1 Match Constraint Language

$\dot{\xi} \;::=\; \top \mid ? \mid \underline{n} \mid \texttt{inl}(\dot{\xi}) \mid \texttt{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$

$\boxed{\dot{\xi} : \tau}$ $\dot{\xi}$ constrains final expressions of type τ

$$\frac{}{\top : \tau}\;\text{CTTruth} \tag{1a}$$

$$\frac{}{? : \tau}\;\text{CTUnknown} \tag{1b}$$

$$\frac{}{\underline{n} : \texttt{num}}\;\text{CTNum} \tag{1c}$$

$$\frac{\dot{\xi}_1 : \tau_1}{\texttt{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)}\;\text{CTInl} \tag{1d}$$

$$\frac{\dot{\xi}_2 : \tau_2}{\texttt{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)}\;\text{CTInr} \tag{1e}$$

$$\frac{\dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)}\;\text{CTPair} \tag{1f}$$

$$\frac{\dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau}\;\text{CTOr} \tag{1g}$$

$\boxed{\dot{\xi}\ \texttt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\;\text{RXNum} \tag{2a}$$

$$\frac{}{?\ \texttt{refutable}_?}\;\text{RXUnknown} \tag{2b}$$

$$\frac{}{\texttt{inl}(\dot{\xi})\ \texttt{refutable}_?}\;\text{RXInl} \tag{2c}$$

$$\frac{}{\texttt{inr}(\dot{\xi})\ \texttt{refutable}_?}\;\text{RXInr} \tag{2d}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\;\text{RXPairL} \tag{2e}$$

$$\frac{\dot{\xi}_2 \ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \ \texttt{refutable}_?} \ \text{RXPairR} \tag{2f}$$

$$\frac{\dot{\xi}_1 \ \texttt{refutable}_? \qquad \dot{\xi}_2 \ \texttt{refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2 \ \texttt{refutable}_?} \ \text{RXOr} \tag{2g}$$

$\boxed{\textit{refutable}_?(\dot{\xi})}$

$$\textit{refutable}_?(\top) = \text{false} \tag{3a}$$
$$\textit{refutable}_?(\underline{n}) = \text{true} \tag{3b}$$
$$\textit{refutable}_?(?) = \text{true} \tag{3c}$$
$$\textit{refutable}_?(\texttt{inl}(\dot{\xi})) = \text{true} \tag{3d}$$
$$\textit{refutable}_?(\texttt{inr}(\dot{\xi})) = \text{true} \tag{3e}$$
$$\textit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \textit{refutable}_?(\dot{\xi}_1) \text{ or } \textit{refutable}_?(\dot{\xi}_2) \tag{3f}$$
$$\textit{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \textit{refutable}_?(\dot{\xi}_1) \text{ and } \textit{refutable}_?(\dot{\xi}_2) \tag{3g}$$

Lemma 1.0.1 (Soundness and Completeness of Refutable Constraints). $\dot{\xi}$ $\texttt{refutable}_?$ *iff* $\textit{refutable}_?(\dot{\xi}) = \textit{true}$.

$\boxed{e \dot{\models} \dot{\xi}}$ $\quad e$ satisfies $\dot{\xi}$

$$\frac{}{e \dot{\models} \top} \ \text{CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \dot{\models} \underline{n}} \ \text{CSNum} \tag{4b}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \dot{\models} \texttt{inl}(\dot{\xi}_1)} \ \text{CSInl} \tag{4c}$$

$$\frac{e_2 \dot{\models} \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \dot{\models} \texttt{inr}(\dot{\xi}_2)} \ \text{CSInr} \tag{4d}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1 \qquad e_2 \dot{\models} \dot{\xi}_2}{(e_1, e_2) \dot{\models} (\dot{\xi}_1, \dot{\xi}_2)} \ \text{CSPair} \tag{4e}$$

$$\frac{e \ \texttt{notintro} \qquad \texttt{fst}(e) \dot{\models} \dot{\xi}_1 \qquad \texttt{snd}(e) \dot{\models} \dot{\xi}_2}{e \dot{\models} (\dot{\xi}_1, \dot{\xi}_2)} \ \text{CSNotIntroPair} \tag{4f}$$

$$\frac{e \dot{\models} \dot{\xi}_1}{e \dot{\models} \dot{\xi}_1 \vee \dot{\xi}_2} \text{ CSOrL} \tag{4g}$$

$$\frac{e \dot{\models} \dot{\xi}_2}{e \dot{\models} \dot{\xi}_1 \vee \dot{\xi}_2} \text{ CSOrR} \tag{4h}$$

$\boxed{\mathit{satisfy}(e, \dot{\xi})}$

$$\mathit{satisfy}(e, \top) = \text{true} \tag{5a}$$

$$\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{5b}$$

$$\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1) \text{ or } \mathit{satisfy}(e, \dot{\xi}_2) \tag{5c}$$

$$\mathit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\dot{\xi}_1)) = \mathit{satisfy}(e_1, \dot{\xi}_1) \tag{5d}$$

$$\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\dot{\xi}_2)) = \mathit{satisfy}(e_2, \dot{\xi}_2) \tag{5e}$$

$$\mathit{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2) \tag{5f}$$

$$\mathit{satisfy}(⦇⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(\mathtt{fst}(⦇⦈^u), \dot{\xi}_1) \text{ and } \mathit{satisfy}(\mathtt{snd}(⦇⦈^u), \dot{\xi}_2) \tag{5g}$$

$$\mathit{satisfy}(⦇e⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(\mathtt{fst}(⦇e⦈^u), \dot{\xi}_1) \text{ and } \mathit{satisfy}(\mathtt{snd}(⦇e⦈^u), \dot{\xi}_2) \tag{5h}$$

$$\begin{aligned}\mathit{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{fst}(e_1(e_2)), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{snd}(e_1(e_2)), \dot{\xi}_2)\end{aligned} \tag{5i}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{match}(e)\{\hat{rs}\}, (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{fst}(\mathtt{match}(e)\{\hat{rs}\}), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{snd}(\mathtt{match}(e)\{\hat{rs}\}), \dot{\xi}_2)\end{aligned} \tag{5j}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{fst}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{fst}(\mathtt{fst}(e)), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{snd}(\mathtt{fst}(e)), \dot{\xi}_2)\end{aligned} \tag{5k}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{snd}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{fst}(\mathtt{snd}(e)), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{snd}(\mathtt{snd}(e)), \dot{\xi}_2)\end{aligned} \tag{5l}$$

$$\text{Otherwise} \quad \mathit{satisfy}(e, \dot{\xi}) = \text{false} \tag{5m}$$

Lemma 1.0.2 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \dot{\models} \dot{\xi}$ iff $\mathit{satisfy}(e, \dot{\xi}) = \mathit{true}$.*

$\boxed{e \dot{\models}_? \dot{\xi}}$ e may satisfy $\dot{\xi}$

$$\frac{}{e \dot{\models}_? ?} \text{ CMSUnknown} \tag{6a}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1}{\mathtt{inl}_{\tau_2}(e_1) \dot{\models}_? \mathtt{inl}(\dot{\xi}_1)} \text{ CMSInl} \tag{6b}$$

$$\frac{e_2 \dot{\models}_? \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \dot{\models}_? \texttt{inr}(\dot{\xi}_2)}\ \text{CMSInr} \tag{6c}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1 \qquad e_2 \dot{\models} \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairL} \tag{6d}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1 \qquad e_2 \dot{\models}_? \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairR} \tag{6e}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1 \qquad e_2 \dot{\models}_? \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPair} \tag{6f}$$

$$\frac{e \dot{\models}_? \dot{\xi}_1 \qquad e \,\dot{\not\models}\, \dot{\xi}_2}{e \dot{\models}_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrL} \tag{6g}$$

$$\frac{e \,\dot{\not\models}\, \dot{\xi}_1 \qquad e \dot{\models}_? \dot{\xi}_2}{e \dot{\models}_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrR} \tag{6h}$$

$$\frac{e\ \texttt{notintro} \qquad \dot{\xi}\ \texttt{refutable}_?}{e \dot{\models}_? \dot{\xi}}\ \text{CMSNotIntro} \tag{6i}$$

$\boxed{\mathit{maysatisfy}(e, \dot{\xi})}$

$$\mathit{maysatisfy}(e, ?) = \text{true} \tag{7a}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \mathit{maysatisfy}(e_1, \dot{\xi}_1) \tag{7b}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \mathit{maysatisfy}(e_2, \dot{\xi}_2) \tag{7c}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inr}(\dot{\xi}_2)) = \text{false} \tag{7d}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inl}(\dot{\xi}_1)) = \text{false} \tag{7e}$$

$$\begin{aligned}\mathit{maysatisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big)\end{aligned} \tag{7f}$$

$$\begin{aligned}\mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = {} & \Big(\mathit{maysatisfy}(e, \dot{\xi}_1) \text{ and } \Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_2)\Big)\Big) \\ & \text{or } \Big(\Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_1)\Big) \text{ and } \mathit{maysatisfy}(e, \dot{\xi}_2)\Big)\end{aligned} \tag{7g}$$

$$\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{notintro}(e) \text{ and } \mathit{refutable}_?(\dot{\xi}) \tag{7h}$$

Lemma 1.0.3 (Soundness and Completeness of Maybe Satisfaction Judgment)**.** *$e \dot{\models}_? \dot{\xi}$ iff $\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{true}$.*

$\boxed{e \dot{\models}^{\dagger}_? \dot{\xi}}$ e satisfies or may satisfy $\dot{\xi}$

$$\text{CSMSMay}\quad \frac{e \dot{\models}_? \dot{\xi}}{e \dot{\models}^{\dagger}_? \dot{\xi}} \tag{8a}$$

$$\text{CSMSSat}\quad \frac{e \dot{\models} \dot{\xi}}{e \dot{\models}^{\dagger}_? \dot{\xi}} \tag{8b}$$

$\boxed{\mathit{satisfyormay}(e, \dot{\xi})}$

$$\mathit{satisfyormay}(e, \dot{\xi}) = \mathit{satisfy}(e, \dot{\xi}) \text{ or } \mathit{maysatisfy}(e, \dot{\xi}) \tag{9}$$

Lemma 1.0.4 (Soundness and Completeness of Satisfaction or Maybe Satisfaction)**.** *$e \dot{\models}^{\dagger}_? \dot{\xi}$ iff $\mathit{satisfyormay}(e, \dot{\xi})$.*

Lemma 1.0.5. *If $\dot{\xi} : \tau$ then there exists e such that e `val` and $\cdot ; \Delta \vdash e : \tau$ and $e \dot{\models}^{\dagger}_? \dot{\xi}$.*

Lemma 1.0.6. *$e \dot{\nvDash}_? \top$*

Lemma 1.0.7. *$e \dot{\nvDash} ?$*

Lemma 1.0.8. *$e \dot{\nvDash}^{\dagger}_? \dot{\xi}_1$ and $e \dot{\nvDash}^{\dagger}_? \dot{\xi}_2$ iff $e \dot{\nvDash}^{\dagger}_? \dot{\xi}_1 \vee \dot{\xi}_2$*

Lemma 1.0.9. *If $e \dot{\models}^{\dagger}_? \dot{\xi}_1 \vee \dot{\xi}_2$ and $e \dot{\nvDash}^{\dagger}_? \dot{\xi}_1$ then $e \dot{\models}^{\dagger}_? \dot{\xi}_2$*

Lemma 1.0.10. *If $e \dot{\models}^{\dagger}_? \dot{\xi}_1$ then $e \dot{\models}^{\dagger}_? \dot{\xi}_1 \vee \dot{\xi}_2$ and $e \dot{\models}^{\dagger}_? \dot{\xi}_2 \vee \dot{\xi}_1$*

Lemma 1.0.11. *$e_1 \dot{\models}^{\dagger}_? \dot{\xi}_1$ iff $\texttt{inl}_{\tau_2}(e_1) \dot{\models}^{\dagger}_? \texttt{inl}(\dot{\xi}_1)$*

Lemma 1.0.12. *$e_2 \dot{\models}^{\dagger}_? \dot{\xi}_2$ iff $\texttt{inr}_{\tau_1}(e_2) \dot{\models}^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$*

Lemma 1.0.13. *$e_1 \dot{\models}^{\dagger}_? \dot{\xi}_1$ and $e_2 \dot{\models}^{\dagger}_? \dot{\xi}_2$ iff $(e_1, e_2) \dot{\models}^{\dagger}_? (\dot{\xi}_1, \dot{\xi}_2)$*

Lemma 1.0.14. *Assume e `notintro`. If $e \dot{\models}_? \dot{\xi}$ or $e \dot{\nvDash} \dot{\xi}$ then $\dot{\xi}$ $\texttt{refutable}_?$.*

Lemma 1.0.15. *If e `notintro` and $e \dot{\models} \dot{\xi}$ then $\cancel{\dot{\xi}\ \texttt{refutable}_?}$.*

Lemma 1.0.16. *$\texttt{inl}_{\tau_2}(e_1) \dot{\models}^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$ is not derivable.*

Lemma 1.0.17. *$\texttt{inr}_{\tau_1}(e_2) \dot{\models}^{\dagger}_? \texttt{inl}(\dot{\xi}_1)$ is not derivable.*

Lemma 1.0.18. *$e \dot{\nvDash} \dot{\xi}$ and $e \dot{\nvDash}_? \dot{\xi}$ iff $e \dot{\nvDash}^{\dagger}_? \dot{\xi}$.*

Theorem 1.1 (Exclusiveness of Satisfaction Judgment). *If $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \dot{\models} \dot{\xi}$
2. $e \dot{\models}_{?} \dot{\xi}$
3. $e \dot{\not\models}^{\dagger}_{?} \dot{\xi}$

Definition 1.1.1 (Entailment of Constraints). *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \dot{\models} \dot{\xi}_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e `val` we have $e \dot{\models}^{\dagger}_{?} \dot{\xi}_1$ implies $e \dot{\models} \dot{\xi}_2$*

Definition 1.1.2 (Potential Entailment of Constraints). *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \dot{\models}^{\dagger}_{?} \dot{\xi}_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \dot{\models}^{\dagger}_{?} \dot{\xi}_1$ implies $e \dot{\models}^{\dagger}_{?} \dot{\xi}_2$*

Corollary 1.1.1. *Suppose that $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e `final`. Then $\top \dot{\models}^{\dagger}_{?} \dot{\xi}$ implies $e \dot{\models}^{\dagger}_{?} \dot{\xi}$*

2 Normal Match Constraint Language

$\xi ::= \top \mid \bot \mid \underline{n} \mid \cancel{\underline{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\text{CTTruth} \quad \frac{}{\top : \tau} \tag{10a}$$

$$\text{CTFalsity} \quad \frac{}{\bot : \tau} \tag{10b}$$

$$\text{CTNum} \quad \frac{}{\underline{n} : \texttt{num}} \tag{10c}$$

$$\text{CTNotNum} \quad \frac{}{\cancel{\underline{n}} : \texttt{num}} \tag{10d}$$

$$\text{CTAnd} \quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \tag{10e}$$

$$\text{CTOr} \quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{10f}$$

$$\text{CTInl} \quad \frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{10g}$$

$$\frac{\xi_2 : \tau_2}{\mathtt{inr}(\xi_2) : (\tau_1 + \tau_2)}\ \text{CTInr} \tag{10h}$$

$$\frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)}\ \text{CTPair} \tag{10i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$
\begin{aligned}
\overline{\top} &= \bot \\
\overline{\bot} &= \top \\
\overline{\underline{n}} &= \cancel{\underline{n}} \\
\overline{\cancel{\underline{n}}} &= \underline{n} \\
\overline{\xi_1 \wedge \xi_2} &= \overline{\xi_1} \vee \overline{\xi_2} \\
\overline{\xi_1 \vee \xi_2} &= \overline{\xi_1} \wedge \overline{\xi_2} \\
\overline{\mathtt{inl}(\xi_1)} &= \mathtt{inl}(\overline{\xi_1}) \vee \mathtt{inr}(\top) \\
\overline{\mathtt{inr}(\xi_2)} &= \mathtt{inr}(\overline{\xi_2}) \vee \mathtt{inl}(\top) \\
\overline{(\xi_1, \xi_2)} &= (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \\
\overline{\overline{\xi}} &= \xi
\end{aligned}
$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\frac{}{e \models \top}\ \text{CSTruth} \tag{12a}$$

$$\frac{}{\underline{n} \models \underline{n}}\ \text{CSNum} \tag{12b}$$

$$\frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}}\ \text{CSNotNum} \tag{12c}$$

$$\frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2}\ \text{CSAnd} \tag{12d}$$

$$\frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2}\ \text{CSOrL} \tag{12e}$$

$$\frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2}\ \text{CSOrR} \tag{12f}$$

$$\frac{e_1 \models \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)} \text{ CSInl} \tag{12g}$$

$$\frac{e_2 \models \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)} \text{ CSInr} \tag{12h}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \text{ CSPair} \tag{12i}$$

Lemma 2.0.1. *Assume e $\texttt{val}$. Then $e \not\models \xi$ iff $e \models \overline{\xi}$.*

Theorem 2.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e $\texttt{val}$ then exactly one of the following holds*

1. $e \models \xi$
2. $e \models \overline{\xi}$

Lemma 2.1.1. *$e \models \overline{\xi_1} \vee \xi_2$ iff $e \models \xi_2$ whenever $e \models \xi_1$.*

Definition 2.1.1 (Entailment of Constraints)**.** *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot;\Delta \vdash e : \tau$ and e $\texttt{val}$ we have $e \models \xi_1$ implies $e \models \xi_2$*

Corollary 2.1.1 (Material Entailment of Complete Constraint)**.** *$\xi_1 \models \xi_2$ iff $\top \models \overline{\xi_1} \vee \xi_2$.*

2.1 Relationship with Incomplete Constraint Language

Lemma 2.1.2. *Assume that e $\texttt{val}$. Then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ iff $e \models \dot{\top}(\dot{\xi})$.*

Lemma 2.1.3. *$e \dot{\models} \dot{\xi}$ iff $e \models \dot{\bot}(\dot{\xi})$*

Lemma 2.1.4. *Suppose $\dot{\xi} : \tau$. Then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\texttt{final}$ **iff** $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\texttt{val}$.*

Theorem 2.2. *$\top \dot{\models}^{\dagger}_{?} \dot{\xi}$ iff $\top \models \dot{\top}(\dot{\xi})$.*

Theorem 2.3. *$\dot{\xi}_1 \dot{\models} \dot{\xi}_2$ iff $\dot{\top}(\dot{\xi}_1) \models \dot{\bot}(\dot{\xi}_2)$.*

3 Static Semantics

$$\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & \lambda x : \tau . e \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
 & \mid & (\!|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid _ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid (\!|\!)^w \mid (\!|p|\!)^w_\tau
\end{array}$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{13a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{13b}$$

$\boxed{\Gamma ; \Delta \vdash e : \tau}$ e is of type τ

$$\frac{}{\Gamma, x : \tau ; \Delta \vdash x : \tau}\ \text{TVar} \tag{14a}$$

$$\frac{}{\Gamma ; \Delta, u :: \tau \vdash (\!|\!)^u : \tau}\ \text{TEHole} \tag{14b}$$

$$\frac{\Gamma ; \Delta, u :: \tau \vdash e : \tau'}{\Gamma ; \Delta, u :: \tau \vdash (\!|e|\!)^u : \tau}\ \text{THole} \tag{14c}$$

$$\frac{}{\Gamma ; \Delta \vdash \underline{n} : \texttt{num}}\ \text{TNum} \tag{14d}$$

$$\frac{\Gamma, x : \tau_1 ; \Delta \vdash e : \tau_2}{\Gamma ; \Delta \vdash \lambda x : \tau_1 . e : (\tau_1 \to \tau_2)}\ \text{TLam} \tag{14e}$$

$$\frac{\Gamma ; \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash e_1(e_2) : \tau}\ \text{TAp} \tag{14f}$$

$$\frac{\Gamma ; \Delta \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)}\ \text{TPair} \tag{14g}$$

$$\frac{\Gamma ; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma ; \Delta \vdash \texttt{fst}(e) : \tau_1}\ \text{TFst} \tag{14h}$$

$$\text{TSnd}\quad \frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \texttt{snd}(e) : \tau_2} \tag{14i}$$

$$\text{TInl}\quad \frac{\Gamma\,;\Delta \vdash e : \tau_1}{\Gamma\,;\Delta \vdash \texttt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \tag{14j}$$

$$\text{TInr}\quad \frac{\Gamma\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash \texttt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \tag{14k}$$

$$\text{TMatchZPre}\quad \frac{\Gamma\,;\Delta \vdash e : \tau \qquad \Gamma\,;\Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \dot{\models}^{\dagger}_{?} \xi}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \tag{14l}$$

$$\text{TMatchNZPre}\quad \frac{\begin{array}{c}\Gamma\,;\Delta \vdash e : \tau \\ \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \qquad \Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \\ \forall e'.e' \in \texttt{values}(e) \Rightarrow e' \dot{\not\models}^{\dagger}_{?} \xi_{pre} \qquad \top \dot{\models}^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}\end{array}}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \tag{14m}$$

$\boxed{p : \tau[\xi] \dashv\!\!\mid \Gamma\,;\Delta}$ p is assigned type τ and emits constraint ξ

$$\text{PTVar}\quad \frac{}{x : \tau[\top] \dashv\!\!\mid x : \tau\,;\cdot} \tag{15a}$$

$$\text{PTWild}\quad \frac{}{\_ : \tau[\top] \dashv\!\!\mid \cdot\,;\cdot} \tag{15b}$$

$$\text{PTEHole}\quad \frac{}{(\!|\!)^{w} : \tau[?] \dashv\!\!\mid \cdot\,; w :: \tau} \tag{15c}$$

$$\text{PTHole}\quad \frac{p : \tau[\xi] \dashv\!\!\mid \Gamma\,;\Delta}{(\!|p|\!)^{w}_{\tau} : \tau'[?] \dashv\!\!\mid \Gamma\,;\Delta, w :: \tau'} \tag{15d}$$

$$\text{PTNum}\quad \frac{}{\underline{n} : \texttt{num}[\underline{n}] \dashv\!\!\mid \cdot\,;\cdot} \tag{15e}$$

$$\text{PTInl}\quad \frac{p : \tau_1[\xi] \dashv\!\!\mid \Gamma\,;\Delta}{\texttt{inl}(p) : (\tau_1 + \tau_2)[\texttt{inl}(\xi)] \dashv\!\!\mid \Gamma\,;\Delta} \tag{15f}$$

$$\text{PTInr}\quad \frac{p : \tau_2[\xi] \dashv\!\!\mid \Gamma\,;\Delta}{\texttt{inr}(p) : (\tau_1 + \tau_2)[\texttt{inr}(\xi)] \dashv\!\!\mid \Gamma\,;\Delta} \tag{15g}$$

$$\frac{p_1 : \tau_1[\xi_1] \dashv\!\!\mid \Gamma_1 \,;\, \Delta_1 \qquad p_2 : \tau_2[\xi_2] \dashv\!\!\mid \Gamma_2 \,;\, \Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\!\!\mid \Gamma_1 \uplus \Gamma_2 \,;\, \Delta_1 \uplus \Delta_2}\ \text{PTPair} \tag{15h}$$

$\boxed{\Gamma \,;\, \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

$$\frac{p : \tau[\xi] \dashv\!\!\mid \Gamma_p \,;\, \Delta_p \qquad \Gamma \uplus \Gamma_p \,;\, \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma \,;\, \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'}\ \text{CTRule} \tag{16a}$$

$\boxed{\Gamma \,;\, \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

$$\frac{\Gamma \,;\, \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \dot{\not\models} \xi_{pre}}{\Gamma \,;\, \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'}\ \text{CTOneRules} \tag{17a}$$

$$\frac{\Gamma \,;\, \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma \,;\, \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \dot{\not\models} \xi_{pre}}{\Gamma \,;\, \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'}\ \text{CTRules} \tag{17b}$$

Lemma 3.0.1. *If* $p : \tau[\xi] \dashv\!\!\mid \Gamma \,;\, \Delta$ *then* $\xi : \tau$.

Lemma 3.0.2. *If* $\cdot \,;\, \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Lemma 3.0.3. *If* $\cdot \,;\, \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Lemma 3.0.4. *If* $\Gamma \,;\, \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma \,;\, \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \dot{\not\models} \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma \,;\, \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^{\diamond} : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Lemma 3.0.5 (Substitution)**.** *If* $\Gamma, x : \tau \,;\, \Delta \vdash e_0 : \tau_0$ *and* $\Gamma \,;\, \Delta \vdash e : \tau$ *then* $\Gamma \,;\, \Delta \vdash [e/x]e_0 : \tau_0$

Lemma 3.0.6 (Simultaneous Substitution)**.** *If* $\Gamma \uplus \Gamma' \,;\, \Delta \vdash e : \tau$ *and* $\theta : \Gamma'$ *then* $\Gamma \,;\, \Delta \vdash [\theta]e : \tau$

Lemma 3.0.7 (Substitution Typing)**.** *If* $e \rhd p \dashv\!\!\mid \theta$ *and* $\cdot \,;\, \Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\!\mid \Gamma \,;\, \Delta$ *then* $\theta : \Gamma$

Proof by induction on the derivation of $e \rhd p \dashv\!\!\mid \theta$.

Theorem 3.1 (Determinism)**.** *If* $\cdot \,;\, \Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

4 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\text{VNum}\quad \frac{}{\underline{n}\ \texttt{val}} \tag{18a}$$

$$\text{VLam}\quad \frac{}{\lambda x:\tau.e\ \texttt{val}} \tag{18b}$$

$$\text{VPair}\quad \frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}} \tag{18c}$$

$$\text{VInl}\quad \frac{e\ \texttt{val}}{\texttt{inl}_\tau(e)\ \texttt{val}} \tag{18d}$$

$$\text{VInr}\quad \frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}} \tag{18e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\text{IEHole}\quad \frac{}{(\!|\,|\!)^u\ \texttt{indet}} \tag{19a}$$

$$\text{IHole}\quad \frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}} \tag{19b}$$

$$\text{IAp}\quad \frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}} \tag{19c}$$

$$\text{IPairL}\quad \frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}} \tag{19d}$$

$$\text{IPairR}\quad \frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}} \tag{19e}$$

$$\text{IPair}\quad \frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}} \tag{19f}$$

$$\text{IFst}\quad \frac{e\ \texttt{final}}{\texttt{fst}(e)\ \texttt{indet}} \tag{19g}$$

$$\text{ISnd}\quad \frac{e\ \texttt{final}}{\texttt{snd}(e)\ \texttt{indet}} \tag{19h}$$

$$\text{IInL}\quad \frac{e\ \texttt{indet}}{\texttt{inl}_\tau(e)\ \texttt{indet}} \tag{19i}$$

$$\text{IInR}\quad \frac{e\ \texttt{indet}}{\texttt{inr}_\tau(e)\ \texttt{indet}} \tag{19j}$$

$$\text{IMatch}\quad \frac{e\ \texttt{final} \qquad e \mathrel{?} p_r}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \texttt{indet}} \tag{19k}$$

$\boxed{e\ \texttt{final}}$ e is final

$$\text{FVal}\quad \frac{e\ \texttt{val}}{e\ \texttt{final}} \tag{20a}$$

$$\text{FIndet}\quad \frac{e\ \texttt{indet}}{e\ \texttt{final}} \tag{20b}$$

Lemma 4.0.1. *If* $\texttt{inl}_\tau(e)\ \texttt{final}$ *then* $e\ \texttt{final}$.

Lemma 4.0.2. *If* $\texttt{inr}_\tau(e)\ \texttt{final}$ *then* $e\ \texttt{final}$.

Lemma 4.0.3. *If* $(e_1, e_2)\ \texttt{final}$ *then* $e_1\ \texttt{final}$ *and* $e_2\ \texttt{final}$.

$\boxed{e\ \texttt{notintro}}$ e cannot be a value syntactically

$$\text{NVEHole}\quad \frac{}{(\!|\,|\!)^u\ \texttt{notintro}} \tag{21a}$$

$$\text{NVHole}\quad \frac{}{(\!|e|\!)^u\ \texttt{notintro}} \tag{21b}$$

$$\text{NVAp}\quad \frac{}{e_1(e_2)\ \texttt{notintro}} \tag{21c}$$

$$\text{NVMatch}\quad \frac{}{\texttt{match}(e)\{\hat{rs}\}\ \texttt{notintro}} \tag{21d}$$

$$\text{NVFst}\quad \frac{}{\texttt{fst}(e)\ \texttt{notintro}} \tag{21e}$$

$$\text{NVSnd}\quad \frac{}{\texttt{snd}(e)\ \texttt{notintro}} \tag{21f}$$

$\boxed{\mathit{notintro}(e)}$

$$\mathit{notintro}(\llparenthesis\rrparenthesis^u) = \text{true} \tag{22a}$$
$$\mathit{notintro}(\llparenthesis e\rrparenthesis^u) = \text{true} \tag{22b}$$
$$\mathit{notintro}(e_1(e_2)) = \text{true} \tag{22c}$$
$$\mathit{notintro}(\texttt{match}(e)\{\hat{rs}\}) = \text{true} \tag{22d}$$
$$\mathit{notintro}(\texttt{fst}(e)) = \text{true} \tag{22e}$$
$$\mathit{notintro}(\texttt{snd}(e)) = \text{true} \tag{22f}$$
$$\text{Otherwise} \quad \mathit{notintro}(e) = \text{false} \tag{22g}$$

Lemma 4.0.4 (Soundness and Completeness of NotIntro Judgment)**.** *e $\texttt{notintro}$ iff $\mathit{notintro}(e)$.*

$\boxed{e' \in \texttt{values}(e)}$ e' is one of the possible values of e

$$\frac{e\ \texttt{val} \qquad \cdot\,;\Delta \vdash e : \tau}{e \in \texttt{values}(e)}\ \text{IVVal} \tag{23a}$$

$$\frac{e\ \texttt{notintro} \qquad \cdot\,;\Delta \vdash e : \tau \qquad e'\ \texttt{val} \qquad \cdot\,;\Delta \vdash e' : \tau}{e' \in \texttt{values}(e)}\ \text{IVIndet} \tag{23b}$$

$$\frac{e' \in \texttt{values}(e)}{\lambda x : \tau.e' \in \texttt{values}(\lambda x : \tau.e)}\ \text{IVInl} \tag{23c}$$

$$\frac{e_1' \in \texttt{values}(e_1)}{\texttt{inl}_{\tau_2}(e_1') \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))}\ \text{IVInl} \tag{23d}$$

$$\frac{e_2' \in \texttt{values}(e_2)}{\texttt{inr}_{\tau_1}(e_2') \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))}\ \text{IVInr} \tag{23e}$$

$$\frac{e_1' \in \texttt{values}(e_1) \qquad e_2' \in \texttt{values}(e_2)}{(e_1', e_2') \in \texttt{values}((e_1, e_2))}\ \text{IVPair} \tag{23f}$$

Lemma 4.0.5. *If $e' \in \texttt{values}(e)$ and $\cdot\,;\Delta \vdash e : \tau$ then $\cdot\,;\Delta \vdash e' : \tau$.*

Lemma 4.0.6. *If $e' \in \texttt{values}(e)$ then e' $\texttt{val}$.*

Lemma 4.0.7. *If e $\texttt{indet}$ then there exists e' such that $e' \in \texttt{values}(e)$.*

Lemma 4.0.8. *Assume e $\texttt{final}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$. Then $e \dot{\not\models}{}^{\dagger}_{?}\dot{\xi}$ iff $\forall e'.e' \in \texttt{values}(e) \implies e' \dot{\not\models}{}^{\dagger}_{?}\dot{\xi}$.*

Lemma 4.0.9. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$ and there exists e' such that $e' \in \texttt{values}(e)$ and $e' \dot{\models}{}^{\dagger}_{?}\dot{\xi}$ then $e \dot{\models}{}^{\dagger}_{?}\dot{\xi}$.*

$\boxed{\theta : \Gamma}$ θ is of type Γ

$$\text{STEmpty}\quad \frac{}{\emptyset : \cdot} \tag{24a}$$

$$\text{STExtend}\quad \frac{\theta : \Gamma_\theta \qquad \Gamma \,;\, \Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau} \tag{24b}$$

$\boxed{p\ \texttt{refutable}_?}$ p is refutable

$$\text{RNum}\quad \frac{}{\underline{n}\ \texttt{refutable}_?} \tag{25a}$$

$$\text{REHole}\quad \frac{}{(\!|\,|\!)^w\ \texttt{refutable}_?} \tag{25b}$$

$$\text{RHole}\quad \frac{}{(\!|p|\!)^w_\tau\ \texttt{refutable}_?} \tag{25c}$$

$$\text{RInl}\quad \frac{}{\texttt{inl}(p)\ \texttt{refutable}_?} \tag{25d}$$

$$\text{RInr}\quad \frac{}{\texttt{inr}(p)\ \texttt{refutable}_?} \tag{25e}$$

$$\text{RPairL}\quad \frac{p_1\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?} \tag{25f}$$

$$\text{RPairR}\quad \frac{p_2\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?} \tag{25g}$$

$\boxed{e \rhd p \dashV \theta}$ e matches p, emitting θ

$$\text{MVar}\quad \frac{}{e \rhd x \dashV e/x} \tag{26a}$$

$$\text{MWild}\quad \frac{}{e \rhd \_ \dashV \cdot} \tag{26b}$$

$$\text{MNum}\quad \frac{}{\underline{n} \rhd \underline{n} \dashV \cdot} \tag{26c}$$

$$\text{MPair}\quad \frac{e_1 \rhd p_1 \dashV \theta_1 \qquad e_2 \rhd p_2 \dashV \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashV \theta_1 \uplus \theta_2} \tag{26d}$$

$$\frac{e \rhd p \dashv\vdash \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\vdash \theta}\ \text{MInl} \tag{26e}$$

$$\frac{e \rhd p \dashv\vdash \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\vdash \theta}\ \text{MInr} \tag{26f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \rhd p_1 \dashv\vdash \theta_1 \qquad \texttt{snd}(e) \rhd p_2 \dashv\vdash \theta_2}{e \rhd (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{26g}$$

$\boxed{e \mathrel{?} p}$ e may match p

$$\frac{}{e \mathrel{?} (\!|\,|\!)^w}\ \text{MMEHole} \tag{27a}$$

$$\frac{}{e \mathrel{?} (\!|p|\!)^w_\tau}\ \text{MMHole} \tag{27b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}_?}{e \mathrel{?} p}\ \text{MMNotIntro} \tag{27c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\vdash \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{27d}$$

$$\frac{e_1 \rhd p_1 \dashv\vdash \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{27e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{27f}$$

$$\frac{e \mathrel{?} p}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)}\ \text{MMInl} \tag{27g}$$

$$\frac{e \mathrel{?} p}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)}\ \text{MMInr} \tag{27h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{28a}$$

$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairL} \tag{28b}$$

$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairR} \tag{28c}$$

$$\frac{}{\mathtt{inr}_\tau(e) \perp \mathtt{inl}(p)}\ \text{NMConfL} \tag{28d}$$

$$\frac{}{\mathtt{inl}_\tau(e) \perp \mathtt{inr}(p)}\ \text{NMConfR} \tag{28e}$$

$$\frac{e \perp p}{\mathtt{inl}_\tau(e) \perp \mathtt{inl}(p)}\ \text{NMInl} \tag{28f}$$

$$\frac{e \perp p}{\mathtt{inr}_\tau(e) \perp \mathtt{inr}(p)}\ \text{NMInr} \tag{28g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

$$\frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u}\ \text{ITHole} \tag{29a}$$

$$\frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)}\ \text{ITApFun} \tag{29b}$$

$$\frac{e_1\ \mathtt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')}\ \text{ITApArg} \tag{29c}$$

$$\frac{e_2\ \mathtt{val}}{\lambda x : \tau . e_1(e_2) \mapsto [e_2/x]e_1}\ \text{ITAP} \tag{29d}$$

$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)}\ \text{ITPairL} \tag{29e}$$

$$\frac{e_1\ \mathtt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')}\ \text{ITPairR} \tag{29f}$$

$$\frac{(e_1, e_2)\ \mathtt{final}}{\mathtt{fst}((e_1, e_2)) \mapsto e_1}\ \text{ITFst} \tag{29g}$$

$$\frac{(e_1, e_2)\ \mathtt{final}}{\mathtt{snd}((e_1, e_2)) \mapsto e_2}\ \text{ITSnd} \tag{29h}$$

ITInl
$$\frac{e \mapsto e'}{\mathtt{inl}_\tau(e) \mapsto \mathtt{inl}_\tau(e')} \tag{29i}$$

ITInr
$$\frac{e \mapsto e'}{\mathtt{inr}_\tau(e) \mapsto \mathtt{inr}_\tau(e')} \tag{29j}$$

ITExpMatch
$$\frac{e \mapsto e'}{\mathtt{match}(e)\{\hat{rs}\} \mapsto \mathtt{match}(e')\{\hat{rs}\}} \tag{29k}$$

ITSuccMatch
$$\frac{e\ \mathtt{final} \qquad e \rhd p_r \dashv\!\mid \theta}{\mathtt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{29l}$$

ITFailMatch
$$\frac{e\ \mathtt{final} \qquad e \perp p_r}{\mathtt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \mathtt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \tag{29m}$$

Lemma 4.0.10. *If* $\mathtt{inl}_{\tau_2}(e_1)$ $\mathtt{final}$ *then* e_1 $\mathtt{final}$*.*

Lemma 4.0.11. *If* $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{final}$ *then* e_2 $\mathtt{final}$*.*

Lemma 4.0.12. *If* (e_1, e_2) $\mathtt{final}$ *then* e_1 $\mathtt{final}$ *and* e_2 $\mathtt{final}$*.*

Lemma 4.0.13. *There doesn't exist* $\underline{n}$ *such that* $\underline{n}$ $\mathtt{notintro}$*.*

Lemma 4.0.14. *There doesn't exist* $\mathtt{inl}_\tau(e)$ *such that* $\mathtt{inl}_\tau(e)$ $\mathtt{notintro}$*.*

Lemma 4.0.15. *There doesn't exist* $\mathtt{inr}_\tau(e)$ *such that* $\mathtt{inr}_\tau(e)$ $\mathtt{notintro}$*.*

Lemma 4.0.16. *There doesn't exist* (e_1, e_2) *such that* (e_1, e_2) $\mathtt{notintro}$*.*

Lemma 4.0.17. *If* e $\mathtt{final}$ *and* e $\mathtt{notintro}$ *then* e $\mathtt{indet}$*.*

Lemma 4.0.18. *There doesn't exist such an expression* e *such that both* e $\mathtt{val}$ *and* e $\mathtt{indet}$*.*

Lemma 4.0.19. *There doesn't exist such an expression* e *such that both* e $\mathtt{val}$ *and* e $\mathtt{notintro}$*.*

Lemma 4.0.20 (Finality)**.** *There doesn't exist such an expression* e *such that both* e $\mathtt{final}$ *and* $e \mapsto e'$ *for some* e'

Lemma 4.0.21 (Matching Determinism)**.** *If* e $\mathtt{final}$ *and* $\cdot\,; \Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\mid \Gamma\,; \Delta$ *then exactly one of the following holds*

1. $e \rhd p \dashv\!\mid \theta$ *for some* θ
2. $e\ ?\ p$
3. $e \perp p$

Lemma 4.0.22 (Matching Coherence of Constraint). *Suppose that* $\cdot ; \Delta_e \vdash e : \tau$ *and* e `final` *and* $p : \tau[\xi] \dashv\Gamma ; \Delta$. *Then we have*

1. $e \dot{\models} \xi$ *iff* $e \rhd p \dashv \theta$
2. $e \dot{\models}_? \xi$ *iff* $e \mathrel{?} p$
3. $e \dot{\not\models}{}^{\dagger}_{?} \xi$ *iff* $e \perp p$

5 Preservation and Progress

Theorem 5.1 (Preservation). *If* $\cdot ; \Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot ; \Delta \vdash e' : \tau$

Theorem 5.2 (Progress). *If* $\cdot ; \Delta \vdash e : \tau$ *then either* e `final` *or* $e \mapsto e'$ *for some* e'.

6 Decidability

$\boxed{\dot{\top}(\dot{\xi}) = \xi}$

$$\dot{\top}(\top) = \top \tag{30a}$$
$$\dot{\top}(?) = \top \tag{30b}$$
$$\dot{\top}(\underline{n}) = \underline{n} \tag{30c}$$
$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{30d}$$
$$\dot{\top}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\top}(\xi)) \tag{30e}$$
$$\dot{\top}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\top}(\xi)) \tag{30f}$$
$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{30g}$$

$\boxed{\dot{\perp}(\dot{\xi}) = \xi}$

$$\dot{\perp}(\top) = \top \tag{31a}$$
$$\dot{\perp}(?) = \perp \tag{31b}$$
$$\dot{\perp}(\underline{n}) = \underline{n} \tag{31c}$$
$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \tag{31d}$$
$$\dot{\perp}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\perp}(\xi)) \tag{31e}$$
$$\dot{\perp}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\perp}(\xi)) \tag{31f}$$
$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \tag{31g}$$

$\boxed{\Xi \ \mathtt{incon}}$ A finite set of constraints, Ξ, is inconsistent

$$\frac{\Xi\ \texttt{incon}}{\Xi, \top\ \texttt{incon}}\ \text{CINCTruth} \tag{32a}$$

$$\frac{}{\Xi, \bot\ \texttt{incon}}\ \text{CINCFalsity} \tag{32b}$$

$$\frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \texttt{incon}}\ \text{CINCNum} \tag{32c}$$

$$\frac{}{\Xi, \underline{n}, \cancel{\underline{n}}\ \texttt{incon}}\ \text{CINCNotNum} \tag{32d}$$

$$\frac{\Xi, \xi_1, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \texttt{incon}}\ \text{CINCAnd} \tag{32e}$$

$$\frac{\Xi, \xi_1\ \texttt{incon} \qquad \Xi, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \vee \xi_2\ \texttt{incon}}\ \text{CINCOr} \tag{32f}$$

$$\frac{}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2)\ \texttt{incon}}\ \text{CINCInj} \tag{32g}$$

$$\frac{\xi_1, \cdots, \xi_n\ \texttt{incon}}{\texttt{inl}(\xi_1), \cdots, \texttt{inl}(\xi_n)\ \texttt{incon}}\ \text{CINCInl} \tag{32h}$$

$$\frac{\xi_1, \cdots, \xi_n\ \texttt{incon}}{\texttt{inr}(\xi_1), \cdots, \texttt{inr}(\xi_n)\ \texttt{incon}}\ \text{CINCInr} \tag{32i}$$

$$\frac{\xi_{11}, \cdots, \xi_{n1}\ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2})\ \texttt{incon}}\ \text{CINCPairL} \tag{32j}$$

$$\frac{\xi_{12}, \cdots, \xi_{n2}\ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2})\ \texttt{incon}}\ \text{CINCPairR} \tag{32k}$$

Lemma 6.0.1 (Decidability of Inconsistency). *It is decidable whether* ξ $\texttt{incon}$.

Lemma 6.0.2 (Inconsistency and Entailment of Constraint). $\overline{\xi}$ $\texttt{incon}$ *iff* $\top \models \xi$

Theorem 6.1 (Decidability of Exhaustiveness). *It is decidable whether* $\top \dot{\models}^{\dagger}_{?} \dot{\xi}$.

Theorem 6.2 (Decidability of Redundancy). *It is decidable whether* $\dot{\xi}_1 \dot{\models} \dot{\xi}_2$.