Aspects of Differential Geometry in HoTT

Felix Wellen

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Smooth Manifolds $\hookrightarrow$ Formal Smooth Sets $= \mathrm{Sh}(\{\mathbb{R}^n \times \mathbb{D} | n \in \mathbb{N}\})$

Schemes $\hookrightarrow$ Zariski-sheaves $= \mathrm{Sh}(\mathrm{Rings}^{\mathrm{op}})$

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Schemes $\hookrightarrow$ Zariski-sheaves $\quad = \quad \mathrm{Sh}(\mathrm{Rings}^{\mathrm{op}})$

Where "Sh" means the topos of set- or ∞-groupoid-valued sheaves on:

1. $\mathbb{R}^n \times \mathbb{D}$ with smooth open good covers (ignoring the $\mathbb{D}$s).
2. Commutative, unital rings with jointly surjective inclusions of Zariski-open affine subsets.

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Let us see, what this functor does to a sheaf S, representing a k-Scheme:

$$\{\text{Tangent vectors at } k\text{-points}\} \cong S(\mathrm{Spec}(k[X]/(X^2)))$$

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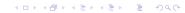$\mathfrak{I}(\mathcal{F})$ is called the infinitesimal shape, coreduction or de Rham stack of $\mathcal{F}$.

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But $(k[X]/(X^2))_{\mathrm{red}}$ is just k, so the tangent vectors at k-points of $\mathfrak{I}(S)$ are just the k-points:

$$\mathfrak{I}(S)(\mathrm{Spec}(k[X]/(X^2))) = S(\mathrm{Spec}(k[X]/(X^2))_{\mathrm{red}}) = S(\mathrm{Spec}(k))$$

So: $\mathfrak{I}$ removes all differential geometric information!

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is fully faithful and let us write Spec for the inverse on its image.

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For all nilpotent $\mathbb{R}$-algebra V, formally extend the left category by

$$\mathbb{R}^n \times \mathbb{D}_V = \mathrm{Spec}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V))$$

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$$\mathbb{R}^n \times \mathbb{D}_V = \mathrm{Spec}(\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V))$$

and call this category FC. For any k, we can restrict the order:

$$\mathrm{FC}_k := \{\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V) | n \in \mathbb{N}, V^{k+1} = 0\}^{\mathrm{op}} \subseteq \mathbb{R}-\mathrm{algebras}^{\mathrm{op}}$$

Now, define $\mathfrak{I} \colon \mathrm{Sh}(\mathrm{FC}) \to \mathrm{Sh}(\mathrm{FC})$ by

$$\mathfrak{I}(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}_V) := \mathcal{F}(\mathbb{R}^n)$$

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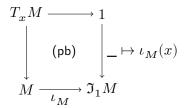$$\mathfrak{I}(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}_V) := \mathcal{F}(\mathbb{R}^n)$$

and, respectively $\mathfrak{I}_k \colon \mathrm{Sh}(\mathrm{FC}_k) \to \mathrm{Sh}(\mathrm{FC}_k)$ by the same equation
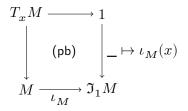
$$\mathfrak{I}_k(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}_V) := \mathcal{F}(\mathbb{R}^n)$$

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For any point $x \in M$, the tangent space is given as a pullback

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\begin{array}{ccc}
T_x M & \longrightarrow & 1 \\
\downarrow & \text{(pb)} & \downarrow {\scriptstyle \_ \,\mapsto\, \iota_M(x)} \\
M & \xrightarrow{\ \iota_M\ } & \mathfrak{I}_1 M
\end{array}
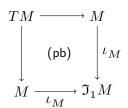$$

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The tangent bundle is also given as a pullback:

$$
\begin{array}{ccc}
TM & \longrightarrow & M \\
\downarrow & \text{(pb)} & \downarrow {\scriptstyle\iota_M} \\
M & \xrightarrow{\iota_M} & \mathfrak{I}_1 M
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3. For any type A, $\Im A$ is coreduced.
4. For any $B \colon \Im A \to \mathcal{U}$, such that $\prod_{a \colon \Im A} B(a)$ is coreduced, a section $s \colon \prod_{a \colon \Im A} B(a)$ is defined by $s_0 \colon \prod_{a \colon A} B(\iota_A(a))$.

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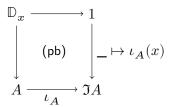
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4. For any $B \colon \mathfrak{I}A \to \mathcal{U}$, such that $\prod_{a \colon \mathfrak{I}A} B(a)$ is coreduced, a section $s \colon \prod_{a \colon \mathfrak{I}A} B(a)$ is defined by $s_0 \colon \prod_{a \colon A} B(\iota_A(a))$.
5. Coreduced types have coreduced identity types.

Internal geometric notions

Definition
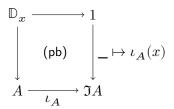
For any point $x \colon A$, $\mathbb{D}_x$ is defined by

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and called *formal disk at x*.

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The *formal disk bundle over A*, $T_\infty A$ is defined by the pullback

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\begin{array}{ccc}
T_\infty A & \longrightarrow & A \\
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Towards a theorem

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3. Proof that the unit is a left and right unit, i.e. a term in each of
$$\prod_{x \,:\, X} \mu(e, x) = x \text{ and } \prod_{x \,:\, X} \mu(x, e) = x.$$

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$$\prod_{x \,:\, X} \mu(e, x) = x \text{ and } \prod_{x \,:\, X} \mu(x, e) = x.$$

4. Proof that for any $a \colon X$ the right-translation $x \mapsto \mu(x, a)$ is an equivalence, i.e. there is a term of type
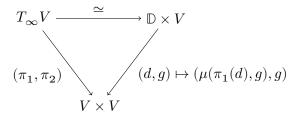$$\prod_{a \,:\, X} (x \mapsto \mu(x, a)) \text{ is an equivalence.}$$

The triviality theorem

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Theorem
Let V be a left invertible H-space and $\mathbb{D}$ the formal disk at the unit in V, then:

$$
\begin{array}{ccc}
T_\infty V & \xrightarrow{\simeq} & \mathbb{D} \times V \\
& & \\
{\scriptstyle(\pi_1, \pi_2)}\searrow & & \swarrow{\scriptstyle(d, g) \mapsto (\mu(\pi_1(d), g), g)} \\
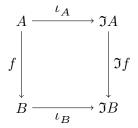& V \times V &
\end{array}
$$

This theorem has a proof similar to its topos theoretic version in [KS17].

Differential structure preserving morphisms

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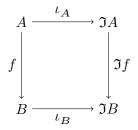A map $f \colon A \to B$ is called *formally étale* if the naturality square

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A & \xrightarrow{\ \iota_A\ } & \mathfrak{I}A \\
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B & \xrightarrow[\ \iota_B\ ]{} & \mathfrak{I}B
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$$

is a pullback square.

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\end{array}
$$

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Remark
For smooth manifolds formally étale maps correspond to local diffeomorphisms.
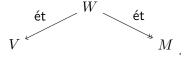For noetherian schemes, they correspond to étale maps.

Structured spaces
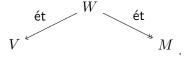
Structured spaces

Definition
Let V be a left invertible H-space. A type M is called a
V-Manifold, if there is a span of formally étale maps

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V & & M
\end{array}
$$

.

Theorem (needs Univalence)
Any V-Manifold has a locally trivial formal disk bundle witnessed
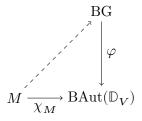by a classifying map

$$\chi_M \colon M \to \mathrm{BAut}(\mathbb{D}_V)$$
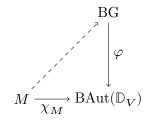
Cartan Geometry

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Remark

If we have a delooping $\mathrm{B}G$ of a group G with a map
$\varphi\colon \mathrm{B}G \to \mathrm{BAut}(\mathbb{D}_V)$, we can ask if there is a lift:

$$
\begin{array}{ccc}
 & & \mathrm{B}G \\
 & \nearrow & \downarrow {\scriptstyle \varphi} \\
M & \xrightarrow{\chi_M} & \mathrm{BAut}(\mathbb{D}_V)
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$$

For example, such a lift for $G = O(n)$ together with another
condition is a Pseudo-Riemannian structure on M.

References

A topos theoretic version of the theorem about the triviality of the formal disk bundle and more on the smooth case may be found in

📄 I. Khavkine and U. Schreiber. "Synthetic geometry of differential equations: I. Jets and comonad structure". In: *ArXiv e-prints* (Jan. 2017). arXiv: 1701.06238 [math.DG].

A thesis titled "Formalizing Cartan Geometry in Modal Homotopy Type Theory" containing more on the topic is to appear very soon.