1 Match Constraint Language

$\xi ::= \top \mid \bot \mid ? \mid \underline{n} \mid \cancel{\underline{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\frac{}{\top : \tau}\ \text{CTTruth} \tag{1a}$$

$$\frac{}{\bot : \tau}\ \text{CTFalsity} \tag{1b}$$

$$\frac{}{? : \tau}\ \text{CTUnknown} \tag{1c}$$

$$\frac{}{\underline{n} : \texttt{num}}\ \text{CTNum} \tag{1d}$$

$$\frac{}{\cancel{\underline{n}} : \texttt{num}}\ \text{CTNotNum} \tag{1e}$$

$$\frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau}\ \text{CTAnd} \tag{1f}$$

$$\frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau}\ \text{CTOr} \tag{1g}$$

$$\frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)}\ \text{CTInl} \tag{1h}$$

$$\frac{\xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)}\ \text{CTInr} \tag{1i}$$

$$\frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)}\ \text{CTPair} \tag{1j}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\overline{\top} = \bot \tag{2a}$$

$$\overline{\bot} = \top \tag{2b}$$

$$\overline{?} = ? \tag{2c}$$

$$\overline{\underline{n}} = \cancel{\underline{n}} \tag{2d}$$

$$\overline{\cancel{\underline{n}}} = \underline{n} \tag{2e}$$

$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \tag{2f}$$

$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \tag{2g}$$

$$\overline{\texttt{inl}(\xi_1)} = \texttt{inl}(\overline{\xi_1}) \vee \texttt{inr}(\top) \tag{2h}$$

$$\overline{\texttt{inr}(\xi_2)} = \texttt{inr}(\overline{\xi_2}) \vee \texttt{inl}(\top) \tag{2i}$$

$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \tag{2j}$$

$\boxed{\xi\ \texttt{refutable}}$ ξ is refutable

$$\text{RXNum}\quad \frac{}{\underline{n}\ \texttt{refutable}} \tag{3a}$$

$$\text{RXUnknown}\quad \frac{}{?\ \texttt{refutable}} \tag{3b}$$

$$\text{RXInl}\quad \frac{}{\texttt{inl}(\xi)\ \texttt{refutable}} \tag{3c}$$

$$\text{RXInr}\quad \frac{}{\texttt{inr}(\xi)\ \texttt{refutable}} \tag{3d}$$

$$\text{RXPairL}\quad \frac{\xi_1\ \texttt{refutable}}{(\xi_1, \xi_2)\ \texttt{refutable}} \tag{3e}$$

$$\text{RXPairR}\quad \frac{\xi_2\ \texttt{refutable}}{(\xi_1, \xi_2)\ \texttt{refutable}} \tag{3f}$$

$$\text{RXOr}\quad \frac{\xi_1\ \texttt{refutable} \qquad \xi_2\ \texttt{refutable}}{\xi_1 \vee \xi_2\ \texttt{refutable}} \tag{3g}$$

$\boxed{\mathit{refutable}(\xi)}$

$$refutable(\underline{n}) = \text{true} \tag{4a}$$
$$refutable(\cancel{\underline{n}}) = \text{true} \tag{4b}$$
$$refutable(?) = \text{true} \tag{4c}$$
$$refutable(\texttt{inl}(\xi)) = refutable(\xi) \tag{4d}$$
$$refutable(\texttt{inr}(\xi)) = refutable(\xi) \tag{4e}$$
$$refutable((\xi_1, \xi_2)) = refutable(\xi_1) \text{ or } refutable(\xi_2) \tag{4f}$$
$$refutable(\xi_1 \vee \xi_2) = refutable(\xi_1) \text{ and } refutable(\xi_2) \tag{4g}$$
$$\text{Otherwise} \quad refutable(\xi) = \text{false} \tag{4h}$$

$\boxed{\dot{\top}(\xi_1) = \xi_2}$

$$\dot{\top}(\top) = \top \tag{5a}$$
$$\dot{\top}(\bot) = \bot \tag{5b}$$
$$\dot{\top}(?) = \top \tag{5c}$$
$$\dot{\top}(\underline{n}) = \underline{n} \tag{5d}$$
$$\dot{\top}(\cancel{\underline{n}}) = \cancel{\underline{n}} \tag{5e}$$
$$\dot{\top}(\xi_1 \wedge \xi_2) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \tag{5f}$$
$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{5g}$$
$$\dot{\top}(\texttt{inl}(\xi)) = \texttt{inl}(\dot{\top}(\xi)) \tag{5h}$$
$$\dot{\top}(\texttt{inr}(\xi)) = \texttt{inr}(\dot{\top}(\xi)) \tag{5i}$$
$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{5j}$$

$\boxed{\dot{\bot}(\xi_1) = \xi_2}$

$$\dot{\bot}(\top) = \top \tag{6a}$$
$$\dot{\bot}(\bot) = \bot \tag{6b}$$
$$\dot{\bot}(?) = \bot \tag{6c}$$
$$\dot{\bot}(\underline{n}) = \underline{n} \tag{6d}$$
$$\dot{\bot}(\cancel{\underline{n}}) = \cancel{\underline{n}} \tag{6e}$$
$$\dot{\bot}(\xi_1 \wedge \xi_2) = \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2) \tag{6f}$$
$$\dot{\bot}(\xi_1 \vee \xi_2) = \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2) \tag{6g}$$
$$\dot{\bot}(\texttt{inl}(\xi)) = \texttt{inl}(\dot{\bot}(\xi)) \tag{6h}$$
$$\dot{\bot}(\texttt{inr}(\xi)) = \texttt{inr}(\dot{\bot}(\xi)) \tag{6i}$$
$$\dot{\bot}((\xi_1, \xi_2)) = (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \tag{6j}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\frac{}{e \models \top}\ \text{CSTruth} \tag{7a}$$

$$\frac{}{\underline{n} \models \underline{n}}\ \text{CSNum} \tag{7b}$$

$$\frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}}\ \text{CSNotNum} \tag{7c}$$

$$\frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2}\ \text{CSAnd} \tag{7d}$$

$$\frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2}\ \text{CSOrL} \tag{7e}$$

$$\frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2}\ \text{CSOrR} \tag{7f}$$

$$\frac{e_1 \models \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)}\ \text{CSInl} \tag{7g}$$

$$\frac{e_2 \models \xi_2}{\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)}\ \text{CSInr} \tag{7h}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)}\ \text{CSPair} \tag{7i}$$

$$\frac{e\ \mathtt{notintro} \qquad \mathtt{prl}(e) \models \xi_1 \qquad \mathtt{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)}\ \text{CSNotIntroPair} \tag{7j}$$

$\boxed{\mathit{satisfy}(e, \xi)}$

$$satisfy(e, \top) = \text{true} \tag{8a}$$

$$satisfy(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{8b}$$

$$satisfy(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) \tag{8c}$$

$$satisfy(e, \xi_1 \wedge \xi_2) = satisfy(e, \xi_1) \text{ and } satisfy(e, \xi_2) \tag{8d}$$

$$satisfy(e, \xi_1 \vee \xi_2) = satisfy(e, \xi_1) \text{ or } satisfy(e, \xi_2) \tag{8e}$$

$$satisfy(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\xi_1)) = satisfy(e_1, \xi_1) \tag{8f}$$

$$satisfy(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\xi_2)) = satisfy(e_2, \xi_2) \tag{8g}$$

$$satisfy((e_1, e_2), (\xi_1, \xi_2)) = satisfy(e_1, \xi_1) \text{ and } satisfy(e_2, \xi_2) \tag{8h}$$

$$satisfy(⦇⦈^u, (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(⦇⦈^u), \xi_1) \text{ and } satisfy(\mathtt{prr}(⦇⦈^u), \xi_2) \tag{8i}$$

$$satisfy(⦇e⦈^u, (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(⦇e⦈^u), \xi_1) \text{ and } satisfy(\mathtt{prr}(⦇e⦈^u), \xi_2) \tag{8j}$$

$$\begin{aligned} satisfy(e_1(e_2), (\xi_1, \xi_2)) &= satisfy(\mathtt{prl}(e_1(e_2)), \xi_1) \\ &\quad \text{and } satisfy(\mathtt{prr}(e_1(e_2)), \xi_2) \end{aligned} \tag{8k}$$

$$\begin{aligned} satisfy(\mathtt{match}(e)\{\hat{rs}\}, (\xi_1, \xi_2)) &= satisfy(\mathtt{prl}(\mathtt{match}(e)\{\hat{rs}\}), \xi_1) \\ &\quad \text{and } satisfy(\mathtt{prr}(\mathtt{match}(e)\{\hat{rs}\}), \xi_2) \end{aligned} \tag{8l}$$

$$\begin{aligned} satisfy(\mathtt{prl}(e), (\xi_1, \xi_2)) &= satisfy(\mathtt{prl}(\mathtt{prl}(e)), \xi_1) \\ &\quad \text{and } satisfy(\mathtt{prr}(\mathtt{prl}(e)), \xi_2) \end{aligned} \tag{8m}$$

$$\begin{aligned} satisfy(\mathtt{prr}(e), (\xi_1, \xi_2)) &= satisfy(\mathtt{prl}(\mathtt{prr}(e)), \xi_1) \\ &\quad \text{and } satisfy(\mathtt{prr}(\mathtt{prr}(e)), \xi_2) \end{aligned} \tag{8n}$$

$$\text{Otherwise} \quad satisfy(e, \xi) = \text{false} \tag{8o}$$

$\boxed{e \models_? \xi}$ e may satisfy ξ

$$\text{CMSUnknown} \quad \frac{}{e \models_? ?} \tag{9a}$$

$$\text{CMSNotIntro} \quad \frac{e \ \mathtt{notintro} \qquad \xi \ \mathtt{refutable}}{e \models_? \xi} \tag{9b}$$

$$\text{CMSOrL} \quad \frac{e \models_? \xi_1 \qquad e \not\models \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{9c}$$

$$\text{CMSOrR} \quad \frac{e \not\models \xi_1 \qquad e \models_? \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{9d}$$

$$\text{CMSInl} \quad \frac{e_1 \models_? \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)} \tag{9e}$$

$$\frac{e_2 \models_? \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)}\ \text{CMSInr} \tag{9f}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)}\ \text{CMSPairL} \tag{9g}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)}\ \text{CMSPairR} \tag{9h}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)}\ \text{CMSPair} \tag{9i}$$

$\boxed{e \models_?^\dagger \xi}$ e satisfies or may satisfy ξ

$$\frac{e \models_? \xi}{e \models_?^\dagger \xi}\ \text{CSMSMay} \tag{10a}$$

$$\frac{e \models \xi}{e \models_?^\dagger \xi}\ \text{CSMSSat} \tag{10b}$$

Lemma 1.0.1. $e \not\models \bot$

Proof. By rule induction over Rules (7), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. □

Lemma 1.0.2. $e \not\models_? \bot$

Proof. Assume $e \models_? \bot$. By rule induction over Rules (9) on $e \models_? \bot$, only one case applies.

Case (9b).

(1) $\bot$ `refutable` by assumption

By rule induction over Rules (3) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. □

Lemma 1.0.3. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (9) on $e \models_? \top$, only one case applies.

Case (9b).

(1) $\top$ `refutable` by assumption

By rule induction over Rules (3) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. □

Lemma 1.0.4. $e \not\models ?$

Proof. By rule induction over Rules (7), we notice that $e \models ?$ is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.5. $e \models_?^\dagger \xi$ *iff* $e \models_?^\dagger \xi \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

   (1) $e \models_?^\dagger \xi$ by assumption

   By rule induction over Rules (10) on (1).

   Case (10a)**.**

   (2) $e \models_? \xi$ by assumption
   (3) $e \models_? \xi \vee \bot$ by Rule (9c) on (2) and Lemma 1.0.1
   (4) $e \models_?^\dagger \xi \vee \bot$ by Rule (10a) on (3)

   Case (10b)**.**

   (2) $e \models \xi$ by assumption
   (3) $e \models \xi \vee \bot$ by Rule (7e) on (2)
   (4) $e \models_?^\dagger \xi \vee \bot$ by Rule (10b) on (3)

2. Necessity:

   (1) $e \models_?^\dagger \xi \vee \bot$ by assumption

   By rule induction over Rules (10) on (1).

   Case (10a)**.**

   (2) $e \models_? \xi \vee \bot$ by assumption

   By rule induction over Rules (9) on (2), only two of them apply.

   Case (9c)**.**

   (3) $e \models_? \xi$ by assumption
   (4) $e \models_?^\dagger \xi$ by Rule (10a) on (3)

Case (9d).

(3) $e \models_? \perp$ by assumption

(4) $e \not\models_? \perp$ by Lemma 1.0.2

(3) contradicts (4).

Case (10b).

(2) $e \models \xi \vee \perp$ by assumption

By rule induction over Rules (7) on (2), only two of them apply.

Case (7e).

(3) $e \models \xi$ by assumption

(4) $e \models_?^\dagger \xi$ by Rule (10b) on (3)

Case (7f).

(3) $e \models \perp$ by assumption

(4) $e \not\models \perp$ by Lemma 1.0.1

(3) contradicts (4).

$\square$

Corollary 1.0.1. $\top \models_?^\dagger \xi$ *iff* $\top \models_?^\dagger \xi \vee \perp$

Proof. By Definition 1.1.2 and Lemma 1.0.5. $\square$

Lemma 1.0.6. *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$. *Then* $\xi_1 \not\models \xi_2$ *iff* $\xi_1 \not\models \xi_2 \vee \perp$

Proof.

(1) $\xi_1 : \tau$ by assumption

(2) $\xi_2 : \tau$ by assumption

(3) $\perp : \tau$ by Rule (1b)

(4) $\xi_2 \vee \perp : \tau$ by Rule (1g) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5) $\xi_1 \not\models \xi_2$ by assumption

To prove $\xi_1 \not\models \xi_2 \vee \perp$, assume $\xi_1 \models \xi_2 \vee \perp$.

(6) $\xi_1 \models \xi_2 \vee \perp$ by assumption

For all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies

(7) $e \models \xi_2 \vee \perp$ by Definition 1.1.1 on (1) and (4) and (6)

By rule induction over Rules (7) on (7).

Case (7e)**.**

(8) $e \models \xi_2$ by assumption

(9) $\xi_1 \models \xi_2$ by Definition 1.1.1 on (8)

(5) contradicts (9).

Case (7f)**.**

(8) $e \models \bot$ by assumption

(9) $e \not\models \bot$ by Lemma 1.0.1

(8) contradicts (9).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2 \vee \bot$

2. Necessity:

(5) $\xi_1 \not\models \xi_2 \vee \bot$ by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

(6) $\xi_1 \models \xi_2$ by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \xi_1$ implies

(7) $e \models \xi_2$ by Definition 1.1.1 on (1) and (2) and (6)

(8) $e \models \xi_2 \vee \bot$ by Rule (7e) on (7)

(9) $\xi_1 \models \xi_2 \vee \bot$ by Definition 1.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2$

$\square$

Lemma 1.0.7. *If* $e \not\models^{\dagger}_{?} \xi_1$ *and* $e \not\models^{\dagger}_{?} \xi_2$ *then* $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

Proof. Assume, for the sake of contradiction, that $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$.

(1) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \xi_1$ by assumption

(3) $e \not\models^{\dagger}_{?} \xi_2$ by assumption

By rule induction over Rules (10) on (1).

Case (10b)**.**

(4) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (7) on (4) and only two of them apply.

Case (7e)**.**

(5) $e \models \xi_1$ by assumption

(6) $e \models^{\dagger}_{?} \xi_1$ by Rule (10b) on (5)

(6) contradicts (2).

Case (7f)**.**

(5) $e \models \xi_2$ by assumption

(6) $e \models^{\dagger}_{?} \xi_2$ by Rule (10b) on (5)

(6) contradicts (3).

Case (10a)**.**

(4) $e \models_{?} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (9) on (4) and only two of them apply.

Case (9c)**.**

(5) $e \models_{?} \xi_1$ by assumption

(6) $e \models^{\dagger}_{?} \xi_1$ by Rule (10a) on (5)

(6) contradicts (2).

Case (9d)**.**

(5) $e \models_{?} \xi_2$ by assumption

(6) $e \models^{\dagger}_{?} \xi_2$ by Rule (10a) on (5)

(6) contradicts (3).

The conclusion holds as follows:

1. $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

□

Lemma 1.0.8. *If* $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ *and* $e \not\models^{\dagger}_{?} \xi_1$ *then* $e \models^{\dagger}_{?} \xi_2$

Proof.

(1) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \xi_1$ by assumption

By rule induction over Rules (10) on (1).

Case (10b)**.**

(3) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (7) on (3) and only two of them apply.

Case (7e)**.**

(4) $e \models \xi_1$ by assumption

(5) $e \models_?^\dagger \xi_1$ by Rule (10b) on (4)

(5) contradicts (2).

Case (7f)**.**

(4) $e \models \xi_2$ by assumption

(5) $e \models_?^\dagger \xi_2$ by Rule (10b) on (4)

Case (10a)**.**

(3) $e \models_? \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (9) on (3) and only two of them apply.

Case (9c)**.**

(4) $e \models_? \xi_1$ by assumption

(5) $e \models_?^\dagger \xi_1$ by Rule (10a) on (4)

(5) contradicts (2).

Case (9d)**.**

(4) $e \models_? \xi_2$ by assumption

(5) $e \models_?^\dagger \xi_2$ by Rule (10a) on (4)

□

Lemma 1.0.9. *If* $e \models_?^\dagger \xi_1$ *and* $e \models_?^\dagger \xi_2$ *then* $e \models_?^\dagger \xi_1 \wedge \xi_2$

Lemma 1.0.10. *If* $e \models_?^\dagger \xi_1$ *then* $e \models_?^\dagger \xi_1 \vee \xi_2$ *and* $e \models_?^\dagger \xi_2 \vee \xi_1$

Proof.

(1) $e \models_?^\dagger \xi_1$ by assumption ,

By rule induction over Rules (10) on (1),

Case (10b)**.**

(2) $e \models \xi_1$ by assumption

(3) $e \models \xi_1 \vee \xi_2$ by Rule (7e) on (2)

(4) $e \models \xi_2 \vee \xi_1$ by Rule (7f) on (2)

(5) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$	by Rule (10b) on (3)
(6) $e \models^{\dagger}_{?} \xi_2 \vee \xi_1$	by Rule (10b) on (4)

Case (10a).

(2) $e \models_{?} \xi_1$	by assumption

By case analysis on the result of $\mathit{satisfy}(e, \xi_2)$.

Case true.

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$	by assumption
(4) $e \models \xi_2$	by Lemma 1.0.19 on (3)
(5) $e \models \xi_1 \vee \xi_2$	by Rule (7f) on (4)
(6) $e \models \xi_2 \vee \xi_1$	by Rule (7e) on (4)
(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$	by Rule (10b) on (5)
(8) $e \models^{\dagger}_{?} \xi_2 \vee \xi_1$	by Rule (10b) on (6)

Case false.

(3) $\mathit{satisfy}(e, \xi_2) = \text{false}$	by assumption
(4) $e \not\models \xi_2$	by Lemma 1.0.19 on (3)
(5) $e \models_{?} \xi_1 \vee \xi_2$	by Rule (9c) on (2) and (4)
(6) $e \models^{\dagger}_{?} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (10a) on (5)

□

Lemma 1.0.11. *If* $e_1 \models^{\dagger}_{?} \xi_1$ *then* $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$

Proof.

(1) $e_1 \models^{\dagger}_{?} \xi_1$	by assumption

By rule induction over Rules (10) on (1).

Case (10b).

(2) $e_1 \models \xi_1$	by assumption
(3) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$	by Rule (7g) on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$	by Rule (10b) on (3)

Case (10a).

(2) $e_1 \models_? \xi_1$	by assumption
(3) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$	by Rule (9e) on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$	by Rule (10a) on (3)

□

Lemma 1.0.12. *If* $e_2 \models_?^\dagger \xi_2$ *then* $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$

Proof.

(1) $e_2 \models_?^\dagger \xi_2$	by assumption

By rule induction over Rules (10) on (1).

Case (10b).

(2) $e_2 \models \xi_2$	by assumption
(3) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$	by Rule (7h) on (2)
(4) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$	by Rule (10b) on (3)

Case (10a).

(2) $e_2 \models_? \xi_2$	by assumption
(3) $\texttt{inl}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$	by Rule (9f) on (2)
(4) $\texttt{inl}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$	by Rule (10a) on (3)

□

Lemma 1.0.13. *If* $e_1 \models_?^\dagger \xi_1$ *and* $e_2 \models_?^\dagger \xi_2$ *then* $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$

Lemma 1.0.14 (Soundness and Completeness of Refutable Constraints)**.** ξ `refutable` *iff* $\mathit{refutable}(\xi) = \mathit{true}$.

Lemma 1.0.15. *If* e `notintro` *and* ξ `refutable` *then either* $\dot{\top}(\xi)$ `refutable` *or* $e \models \dot{\top}(\xi)$.

Proof. By structural induction on ξ. □

Lemma 1.0.16. *There does not exist such a constraint* $\xi_1 \wedge \xi_2$ *such that* $\xi_1 \wedge \xi_2$ `refutable`*.*

Proof. By rule induction over Rules (3), we notice that $\xi_1 \wedge \xi_2$ `refutable` is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.17. *There does not exist such a constraint* $\xi_1 \vee \xi_2$ *such that* $\xi_1 \vee \xi_2$ `refutable`*.*

Proof. By rule induction over Rules (3), we notice that $\xi_1 \vee \xi_2$ `refutable` is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 1.0.18. *If* e `notintro` *and* $e \models \xi$ *then* $\cancel{\xi\ \texttt{refutable}}$.

Proof.

(1)	e `notintro`	by assumption
(2)	$e \models \xi$	by assumption

By rule induction over Rules (7) on (2).

Case (7a)**.**

(3)	$\xi = \top$	by assumption

Assume $\top$ `refutable`. By rule induction over Rules (3), no case applies due to syntactic contradiction.
Therefore, $\cancel{\top\ \texttt{refutable}}$.

Case (7e),(7f)**.**

(3)	$\xi = \xi_1 \vee \xi_2$	by assumption
(4)	$\cancel{\xi_1 \vee \xi_2\ \texttt{refutable}}$	by Lemma 1.0.17

Case (7d)**.**

(3)	$\xi = \xi_1 \wedge \xi_2$	by assumption
(4)	$\cancel{\xi_1 \wedge \xi_2\ \texttt{refutable}}$	by Lemma 1.0.16

Case (7j)**.**

(3)	$\xi = (\xi_1, \xi_2)$	by assumption
(4)	$\texttt{prl}(e) \models \xi_1$	by assumption
(5)	$\texttt{prr}(e) \models \xi_2$	by assumption
(6)	$\texttt{prl}(e)$ `notintro`	by Rule (19e)
(7)	$\texttt{prr}(e)$ `notintro`	by Rule (19f)
(8)	$\cancel{\xi_1\ \texttt{refutable}}$	by IH on (6) and (4)
(9)	$\cancel{\xi_2\ \texttt{refutable}}$	by IH on (7) and (5)

Assume (ξ_1, ξ_2) `refutable`. By rule induction over Rules (3) on it, only two cases apply.

Case (3e)**.**

(10)	ξ_1 `refutable`	by assumption

Contradicts (8).

Case (3f)**.**

(10) ξ_2 `refutable` by assumption

Contradicts (9).

Therefore, $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}}$.

Otherwise.

(3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (19) on (1), no case applies due to syntactic contradiction.

□

Lemma 1.0.19 (Soundness and Completeness of Satisfaction Judgment)**.** $e \models \xi$ *iff* $\mathit{satisfy}(e, \xi) = \mathit{true}$.

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models \xi$ by assumption

   By rule induction over Rules (7) on (1).

   Case (7a)**.**

   (2) $\xi = \top$ by assumption
   (3) $\mathit{satisfy}(e, \top) = \text{true}$ by Definition 8a

   Case (7b)**.**

   (2) $e = \underline{n}$ by assumption
   (3) $\xi = \underline{n}$ by assumption
   (4) $\mathit{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$ by Definition 8b

   Case (7c)**.**

   (2) $e = \underline{n_1}$ by assumption
   (3) $\xi = \cancel{\underline{n_2}}$ by assumption
   (4) $n_1 \neq n_2$ by assumption
   (5) $\mathit{satisfy}(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) = \text{true}$ by Definition 8c on (4)

   Case (7d)**.**

   (2) $\xi = \xi_1 \wedge \xi_2$ by assumption
   (3) $e \models \xi_1$ by assumption

(4) $e \models \xi_2$ by assumption

(5) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)

(6) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e, \xi_1 \wedge \xi_2) = \mathit{satisfy}(e, \xi_1)$ and $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 8d on (5) and (6)

Case (7e).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_1$ by assumption

(4) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)

(5) $\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 8e on (4)

Case (7f).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_2$ by assumption

(4) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by IH on (3)

(5) $\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 8e on (4)

Case (7g).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\xi = \texttt{inl}(\xi_1)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\xi_1)) = \mathit{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 8f on (5)

Case (7h).

(2) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\xi = \texttt{inl}(\xi_2)$ by assumption

(4) $e_2 \models \xi_2$ by assumption

(5) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\xi_2)) = \mathit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 8g on (5)

Case (7i).

(2) $e = (e_1, e_2)$ by assumption

(3) $\xi = (\xi_1, \xi_2)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $e_2 \models \xi_2$ by assumption

(6) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (5)

(8) $\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$
$\mathit{satisfy}(e_1, \xi_1)$ and $\mathit{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 8h on (6) and (7)

Case (7j)**.**

(2) $\xi = (\xi_1, \xi_2)$ by assumption

(3) e `notintro` by assumption

(4) $\texttt{prl}(e) \models \xi_1$ by assumption

(5) $\texttt{prr}(e) \models \xi_2$ by assumption

(6) $\mathit{satisfy}(\texttt{prl}(e), \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$ by IH on (5)

By rule induction over Rules (19) on (3).

Otherwise.

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(9) $\mathit{satisfy}(e, (\xi_1, \xi_2)) =$
$\mathit{satisfy}(\texttt{prl}(e), \xi_1)$ and $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$
by Definition 8 on (6) and (7)

2. Completeness:

(1) $\mathit{satisfy}(e, \xi) = \text{true}$ by assumption

By structural induction on ξ.

Case $\xi = \top$**.**

(2) $e \models \top$ by Rule (7a)

Case $\xi = \bot, ?$**.**

(2) $\mathit{satisfy}(e, \xi) = \text{false}$ by Definition 8o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$**.**

By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' = n$ — by Definition 8b on (1)

(3) $\underline{n'} \models \underline{n}$ — by Rule (7b) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \underline{n}) = \text{false}$ — by Definition 8o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \cancel{\underline{n}}$**.**

By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' \neq n$ — by Definition 8c on (1)

(3) $\underline{n'} \models \cancel{\underline{n}}$ — by Rule (7c) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \cancel{\underline{n}}) = \text{false}$ — by Definition 8o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $\mathit{satisfy}(e, \xi_1) = \text{true}$ — by Definition 8d on (1)

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ — by Definition 8d on (1)

(4) $e \models \xi_1$ — by IH on (2)

(5) $e \models \xi_2$ — by IH on (3)

(6) $e \models \xi_1 \wedge \xi_2$ — by Rule (7d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $\mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ — by Definition 8e on (1)

By case analysis on (2).

Case $\mathit{satisfy}(e, \xi_1) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_1) = \text{true}$ — by assumption

(4) $e \models \xi_1$ — by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ — by Rule (7e) on (4)

Case $\mathit{satisfy}(e, \xi_2) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ — by assumption

(4) $e \models \xi_2$ — by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ — by Rule (7f) on (4)

Case $\xi = \texttt{inl}(\xi_1)$**.**

By structural induction on e.

Case $e = \texttt{inl}_{\tau_2}(e_1)$.

(2) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ — by Definition 8f on (1)

(3) $e_1 \models \xi_1$ — by IH on (2)

(4) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ — by Rule (7g) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \texttt{inl}(\xi_1)) = \text{false}$ — by Definition 8o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \texttt{inr}(\xi_2)$.

By structural induction on e.

Case $e = \texttt{inr}_{\tau_1}(e_2)$.

(2) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ — by Definition 8g on (1)

(3) $e_2 \models \xi_2$ — by IH on (2)

(4) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ — by Rule (7h) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \texttt{inr}(\xi_2)) = \text{false}$ — by Definition 8o

(2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$.

By structural induction on e.

Case $e = (e_1, e_2)$.

(2) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ — by Definition 8h on (1)

(3) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ — by Definition 8h on (1)

(4) $e_1 \models \xi_1$ — by IH on (2)

(5) $e_2 \models \xi_2$ — by IH on (3)

(6) $(e_1, e_2) \models (\xi_1, \xi_2)$ — by Rule (7i) on (4) and (5)

Case $e = \llparenthesis\rrparenthesis^u, \llparenthesis e_0 \rrparenthesis^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$.

(2) $\mathit{satisfy}(\texttt{prl}(e), \xi_1) = \text{true}$ — by Definition 8h on (1)

(3) $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$ — by Definition 8h on (1)

(4) $\texttt{prl}(e) \models \xi_1$ — by IH on (2)

(5) $\texttt{prr}(e) \models \xi_2$ — by IH on (3)

(6) $e\ \texttt{notintro}$ — by each rule in Rules (19)

(7) $(e_1, e_2) \models (\xi_1, \xi_2)$ — by Rule (7j) on (6) and (4) and (5)

Otherwise.

(2) $\mathit{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$ — by Definition 8o

(2) contradicts (1) and thus vacuously true.

$\square$

Lemma 1.0.20. *$e \not\models \xi$ and $e \not\models_? \xi$ iff $e \not\models_?^\dagger \xi$.*

Proof. 1. Sufficiency:

(1) $e \not\models \xi$ by assumption

(2) $e \not\models_? \xi$ by assumption

Assume $e \models_?^\dagger \xi$. By rule induction over Rules (10) on it.

Case (10a)**.**

(3) $e \models \xi$ by assumption

Contradicts (1).

Case (10b)**.**

(3) $e \models_? \xi$ by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \xi$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \xi$ by assumption

Assume $e \models \xi$.

(2) $e \models_?^\dagger \xi$ by Rule (10b) on assumption

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_? \xi$.

(3) $e \models_?^\dagger \xi$ by Rule (10a) on assumption

Contradicts (1). Therefore, $e \not\models_? \xi$.

$\square$

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \models \xi$

2. $e \models_? \xi$

3. $e \not\models_?^\dagger \xi$

Proof.

(4) $\xi : \tau$ by assumption

(5) $\cdot\,;\Delta \vdash e : \tau$ by assumption

(6) e $\texttt{final}$ by assumption

By rule induction over Rules (1) on (4), we would show one conclusion is derivable while the other two are not.

Case (1a).

(7) $\xi = \top$ by assumption

(8) $e \models \top$ by Rule (7a)

(9) $e \not\models_? \top$ by Lemma 1.0.3

(10) $e \models_?^\dagger \top$ by Rule (10b) on (8)

Case (1b).

(7) $\xi = \bot$ by assumption

(8) $e \not\models \bot$ by Lemma 1.0.1

(9) $e \not\models_? \bot$ by Lemma 1.0.2

(10) $e \not\models_?^\dagger \bot$ by Lemma 1.0.20 on (8) and (9)

Case (1c).

(7) $\xi = ?$ by assumption

(8) $e \not\models ?$ by Lemma 1.0.4

(9) $e \models_? ?$ by Rule (9a)

(10) $e \models_?^\dagger ?$ by Rule (10a) on (9)

Case (1d).

(7) $\xi = \underline{n_2}$ by assumption

(8) $\tau = \texttt{num}$ by assumption

By rule induction over Rules (12) on (5), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (7) on it, no case applies due to syntactic contradiction on ξ.

(11) $e \not\models \underline{n_2}$	by contradiction
(12) $\underline{n_2}$ `refutable`	by Rule (3a)
(13) $e \models_? \underline{n_2}$	by Rule (9b) on (10) and (12)
(14) $e \models_?^\dagger \underline{n_2}$	by Rule (10a) on (13)

Case (12d)**.**

(9) $e = \underline{n_1}$	by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (9), only one case applies.

Case (9b)**.**

(10) $\underline{n_1}$ `notintro`	by assumption

Contradicts Lemma 3.0.4.

(11) $\underline{n_1} \not\models_? \underline{n_2}$	by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$**.**

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$	by Definition 8
(13) $\underline{n_1} \models \underline{n_2}$	by Lemma 1.0.19 on (12)
(14) $e \models_?^\dagger \underline{n_2}$	by Rule (10b) on (13)

Case $n_1 \neq n_2$**.**

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$	by Definition 8
(13) $\underline{n_1} \not\models \underline{n_2}$	by Lemma 1.0.19 on (12)
(14) $e \not\models_?^\dagger \underline{n_2}$	by Lemma 1.0.20 on (11) and (13)

Case (1g)**.**

(7) $\xi = \xi_1 \vee \xi_2$	by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models_? \xi_1$, and $e \not\models_?^\dagger \xi_1$ holds. The same goes for ξ_2. By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e \models \xi_1, e \models \xi_2$**.**

(8) $e \models \xi_1$	by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (7e) on (8)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (10b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 1.0.17.

Case (9c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (9d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \models_? \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (7e) on (8)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (10b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 1.0.17.

Case (9c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (9d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (7e) on (8)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (10b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 1.0.17.

Case (9c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (9d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (7f) on (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (10b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 1.0.17.

Case (9c).

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (9d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models_? \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (9c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (10a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (7), only two cases apply.

Case (7e)**.**

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (7f)**.**

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \not\models_?^\dagger \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (9c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (10a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (7), only two cases apply.

Case (7e)**.**

(14) $e \models \xi_1$ by assumption

Contradicts (8).

Case (7f)**.**

(14) $e \models \xi_2$ by assumption

Contradicts (10).

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \xi_1, e \models \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{?} \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_{?} \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (7f) on (10)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (10b) on (12)

Assume $e \models_{?} \xi_1 \vee \xi_2$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b)**.**

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 1.0.17.

Case (9c)**.**

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (9d)**.**

(14) $e \models_{?} \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_{?} \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \xi_1, e \models_{?} \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{?} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_{?} \xi_2$ by assumption

(12) $e \models_{?} \xi_1 \vee \xi_2$ by Rule (9d) on (11) and (8)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (10a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (7), only two cases apply.

Case (7e)**.**

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (7f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \xi_1, e \not\models^{\dagger}_{?} \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{?} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_{?} \xi_2$ by assumption

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (7) on it, only two cases apply.

Case (7e).

(12) $e \models \xi_1$ by assumption

Contradicts (8).

Case (7f).

(12) $e \models \xi_2$ by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Assume $e \models_{?} \xi_1 \vee \xi_2$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 1.0.17.

Case (9c).

(14) $e \models_{?} \xi_1$ by assumption

Contradicts (9).

Case (9d).

(14) $e \models_{?} \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_{?} \xi_1 \vee \xi_2$ by contradiction

(16) $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 1.0.20 on (13) and (15)

Case (1h).

(7) $\xi = \texttt{inl}(\xi_1)$ — by assumption

(8) $\tau = (\tau_1 + \tau_2)$ — by assumption

(9) $\xi_1 : \tau_1$ — by assumption

By rule induction over Rules (12) on (5), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ — by assumption

(11) $e\ \texttt{notintro}$ — by Rule (19a),(19b),(19c),(19d),(19e),(19f)

Assume $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (7) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inl}(\xi_1)$ — by contradiction

By case analysis on the value of $\mathit{refutable}(\texttt{inl}(\xi_1))$.

Case $\mathit{refutable}(\texttt{inl}(\xi_1)) = \text{true}$.

(13) $\mathit{refutable}(\texttt{inl}(\xi_1)) = \text{true}$ — by assumption

(14) $\texttt{inl}(\xi_1)\ \texttt{refutable}$ — by Lemma 1.0.14 on (13)

(15) $e \models_? \texttt{inl}(\xi_1)$ — by Rule (9b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inl}(\xi_1)$ — by Rule (10a) on (15)

Case $\mathit{refutable}(\texttt{inl}(\xi_1)) = \text{false}$.

(13) $\mathit{refutable}(\texttt{inl}(\xi_1)) = \text{false}$ — by assumption

(14) $\cancel{\texttt{inl}(\xi_1)\ \texttt{refutable}}$ — by Lemma 1.0.14 on (13)

Assume $e \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b).

(15) $\texttt{inl}(\xi_1)\ \texttt{refutable}$ — by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inl}(\xi_1)$ — by contradiction

(17) $e \not\models_?^\dagger \texttt{inl}(\xi_1)$ — by Lemma 1.0.20 on (12) and (16)

Case (12j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ — by assumption

(11) $\cdot\,; \Delta \vdash e_1 : \tau_1$ — by assumption

(12) $e_1\ \texttt{final}$ — by Lemma 3.0.1 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \not\models_?^\dagger \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$**.**

(13) $e_1 \models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ by Rule (7g) on (13)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (10b) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (9) on it, only two cases apply.

Case (9b)**.**

(17) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (19) on (17), no case applies due to syntactic contradiction.

Case (9e)**.**

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \models_? \xi_1$**.**

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ by Rule (9e) on (14)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (10a) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (7) on it, only one case applies.

Case (7g)**.**

(17) $e_1 \models \xi_1$

Contradicts (13).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \not\models_?^\dagger \xi_1$**.**

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (7) on it, only one case applies.

Case (7g)**.**

(15) $e_1 \models \xi_1$

Contradicts (13).

(16) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (19) on (17), no case applies due to syntactic contradiction.

Case (9e).

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 1.0.20 on (16) and (18)

Case (12k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (7) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inl}(\xi_1)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b).

(12) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (19) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(14) $\texttt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 1.0.20 on (11) and (13)

Case (1i).

(7) $\xi = \texttt{inr}(\xi_2)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (12) on (5), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) $e\ \texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

Assume $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (7) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inr}(\xi_2)$ by contradiction

By case analysis on the value of $\mathit{refutable}(\texttt{inr}(\xi_2))$. inr is refutable

Case $\mathit{refutable}(\texttt{inr}(\xi_2)) = \text{true}$**.**

(13) $\mathit{refutable}(\texttt{inr}(\xi_2)) = \text{true}$ by assumption

(14) $\texttt{inr}(\xi_2)\ \texttt{refutable}$ by Lemma 1.0.14 on (13)

(15) $e \models_? \texttt{inr}(\xi_2)$ by Rule (9b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inr}(\xi_2)$ by Rule (10a) on (15)

Case $\mathit{refutable}(\texttt{inr}(\xi_2)) = \text{false}$**.**

(13) $\mathit{refutable}(\texttt{inr}(\xi_2)) = \text{false}$ by assumption

(14) $\cancel{\texttt{inr}(\xi_2)\ \texttt{refutable}}$ by Lemma 1.0.14 on (13)

Assume $e \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b)**.**

(15) $\texttt{inr}(\xi_2)\ \texttt{refutable}$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inr}(\xi_2)$ by Lemma 1.0.20 on (12) and (16)

Case (12j)**.**

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (7) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b)**.**

(12) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (19) on (12), no case applies due to syntactic contradiction.

(13) $\mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inr}(\xi_2)$ by contradiction

(14) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inr}(\xi_2)$ by Lemma 1.0.20 on (11) and (13)

Case (12k).

(10) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot\,;\Delta \vdash e_2 : \tau_2$ by assumption

(12) e_2 $\mathtt{final}$ by Lemma 3.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \not\models_?^\dagger \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

(13) $e_2 \models \xi_2$ by assumption

(14) $e_2 \not\models_? \xi_2$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (7g) on (13)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (10b) on (15)

Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$. By rule induction over Rules (9) on it, only two cases apply.

Case (9b).

(17) $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{notintro}$ by assumption

By rule induction over Rules (19) on (17), no case applies due to syntactic contradiction.

Case (9f).

(17) $e_2 \models_? \xi_2$

Contradicts (14).

(18) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inr}(\xi_2)$ by contradiction

Case $e_2 \models_? \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \models_? \xi_2$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (9f) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (10a) on (15)

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (7) on it, only one case applies.

Case (7h).

(17) $e_2 \models \xi_2$

Contradicts (13).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Case $e_2 \not\models^{\dagger}_{?} \xi_2$**.**

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \not\models_? \xi_2$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (7) on it, only one case applies.

Case (7h)**.**

(15) $e_2 \models \xi_2$

Contradicts (13).

(16) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b)**.**

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (19) on (17), no case applies due to syntactic contradiction.

Case (9f)**.**

(17) $e_2 \models_? \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models^{\dagger}_{?} \texttt{inl}(\xi_1)$ by Lemma 1.0.20 on (16) and (18)

Case (7i)**.**

(7) $\xi = (\xi_1, \xi_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

(10) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (12) on (5), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m)**.**

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) $e\ \texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

(13) $e\ \texttt{indet}$ by Lemma 3.0.8 on (6) and (12)

(14) $\texttt{prl}(e)\ \texttt{indet}$ by Rule (17g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (18b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (17h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (18b) on (16)

(18) $\cdot\,;\Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (12h) on (5)

(19) $\cdot\,;\Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (12i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\texttt{prl}(e) \models \xi_1$, $\texttt{prl}(e) \models_? \xi_1$, and $\texttt{prl}(e) \not\models_?^\dagger \xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\texttt{prr}(e) \models \xi_2$, $\texttt{prr}(e) \models_? \xi_2$, and $\texttt{prr}(e) \not\models_?^\dagger \xi_2$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models \xi_2$**.**

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

(24) $e \models (\xi_1, \xi_2)$ by Rule (7j) on (12) and (20) and (22)

(25) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10b) on (24)

(26) $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}}$ by Lemma 1.0.18 on (12) and (24)

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b)**.**

(27) (ξ_1, ξ_2) $\texttt{refutable}$ by assumption

Contradicts (26).

(28) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models_? \xi_2$**.**

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7), only one case applies.

Case (7j)**.**

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (9) on (23), only one case applies.

assume no "or" and "and" in pair

Case (9b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (3f) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10a) on (28)

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only one case applies.

Case (7j).

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b).

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (3) on (26), only two cases apply.

Case (3e).

(27) ξ_1 `refutable` by assumption

(28) $\texttt{prl}(e)$ `notintro` by Rule (19e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (9b) on (28) and (27)

Contradicts (21).

Case (3f).

(27) ξ_2 `refutable` by assumption

(28) $\texttt{prr}(e)$ `notintro` by Rule (19f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (9b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.20 on (25) and (30)

Case $\texttt{prl}(e) \models_? \xi_1, \texttt{prr}(e) \models \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7), only one case applies.

Case (7j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (9) on (21), only one case applies.

assume no "or" and "and" in pair

Case (9b).

(26) ξ_1 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (3f) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (27)

(29) $e \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Rule (10a) on (28)

Case $\texttt{prl}(e) \models_? \xi_1, \texttt{prr}(e) \models_? \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7), only one case applies.

Case (7j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (9) on (23), only one case applies.

assume no "or" and "and" in pair

Case (9b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (3f) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (27)

(29) $e \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Rule (10a) on (28)

Case $\texttt{prl}(e) \models_{?} \xi_1, \texttt{prr}(e) \not\models^{\dagger}_{?} \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_{?} \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_{?} \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7), only one case applies.

Case (7j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (9) on (21), only one case applies.

assume no "or" and "and" in pair

Case (9b)**.**

(26) ξ_1 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (3f) on (26)

(28) $e \models_{?} (\xi_1, \xi_2)$ by Rule (9b) on (12) and (27)

(29) $e \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Rule (10a) on (28)

Case $\texttt{prl}(e) \not\models^{\dagger}_{?} \xi_1, \texttt{prr}(e) \models \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_{?} \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_{?} \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only one case applies.

Case (7j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_{?} (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b)**.**

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (3) on (26), only two cases apply.

Case (3e)**.**

(27) ξ_1 `refutable` by assumption

(28) `prl`(e) `notintro` by Rule (19e)

(29) `prl`$(e) \models_? \xi_1$ by Rule (9b) on (28) and (27)

Contradicts (21).

Case (3f).

(27) ξ_2 `refutable` by assumption

(28) `prr`(e) `notintro` by Rule (19f)

(29) `prr`$(e) \models_? \xi_2$ by Rule (9b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 1.0.20 on (25) and (30)

Case `prl`$(e) \not\models_?^\dagger \xi_1,$ `prr`$(e) \models_? \xi_2$**.**

(20) `prl`$(e) \not\models \xi_1$ by assumption

(21) `prl`$(e) \not\models_? \xi_1$ by assumption

(22) `prr`$(e) \not\models \xi_2$ by assumption

(23) `prr`$(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7), only one case applies.

Case (7j).

(24) `prl`$(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (9) on (23), only one case applies.

assume no "or" and "and" in pair

Case (9b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (3f) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10a) on (28)

Case `prl`$(e) \not\models_?^\dagger \xi_1,$ `prr`$(e) \not\models_?^\dagger \xi_2$**.**

(20) `prl`$(e) \not\models \xi_1$ by assumption

(21) `prl`$(e) \not\models_? \xi_1$ by assumption

(22) `prr`$(e) \not\models \xi_2$ by assumption

(23) `prr`$(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only one case applies.

Case (7j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, only one case applies.

Case (9b).

(26) (ξ_1, ξ_2) $\texttt{refutable}$ by assumption

By rule induction over Rules (3) on (26), only two cases apply.

Case (3e).

(27) ξ_1 $\texttt{refutable}$ by assumption

(28) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (19e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (9b) on (28) and (27)

Contradicts (21).

Case (3f).

(27) ξ_2 $\texttt{refutable}$ by assumption

(28) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (19f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (9b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^{\dagger} (\xi_1, \xi_2)$ by Lemma 1.0.20 on (25) and (30)

Case (12g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(14) e_1 $\texttt{final}$ by Lemma 3.0.3 on (6)

(15) e_2 $\texttt{final}$ by Lemma 3.0.3 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \models \overline{\xi_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \models \overline{\xi_2}$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e_1 \models \xi_1, e_2 \models \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (7i) on (16) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10b) on (20)

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (9h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (9i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (9h) on (16) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models^{\dagger}_{?} \xi_2$**.**

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_{?} \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_{?} \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i)**.**

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_{?} (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g)**.**

(22) $e_1 \models_{?} \xi_1$ by assumption

Contradicts (17).

Case (9h)**.**

(22) $e_2 \models_{?} \xi_2$ by assumption

Contradicts (19).

Case (9i)**.**

(22) $e_1 \models_{?} \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_{?} (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.20 on (21) and (23)

Case $e_1 \models_{?} \xi_1, e_2 \models \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_{?} \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (9g) on (17) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (9i) on (17) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (10a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

Case (9h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (9i).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 1.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (9h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (9i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 1.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (9h).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

Case (9i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 1.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (7) on it, only two cases apply.

Case (7j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (7i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (9) on it, the following cases apply.

Case (9b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (9h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (9i)**.**

(22) $e_1 \models_? \xi_1$ — by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ — by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ — by Lemma 1.0.20 on (21) and (23)

$\square$

Definition 1.1.1 (Entailment of Constraints)**.** *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e `val` we have $e \models_?^\dagger \xi_1$ implies $e \models \xi_2$*

Definition 1.1.2 (Potential Entailment of Constraints)**.** *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models_?^\dagger \xi_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies $e \models_?^\dagger \xi_2$*

Corollary 1.1.1. *Suppose that $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e `final`. Then $\top \models_?^\dagger \xi$ implies $e \models_?^\dagger \xi$*

Proof.

(1) $\xi : \tau$	by assumption
(2) $\cdot ; \Gamma \vdash e : \tau$	by assumption
(3) e `final`	by assumption
(4) $\top \models_?^\dagger \xi$	by assumption
(5) $e_1 \models \top$	by Rule (7a)
(6) $e_1 \models_?^\dagger \top$	by Rule (10b) on (5)
(7) $\top : \tau$	by Rule (1a)
(8) $e_1 \models_?^\dagger \xi_r$	by Definition 1.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

$\square$

2 Static Semantics

$$
\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & (\lambda x : \tau . e) \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
 & \mid & (\!|\,|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid (\!|\,|\!)^w \mid (\!|p|\!)^w
\end{array}
$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{11a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{11b}$$

$\boxed{\Gamma ; \Delta \vdash e : \tau}$ e is of type τ

$$\text{TVar}\quad \frac{}{\Gamma, x : \tau ; \Delta \vdash x : \tau} \tag{12a}$$

$$\text{TEHole}\quad \frac{}{\Gamma ; \Delta, u :: \tau \vdash (\!|\,|\!)^u : \tau} \tag{12b}$$

$$\text{THole}\quad \frac{\Gamma ; \Delta, u :: \tau \vdash e : \tau'}{\Gamma ; \Delta, u :: \tau \vdash (\!|e|\!)^u : \tau} \tag{12c}$$

$$\text{TNum}\quad \frac{}{\Gamma ; \Delta \vdash \underline{n} : \texttt{num}} \tag{12d}$$

$$\text{TLam}\quad \frac{\Gamma, x : \tau_1 ; \Delta \vdash e : \tau_2}{\Gamma ; \Delta \vdash (\lambda x : \tau_1 . e) : (\tau_1 \to \tau_2)} \tag{12e}$$

$$\text{TAp}\quad \frac{\Gamma ; \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash e_1(e_2) : \tau} \tag{12f}$$

$$\text{TPair}\quad \frac{\Gamma ; \Delta \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \tag{12g}$$

$$\text{TPrl}\quad \frac{\Gamma ; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma ; \Delta \vdash \texttt{prl}(e) : \tau_1} \tag{12h}$$

$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \mathtt{prr}(e) : \tau_2}\ \text{TPrr} \tag{12i}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau_1}{\Gamma\,;\Delta \vdash \mathtt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)}\ \text{TInl} \tag{12j}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash \mathtt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)}\ \text{TInr} \tag{12k}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad \Gamma\,;\Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \models^{\dagger}_{?} \xi}{\Gamma\,;\Delta \vdash \mathtt{match}(e)\{\cdot \mid r \mid rs\} : \tau'}\ \text{TMatchZPre} \tag{12l}$$

$$\frac{\begin{array}{c}\Gamma\,;\Delta \vdash e : \tau \qquad e\ \mathtt{final} \qquad \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \\ \Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \qquad e \not\models^{\dagger}_{?} \xi_{pre} \qquad \top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}\end{array}}{\Gamma\,;\Delta \vdash \mathtt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'}\ \text{TMatchNZPre} \tag{12m}$$

$\boxed{p : \tau[\xi] \dashv\!\vdash \Gamma\,;\Delta}$ p is assigned type τ and emits constraint ξ

$$\frac{}{x : \tau[\top] \dashv\!\vdash \cdot\,; x : \tau}\ \text{PTVar} \tag{13a}$$

$$\frac{}{\_ : \tau[\top] \dashv\!\vdash \cdot\,;\cdot}\ \text{PTWild} \tag{13b}$$

$$\frac{}{(\!|\,|\!)^{w} : \tau[?] \dashv\!\vdash \cdot\,; w :: \tau}\ \text{PTEHole} \tag{13c}$$

$$\frac{p : \tau[\xi] \dashv\!\vdash \Gamma\,;\Delta}{(\!|p|\!)^{w} : \tau'[?] \dashv\!\vdash \Gamma\,;\Delta, w :: \tau'}\ \text{PTHole} \tag{13d}$$

$$\frac{}{\underline{n} : \mathtt{num}[\underline{n}] \dashv\!\vdash \cdot\,;\cdot}\ \text{PTNum} \tag{13e}$$

$$\frac{p : \tau_1[\xi] \dashv\!\vdash \Gamma\,;\Delta}{\mathtt{inl}(p) : (\tau_1 + \tau_2)[\mathtt{inl}(\xi)] \dashv\!\vdash \Gamma\,;\Delta}\ \text{PTInl} \tag{13f}$$

$$\frac{p : \tau_2[\xi] \dashv\!\vdash \Gamma\,;\Delta}{\mathtt{inr}(p) : (\tau_1 + \tau_2)[\mathtt{inr}(\xi)] \dashv\!\vdash \Gamma\,;\Delta}\ \text{PTInr} \tag{13g}$$

$$\frac{p_1 : \tau_1[\xi_1] \dashv\!\vdash \Gamma_1\,;\Delta_1 \qquad p_2 : \tau_2[\xi_2] \dashv\!\vdash \Gamma_2\,;\Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\!\vdash \Gamma_1 \uplus \Gamma_2\,;\Delta_1 \uplus \Delta_2}\ \text{PTPair} \tag{13h}$$

$\boxed{\Gamma\,;\Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

CTRule
$$\frac{p : \tau[\xi] \dashv\!\vert\, \Gamma_p\,;\Delta_p \qquad \Gamma \uplus \Gamma_p\,;\Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma\,;\Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \tag{14a}$$

$\boxed{\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

CTOneRules
$$\frac{\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,;\Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \tag{15a}$$

CTRules
$$\frac{\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,;\Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \tag{15b}$$

Lemma 2.0.1. *If* $p : \tau[\xi] \dashv\!\vert\, \Gamma\,;\Delta$ *then* $\xi : \tau$.

Proof. By rule induction over Rules (13). ☐

Lemma 2.0.2. *If* $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Proof. By rule induction over Rules (14). ☐

Lemma 2.0.3. *If* $\cdot\,;\Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Proof. By rule induction over Rules (15). ☐

Lemma 2.0.4. *If* $\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma\,;\Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^{\diamond} : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Proof.

(1)	$\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$	by assumption
(2)	$\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$	by assumption
(3)	$\xi_r \not\models \xi_{pre} \vee \xi_{rs}$	by assumption

By rule induction over Rules (15) on (1).

Case (15a)**.**

(4)	$rs = r' \mid \cdot$	by assumption
(5)	$\xi_{rs} = \xi_r'$	by assumption
(6)	$\Gamma\,;\Delta \vdash r' : \tau[\xi_r'] \Rightarrow \tau'$	by assumption
(7)	$\xi_r' \not\models \xi_{pre}$	by assumption
(8)	$\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi_r']r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$	by Rule (15a) on (2) and (3)
(9)	$\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi_r' \vee \xi_r] \Rightarrow \tau'$	by Rule (15b) on (6) and (8) and (7)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$
by Definition 11 on (9)

Case (15b).

(4) $rs = r' \mid rs'$ by assumption

(5) $\xi_{rs} = \xi'_r \vee \xi'_{rs}$ by assumption

(6) $\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$ by assumption

(7) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] \Rightarrow \tau'$ by assumption

(8) $\xi'_r \not\models \xi_{pre}$ by assumption

(9) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^{\diamond} : \tau[\xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by IH on (7) and (2) and (3)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^{\diamond}) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by Rule (15b) on (6) and (9) and (8)

(11) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by Definition 11 on (10)

$\square$

Lemma 2.0.5 (Substitution)**.** *If* $\Gamma, x : \tau\,;\Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta \vdash e : \tau$ *then* $\Gamma\,;\Delta \vdash [e/x]e_0 : \tau_0$

Lemma 2.0.6 (Simultaneous Substitution)**.** *If* $\Gamma \uplus \Gamma'\,;\Delta \vdash e : \tau$ *and* $\theta : \Gamma'$ *then* $\Gamma\,;\Delta \vdash [\theta]e : \tau$

Lemma 2.0.7 (Substitution Typing)**.** *If* $e \rhd p \dashv\!\mid \theta$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\mid \Gamma\,;\Delta$ *then* $\theta : \Gamma$

Proof by induction on the derivation of $e \rhd p \dashv\!\mid \theta$.

Theorem 2.1 (Determinism)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

3 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\frac{}{\underline{n}\ \texttt{val}}\ \text{VNum} \tag{16a}$$

$$\frac{}{(\lambda x:\tau.e)\ \texttt{val}}\ \text{VLam} \tag{16b}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}}\ \text{VPair} \tag{16c}$$

$$\frac{e\ \texttt{val}}{\texttt{inl}_\tau(e)\ \texttt{val}}\ \text{VInl} \tag{16d}$$

$$\frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}}\ \text{VInr} \tag{16e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{}{(\!|\,|\!)^u\ \texttt{indet}}\ \text{IEHole} \tag{17a}$$

$$\frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}}\ \text{IHole} \tag{17b}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}}\ \text{IAp} \tag{17c}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairL} \tag{17d}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairR} \tag{17e}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPair} \tag{17f}$$

$$\frac{e\ \texttt{indet}}{\texttt{prl}(e)\ \texttt{indet}}\ \text{IPrl} \tag{17g}$$

$$\frac{e \ \texttt{indet}}{\texttt{prr}(e) \ \texttt{indet}} \text{ IPrr} \tag{17h}$$

$$\frac{e \ \texttt{indet}}{\texttt{inl}_\tau(e) \ \texttt{indet}} \text{ IInL} \tag{17i}$$

$$\frac{e \ \texttt{indet}}{\texttt{inr}_\tau(e) \ \texttt{indet}} \text{ IInR} \tag{17j}$$

$$\frac{e \ \texttt{final} \qquad e \mathrel{?} p_r}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \ \texttt{indet}} \text{ IMatch} \tag{17k}$$

$\boxed{e \ \texttt{final}}$ e is final

$$\frac{e \ \texttt{val}}{e \ \texttt{final}} \text{ FVal} \tag{18a}$$

$$\frac{e \ \texttt{indet}}{e \ \texttt{final}} \text{ FIndet} \tag{18b}$$

$\boxed{e \ \texttt{notintro}}$ e cannot be a value syntactically

$$\frac{}{(\!|\,|\!)^u \ \texttt{notintro}} \text{ NVEHole} \tag{19a}$$

$$\frac{}{(\!|e|\!)^u \ \texttt{notintro}} \text{ NVHole} \tag{19b}$$

$$\frac{}{e_1(e_2) \ \texttt{notintro}} \text{ NVAp} \tag{19c}$$

$$\frac{}{\texttt{match}(e)\{\hat{rs}\} \ \texttt{notintro}} \text{ NVMatch} \tag{19d}$$

$$\frac{}{\texttt{prl}(e) \ \texttt{notintro}} \text{ NVPrl} \tag{19e}$$

$$\frac{}{\texttt{prr}(e) \ \texttt{notintro}} \text{ NVPrr} \tag{19f}$$

$\boxed{\theta : \Gamma}$ θ is of type Γ

$$\frac{}{\emptyset : \cdot} \text{ STEmpty} \tag{20a}$$

$$\frac{\theta : \Gamma_\theta \qquad \Gamma\,;\Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau}\ \text{STExtend} \tag{20b}$$

$\boxed{p\ \texttt{refutable}}$ p is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}}\ \text{RNum} \tag{21a}$$

$$\frac{}{(\!|\!)^w\ \texttt{refutable}}\ \text{REHole} \tag{21b}$$

$$\frac{}{(\!|p|\!)^w\ \texttt{refutable}}\ \text{RHole} \tag{21c}$$

$$\frac{}{\texttt{inl}(p)\ \texttt{refutable}}\ \text{RInl} \tag{21d}$$

$$\frac{}{\texttt{inr}(p)\ \texttt{refutable}}\ \text{RInr} \tag{21e}$$

$$\frac{p_1\ \texttt{refutable}}{(p_1, p_2)\ \texttt{refutable}}\ \text{RPairL} \tag{21f}$$

$$\frac{p_2\ \texttt{refutable}}{(p_1, p_2)\ \texttt{refutable}}\ \text{RPairR} \tag{21g}$$

$\boxed{e \rhd p \dashv\!\vdash \theta}$ e matches p, emitting θ

$$\frac{}{e \rhd x \dashv\!\vdash e/x}\ \text{MVar} \tag{22a}$$

$$\frac{}{e \rhd \_ \dashv\!\vdash \cdot}\ \text{MWild} \tag{22b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashv\!\vdash \cdot}\ \text{MNum} \tag{22c}$$

$$\frac{e_1 \rhd p_1 \dashv\!\vdash \theta_1 \qquad e_2 \rhd p_2 \dashv\!\vdash \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2}\ \text{MPair} \tag{22d}$$

$$\frac{e \rhd p \dashv\!\vdash \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\vdash \theta}\ \text{MInl} \tag{22e}$$

$$\frac{e \rhd p \dashv\!\vdash \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\vdash \theta}\ \text{MInr} \tag{22f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1 \qquad \texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2}{e \rhd (p_1, p_2) \dashv\!\!\mid \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{22g}$$

$\boxed{e \mathrel{?} p}$ e may match p

$$\frac{}{e \mathrel{?} \llparenthesis\rrparenthesis^w}\ \text{MMEHole} \tag{23a}$$

$$\frac{}{e \mathrel{?} \llparenthesis p \rrparenthesis^w}\ \text{MMHole} \tag{23b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}}{e \mathrel{?} p}\ \text{MMNotIntro} \tag{23c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\!\!\mid \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{23d}$$

$$\frac{e_1 \rhd p_1 \dashv\!\!\mid \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{23e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{23f}$$

$$\frac{e \mathrel{?} p}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)}\ \text{MMInl} \tag{23g}$$

$$\frac{e \mathrel{?} p}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)}\ \text{MMInr} \tag{23h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{24a}$$

$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairL} \tag{24b}$$

$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairR} \tag{24c}$$

$$\frac{}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)}\ \text{NMConfL} \tag{24d}$$

NMConfR

$$\frac{}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{24e}$$

NMInl

$$\frac{e \perp p}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)} \tag{24f}$$

NMInr

$$\frac{e \perp p}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{24g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

ITHole

$$\frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{25a}$$

ITApFun

$$\frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \tag{25b}$$

ITApArg

$$\frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')} \tag{25c}$$

ITAP

$$\frac{e_2\ \texttt{val}}{(\lambda x : \tau.e_1)(e_2) \mapsto [e_2/x]e_1} \tag{25d}$$

ITPairL

$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \tag{25e}$$

ITPairR

$$\frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \tag{25f}$$

ITPrl

$$\frac{(e_1, e_2)\ \texttt{final}}{\texttt{prl}((e_1, e_2)) \mapsto e_1} \tag{25g}$$

ITPrr

$$\frac{(e_1, e_2)\ \texttt{final}}{\texttt{prr}((e_1, e_2)) \mapsto e_2} \tag{25h}$$

ITInl

$$\frac{e \mapsto e'}{\texttt{inl}_\tau(e) \mapsto \texttt{inl}_\tau(e')} \tag{25i}$$

ITInr

$$\frac{e \mapsto e'}{\texttt{inr}_\tau(e) \mapsto \texttt{inr}_\tau(e')} \tag{25j}$$

ITExpMatch
$$\frac{e \mapsto e'}{\mathtt{match}(e)\{\hat{rs}\} \mapsto \mathtt{match}(e')\{\hat{rs}\}} \tag{25k}$$

ITSuccMatch
$$\frac{e\ \mathtt{final} \qquad e \triangleright p_r \dashv\!\!\mid \theta}{\mathtt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{25l}$$

ITFailMatch
$$\frac{e\ \mathtt{final} \qquad e \perp p_r}{\mathtt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \mathtt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^{\diamond} \mid r' \mid rs'\}} \tag{25m}$$

Lemma 3.0.1. *If* $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{final}$ *then* $e_1\ \mathtt{final}$.

Proof. By rule induction over Rules (18) on $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{final}$.

Case (18a).

(1) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{val}$ — by assumption

By rule induction over Rules (16) on (1), only one case applies.

Case (16d).

(2) $e_1\ \mathtt{val}$ — by assumption

(3) $e_1\ \mathtt{final}$ — by Rule (18a) on (2)

Case (18b).

(1) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{indet}$ — by assumption

By rule induction over Rules (17) on (1), only one case applies.

Case (17i).

(2) $e_1\ \mathtt{indet}$ — by assumption

(3) $e_1\ \mathtt{final}$ — by Rule (18b) on (2)

□

Lemma 3.0.2. *If* $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{final}$ *then* $e_2\ \mathtt{final}$.

Proof. By rule induction over Rules (18) on $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{final}$.

Case (18a).

(1) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{val}$ — by assumption

By rule induction over Rules (16) on (1), only one case applies.

Case (16d).

(2) e_2 `val` by assumption
(3) e_2 `final` by Rule (18a) on (2)

Case (18b).

(1) $\texttt{inr}_{\tau_1}(e_2)$ `indet` by assumption

By rule induction over Rules (17) on (1), only one case applies.

Case (17i).

(2) e_2 `indet` by assumption
(3) e_2 `final` by Rule (18b) on (2)

□

Lemma 3.0.3. *If* (e_1, e_2) `final` *then* e_1 `final` *and* e_2 `final`*.*

Proof. By rule induction over Rules (18) on (e_1, e_2) `final`.

Case (18a).

(1) (e_1, e_2) `val` by assumption

By rule induction over Rules (16) on (1), only one case applies.

Case (16c).

(2) e_1 `val` by assumption
(3) e_2 `val` by assumption
(4) e_1 `final` by Rule (18a) on (2)
(5) e_2 `final` by Rule (18a) on (3)

Case (18b).

(1) (e_1, e_2) `indet` by assumption

By rule induction over Rules (17) on (1), only three cases apply.

Case (17d).

(2) e_1 `indet` by assumption
(3) e_2 `val` by assumption
(4) e_1 `final` by Rule (18b) on (2)
(5) e_1 `final` by Rule (18a) on (3)

Case (17e).

(2) e_1 `val` by assumption
(3) e_2 `indet` by assumption
(4) e_1 `final` by Rule (18a) on (2)

(5) e_1 `final` by Rule (18b) on (3)

Case (17f)**.**

(2) e_1 `indet` by assumption

(3) e_2 `indet` by assumption

(4) e_1 `final` by Rule (18b) on (2)

(5) e_1 `final` by Rule (18b) on (3)

□

Lemma 3.0.4. *There doesn't exist* $\underline{n}$ *such that* $\underline{n}$ `notintro`*.*

Proof. By rule induction over Rules (19) on $\underline{n}$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.5. *There doesn't exist* $\texttt{inl}_\tau(e)$ *such that* $\texttt{inl}_\tau(e)$ `notintro`*.*

Proof. By rule induction over Rules (19) on $\texttt{inl}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.6. *There doesn't exist* $\texttt{inr}_\tau(e)$ *such that* $\texttt{inr}_\tau(e)$ `notintro`*.*

Proof. By rule induction over Rules (19) on $\texttt{inr}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.7. *There doesn't exist* (e_1, e_2) *such that* (e_1, e_2) `notintro`*.*

Proof. By rule induction over Rules (19) on (e_1, e_2) `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.8. *If* e `final` *and* e `notintro` *then* e `indet`*.*

Proof Sketch. By rule induction over Rules (19) on e `notintro`, for each case, by rule induction over Rules (16) on e `val` and we notice that e `val` is not derivable. By rule induction over Rules (18) on e `final`, Rule (18a) result in a contradiction with the fact that e `val` is not derivable while Rule (18b) tells us e `indet`. □

Lemma 3.0.9 (Finality)**.** *There doesn't exist such an expression* e *such that both* e `final` *and* $e \mapsto e'$ *for some* e'

Proof. Assume there exists such an e such that both e `final` and $e \mapsto e'$ for some e' then proof by contradiction.

By rule induction over Rules (18) and Rules (25), *i.e.*, over Rules (16) and Rules (25) and over Rules (17) and Rules (25) respectively. The proof can be done by straightforward observation of syntactic contradictions. □

Lemma 3.0.10 (Matching Determinism)**.** *If* e `final` *and* $\cdot\,; \Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\vert\, \Gamma\,; \Delta$ *then exactly one of the following holds*

1. $e \rhd p \dashv\!\vert\, \theta$ *for some* θ

2. $e \,?\, p$

3. $e \perp p$

Proof.

(1) e `final` by assumption

(2) $\cdot\,; \Delta_e \vdash e : \tau$ by assumption

(3) $p : \tau[\xi] \dashv\!\vert\, \Gamma\,; \Delta$ by assumption

By rule induction over Rules (13) on (3), we would show one conclusion is derivable while the other two are not.

Case (13a)**.**

(4) $p = x$ by assumption

(5) $e \rhd x \dashv\!\vert\, e/x$ by Rule (22a)

Assume $e \,?\, x$. By rule induction over Rules (23) on it, only one case applies.

Case (23c)**.**

(6) x `refutable` by assumption

By rule induction over Rules (21) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \,?\, x}$ by contradiction

Assume $e \perp x$. By rule induction over Rules (24) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp x}$ by contradiction

Case (13b)**.**

(4) $p =$ _ by assumption

(5) $e \rhd$ _ $\dashv\!\vert\, \cdot$ by Rule (22b)

Assume e ? _. By rule induction over Rules (23) on it, only one case applies.

Case (23c)**.**

(6) _ `refutable` by assumption

By rule induction over Rules (21) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \mathrel{?} \_}$ by contradiction

Assume $e \perp \_$. By rule induction over Rules (24) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp \_}$ by contradiction

Case (13c)**.**

(4) $p = \llparenthesis\rrparenthesis^w$ by assumption

(5) $e \mathrel{?} \llparenthesis\rrparenthesis^w$ by Rule (23a)

Assume $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (23) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd \llparenthesis\rrparenthesis^w \dashv\!\vert\, \theta}$ by contradiction

Assume $e \perp \llparenthesis\rrparenthesis^w$. By rule induction over Rules (24) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp \llparenthesis\rrparenthesis^w}$ by contradiction

Case (13d)**.**

(4) $p = \llparenthesis p_0 \rrparenthesis^w$ by assumption

(5) $e \mathrel{?} \llparenthesis p_0 \rrparenthesis^w$ by Rule (23b)

Assume $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (23) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\!\vert\, \theta}$ by contradiction

Assume $e \perp \llparenthesis p_0 \rrparenthesis^w$. By rule induction over Rules (24) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp \llparenthesis p_0 \rrparenthesis^w}$ by contradiction

Case (13e)**.**

(4) $p = \underline{n_2}$ by assumption

(5) $\tau = \texttt{num}$ by assumption

(6) $\xi = \underline{n_2}$ by assumption

(7) $\underline{n_2}$ `refutable` by Rule (21a)

By rule induction over Rules (12) on (2), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e `notintro` by Rule (19a),(19b),(19c),(19d),(19e),(19f)

(10) $e \mathrel{?} \underline{n_2}$ by Rule (9b) on (7) and (9)

Assume $e \rhd \underline{n_2} \dashv\!\mid \theta$ for some θ. By rule induction over it, no case applies due to syntactic contradiction.

(11) $\cancel{e \rhd \underline{n_2} \dashv\!\mid \theta}$ by contradiction

Assume $e \perp \underline{n_2}$. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \perp \underline{n_2}}$ by contradiction

Case (12d).

(8) $e = \underline{n_1}$

Assume $\underline{n_1} \mathrel{?} \underline{n_2}$. By rule induction over Rules (23) on it, only two cases apply.

Case (23c).

(9) $\underline{n_1}$ `notintro` by assumption

Contradicts Lemma 3.0.4.

(10) $\cancel{\underline{n_1} \mathrel{?} \underline{n_2}}$ by contradiction

By case analysis on whether $n_1 = n_2$.

Case $n_1 = n_2$.

(11) $n_1 = n_2$ by assumption

(12) $\underline{n_1} \rhd \underline{n_2} \dashv\!\mid \cdot$ by Rule (22c)

Assume $\underline{n_1} \perp \underline{n_2}$. By rule induction over Rules (24) on it, only one case applies.

Case (24a).

(13) $n_1 \neq n_2$ by assumption

Contradicts (11).

(14) $\cancel{\underline{n_1} \perp \underline{n_2}}$ by contradiction

Case $n_1 \neq n_2$.

(11) $n_1 \neq n_2$ by assumption

(12) $\underline{n_1} \perp \underline{n_2}$ by Rule (24a) on (11)

Assume $\underline{n_1} \rhd \underline{n_2} \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (22) on it, no case applies due to syntactic contradiction.

(13) $\cancel{\underline{n_1} \rhd \underline{n_2} \dashv\!\vert\, \theta}$ by contradiction

Case (13f).

(4) $p = \texttt{inl}(p_1)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \texttt{inl}(\xi_1)$ by assumption

(7) $p_1 : \tau_1[\xi_1] \dashv\!\vert\, \Gamma\,;\Delta$ by assumption

(8) $\texttt{inl}(p_1)$ $\texttt{refutable}$ by Rule (21d)

By rule induction over Rules (12) on (2), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(9) $e = \llparenthesis\rrparenthesis^u, \llparenthesis e_0 \rrparenthesis^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(10) e $\texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

(11) $e \,?\, \texttt{inl}(p_1)$ by Rule (9b) on (8) and (10)

Assume $e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1$ for some θ_1. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1}$ by contradiction

Assume $e \perp \texttt{inl}(p_1)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inl}(p_1)}$ by contradiction

Case (12j).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(10) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$ by assumption

(11) e_1 $\texttt{final}$ by Lemma 3.0.1 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1, $e_1 \,?\, p_1$, and $e_1 \perp p_1$ holds.
By case analysis on which one holds.

Case $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$.

(12) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

(13) $\cancel{e_1 \,?\, p_1}$ by assumption

(14) $\cancel{e_1 \perp p_1}$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1$ by Rule (22e) on (12)

Assume $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$. By rule induction over Rules (23) on it, only two cases apply.

Case (23c).

(16) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ — by assumption

Contradicts Lemma 3.0.5.

Case (23g).

(16) $e_1 \,?\, p_1$ — by assumption

Contradicts (13).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)}$ — by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (24) on it, only one case applies.

Case (24f).

(18) $e_1 \perp p_1$ — by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ — by contradiction

Case $e_1 \,?\, p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ — by assumption

(13) $e_1 \,?\, p_1$ — by assumption

(14) $\cancel{e_1 \perp p_1}$ — by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$ — by Rule (23g) on (13)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (22) on it, only one case applies.

Case (22e).

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta$ — by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta}$ — by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (24) on it, only one case applies.

Case (24f).

(18) $e_1 \perp p_1$ — by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ — by contradiction

Case $e_1 \perp p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ — by assumption

(13) $\cancel{e_1 \,?\, p_1}$ — by assumption

(14) $e_1 \perp p_1$ — by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$ by Rule (24f) on (14)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (22) on it, only one case applies.

Case (22e).

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1)\,?\,\texttt{inl}(p_1)$. By rule induction over Rules (23) on it, only two cases apply.

Case (23c).

(18) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 3.0.5.

Case (23g).

(18) $e_1\,?\,p_1$ by assumption

Contradicts (13).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1)\,?\,\texttt{inl}(p_1)}$ by contradiction

Case (13g).

(4) $p = \texttt{inr}(p_2)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \texttt{inr}(\xi_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\vert\, \Gamma\,;\Delta$ by assumption

(8) $\texttt{inr}(p_2)$ $\texttt{refutable}$ by Rule (21e)

By rule induction over Rules (12) on (2), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(10) e $\texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

(11) $e\,?\,\texttt{inr}(p_2)$ by Rule (9b) on (8) and (10)

Assume $e \rhd \texttt{inr}(p_2) \dashv\!\vert\, \theta_2$ for some θ_2. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inr}(p_2) \dashv\!\vert\, \theta_2}$ by contradiction

Assume $e \perp \texttt{inr}(p_2)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inr}(p_2)}$ by contradiction

Case (12k).

(9) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(10) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ by assumption

(11) e_2 $\texttt{final}$ by Lemma 3.0.2 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2, $e_2 \,?\, p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which one holds.

Case $e_2 \rhd p_2 \dashv\!\mid \theta_2$**.**

(12) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(13) $\cancel{e_2 \,?\, p_2}$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\mid \theta_2$ by Rule (22f) on (12)

Assume $\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)$. By rule induction over Rules (23) on it, only two cases apply.

Case (23c)**.**

(16) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 3.0.6.

Case (23h)**.**

(16) $e_2 \,?\, p_2$ by assumption

Contradicts (13).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$. By rule induction over Rules (24) on it, only one case applies.

Case (24g)**.**

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)}$ by contradiction

Case $e_2 \,?\, p_2$**.**

(12) $\cancel{e_2 \rhd p_2 \dashv\!\mid \theta}$ by assumption

(13) $e_2 \,?\, p_2$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)$ by Rule (23h) on (13)

Assume $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\mid \theta$ for some θ. By rule induction over Rules (22) on it, only one case applies.

Case (22f)**.**

(16) $e_2 \rhd p_2 \dashv\!\mid \theta$ by assumption

Contradicts (12).

(17) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\!\!\!\dashv \theta}$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)$. By rule induction over Rules (24) on it, only one case applies.

Case (24g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)}$ by contradiction

Case $e_2 \perp p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\!\!\!\dashv \theta}$ by assumption

(13) $\cancel{e_2 \,?\, p_2}$ by assumption

(14) $e_2 \perp p_2$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)$ by Rule (24g) on (14)

Assume $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\!\!\!\dashv \theta$ for some θ. By rule induction over Rules (22) on it, only one case applies.

Case (22f).

(16) $e_2 \rhd p_2 \dashv\!\!\!\!\dashv \theta$ by assumption

Contradicts (12).

(17) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\!\!\!\dashv \theta}$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$. By rule induction over Rules (23) on it, only two cases apply.

Case (23c).

(18) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.6.

Case (23h).

(18) $e_2 \,?\, p_2$ by assumption

Contradicts (13).

(19) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)}$ by contradiction

Case (13h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\tau = (\tau_1 \times \tau_2)$ by assumption

(6) $\xi = (\xi_1, \xi_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\mid \Gamma_1 \,;\, \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\mid \Gamma_2 \,;\, \Delta_2$ by assumption

By rule induction over Rules (12) on (2), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

(13) e $\texttt{indet}$ by Lemma 3.0.8 on (1) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (17g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (18b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (17h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (18b) on (16)

(18) $\cdot \,;\, \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (12h) on (2)

(19) $\cdot \,;\, \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (12i) on (2)

Assume $e \perp (p_1, p_2)$. By rule induction on it, no case applies due to syntactic contradiction.

(20) $\cancel{e \perp (p_1, p_2)}$ by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of $\texttt{prl}(e) \rhd p_1 \dashv\mid \theta_1$, $\texttt{prl}(e) \,?\, p_1$, and $\texttt{prl}(e) \perp p_1$ holds.
By inductive hypothesis on (17) and (19) and (10), exactly one of $\texttt{prr}(e) \rhd p_2 \dashv\mid \theta_2$, $\texttt{prr}(e) \,?\, p_2$, and $\texttt{prr}(e) \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2. Note that we have already shown $\cancel{e \perp (p_1, p_2)}$.

Case $\texttt{prl}(e) \rhd p_1 \dashv\mid \theta_1, \texttt{prr}(e) \rhd p_2 \dashv\mid \theta_2$.

(21) $\texttt{prl}(e) \rhd p_1 \dashv\mid \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashv\mid \theta_2$ by assumption

(25) $\cancel{\texttt{prr}(e) \,?\, p_2}$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

(27) $e \rhd (p_1, p_2) \dashv\mid \theta_1 \uplus \theta_2$ by Rule (22g) on (12) and (21) and (24)

Assume $e \,?\, (p_1, p_2)$. By rule induction over Rules (23) on it, only one case applies.

Case (23c).

(28) (p_1, p_2) $\texttt{refutable}$ by assumption

By rule induction over Rules (21), only two cases apply.

Case (21f).

(29) p_1 `refutable` by assumption

(30) `prl`(e) `notintro` by Rule (19e)

(31) `prl`$(e)\,?\,p_1$ by Rule (23c) on (29) and (30)

Contradicts (22).

Case (21g).

(29) p_2 `refutable` by assumption

(30) `prr`(e) `notintro` by Rule (19f)

(31) `prl`$(e)\,?\,p_1$ by Rule (23c) on (29) and (30)

Contradicts (22).

(32) $\cancel{e\,?\,(p_1, p_2)}$ by contradiction

Case `prl`$(e) \rhd p_1 \dashv\!\vert\, \theta_1$, `prr`$(e)\,?\,p_2$.

(21) `prl`$(e) \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e)\,?\,p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(25) `prr`$(e)\,?\,p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (22), only one case applies.

Case (22g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) `prr`$(e) \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

Contradicts (24).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\vert\, \theta}$ by contradiction

By rule induction over Rules (23) on (25), the following cases apply.

Case (23a),(23b).

(30) $p_2 = (\!|\!)^w, (\!|p|\!)^w$ by assumption

(31) p_2 `refutable` by Rule (21b) and Rule (21c)

(32) (p_1, p_2) `refutable` by Rule (21g) on (31)

(33) $e\,?\,(p_1, p_2)$ by Rule (23c) on (12) and (32)

Case (23c).

(30) p_2 `refutable` by assumption

(31) (p_1, p_2) $\texttt{refutable}$ — by Rule (21g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ — by Rule (23c) on (12) and (31)

Case $\texttt{prl}(e) \rhd p_1 \dashV \theta_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\texttt{prl}(e) \rhd p_1 \dashV \theta_1$ — by assumption

(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ — by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ — by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ — by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ — by assumption

(26) $\texttt{prr}(e) \perp p_2$ — by assumption

By rule induction over Rules (24) on (26), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \rhd p_2 \dashV \theta_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ — by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ — by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ — by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$ — by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ — by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ — by assumption

Assume $e \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (22), only one case applies.

Case (22g)**.**

(27) $\theta = \theta_1 \uplus \theta_2$ — by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashV \theta_1$ — by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashV \theta}$ — by contradiction

By rule induction over Rules (23) on (22), the following cases apply.

Case (23a),(23b)**.**

(30) $p_1 = \llparenthesis\rrparenthesis^w, \llparenthesis p \rrparenthesis^w$ — by assumption

(31) p_1 $\texttt{refutable}$ — by Rule (21b) and Rule (21c)

(32) (p_1, p_2) $\texttt{refutable}$ — by Rule (21g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ — by Rule (23c) on (12) and (32)

Case (23c)**.**

(30) p_1 $\texttt{refutable}$ — by assumption

(31) (p_1, p_2) $\texttt{refutable}$ — by Rule (21g) on (30)

(32) $e\ ?\ (p_1, p_2)$ — by Rule (23c) on (12) and (31)

Case $\texttt{prl}(e)\ ?\ p_1, \texttt{prr}(e)\ ?\ p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1}$ — by assumption

(22) $\texttt{prl}(e)\ ?\ p_1$ — by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ — by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2}$ — by assumption

(25) $\texttt{prr}(e)\ ?\ p_2$ — by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ — by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\vdash \theta$. By rule induction over Rules (22), only one case applies.

Case (22g)**.**

(27) $\theta = \theta_1 \uplus \theta_2$ — by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1$ — by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\vdash \theta}$ — by contradiction

By rule induction over Rules (23) on (22), the following cases apply.

Case (23a),(23b)**.**

(30) $p_1 = (\!|\,|\!)^w, (\!|p|\!)^w$ — by assumption

(31) $p_1\ \texttt{refutable}$ — by Rule (21b) and Rule (21c)

(32) $(p_1, p_2)\ \texttt{refutable}$ — by Rule (21g) on (31)

(33) $e\ ?\ (p_1, p_2)$ — by Rule (23c) on (12) and (32)

Case (23c)**.**

(30) $p_1\ \texttt{refutable}$ — by assumption

(31) $(p_1, p_2)\ \texttt{refutable}$ — by Rule (21g) on (30)

(32) $e\ ?\ (p_1, p_2)$ — by Rule (23c) on (12) and (31)

Case $\texttt{prl}(e)\ ?\ p_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1}$ — by assumption

(22) $\texttt{prl}(e)\ ?\ p_1$ — by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ — by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2}$ — by assumption

(25) $\cancel{\texttt{prr}(e)\ ?\ p_2}$ — by assumption

(26) $\texttt{prr}(e) \perp p_2$ — by assumption

By rule induction over Rules (24) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \rhd p_2 \dashV \theta_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$ by assumption
(25) $\cancel{\texttt{prr}(e) \,?\, p_2}$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (24) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \,?\, p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ by assumption
(25) $\texttt{prr}(e) \,?\, p_2$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (24) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \perp p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ by assumption
(25) $\texttt{prr}(e) \,?\, p_2$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (24) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case (12g).

(11) $e = (e_1, e_2)$ by assumption
(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption
(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption
(14) e_1 $\texttt{final}$ by Lemma 3.0.3 on (1)
(15) e_2 $\texttt{final}$ by Lemma 3.0.3 on (1)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \rhd p_1 \dashV \theta_1$, $e_1 \,?\, p_1$, and $e_1 \perp p_1$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \rhd p_2 \dashV \theta_2$, $e_2 \,?\, p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2.

Case $e_1 \rhd p_1 \dashv\!\mid \theta_1, e_2 \rhd p_2 \dashv\!\mid \theta_2$**.**

(16) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2$ by Rule (22d) on (16) and (19)

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (23) on it, only four cases apply.

Case (23c)**.**

(23) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (23d)**.**

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (23e)**.**

(23) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (23f)**.**

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(24) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24b)**.**

(25) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (24c)**.**

(25) $e_2 \perp p_2$ by assumption

Contradicts (21).

(26) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\mid \theta_1, e_2 \mathrel{?} p_2$**.**

(16) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\mid \theta_2}$ by assumption

(20) $e_2 \mathbin{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathbin{?} (p_1, p_2)$ by Rule (23e) on (16) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (24c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\vert\, \theta_1, e_2 \perp p_2$**.**

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

(17) $\cancel{e_1 \mathbin{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(20) $\cancel{e_2 \mathbin{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (24c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta}$ by contradiction

Assume $(e_1, e_2) \mathbin{?} (p_1, p_2)$. By rule induction over Rules (23) on it, only four cases apply.

Case (23c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (23d).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (23e).

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (23f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \rhd p_2 \dashv\!\mid \theta_2$.

(16) $\cancel{e_1 \rhd p_1 \dashv\!\mid \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (23d) on (17) and (19)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24b).

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (24c).

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \mathrel{?} p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (23f) on (17) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (24c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (24c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\!\!\,\dashv \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\!\,\dashv \theta}$ by contradiction

Assume $(e_1, e_2)\ ?\ (p_1, p_2)$. By rule induction over Rules (23) on it, only four cases apply.

Case (23c)**.**

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (23d)**.**

(26) $e_2 \rhd p_2 \dashv\!\!\!\,\dashv \theta_2$ by assumption

Contradicts (19).

Case (23e)**.**

(26) $e_2\ ?\ p_2$ by assumption

Contradicts (20).

Case (23f)**.**

(26) $e_2\ ?\ p_2$ by assumption

Contradicts (20).

(27) $\cancel{(e_1, e_2)\ ?\ (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \rhd p_2 \dashv\!\!\!\,\dashv \theta_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\!\!\,\dashv \theta_1}$ by assumption

(17) $\cancel{e_1\ ?\ p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\!\!\,\dashv \theta_2$ by assumption

(20) $\cancel{e_2\ ?\ p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (24b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\!\,\dashv \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\!\!\,\dashv \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\!\,\dashv \theta}$ by contradiction

Assume $(e_1, e_2)\ ?\ (p_1, p_2)$. By rule induction over Rules (23) on it, only four cases apply.

Case (23c)**.**

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (23d).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (23e).

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (23f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \mathrel{?} p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\Vdash \theta_1}$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\Vdash \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (24b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\Vdash \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\Vdash \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\Vdash \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (23) on it, only four cases apply.

Case (23c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (23d).

(26) $e_2 \rhd p_2 \dashv\!\Vdash \theta_2$ by assumption

Contradicts (19).

Case (23e).

(26) $e_1 \rhd p_1 \dashv\!\Vdash \theta_1$ by assumption

Contradicts (16).

Case (23f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \perp p_2$.

(16) $\cancel{e_1 \rhd p_1 \dashv\!\!\mid \theta_1}$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (24b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (22) on it, only one case applies.

Case (22d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (23) on it, only four cases apply.

Case (23c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (23d).

(26) $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

Contradicts (19).

Case (23e).

(26) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

Contradicts (16).

Case (23f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

□

Lemma 3.0.11 (Matching Coherence of Constraint). *Suppose that* $\cdot ; \Delta_e \vdash e : \tau$ *and* e `final` *and* $p : \tau[\xi] \dashv\!\!\mid \Gamma ; \Delta$. *Then we have*

1. $e \models \xi$ *iff* $e \rhd p \dashv\vdash \theta$

2. $e \models_{?} \xi$ *iff* $e \mathrel{?} p$

3. $e \not\models_{?}^{\dagger} \xi$ *iff* $e \perp p$

Proof.

(1) $\cdot\,;\Delta_e \vdash e : \tau$	by assumption
(2) e `final`	by assumption
(3) $p : \tau[\xi] \dashv\vdash \Gamma\,;\Delta$	by assumption

Given Lemma 2.0.1, Theorem 1.1, and Lemma 3.0.10, it is sufficient to prove

1. $e \models \xi$ iff $e \rhd p \dashv\vdash \theta$
2. $e \models_{?} \xi$ iff $e \mathrel{?} p$

By rule induction over Rules (13) on (3).

Case (13a)**.**

(4) $p = x$	by assumption
(5) $\xi = \top$	by assumption

1. Prove $e \models \top$ implies $e \rhd x \dashv\vdash \theta$ for some θ.

   (6) $e \rhd x \dashv\vdash e/x$ — by Rule (22a)

2. Prove $e \rhd x \dashv\vdash \theta$ implies $e \models \top$.

   (6) $e \models \top$ — by Rule (7a)

3. Prove $e \models_{?} \top$ implies $e \mathrel{?} x$.

   (6) $e \not\models_{?} \top$ — by Lemma 1.0.3

   Vacuously true.

4. Prove $e \mathrel{?} x$ implies $e \models_{?} \top$.
   By rule induction over Rules (23), we notice that either, $e \mathrel{?} x$ is in syntactic contradiction with all the cases, or the premise x `refutable` is not derivable. Hence, $e \mathrel{?} x$ are not derivable. And thus vacuously true.

Case (13b)**.**

(4) $p = \_$	by assumption
(5) $\xi = \top$	by assumption

1. Prove $e \models \top$ implies $e \rhd \_ \dashv\vdash \theta$ for some θ.

   (6) $e \rhd \_ \dashv\vdash \cdot$ — by Rule (22a)

2. Prove $e \rhd \_ \dashv\!\mid \theta$ implies $e \models \top$.

   (6) $e \models \top$ — by Rule (7a)

3. Prove $e \models_{?} \top$ implies $e \mathrel{?} \_$.

   (6) $e \not\models_{?} \top$ — by Lemma 1.0.3

   Vacuously true.

4. Prove $e \mathrel{?} \_$ implies $e \models_{?} \xi$.

   By rule induction over Rules (23), we notice that either, $e ? \_$ is in syntactic contradiction with all the cases, or the premise $\_$ `refutable` is not derivable. Hence, $e \mathrel{?} \_$ are not derivable. And thus vacuously true.

Case (13c)**.**

(4) $p = \llparenthesis\rrparenthesis^w$ — by assumption

(5) $\xi = ?$ — by assumption

(6) $\overline{\xi} = ?$ — by Definition 2

1. Prove $e \models ?$ implies $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\mid \theta$ for some θ.

   (7) $e \not\models ?$ — by Rule (22a)

   Vacuously true.

2. Prove $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\mid \theta$ implies $e \models ?$.
   By rule induction over Rules (22), we notice that $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\mid \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_{?} ?$ implies $e \mathrel{?} \llparenthesis\rrparenthesis^w$.

   (7) $e \mathrel{?} \llparenthesis\rrparenthesis^w$ — by Rule (23a)

4. Prove $e \mathrel{?} \llparenthesis\rrparenthesis^w$ implies $e \models_{?} ?$.

   (7) $e \models_{?} ?$ — by Rule (9a)

Case (13d)**.**

(4) $p = \llparenthesis p_0 \rrparenthesis^w$ — by assumption

(5) $\xi = ?$ — by assumption

1. Prove $e \models ?$ implies $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\!\mid \theta$ for some θ.

   (6) $e \not\models ?$ — by Rule (22a)

   Vacuously true.

2. Prove $e \rhd (\!|p_0|\!)^w \dashv\!\!\mid \theta$ implies $e \models ?$.
   By rule induction over Rules (22), we notice that $e \rhd (\!|p_0|\!)^w \dashv\!\!\mid \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? ?$ implies $e \,?\, (\!|p_0|\!)^w$.

   (6) $e \,?\, (\!|p_0|\!)^w$ — by Rule (23b)

4. Prove $e \,?\, (\!|p_0|\!)^w$ implies $e \models_? ?$.

   (6) $e \models_? ?$ — by Rule (9a)

Case (13e).

(4) $p = \underline{n}$ — by assumption

(5) $\xi = \underline{n}$ — by assumption

1. Prove $e \models \underline{n}$ implies $e \rhd \underline{n} \dashv\!\!\mid \theta$ for some θ.

   (6) $e \models \underline{n}$ — by assumption

   By rule induction over Rules (7) on (6), only one case applies.

   Case (7b).

   (7) $e = \underline{n}$ — by assumption

   (8) $\underline{n} \rhd \underline{n} \dashv\!\!\mid \cdot$ — by Rule (22c)

2. Prove $e \rhd \underline{n} \dashv\!\!\mid \theta$ implies $e \models \underline{n}$.

   (6) $e \rhd \underline{n} \dashv\!\!\mid \theta$ — by assumption

   By rule induction over Rules (22) on (6), only one case applies.

   Case (22c).

   (7) $e = \underline{n}$ — by assumption

   (8) $\theta = \cdot$ — by assumption

   (9) $\underline{n} \models \underline{n}$ — by Rule (7b)

3. Prove $e \models_? \underline{n}$ implies $e \,?\, \underline{n}$.

   (6) $e \models_? \underline{n}$ — by assumption

   By rule induction over Rules (9) on (6), only one case applies.

   Case (9b).

   (7) e `notintro` — by assumption

   (8) $\underline{n}$ `refutable` — by Rule (21a)

   (9) $e \,?\, \underline{n}$ — by Rule (23c) on (7) and (8)

4. Prove $e \,?\, \underline{n}$ implies $e \models_? \underline{n}$.

   (6) $e \,?\, \underline{n}$ — by assumption

By rule induction over Rules (23) on (6), only one case applies.

Case (23c).

(7)	e `notintro`	by assumption
(8)	$\underline{n}$ `refutable`	by Rule (3a)
(9)	$e \models_{?} \underline{n}$	by Rule (9) on (7) and (8)

Case (13f).

(4)	$p = \texttt{inl}(p_1)$	by assumption
(5)	$\xi = \texttt{inl}(\xi_1)$	by assumption
(6)	$\tau = (\tau_1 + \tau_2)$	by assumption
(7)	$p_1 : \tau_1[\xi_1] \dashv\vert \Gamma ; \Delta$	by assumption

By rule induction over Rules (12) on (1), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m).

(8)	$e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$	by assumption
(9)	e `notintro`	by Rule (19a),(19b),(19c),(19d),(19e),(19f)

1. Prove $e \models \texttt{inl}(\xi_1)$ implies $e \rhd \texttt{inl}(p_1) \dashv\vert \theta$ for some θ. By rule induction over Rules (7) on $e \models \texttt{inl}(\xi_1)$, no case applies due to syntactic contradiction.
Therefore, vacuously true.
2. Prove $e \rhd \texttt{inl}(p_1) \dashv\vert \theta$ implies $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (22) on $e \rhd \texttt{inl}(p_1) \dashv\vert \theta$, no case applies due to syntactic contradiction.
Therefore, vacuously true.
3. Prove $e \models_{?} \texttt{inl}(\xi_1)$ implies $e \,?\, \texttt{inl}(p_1)$.

(10)	$\texttt{inl}(p_1)$ `refutable`	by Rule (21d)
(11)	$e \,?\, \texttt{inl}(p_1)$	by Rule (23c) on (9) and (10)

4. Prove $e \,?\, \texttt{inl}(p_1)$ implies $e \models_{?} \texttt{inl}(\xi_1)$.

(10)	$\texttt{inl}(\xi_1)$ `refutable`	by Rule (3c)
(11)	$e \models_{?} \texttt{inl}(\xi_1)$	by Rule (9b) on (9) and (10)

Case (12j).

(8)	$e = \texttt{inl}_{\tau_2}(e_1)$	by assumption
(9)	$\cdot\, ; \Delta_e \vdash e_1 : \tau_1$	by assumption
(10)	e_1 `final`	by Lemma 3.0.1 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\!\mid \theta$ for some θ

(12) $e_1 \models_? \xi_1$ iff $e_1 \,?\, p_1$

1. Prove $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta$ for some θ.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by assumption

   By rule induction over Rules (7) on (13), only one case applies.

   Case (7g).

   (14) $e_1 \models \xi_1$ by assumption

   (15) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ for some θ_1 by (11) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta_1$ by Rule (22e) on (15)

2. Prove $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta$ implies $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta$ by assumption

   By rule induction over Rules (22) on (13), only one case applies.

   Case (22e).

   (14) $e_1 \rhd p_1 \dashv\!\!\mid \theta$ by assumption

   (15) $e_1 \models \xi_1$ by (11) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by Rule (7g) on (15)

3. Prove $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ by assumption

   By rule induction over Rules (9) on (13), only two cases apply.

   Case (9b).

   (14) $\mathtt{inl}_{\tau_2}(e_1)$ $\mathtt{notintro}$ by assumption

   Contradicts Lemma 3.0.5.

   Case (9e).

   (14) $e_1 \models_? \xi_1$ by assumption

   (15) $e_1 \,?\, p_1$ by (12) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$ by Rule (23g) on (15)

4. Prove $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$ by assumption

   By rule induction over Rules (23) on (13), only two cases apply.

   Case (23c).

   (14) $\mathtt{inl}_{\tau_2}(e_1)$ $\mathtt{notintro}$ by assumption

   Contradicts Lemma 3.0.5.

   Case (23g).

   (14) $e_1 \,?\, p_1$ by assumption

   (15) $e_1 \models_? \xi_1$ by (12) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ by Rule (9e) on (15)

Case (13g)**.**

(4) $p = \texttt{inr}(p_2)$ by assumption

(5) $\xi = \texttt{inr}(\xi_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\Vert\, \Gamma ; \Delta$ by assumption

By rule induction over Rules (12) on (1), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m)**.**

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e $\texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

1. Prove $e \models \texttt{inr}(\xi_2)$ implies $e \rhd \texttt{inr}(p_2) \dashv\!\Vert\, \theta$ for some θ. By rule induction over Rules (7) on $e \models \texttt{inr}(\xi_2)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \texttt{inr}(p_2) \dashv\!\Vert\, \theta$ implies $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (22) on $e \rhd \texttt{inr}(p_2) \dashv\!\Vert\, \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \texttt{inr}(\xi_2)$ implies $e \,?\, \texttt{inr}(p_2)$.

   (10) $\texttt{inr}(p_2)$ $\texttt{refutable}$ by Rule (21e)

   (11) $e \,?\, \texttt{inr}(p_2)$ by Rule (23c) on (9) and (10)
4. Prove $e \,?\, \texttt{inr}(p_2)$ implies $e \models_? \texttt{inr}(\xi_2)$.

   (10) $\texttt{inr}(\xi_2)$ $\texttt{refutable}$ by Rule (3d)

   (11) $e \models_? \texttt{inr}(\xi_2)$ by Rule (9b) on (9) and (10)

Case (12k)**.**

(8) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(9) $\cdot\, ; \Delta_e \vdash e_2 : \tau_2$ by assumption

(10) e_2 $\texttt{final}$ by Lemma 3.0.1 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\Vert\, \theta$ for some θ

(12) $e_2 \models_? \xi_2$ iff $e_2 \,?\, p_2$

1. Prove $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ implies $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\Vert\, \theta$ for some θ.

   (13) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ by assumption

By rule induction over Rules (7) on (13), only one case applies.

Case (7g).

(14) $e_2 \models \xi_2$ by assumption

(15) $e_2 \rhd p_2 \dashV \theta_1$ for some θ_1 by (11) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashV \theta_1$ by Rule (22e) on (15)

2. Prove $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashV \theta$ implies $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashV \theta$ by assumption

By rule induction over Rules (22) on (13), only one case applies.

Case (22e).

(14) $e_2 \rhd p_2 \dashV \theta$ by assumption

(15) $e_2 \models \xi_2$ by (11) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (7g) on (15)

3. Prove $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by assumption

By rule induction over Rules (9) on (13), only two cases apply.

Case (9b).

(14) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.5.

Case (9e).

(14) $e_2 \models_? \xi_2$ by assumption

(15) $e_2 \,?\, p_2$ by (12) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ by Rule (23g) on (15)

4. Prove $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ by assumption

By rule induction over Rules (23) on (13), only two cases apply.

Case (23c).

(14) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.5.

Case (23g).

(14) $e_2 \,?\, p_2$ by assumption

(15) $e_2 \models_? \xi_2$ by (12) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (9e) on (15)

Case (13h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\xi = (\xi_1, \xi_2)$ by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\!\vdash \Gamma_1 ; \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\!\vdash \Gamma_2 ; \Delta_2$ by assumption

By rule induction over Rules (12) on (1), the following cases apply.

Case (12b),(12c),(12f),(12h),(12i),(12l),(12m)**.**

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (19a),(19b),(19c),(19d),(19e),(19f)

(13) e $\texttt{indet}$ by Lemma 3.0.8 on (2) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (17g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (18b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (17h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (18b) on (16)

(18) $\cdot ; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (12h) on (1)

(19) $\cdot ; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (12i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

(20) $\texttt{prl}(e) \models \xi_1$ iff $\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1$ for some θ_1

(21) $\texttt{prl}(e) \models_? \xi_1$ iff $\texttt{prl}(e)\ ?\ p_1$

(22) $\texttt{prr}(e) \models \xi_2$ iff $\texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2$ for some θ_2

(23) $\texttt{prr}(e) \models_? \xi_2$ iff $\texttt{prr}(e)\ ?\ p_2$

1. Prove $e \models (\xi_1, \xi_2)$ implies $e \rhd (p_1, p_2) \dashv\!\vdash \theta$ for some θ.

   (24) $e \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (7) on (24), only one case applies.

   Case (7j)**.**

   (25) $\texttt{prl}(e) \models \xi_1$ by assumption

   (26) $\texttt{prr}(e) \models \xi_2$ by assumption

   (27) $\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1$ by (20) on (25)

   (28) $\texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2$ by (22) on (26)

   (29) $e \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2$ by Rule (22g) on (12) and (27) and (28)

2. Prove $e \rhd (p_1, p_2) \dashv\!\vdash \theta$ implies $e \models (\xi_1, \xi_2)$.

   (24) $e \rhd (p_1, p_2) \dashv\!\vdash \theta$ by assumption

   By rule induction over Rules (22) on (24), only one case applies.

   Case (22g)**.**

   (25) $\theta = \theta_1 \uplus \theta_2$ by assumption

   (26) $\texttt{prl}(e) \rhd \xi_1 \dashv\!\vdash \theta_1$ by assumption

(27) $\mathtt{prr}(e) \rhd \xi_2 \dashv\!\vert\, \theta_2$ by assumption
(28) $\mathtt{prl}(e) \models \xi_1$ by (20) on (26)
(29) $\mathtt{prr}(e) \models \xi_2$ by (22) on (27)
(30) $e \models (\xi_1, \xi_2)$ by Rule (7j) on (12) and (28) and (29)

3. Prove $e \models_? (\xi_1, \xi_2)$ implies $e \,?\, (p_1, p_2)$.

(24) $e \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (9) on (24), only one case applies.

Case (9b)**.**

(25) (ξ_1, ξ_2) $\mathtt{refutable}$ by assumption

By rule induction over Rules (3) on (25), only two cases apply.

Case (3e)**.**

(26) ξ_1 $\mathtt{refutable}$ by assumption
(27) $\mathtt{prl}(e)$ $\mathtt{notintro}$ by Rule (19e)
(28) $\mathtt{prl}(e) \models_? \xi_1$ by Rule (9b) on (26) and (27)
(29) $\mathtt{prl}(e) \,?\, p_1$ by (21) on (28)

By rule induction over Rules (23) on (29), only three cases apply.

Case (23a),(23b)**.**

(30) $p_1 = (\!|\,|\!)^w, (\!|p_0|\!)^w$ by assumption
(31) p_1 $\mathtt{refutable}$ by Rule (21b) and Rule (21c)
(32) (p_1, p_2) $\mathtt{refutable}$ by Rule (21f) on (31)
(33) $e \,?\, (p_1, p_2)$ by Rule (23c) on (12) and (32)

Case (23c)**.**

(30) p_1 $\mathtt{refutable}$ by assumption
(31) (p_1, p_2) $\mathtt{refutable}$ by Rule (21f) on (30)
(32) $e \,?\, (p_1, p_2)$ by Rule (23c) on (12) and (31)

Case (3f)**.**

(26) ξ_2 $\mathtt{refutable}$ by assumption
(27) $\mathtt{prr}(e)$ $\mathtt{notintro}$ by Rule (19e)
(28) $\mathtt{prr}(e) \models_? \xi_2$ by Rule (9b) on (26) and (27)
(29) $\mathtt{prr}(e) \,?\, p_2$ by (23) on (28)

By rule induction over Rules (23) on (29), only three cases apply.

Case (23a),(23b)**.**

(30) $p_2 = (\!|\!)^w, (\!|p_0|\!)^w$ by assumption

(31) p_2 `refutable` by Rule (21b) and Rule (21c)

(32) (p_1, p_2) `refutable` by Rule (21g) on (31)

(33) $e\,?\,(p_1, p_2)$ by Rule (23c) on (12) and (32)

Case (23c).

(30) p_2 `refutable` by assumption

(31) (p_1, p_2) `refutable` by Rule (21g) on (30)

(32) $e\,?\,(p_1, p_2)$ by Rule (23c) on (12) and (31)

4. Prove $e\,?\,(p_1, p_2)$ implies $e \models_? (\xi_1, \xi_2)$.

(24) $e\,?\,(p_1, p_2)$ by assumption

By rule induction over Rules (23) on (24), only one case applies.

Case (23c).

(25) (p_1, p_2) `refutable` by assumption

By rule induction over Rules (21) on (25), only two cases apply.

Case (21f).

(26) p_1 `refutable` by assumption

(27) `prl`(e) `notintro` by Rule (19e)

(28) `prl`$(e)\,?\,p_1$ by Rule (23c) on (26) and (27)

(29) `prl`$(e) \models_? \xi_1$ by (21) on (28)

By rule induction over Rules (9) on (29), only three cases apply.

Case (9a).

(30) $\xi_1 = ?$ by assumption

(31) ξ_1 `refutable` by Rule (3b)

(32) (ξ_1, ξ_2) `refutable` by Rule (3e) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (32)

Case (9b).

(30) ξ_1 `refutable` by assumption

(31) (ξ_1, ξ_2) `refutable` by Rule (3e) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (31)

Case (21g).

(26) p_2 `refutable` by assumption

(27) `prr`(e) `notintro` by Rule (19e)

(28) $\texttt{prr}(e)\,?\,p_2$ by Rule (23c) on (26) and (27)

(29) $\texttt{prr}(e) \models_? \xi_2$ by (23) on (28)

By rule induction over Rules (9) on (29), only three cases apply.

Case (9a).

(30) $\xi_2 = ?$ by assumption

(31) ξ_2 $\texttt{refutable}$ by Rule (3b)

(32) (ξ_1, ξ_2) $\texttt{refutable}$ by Rule (3f) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (32)

Case (9b).

(30) ξ_2 $\texttt{refutable}$ by assumption

(31) (ξ_1, ξ_2) $\texttt{refutable}$ by Rule (3f) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (9b) on (12) and (31)

Case (12g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ by assumption

(14) e_1 $\texttt{final}$ by Lemma 3.0.3 on (2)

(15) e_2 $\texttt{final}$ by Lemma 3.0.3 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1

(17) $e_1 \models_? \xi_1$ iff $e_1\,?\,p_1$

(18) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ for some θ_2

(19) $e_2 \models_? \xi_2$ iff $e_2\,?\,p_2$

1. Prove $(e_1, e_2) \models (\xi_1, \xi_2)$ implies $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$ for some θ.

   (20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (7) on (20), only two cases apply.

   Case (7i).

   (21) $e_1 \models \xi_1$ by assumption

   (22) $e_2 \models \xi_2$ by assumption

   (23) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1 by (16) on (21)

   (24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ for some θ_2 by (18) on (22)

   (25) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2$ by Rule (22d) on (23) and (24)

Case (7j).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

2. Prove $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$ implies $(e_1, e_2) \models (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$ by assumption

By rule induction over Rules (22) on (20), only two cases apply.

Case (22d).

(21) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1 by assumption

(22) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2 by assumption

(23) $e_1 \models \xi_1$ by (16) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (7i) on (23) and (24)

Case (22g).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

3. Prove $(e_1, e_2) \models_? (\xi_1, \xi_2)$ implies $(e_1, e_2)\ ?\ (p_1, p_2)$.

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (9) on (20), only four cases apply.

Case (9b).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.7.

Case (9g).

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models \xi_2$ by assumption

(23) $e_1\ ?\ p_1$ by (17) on (21)

(24) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by (18) on (22)

(25) $(e_1, e_2)\ ?\ (p_1, p_2)$ by Rule (23d) on (23) and (24)

Case (9h).

(21) $e_1 \models \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by (16) on (21)

(24) $e_2\ ?\ p_2$ by (19) on (22)

(25) $(e_1, e_2)\ ?\ (p_1, p_2)$ by Rule (23e) on (23) and (24)

Case (9i).

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1\ ?\ p_1$ by (17) on (21)

(24) $e_2 \mathrel{?} p_2$ by (19) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (23f) on (23) and (24)

4. Prove $(e_1, e_2) \mathrel{?} (p_1, p_2)$ implies $(e_1, e_2) \models_? (\xi_1, \xi_2)$.

   (20) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by assumption

   By rule induction over Rules (23) on (20), only four cases apply.

   Case (23c).

   (21) (e_1, e_2) `notintro` by assumption

   Contradicts Lemma 3.0.7.

   Case (23d).

   (21) $e_1 \mathrel{?} p_1$ by assumption

   (22) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

   (23) $e_1 \models_? \xi_1$ by (17) on (21)

   (24) $e_2 \models \xi_2$ by (18) on (22)

   (25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (9g) on (23) and (24)

   Case (23e).

   (21) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

   (22) $e_2 \mathrel{?} p_2$ by assumption

   (23) $e_1 \models \xi_1$ by (16) on (21)

   (24) $e_2 \models_? \xi_2$ by (19) on (22)

   (25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (9h) on (23) and (24)

   Case (23f).

   (21) $e_1 \mathrel{?} p_1$ by assumption

   (22) $e_2 \mathrel{?} p_2$ by assumption

   (23) $e_1 \models_? \xi_1$ by (17) on (21)

   (24) $e_2 \models_? \xi_2$ by (19) on (22)

   (25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (9i) on (23) and (24)

□

4 Preservation and Progress

Theorem 4.1 (Preservation). *If* $\cdot\,; \Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,; \Delta \vdash e' : \tau$

Proof. By rule induction over Rules (12) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (12l).

(1) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

(2) $\mathtt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$	by assumption
(3) $\cdot\,;\Delta \vdash e_1 : \tau_1$	by assumption
(4) $\cdot\,;\Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$	by assumption
(5) $\top \models^{\dagger}_{?} \xi$	by assumption

By rule induction over Rules (25) on (2).

Case (25k)**.**

(6) $e' = \mathtt{match}(e_1')\{\cdot \mid r \mid rs\}$	by assumption
(7) $e_1 \mapsto e_1'$	by assumption
(8) $\cdot\,;\Delta \vdash e_1' : \tau_1$	by IH on (3) and (7)
(9) $\cdot\,;\Delta \vdash \mathtt{match}(e_1')\{\cdot \mid r \mid rs\} : \tau$	by Rule (12l) on (8) and (4) and (5)

Case (25l)**.**

(6) $r = p_r \Rightarrow e_r$	by assumption
(7) $e' = [\theta](e_r)$	by assumption
(8) $e_1 \rhd p_r \dashv\!\!\vert\, \theta$	by assumption

By rule induction over Rules (15) on (4).

Case (15a)**.**

(9) $\xi = \xi_r$	by assumption
(10) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$	by assumption
(11) $p_r : \tau_1[\xi_r] \dashv\!\!\vert\, \Gamma_r\,;\Delta_r$	by Inversion of Rule (14a) on (10)
(12) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$	by Inversion of Rule (14a) on (10)
(13) $\theta : \Gamma_r$	by Lemma 2.0.7 on (3) and (11) and (8)
(14) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$	by Lemma 2.0.6 on (12) and (13)

Case (15b)**.**

(9) $\xi = \xi_r \vee \xi_{rs}$	by assumption
(10) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$	by assumption
(11) $p_r : \tau_1[\xi_r] \dashv\!\!\vert\, \Gamma_r\,;\Delta_r$	by Inversion of Rule (14a) on (10)
(12) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$	by Inversion of Rule (14a) on (10)
(13) $\theta : \Gamma_r$	by Lemma 2.0.7 on (3) and (11) and (8)
(14) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$	by Lemma 2.0.6 on (12) and (13)

Case (25m).

(6) $rs = r' \mid rs'$ by assumption

(7) $e' = \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by assumption

(8) e_1 $\texttt{final}$ by assumption

(9) $e_1 \perp p_r$ by assumption

By rule induction over Rules (15) on (4).

Case (15a). Syntactic contradiction of rs.

Case (15b).

(10) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(11) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(12) $\cdot\,;\Delta \vdash [\bot \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$ by assumption

(13) $\xi_r \not\models \bot$ by assumption

(14) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r\,;\Delta_r$ by Inversion of Rule (14a) on (11)

(15) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (14a) on (11)

(16) $\cdot\,;\Delta \vdash [\bot](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$ by Rule (15a) on (11) and (13)

(17) $e_1 \not\models^{\dagger}_{?} \xi_r$ by Lemma 3.0.11 on (3) and (8) and (14) and (9)

(18) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$ by Rule (12m) on (3) and (8) and (16) and (12) and (17) and (5)

Case (12m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$ by assumption

(4) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(5) e_1 $\texttt{final}$ by assumption

(6) $\cdot\,;\Delta \vdash [\bot]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$ by assumption

(7) $\cdot\,;\Delta \vdash [\bot \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(8) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(9) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (25) on (3).

Case (25k).

(10) $e' = \texttt{match}(e_1')\{rs_{pre} \mid r \mid rs_{post}\}$ by assumption

(11) $e_1 \mapsto e_1'$ by assumption

By Lemma 3.0.9, (11) contradicts (5).

Case (25l).

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $e' = [\theta](e_r)$ by assumption

(12) $e_1 \rhd p_r \dashv\!\vert\, \theta$ by assumption

By rule induction over Rules (15) on (7).

Case (15a).

(13) $\xi_{rest} = \xi_r$ by assumption

(14) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv\!\vert\, \Gamma_r\,;\Delta_r$ by Inversion of Rule (14a) on (14)

(16) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (14a) on (14)

(17) $\theta : \Gamma_r$ by Lemma 2.0.7 on (4) and (15) and (12)

(18) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 2.0.6 on (16) and (17)

Case (15b).

(13) $\xi_{rest} = \xi_r \vee \xi_{rs}$ by assumption

(14) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv\!\vert\, \Gamma_r\,;\Delta_r$ by assumption

(16) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by assumption

(17) $\theta : \Gamma_r$ by Lemma 2.0.7 on (4) and (15) and (12)

(18) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 2.0.6 on (16) and (17)

Case (25m).

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $rs_{post} = r' \mid rs'$ by assumption

(12) $e' = \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\}$ by assumption

(13) $e_1 \perp p_r$ by assumption

By rule induction over Rules (15) on (7).

Case (15a)**.** Syntactic contradiction of rs_{post}.

Case (15b)**.**

(14) $\xi_{rest} = \xi_r \vee \xi_{post}$ by assumption

(15) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(16) $\cdot\,;\Delta \vdash [\bot \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$ by assumption

(17) $\xi_r \not\models \xi_{pre}$ by assumption

(18) $p_r : \tau_1[\xi_r] \dashv\!\vdash \Gamma_r\,;\Delta_r$ by Inversion of Rule (14a) on (15)

(19) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (14a) on (15)

(20) $\xi_r : \tau_1$ by Lemma 2.0.2 on (15)

(21) $\xi_{pre} : \tau_1$ by Lemma 2.0.3 on (6)

(22) $\xi_r \not\models \bot \vee \xi_{pre}$ by Lemma 1.0.6 on (20) and (21) and (17)

(23) $\cdot\,;\Delta \vdash [\bot](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$ by Lemma 2.0.4 on (6) and (15) and (22)

(24) $e_1 \not\models^{\dagger}_{?} \xi_r$ by Lemma 3.0.11 on (4) and (5) and (18) and (13)

(25) $e_1 \not\models^{\dagger}_{?} \xi_{pre} \vee \xi_r$ by Lemma 1.0.7 on (8) and (24)

(26) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\} : \tau$ by Rule (12m) on (4) and (5) and (23) and (16) and (25) and (9)

$\square$

Theorem 4.2 (Progress)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then either* e $\texttt{final}$ *or* $e \mapsto e'$ *for some* e'.

Proof. By rule induction over Rules (12) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (12l)**.**

(1) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

(2) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(3) $\cdot\,;\Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ by assumption

(4) $\top \models^{\dagger}_{?} \xi$ by assumption

By IH on (2).

Case Scrutinee takes a step.

(5) $e_1 \mapsto e_1'$ by assumption

(6) $\texttt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \texttt{match}(e_1')\{\cdot \mid r \mid rs\}$ by Rule (25k) on (5)

Case Scrutinee is final.

(5) e_1 $\texttt{final}$ by assumption

By rule induction over Rules (15) on (3).

Case (15a).

(6) $rs = \cdot$ by assumption

(7) $\xi = \xi_r$ by assumption

(8) $\cdot \,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (14a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r \,; \Delta_r$ by Inversion of Rule (14a) on (8)

(11) $e_1 \models_?^\dagger \xi_r$ by Corollary 1.1.1 on (5) and (4)

By rule induction over Rules (10) on (11).

Case (10a).

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \mathrel{?} p_r$ by Lemma 3.0.11 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{indet}$ by Rule (17k) on (5) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{final}$ by Rule (18b) on (14)

Case (10b).

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\vert\, \theta$ by Lemma 3.0.11 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (25l) on (5) and (13)

Case (15b).

(6) $rs = r' \mid rs'$ by assumption

(7) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(8) $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (14a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\!\mid \Gamma_r \,;\Delta_r$ by Inversion of Rule (14a) on (8)

By Lemma 3.0.10 on (2) and (5) and (10).

Case Scrutinee matches pattern.

(11) $e_1 \rhd p_r \dashv\!\mid \theta$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$ by Rule (25l) on (5) and (11)

Case Scrutinee may matches pattern.

(11) $e_1 \,?\, p_r$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}\ \texttt{indet}$ by Rule (17k) on (5) and (11)

(13) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}\ \texttt{final}$ by Rule (18b) on (12)

Case Scrutinee doesn't matche pattern.

(11) $e_1 \perp p_r$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$ $\mapsto \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by Rule (25m) on (5) and (11)

Case (12m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(4) $e_1\ \texttt{final}$ by assumption

(5) $\cdot\,;\Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(6) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(7) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (15) on (5).

Case (15a).

(5) $rs_{post} = \cdot$ by assumption

(6) $\xi_{rest} = \xi_r$ by assumption

(7) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(8) $r = p_r \Rightarrow e_r$ by Inversion of Rule (14a) on (7)

(9) $p_r : \tau_1[\xi_r] \dashv\!\mid \Gamma_r \,; \Delta_r$ by Inversion of Rule (14a) on (7)

(10) $e_1 \models_?^\dagger \xi_{pre} \vee \xi_r$ by Corollary 1.1.1 on (4) and (7)

(11) $e_1 \models_?^\dagger \xi_r$ by Lemma 1.0.8 on (10) and (6)

By rule induction over Rules (10) on (11).

Case (10a)**.**

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \,?\, p_r$ by Lemma 3.0.11 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{indet}$ by Rule (17k) on (4) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{final}$ by Rule (18b) on (14)

Case (10b)**.**

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\!\mid \theta$ by Lemma 3.0.11 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (25l) on (4) and (13)

Case (15b)**.**

(5) $rs_{post} = r' \mid rs'_{post}$ by assumption

(6) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(7) $r = p_r \Rightarrow e_r$ by Inversion of Rule (14a) on (6)

(8) $p_r : \tau_1[\xi_r] \dashv\!\mid \Gamma_r \,; \Delta_r$ by Inversion of Rule (14a) on (6)

By Lemma 3.0.10 on (3) and (4) and (8).

Case Scrutinee matches pattern**.**

(9) $e_1 \rhd p_r \dashv\!\mid \theta$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$
by Rule (25l) on (4) and (9)

Case Scrutinee may matches pattern.

(9) $e_1 \,?\, p_r$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}$ $\mathtt{indet}$
by Rule (17k) on (4) and (9)

(11) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}$ $\mathtt{final}$
by Rule (18b) on (10)

Case Scrutinee doesn't matche pattern.

(9) $e_1 \perp p_r$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\}$
$\mapsto \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$
by Rule (25m) on (4) and (9)

□

5 Decidability

$\boxed{\Xi\ \mathtt{incon}}$ A finite set of constraints, Ξ, is inconsistent

$$\text{CINCTruth}\quad \frac{\Xi\ \mathtt{incon}}{\Xi, \top\ \mathtt{incon}} \tag{26a}$$

$$\text{CINCFalsity}\quad \frac{}{\Xi, \bot\ \mathtt{incon}} \tag{26b}$$

$$\text{CINCNum}\quad \frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \mathtt{incon}} \tag{26c}$$

$$\text{CINCNotNum}\quad \frac{}{\Xi, \underline{n}, \cancel{n}\ \mathtt{incon}} \tag{26d}$$

$$\text{CINCAnd}\quad \frac{\Xi, \xi_1, \xi_2\ \mathtt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \mathtt{incon}} \tag{26e}$$

$$\text{CINCOr}\quad \frac{\Xi, \xi_1\ \mathtt{incon} \qquad \Xi, \xi_2\ \mathtt{incon}}{\Xi, \xi_1 \vee \xi_2\ \mathtt{incon}} \tag{26f}$$

$$\frac{\text{CINCInj}}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2)\ \texttt{incon}} \tag{26g}$$

$$\text{CINCInl}\ \frac{\Xi\ \texttt{incon}}{\texttt{inl}(\Xi)\ \texttt{incon}} \tag{26h}$$

$$\text{CINCInr}\ \frac{\Xi\ \texttt{incon}}{\texttt{inr}(\Xi)\ \texttt{incon}} \tag{26i}$$

$$\text{CINCPairL}\ \frac{\Xi_1\ \texttt{incon}}{(\Xi_1, \Xi_2)\ \texttt{incon}} \tag{26j}$$

$$\text{CINCPairR}\ \frac{\Xi_2\ \texttt{incon}}{(\Xi_1, \Xi_2)\ \texttt{incon}} \tag{26k}$$

Lemma 5.0.1 (Decidability of Inconsistency)**.** *Suppose* $\dot{\top}(\xi) = \xi$*. It is decidable whether* ξ `incon`*.*

Lemma 5.0.2 (Inconsistency and Entailment of Constraint)**.** *Suppose that* $\dot{\top}(\xi) = \xi$*. Then* $\overline{\xi}$ `incon` *iff* $\top \models \xi$

Lemma 5.0.3. *If* $e \models \xi$ *then* $e \models \dot{\top}(\xi)$

Proof. By rule induction over Rules (7), it is obvious to see that $\dot{\top}(\xi) = \xi$. □

Lemma 5.0.4. *If* $e \models_? \xi$ *then* $e \models_?^\dagger \dot{\top}(\xi)$*.*

Proof.

(11) $e \models_? \xi$ — by assumption

By Rule Induction over Rules (9) on (11).

Case (9a)**.**

(12) $\xi = ?$ — by assumption

(13) $e \models \top$ — by Rule (7a)

(14) $e \models_?^\dagger \top$ — by Rule (10b) on (13)

Case (9b)**.**

(12) e `notintro` — by assumption

(13) ξ `refutable` — by assumption

By Lemma 1.0.15 on (12) and (13) and case analysis on its conclusion. By rule induction over Rules (3).

Case $\dot{\top}(\xi)$ `refutable`**.**

(14) $\dot{\top}(\xi)$ `refutable`	by assumption
(15) $e \models_? \dot{\top}(\xi)$	by Rule (9b) on (12) and (14)
(16) $e \models_?^{\dagger} \dot{\top}(\xi)$	by Rule (10b) on (15)

Case $e \models \dot{\top}(\xi)$**.**

(14) $e \models \dot{\top}(\xi)$	by assumption
(15) $e \models_?^{\dagger} \top$	by Rule (10b) on (14)

Case (9c)**.**

(12) $\xi = \xi_1 \vee \xi_2$	by assumption
(13) $e \models_? \xi_1$	by assumption
(14) $e \models_?^{\dagger} \dot{\top}(\xi_1)$	by IH on (13)
(15) $e \models_?^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 1.0.10 on (14)

Case (9d)**.**

(12) $\xi = \xi_1 \vee \xi_2$	by assumption
(13) $e \models_? \xi_2$	by assumption
(14) $e \models_?^{\dagger} \dot{\top}(\xi_2)$	by IH on (13)
(15) $e \models_?^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 1.0.10 on (14)

Case (9e)**.**

(12) $e = \texttt{inl}_{\tau_2}(e_1)$	by assumption
(13) $\xi = \texttt{inl}(\xi_1)$	by assumption
(14) $e_1 \models_? \xi_1$	by assumption
(15) $e_1 \models_?^{\dagger} \dot{\top}(\xi_1)$	by IH on (14)
(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^{\dagger} \texttt{inl}(\dot{\top}(\xi_1))$	by Lemma 1.0.11 on (15)

Case (9f)**.**

(12) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption
(13) $\xi = \texttt{inr}(\xi_2)$	by assumption

(14)	$e_2 \models_? \xi_2$	by assumption
(15)	$e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (14)
(16)	$\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\top}(\xi_2))$	by Lemma 1.0.12 on (15)

Case (9g).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_2 \models \xi_2$	by assumption
(16)	$e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (14)
(17)	$e_2 \models \dot{\top}(\xi_2)$	by Lemma 5.0.3 on (15)
(18)	$e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by Rule (10b) on (17)
(19)	$(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 1.0.13 on (16) and (18)

Case (9h).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models \xi_1$	by assumption
(15)	$e_2 \models_? \xi_2$	by assumption
(16)	$e_1 \models \dot{\top}(\xi_1)$	by Lemma 5.0.3 on (14)
(17)	$e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by Rule (10b) on (16)
(18)	$e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (15)
(19)	$(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 1.0.13 on (17) and (18)

Case (9i).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_2 \models_? \xi_2$	by assumption

(16) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by IH on (14)

(17) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by IH on (15)

(18) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Lemma 1.0.13 on (16) and (17)

$\square$

Lemma 5.0.5. $e \models^{\dagger}_{?} \xi$ *iff* $e \models^{\dagger}_{?} \dot{\top}(\xi)$

Proof. 1. Sufficiency:

(1) $e \models^{\dagger}_{?} \xi$ by assumption

By rule induction over Rules (10) on (1)

Case (10b)**.**

(2) $e \models \xi$ by assumption

(3) $e \models \dot{\top}(\xi)$ by Lemma 5.0.3 on (2)

(4) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by Rule (10b) on (3)

Case (10a)**.**

(2) $e \models_{?} \xi$ by assumption

(3) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by Lemma 5.0.4 on (2)

2. Necessity:

(1) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by assumption

By structural induction on ξ,

Case $\xi = \top, \bot, \underline{n}, \underline{\cancel{n}}$**.**

(2) $e \models^{\dagger}_{?} \xi$ by (1) and Definition 5

Case $\xi = ?$**.**

(2) $e \models_{?} ?$ by Rule (9a)

(3) $e \models^{\dagger}_{?} ?$ by Rule (10a) on (2)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Definition 5

By rule induction over Rules (10) on (1),

Case (10b).

(3) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (7) on (3) and two cases apply,

Case (7e).

(4) $e \models \dot{\top}(\xi_1)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (10b) on (4)

(6) $e \models_?^\dagger \xi_1$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.10 on (6)

Case (7f).

(4) $e \models \dot{\top}(\xi_2)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (10b) on (4)

(6) $e \models_?^\dagger \xi_2$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.10 on (6)

Case (10a).

(3) $e \models_? \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (9) on (3) and two cases apply,

Case (9c).

(4) $e \models_? \dot{\top}(\xi_1)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (10a) on (4)

(6) $e \models_?^\dagger \xi_1$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.10 on (6)

Case (9d).

(4) $e \models_? \dot{\top}(\xi_2)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (10a) on (4)

(6) $e \models_?^\dagger \xi_2$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.10 on (6)

Case $\xi = \texttt{inl}(\xi_1)$.

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (10) on (1),

Case (10b).

(4) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (7) and only one case applies,

Case (7g).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (10b) on (5)

(7) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (6)

(8) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\xi_1)$ by Lemma 1.0.11 on (7)

Case (10a).

(4) $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (9) and only one case applies,

Case (9e).

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (10a) on (5)

(7) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (6)

(8) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\xi_1)$ by Lemma 1.0.11 on (7)

Case $\xi = \mathtt{inr}(\xi_2)$.

(2) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\dot{\top}(\xi) = \mathtt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (10) on (1),

Case (10b).

(4) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (7) and only one case applies,

Case (7h).

(5) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (10b) on (5)

(7) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (6)

(8) $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inr}(\xi_2)$ by Lemma 1.0.12 on (7)

Case (10a).

(4) $\mathtt{inr}_{\tau_1}(e_2) \models_{?} \mathtt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (9) and only one case applies,

Case (9f).

(5) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (10a) on (5)

(7) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (6)

(8) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 1.0.12 on (7)

Case $\xi = (\xi_1, \xi_2)$.

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by Definition 5

By rule induction over Rules (10) on (1),

Case (10b).

(4) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (7) on (4) and only one case applies,

Case (7i).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (10b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (10b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.13 on (9) and (10)

Case (10a).

(4) $(e_1, e_2) \models_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (9) on (4) and three cases apply,

Case (9g).

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (10a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (10b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.13 on (9) and (10)

Case (9h).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (10b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (10a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.13 on (9) and (10)

Case (9i)**.**

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (10a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (10a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.13 on (9) and (10)

$\square$

Lemma 5.0.6. *Assume* $\dot{\top}(\xi) = \xi$*. Then* $\top \models^{\dagger}_{?} \xi$ *iff* $\top \models \xi$*.*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

2. Necessity:

$\square$

Theorem 5.1. $\top \models^{\dagger}_{?} \xi$ *iff* $\top \models \dot{\top}(\xi)$*.*

Lemma 5.1.1. *Assume that* e `val`*. Then* $e \models^{\dagger}_{?} \xi$ *iff* $e \models \dot{\top}(\xi)$

Proof.

(1) e `val` by assumption

We prove sufficiency and necessity separately.

1. Sufficiency:

   (2) $e \models^{\dagger}_{?} \xi$ by assumption

   By rule induction over Rules (10) on (2).

   Case (10b)**.**

   (3) $e \models \xi$ by assumption

   (4) $e \models \dot{\top}(\xi)$ by Lemma 5.0.3 on (3)

   Case (10a)**.**

   (3) $e \models_{?} \xi$ by assumption

By rule induction over Rules (9) on (3).

Case (9a).

(4) $\xi = ?$	by assumption
(5) $e \models \dot{\top}(\xi)$	by Rule (7a) and Definition 5

Case (9b).

(4) e `notintro`	by assumption

By rule induction over Rules (19) on (4), for each case, by rule induction over Rules (16) on (1), no case applies due to syntactic contradiction.

Case (9c).

(4) $\xi = \xi_1 \vee \xi_2$	by assumption
(5) $e \models_? \xi_1$	by assumption
(6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Equation 5
(7) $e \models_?^\dagger \xi_1$	by Rule (10a) on (5)
(8) $e \models \dot{\top}(\xi_1)$	by IH on (7)
(9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (7e) on (8)

Case (9d).

(4) $\xi = \xi_1 \vee \xi_2$	by assumption
(5) $e \models_? \xi_2$	by assumption
(6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Equation 5
(7) $e \models_?^\dagger \xi_2$	by Rule (10a) on (5)
(8) $e \models \dot{\top}(\xi_2)$	by IH on (7)
(9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (7f) on (8)

Case (9e).

(4) $\xi = \texttt{inl}(\xi_1)$	by assumption
(5) $e \models_? \xi_1$	by assumption
(6) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$	by Equation 5
(7) $e \models_?^\dagger \xi_1$	by Rule (10a) on (5)
(8) $e \models \dot{\top}(\xi_1)$	by IH on (7)
(9) $e \models \texttt{inl}(\dot{\top}(\xi_1))$	by Rule (7g) on (8)

Case (9f).

(4) $\xi = \texttt{inr}(\xi_2)$	by assumption
(5) $e \models_? \xi_2$	by assumption
(6) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$	by Equation 5
(7) $e \models_?^\dagger \xi_2$	by Rule (10a) on (5)

(8)	$e \models \dot{\top}(\xi_2)$	by IH on (7)
(9)	$e \models \texttt{inr}(\dot{\top}(\xi_2))$	by Rule (7h) on (8)

Case (9g).

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models_? \xi_1$	by assumption
(7)	$e_2 \models \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 5
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (10a) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (10b) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (7i) on (11) and (12)

Case (9h).

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models \xi_1$	by assumption
(7)	$e_2 \models_? \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 5
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (10b) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (10a) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (7i) on (11) and (12)

Case (9i).

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models_? \xi_1$	by assumption
(7)	$e_2 \models_? \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 5
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (10a) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (10a) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (7i) on (11) and (12)

2. Necessity:

(2) $e \models \dot{\top}(\xi)$	by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\cancel{n}}$**.**

(3) $\xi = \dot{\top}(\xi)$	by Equation 5
(4) $e \models^{\dagger}_{?} \xi$	by Rule (10b) on (2)

Case $\xi = ?$**.**

(3) $e \models_{?} ?$	by Rule (9a)
(4) $e \models^{\dagger}_{?} ?$	by Rule (10a) on (3)

Case $\xi = \xi_1 \wedge \xi_2$**.**

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$	by Equation 5

By rule induction over Rules (7) on (2), only one case applies.

Case (7d)**.**

(4) $e \models \dot{\top}(\xi_1)$	by assumption
(5) $e \models \dot{\top}(\xi_2)$	by assumption
(6) $e \models^{\dagger}_{?} \xi_1$	by IH on (4)
(7) $e \models^{\dagger}_{?} \xi_2$	by IH on (5)
(8) $e \models \xi_1 \wedge \xi_2$	by Lemma 1.0.9 on (6) and (7)

Case $\xi = \xi_1 \vee \xi_2$**.**

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Equation 5

By rule induction over Rules (7) on (2) and only two cases apply.

Case (7e)**.**

(4) $e \models \dot{\top}(\xi_1)$	by assumption
(5) $e \models^{\dagger}_{?} \xi_1$	by IH on (4)
(6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$	by Lemma 1.0.10 on (5)

Case (7f)**.**

(4) $e \models \dot{\top}(\xi_2)$	by assumption
(5) $e \models^{\dagger}_{?} \xi_2$	by IH on (4)

(6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 1.0.10 on (5)

Case $\xi = \texttt{inl}(\xi_1)$**.**

(3) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ by Equation 5

By rule induction over Rules (7) on (2) and only one case applies.

Case (7g)**.**

(4) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (5)

(7) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$ by Lemma 1.0.11 on (6)

Case $\xi = \texttt{inr}(\xi_2)$**.**

(3) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$ by Equation 5

By rule induction over Rules (7) on (2) and only one case applies.

Case (7h)**.**

(4) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(5) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (5)

(7) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 1.0.12 on (6)

Case $\xi = (\xi_1, \xi_2)$**.**

(3) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Equation 5

By rule induction over Rules (7) on (2) and only one case applies.

Case (7i)**.**

(4) $e = (e_1, e_2)$ by assumption

(5) $e_1 \models \dot{\bot}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\bot}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (5)

(8) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (6)

(9) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.13 on (7) and (8)

□

Lemma 5.1.2. *$e \models \xi$ iff $e \models \dot{\bot}(\xi)$*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models \xi$ — by assumption

By rule induction over Rules (7) on (1).

Case (7a)**.**

(2)	$\xi = \top$	by assumption
(3)	$e \models \dot{\bot}(\top)$	by (1) and Definition 6

Case (7b)**.**

(2)	$\xi = \underline{n}$	by assumption
(3)	$e \models \dot{\bot}(\underline{n})$	by (1) and Definition 6

Case (7c)**.**

(2)	$\xi = \underline{\not{n}}$	by assumption
(3)	$e \models \dot{\bot}(\underline{\not{n}})$	by (1) and Definition 6

Case (7d)**.**

(2)	$\xi = \xi_1 \wedge \xi_2$	by assumption
(3)	$e \models \xi_1$	by assumption
(4)	$e \models \xi_2$	by assumption
(5)	$e \models \dot{\bot}(\xi_1)$	by IH on (3)
(6)	$e \models \dot{\bot}(\xi_2)$	by IH on (4)
(7)	$e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$	by Rule (7d) on (5) and (6)
(8)	$e \models \dot{\bot}(\xi_1 \wedge \xi_2)$	by (7) and Definition 6

Case (7e)**.**

(2)	$\xi = \xi_1 \vee \xi_2$	by assumption
(3)	$e \models \xi_1$	by assumption
(4)	$e \models \dot{\bot}(\xi_1)$	by IH on (3)
(5)	$e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (7e) on (4)
(6)	$e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 6

Case (7f)**.**

(2)	$\xi = \xi_1 \vee \xi_2$	by assumption
(3)	$e \models \xi_2$	by assumption
(4)	$e \models \dot{\bot}(\xi_2)$	by IH on (3)
(5)	$e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (7f) on (4)

(6) $e \models \dot{\bot}(\xi_1 \vee \xi_2)$ by (5) and Definition 6

Case (7g)**.**

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\xi = \texttt{inl}(\xi_1)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $e_1 \models \dot{\bot}(\xi_1)$ by IH on (4)

(6) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\bot}(\xi_1))$ by Rule (7g) on (5)

(7) $\texttt{inl}_{\tau_2}(e_1) \models \dot{\bot}(\texttt{inl}(\xi_1))$ by (6) and Definition 6

Case (7h)**.**

(2) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\xi = \texttt{inr}(\xi_2)$ by assumption

(4) $e_2 \models \xi_2$ by assumption

(5) $e_2 \models \dot{\bot}(\xi_2)$ by IH on (4)

(6) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\bot}(\xi_2))$ by Rule (7h) on (5)

(7) $\texttt{inr}_{\tau_1}(e_2) \models \dot{\bot}(\texttt{inr}(\xi_2))$ by (6) and Definition 6

Case (7i)**.**

(2) $e = (e_1, e_2)$ by assumption

(3) $\xi = (\xi_1, \xi_2)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $e_2 \models \xi_2$ by assumption

(6) $e_1 \models \dot{\bot}(\xi_1)$ by IH on (4)

(7) $e_2 \models \dot{\bot}(\xi_2)$ by IH on (5)

(8) $(e_1, e_2) \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$ by Rule (7i) on (6) and (7)

(9) $(e_1, e_2) \models \dot{\bot}((\xi_1, \xi_2))$ by (8) and Definition 6

2. Necessity:

(1) $e \models \dot{\bot}(\xi)$ by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \not{\underline{n}}$**.**

(2) $e \models \xi$ by (1) and Definition 6

Case $\xi = ?$**.**

(2) $e \models \bot$ by (1) and Definition 6

(3) $e \not\models \bot$ by Lemma 1.0.1

(3) contradicts (2).

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$ by (1) and Definition 6

By rule induction over Rules (7) on (2) and only case applies.

Case (7d)**.**

(3) $e \models \dot{\bot}(\xi_1)$ by assumption

(4) $e \models \dot{\bot}(\xi_2)$ by assumption

(5) $e \models \xi_1$ by IH on (3)

(6) $e \models \xi_2$ by IH on (4)

(7) $e \models \xi_1 \wedge \xi_2$ by Rule (7d) on (5) and (6)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$ by (1) and Definition 6

By rule induction over Rules (7) on (2) and only two cases apply.

Case (7e)**.**

(3) $e \models \dot{\bot}(\xi_1)$ by assumption

(4) $e \models \xi_1$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (7e) on (4)

Case (7f)**.**

(3) $e \models \dot{\bot}(\xi_2)$ by assumption

(4) $e \models \xi_2$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (7f) on (4)

Case $\xi = \texttt{inl}(\xi_1)$**.**

(2) $e \models \texttt{inl}(\dot{\bot}(\xi_1))$ by (1) and Definition 6

By rule induction over Rules (7) on (2) and only one case applies.

Case (7g)**.**

(3) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(4) $e_1 \models \dot{\bot}(\xi_1)$ by assumption

(5) $e_1 \models \xi_1$ by IH on (4)

(6) $e \models \texttt{inl}(\xi_1)$ by Rule (7g) on (5)

Case $\xi = \texttt{inr}(\xi_2)$**.**

(2) $e \models \texttt{inr}(\dot{\bot}(\xi_2))$ by (1) and Definition 6

By rule induction over Rules (7) on (2) and only one case applies.

Case (7h).

(3) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption
(4) $e_2 \models \dot{\bot}(\xi_2)$	by assumption
(5) $e_2 \models \xi_2$	by IH on (4)
(6) $e \models \texttt{inr}(\xi_2)$	by Rule (7h) on (5)

Case $\xi = (\xi_1, \xi_2)$**.**

(2) $e \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$	by (1) and Definition 6

By rule induction over Rules (7) on (2) and only case applies.

Case (7i).

(3) $e = (e_1, e_2)$	by assumption
(4) $e_1 \models \dot{\bot}(\xi_1)$	by assumption
(5) $e_2 \models \dot{\bot}(\xi_2)$	by assumption
(6) $e_1 \models \xi_1$	by IH on (4)
(7) $e_2 \models \xi_2$	by IH on (5)
(8) $e \models (\xi_1, \xi_2)$	by Rule (7i) on (6) and (7)

□

Lemma 5.1.3. *Assume* e `val` *and* $\dot{\top}(\xi) = \xi$*. Then* $e \not\models \xi$ *iff* $e \models \overline{\xi}$*.*

Theorem 5.2. $\xi_r \models \xi_{rs}$ *iff* $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\bot}(\xi_{rs})$.