SPATIAL ECOLOGY

Perceptron & Neural Nets

Antonio Fonseca

Agenda

1) Recap

- Linear regression
- Loss minimization and regularization

2) Perceptron

- Architecture
- Intro to optimizers (gradient descent)
- Hands-on tutorial

3) Feedforward Neural Networks

- The limitations of Perceptrons
- Multi-layer Perceptron
- Training: the forward and back-propagation
- Debugging tips

Review on Linear Regression

Task (T)

Input $x \in \mathbb{R}^n$

Weights $w \in \mathbb{R}^n$

$$\hat{y} = w^T x$$

$$f(x, w) = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n$$

Performance (P)

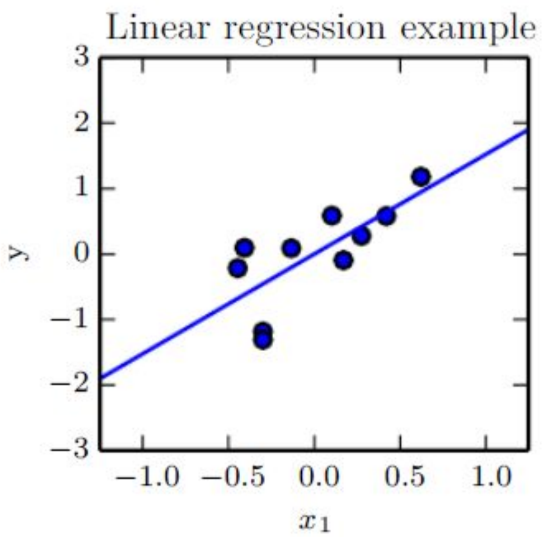
$$MSE_{test} = \frac{1}{m} \sum_i (\hat{y}_{test} - y_{test})_i^2$$

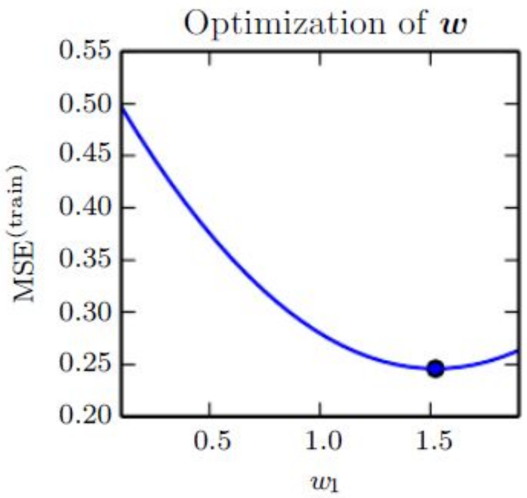
Dataset

$$(X, y) \begin{cases} (X_{train}, y_{train}) \\ (X_{test}, y_{test}) \end{cases}$$

Training

$$\nabla_w \left(\frac{1}{m} \sum_i (w^T X_{train} - y_{train})_i^2 \right) = 0$$

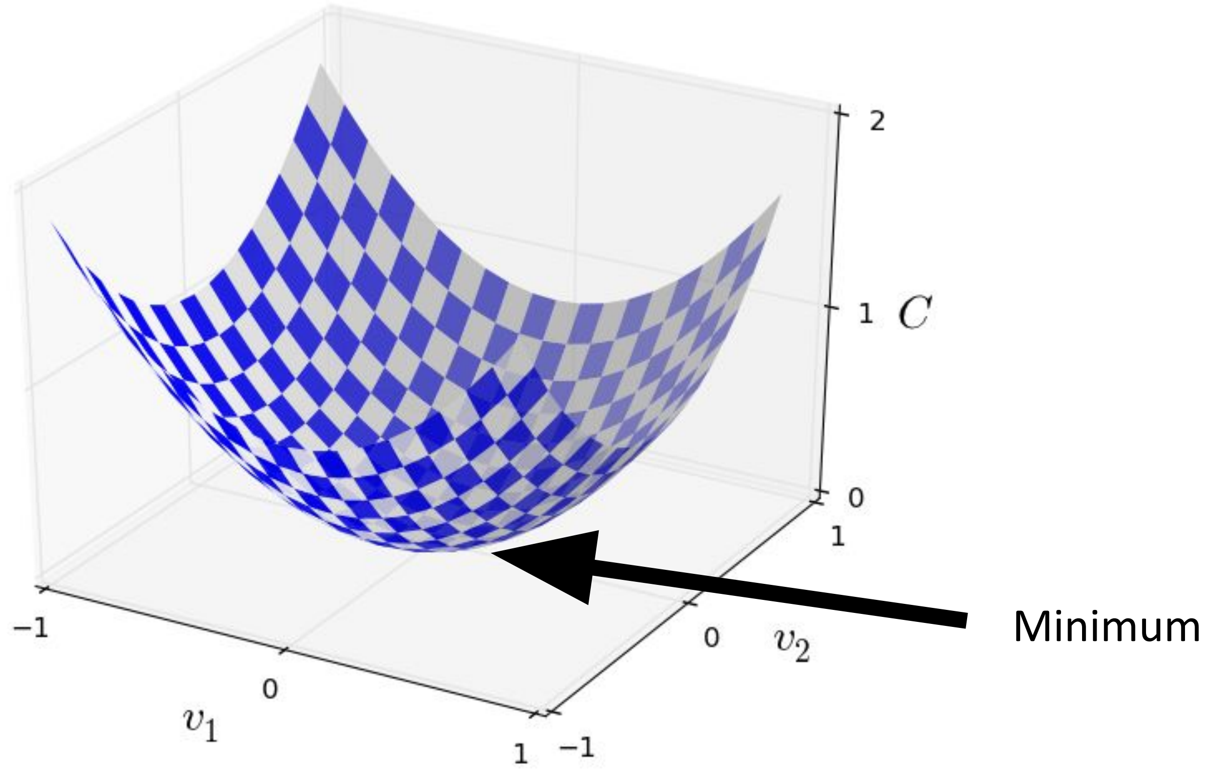




Solves linear problems

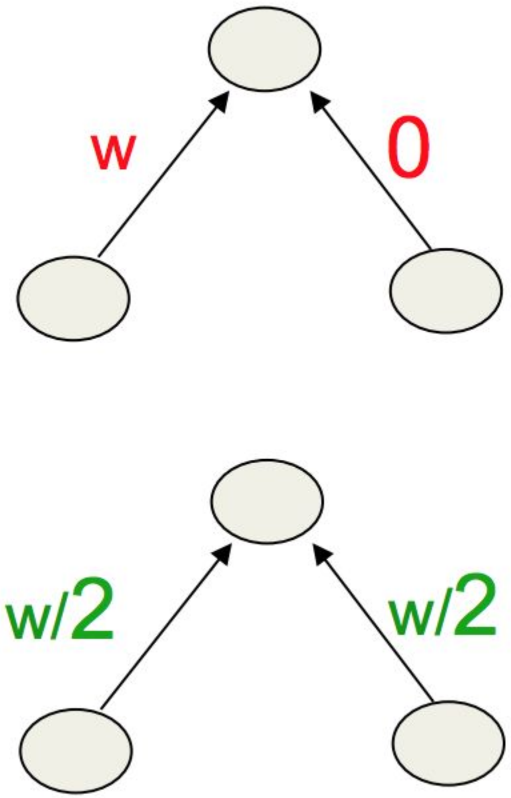
Can't solve more complex problems (e.g., XOR problem)

Loss Minimization



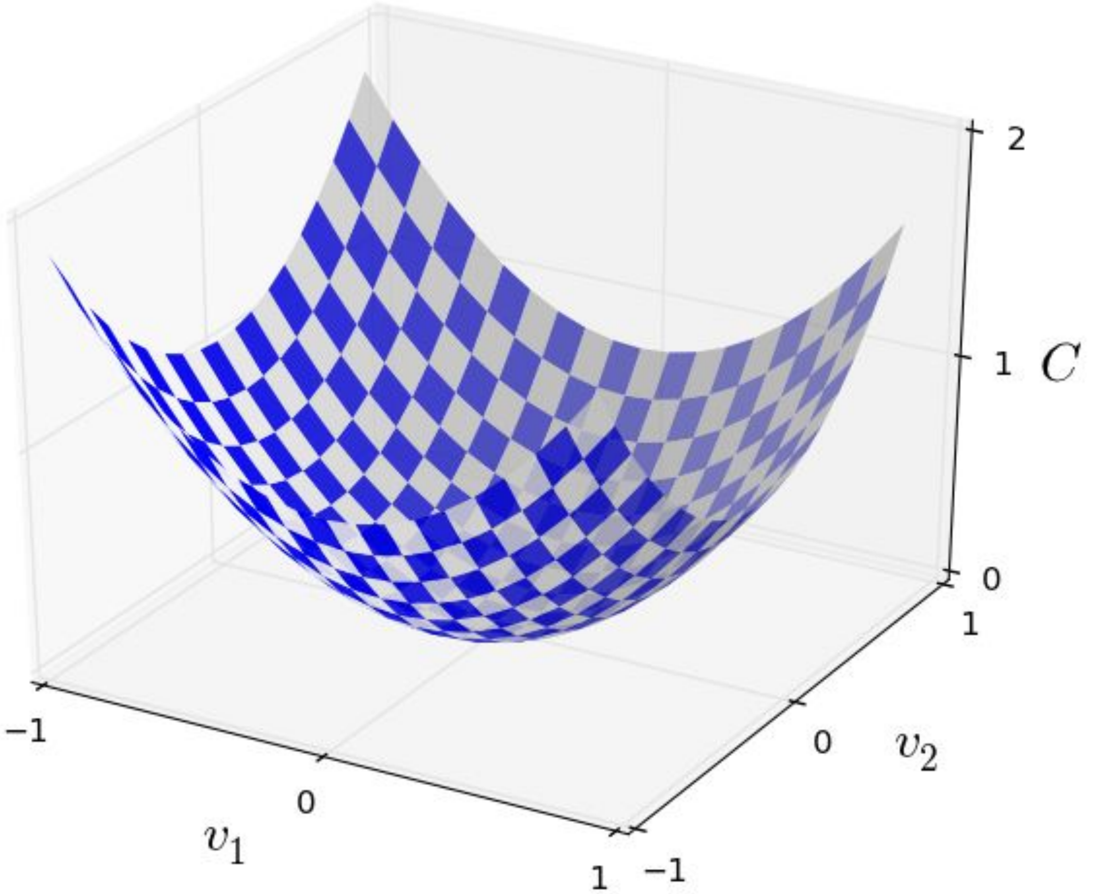
Convex loss functions can be solved by differentiation, at the point where Loss is minimum the derivative wrt to parameters should be 0!

Regularization

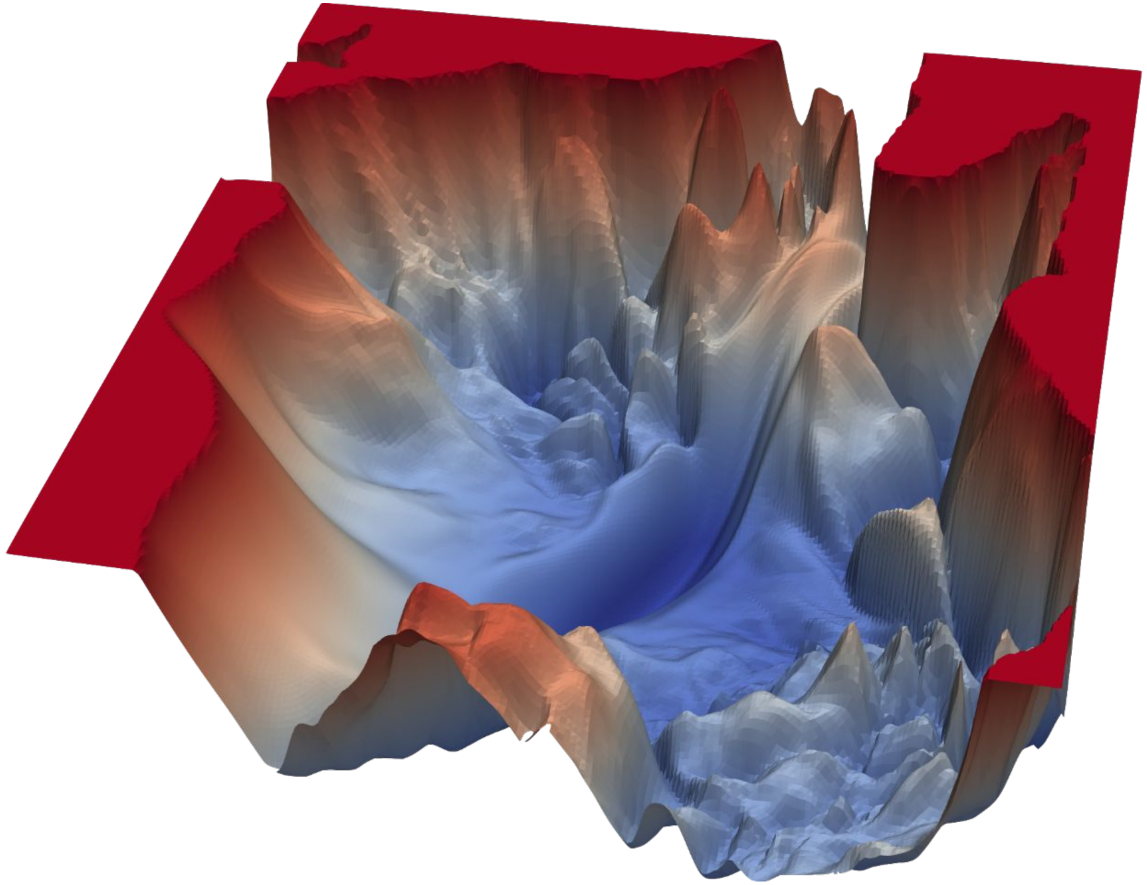


- Prefers to share smaller weights
- Makes model smoother
- More Convex

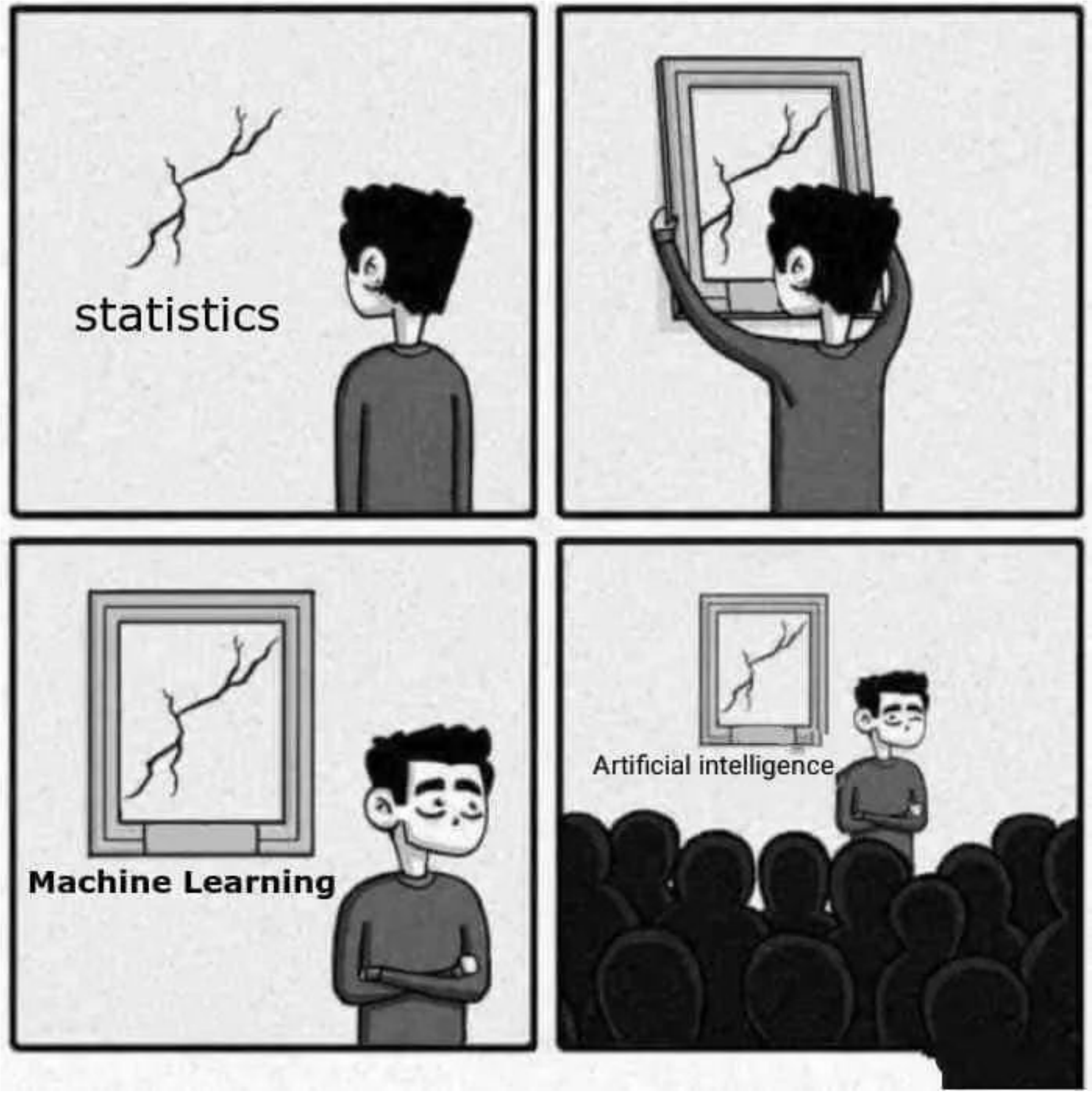
Expectation



Reality

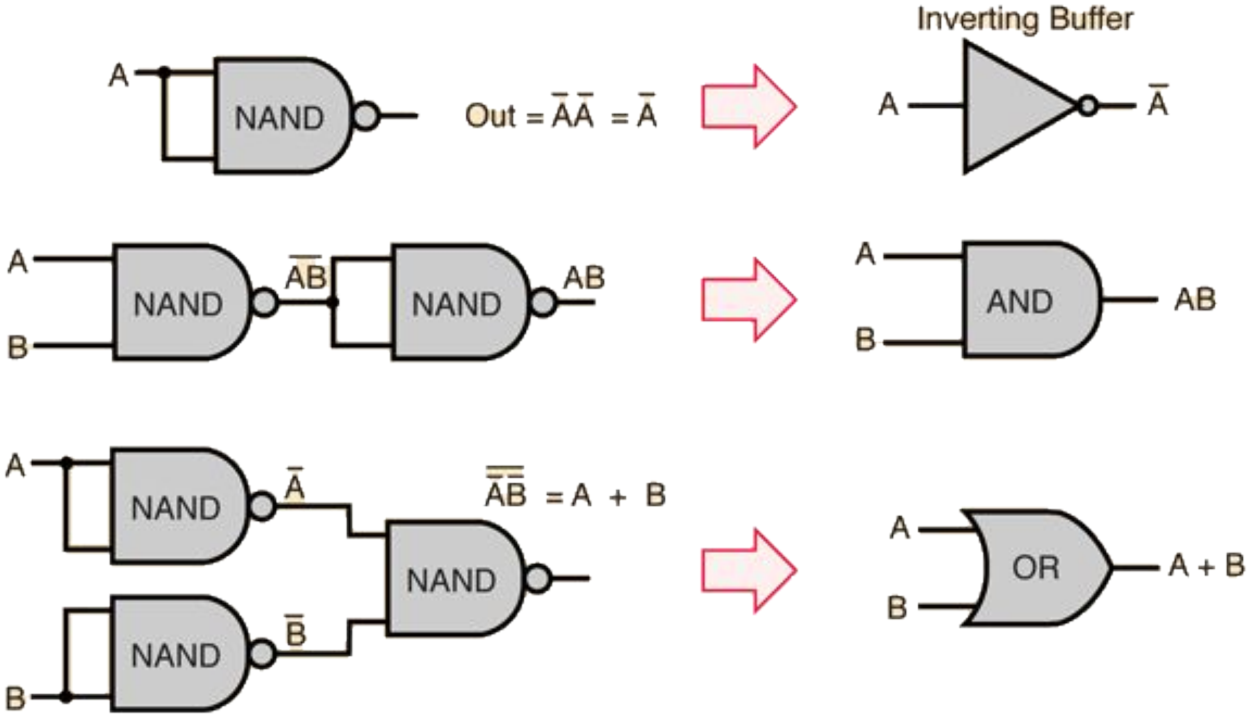


Summary of Class 1 and 2



Logic circuits with perceptrons

- NAND gates can be constructed from perceptrons
- NAND gates are universal for computation
  - Any computation can be built from NAND gates
  - Therefore, perceptrons are universal for computation

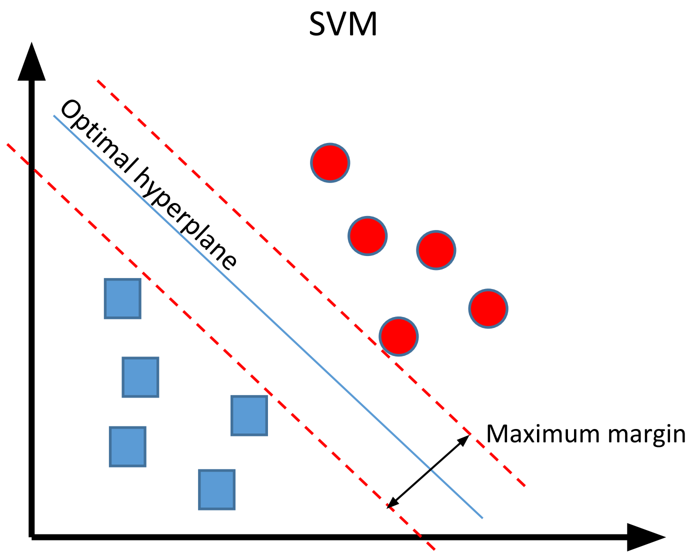


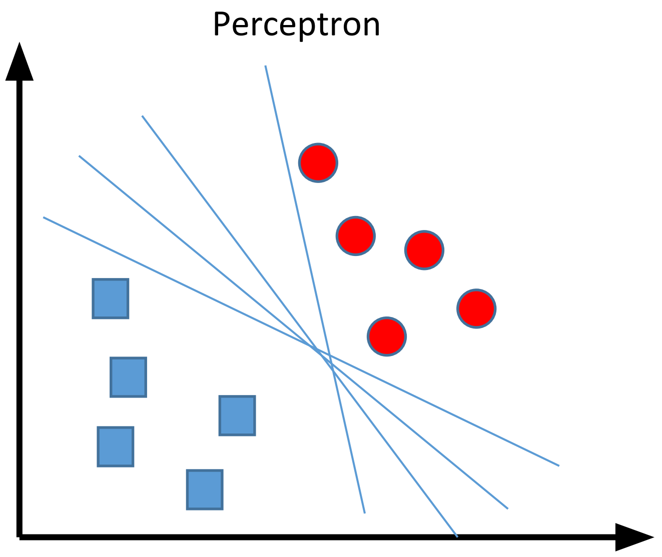
Nielsen, 2015

Perceptron

SVM vs Perceptron

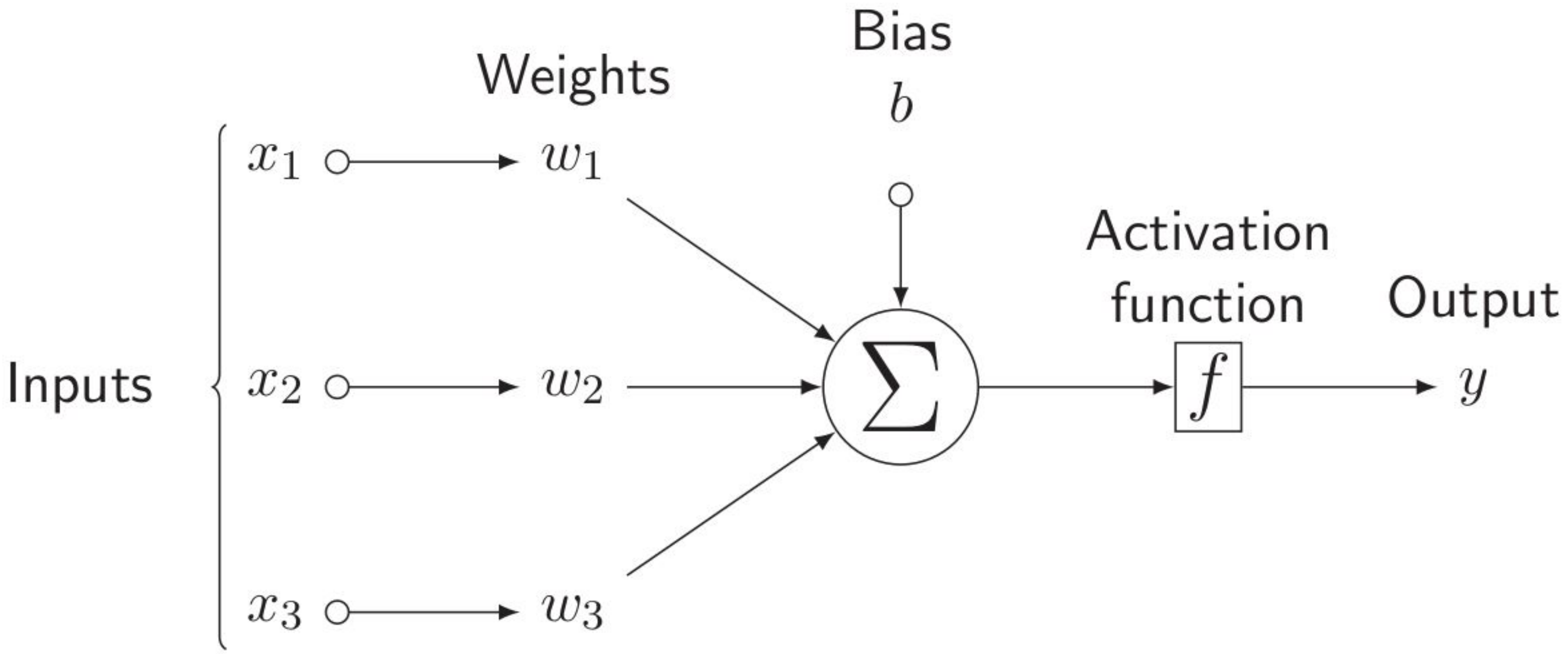
- SVM: Find the optimal hyperplane in an N-dimensional space that distinctly classifies the data points.
- Perceptron: Any hyperplane that can classify the points





Perceptron: Threshold Logic

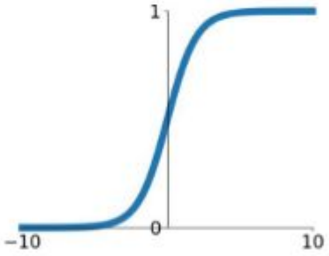
$$\mathcal{L}_{\text{perc}}(\mathbf{x}, y) = \begin{cases} 0 & \text{if } y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) > 0 \\ -y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) & \text{if } y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) \leq 0 \end{cases}$$



Activation functions

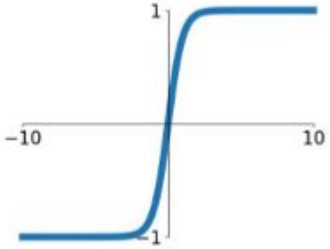
Sigmoid

$\sigma(x) = \frac{1}{1+e^{-x}}$



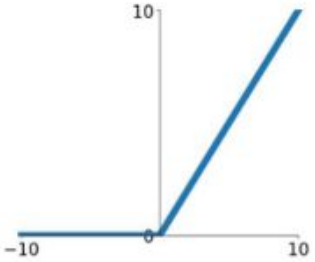
tanh

$\tanh(x)$



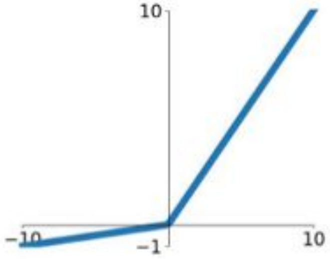
ReLU

$\max(0, x)$



Leaky ReLU

$\max(0.1x, x)$

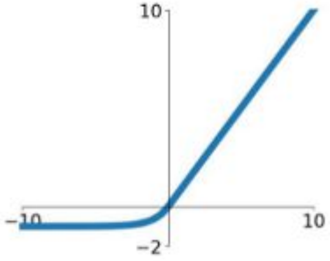


Maxout

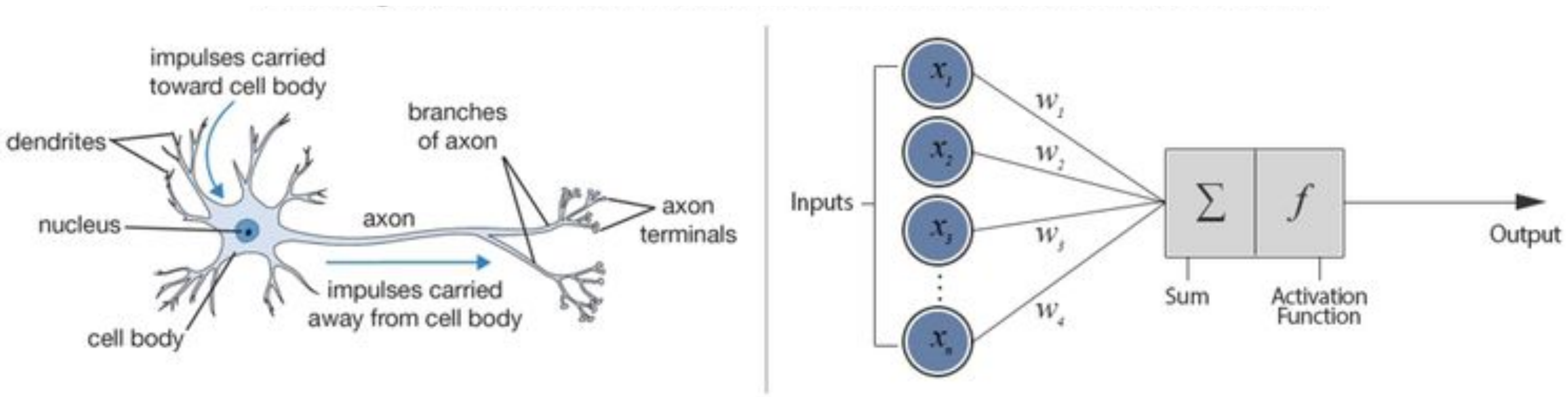
$\max(w_1^T x + b_1, w_2^T x + b_2)$

ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



Perceptrons and neurons



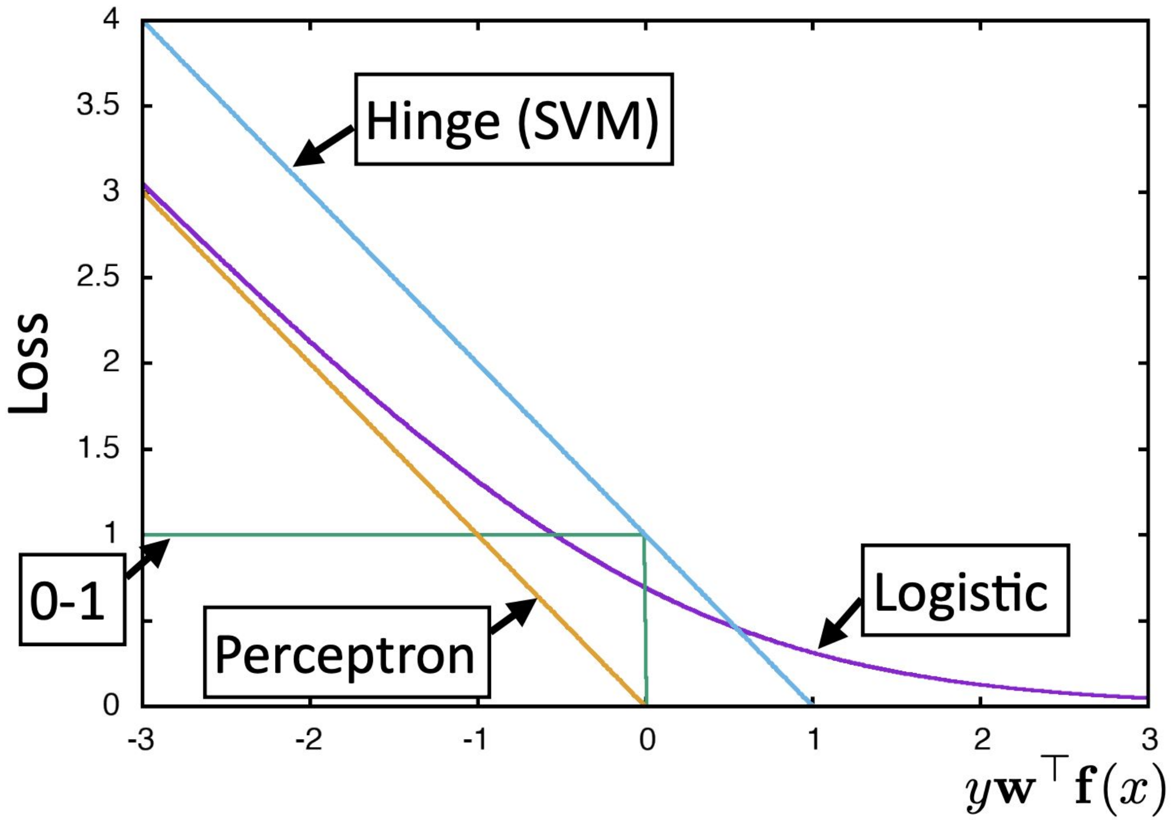
(Putting things in perspective)

$$\mathcal{L}_{\mathrm{lr}}(\mathbf{x}, y) = \begin{cases} -y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) + \log\left(1 + \exp\left(y\mathbf{w}^\top \mathbf{f}(\mathbf{x})\right)\right) & \text{if } y = +1 \text{ (positive)} \\ \log\left(1 + \exp\left(-y\mathbf{w}^\top \mathbf{f}(\mathbf{x})\right)\right) & \text{if } y = -1 \text{ (negative)} \end{cases}$$

$$\mathcal{L}_{\mathrm{perc}}(\mathbf{x}, y) = \begin{cases} 0 & \text{if } y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) > 0 \\ -y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) & \text{if } y\mathbf{w}^\top \mathbf{f}(\mathbf{x}) \le 0 \end{cases}$$

Main differences:

- Perceptron: gradient-based optimization
- LR: probabilistic model
- Perceptron: if the data are linearly separable, perceptron is guaranteed to converge.
- LR: likelihood can never truly be maximized with a finite w vector.



Optimizers

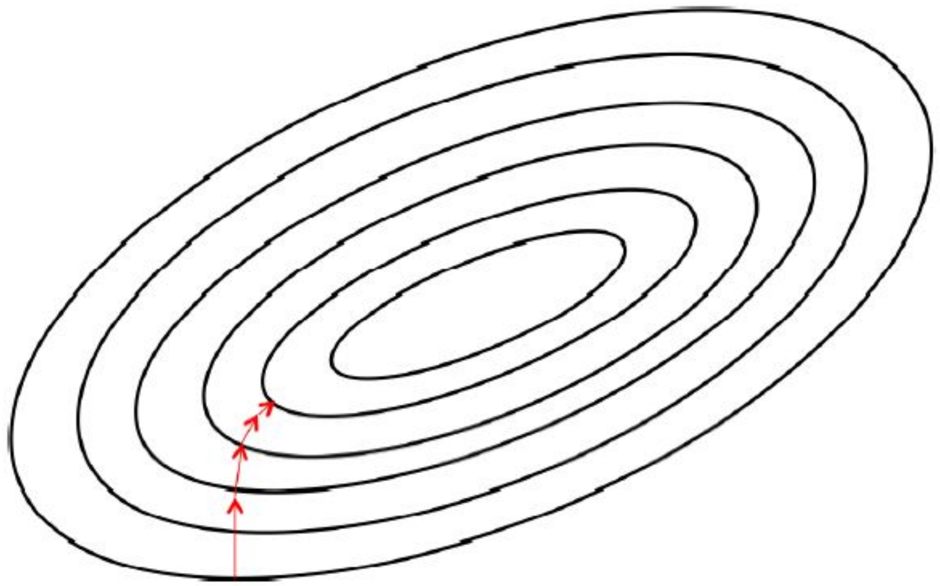
Gradient

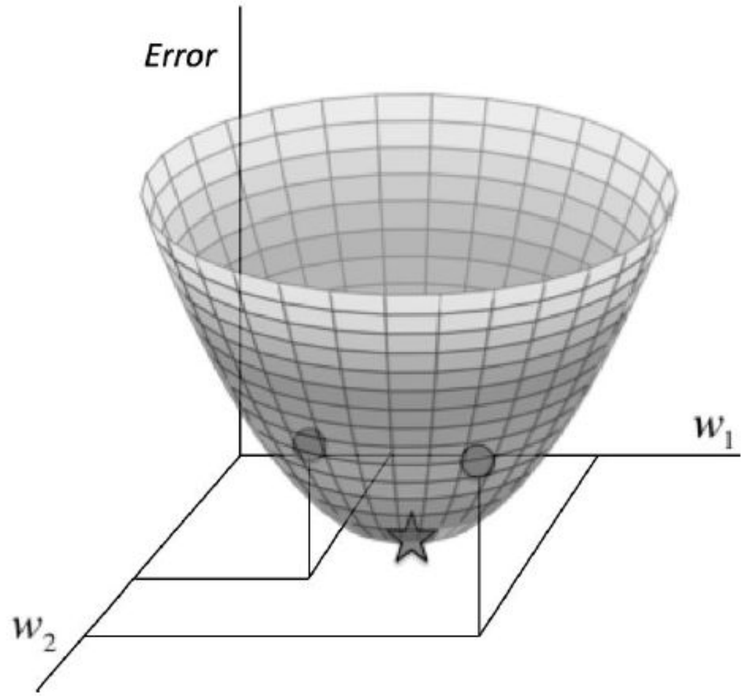
$$\Delta w_k = -\frac{\partial E}{\partial w_k}$$

$$= -\frac{\partial}{\partial w_k}\left(\frac{1}{m}\sum_i (w^T X_i - y_i)_i^2\right)$$

$$w_{i+1} = w_i + \Delta w_k$$

Stochastic gradient descent (**SGD**)





Optimizers

Hyperparameters

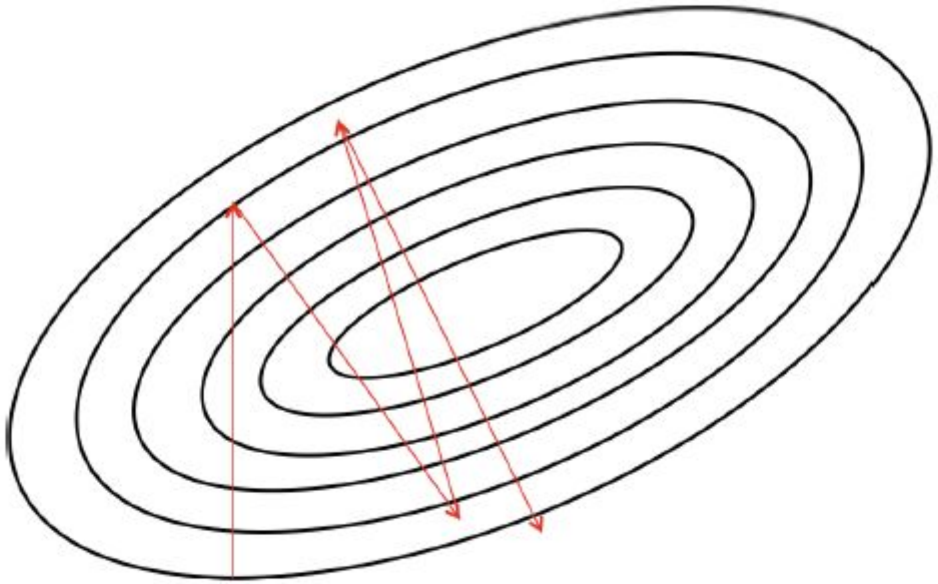
- Learning rate (α)

$$\Delta w_k = -\alpha \frac{\partial E}{\partial w_k}$$

$$= -\alpha \frac{\partial}{\partial w_k}\left(\frac{1}{m}\sum_i (w^T X_i - y_i)_i^2\right)$$

$$w_{i+1} = w_i + \Delta w_k$$

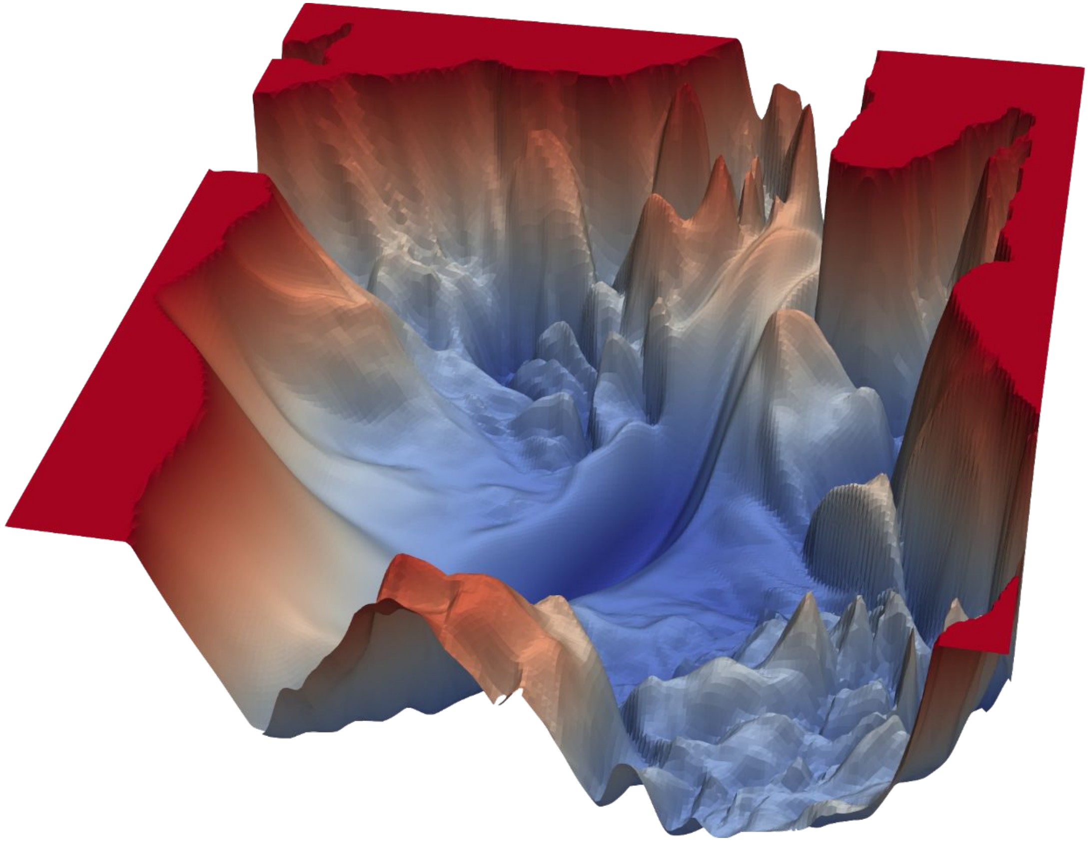
Stochastic gradient descent (**SGD**)



Result of a large learning rate α

Optimizers

Watch out for local minimal areas



Hyperparameters

- Learning rate (α)

$$\Delta w_k = -\alpha \frac{\partial E}{\partial w_k}$$

$$= -\alpha \frac{\partial}{\partial w_k}\left(\frac{1}{m}\sum_i (w^T X_i - y_i)_i^2\right)$$

$$w_{i+1} = w_i + \Delta w_k$$

Stochastic gradient descent (**SGD**)

Gradient Descent

- Gradient descent refers to taking a step in the direction of the ***gradient (partial derivative)*** of a weight or bias with respect to the cost function
- Gradients are propagated backwards through the network in a process known as ***backpropagation***
- The size of the step taken in the direction of the gradient is called the ***learning rate***

Time for a quiz and tutorial!

https://tinyurl.com/GeoComp2023

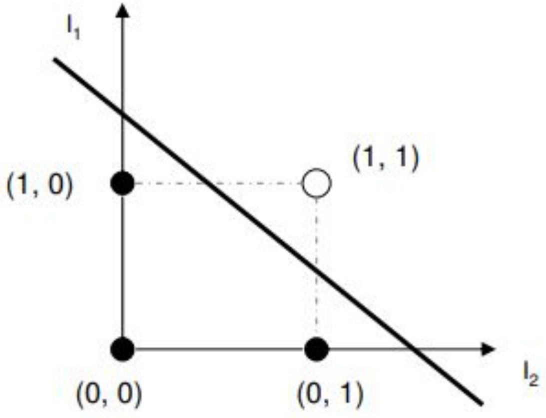
Now let's get our hands dirty!

Open: - Perceptron_Intro_Class3.ipynb
- Perceptron_tree_height_Class3.ipynb

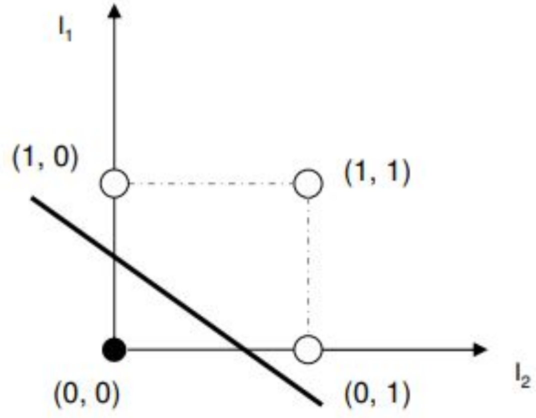
Multi-layer Perceptron

Limitations of the Perceptron

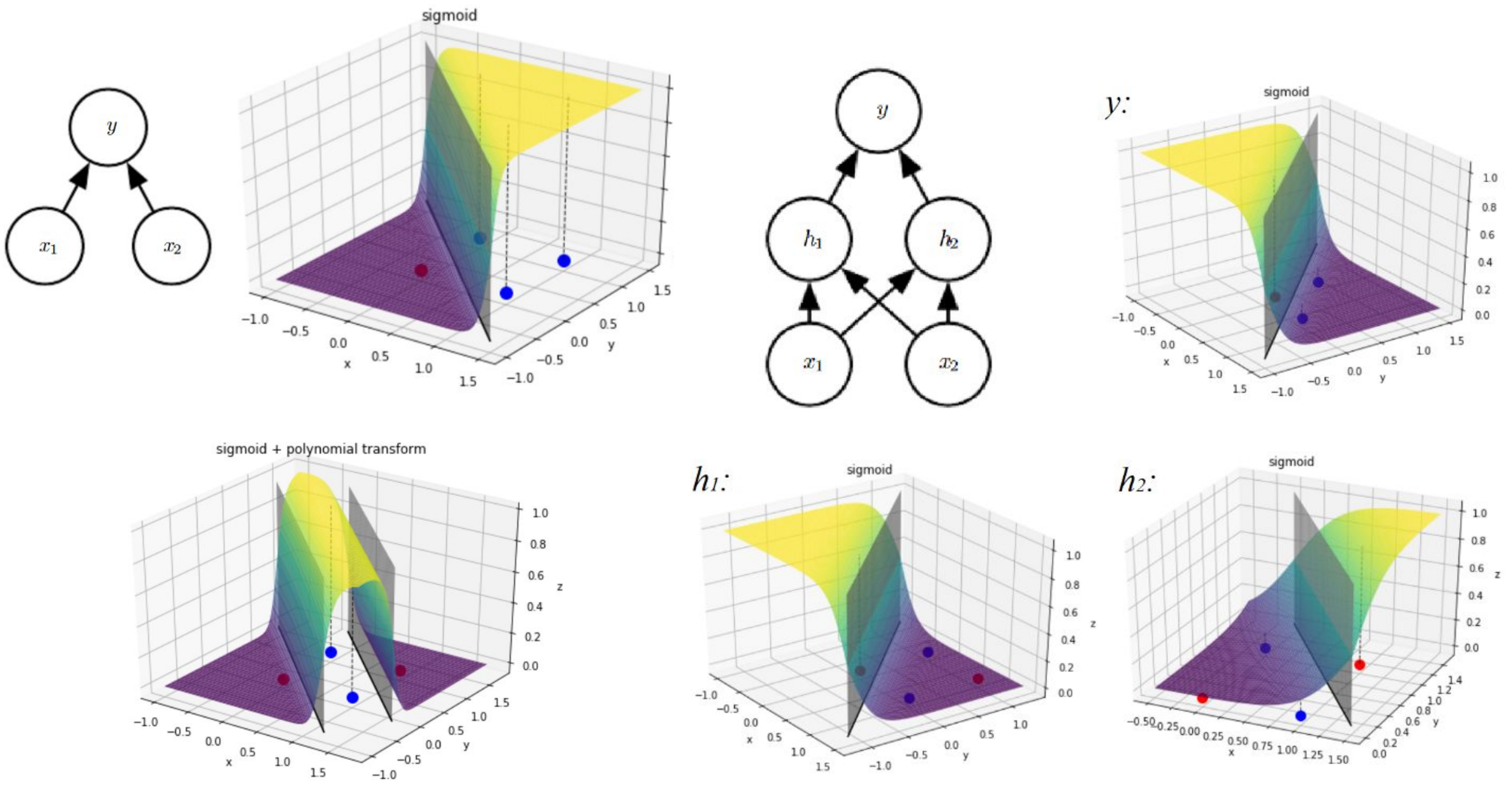
AND		
I_1	I_2	out
0	0	0
0	1	0
1	0	0
1	1	1



OR		
I_1	I_2	out
0	0	0
0	1	1
1	0	1
1	1	1

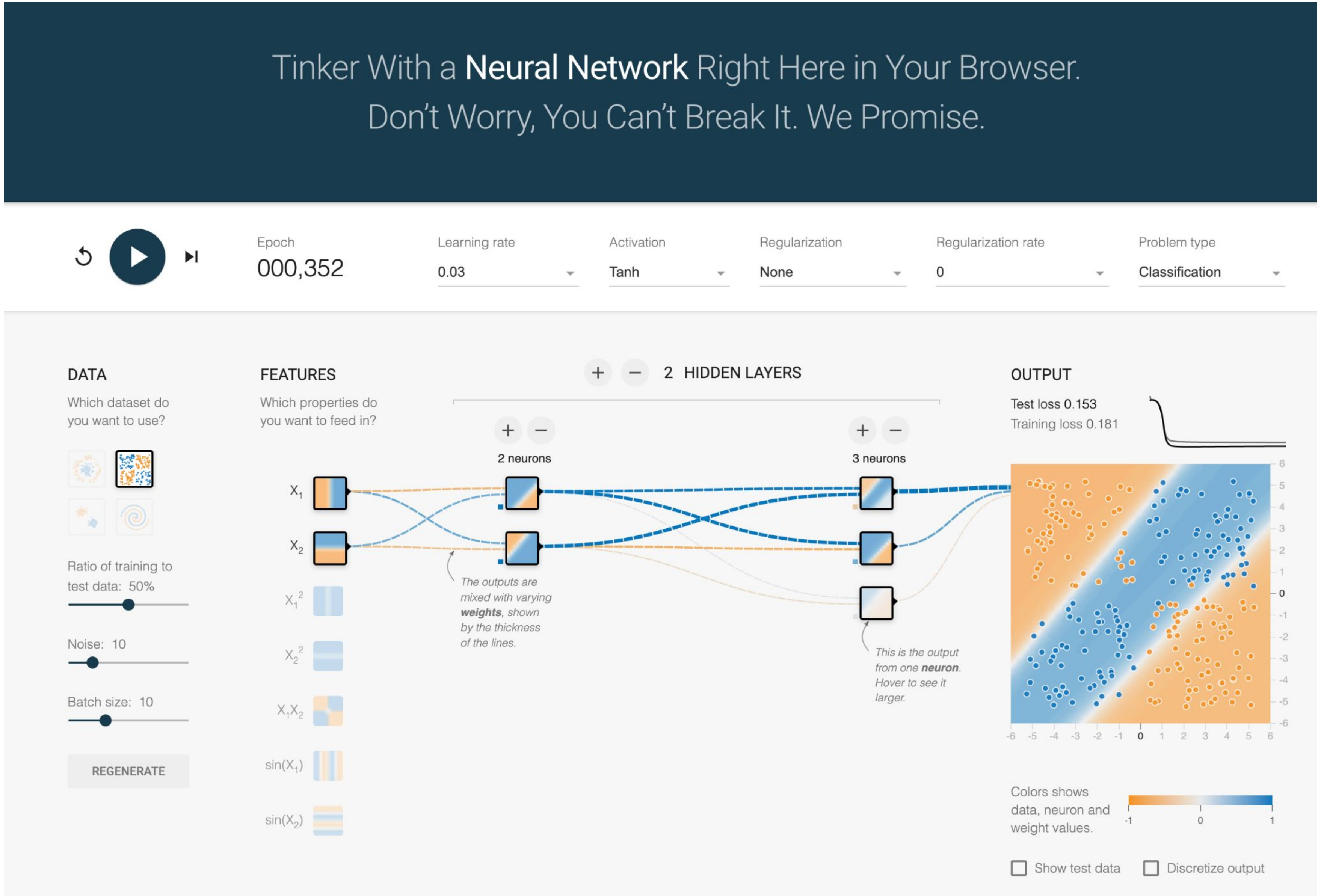


Perceptron



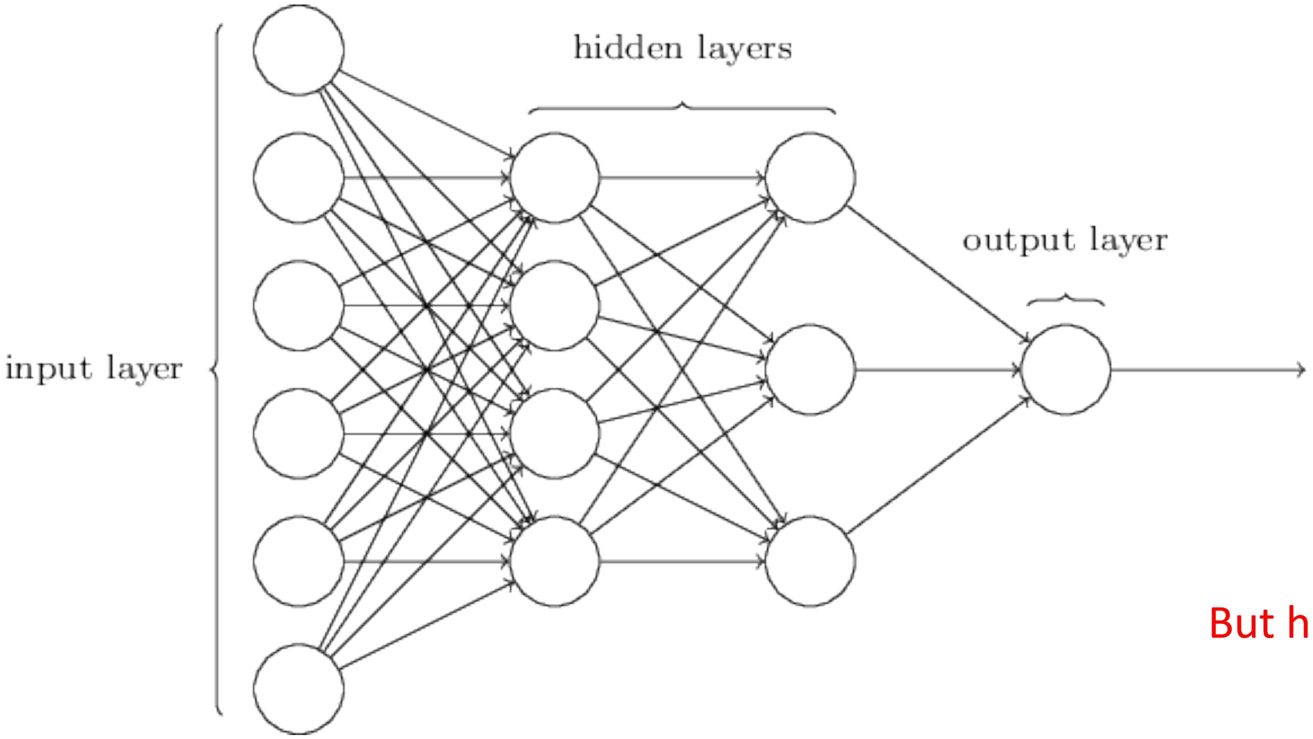
y
x1
x2
sigmoid
x
y
y
h1
h2
x1
x2
y:
sigmoid
x
y
sigmoid + polynomial transform
x
y
z
h1:
sigmoid
x
y
z
h2:
sigmoid
x
y
z

Let's play with it!



Try it [here]

Architecture of Neural Networks



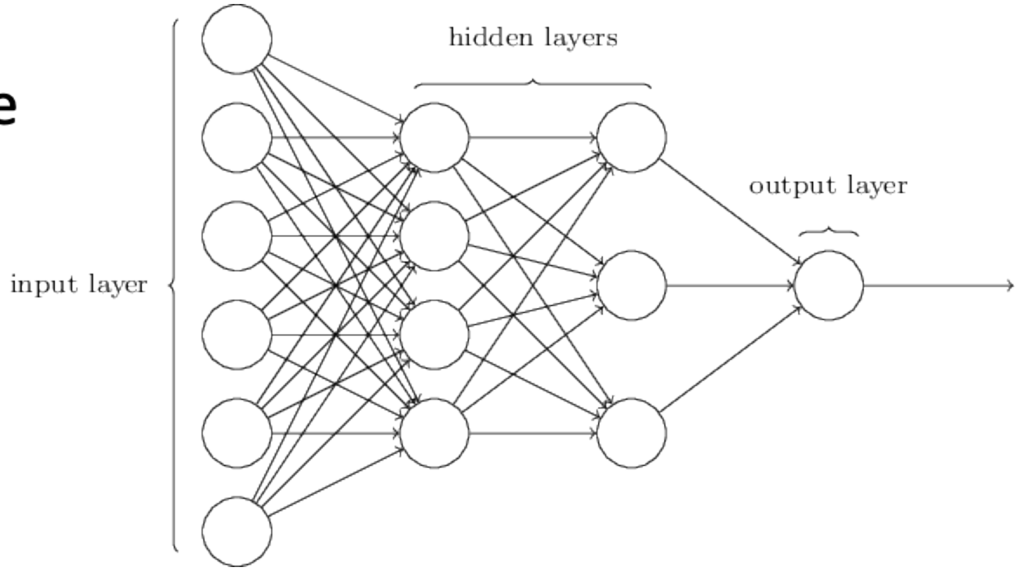
But how do we train it?

- Sometimes called multi-layer perceptron (MLP)
- Output from one layer is used as input for the next (Feedforward network)

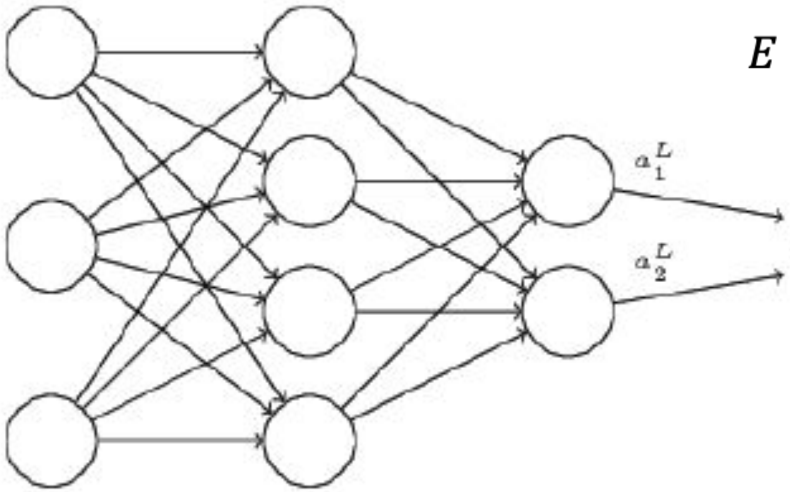
Forward Propagation

- Store weights and biases as matrices
- Suppose we are considering the weights from the second (hidden) layer to the third (output) layer
  - w is the weight matrix with w_{ji} the weight for the connection between the ith neuron in the second layer and the jth neuron in the third layer
  - b is the vector of biases in the third layer
  - a is the vector of activations (output) of the 2nd layer
  - a' the vector of activations (output) of the third layer

$$\boldsymbol{a'} = \boldsymbol{\sigma}(\boldsymbol{wa} + \boldsymbol{b})$$



Backpropagation



$$E = \frac{1}{2}\sum_i \left(a_i^L - y_i\right)^2$$

$$cost\ E = E(a^L)$$

1. **Input x:** Set the corresponding activation a^1 for the input layer.
2. **Feedforward:** For each $l = 2, 3, \ldots, L$ compute $z^l = w^l a^{l-1} + b^l$ and $a^l = \sigma(z^l)$.
3. **Output error δ^L:** Compute the vector $\delta^L = \nabla_a C \odot \sigma'(z^L)$.
4. **Backpropagate the error:** For each $l = L-1, L-2, \ldots, 2$ compute $\delta^l = ((w^{l+1})^T \delta^{l+1}) \odot \sigma'(z^l)$.
5. **Output:** The gradient of the cost function is given by $\frac{\partial C}{\partial w^l_{jk}} = a^{l-1}_k \delta^l_j$ and $\frac{\partial C}{\partial b^l_j} = \delta^l_j$.

$$z_j^l = \sum_i w_{ji}^l a_i^{l-1} + b_j^l \qquad a_j^l = \sigma\left(\sum_i w_{ji}^l a_i^{l-1} + b_j^l\right) = \sigma(z_j^l)$$

$$\delta_j^L \equiv \frac{\partial E}{\partial z_j^L} = \frac{\partial E}{\partial a_i^L}\frac{\partial a_i^L}{\partial z_j^L} = \frac{\partial E}{\partial a_j^L}\sigma'(z_j^L) \qquad (1)$$

$$\delta_j^l \equiv \frac{\partial E}{\partial z_j^l} = \frac{\partial E}{\partial z_i^{l+1}}\frac{\partial z_i^{l+1}}{\partial z_j^l} = \frac{\partial z_i^{l+1}}{\partial z_j^l}\delta_i^{l+1}$$

$$= \frac{\partial(\sum_i w_{ij}^{l+1} a_j^l + b_i^{l+1})}{\partial z_j^l}\delta_j^{l+1} = \sum_i w_{ij}^{l+1}\delta_i^{l+1}\sigma'(z_j^l) \qquad (2)$$

$$\frac{\partial E}{\partial w_{ji}^l} = \frac{\partial E}{\partial a_i^l}\frac{\partial a_i^l}{\partial z_j^l}\frac{\partial (w_{ji}^l a_i^{l-1})}{\partial w_{ji}^l}$$

Optimizers

Hyperparameters

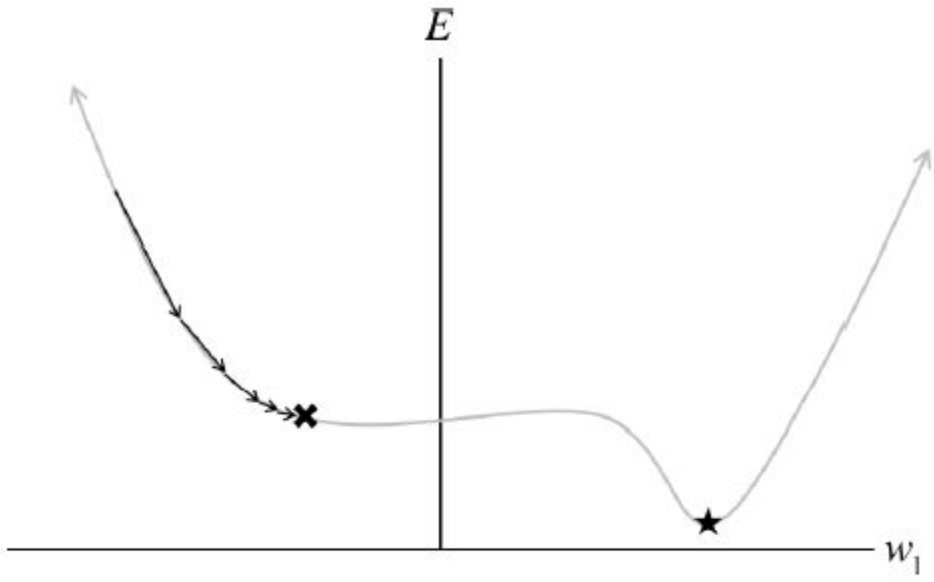
- Learning rate (α)

$$\Delta w_k = -\alpha \frac{\partial E}{\partial w_k}$$

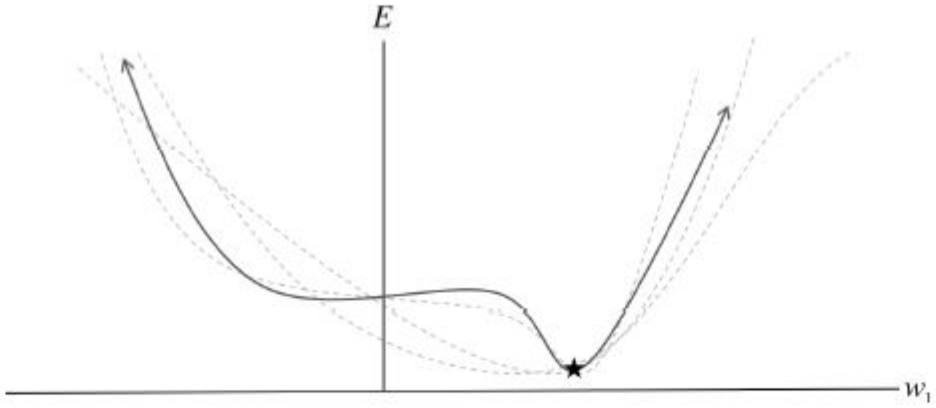
$$= -\alpha \frac{\partial}{\partial w_k} \left(\frac{1}{m} \sum_i (w^T X_i - y_i)_i^2 \right)$$

$$w_{i+1} = w_i + \Delta w_k$$

Stochastic gradient descent (**SGD**)



Local Minima



Multiple samples

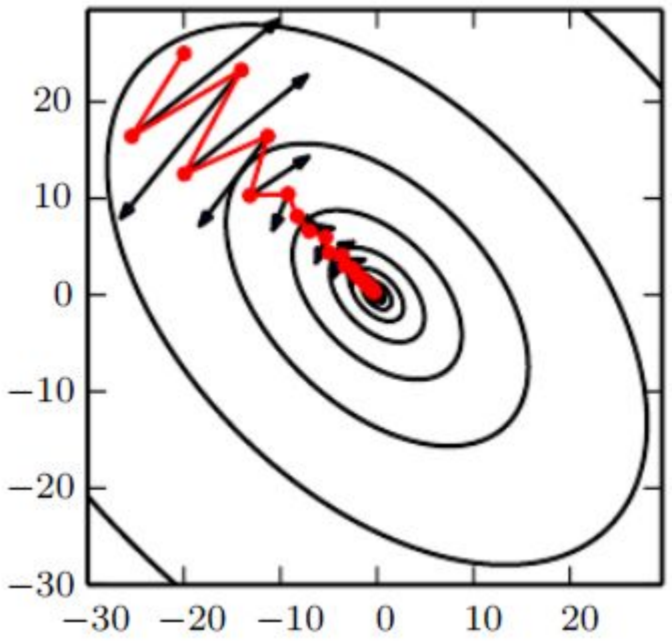
Optimizers

Hyperparameters

- Learning rate (α)
- Momentum (β)

$$v_{i+1} = v\beta - \alpha \frac{\partial}{\partial w_k}\left(\frac{1}{m}\sum_i (w^T X_i - y_i)_i^2\right)$$

$$w_{i+1} = w_i + v$$



SGD

SGD+Momentum

Stochastic gradient descent with momentum (**SGD+Momentum**)

Optimizers

Hard to pick right hyperparameters

- Small learning rate: long convergence time
- Large learning rate: convergence problems

Adagrad: adapts learning rate to each parameter

$$\Delta w_{k,t} = -\alpha \frac{\partial E_t}{\partial w_{k,t}} = -\alpha \nabla_w E(w_t)$$

- Learning rate might decrease too fast
- Might not converge

$$g_{t,i} = \nabla_w E(w_{t,i})$$

$$G_{t+1,i} = G_{t,i} + g_{t,i} \odot g_{t,i}$$

$$w_{t+1,i} = w_{t,i} - \frac{\alpha}{\sqrt{G_{t,i} + \epsilon}} g_{t,i}$$

Optimizers

RMSprop: decaying average of the past squared gradients

$$E[g^2]_t = \gamma E[g^2]_{t-1} + (1-\gamma) g_t^2$$

Decaying average

Adadelta

$$\boxed{\begin{aligned} E[\Delta_w^2]_t &= \gamma E[\Delta_w^2]_{t-1} + (1-\gamma)\Delta_w^2 \\ \Delta w_t &= \frac{\sqrt{E[\Delta_w^2]_t + \epsilon}}{\sqrt{G_{t,i} + \epsilon}} g_t \end{aligned}}$$

$$\Delta w_{k,t} = -\alpha \frac{\partial E_t}{\partial w_{k,t}} = -\alpha \nabla_w E(w_t) = -\alpha g_{t,i}$$

$$g_{t,i} = \nabla_w E(w_{t,i})$$

$$G_{t+1,i} = \gamma G_{t,i} + (1-\gamma) g_{t,i} \odot g_{t,i}$$

$$\Longrightarrow$$

$$w_{t+1,i} = w_{t,i} - \frac{\alpha}{\sqrt{G_{t,i} + \epsilon}} g_{t,i}$$

Optimizers

ADAM: decaying average of the past squared gradients and momentum

RMSprop / Adadelta

$$g_{t,i} = \nabla_w E(w_{t,i})$$

$$G_{t+1,i} = \gamma G_{t,i} + (1-\gamma) g_{t,i} \odot g_{t,i}$$

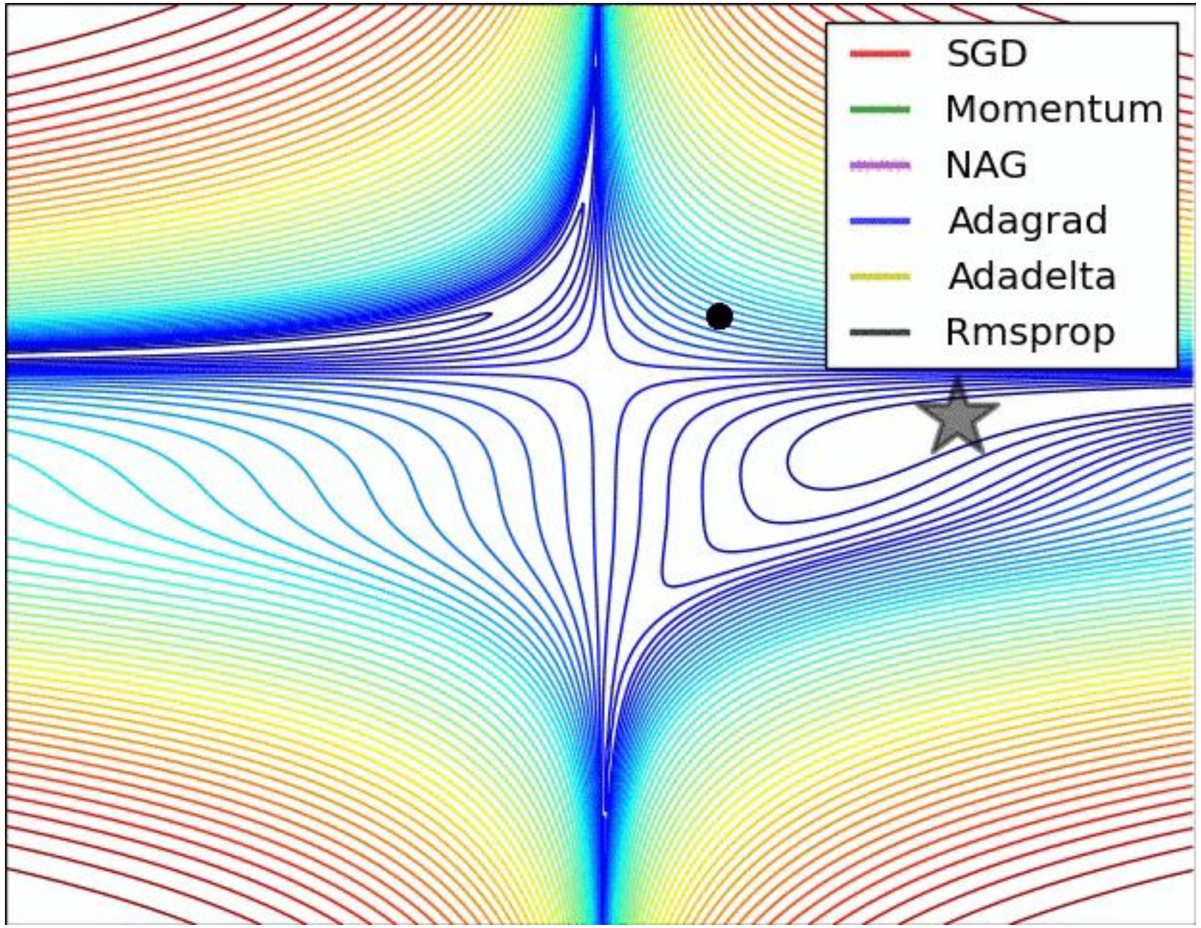
$$v_t = \beta_2 v_{t-1} + (1-\beta_2) g_t^2$$

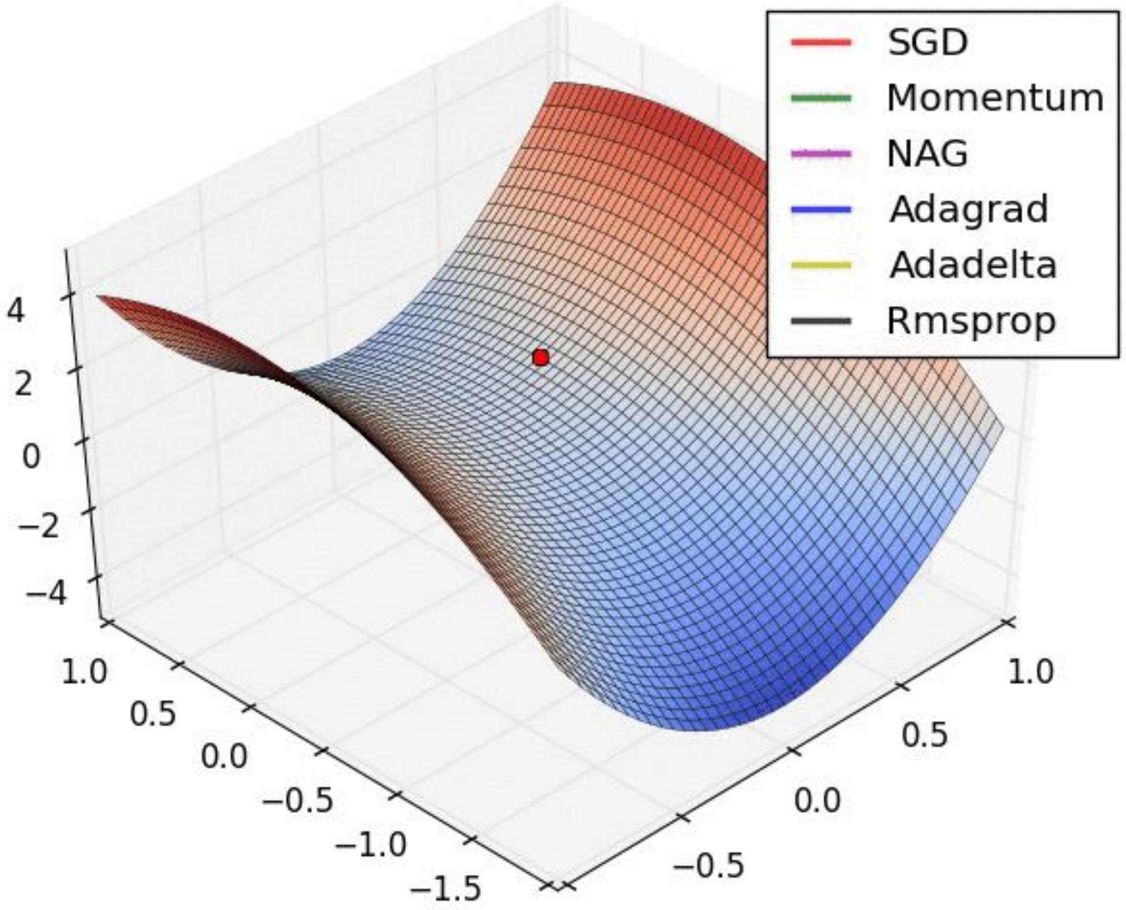
$$m_t = \beta_1 m_{t-1} + (1-\beta_1) g_t$$

$$\hat{v}_t = \frac{v_t}{1-\beta_2^t}$$

$$\hat{m}_t = \frac{m_t}{1-\beta_1^t}$$

$$w_{t+1,i} = w_{t,i} - \frac{\alpha}{\sqrt{\hat{v}_t} + \epsilon} \hat{m}_t$$



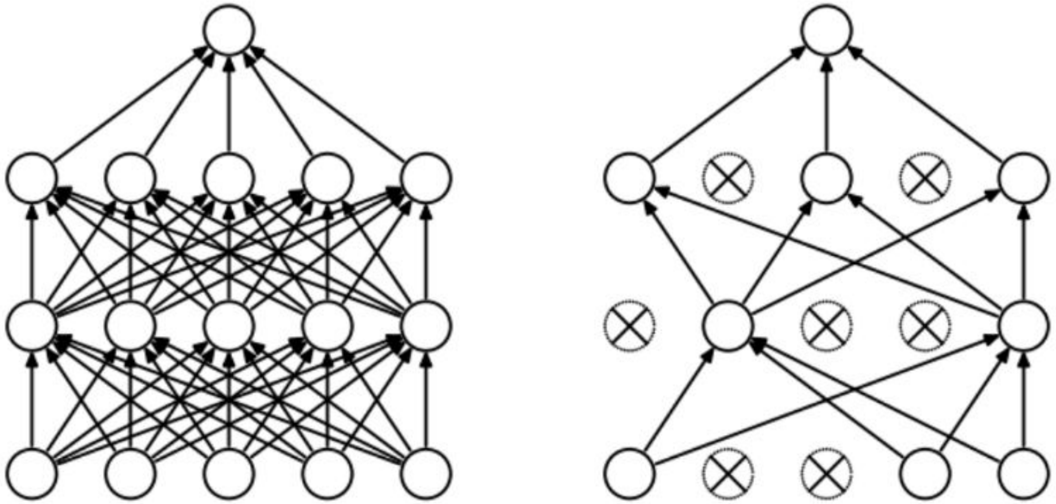


Which optimizer is the best?

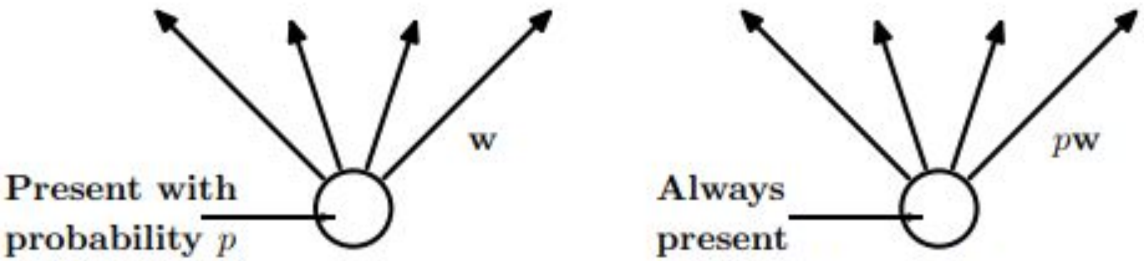
Extra Regularization for Neural Nets

Dropout: accuracy in the absence of certain information

- Prevent dependence on any one (or any small combination) of neurons



(a) Standard Neural Net (b) After applying dropout.



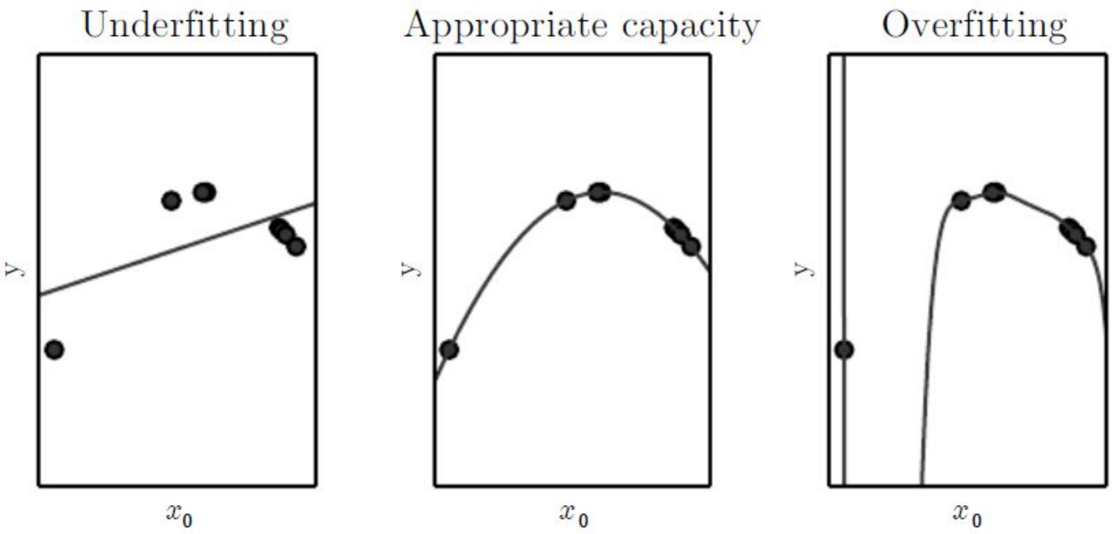
(a) At training time (b) At test time

Capacity, Overfitting and Underfitting

1) Make training error small
2) Make the gap between training and test error small

Capacity (fitting ability)

Low capacity ← $\hat{y} = \sum_{i}^{1} w_i x^i$ $\quad \hat{y} = \sum_{i}^{2} w_i x^i$ $\quad \hat{y} = \sum_{i}^{9} w_i x^i$ → High capacity



Back to the code

Open: - FeedForward_Networks_Class3.ipynb

When people want to use Machine Learning without math

How training works

1. In each ***epoch***, randomly shuffle the training data
2. Partition the shuffled training data into ***mini-batches***
3. For each mini-batch, apply a single step of **gradient descent**
   - **Gradients** are calculated via ***backpropagation*** (the next topic)
4. Train for multiple epochs

Debugging a neural network

- What can we do?
  - Should we change the learning rate?
  - Should we initialize differently?
  - Do we need more training data?
  - Should we change the architecture?
  - Should we run for more epochs?
  - Are the features relevant for the problem?
- Debugging is an art
  - We’ll develop good heuristics for choosing good architectures and hyper parameters

Extra readings

Deep Learning book:

- Chapter 5.9: Intro to Stochastic Gradient Descent (SGD)
- Chapter 6: Multilayer perceptrons
- Chapter 6.2.2: Output Units (Activation functions)
- Chapter 6.5: Back-Propagation
- Chapter 8.3: Basic Algorithms (Optimizers)