ Sets} = {\sf Sh}(\{\mathbb{R}^n \times \mathbb{D} | n \in \mathbb{N}\})$ 

 $\mathsf{Schemes} \overset{\longleftarrow}{\longrightarrow} \mathsf{Zariski\text{-}sheaves} \quad = \quad \operatorname{Sh}(\operatorname{Rings}^{\mathrm{op}})$ 

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Schemes 
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 Zariski-sheaves =  $Sh(Rings^{op})$ 

Where " $\mathrm{Sh}$ " means the topos of set- or  $\infty$ -groupoid-valued sheaves on:

- 1.  $\mathbb{R}^n \times \mathbb{D}$  with smooth open good covers (ignoring the  $\mathbb{D}$ s).
- 2. Commutative, unital rings with jointly surjective inclusions of Zariski-open affine subsets.

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 $\Rightarrow$  Applicable in any context.

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Let us see, what this functor does to a sheaf S, representing a k-Scheme:

 $\{\mathsf{Tangent}\ \mathsf{vectors}\ \mathsf{at}\ k\mathsf{-points}\} \cong S(\mathrm{Spec}(k[X]/(X^2)))$ 



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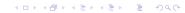
Let us see, what this functor does to a sheaf S, representing a k-Scheme:

$$\{\mathsf{Tangent}\ \mathsf{vectors}\ \mathsf{at}\ k\mathsf{-points}\} \cong S(\mathrm{Spec}(k[X]/(X^2)))$$

But  $(k[X]/(X^2))_{\mathrm{red}}$  is just k, so the tangent vectors at k-points of  $\Im(S)$  are just the k-points:

$$\Im(S)(\operatorname{Spec}(k[X]/(X^2))) = S(\operatorname{Spec}(k[X]/(X^2))_{\operatorname{red}}) = S(\operatorname{Spec}(k))$$

So:  $\Im$  removes all differential geometric information!



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$$\mathbb{R}^n \times \mathbb{D}_V = \operatorname{Spec}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V))$$

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and call this category FC. For any k, we can restrict the order:

$$\mathrm{FC}_k \coloneqq \{\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V) | n \in \mathbb{N}, V^{k+1} = 0 \}^\mathrm{op} \subseteq \mathbb{R} - \mathrm{algebras}^\mathrm{op}$$



Now, define 
$$\Im\colon\mathrm{Sh}(\mathrm{FC}) o\mathrm{Sh}(\mathrm{FC})$$
 by 
$$\Im(\mathcal{F})(\mathbb{R}^n\times\mathbb{D}_V)\coloneqq\mathcal{F}(\mathbb{R}^n)$$

Now, define  $\mathfrak{I} \colon \mathrm{Sh}(\mathrm{FC}) \to \mathrm{Sh}(\mathrm{FC})$  by

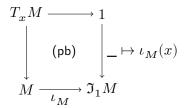
$$\Im(\mathcal{F})(\mathbb{R}^n\times\mathbb{D}_V)\coloneqq\mathcal{F}(\mathbb{R}^n)$$

and, respectively  $\mathfrak{I}_k \colon \mathrm{Sh}(\mathrm{FC}_k) \to \mathrm{Sh}(\mathrm{FC}_k)$  by the same equation

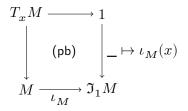
$$\mathfrak{I}_k(\mathcal{F})(\mathbb{R}^n\times \mathbb{D}_V)\coloneqq \mathcal{F}(\mathbb{R}^n)$$

Let M be a sheaf in  $\mathrm{Sh}(\mathrm{FC}_1)$  representing a smooth manifold.

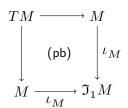
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The tangent bundle is also given as a pullback:



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- 3. For any type A,  $\Im A$  is coreduced.
- 4. For any  $B\colon \Im A \to \mathcal{U}$ , such that  $\prod_{a\colon \Im A} B(a)$  is coreduced, a section  $s\colon \prod_{a\colon \Im A} B(a)$  is defined by  $s_0\colon \prod_{a\colon A} B(\iota_A(a))$ .

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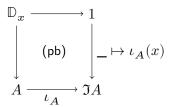
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- 5. Coreduced types have coreduced identity types.

# Internal geometric notions

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#### **Definition**

For any point  $x \colon A$ ,  $\mathbb{D}_x$  is defined by

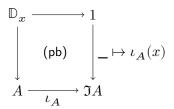


and called formal disk at x.

## Internal geometric notions

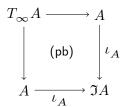
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The formal disk bundle over A,  $T_{\infty}A$  is defined by the pullback



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$$\prod_{x:X} \mu(e,x) = x \text{ and } \prod_{x:X} \mu(x,e) = x.$$

4. Proof that for any  $a\!:\!X$  the right-translation  $x\mapsto \mu(x,a)$  is an equivalence, i.e. there is a term of type

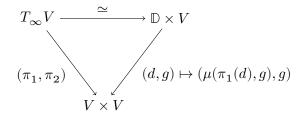
$$\prod_{a \in X} (x \mapsto \mu(x, a)) \text{ is an equivalence.}$$

# The triviality theorem

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#### Theorem

Let V be a left invertible H-space and  $\mathbb D$  the formal disk at the unit in V, then:

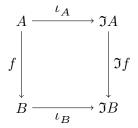


# Differential structure preserving morphisms

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A map  $f \colon A \to B$  is called *formally étale* if the naturality square

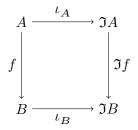


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#### Remark

For smooth manifolds formally étale maps correspond to local diffeomorphisms.

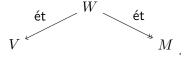
For noetherian schemes, they correspond to étale maps.

# Structured spaces

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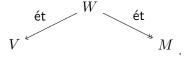
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## **Theorem** (needs Univalence)

Any  $V\operatorname{-Manifold}$  has a locally trivial formal disk bundle witnessed by a classifying map

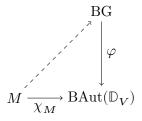
$$\chi_M \colon M \to \mathrm{BAut}(\mathbb{D}_V)$$

# Cartan Geometry

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#### Remark

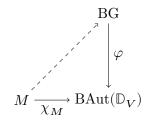
If we have a delooping BG of a group G with a map  $\varphi\colon BG\to \mathrm{BAut}(\mathbb{D}_V)$ , we can ask if there is a lift:



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For example, such a lift for G = O(n) together with another condition is a Pseudo-Riemannian structure on M.