

# 1 Match Constraint Language

$\dot{\xi} ::= \top \mid \perp \mid ? \mid \underline{n} \mid \text{inl}(\dot{\xi}) \mid \text{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$

$\boxed{\dot{\xi} : \tau}$      $\dot{\xi}$  constrains final expressions of type  $\tau$

CTTruth

$\frac{}{\top : \tau}$

(1a)

CTUnknown

$\frac{}{? : \tau}$

(1b)

CTNum

$\frac{}{\underline{n} : \text{num}}$

(1c)

CTInl

$\frac{\dot{\xi}_1 : \tau_1}{\text{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)}$

(1d)

CTInr

$\frac{\dot{\xi}_2 : \tau_2}{\text{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)}$

(1e)

CTPair

$\frac{\dot{\xi}_1 : \tau_1 \quad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)}$

(1f)

CTOr

$\frac{\dot{\xi}_1 : \tau \quad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau}$

(1g)

$\boxed{\dot{\xi} \text{ refutable?}}$

$\dot{\xi}$  is refutable

RXNum

$\frac{}{\underline{n} \text{ refutable?}}$

(2a)

RXUnknown

$\frac{}{? \text{ refutable?}}$

(2b)

RXInl

$\frac{}{\text{inl}(\dot{\xi}) \text{ refutable?}}$

(2c)

RXInr

$\frac{}{\text{inr}(\dot{\xi}) \text{ refutable?}}$

(2d)

RXPairL

$\frac{\dot{\xi}_1 \text{ refutable?}}{(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}}$

(2e)

$$\frac{\text{RXPairR} \quad \dot{\xi}_2 \text{ refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_?} \quad (2f)$$

$$\frac{\text{RXOr} \quad \dot{\xi}_1 \text{ refutable}_? \quad \dot{\xi}_2 \text{ refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2 \text{ refutable}_?} \quad (2g)$$

$$\boxed{\text{refutable}_?(\dot{\xi})}$$

$$\text{refutable}_?(\underline{n}) = \text{true} \quad (3a)$$

$$\text{refutable}_?(?) = \text{true} \quad (3b)$$

$$\text{refutable}_?(\text{inl}(\dot{\xi})) = \text{refutable}_?(\dot{\xi}) \quad (3c)$$

$$\text{refutable}_?(\text{inr}(\dot{\xi})) = \text{refutable}_?(\dot{\xi}) \quad (3d)$$

$$\text{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{refutable}_?(\dot{\xi}_1) \text{ or } \text{refutable}_?(\dot{\xi}_2) \quad (3e)$$

$$\text{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \text{refutable}_?(\dot{\xi}_1) \text{ and } \text{refutable}_?(\dot{\xi}_2) \quad (3f)$$

$$\text{Otherwise} \quad \text{refutable}_?(\dot{\xi}) = \text{false} \quad (3g)$$

$$\boxed{e \models \dot{\xi}} \quad e \text{ satisfies } \dot{\xi}$$

$$\frac{\text{CSTruth}}{e \models \top} \quad (4a)$$

$$\frac{\text{CSNum}}{\underline{n} \models n} \quad (4b)$$

$$\frac{\text{CSInl} \quad e_1 \models \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)} \quad (4c)$$

$$\frac{\text{CSInr} \quad e_2 \models \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)} \quad (4d)$$

$$\frac{\text{CSPair} \quad e_1 \models \dot{\xi}_1 \quad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)} \quad (4e)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \dot{\xi}_1 \quad \text{prl}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)} \quad (4f)$$

$$\frac{\text{CSOrL} \quad e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \quad (4g)$$

$$\text{CSOrR} \quad \frac{e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \quad (4h)$$

$$\boxed{\text{satisfy}(e, \dot{\xi})}$$

$$\text{satisfy}(e, \top) = \text{true} \quad (5a)$$

$$\text{satisfy}(n_1, n_2) = (n_1 = n_2) \quad (5b)$$

$$\text{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{satisfy}(e, \dot{\xi}_1) \text{ or } \text{satisfy}(e, \dot{\xi}_2) \quad (5c)$$

$$\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{satisfy}(e_1, \dot{\xi}_1) \quad (5d)$$

$$\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = \text{satisfy}(e_2, \dot{\xi}_2) \quad (5e)$$

$$\text{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(e_1, \dot{\xi}_1) \text{ and } \text{satisfy}(e_2, \dot{\xi}_2) \quad (5f)$$

$$\text{satisfy}(\llbracket \cdot \rrbracket^u, (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(\text{prl}(\llbracket \cdot \rrbracket^u), \dot{\xi}_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket \cdot \rrbracket^u), \dot{\xi}_2) \quad (5g)$$

$$\text{satisfy}(\llbracket e \rrbracket^u, (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(\text{prl}(\llbracket e \rrbracket^u), \dot{\xi}_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket e \rrbracket^u), \dot{\xi}_2) \quad (5h)$$

$$\begin{aligned} \text{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(e_1(e_2)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(e_1(e_2)), \dot{\xi}_2) \end{aligned} \quad (5i)$$

$$\begin{aligned} \text{satisfy}(\text{match}(e)\{\hat{r}s\}, (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{match}(e)\{\hat{r}s\}), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{match}(e)\{\hat{r}s\}), \dot{\xi}_2) \end{aligned} \quad (5j)$$

$$\begin{aligned} \text{satisfy}(\text{prl}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{prl}(e)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{prl}(e)), \dot{\xi}_2) \end{aligned} \quad (5k)$$

$$\begin{aligned} \text{satisfy}(\text{pr}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{pr}(e)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{pr}(e)), \dot{\xi}_2) \end{aligned} \quad (5l)$$

$$\text{Otherwise } \text{satisfy}(e, \dot{\xi}) = \text{false} \quad (5m)$$

$$\boxed{e \models_{\text{?}} \dot{\xi}} \quad e \text{ may satisfy } \dot{\xi}$$

$$\text{CMSUnknown} \quad \frac{}{e \models_{\text{?}} ?} \quad (6a)$$

$$\text{CMSInl} \quad \frac{e_1 \models_{\text{?}} \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models_{\text{?}} \text{inl}(\dot{\xi}_1)} \quad (6b)$$

$$\text{CMSInr} \quad \frac{e_2 \models_{\text{?}} \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\dot{\xi}_2)} \quad (6c)$$

$$\text{CMSPairL} \quad \frac{e_1 \models_{\text{?}} \dot{\xi}_1 \quad e_2 \models_{\text{?}} \dot{\xi}_2}{(e_1, e_2) \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6d)$$

$$\frac{\text{CMSPairR} \quad e_1 \models \dot{\xi}_1 \quad e_2 \models_{?} \dot{\xi}_2}{(e_1, e_2) \models_{?} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6e)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{?} \dot{\xi}_1 \quad e_2 \models_{?} \dot{\xi}_2}{(e_1, e_2) \models_{?} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6f)$$

$$\frac{\text{CMSOrL} \quad e \models_{?} \dot{\xi}_1 \quad e \not\models \dot{\xi}_2}{e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2} \quad (6g)$$

$$\frac{\text{CMSOrR} \quad e \not\models \dot{\xi}_1 \quad e \models_{?} \dot{\xi}_2}{e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2} \quad (6h)$$

$$\frac{\text{CMSNotIntro} \quad e \text{ notintro} \quad \dot{\xi} \text{ refutable}_{?}}{e \models_{?} \dot{\xi}} \quad (6i)$$

$$\boxed{e \models_{?}^{\dagger} \dot{\xi}} \quad e \text{ satisfies or may satisfy } \dot{\xi}$$

$$\frac{\text{CSMSMay} \quad e \models_{?} \dot{\xi}}{e \models_{?}^{\dagger} \dot{\xi}} \quad (7a)$$

$$\frac{\text{CSMSSat} \quad e \models \dot{\xi}}{e \models_{?}^{\dagger} \dot{\xi}} \quad (7b)$$

**Lemma 1.0.1.**  $e \not\models \perp$

*Proof.* By rule induction over Rules (14), we notice that  $e \models \perp$  is in syntactic contradiction with all rules, hence not derivable.  $\square$

**Lemma 1.0.2.**  $e \not\models_{?} \perp$

*Proof.* Assume  $e \models_{?} \perp$ . By rule induction over Rules (16) on  $e \models_{?} \perp$ , only one case applies.

**Case (16b).**

$$(1) \quad \perp \text{ refutable}_{?} \quad \text{by assumption}$$

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{?} \perp$  is not derivable.  $\square$

**Lemma 1.0.3.**  $e \not\models_{?} \top$

*Proof.* Assume  $e \models_{\text{?}} \top$ . By rule induction over Rules (16) on  $e \models_{\text{?}} \top$ , only one case applies.

**Case (16b).**

(1)  $\top$  **refutable**<sub>?</sub> by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \top$  is not derivable.  $\square$

**Lemma 1.0.4.**  $e \not\models_{\text{?}} ?$

*Proof.* By rule induction over Rules (14), we notice that  $e \models_{\text{?}} ?$  is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 1.0.5.**  $e \models_{\text{?}}^{\dagger} \dot{\xi}$  *iff*  $e \models_{\text{?}}^{\dagger} \dot{\xi} \vee \perp$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

(1)  $e \models_{\text{?}}^{\dagger} \dot{\xi}$  by assumption

By rule induction over Rules (17) on (1).

**Case (17a).**

(2)  $e \models_{\text{?}} \dot{\xi}$  by assumption  
(3)  $e \models_{\text{?}} \dot{\xi} \vee \perp$  by Rule (16c) on (2)  
and Lemma 2.0.1  
(4)  $e \models_{\text{?}}^{\dagger} \dot{\xi} \vee \perp$  by Rule (17a) on (3)

**Case (17b).**

(2)  $e \models \dot{\xi}$  by assumption  
(3)  $e \models \dot{\xi} \vee \perp$  by Rule (14e) on (2)  
(4)  $e \models_{\text{?}}^{\dagger} \dot{\xi} \vee \perp$  by Rule (17b) on (3)

2. Necessity:

(1)  $e \models_{\text{?}}^{\dagger} \dot{\xi} \vee \perp$  by assumption

By rule induction over Rules (17) on (1).

**Case (17a).**

(2)  $e \models_{\text{?}} \dot{\xi} \vee \perp$  by assumption

By rule induction over Rules (16) on (2), only two of them apply.

**Case (16c).**

- |  |                      |
|--|----------------------|
| (3) $e \models_{\tau} \dot{\xi}$           | by assumption        |
| (4) $e \models_{\tau}^{\dagger} \dot{\xi}$ | by Rule (17a) on (3) |

**Case (16d).**

- |                                  |                |
|----------------------------------|----------------|
| (3) $e \models_{\tau} \perp$     | by assumption  |
| (4) $e \not\models_{\tau} \perp$ | by Lemma 2.0.2 |
- (3) contradicts (4).

**Case (17b).**

- |                                      |               |
|--------------------------------------|---------------|
| (2) $e \models \dot{\xi} \vee \perp$ | by assumption |
|--------------------------------------|---------------|

By rule induction over Rules (14) on (2), only two of them apply.

**Case (14e).**

- |  |                      |
|--|----------------------|
| (3) $e \models \dot{\xi}$                  | by assumption        |
| (4) $e \models_{\tau}^{\dagger} \dot{\xi}$ | by Rule (17b) on (3) |

**Case (14f).**

- |                           |                |
|---------------------------|----------------|
| (3) $e \models \perp$     | by assumption  |
| (4) $e \not\models \perp$ | by Lemma 2.0.1 |
- (3) contradicts (4).

□

**Corollary 1.0.1.**  $\top \models_{\tau}^{\dagger} \dot{\xi} \text{ iff } \top \models_{\tau} \dot{\xi} \vee \perp$

*Proof.* Follows directly from Definition 2.1.2 and Lemma 2.0.5. □

**Lemma 1.0.6.** *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \not\models \dot{\xi}_2$  iff  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$*

*Proof.*

- |                                     |                             |
|-------------------------------------|-----------------------------|
| (1) $\dot{\xi}_1 : \tau$            | by assumption               |
| (2) $\dot{\xi}_2 : \tau$            | by assumption               |
| (3) $\perp : \tau$                  | by Rule (8b)                |
| (4) $\dot{\xi}_2 \vee \perp : \tau$ | by Rule (8f) on (2) and (3) |

Then we prove sufficiency and necessity separately.

1. Sufficiency:

- |   |               |
|---|---------------|
| (5) $\dot{\xi}_1 \not\models \dot{\xi}_2$ | by assumption |
|---|---------------|

To prove  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$ , assume  $\dot{\xi}_1 \models \dot{\xi}_2 \vee \perp$ .

(6)  $\dot{\xi}_1 \models \dot{\xi}_2 \vee \perp$  by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies

(7)  $e \models \dot{\xi}_2 \vee \perp$  by Definition 2.1.1 on  
(1) and (4) and (6)

By rule induction over Rules (14) on (7).

**Case (14e).**

(8)  $e \models \dot{\xi}_2$  by assumption  
 (9)  $\dot{\xi}_1 \models \dot{\xi}_2$  by Definition 2.1.1 on  
(8)

(5) contradicts (9).

**Case (14f).**

(8)  $e \models \perp$  by assumption  
 (9)  $e \not\models \perp$  by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

(a)  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$

2. Necessity:

(5)  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$  by assumption

To prove  $\dot{\xi}_1 \not\models \dot{\xi}_2$ , assume  $\dot{\xi}_1 \models \dot{\xi}_2$ .

(6)  $\dot{\xi}_1 \models \dot{\xi}_2$  by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies

(7)  $e \models \dot{\xi}_2$  by Definition 2.1.1 on  
(1) and (2) and (6)  
 (8)  $e \models \dot{\xi}_2 \vee \perp$  by Rule (14e) on (7)  
 (9)  $\dot{\xi}_1 \models \dot{\xi}_2 \vee \perp$  by Definition 2.1.1 on  
(8)

(9) contradicts (5).

The conclusion holds as follows:

(a)  $\dot{\xi}_1 \not\models \dot{\xi}_2$

□

**Lemma 1.0.7.**  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  iff  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency: to show  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ , we assume  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ .

- (1)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption
- (2)  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by assumption
- (3)  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

- (4)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (14) on (4) and only two of them apply.

**Case (14e).**

- (5)  $e \models \dot{\xi}_1$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by Rule (17b) on (5)
- (6) contradicts (2).

**Case (14f).**

- (5)  $e \models \dot{\xi}_2$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Rule (17b) on (5)
- (6) contradicts (3).

**Case (17a).**

- (4)  $e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (16) on (4) and only two of them apply.

**Case (16c).**

- (5)  $e \models_{\dot{?}} \dot{\xi}_1$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by Rule (17a) on (5)
- (6) contradicts (2).

**Case (16d).**

- (5)  $e \models_{\dot{?}} \dot{\xi}_2$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Rule (17a) on (5)
- (6) contradicts (3).

The conclusion holds as follows:

- (a)  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$

2. Necessity:



(1)  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

We show  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_2$  separately.

(a) To show  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_1$ , we assume  $e \models_{\gamma}^{\dagger} \dot{\xi}_1$ .

(2)  $e \models_{\gamma}^{\dagger} \dot{\xi}_1$  by assumption

(3)  $e \models_{\gamma}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Lemma 2.0.10 on  
(2)

Contradicts (1).

(b) To show  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_2$ , we assume  $e \models_{\gamma}^{\dagger} \dot{\xi}_2$ .

(2)  $e \models_{\gamma}^{\dagger} \dot{\xi}_2$  by assumption

(3)  $e \models_{\gamma}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Lemma 2.0.10 on  
(2)

Contradicts (1).

In conclusion,  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_2$ .

□

**Lemma 1.0.8.** *If  $e \models_{\gamma}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  and  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_1$  then  $e \models_{\gamma}^{\dagger} \dot{\xi}_2$*

*Proof.*

(4)  $e \models_{\gamma}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

(5)  $e \not\models_{\gamma}^{\dagger} \dot{\xi}_1$  by assumption

By rule induction over Rules (17) on (4).

**Case (17b).**

(6)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (14) on (6) and only two of them apply.

**Case (14e).**

(7)  $e \models \dot{\xi}_1$  by assumption

(8)  $e \models_{\gamma}^{\dagger} \dot{\xi}_1$  by Rule (17b) on (7)

(8) contradicts (5).

**Case (14f).**

(7)  $e \models \dot{\xi}_2$  by assumption

(8)  $e \models_{\gamma}^{\dagger} \dot{\xi}_2$  by Rule (17b) on (7)

**Case (17a).**

(6)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (16) on (6) and only two of them apply.

**Case (16c).**

(7)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption

(8)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1$  by Rule (17a) on (7)

(8) contradicts (5).

**Case (16d).**

(7)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption

(8)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_2$  by Rule (17a) on (7)

□

**Lemma 1.0.9.** *If  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1$  then  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  and  $e \models_{\text{?}}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$*

*Proof.*

(1)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1$  by assumption ,

By rule induction over Rules (17) on (1),

**Case (17b).**

(2)  $e \models \dot{\xi}_1$  by assumption

(3)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (14e) on (2)

(4)  $e \models \dot{\xi}_2 \vee \dot{\xi}_1$  by Rule (14f) on (2)

(5)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17b) on (3)

(6)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$  by Rule (17b) on (4)

**Case (17a).**

(2)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption

By case analysis on the result of  $\text{satisfy}(e, \dot{\xi}_2)$ .

**Case true.**

(3)  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$  by assumption

(4)  $e \models \dot{\xi}_2$  by Lemma 2.0.19 on (3)

(5)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (14f) on (4)

(6)  $e \models \dot{\xi}_2 \vee \dot{\xi}_1$  by Rule (14e) on (4)

- (7)  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17b) on (5)  
 (8)  $e \models_{\tau}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$  by Rule (17b) on (6)

**Case false.**

- (3)  $\text{satisfy}(e, \dot{\xi}_2) = \text{false}$  by assumption  
 (4)  $e \not\models \dot{\xi}_2$  by Lemma 2.0.19 on (3)  
 (5)  $e \models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (16c) on (2) and (4)  
 (6)  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17a) on (5)

□

**Lemma 1.0.10.**  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  iff  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

- (1)  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

- (2)  $e_1 \models \dot{\xi}_1$  by assumption  
 (3)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$  by Rule (14g) on (2)  
 (4)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (17b) on (3)

**Case (17a).**

- (2)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption  
 (3)  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$  by Rule (16e) on (2)  
 (4)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (17a) on (3)

2. Necessity:

- (1)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

- (2)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$  by assumption

By rule induction over Rules (14) on (2), only one rule applies.

**Case (14g).**

- (3)  $e_1 \models \dot{\xi}_1$  by assumption
- (4)  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  by Rule (17b) on (3)

**Case (17a).**

- (2)  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$  by assumption

By rule induction over Rules (16) on (2), only two rules apply.

**Case (16e).**

- (3)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption
- (4)  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  by Rule (17a) on (3)

**Case (16b).**

- (3)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.11.**  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2 \text{ iff } \text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

- (1)  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

- (2)  $e_2 \models \dot{\xi}_2$  by assumption
- (3)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$  by Rule (14h) on (2)
- (4)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17b) on (3)

**Case (17a).**

- (2)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption
- (3)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16f) on (2)
- (4)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (3)

2. Necessity:

- (1)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

(2)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$  by assumption

By rule induction over Rules (14) on (2), only one rule applies.

**Case (14h).**

(3)  $e_2 \models \dot{\xi}_2$  by assumption

(4)  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  by Rule (17b) on (3)

**Case (17a).**

(2)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by assumption

By rule induction over Rules (16) on (2), only two rules apply.

**Case (16f).**

(3)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption

(4)  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  by Rule (17a) on (3)

**Case (16b).**

(3)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.12.**  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  and  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  iff  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

(1)  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  by assumption

(2)  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

(3)  $e_1 \models \dot{\xi}_1$  by assumption

By rule induction over Rules (17) on (2).

**Case (17b).**

(4)  $e_2 \models \dot{\xi}_2$  by assumption

(5)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (14i) on (3) and (4)

(6)  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17b) on (5)

**Case (17a).**

(4)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption

(5)  $(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16h) on (3) and (4)

$$(6) \quad (e_1, e_2) \models_{\vdash}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by Rule (17a) on (5)}$$

**Case (17a).**

$$(4) \quad e_1 \models_{\vdash} \dot{\xi}_1 \quad \text{by assumption}$$

By rule induction over Rules (17) on (2).

**Case (17b).**

$$(5) \quad e_2 \models \dot{\xi}_2 \quad \text{by assumption}$$

$$(6) \quad (e_1, e_2) \models_{\vdash} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by Rule (16g) on (4) and (5)}$$

$$(7) \quad (e_1, e_2) \models_{\vdash}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by Rule (17a) on (6)}$$

**Case (17a).**

$$(5) \quad e_2 \models_{\vdash} \dot{\xi}_2 \quad \text{by assumption}$$

$$(6) \quad (e_1, e_2) \models_{\vdash} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by Rule (16h) on (4) and (5)}$$

$$(7) \quad (e_1, e_2) \models_{\vdash}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by Rule (17a) on (6)}$$

2. Necessity:

$$(1) \quad (e_1, e_2) \models_{\vdash}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (17) on (1).

**Case (17b).**

$$(2) \quad (e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (14) on (2), only one rule applies.

**Case (14i).**

$$(3) \quad e_1 \models \dot{\xi}_1 \quad \text{by assumption}$$

$$(4) \quad e_2 \models \dot{\xi}_2 \quad \text{by assumption}$$

$$(5) \quad e_1 \models_{\vdash}^{\dagger} \dot{\xi}_1 \quad \text{by Rule (17b) on (3)}$$

$$(6) \quad e_2 \models_{\vdash}^{\dagger} \dot{\xi}_2 \quad \text{by Rule (17b) on (4)}$$

**Case (17a).**

$$(2) \quad (e_1, e_2) \models_{\vdash} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (16) on (2), only three rules apply.

**Case (16g).**

$$(3) \quad e_1 \models_{\vdash} \dot{\xi}_1 \quad \text{by assumption}$$

$$(4) \quad e_2 \models \dot{\xi}_2 \quad \text{by assumption}$$

$$(5) \quad e_1 \models_{\vdash}^{\dagger} \dot{\xi}_1 \quad \text{by Rule (17a) on (3)}$$

$$(6) \quad e_2 \models_{\vdash}^{\dagger} \dot{\xi}_2 \quad \text{by Rule (17b) on (4)}$$

**Case (16h).**

- |  |                      |
|--|----------------------|
| (3) $e_1 \models \dot{\xi}_1$                      | by assumption        |
| (4) $e_2 \models_{\text{?}} \dot{\xi}_2$           | by assumption        |
| (5) $e_1 \models_{\text{?}}^{\dagger} \dot{\xi}_1$ | by Rule (17b) on (3) |
| (6) $e_2 \models_{\text{?}}^{\dagger} \dot{\xi}_2$ | by Rule (17a) on (4) |

**Case (16i).**

- |  |                      |
|--|----------------------|
| (3) $e_1 \models_{\text{?}} \dot{\xi}_1$           | by assumption        |
| (4) $e_2 \models_{\text{?}} \dot{\xi}_2$           | by assumption        |
| (5) $e_1 \models_{\text{?}}^{\dagger} \dot{\xi}_1$ | by Rule (17a) on (3) |
| (6) $e_2 \models_{\text{?}}^{\dagger} \dot{\xi}_2$ | by Rule (17a) on (4) |

□

**Lemma 1.0.13** (Soundness and Completeness of Refutable Constraints).  $\dot{\xi} \text{ refutable}_{\text{?}}$  iff  $\text{refutable}_{\text{?}}(\dot{\xi}) = \text{true}$ .

**Lemma 1.0.14.** If  $e \text{ notintro}$  and  $e \models_{\text{?}} \xi$  then  $\xi \text{ refutable}_{\text{?}}$ .

**Lemma 1.0.15.** There does not exist such a constraint  $\dot{\xi}_1 \wedge \dot{\xi}_2$  such that  $\dot{\xi}_1 \wedge \dot{\xi}_2 \text{ refutable}_{\text{?}}$ .

*Proof.* By rule induction over Rules (10), we notice that  $\dot{\xi}_1 \wedge \dot{\xi}_2 \text{ refutable}_{\text{?}}$  is in syntactic contradiction with all the cases, hence not derivable. □

**Lemma 1.0.16.** There does not exist such a constraint  $\dot{\xi}_1 \vee \dot{\xi}_2$  such that  $\dot{\xi}_1 \vee \dot{\xi}_2 \text{ refutable}_{\text{?}}$ .

*Proof.* By rule induction over Rules (10), we notice that  $\dot{\xi}_1 \vee \dot{\xi}_2 \text{ refutable}_{\text{?}}$  is in syntactic contradiction with all the cases, hence not derivable. □

**Lemma 1.0.17.** If  $e \text{ notintro}$  and  $e \models \dot{\xi}$  then  $\dot{\xi} \text{ ~~refutable}_{\text{?}}~~$ .

*Proof.*

- |                           |               |
|---------------------------|---------------|
| (1) $e \text{ notintro}$  | by assumption |
| (2) $e \models \dot{\xi}$ | by assumption |

By rule induction over Rules (14) on (2).

**Case (14a).**

- |                        |               |
|------------------------|---------------|
| (3) $\dot{\xi} = \top$ | by assumption |
|------------------------|---------------|

Assume  $\top \text{ refutable}_{\text{?}}$ . By rule induction over Rules (10), no case applies due to syntactic contradiction.

Therefore,  $\top \text{ ~~refutable}_{\text{?}}~~$ .

**Case (14e),(14f).**

- |  |               |
|--|---------------|
| (3) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ | by assumption |
|--|---------------|

(4)  $\dot{\xi}_1 \vee \dot{\xi}_2 \text{ refutable?}$  by Lemma 2.0.17

**Case (14d).**

(3)  $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$  by assumption  
 (4)  $\dot{\xi}_1 \wedge \dot{\xi}_2 \text{ refutable?}$  by Lemma 2.0.16

**Case (14j).**

(3)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption  
 (4)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption  
 (5)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption  
 (6)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)  
 (7)  $\text{prr}(e) \text{ notintro}$  by Rule (26f)  
 (8)  $\dot{\xi}_1 \text{ refutable?}$  by IH on (6) and (4)  
 (9)  $\dot{\xi}_2 \text{ refutable?}$  by IH on (7) and (5)

Assume  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$ . By rule induction over Rules (10) on it, only two cases apply.

**Case (10d).**

(10)  $\dot{\xi}_1 \text{ refutable?}$  by assumption

Contradicts (8).

**Case (10e).**

(10)  $\dot{\xi}_2 \text{ refutable?}$  by assumption

Contradicts (9).

Therefore,  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$ .

**Otherwise.**

(3)  $e = \underline{n}, \text{inl}_{\tau_2}(e_1), \text{inr}_{\tau_1}(e_2), (e_1, e_2)$  by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

□

**Lemma 1.0.18.**  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  is not derivable.

*Proof.* We prove by assuming  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  and obtaining a contradiction.

(1)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by assumption



By rule induction over Rules (17) on (1).

**Case (17b).**

$$(2) \text{ inl}_{\tau_2}(e_1) \models \text{inr}(\dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (14) on (2), no rule applies due to syntactic contradiction.

**Case (17a).**

$$(2) \text{ inl}_{\tau_2}(e_1) \models_{\tau} \text{inr}(\dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (16) on (2), only one rule applies.

**Case (16b).**

$$(3) \text{ inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.19.**  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  is not derivable.

*Proof.* We prove by assuming  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  and obtaining a contradiction.

$$(1) \text{ inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

By rule induction over Rules (17) on (1).

**Case (17b).**

$$(2) \text{ inr}_{\tau_1}(e_2) \models \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

By rule induction over Rules (14) on (2), no rule applies due to syntactic contradiction.

**Case (17a).**

$$(2) \text{ inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

By rule induction over Rules (16) on (2), only one rule applies.

**Case (16b).**

$$(3) \text{ inr}_{\tau_1}(e_2) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.20** (Soundness and Completeness of Satisfaction Judgment).  $e \models \dot{\xi}$  iff  $\text{satisfy}(e, \dot{\xi}) = \text{true}$ .

*Proof.* We prove soundness and completeness separately.

1. Soundness:

(1)  $e \models \dot{\xi}$  by assumption

By rule induction over Rules (14) on (1).

**Case (14a).**

(2)  $\dot{\xi} = \top$  by assumption  
(3)  $\text{satisfy}(e, \top) = \text{true}$  by Definition 15a

**Case (14b).**

(2)  $e = \underline{n}$  by assumption  
(3)  $\dot{\xi} = \underline{n}$  by assumption  
(4)  $\text{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$  by Definition 15b

**Case (14c).**

(2)  $e = \underline{n_1}$  by assumption  
(3)  $\dot{\xi} = \underline{\underline{n_2}}$  by assumption  
(4)  $n_1 \neq n_2$  by assumption  
(5)  $\text{satisfy}(\underline{n_1}, \underline{\underline{n_2}}) = (n_1 \neq n_2) = \text{true}$  by Definition 15c on (4)

**Case (14d).**

(2)  $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$  by assumption  
(3)  $e \models \dot{\xi}_1$  by assumption  
(4)  $e \models \dot{\xi}_2$  by assumption  
(5)  $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$  by IH on (3)  
(6)  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$  by IH on (4)  
(7)  $\text{satisfy}(e, \dot{\xi}_1 \wedge \dot{\xi}_2) = \text{satisfy}(e, \dot{\xi}_1)$  and  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$   
by Definition 15d on (5) and (6)

**Case (14e).**

(2)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption  
(3)  $e \models \dot{\xi}_1$  by assumption  
(4)  $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$  by IH on (3)

- (5)  $\text{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{satisfy}(e, \dot{\xi}_1)$  or  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$   
by Definition 15e on (4)

**Case (14f).**

- (2)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption  
(3)  $e \models \dot{\xi}_2$  by assumption  
(4)  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$  by IH on (3)  
(5)  $\text{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{satisfy}(e, \dot{\xi}_1)$  or  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$   
by Definition 15e on (4)

**Case (14g).**

- (2)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption  
(3)  $\dot{\xi} = \text{inl}(\dot{\xi}_1)$  by assumption  
(4)  $e_1 \models \dot{\xi}_1$  by assumption  
(5)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by IH on (4)  
(6)  $\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$   
by Definition 15f on (5)

**Case (14h).**

- (2)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption  
(3)  $\dot{\xi} = \text{inl}(\dot{\xi}_2)$  by assumption  
(4)  $e_2 \models \dot{\xi}_2$  by assumption  
(5)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by IH on (4)  
(6)  $\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\dot{\xi}_2)) = \text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$   
by Definition 15g on (5)

**Case (14i).**

- (2)  $e = (e_1, e_2)$  by assumption  
(3)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption  
(4)  $e_1 \models \dot{\xi}_1$  by assumption  
(5)  $e_2 \models \dot{\xi}_2$  by assumption  
(6)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by IH on (4)  
(7)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by IH on (5)  
(8)  $\text{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) =$   
 $\text{satisfy}(e_1, \dot{\xi}_1)$  and  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$   
by Definition 15h on (6) and (7)

**Case (14j).**

- (2)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption
- (3)  $e \text{ notintro}$  by assumption
- (4)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption
- (5)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption
- (6)  $\text{satisfy}(\text{prl}(e), \dot{\xi}_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(\text{prr}(e), \dot{\xi}_2) = \text{true}$  by IH on (5)

By rule induction over Rules (26) on (3).

**Otherwise.**

- (8)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (9)  $\text{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) =$   
 $\text{satisfy}(\text{prl}(e), \dot{\xi}_1)$  and  $\text{satisfy}(\text{prr}(e), \dot{\xi}_2) = \text{true}$   
by Definition 15 on (6)  
and (7)

2. Completeness:

- (1)  $\text{satisfy}(e, \dot{\xi}) = \text{true}$  by assumption

By structural induction on  $\dot{\xi}$ .

**Case  $\dot{\xi} = \top$ .**

- (2)  $e \models \top$  by Rule (14a)

**Case  $\dot{\xi} = \perp, ?$ .**

- (2)  $\text{satisfy}(e, \dot{\xi}) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

**Case  $\dot{\xi} = \underline{n}$ .**

By structural induction on  $e$ .

**Case  $e = \underline{n'}$ .**

- (2)  $n' = n$  by Definition 15b on (1)
- (3)  $\underline{n'} \models \underline{n}$  by Rule (14b) on (2)

**Otherwise.**

- (2)  $\text{satisfy}(e, \underline{n}) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

**Case  $\dot{\xi} = \underline{\neg}$ .**

By structural induction on  $e$ .

**Case**  $e = \underline{n}'$ .

(2)  $n' \neq n$

by Definition 15c on (1)

(3)  $\underline{n}' \models \underline{\mathcal{A}}$

by Rule (14c) on (2)

**Otherwise.**

(2)  $\text{satisfy}(e, \underline{\mathcal{A}}) = \text{false}$

by Definition 15o

(2) contradicts (1) and thus vacuously true.

**Case**  $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$ .

(2)  $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$

by Definition 15d on (1)

(3)  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$

by Definition 15d on (1)

(4)  $e \models \dot{\xi}_1$

by IH on (2)

(5)  $e \models \dot{\xi}_2$

by IH on (3)

(6)  $e \models \dot{\xi}_1 \wedge \dot{\xi}_2$

by Rule (14d) on (4) and (5)

**Case**  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ .

(2)  $\text{satisfy}(e, \dot{\xi}_1) \text{ or } \text{satisfy}(e, \dot{\xi}_2) = \text{true}$

by Definition 15e on (1)

By case analysis on (2).

**Case**  $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$ .

(3)  $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$

by assumption

(4)  $e \models \dot{\xi}_1$

by IH on (3)

(5)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$

by Rule (14e) on (4)

**Case**  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$ .

(3)  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$

by assumption

(4)  $e \models \dot{\xi}_2$

by IH on (3)

(5)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$

by Rule (14f) on (4)

**Case**  $\dot{\xi} = \text{inl}(\dot{\xi}_1)$ .

By structural induction on  $e$ .

**Case**  $e = \text{inl}_{\tau_2}(e_1)$ .

(2)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$

by Definition 15f on (1)

(3)  $e_1 \models \dot{\xi}_1$

by IH on (2)

(4)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$

by Rule (14g) on (3)

**Otherwise.**

- (2)  $\text{satisfy}(e, \text{inl}(\dot{\xi}_1)) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

**Case  $\dot{\xi} = \text{inr}(\dot{\xi}_2)$ .**

By structural induction on  $e$ .

**Case  $e = \text{inr}_{\tau_1}(e_2)$ .**

- (2)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by Definition 15g on (1)
- (3)  $e_2 \models \dot{\xi}_2$  by IH on (2)
- (4)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$  by Rule (14h) on (3)

**Otherwise.**

- (2)  $\text{satisfy}(e, \text{inr}(\dot{\xi}_2)) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

**Case  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ .**

By structural induction on  $e$ .

**Case  $e = (e_1, e_2)$ .**

- (2)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by Definition 15h on (1)
- (3)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by Definition 15h on (1)
- (4)  $e_1 \models \dot{\xi}_1$  by IH on (2)
- (5)  $e_2 \models \dot{\xi}_2$  by IH on (3)
- (6)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (14i) on (4) and (5)

**Case  $e = \llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$ .**

- (2)  $\text{satisfy}(\text{prl}(e), \dot{\xi}_1) = \text{true}$  by Definition 15h on (1)
- (3)  $\text{satisfy}(\text{prr}(e), \dot{\xi}_2) = \text{true}$  by Definition 15h on (1)
- (4)  $\text{prl}(e) \models \dot{\xi}_1$  by IH on (2)
- (5)  $\text{prr}(e) \models \dot{\xi}_2$  by IH on (3)
- (6)  $e \text{ notintro}$  by each rule in Rules (26)
- (7)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (14j) on (6) and (4) and (5)

**Otherwise.**

- (2)  $\text{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

□

**Lemma 1.0.21.**

satormay  
soundness  
and com-  
pleteness

**Lemma 1.0.22.**  $e \not\models \dot{\xi}$  and  $e \not\models_{\text{?}} \dot{\xi}$  iff  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}$ .

*Proof.* 1. Sufficiency:

- (1)  $e \not\models \dot{\xi}$  by assumption
- (2)  $e \not\models_{\text{?}} \dot{\xi}$  by assumption

Assume  $e \models_{\text{?}}^{\dagger} \dot{\xi}$ . By rule induction over Rules (17) on it.

**Case (17a).**

- (3)  $e \models \dot{\xi}$  by assumption

Contradicts (1).

**Case (17b).**

- (3)  $e \models_{\text{?}} \dot{\xi}$  by assumption

Contradicts (2).

Therefore,  $e \models_{\text{?}}^{\dagger} \dot{\xi}$  is not derivable.

2. Necessity:

- (1)  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}$  by assumption

Assume  $e \models \dot{\xi}$ .

- (2)  $e \models_{\text{?}}^{\dagger} \dot{\xi}$  by Rule (17b) on assumption

Contradicts (1). Therefore,  $e \not\models \dot{\xi}$ . Assume  $e \models_{\text{?}} \dot{\xi}$ .

- (3)  $e \models_{\text{?}}^{\dagger} \dot{\xi}$  by Rule (17a) on assumption

Contradicts (1). Therefore,  $e \not\models_{\text{?}} \dot{\xi}$ .

□

**Theorem 1.1** (Exclusiveness of Satisfaction Judgment). *If  $\dot{\xi} : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e$  final then exactly one of the following holds*

1.  $e \models \dot{\xi}$
2.  $e \models_{\text{?}} \dot{\xi}$
3.  $e \models_{\text{?}}^{\dagger} \dot{\xi}$

*Proof.*

- |                                     |               |
|-------------------------------------|---------------|
| (4) $\dot{\xi} : \tau$              | by assumption |
| (5) $\cdot; \Delta \vdash e : \tau$ | by assumption |
| (6) $e \text{ final}$               | by assumption |

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

**Case (8a).**

- |  |                      |
|--|----------------------|
| (7) $\dot{\xi} = \top$                     | by assumption        |
| (8) $e \models \top$                       | by Rule (14a)        |
| (9) $e \not\models_{\text{?}} \top$        | by Lemma 2.0.3       |
| (10) $e \models_{\text{?}}^{\dagger} \top$ | by Rule (17b) on (8) |

**Case (8b).**

- |   |                                |
|---|--------------------------------|
| (7) $\dot{\xi} = \perp$                         | by assumption                  |
| (8) $e \not\models \perp$                       | by Lemma 2.0.1                 |
| (9) $e \not\models_{\text{?}} \perp$            | by Lemma 2.0.2                 |
| (10) $e \not\models_{\text{?}}^{\dagger} \perp$ | by Lemma 2.0.20 on (8) and (9) |

**Case (1b).**

- |   |                      |
|---|----------------------|
| (7) $\dot{\xi} = ?$                     | by assumption        |
| (8) $e \not\models ?$                   | by Lemma 2.0.4       |
| (9) $e \models_{\text{?}} ?$            | by Rule (16a)        |
| (10) $e \models_{\text{?}}^{\dagger} ?$ | by Rule (17a) on (9) |

**Case (8c).**

- |                                   |               |
|-----------------------------------|---------------|
| (7) $\dot{\xi} = \underline{n_2}$ | by assumption |
| (8) $\tau = \text{num}$           | by assumption |

By rule induction over Rules (19) on (5), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- |   |               |
|---|---------------|
| (9) $e = \textcolor{violet}{\emptyset}^u, \textcolor{violet}{(e_0)}^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$ | by assumption |
|---|---------------|



(10)  $e$  **notintro** by Rule  
(26a),(26b),(26c),(26d),(26e),(26f)

Assume  $e \models \underline{n}_2$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on  $\dot{\xi}$ .

(11)  $e \not\models \underline{n}_2$  by contradiction  
(12)  $\underline{n}_2$  **refutable?** by Rule (10a)  
(13)  $e \models_{\dot{?}} \underline{n}_2$  by Rule (16b) on (10) and (12)  
(14)  $e \models_{\dot{?}}^{\dagger} \underline{n}_2$  by Rule (17a) on (13)

**Case (19d).**

(9)  $e = \underline{n}_1$  by assumption

Assume  $\underline{n}_1 \models_{\dot{?}} \underline{n}_2$ . By rule induction over Rules (16), only one case applies.

**Case (16b).**

(10)  $\underline{n}_1$  **notintro** by assumption  
Contradicts Lemma 4.0.5.

(11)  $\underline{n}_1 \not\models_{\dot{?}} \underline{n}_2$  by contradiction

By case analysis on whether  $n_1$  is equal to  $n_2$ .

**Case  $n_1 = n_2$ .**

(12)  $\text{satisfy}(\underline{n}_1, \underline{n}_2) = \text{true}$  by Definition 15  
(13)  $\underline{n}_1 \models \underline{n}_2$  by Lemma 2.0.19 on (12)  
(14)  $e \models_{\dot{?}}^{\dagger} \underline{n}_2$  by Rule (17b) on (13)

**Case  $n_1 \neq n_2$ .**

(12)  $\text{satisfy}(\underline{n}_1, \underline{n}_2) = \text{false}$  by Definition 15  
(13)  $\underline{n}_1 \not\models \underline{n}_2$  by Lemma 2.0.19 on (12)  
(14)  $e \not\models_{\dot{?}}^{\dagger} \underline{n}_2$  by Lemma 2.0.20 on (11) and (13)

**Case (8f).**

(7)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By inductive hypothesis on (5) and (6), exactly one of  $e \models \dot{\xi}_1$ ,  $e \models_{\dot{?}} \dot{\xi}_1$ , and  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  holds. The same goes for  $\dot{\xi}_2$ . By case analysis on which conclusion holds for  $\dot{\xi}_1$  and  $\dot{\xi}_2$ .

**Case**  $e \models \dot{\xi}_1, e \models \dot{\xi}_2$ .

- |   |                       |
|---|-----------------------|
| (8) $e \models \dot{\xi}_1$                           | by assumption         |
| (9) $e \not\models \dot{\xi}_1$                       | by assumption         |
| (10) $e \models \dot{\xi}_2$                          | by assumption         |
| (11) $e \not\models \dot{\xi}_2$                      | by assumption         |
| (12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$         | by Rule (14e) on (8)  |
| (13) $e \models^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ | by Rule (17b) on (12) |

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

- |   |               |
|---|---------------|
| (14) $\dot{\xi}_1 \vee \dot{\xi}_2$ <b>refutable</b> <sub>?</sub> | by assumption |
|---|---------------|

Contradicts Lemma 2.0.17.

**Case** (16c).

- |                              |               |
|------------------------------|---------------|
| (14) $e \models \dot{\xi}_1$ | by assumption |
|------------------------------|---------------|

Contradicts (9).

**Case** (16d).

- |                              |               |
|------------------------------|---------------|
| (14) $e \models \dot{\xi}_2$ | by assumption |
|------------------------------|---------------|

Contradicts (11).

- |   |                  |
|---|------------------|
| (15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ | by contradiction |
|---|------------------|

**Case**  $e \models \dot{\xi}_1, e \models \dot{\xi}_2$ .

- |   |                       |
|---|-----------------------|
| (8) $e \models \dot{\xi}_1$                           | by assumption         |
| (9) $e \not\models \dot{\xi}_1$                       | by assumption         |
| (10) $e \not\models \dot{\xi}_2$                      | by assumption         |
| (11) $e \models \dot{\xi}_2$                          | by assumption         |
| (12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$         | by Rule (14e) on (8)  |
| (13) $e \models^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ | by Rule (17b) on (12) |

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

- |   |               |
|---|---------------|
| (14) $\dot{\xi}_1 \vee \dot{\xi}_2$ <b>refutable</b> <sub>?</sub> | by assumption |
|---|---------------|

Contradicts Lemma 2.0.17.

**Case** (16c).

- |                              |               |
|------------------------------|---------------|
| (14) $e \models \dot{\xi}_1$ | by assumption |
|------------------------------|---------------|

Contradicts (9).

**Case** (16d).

(14)  $e \not\models \dot{\xi}_1$  by assumption  
 Contradicts (8).

(15)  $e \not\models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case**  $e \models \dot{\xi}_1, e \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .

(8)  $e \models \dot{\xi}_1$  by assumption  
 (9)  $e \not\models_{\text{?}} \dot{\xi}_1$  by assumption  
 (10)  $e \not\models \dot{\xi}_2$  by assumption  
 (11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption  
 (12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (14e) on (8)  
 (13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17b) on (12)

Assume  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

(14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable**<sub>?</sub> by assumption  
 Contradicts Lemma 2.0.17.

**Case** (16c).

(14)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption  
 Contradicts (9).

**Case** (16d).

(14)  $e \not\models \dot{\xi}_1$  by assumption  
 Contradicts (8).

(15)  $e \not\models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case**  $e \models_{\text{?}} \dot{\xi}_1, e \models \dot{\xi}_2$ .

(8)  $e \not\models \dot{\xi}_1$  by assumption  
 (9)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption  
 (10)  $e \models \dot{\xi}_2$  by assumption  
 (11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption  
 (12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (14f) on (10)  
 (13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17b) on (12)

Assume  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

(14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable**<sub>?</sub> by assumption  
 Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \not\models \dot{\xi}_2$  by assumption

Contradicts (10).

**Case (16d).**

(14)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption

Contradicts (11).

(15)  $e \not\models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \models_{\text{?}} \dot{\xi}_1, e \models_{\text{?}} \dot{\xi}_2$ .**

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \not\models \dot{\xi}_2$  by assumption

(11)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption

(12)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (16c) on (9) and (10)

(13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17a) on (12)

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (14), only two cases apply.

**Case (14e).**

(14)  $e \models \dot{\xi}_1$  by assumption

Contradicts (8)

**Case (14f).**

(14)  $e \models \dot{\xi}_2$  by assumption

Contradicts (10)

(15)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \models_{\text{?}} \dot{\xi}_1, e \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .**

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \not\models \dot{\xi}_2$  by assumption

(11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption

(12)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (16c) on (9) and (10)

(13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17a) on (12)

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (14), only two cases apply.

**Case (14e).**

(14)  $e \models \dot{\xi}_1$  by assumption

Contradicts (8).

**Case (14f).**

(14)  $e \models \dot{\xi}_2$  by assumption

Contradicts (10).

(15)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e \models \dot{\xi}_2$ .**

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \not\models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \models \dot{\xi}_2$  by assumption

(11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption

(12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (14f) on (10)

(13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17b) on (12)

Assume  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable**<sub>?</sub> by assumption

Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \not\models \dot{\xi}_2$  by assumption

Contradicts (10).

**Case (16d).**

(14)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption

Contradicts (11).

(15)  $e \not\models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e \models_{\text{?}} \dot{\xi}_2$ .**

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \not\models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \not\models \dot{\xi}_2$  by assumption

(11)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption

(12)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (16d) on (11) and (8)

(13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17a) on (12)

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (14), only two cases apply.

**Case (14e).**

(14)  $e \models \dot{\xi}_1$  by assumption

Contradicts (8)

**Case (14f).**

(14)  $e \models \dot{\xi}_2$  by assumption

Contradicts (10)

(15)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .**

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \not\models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \not\models \dot{\xi}_2$  by assumption

(11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14e).**

(12)  $e \models \dot{\xi}_1$  by assumption

Contradicts (8).

**Case (14f).**

(12)  $e \models \dot{\xi}_2$  by assumption

Contradicts (10).

(13)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

Assume  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable**<sub>?</sub> by assumption

Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption

Contradicts (9).

**Case (16d).**

(14)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption

Contradicts (11).

- |  |                                     |
|--|-------------------------------------|
| (15) $e \not\models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2$           | by contradiction                    |
| (16) $e \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ | by Lemma 2.0.20 on<br>(13) and (15) |

**Case (8g).**

- |   |               |
|---|---------------|
| (7) $\dot{\xi} = \text{inl}(\dot{\xi}_1)$ | by assumption |
| (8) $\tau = (\tau_1 + \tau_2)$            | by assumption |
| (9) $\dot{\xi}_1 : \tau_1$                | by assumption |

By rule induction over Rules (19) on (5), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- |  |  |
|--|--|
| (10) $e = \mathbb{0}^u, \mathbb{0}^{e_0}{}^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$ | by assumption                                  |
| (11) $e \text{ notintro}$  | by Rule<br>(26a),(26b),(26c),(26d),(26e),(26f) |

Assume  $e \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- |  |                  |
|--|------------------|
| (12) $e \not\models \text{inl}(\dot{\xi}_1)$ | by contradiction |
|--|------------------|

By case analysis on the value of  $\text{refutable}_{\tau}(\text{inl}(\dot{\xi}_1))$ .

**Case  $\text{refutable}_{\tau}(\text{inl}(\dot{\xi}_1)) = \text{true}$ .**

- |   |                                   |
|---|-----------------------------------|
| (13) $\text{refutable}_{\tau}(\text{inl}(\dot{\xi}_1)) = \text{true}$ | by assumption                     |
| (14) $\text{inl}(\dot{\xi}_1) \text{ refutable}_{\tau}$               | by Lemma 2.0.14 on<br>(13)        |
| (15) $e \models_{\tau} \text{inl}(\dot{\xi}_1)$                       | by Rule (16b) on (11)<br>and (14) |
| (16) $e \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$             | by Rule (17a) on (15)             |

**Case  $\text{refutable}_{\tau}(\text{inl}(\dot{\xi}_1)) = \text{false}$ .**

- |   |                            |
|---|----------------------------|
| (13) $\text{refutable}_{\tau}(\text{inl}(\dot{\xi}_1)) = \text{false}$        | by assumption              |
| (14) <del><math>\text{inl}(\dot{\xi}_1) \text{ refutable}_{\tau}</math></del> | by Lemma 2.0.14 on<br>(13) |

Assume  $e \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

- |   |                  |
|---|------------------|
| (15) $\text{inl}(\dot{\xi}_1) \text{ refutable}_{\tau}$ | by assumption    |
| Contradicts (14).                                       |                  |
| (16) $e \not\models_{\tau} \text{inl}(\dot{\xi}_1)$     | by contradiction |

(17)  $e \not\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (12) and (16)

**Case (19j).**

(10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption  
 (11)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption  
 (12)  $e_1$  **final** by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_1 \models \dot{\xi}_1$ ,  $e_1 \models_{\tau} \dot{\xi}_1$ , and  $e_1 \not\models_{\tau}^{\dagger} \dot{\xi}_1$  holds. By case analysis on which one holds.

**Case  $e_1 \models \dot{\xi}_1$ .**

(13)  $e_1 \models \dot{\xi}_1$  by assumption  
 (14)  $e_1 \not\models_{\tau} \dot{\xi}_1$  by assumption  
 (15)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$  by Rule (14g) on (13)  
 (16)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (17b) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (16) on it, only two cases apply.

**Case (16b).**

(17)  $\text{inl}_{\tau_2}(e_1)$  **notintro** by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16e).**

(17)  $e_1 \models_{\tau} \dot{\xi}_1$

Contradicts (14).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\dot{\xi}_1)$  by contradiction

**Case  $e_1 \models_{\tau} \dot{\xi}_1$ .**

(13)  $e_1 \not\models \dot{\xi}_1$  by assumption  
 (14)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption  
 (15)  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$  by Rule (16e) on (14)  
 (16)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (17a) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14g).**

(17)  $e_1 \models \dot{\xi}_1$

Contradicts (13).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\dot{\xi}_1)$  by contradiction



**Case**  $e_1 \not\models_{\tau}^{\dagger} \dot{\xi}_1$ .

(13)  $e_1 \not\models \dot{\xi}_1$  by assumption

(14)  $e_1 \not\models_{\tau} \dot{\xi}_1$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (14) on it, only one case applies.

**Case** (14g).

(15)  $e_1 \models \dot{\xi}_1$

Contradicts (13).

(16)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\dot{\xi}_1)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (16) on it, only one case applies.

**Case** (16b).

(17)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case** (16e).

(17)  $e_1 \models_{\tau} \dot{\xi}_1$

Contradicts (14).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\dot{\xi}_1)$  by contradiction

(19)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (16) and (18)

**Case** (19k).

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\dot{\xi}_1)$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (16) on it, only one case applies.

**Case** (16b).

(12)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inl}(\dot{\xi}_1)$  by contradiction

(14)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (11) and (13)

**Case (8h).**

- (7)  $\dot{\xi} = \text{inr}(\dot{\xi}_2)$  by assumption
- (8)  $\tau = (\tau_1 + \tau_2)$  by assumption
- (9)  $\dot{\xi}_2 : \tau_2$  by assumption

By rule induction over Rules (19) on (5), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- (10)  $e = \llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{rs\}$  by assumption
- (11)  $e \text{ notintro}$  by Rule  
(26a),(26b),(26c),(26d),(26e),(26f)

Assume  $e \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (12)  $e \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

By case analysis on the value of  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) )$ .

inr is  
refutable

**Case  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{true}$ .**

- (13)  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{true}$  by assumption
- (14)  $\text{inr}(\dot{\xi}_2) \text{ refutable}_?$  by Lemma 2.0.14 on  
(13)
- (15)  $e \models_? \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (11)  
and (14)
- (16)  $e \models_?^\dagger \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (15)

**Case  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{false}$ .**

- (13)  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{false}$  by assumption
- (14)  ~~$\text{inr}(\dot{\xi}_2) \text{ refutable}_?$~~  by Lemma 2.0.14 on  
(13)

Assume  $e \models_? \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

- (15)  $\text{inr}(\dot{\xi}_2) \text{ refutable}_?$  by assumption
- Contradicts (14).
- (16)  $e \not\models_? \text{inr}(\dot{\xi}_2)$  by contradiction
- (17)  $e \not\models_?^\dagger \text{inr}(\dot{\xi}_2)$  by Lemma 2.0.20 on  
(12) and (16)

**Case (19j).**

- (10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(12)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inr}(\dot{\xi}_2)$  by contradiction

(14)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Lemma 2.0.20 on (11) and (13)

**Case (19k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

(11)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(12)  $e_2 \text{ final}$  by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_2 \models \dot{\xi}_2$ ,  $e_2 \models_{\tau} \dot{\xi}_2$ , and  $e_2 \not\models_{\tau}^{\dagger} \dot{\xi}_2$  holds. By case analysis on which one holds.

**Case  $e_2 \models \dot{\xi}_2$ .**

(13)  $e_2 \models \dot{\xi}_2$  by assumption

(14)  $e_2 \not\models_{\tau} \dot{\xi}_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$  by Rule (14g) on (13)

(16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17b) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (16) on it, only two cases apply.

**Case (16b).**

(17)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16f).**

(17)  $e_2 \models_{\tau} \dot{\xi}_2$

Contradicts (14).

(18)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\dot{\xi}_2)$  by contradiction

**Case  $e_2 \models_{\tau} \dot{\xi}_2$ .**

(13)  $e_2 \not\models \dot{\xi}_2$  by assumption

- (14)  $e_2 \models_{\tau_1} \dot{\xi}_2$  by assumption  
 (15)  $\text{inr}_{\tau_1}(e_2) \models_{\tau_1} \text{inr}(\dot{\xi}_2)$  by Rule (16f) on (14)  
 (16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14h).**

- (17)  $e_2 \models \dot{\xi}_2$   
 Contradicts (13).

- (18)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

**Case  $e_2 \not\models_{\tau_1}^{\dagger} \dot{\xi}_2$ .**

- (13)  $e_2 \not\models \dot{\xi}_2$  by assumption  
 (14)  $e_2 \not\models_{\tau_1} \dot{\xi}_2$  by assumption

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14h).**

- (15)  $e_2 \models \dot{\xi}_2$   
 Contradicts (13).

- (16)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau_1} \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

- (17)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16f).**

- (17)  $e_2 \models_{\tau_1} \dot{\xi}_2$   
 Contradicts (14).

- (18)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau_1} \text{inr}(\dot{\xi}_2)$  by contradiction  
 (19)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau_2}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (16) and (18)

**Case (14i).**

- (7)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption  
 (8)  $\tau = (\tau_1 \times \tau_2)$  by assumption  
 (9)  $\dot{\xi}_1 : \tau_1$  by assumption  
 (10)  $\dot{\xi}_2 : \tau_2$  by assumption

By rule induction over Rules (19) on (5), the following cases apply.

**Case** (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (11)  $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (12)  $e$  **notintro**  
by Rule (26a),(26b),(26c),(26d),(26e),(26f)
- (13)  $e$  **indet**  
by Lemma 4.0.9 on (6) and (12)
- (14)  $\text{prl}(e)$  **indet**  
by Rule (24g) on (13)
- (15)  $\text{prl}(e)$  **final**  
by Rule (25b) on (14)
- (16)  $\text{prr}(e)$  **indet**  
by Rule (24h) on (13)
- (17)  $\text{prr}(e)$  **final**  
by Rule (25b) on (16)
- (18)  $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$   
by Rule (19h) on (5)
- (19)  $\cdot; \Delta \vdash \text{prr}(e) : \tau_2$   
by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of  $\text{prl}(e) \models \dot{\xi}_1$ ,  $\text{prl}(e) \models? \dot{\xi}_1$ , and  $\text{prl}(e) \not\models?^\dagger \dot{\xi}_1$  holds.

By inductive hypothesis on (10) and (19) and (17), exactly one of  $\text{prr}(e) \models \dot{\xi}_2$ ,  $\text{prr}(e) \models? \dot{\xi}_2$ , and  $\text{prr}(e) \not\models?^\dagger \dot{\xi}_2$  holds.

By case analysis on which conclusion holds for  $\dot{\xi}_1$  and  $\dot{\xi}_2$ .

**Case**  $\text{prl}(e) \models \dot{\xi}_1, \text{prr}(e) \models \dot{\xi}_2$ .

- (20)  $\text{prl}(e) \models \dot{\xi}_1$   
by assumption
- (21)  $\text{prl}(e) \not\models? \dot{\xi}_1$   
by assumption
- (22)  $\text{prr}(e) \models \dot{\xi}_2$   
by assumption
- (23)  $\text{prr}(e) \not\models? \dot{\xi}_2$   
by assumption
- (24)  $e \models (\dot{\xi}_1, \dot{\xi}_2)$   
by Rule (14j) on (12) and (20) and (22)
- (25)  $e \models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$   
by Rule (17b) on (24)
- (26)  ~~$(\dot{\xi}_1, \dot{\xi}_2)$  **refutable?**~~  
by Lemma 2.0.18 on (12) and (24)

Assume  $e \models? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case** (16b).

- (27)  $(\dot{\xi}_1, \dot{\xi}_2)$  **refutable?**  
by assumption
- Contradicts (26).

- (28)  $e \not\models? (\dot{\xi}_1, \dot{\xi}_2)$   
by contradiction

**Case**  $\text{prl}(e) \models \dot{\xi}_1, \text{prr}(e) \models? \dot{\xi}_2$ .

- (20)  $\text{prl}(e) \models \dot{\xi}_1$   
by assumption

(21)  $\text{prl}(e) \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

(24)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption

Contradicts (22)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (16) on (23), only one case applies.

**Case (16b).**

(26)  $\dot{\xi}_2 \text{ refutable}_{\dot{?}}$  by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by Rule (10e) on (26)

(28)  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16b) on (12) and (27)

(29)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (28)

**Case  $\text{prl}(e) \models \dot{\xi}_1, \text{prr}(e) \not\models_{\dot{?}} \dot{\xi}_2$ .**

(20)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14j).**

(24)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption

Contradicts (22).

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(26)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by assumption

By rule induction over Rules (10) on (26), only two cases apply.

**Case (10d).**

(27)  $\dot{\xi}_1 \text{ refutable}_{\dot{?}}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)

assume no  
"or" and  
"and" in  
pair

(29)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by Rule (16b) on (28) and (27)

Contradicts (21).

**Case (10e).**

(27)  $\dot{\xi}_2 \text{ refutable}_{\dot{?}}$  by assumption  
 (28)  $\text{prr}(e) \text{ notintro}$  by Rule (26f)  
 (29)  $\text{prr}(e) \models_{\dot{?}} \dot{\xi}_2$  by Rule (16b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction  
 (31)  $e \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (25) and (30)

**Case  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1, \text{prr}(e) \models \dot{\xi}_2$ .**

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption  
 (21)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by assumption  
 (22)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption  
 (23)  $\text{prr}(e) \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20).

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (16) on (21), only one case applies.

**Case (16b).**

(26)  $\dot{\xi}_1 \text{ refutable}_{\dot{?}}$  by assumption  
 (27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by Rule (10e) on (26)  
 (28)  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16b) on (12) and (27)  
 (29)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (28)

**Case  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1, \text{prr}(e) \models_{\dot{?}} \dot{\xi}_2$ .**

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption  
 (21)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by assumption  
 (22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption  
 (23)  $\text{prr}(e) \models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14), only one case applies.

assume no  
"or" and  
"and" in  
pair

**Case (14j).**

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20).

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (16) on (23), only one case applies.

**Case (16b).**

(26)  $\dot{\xi}_2 \text{ refutable?}$  by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$  by Rule (10e) on (26)

(28)  $e \models? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16b) on (12) and (27)

(29)  $e \models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (28)

**Case  $\text{prl}(e) \models? \dot{\xi}_1, \text{prr}(e) \not\models?^\dagger \dot{\xi}_2$ .**

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \models? \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \not\models? \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (16) on (21), only one case applies.

**Case (16b).**

(26)  $\dot{\xi}_1 \text{ refutable?}$  by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$  by Rule (10e) on (26)

(28)  $e \models? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16b) on (12) and (27)

(29)  $e \models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (28)

**Case  $\text{prl}(e) \not\models?^\dagger \dot{\xi}_1, \text{prr}(e) \models \dot{\xi}_2$ .**

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \not\models? \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \not\models? \dot{\xi}_2$  by assumption

assume no  
"or" and  
"and" in  
pair

assume no  
"or" and  
"and" in  
pair



Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14j).**

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption  
Contradicts (20)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $e \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(26)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\text{?}}$  by assumption

By rule induction over Rules (10) on (26), only two cases apply.

**Case (10d).**

(27)  $\dot{\xi}_1 \text{ refutable}_{\text{?}}$  by assumption  
(28)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)  
(29)  $\text{prl}(e) \models_{\text{?}} \dot{\xi}_1$  by Rule (16b) on (28) and (27)

Contradicts (21).

**Case (10e).**

(27)  $\dot{\xi}_2 \text{ refutable}_{\text{?}}$  by assumption  
(28)  $\text{prr}(e) \text{ notintro}$  by Rule (26f)  
(29)  $\text{prr}(e) \models_{\text{?}} \dot{\xi}_2$  by Rule (16b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(31)  $e \not\models_{\text{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (25) and (30)

**Case  $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, \text{prr}(e) \models_{\text{?}} \dot{\xi}_2$ .**

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption  
(21)  $\text{prl}(e) \not\models_{\text{?}} \dot{\xi}_1$  by assumption  
(22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption  
(23)  $\text{prr}(e) \models_{\text{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption  
Contradicts (20).

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$

by contradiction

By rule induction over Rules (16) on (23), only one case applies.

**Case (16b).**

(26)  $\dot{\xi}_2 \text{ refutable?}$

by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$

by Rule (10e) on (26)

(28)  $e \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$

by Rule (16b) on (12)  
and (27)

(29)  $e \models_{\text{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$

by Rule (17a) on (28)

**Case**  $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, \text{pr}(e) \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$

by assumption

(21)  $\text{prl}(e) \not\models_{\text{?}} \dot{\xi}_1$

by assumption

(22)  $\text{pr}(e) \not\models \dot{\xi}_2$

by assumption

(23)  $\text{pr}(e) \not\models_{\text{?}} \dot{\xi}_2$

by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it,  
only one case applies.

**Case (14j).**

(24)  $\text{prl}(e) \models \dot{\xi}_1$

by assumption

Contradicts (20)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$

by contradiction

Assume  $e \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it,  
only one case applies.

**Case (16b).**

(26)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$

by assumption

By rule induction over Rules (10) on (26), only two cases  
apply.

**Case (10d).**

(27)  $\dot{\xi}_1 \text{ refutable?}$

by assumption

(28)  $\text{prl}(e) \text{ notintro}$

by Rule (26e)

(29)  $\text{prl}(e) \models_{\text{?}} \dot{\xi}_1$

by Rule (16b) on (28)  
and (27)

Contradicts (21).

**Case (10e).**

(27)  $\dot{\xi}_2 \text{ refutable?}$

by assumption

(28)  $\text{pr}(e) \text{ notintro}$

by Rule (26f)

(29)  $\text{pr}(e) \models_{\text{?}} \dot{\xi}_2$

by Rule (16b) on (28)  
and (27)

Contradicts (23).

assume no  
"or" and  
"and" in  
pair

- |  |                                     |
|--|-------------------------------------|
| (30) $e \not\models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$           | by contradiction                    |
| (31) $e \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ | by Lemma 2.0.20 on<br>(25) and (30) |

**Case (19g).**

- |  |                       |
|--|-----------------------|
| (11) $e = (e_1, e_2)$                    | by assumption         |
| (12) $\cdot; \Delta \vdash e_1 : \tau_1$ | by assumption         |
| (13) $\cdot; \Delta \vdash e_2 : \tau_2$ | by assumption         |
| (14) $e_1$ <b>final</b>                  | by Lemma 4.0.4 on (6) |
| (15) $e_2$ <b>final</b>                  | by Lemma 4.0.4 on (6) |

By inductive hypothesis on (9) and (12) and (14), exactly one of  $e_1 \models \dot{\xi}_1$ ,  $e_1 \models_{\tau} \dot{\xi}_1$ , and  $e_1 \models \overline{\dot{\xi}_1}$  holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of  $e_2 \models \dot{\xi}_2$ ,  $e_2 \models_{\tau} \dot{\xi}_2$ , and  $e_2 \models \overline{\dot{\xi}_2}$  holds.

By case analysis on which conclusion holds for  $\dot{\xi}_1$  and  $\dot{\xi}_2$ .

**Case**  $e_1 \models \dot{\xi}_1, e_2 \models \dot{\xi}_2$ .

- |   |                                   |
|---|-----------------------------------|
| (16) $e_1 \models \dot{\xi}_1$  | by assumption                     |
| (17) $e_1 \not\models_{\tau} \dot{\xi}_1$                             | by assumption                     |
| (18) $e_2 \models \dot{\xi}_2$  | by assumption                     |
| (19) $e_2 \not\models_{\tau} \dot{\xi}_2$                             | by assumption                     |
| (20) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$                  | by Rule (14i) on (16)<br>and (18) |
| (21) $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ | by Rule (17b) on (20)             |

Assume  $(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

- |                                   |               |
|-----------------------------------|---------------|
| (22) $(e_1, e_2)$ <b>notintro</b> | by assumption |
|-----------------------------------|---------------|

Contradicts Lemma 4.0.8.

**Case (16g).**

- |                                       |               |
|---------------------------------------|---------------|
| (22) $e_1 \models_{\tau} \dot{\xi}_1$ | by assumption |
|---------------------------------------|---------------|

Contradicts (17).

**Case (16h).**

- |                                       |               |
|---------------------------------------|---------------|
| (22) $e_2 \models_{\tau} \dot{\xi}_2$ | by assumption |
|---------------------------------------|---------------|

Contradicts (19).

**Case (16i).**

- |                                       |               |
|---------------------------------------|---------------|
| (22) $e_1 \models_{\tau} \dot{\xi}_1$ | by assumption |
|---------------------------------------|---------------|

Contradicts (17).

- |   |                  |
|---|------------------|
| (23) $(e_1, e_2) \not\models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$ | by contradiction |
|---|------------------|

**Case**  $e_1 \models \dot{\xi}_1, e_2 \models? \dot{\xi}_2$ .

- (16)  $e_1 \models \dot{\xi}_1$  by assumption
- (17)  $e_1 \not\models? \dot{\xi}_1$  by assumption
- (18)  $e_2 \not\models \dot{\xi}_2$  by assumption
- (19)  $e_2 \models? \dot{\xi}_2$  by assumption
- (20)  $(e_1, e_2) \models? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16h) on (16) and (19)
- (21)  $(e_1, e_2) \models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (20)

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case** (14j).

- (22)  $(e_1, e_2)$  **notintro** by assumption
- Contradicts Lemma 4.0.8.

**Case** (14i).

- (22)  $e_2 \models \dot{\xi}_2$  by assumption
- Contradicts (18).

- (23)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case**  $e_1 \models \dot{\xi}_1, e_2 \not\models? \dot{\xi}_2$ .

- (16)  $e_1 \models \dot{\xi}_1$  by assumption
- (17)  $e_1 \not\models? \dot{\xi}_1$  by assumption
- (18)  $e_2 \not\models \dot{\xi}_2$  by assumption
- (19)  $e_2 \not\models? \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case** (14j).

- (20)  $(e_1, e_2)$  **notintro** by assumption
- Contradicts Lemma 4.0.8.

**Case** (14i).

- (20)  $e_2 \models \dot{\xi}_2$  by assumption
- Contradicts (18).

- (21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

- (22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (16g).**

(22)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

Contradicts (17).

**Case (16h).**

(22)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

Contradicts (19).

**Case (16i).**

(22)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

Contradicts (17).

(23)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \models_{\dot{?}} \dot{\xi}_1, e_2 \models \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \models \dot{\xi}_2$  by assumption

(19)  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

(20)  $(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16g) on (17) and (18)

(21)  $(e_1, e_2) \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (20)

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(22)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (14i).**

(22)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(23)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case**  $e_1 \models_{\dot{?}} \dot{\xi}_1, e_2 \models_{\dot{?}} \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption

(19)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

(20)  $(e_1, e_2) \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16i) on (17) and (19)

(21)  $(e_1, e_2) \models_{\text{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (20)

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (14i).**

(22)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(23)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case**  $e_1 \models_{\text{?}} \dot{\xi}_1, e_2 \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \models_{\text{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption

(19)  $e_2 \not\models_{\text{?}} \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (14i).**

(20)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (16g).**

(22)  $e_2 \models \dot{\xi}_2$  by assumption

Contradicts (18).

**Case (16h).**

(22)  $e_2 \models_{\text{?}} \dot{\xi}_2$  by assumption

Contradicts (19).

**Case (16i).**

(22)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

Contradicts (19).

(23)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1, e_2 \models \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \models \dot{\xi}_2$  by assumption

(19)  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (14i).**

(20)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (16g).**

(22)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

Contradicts (17).

**Case (16h).**

(22)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

Contradicts (19).

**Case (16i).**

(22)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

Contradicts (17).

(23)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models_{\dot{?}} \dot{\xi}_1, e_2 \models_{\dot{?}} \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption

(19)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case** (14j).

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case** (14i).

(20)  $e_2 \models \dot{\xi}_2$  by assumption

Contradicts (18).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case** (16g).

(22)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

Contradicts (17).

**Case** (16h).

(22)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

**Case** (16i).

(22)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

Contradicts (17).

(23)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1, e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption



(19)  $e_2 \not\models_{\tau} \dot{\xi}_2$  by assumption  
 Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (14) on it, only two cases apply.  
**Case (14j).**  
 (20)  $(e_1, e_2) \text{ notintro}$  by assumption  
 Contradicts Lemma 4.0.8.  
**Case (14i).**  
 (20)  $e_2 \models \dot{\xi}_2$  by assumption  
 Contradicts (18).  
 (21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction  
 Assume  $(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (16) on it, the following cases apply.  
**Case (16b).**  
 (22)  $(e_1, e_2) \text{ notintro}$  by assumption  
 Contradicts Lemma 4.0.8.  
**Case (16g).**  
 (22)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption  
 Contradicts (17).  
**Case (16h).**  
 (22)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption  
 Contradicts (19).  
**Case (16i).**  
 (22)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption  
 Contradicts (17).  
 (23)  $(e_1, e_2) \not\models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction  
 (24)  $(e_1, e_2) \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

□

**Definition 1.1.1** (Entailment of Constraints). *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \models \dot{\xi}_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e \text{ val}$  we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies  $e \models \dot{\xi}_2$*

**Definition 1.1.2** (Potential Entailment of Constraints). *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \models_{\tau}^{\dagger} \dot{\xi}_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e \text{ final}$  we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies  $e \models_{\tau}^{\dagger} \dot{\xi}_2$*

**Corollary 1.1.1.** *Suppose that  $\dot{\xi} : \tau$  and  $\cdot ; \Delta \vdash e : \tau$  and  $e \text{ final}$ . Then  $\top \models_{\tau}^{\dagger} \dot{\xi}$  implies  $e \models_{\tau}^{\dagger} \dot{\xi}$*

*Proof.*

- |  |   |
|--|---|
| (1) $\dot{\xi} : \tau$                         | by assumption   |
| (2) $\cdot; \Gamma \vdash e : \tau$            | by assumption   |
| (3) $e \text{ final}$                          | by assumption   |
| (4) $\top \models_{\tau}^{\dagger} \dot{\xi}$  | by assumption   |
| (5) $e_1 \models \top$                         | by Rule (14a)   |
| (6) $e_1 \models_{\tau}^{\dagger} \top$        | by Rule (17b) on (5)  |
| (7) $\top : \tau$                              | by Rule (8a)  |
| (8) $e_1 \models_{\tau}^{\dagger} \dot{\xi}_r$ | by Definition 2.1.2 of<br>(4) on (7) and (1) and<br>(2) and (3) and (6) |

□

## 2 Match Constraint Language

$\xi ::= \top \mid \perp \mid \underline{n} \mid \underline{\mathcal{N}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \text{inl}(\xi) \mid \text{inr}(\xi) \mid (\xi_1, \xi_2)$   
 $\boxed{\xi : \tau}$      $\xi$  constrains final expressions of type  $\tau$

$$\frac{\text{CTTruth}}{\top : \tau} \quad (8a)$$

$$\frac{\text{CTFalsity}}{\perp : \tau} \quad (8b)$$

$$\frac{\text{CTNum}}{\underline{n} : \text{num}} \quad (8c)$$

$$\frac{\text{CTNotNum}}{\underline{\mathcal{N}} : \text{num}} \quad (8d)$$

$$\frac{\text{CTAnd} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \quad (8e)$$

$$\frac{\text{CTOr} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \quad (8f)$$

$$\frac{\text{CTInl} \quad \xi_1 : \tau_1}{\text{inl}(\xi_1) : (\tau_1 + \tau_2)} \quad (8g)$$

$$\frac{\text{CTInr} \quad \xi_2 : \tau_2}{\text{inr}(\xi_2) : (\tau_1 + \tau_2)} \quad (8h)$$

$$\frac{\text{CTPair} \quad \xi_1 : \tau_1 \quad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \quad (8i)$$

$$\boxed{\overline{\xi_1} = \xi_2} \quad \text{dual of } \xi_1 \text{ is } \xi_2$$

$$\overline{\top} = \perp \quad (9a)$$

$$\overline{\perp} = \top \quad (9b)$$

$$\overline{\underline{n}} = \overline{\mathcal{N}} \quad (9c)$$

$$\overline{\overline{\mathcal{N}}} = \underline{n} \quad (9d)$$

$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \quad (9e)$$

$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \quad (9f)$$

$$\overline{\text{inl}(\xi_1)} = \text{inl}(\overline{\xi_1}) \vee \text{inr}(\top) \quad (9g)$$

$$\overline{\text{inr}(\xi_2)} = \text{inr}(\overline{\xi_2}) \vee \text{inl}(\top) \quad (9h)$$

$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \quad (9i)$$

$$\boxed{\xi \text{ refutable}_?} \quad \xi \text{ is refutable}$$

$$\frac{\text{RXNum}}{\underline{n} \text{ refutable}_?} \quad (10a)$$

$$\frac{\text{RXInl}}{\text{inl}(\xi) \text{ refutable}_?} \quad (10b)$$

$$\frac{\text{RXInr}}{\text{inr}(\xi) \text{ refutable}_?} \quad (10c)$$

$$\frac{\text{RXPairL} \quad \xi_1 \text{ refutable}_?}{(\xi_1, \xi_2) \text{ refutable}_?} \quad (10d)$$

$$\frac{\text{RXPairR} \quad \xi_2 \text{ refutable}_?}{(\xi_1, \xi_2) \text{ refutable}_?} \quad (10e)$$

$$\frac{\text{RXOr} \quad \xi_1 \text{ refutable}_? \quad \xi_2 \text{ refutable}_?}{\xi_1 \vee \xi_2 \text{ refutable}_?} \quad (10f)$$

$$\boxed{\text{refutable}_?(\xi)}$$

$$refutable_?(n) = \text{true} \quad (11a)$$

$$refutable_?(\underline{n}) = \text{true} \quad (11b)$$

$$refutable_?(?) = \text{true} \quad (11c)$$

$$refutable_?(inl(\xi)) = refutable_?(\xi) \quad (11d)$$

$$refutable_?(inr(\xi)) = refutable_?(\xi) \quad (11e)$$

$$refutable_?((\xi_1, \xi_2)) = refutable_?(\xi_1) \text{ or } refutable_?(\xi_2) \quad (11f)$$

$$refutable_?(\xi_1 \vee \xi_2) = refutable_?(\xi_1) \text{ and } refutable_?(\xi_2) \quad (11g)$$

$$\text{Otherwise } refutable_?(\xi) = \text{false} \quad (11h)$$

$$\boxed{\dot{\top}(\xi_1) = \xi_2}$$

$$\dot{\top}(\top) = \top \quad (12a)$$

$$\dot{\top}(\perp) = \perp \quad (12b)$$

$$\dot{\top}(?) = \top \quad (12c)$$

$$\dot{\top}(n) = \underline{n} \quad (12d)$$

$$\dot{\top}(\underline{n}) = \underline{n} \quad (12e)$$

$$\dot{\top}(\xi_1 \wedge \xi_2) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \quad (12f)$$

$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad (12g)$$

$$\dot{\top}(inl(\xi)) = inl(\dot{\top}(\xi)) \quad (12h)$$

$$\dot{\top}(inr(\xi)) = inr(\dot{\top}(\xi)) \quad (12i)$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \quad (12j)$$

$$\boxed{\dot{\perp}(\xi_1) = \xi_2}$$

$$\dot{\perp}(\top) = \top \quad (13a)$$

$$\dot{\perp}(\perp) = \perp \quad (13b)$$

$$\dot{\perp}(?) = \perp \quad (13c)$$

$$\dot{\perp}(n) = \underline{n} \quad (13d)$$

$$\dot{\perp}(\underline{n}) = \underline{n} \quad (13e)$$

$$\dot{\perp}(\xi_1 \wedge \xi_2) = \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2) \quad (13f)$$

$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \quad (13g)$$

$$\dot{\perp}(inl(\xi)) = inl(\dot{\perp}(\xi)) \quad (13h)$$

$$\dot{\perp}(inr(\xi)) = inr(\dot{\perp}(\xi)) \quad (13i)$$

$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \quad (13j)$$

$e \models \xi$       $e$  satisfies  $\xi$

$$\frac{\text{CSTruth}}{e \models \top} \quad (14a)$$

$$\frac{\text{CSNum}}{\underline{n} \models n} \quad (14b)$$

$$\frac{\text{CSNotNum} \quad n_1 \neq n_2}{\underline{n_1} \models \underline{\text{not}} n_2} \quad (14c)$$

$$\frac{\text{CSAnd} \quad e \models \xi_1 \quad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \quad (14d)$$

$$\frac{\text{CSOrL} \quad e \models \xi_1}{e \models \xi_1 \vee \xi_2} \quad (14e)$$

$$\frac{\text{CSOrR} \quad e \models \xi_2}{e \models \xi_1 \vee \xi_2} \quad (14f)$$

$$\frac{\text{CSInl} \quad e_1 \models \xi_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)} \quad (14g)$$

$$\frac{\text{CSInr} \quad e_2 \models \xi_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)} \quad (14h)$$

$$\frac{\text{CSPair} \quad e_1 \models \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \quad (14i)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \xi_1 \quad \text{prl}(e) \models \xi_2}{e \models (\xi_1, \xi_2)} \quad (14j)$$

$\text{satisfy}(e, \xi)$

$$\text{satisfy}(e, \top) = \text{true} \quad (15a)$$

$$\text{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \quad (15b)$$

$$\text{satisfy}(\underline{n_1}, \underline{\neg n_2}) = (n_1 \neq n_2) \quad (15c)$$

$$\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1) \text{ and } \text{satisfy}(e, \xi_2) \quad (15d)$$

$$\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1) \text{ or } \text{satisfy}(e, \xi_2) \quad (15e)$$

$$\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) \quad (15f)$$

$$\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\xi_2)) = \text{satisfy}(e_2, \xi_2) \quad (15g)$$

$$\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) \quad (15h)$$

$$\text{satisfy}(\llbracket e \rrbracket^u, (\xi_1, \xi_2)) = \text{satisfy}(\text{prl}(\llbracket e \rrbracket^u), \xi_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket e \rrbracket^u), \xi_2) \quad (15i)$$

$$\text{satisfy}(\llbracket e \rrbracket^u, (\xi_1, \xi_2)) = \text{satisfy}(\text{prl}(\llbracket e \rrbracket^u), \xi_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket e \rrbracket^u), \xi_2) \quad (15j)$$

$$\begin{aligned} \text{satisfy}(e_1(e_2), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(e_1(e_2)), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(e_1(e_2)), \xi_2) \end{aligned} \quad (15k)$$

$$\begin{aligned} \text{satisfy}(\text{match}(e)\{\hat{r}s\}, (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{match}(e)\{\hat{r}s\}), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{match}(e)\{\hat{r}s\}), \xi_2) \end{aligned} \quad (15l)$$

$$\begin{aligned} \text{satisfy}(\text{prl}(e), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{prl}(e)), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{prl}(e)), \xi_2) \end{aligned} \quad (15m)$$

$$\begin{aligned} \text{satisfy}(\text{pr}(e), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{pr}(e)), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{pr}(e)), \xi_2) \end{aligned} \quad (15n)$$

$$\text{Otherwise } \text{satisfy}(e, \xi) = \text{false} \quad (15o)$$

$$\boxed{e \models? \xi} \quad e \text{ may satisfy } \xi$$

$$\frac{\text{CMSUnknown}}{e \models? ?} \quad (16a)$$

$$\frac{\text{CMSNotIntro} \quad e \text{ notintro} \quad \xi \text{ refutable?}}{e \models? \xi} \quad (16b)$$

$$\frac{\text{CMSOrL} \quad \frac{e \models? \xi_1 \quad e \not\models \xi_2}{e \models? \xi_1 \vee \xi_2}}{e \models? \xi_1 \vee \xi_2} \quad (16c)$$

$$\frac{\text{CMSOrR} \quad \frac{e \not\models \xi_1 \quad e \models? \xi_2}{e \models? \xi_1 \vee \xi_2}}{e \models? \xi_1 \vee \xi_2} \quad (16d)$$

$$\frac{\text{CMSInl} \quad \frac{e_1 \models? \xi_1}{\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)}}{\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)} \quad (16e)$$

$$\frac{\text{CMSInr} \quad e_2 \models_{\text{?}} \xi_2}{\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\xi_2)} \quad (16f)$$

$$\frac{\text{CMSPairL} \quad e_1 \models_{\text{?}} \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (16g)$$

$$\frac{\text{CMSPairR} \quad e_1 \models \xi_1 \quad e_2 \models_{\text{?}} \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (16h)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{\text{?}} \xi_1 \quad e_2 \models_{\text{?}} \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (16i)$$

$$\boxed{e \models_{\text{?}}^{\dagger} \xi} \quad e \text{ satisfies or may satisfy } \xi$$

$$\frac{\text{CSMSMay} \quad e \models_{\text{?}} \xi}{e \models_{\text{?}}^{\dagger} \xi} \quad (17a)$$

$$\frac{\text{CSMSSat} \quad e \models \xi}{e \models_{\text{?}}^{\dagger} \xi} \quad (17b)$$

**Lemma 2.0.1.**  $e \not\models \perp$

*Proof.* By rule induction over Rules (14), we notice that  $e \models \perp$  is in syntactic contradiction with all rules, hence not derivable.  $\square$

**Lemma 2.0.2.**  $e \not\models_{\text{?}} \perp$

*Proof.* Assume  $e \models_{\text{?}} \perp$ . By rule induction over Rules (16) on  $e \models_{\text{?}} \perp$ , only one case applies.

**Case (16b).**

$$(1) \quad \perp \text{ refutable}_{\text{?}} \quad \text{by assumption}$$

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \perp$  is not derivable.  $\square$

**Lemma 2.0.3.**  $e \not\models_{\text{?}} \top$

*Proof.* Assume  $e \models_{\text{?}} \top$ . By rule induction over Rules (16) on  $e \models_{\text{?}} \top$ , only one case applies.

**Case (16b).**

$$(1) \quad \top \text{ refutable}_{\text{?}} \quad \text{by assumption}$$

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \top$  is not derivable.  $\square$

**Lemma 2.0.4.**  $e \not\models_{\text{?}}$

*Proof.* By rule induction over Rules (14), we notice that  $e \models_{\text{?}}$  is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 2.0.5.**  $e \models_{\text{?}}^{\dagger} \xi$  iff  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

(1)  $e \models_{\text{?}}^{\dagger} \xi$  by assumption

By rule induction over Rules (17) on (1).

**Case (17a).**

(2)  $e \models_{\text{?}} \xi$  by assumption  
(3)  $e \models_{\text{?}} \xi \vee \perp$  by Rule (16c) on (2)  
and Lemma 2.0.1  
(4)  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$  by Rule (17a) on (3)

**Case (17b).**

(2)  $e \models \xi$  by assumption  
(3)  $e \models \xi \vee \perp$  by Rule (14e) on (2)  
(4)  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$  by Rule (17b) on (3)

2. Necessity:

(1)  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$  by assumption

By rule induction over Rules (17) on (1).

**Case (17a).**

(2)  $e \models_{\text{?}} \xi \vee \perp$  by assumption

By rule induction over Rules (16) on (2), only two of them apply.

**Case (16c).**

(3)  $e \models_{\text{?}} \xi$  by assumption  
(4)  $e \models_{\text{?}}^{\dagger} \xi$  by Rule (17a) on (3)



**Case (16d).**

(3)  $e \models_{\tau} \perp$

by assumption

(4)  $e \not\models_{\tau} \perp$

by Lemma 2.0.2

(3) contradicts (4).

**Case (17b).**

(2)  $e \models \xi \vee \perp$

by assumption

By rule induction over Rules (14) on (2), only two of them apply.

**Case (14e).**

(3)  $e \models \xi$

by assumption

(4)  $e \models_{\tau}^{\dagger} \xi$

by Rule (17b) on (3)

**Case (14f).**

(3)  $e \models \perp$

by assumption

(4)  $e \not\models \perp$

by Lemma 2.0.1

(3) contradicts (4).

□

**Corollary 2.0.1.**  $\top \models_{\tau}^{\dagger} \xi \text{ iff } \top \models_{\tau}^{\dagger} \xi \vee \perp$

*Proof.* By Definition 2.1.2 and Lemma 2.0.5. □

**Lemma 2.0.6.** *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \not\models \xi_2$  iff  $\xi_1 \not\models \xi_2 \vee \perp$*

*Proof.*

(1)  $\xi_1 : \tau$

by assumption

(2)  $\xi_2 : \tau$

by assumption

(3)  $\perp : \tau$

by Rule (8b)

(4)  $\xi_2 \vee \perp : \tau$

by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5)  $\xi_1 \not\models \xi_2$

by assumption

To prove  $\xi_1 \not\models \xi_2 \vee \perp$ , assume  $\xi_1 \models \xi_2 \vee \perp$ .

(6)  $\xi_1 \models \xi_2 \vee \perp$

by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies

(7)  $e \models \xi_2 \vee \perp$

by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

**Case (14e).**

- (8)  $e \models \xi_2$  by assumption
- (9)  $\xi_1 \models \xi_2$  by Definition 2.1.1 on (8)

(5) contradicts (9).

**Case (14f).**

- (8)  $e \models \perp$  by assumption
- (9)  $e \not\models \perp$  by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

- (a)  $\xi_1 \not\models \xi_2 \vee \perp$

2. Necessity:

- (5)  $\xi_1 \not\models \xi_2 \vee \perp$  by assumption

To prove  $\xi_1 \not\models \xi_2$ , assume  $\xi_1 \models \xi_2$ .

- (6)  $\xi_1 \models \xi_2$  by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies

- (7)  $e \models \xi_2$  by Definition 2.1.1 on (1) and (2) and (6)
- (8)  $e \models \xi_2 \vee \perp$  by Rule (14e) on (7)
- (9)  $\xi_1 \models \xi_2 \vee \perp$  by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

- (a)  $\xi_1 \not\models \xi_2$

□

**Lemma 2.0.7.** *If  $e \not\models_{\tau}^{\dagger} \xi_1$  and  $e \not\models_{\tau}^{\dagger} \xi_2$  then  $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$*

*Proof.* Assume, for the sake of contradiction, that  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ .

- (1)  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by assumption
- (2)  $e \not\models_{\tau}^{\dagger} \xi_1$  by assumption
- (3)  $e \not\models_{\tau}^{\dagger} \xi_2$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

$$(4) \quad e \models \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (14) on (4) and only two of them apply.

**Case (14e).**

$$(5) \quad e \models \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (17b) on (5)}$$

(6) contradicts (2).

**Case (14f).**

$$(5) \quad e \models \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (17b) on (5)}$$

(6) contradicts (3).

**Case (17a).**

$$(4) \quad e \models_{\vdash} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (16) on (4) and only two of them apply.

**Case (16c).**

$$(5) \quad e \models_{\vdash} \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (17a) on (5)}$$

(6) contradicts (2).

**Case (16d).**

$$(5) \quad e \models_{\vdash} \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (17a) on (5)}$$

(6) contradicts (3).

The conclusion holds as follows:

$$1. \quad e \not\models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$$

□

**Lemma 2.0.8.** *If  $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$  and  $e \not\models_{\vdash}^{\dagger} \xi_1$  then  $e \models_{\vdash}^{\dagger} \xi_2$*

*Proof.*

$$(1) \quad e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

$$(2) \quad e \not\models_{\vdash}^{\dagger} \xi_1 \quad \text{by assumption}$$

By rule induction over Rules (17) on (1).

**Case (17b).**

(3)  $e \models \xi_1 \vee \xi_2$  by assumption

By rule induction over Rules (14) on (3) and only two of them apply.

**Case (14e).**

(4)  $e \models \xi_1$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_1$  by Rule (17b) on (4)

(5) contradicts (2).

**Case (14f).**

(4)  $e \models \xi_2$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_2$  by Rule (17b) on (4)

**Case (17a).**

(3)  $e \models_{\neg} \xi_1 \vee \xi_2$  by assumption

By rule induction over Rules (16) on (3) and only two of them apply.

**Case (16c).**

(4)  $e \models_{\neg} \xi_1$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_1$  by Rule (17a) on (4)

(5) contradicts (2).

**Case (16d).**

(4)  $e \models_{\neg} \xi_2$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_2$  by Rule (17a) on (4)

□

**Lemma 2.0.9.** *If  $e \models_{\neg}^{\dagger} \xi_1$  and  $e \models_{\neg}^{\dagger} \xi_2$  then  $e \models_{\neg}^{\dagger} \xi_1 \wedge \xi_2$*

**Lemma 2.0.10.** *If  $e \models_{\neg}^{\dagger} \xi_1$  then  $e \models_{\neg}^{\dagger} \xi_1 \vee \xi_2$  and  $e \models_{\neg}^{\dagger} \xi_2 \vee \xi_1$*

*Proof.*

(1)  $e \models_{\neg}^{\dagger} \xi_1$  by assumption ,

By rule induction over Rules (17) on (1),

**Case (17b).**

(2)  $e \models \xi_1$  by assumption

(3)  $e \models \xi_1 \vee \xi_2$  by Rule (14e) on (2)

(4)  $e \models \xi_2 \vee \xi_1$  by Rule (14f) on (2)

- (5)  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by Rule (17b) on (3)
- (6)  $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$  by Rule (17b) on (4)

**Case (17a).**

- (2)  $e \models_{\tau} \xi_1$  by assumption

By case analysis on the result of  $satisfy(e, \xi_2)$ .

**Case true.**

- (3)  $satisfy(e, \xi_2) = \text{true}$  by assumption
- (4)  $e \models \xi_2$  by Lemma 2.0.19 on (3)
- (5)  $e \models \xi_1 \vee \xi_2$  by Rule (14f) on (4)
- (6)  $e \models \xi_2 \vee \xi_1$  by Rule (14e) on (4)
- (7)  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by Rule (17b) on (5)
- (8)  $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$  by Rule (17b) on (6)

**Case false.**

- (3)  $satisfy(e, \xi_2) = \text{false}$  by assumption
- (4)  $e \not\models \xi_2$  by Lemma 2.0.19 on (3)
- (5)  $e \models_{\tau} \xi_1 \vee \xi_2$  by Rule (16c) on (2) and (4)
- (6)  $e \models_{\tau}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by Rule (17a) on (5)

□

**Lemma 2.0.11.** *If  $e_1 \models_{\tau}^{\dagger} \xi_1$  then  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$*

*Proof.*

- (1)  $e_1 \models_{\tau}^{\dagger} \xi_1$  by assumption

By rule induction over Rules (17) on (1).

**Case (17b).**

- (2)  $e_1 \models \xi_1$  by assumption
- (3)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (14g) on (2)
- (4)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Rule (17b) on (3)

**Case (17a).**

- |   |                      |
|---|----------------------|
| (2) $e_1 \models_{\tau} \xi_1$  | by assumption        |
| (3) $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\xi_1)$           | by Rule (16e) on (2) |
| (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ | by Rule (17a) on (3) |

□

**Lemma 2.0.12.** *If  $e_2 \models_{\tau}^{\dagger} \xi_2$  then  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$*

*Proof.*

- |  |               |
|--|---------------|
| (1) $e_2 \models_{\tau}^{\dagger} \xi_2$ | by assumption |
|--|---------------|

By rule induction over Rules (17) on (1).

**Case (17b).**

- |   |                      |
|---|----------------------|
| (2) $e_2 \models \xi_2$   | by assumption        |
| (3) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$                  | by Rule (14h) on (2) |
| (4) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ | by Rule (17b) on (3) |

**Case (17a).**

- |   |                      |
|---|----------------------|
| (2) $e_2 \models_{\tau} \xi_2$  | by assumption        |
| (3) $\text{inl}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$           | by Rule (16f) on (2) |
| (4) $\text{inl}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ | by Rule (17a) on (3) |

□

**Lemma 2.0.13.** *If  $e_1 \models_{\tau}^{\dagger} \xi_1$  and  $e_2 \models_{\tau}^{\dagger} \xi_2$  then  $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$*

**Lemma 2.0.14** (Soundness and Completeness of Refutable Constraints).  $\xi \text{ refutable}_{\tau}$  iff  $\text{refutable}_{\tau}(\xi) = \text{true}$ .

**Lemma 2.0.15.** *If  $e \text{ notintro}$  and  $\xi \text{ refutable}_{\tau}$  then either  $\dagger(\xi) \text{ refutable}_{\tau}$  or  $e \models \dagger(\xi)$ .*

*Proof.* By structural induction on  $\xi$ . □

**Lemma 2.0.16.** *There does not exist such a constraint  $\xi_1 \wedge \xi_2$  such that  $\xi_1 \wedge \xi_2 \text{ refutable}_{\tau}$ .*

*Proof.* By rule induction over Rules (10), we notice that  $\xi_1 \wedge \xi_2 \text{ refutable}_{\tau}$  is in syntactic contradiction with all the cases, hence not derivable. □

**Lemma 2.0.17.** *There does not exist such a constraint  $\xi_1 \vee \xi_2$  such that  $\xi_1 \vee \xi_2 \text{ refutable}_{\tau}$ .*

*Proof.* By rule induction over Rules (10), we notice that  $\xi_1 \vee \xi_2$  **refutable?** is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 2.0.18.** *If  $e$  **notintro** and  $e \models \xi$  then  $\xi$  ~~**refutable?**~~.*

*Proof.*

- |                         |               |
|-------------------------|---------------|
| (1) $e$ <b>notintro</b> | by assumption |
| (2) $e \models \xi$     | by assumption |

By rule induction over Rules (14) on (2).

**Case (14a).**

- |                  |               |
|------------------|---------------|
| (3) $\xi = \top$ | by assumption |
|------------------|---------------|

Assume  $\top$  **refutable?**. By rule induction over Rules (10), no case applies due to syntactic contradiction.  
Therefore,  $\top$  ~~**refutable?**~~.

**Case (14e),(14f).**

- |   |                 |
|---|-----------------|
| (3) $\xi = \xi_1 \vee \xi_2$                        | by assumption   |
| (4) $\xi_1 \vee \xi_2$ <del><b>refutable?</b></del> | by Lemma 2.0.17 |

**Case (14d).**

- |   |                 |
|---|-----------------|
| (3) $\xi = \xi_1 \wedge \xi_2$                        | by assumption   |
| (4) $\xi_1 \wedge \xi_2$ <del><b>refutable?</b></del> | by Lemma 2.0.16 |

**Case (14j).**

- |  |                      |
|--|----------------------|
| (3) $\xi = (\xi_1, \xi_2)$               | by assumption        |
| (4) $\text{prl}(e) \models \xi_1$        | by assumption        |
| (5) $\text{prr}(e) \models \xi_2$        | by assumption        |
| (6) $\text{prl}(e)$ <b>notintro</b>      | by Rule (26e)        |
| (7) $\text{prr}(e)$ <b>notintro</b>      | by Rule (26f)        |
| (8) $\xi_1$ <del><b>refutable?</b></del> | by IH on (6) and (4) |
| (9) $\xi_2$ <del><b>refutable?</b></del> | by IH on (7) and (5) |

Assume  $(\xi_1, \xi_2)$  **refutable?**. By rule induction over Rules (10) on it, only two cases apply.

**Case (10d).**

- |                                |               |
|--------------------------------|---------------|
| (10) $\xi_1$ <b>refutable?</b> | by assumption |
|--------------------------------|---------------|

Contradicts (8).

**Case (10e).**

(10)  $\xi_2$  **refutable?** by assumption

Contradicts (9).

Therefore,  $(\xi_1, \xi_2)$  ~~**refutable?**~~.

**Otherwise.**

(3)  $e = \underline{n}, \text{inl}_{\tau_2}(e_1), \text{inr}_{\tau_1}(e_2), (e_1, e_2)$  by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

□

**Lemma 2.0.19** (Soundness and Completeness of Satisfaction Judgment).  $e \models \xi$  iff  $\text{satisfy}(e, \xi) = \text{true}$ .

*Proof.* We prove soundness and completeness separately.

1. Soundness:

(1)  $e \models \xi$  by assumption

By rule induction over Rules (14) on (1).

**Case (14a).**

(2)  $\xi = \top$  by assumption

(3)  $\text{satisfy}(e, \top) = \text{true}$  by Definition 15a

**Case (14b).**

(2)  $e = \underline{n}$  by assumption

(3)  $\xi = \underline{n}$  by assumption

(4)  $\text{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$  by Definition 15b

**Case (14c).**

(2)  $e = \underline{n_1}$  by assumption

(3)  $\xi = \underline{\neg \bot}$  by assumption

(4)  $n_1 \neq n_2$  by assumption

(5)  $\text{satisfy}(\underline{n_1}, \underline{\neg \bot}) = (n_1 \neq n_2) = \text{true}$  by Definition 15c on (4)

**Case (14d).**

(2)  $\xi = \xi_1 \wedge \xi_2$  by assumption



- (3)  $e \models \xi_1$  by assumption
- (4)  $e \models \xi_2$  by assumption
- (5)  $\text{satisfy}(e, \xi_1) = \text{true}$  by IH on (3)
- (6)  $\text{satisfy}(e, \xi_2) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1)$  and  $\text{satisfy}(e, \xi_2) = \text{true}$   
by Definition 15d on (5) and (6)

**Case (14e).**

- (2)  $\xi = \xi_1 \vee \xi_2$  by assumption
- (3)  $e \models \xi_1$  by assumption
- (4)  $\text{satisfy}(e, \xi_1) = \text{true}$  by IH on (3)
- (5)  $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$  or  $\text{satisfy}(e, \xi_2) = \text{true}$   
by Definition 15e on (4)

**Case (14f).**

- (2)  $\xi = \xi_1 \vee \xi_2$  by assumption
- (3)  $e \models \xi_2$  by assumption
- (4)  $\text{satisfy}(e, \xi_2) = \text{true}$  by IH on (3)
- (5)  $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$  or  $\text{satisfy}(e, \xi_2) = \text{true}$   
by Definition 15e on (4)

**Case (14g).**

- (2)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption
- (3)  $\xi = \text{inl}(\xi_1)$  by assumption
- (4)  $e_1 \models \xi_1$  by assumption
- (5)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by IH on (4)
- (6)  $\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) = \text{true}$   
by Definition 15f on (5)

**Case (14h).**

- (2)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption
- (3)  $\xi = \text{inl}(\xi_2)$  by assumption
- (4)  $e_2 \models \xi_2$  by assumption
- (5)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by IH on (4)
- (6)  $\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\xi_2)) = \text{satisfy}(e_2, \xi_2) = \text{true}$   
by Definition 15g on (5)

**Case (14i).**

- (2)  $e = (e_1, e_2)$  by assumption
- (3)  $\xi = (\xi_1, \xi_2)$  by assumption
- (4)  $e_1 \models \xi_1$  by assumption
- (5)  $e_2 \models \xi_2$  by assumption
- (6)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by IH on (5)
- (8)  $\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$   
 $\text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) = \text{true}$   
by Definition 15h on (6) and (7)

**Case (14j).**

- (2)  $\xi = (\xi_1, \xi_2)$  by assumption
- (3)  $e \text{ notintro}$  by assumption
- (4)  $\text{prl}(e) \models \xi_1$  by assumption
- (5)  $\text{prr}(e) \models \xi_2$  by assumption
- (6)  $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$  by IH on (5)

By rule induction over Rules (26) on (3).

**Otherwise.**

- (8)  $e = (\emptyset^u, (\emptyset_0)^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\})$   
by assumption
- (9)  $\text{satisfy}(e, (\xi_1, \xi_2)) =$   
 $\text{satisfy}(\text{prl}(e), \xi_1) \text{ and } \text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$   
by Definition 15 on (6) and (7)

2. Completeness:

- (1)  $\text{satisfy}(e, \xi) = \text{true}$  by assumption

By structural induction on  $\xi$ .

**Case  $\xi = \top$ .**

- (2)  $e \models \top$  by Rule (14a)

**Case  $\xi = \perp, ?$ .**

- (2)  $\text{satisfy}(e, \xi) = \text{false}$  by Definition 15o

(2) contradicts (1) and thus vacuously true.

**Case  $\xi = \underline{n}$ .**

By structural induction on  $e$ .

**Case**  $e = \underline{n'}$ .

$$(2) \quad n' = n$$

by Definition 15b on

(1)

$$(3) \quad \underline{n'} \models \underline{n}$$

by Rule (14b) on (2)

**Otherwise.**

$$(2) \quad \textit{satisfy}(e, \underline{n}) = \text{false}$$

by Definition 15o

(2) contradicts (1) and thus vacuously true.

**Case**  $\xi = \underline{\mathcal{N}}$ .

By structural induction on  $e$ .

**Case**  $e = \underline{n'}$ .

$$(2) \quad n' \neq n$$

by Definition 15c on (1)

$$(3) \quad \underline{n'} \models \underline{\mathcal{N}}$$

by Rule (14c) on (2)

**Otherwise.**

$$(2) \quad \textit{satisfy}(e, \underline{\mathcal{N}}) = \text{false}$$

by Definition 15o

(2) contradicts (1) and thus vacuously true.

**Case**  $\xi = \xi_1 \wedge \xi_2$ .

$$(2) \quad \textit{satisfy}(e, \xi_1) = \text{true}$$

by Definition 15d on

(1)

$$(3) \quad \textit{satisfy}(e, \xi_2) = \text{true}$$

by Definition 15d on

(1)

$$(4) \quad e \models \xi_1$$

by IH on (2)

$$(5) \quad e \models \xi_2$$

by IH on (3)

$$(6) \quad e \models \xi_1 \wedge \xi_2$$

by Rule (14d) on (4)

and (5)

**Case**  $\xi = \xi_1 \vee \xi_2$ .

$$(2) \quad \textit{satisfy}(e, \xi_1) \text{ or } \textit{satisfy}(e, \xi_2) = \text{true}$$

by Definition 15e on (1)

By case analysis on (2).

**Case**  $\textit{satisfy}(e, \xi_1) = \text{true}$ .

$$(3) \quad \textit{satisfy}(e, \xi_1) = \text{true}$$

by assumption

$$(4) \quad e \models \xi_1$$

by IH on (3)

$$(5) \quad e \models \xi_1 \vee \xi_2$$

by Rule (14e) on (4)

**Case**  $\textit{satisfy}(e, \xi_2) = \text{true}$ .

$$(3) \quad \textit{satisfy}(e, \xi_2) = \text{true}$$

by assumption

$$(4) \quad e \models \xi_2$$

by IH on (3)

$$(5) \quad e \models \xi_1 \vee \xi_2$$

by Rule (14f) on (4)

**Case  $\xi = \text{inl}(\xi_1)$ .**

By structural induction on  $e$ .

**Case  $e = \text{inl}_{\tau_2}(e_1)$ .**

- (2)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by Definition 15f on (1)
- (3)  $e_1 \models \xi_1$  by IH on (2)
- (4)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (14g) on (3)

**Otherwise.**

- (2)  $\text{satisfy}(e, \text{inl}(\xi_1)) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

**Case  $\xi = \text{inr}(\xi_2)$ .**

By structural induction on  $e$ .

**Case  $e = \text{inr}_{\tau_1}(e_2)$ .**

- (2)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by Definition 15g on (1)
- (3)  $e_2 \models \xi_2$  by IH on (2)
- (4)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by Rule (14h) on (3)

**Otherwise.**

- (2)  $\text{satisfy}(e, \text{inr}(\xi_2)) = \text{false}$  by Definition 15o
- (2) contradicts (1) and thus vacuously true.

**Case  $\xi = (\xi_1, \xi_2)$ .**

By structural induction on  $e$ .

**Case  $e = (e_1, e_2)$ .**

- (2)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by Definition 15h on (1)
- (3)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by Definition 15h on (1)
- (4)  $e_1 \models \xi_1$  by IH on (2)
- (5)  $e_2 \models \xi_2$  by IH on (3)
- (6)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (14i) on (4) and (5)

**Case  $e = (\llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{rs\})$ .**

- (2)  $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$  by Definition 15h on (1)
- (3)  $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$  by Definition 15h on (1)
- (4)  $\text{prl}(e) \models \xi_1$  by IH on (2)
- (5)  $\text{prr}(e) \models \xi_2$  by IH on (3)
- (6)  $e \text{ notintro}$  by each rule in Rules (26)

(7)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (14j) on (6) and (4) and (5)

**Otherwise.**

(2)  $\text{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$  by Definition 15o  
 (2) contradicts (1) and thus vacuously true.

□

**Lemma 2.0.20.**  $e \not\models \xi$  and  $e \not\models_{\text{?}} \xi$  iff  $e \not\models_{\text{?}}^{\dagger} \xi$ .

*Proof.* 1. Sufficiency:

(1)  $e \not\models \xi$  by assumption  
 (2)  $e \not\models_{\text{?}} \xi$  by assumption

Assume  $e \models_{\text{?}}^{\dagger} \xi$ . By rule induction over Rules (17) on it.

**Case (17a).**

(3)  $e \models \xi$  by assumption

Contradicts (1).

**Case (17b).**

(3)  $e \models_{\text{?}} \xi$  by assumption

Contradicts (2).

Therefore,  $e \models_{\text{?}}^{\dagger} \xi$  is not derivable.

2. Necessity:

(1)  $e \not\models_{\text{?}}^{\dagger} \xi$  by assumption

Assume  $e \models \xi$ .

(2)  $e \models_{\text{?}}^{\dagger} \xi$  by Rule (17b) on assumption

Contradicts (1). Therefore,  $e \not\models \xi$ . Assume  $e \models_{\text{?}} \xi$ .

(3)  $e \models_{\text{?}}^{\dagger} \xi$  by Rule (17a) on assumption

Contradicts (1). Therefore,  $e \not\models_{\text{?}} \xi$ .

□

**Theorem 2.1** (Exclusiveness of Satisfaction Judgment). *If  $\xi : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e$  final then exactly one of the following holds*

1.  $e \models \xi$

2.  $e \models_{\tau} \xi$

3.  $e \not\models_{\tau}^{\dagger} \xi$

*Proof.*

(4)  $\xi : \tau$  by assumption

(5)  $\cdot; \Delta \vdash e : \tau$  by assumption

(6)  $e$  **final** by assumption

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

**Case (8a).**

(7)  $\xi = \top$  by assumption

(8)  $e \models \top$  by Rule (14a)

(9)  $e \not\models_{\tau} \top$  by Lemma 2.0.3

(10)  $e \models_{\tau}^{\dagger} \top$  by Rule (17b) on (8)

**Case (8b).**

(7)  $\xi = \perp$  by assumption

(8)  $e \not\models \perp$  by Lemma 2.0.1

(9)  $e \not\models_{\tau} \perp$  by Lemma 2.0.2

(10)  $e \not\models_{\tau}^{\dagger} \perp$  by Lemma 2.0.20 on (8) and (9)

**Case (1b).**

(7)  $\xi = ?$  by assumption

(8)  $e \not\models ?$  by Lemma 2.0.4

(9)  $e \models_{\tau} ?$  by Rule (16a)

(10)  $e \models_{\tau}^{\dagger} ?$  by Rule (17a) on (9)

**Case (8c).**

(7)  $\xi = \underline{n_2}$  by assumption

(8)  $\tau = \mathbf{num}$  by assumption

By rule induction over Rules (19) on (5), the following cases apply.

**Case** (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (9)  $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (10)  $e$  **notintro** by Rule  
(26a),(26b),(26c),(26d),(26e),(26f)

Assume  $e \models \underline{n_2}$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on  $\xi$ .

- (11)  $e \not\models \underline{n_2}$  by contradiction
- (12)  $\underline{n_2}$  **refutable?** by Rule (10a)
- (13)  $e \models_{\text{?}} \underline{n_2}$  by Rule (16b) on (10)  
and (12)
- (14)  $e \models_{\text{?}}^{\dagger} \underline{n_2}$  by Rule (17a) on (13)

**Case** (19d).

- (9)  $e = \underline{n_1}$  by assumption

Assume  $\underline{n_1} \models_{\text{?}} \underline{n_2}$ . By rule induction over Rules (16), only one case applies.

**Case** (16b).

- (10)  $\underline{n_1}$  **notintro** by assumption
- Contradicts Lemma 4.0.5.

- (11)  $\underline{n_1} \not\models_{\text{?}} \underline{n_2}$  by contradiction

By case analysis on whether  $n_1$  is equal to  $n_2$ .

**Case**  $n_1 = n_2$ .

- (12)  $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$  by Definition 15
- (13)  $\underline{n_1} \models \underline{n_2}$  by Lemma 2.0.19 on  
(12)
- (14)  $e \models_{\text{?}}^{\dagger} \underline{n_2}$  by Rule (17b) on (13)

**Case**  $n_1 \neq n_2$ .

- (12)  $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$  by Definition 15
- (13)  $\underline{n_1} \not\models \underline{n_2}$  by Lemma 2.0.19 on  
(12)
- (14)  $e \not\models_{\text{?}}^{\dagger} \underline{n_2}$  by Lemma 2.0.20 on  
(11) and (13)

**Case** (8f).

- (7)  $\xi = \xi_1 \vee \xi_2$  by assumption

By inductive hypothesis on (5) and (6), exactly one of  $e \models \xi_1$ ,  $e \models? \xi_1$ , and  $e \not\models?^\dagger \xi_1$  holds. The same goes for  $\xi_2$ . By case analysis on which conclusion holds for  $\xi_1$  and  $\xi_2$ .

**Case**  $e \models \xi_1, e \models \xi_2$ .

- |  |                       |
|--|-----------------------|
| (8) $e \models \xi_1$                      | by assumption         |
| (9) $e \not\models? \xi_1$                 | by assumption         |
| (10) $e \models \xi_2$                     | by assumption         |
| (11) $e \not\models? \xi_2$                | by assumption         |
| (12) $e \models \xi_1 \vee \xi_2$          | by Rule (14e) on (8)  |
| (13) $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (17b) on (12) |

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

- |   |               |
|---|---------------|
| (14) $\xi_1 \vee \xi_2$ <b>refutable?</b> | by assumption |
| Contradicts Lemma 2.0.17.                 |               |

**Case** (16c).

- |                         |               |
|-------------------------|---------------|
| (14) $e \models? \xi_1$ | by assumption |
| Contradicts (9).        |               |

**Case** (16d).

- |                         |               |
|-------------------------|---------------|
| (14) $e \models? \xi_2$ | by assumption |
| Contradicts (11).       |               |

- |  |                  |
|--|------------------|
| (15) $e \not\models? \xi_1 \vee \xi_2$ | by contradiction |
|--|------------------|

**Case**  $e \models \xi_1, e \models? \xi_2$ .

- |  |                       |
|--|-----------------------|
| (8) $e \models \xi_1$                      | by assumption         |
| (9) $e \not\models? \xi_1$                 | by assumption         |
| (10) $e \not\models \xi_2$                 | by assumption         |
| (11) $e \models? \xi_2$                    | by assumption         |
| (12) $e \models \xi_1 \vee \xi_2$          | by Rule (14e) on (8)  |
| (13) $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (17b) on (12) |

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

- |   |               |
|---|---------------|
| (14) $\xi_1 \vee \xi_2$ <b>refutable?</b> | by assumption |
| Contradicts Lemma 2.0.17.                 |               |

**Case** (16c).

- |                         |               |
|-------------------------|---------------|
| (14) $e \models? \xi_1$ | by assumption |
|-------------------------|---------------|



Contradicts (9).

**Case (16d).**

(14)  $e \not\models \xi_1$

by assumption

Contradicts (8).

(15)  $e \not\models? \xi_1 \vee \xi_2$

by contradiction

**Case**  $e \models \xi_1, e \not\models?^\dagger \xi_2$ .

(8)  $e \models \xi_1$

by assumption

(9)  $e \not\models? \xi_1$

by assumption

(10)  $e \not\models \xi_2$

by assumption

(11)  $e \not\models? \xi_2$

by assumption

(12)  $e \models \xi_1 \vee \xi_2$

by Rule (14e) on (8)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$

by Rule (17b) on (12)

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(14)  $\xi_1 \vee \xi_2$  **refutable?**

by assumption

Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \models? \xi_1$

by assumption

Contradicts (9).

**Case (16d).**

(14)  $e \not\models \xi_1$

by assumption

Contradicts (8).

(15)  $e \not\models? \xi_1 \vee \xi_2$

by contradiction

**Case**  $e \models? \xi_1, e \models \xi_2$ .

(8)  $e \not\models \xi_1$

by assumption

(9)  $e \models? \xi_1$

by assumption

(10)  $e \models \xi_2$

by assumption

(11)  $e \not\models? \xi_2$

by assumption

(12)  $e \models \xi_1 \vee \xi_2$

by Rule (14f) on (10)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$

by Rule (17b) on (12)

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(14)  $\xi_1 \vee \xi_2$  **refutable?**

by assumption

Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \not\models \xi_2$  by assumption

Contradicts (10).

**Case (16d).**

(14)  $e \models? \xi_2$  by assumption

Contradicts (11).

(15)  $e \not\models? \xi_1 \vee \xi_2$  by contradiction

**Case  $e \models? \xi_1, e \models? \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \models? \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \models? \xi_2$  by assumption

(12)  $e \models? \xi_1 \vee \xi_2$  by Rule (16c) on (9) and (10)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$  by Rule (17a) on (12)

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (14), only two cases apply.

**Case (14e).**

(14)  $e \models \xi_1$  by assumption

Contradicts (8)

**Case (14f).**

(14)  $e \models \xi_2$  by assumption

Contradicts (10)

(15)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

**Case  $e \models? \xi_1, e \not\models?^\dagger \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \models? \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \not\models? \xi_2$  by assumption

(12)  $e \models? \xi_1 \vee \xi_2$  by Rule (16c) on (9) and (10)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$  by Rule (17a) on (12)

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (14), only two cases apply.

**Case (14e).**

(14)  $e \models \xi_1$  by assumption

Contradicts (8).

**Case (14f).**

(14)  $e \models \xi_2$  by assumption

Contradicts (10).

(15)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

**Case  $e \not\models_{\vdash}^{\dagger} \xi_1, e \models \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \not\models_{\vdash} \xi_1$  by assumption

(10)  $e \models \xi_2$  by assumption

(11)  $e \not\models_{\vdash} \xi_2$  by assumption

(12)  $e \models \xi_1 \vee \xi_2$  by Rule (14f) on (10)

(13)  $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$  by Rule (17b) on (12)

Assume  $e \models_{\vdash} \xi_1 \vee \xi_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(14)  $\xi_1 \vee \xi_2$  **refutable**<sub>?</sub> by assumption

Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \not\models \xi_2$  by assumption

Contradicts (10).

**Case (16d).**

(14)  $e \models_{\vdash} \xi_2$  by assumption

Contradicts (11).

(15)  $e \not\models_{\vdash} \xi_1 \vee \xi_2$  by contradiction

**Case  $e \not\models_{\vdash}^{\dagger} \xi_1, e \models_{\vdash} \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \not\models_{\vdash} \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \models_{\vdash} \xi_2$  by assumption

(12)  $e \models_{\vdash} \xi_1 \vee \xi_2$  by Rule (16d) on (11) and (8)

(13)  $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$  by Rule (17a) on (12)

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (14), only two cases apply.

**Case (14e).**

(14)  $e \models \xi_1$  by assumption

Contradicts (8)

**Case (14f).**

(14)  $e \models \xi_2$  by assumption

Contradicts (10)

(15)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

**Case  $e \not\models_{\text{?}} \xi_1, e \not\models_{\text{?}} \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \not\models_{\text{?}} \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \not\models_{\text{?}} \xi_2$  by assumption

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14e).**

(12)  $e \models \xi_1$  by assumption

Contradicts (8).

**Case (14f).**

(12)  $e \models \xi_2$  by assumption

Contradicts (10).

(13)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

Assume  $e \models_{\text{?}} \xi_1 \vee \xi_2$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(14)  $\xi_1 \vee \xi_2$  **refutable**<sub>?</sub> by assumption

Contradicts Lemma 2.0.17.

**Case (16c).**

(14)  $e \models_{\text{?}} \xi_1$  by assumption

Contradicts (9).

**Case (16d).**

(14)  $e \models_{\text{?}} \xi_2$  by assumption

Contradicts (11).

(15)  $e \not\models_{\text{?}} \xi_1 \vee \xi_2$  by contradiction

(16)  $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by Lemma 2.0.20 on (13) and (15)

**Case (8g).**

(7)  $\xi = \text{inl}(\xi_1)$  by assumption  
 (8)  $\tau = (\tau_1 + \tau_2)$  by assumption  
 (9)  $\xi_1 : \tau_1$  by assumption

By rule induction over Rules (19) on (5), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

(10)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$  by assumption  
 (11)  $e \text{ notintro}$  by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume  $e \models \text{inl}(\xi_1)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12)  $e \not\models \text{inl}(\xi_1)$  by contradiction

By case analysis on the value of  $\text{refutable}_{\tau}(\text{inl}(\xi_1))$ .

**Case  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{true}$ .**

(13)  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{true}$  by assumption  
 (14)  $\text{inl}(\xi_1) \text{ refutable}_{\tau}$  by Lemma 2.0.14 on (13)  
 (15)  $e \models_{\tau} \text{inl}(\xi_1)$  by Rule (16b) on (11) and (14)  
 (16)  $e \models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Rule (17a) on (15)

**Case  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{false}$ .**

(13)  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{false}$  by assumption  
 (14)  ~~$\text{inl}(\xi_1) \text{ refutable}_{\tau}$~~  by Lemma 2.0.14 on (13)

Assume  $e \models_{\tau} \text{inl}(\xi_1)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(15)  $\text{inl}(\xi_1) \text{ refutable}_{\tau}$  by assumption  
 Contradicts (14).

(16)  $e \not\models_{\tau} \text{inl}(\xi_1)$  by contradiction  
 (17)  $e \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.20 on (12) and (16)

**Case (19j).**

- (10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption
- (11)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (12)  $e_1$  **final** by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_1 \models \xi_1$ ,  $e_1 \models? \xi_1$ , and  $e_1 \not\models? \xi_1$  holds. By case analysis on which one holds.

**Case  $e_1 \models \xi_1$ .**

- (13)  $e_1 \models \xi_1$  by assumption
- (14)  $e_1 \not\models? \xi_1$  by assumption
- (15)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (14g) on (13)
- (16)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (17b) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ . By rule induction over Rules (16) on it, only two cases apply.

**Case (16b).**

- (17)  $\text{inl}_{\tau_2}(e_1)$  **notintro** by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16e).**

- (17)  $e_1 \models? \xi_1$

Contradicts (14).

- (18)  $\text{inl}_{\tau_2}(e_1) \not\models? \text{inl}(\xi_1)$  by contradiction

**Case  $e_1 \models? \xi_1$ .**

- (13)  $e_1 \not\models \xi_1$  by assumption
- (14)  $e_1 \models? \xi_1$  by assumption
- (15)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (16e) on (14)
- (16)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (17a) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14g).**

- (17)  $e_1 \models \xi_1$

Contradicts (13).

- (18)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$  by contradiction

**Case  $e_1 \not\models? \xi_1$ .**

- (13)  $e_1 \not\models \xi_1$  by assumption

(14)  $e_1 \not\models_{\tau} \xi_1$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14g).**

(15)  $e_1 \models \xi_1$

Contradicts (13).

(16)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\xi_1)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(17)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16e).**

(17)  $e_1 \models_{\tau} \xi_1$

Contradicts (14).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\xi_1)$  by contradiction

(19)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.20 on (16) and (18)

**Case (19k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\xi_1)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\xi_1)$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\xi_1)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(12)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inl}(\xi_1)$  by contradiction

(14)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.20 on (11) and (13)

**Case (8h).**

(7)  $\xi = \text{inr}(\xi_2)$  by assumption

- (8)  $\tau = (\tau_1 + \tau_2)$  by assumption  
 (9)  $\xi_2 : \tau_2$  by assumption

By rule induction over Rules (19) on (5), the following cases apply.

**Case** (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (10)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
 by assumption  
 (11)  $e$  **notintro** by Rule  
 (26a),(26b),(26c),(26d),(26e),(26f)

Assume  $e \models \text{inr}(\xi_2)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (12)  $e \not\models \text{inr}(\xi_2)$  by contradiction

By case analysis on the value of  $\text{refutable}_?( \text{inr}(\xi_2) )$ .

inr is  
refutable

**Case**  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{true}$ .

- (13)  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{true}$  by assumption  
 (14)  $\text{inr}(\xi_2)$  **refutable?** by Lemma 2.0.14 on  
 (13)  
 (15)  $e \models? \text{inr}(\xi_2)$  by Rule (16b) on (11)  
 and (14)  
 (16)  $e \models?^\dagger \text{inr}(\xi_2)$  by Rule (17a) on (15)

**Case**  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{false}$ .

- (13)  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{false}$  by assumption  
 (14)  ~~$\text{inr}(\xi_2)$  **refutable?**~~ by Lemma 2.0.14 on  
 (13)

Assume  $e \models? \text{inr}(\xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case** (16b).

- (15)  $\text{inr}(\xi_2)$  **refutable?** by assumption  
 Contradicts (14).

- (16)  $e \not\models? \text{inr}(\xi_2)$  by contradiction  
 (17)  $e \not\models?^\dagger \text{inr}(\xi_2)$  by Lemma 2.0.20 on  
 (12) and (16)

**Case** (19j).

- (10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\xi_2)$ . By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (11)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inr}(\xi_2)$  by contradiction



Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inr}(\xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(12)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inr}(\xi_2)$  by contradiction

(14)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inr}(\xi_2)$  by Lemma 2.0.20 on (11) and (13)

**Case (19k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

(11)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(12)  $e_2 \text{ final}$  by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_2 \models \xi_2$ ,  $e_2 \models_{\tau} \xi_2$ , and  $e_2 \not\models_{\tau}^{\dagger} \xi_2$  holds. By case analysis on which one holds.

**Case  $e_2 \models \xi_2$ .**

(13)  $e_2 \models \xi_2$  by assumption

(14)  $e_2 \not\models_{\tau} \xi_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by Rule (14g) on (13)

(16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$  by Rule (17b) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$ . By rule induction over Rules (16) on it, only two cases apply.

**Case (16b).**

(17)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16f).**

(17)  $e_2 \models_{\tau} \xi_2$

Contradicts (14).

(18)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\xi_2)$  by contradiction

**Case  $e_2 \models_{\tau} \xi_2$ .**

(13)  $e_2 \not\models \xi_2$  by assumption

(14)  $e_2 \models_{\tau} \xi_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$  by Rule (16f) on (14)

(16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$  by Rule (17a) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14h).**

$$(17) \quad e_2 \models \xi_2$$

Contradicts (13).

$$(18) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2) \quad \text{by contradiction}$$

**Case  $e_2 \not\models_{\tau_1}^{\dagger} \xi_2$ .**

$$(13) \quad e_2 \not\models \xi_2 \quad \text{by assumption}$$

$$(14) \quad e_2 \not\models_{\tau_1} \xi_2 \quad \text{by assumption}$$

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14h).**

$$(15) \quad e_2 \models \xi_2$$

Contradicts (13).

$$(16) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2) \quad \text{by contradiction}$$

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau_1} \text{inr}(\xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

$$(17) \quad \text{inr}_{\tau_1}(e_2) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

**Case (16f).**

$$(17) \quad e_2 \models_{\tau_1} \xi_2$$

Contradicts (14).

$$(18) \quad \text{inr}_{\tau_1}(e_2) \not\models_{\tau_1} \text{inr}(\xi_2) \quad \text{by contradiction}$$

$$(19) \quad \text{inl}_{\tau_2}(e_1) \not\models_{\tau_2}^{\dagger} \text{inl}(\xi_1) \quad \text{by Lemma 2.0.20 on (16) and (18)}$$

**Case (14i).**

$$(7) \quad \xi = (\xi_1, \xi_2) \quad \text{by assumption}$$

$$(8) \quad \tau = (\tau_1 \times \tau_2) \quad \text{by assumption}$$

$$(9) \quad \xi_1 : \tau_1 \quad \text{by assumption}$$

$$(10) \quad \xi_2 : \tau_2 \quad \text{by assumption}$$

By rule induction over Rules (19) on (5), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- (11)  $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (12)  $e$  notintro  
by Rule (26a),(26b),(26c),(26d),(26e),(26f)
- (13)  $e$  indet  
by Lemma 4.0.9 on (6) and (12)
- (14)  $\text{prl}(e)$  indet  
by Rule (24g) on (13)
- (15)  $\text{prl}(e)$  final  
by Rule (25b) on (14)
- (16)  $\text{prr}(e)$  indet  
by Rule (24h) on (13)
- (17)  $\text{prr}(e)$  final  
by Rule (25b) on (16)
- (18)  $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$   
by Rule (19h) on (5)
- (19)  $\cdot; \Delta \vdash \text{prr}(e) : \tau_2$   
by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of  $\text{prl}(e) \models \xi_1$ ,  $\text{prl}(e) \models? \xi_1$ , and  $\text{prl}(e) \not\models?^\dagger \xi_1$  holds.

By inductive hypothesis on (10) and (19) and (17), exactly one of  $\text{prr}(e) \models \xi_2$ ,  $\text{prr}(e) \models? \xi_2$ , and  $\text{prr}(e) \not\models?^\dagger \xi_2$  holds.

By case analysis on which conclusion holds for  $\xi_1$  and  $\xi_2$ .

**Case**  $\text{prl}(e) \models \xi_1, \text{prr}(e) \models \xi_2$ .

- (20)  $\text{prl}(e) \models \xi_1$   
by assumption
- (21)  $\text{prl}(e) \not\models? \xi_1$   
by assumption
- (22)  $\text{prr}(e) \models \xi_2$   
by assumption
- (23)  $\text{prr}(e) \not\models? \xi_2$   
by assumption
- (24)  $e \models (\xi_1, \xi_2)$   
by Rule (14j) on (12) and (20) and (22)
- (25)  $e \models?^\dagger (\xi_1, \xi_2)$   
by Rule (17b) on (24)
- (26)  ~~$(\xi_1, \xi_2)$  refutable?~~  
by Lemma 2.0.18 on (12) and (24)

Assume  $e \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case** (16b).

- (27)  $(\xi_1, \xi_2)$  refutable?  
by assumption
- Contradicts (26).

- (28)  $e \not\models? (\xi_1, \xi_2)$   
by contradiction

**Case**  $\text{prl}(e) \models \xi_1, \text{prr}(e) \models? \xi_2$ .

- (20)  $\text{prl}(e) \models \xi_1$   
by assumption
- (21)  $\text{prl}(e) \not\models? \xi_1$   
by assumption
- (22)  $\text{prr}(e) \not\models \xi_2$   
by assumption
- (23)  $\text{prr}(e) \models? \xi_2$   
by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

(24)  $\text{prr}(e) \models \xi_2$  by assumption  
Contradicts (22)

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

By rule induction over Rules (16) on (23), only one case applies.

**Case (16b).**

(26)  $\xi_2 \text{ refutable?}$  by assumption  
(27)  $(\xi_1, \xi_2) \text{ refutable?}$  by Rule (10e) on (26)  
(28)  $e \models? (\xi_1, \xi_2)$  by Rule (16b) on (12) and (27)  
(29)  $e \models?^\dagger (\xi_1, \xi_2)$  by Rule (17a) on (28)

**Case  $\text{prl}(e) \models \xi_1, \text{prr}(e) \not\models?^\dagger \xi_2$ .**

(20)  $\text{prl}(e) \models \xi_1$  by assumption  
(21)  $\text{prl}(e) \not\models? \xi_1$  by assumption  
(22)  $\text{prr}(e) \not\models \xi_2$  by assumption  
(23)  $\text{prr}(e) \not\models? \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14j).**

(24)  $\text{prr}(e) \models \xi_2$  by assumption  
Contradicts (22).

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $e \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(26)  $(\xi_1, \xi_2) \text{ refutable?}$  by assumption

By rule induction over Rules (10) on (26), only two cases apply.

**Case (10d).**

(27)  $\xi_1 \text{ refutable?}$  by assumption  
(28)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)  
(29)  $\text{prl}(e) \models? \xi_1$  by Rule (16b) on (28) and (27)

Contradicts (21).

**Case (10e).**

assume no  
"or" and  
"and" in  
pair

(27) $\xi_2$ <b>refutable</b> <sub>?</sub>	by assumption
(28) <b>pr</b> $r(e)$ <b>notintro</b>	by Rule (26f)
(29) <b>pr</b> $r(e) \models_? \xi_2$	by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$	by contradiction
(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$	by Lemma 2.0.20 on (25) and (30)

**Case**  $\text{prl}(e) \models_? \xi_1, \text{pr}r(e) \models \xi_2$ .

(20) $\text{prl}(e) \not\models \xi_1$	by assumption
(21) $\text{prl}(e) \models_? \xi_1$	by assumption
(22) $\text{pr}r(e) \models \xi_2$	by assumption
(23) $\text{pr}r(e) \not\models_? \xi_2$	by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14), only one case applies.

**Case** (14j).

(24) $\text{prl}(e) \models \xi_1$	by assumption
------------------------------------	---------------

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$	by contradiction
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By rule induction over Rules (16) on (21), only one case applies.

**Case** (16b).

(26) $\xi_1$ <b>refutable</b> <sub>?</sub>	by assumption
(27) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub>	by Rule (10e) on (26)
(28) $e \models_? (\xi_1, \xi_2)$	by Rule (16b) on (12) and (27)
(29) $e \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (28)

assume no  
"or" and  
"and" in  
pair

**Case**  $\text{prl}(e) \models_? \xi_1, \text{pr}r(e) \models_? \xi_2$ .

(20) $\text{prl}(e) \not\models \xi_1$	by assumption
(21) $\text{prl}(e) \models_? \xi_1$	by assumption
(22) $\text{pr}r(e) \not\models \xi_2$	by assumption
(23) $\text{pr}r(e) \models_? \xi_2$	by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14), only one case applies.

**Case** (14j).

(24) $\text{prl}(e) \models \xi_1$	by assumption
------------------------------------	---------------

Contradicts (20).

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction  
 By rule induction over Rules (16) on (23), only one case applies.

**Case (16b).**

- |   |                                |
|---|--------------------------------|
| (26) $\xi_2$ <b>refutable</b> <sub>?</sub>          | by assumption                  |
| (27) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (10e) on (26)          |
| (28) $e \models_{?} (\xi_1, \xi_2)$                 | by Rule (16b) on (12) and (27) |
| (29) $e \models_{?}^{\dagger} (\xi_1, \xi_2)$       | by Rule (17a) on (28)          |

assume no  
"or" and  
"and" in  
pair

**Case**  $\text{prl}(e) \models_{?} \xi_1, \text{prr}(e) \not\models_{?}^{\dagger} \xi_2$ .

- |  |               |
|--|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$     | by assumption |
| (21) $\text{prl}(e) \models_{?} \xi_1$     | by assumption |
| (22) $\text{prr}(e) \not\models \xi_2$     | by assumption |
| (23) $\text{prr}(e) \not\models_{?} \xi_2$ | by assumption |

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

- |                                    |               |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
| Contradicts (20)                   |               |

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction  
 By rule induction over Rules (16) on (21), only one case applies.

**Case (16b).**

- |   |                                |
|---|--------------------------------|
| (26) $\xi_1$ <b>refutable</b> <sub>?</sub>          | by assumption                  |
| (27) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (10e) on (26)          |
| (28) $e \models_{?} (\xi_1, \xi_2)$                 | by Rule (16b) on (12) and (27) |
| (29) $e \models_{?}^{\dagger} (\xi_1, \xi_2)$       | by Rule (17a) on (28)          |

assume no  
"or" and  
"and" in  
pair

**Case**  $\text{prl}(e) \not\models_{?}^{\dagger} \xi_1, \text{prr}(e) \models \xi_2$ .

- |  |               |
|--|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$     | by assumption |
| (21) $\text{prl}(e) \not\models_{?} \xi_1$ | by assumption |
| (22) $\text{prr}(e) \models \xi_2$         | by assumption |
| (23) $\text{prr}(e) \not\models_{?} \xi_2$ | by assumption |

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case (14j).**

- |                                    |               |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
| Contradicts (20)                   |               |

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $e \models_{\text{?}} (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case (16b).**

(26)  $(\xi_1, \xi_2) \text{ refutable}_{\text{?}}$  by assumption

By rule induction over Rules (10) on (26), only two cases apply.

**Case (10d).**

(27)  $\xi_1 \text{ refutable}_{\text{?}}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)

(29)  $\text{prl}(e) \models_{\text{?}} \xi_1$  by Rule (16b) on (28) and (27)

Contradicts (21).

**Case (10e).**

(27)  $\xi_2 \text{ refutable}_{\text{?}}$  by assumption

(28)  $\text{prr}(e) \text{ notintro}$  by Rule (26f)

(29)  $\text{prr}(e) \models_{\text{?}} \xi_2$  by Rule (16b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models_{\text{?}} (\xi_1, \xi_2)$  by contradiction

(31)  $e \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$  by Lemma 2.0.20 on (25) and (30)

**Case  $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \xi_1, \text{prr}(e) \models_{\text{?}} \xi_2$ .**

(20)  $\text{prl}(e) \not\models \xi_1$  by assumption

(21)  $\text{prl}(e) \not\models_{\text{?}} \xi_1$  by assumption

(22)  $\text{prr}(e) \not\models \xi_2$  by assumption

(23)  $\text{prr}(e) \models_{\text{?}} \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14), only one case applies.

**Case (14j).**

(24)  $\text{prl}(e) \models \xi_1$  by assumption

Contradicts (20).

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

By rule induction over Rules (16) on (23), only one case applies.

**Case (16b).**

(26)  $\xi_2 \text{ refutable}_{\text{?}}$  by assumption

(27)  $(\xi_1, \xi_2) \text{ refutable}_{\text{?}}$  by Rule (10e) on (26)

assume no  
"or" and  
"and" in  
pair

(28)  $e \models_{\tau} (\xi_1, \xi_2)$  by Rule (16b) on (12) and (27)

(29)  $e \models_{\tau}^{\dagger} (\xi_1, \xi_2)$  by Rule (17a) on (28)

**Case**  $\text{prl}(e) \not\models_{\tau}^{\dagger} \xi_1, \text{prl}(e) \not\models_{\tau}^{\dagger} \xi_2$ .

(20)  $\text{prl}(e) \not\models \xi_1$  by assumption

(21)  $\text{prl}(e) \not\models_{\tau} \xi_1$  by assumption

(22)  $\text{prl}(e) \not\models \xi_2$  by assumption

(23)  $\text{prl}(e) \not\models_{\tau} \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only one case applies.

**Case** (14j).

(24)  $\text{prl}(e) \models \xi_1$  by assumption

Contradicts (20)

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $e \models_{\tau} (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, only one case applies.

**Case** (16b).

(26)  $(\xi_1, \xi_2) \text{ refutable}_{\tau}$  by assumption

By rule induction over Rules (10) on (26), only two cases apply.

**Case** (10d).

(27)  $\xi_1 \text{ refutable}_{\tau}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)

(29)  $\text{prl}(e) \models_{\tau} \xi_1$  by Rule (16b) on (28) and (27)

Contradicts (21).

**Case** (10e).

(27)  $\xi_2 \text{ refutable}_{\tau}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (26f)

(29)  $\text{prl}(e) \models_{\tau} \xi_2$  by Rule (16b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models_{\tau} (\xi_1, \xi_2)$  by contradiction

(31)  $e \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$  by Lemma 2.0.20 on (25) and (30)

**Case** (19g).

(11)  $e = (e_1, e_2)$  by assumption

(12)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption



- (13)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption
- (14)  $e_1$  **final** by Lemma 4.0.4 on (6)
- (15)  $e_2$  **final** by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of  $e_1 \models \xi_1$ ,  $e_1 \models? \xi_1$ , and  $e_1 \models \overline{\xi_1}$  holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of  $e_2 \models \xi_2$ ,  $e_2 \models? \xi_2$ , and  $e_2 \models \overline{\xi_2}$  holds.

By case analysis on which conclusion holds for  $\xi_1$  and  $\xi_2$ .

**Case**  $e_1 \models \xi_1, e_2 \models \xi_2$ .

- (16)  $e_1 \models \xi_1$  by assumption
- (17)  $e_1 \not\models? \xi_1$  by assumption
- (18)  $e_2 \models \xi_2$  by assumption
- (19)  $e_2 \not\models? \xi_2$  by assumption
- (20)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (14i) on (16)  
and (18)
- (21)  $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$  by Rule (17b) on (20)

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

- (22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case** (16g).

- (22)  $e_1 \models? \xi_1$  by assumption

Contradicts (17).

**Case** (16h).

- (22)  $e_2 \models? \xi_2$  by assumption

Contradicts (19).

**Case** (16i).

- (22)  $e_1 \models? \xi_1$  by assumption

Contradicts (17).

- (23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction

**Case**  $e_1 \models \xi_1, e_2 \models? \xi_2$ .

- (16)  $e_1 \models \xi_1$  by assumption
- (17)  $e_1 \not\models? \xi_1$  by assumption
- (18)  $e_2 \not\models \xi_2$  by assumption
- (19)  $e_2 \models? \xi_2$  by assumption
- (20)  $(e_1, e_2) \models? (\xi_1, \xi_2)$  by Rule (16h) on (16)  
and (19)

(21)  $(e_1, e_2) \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$  by Rule (17a) on (20)

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (14i).**

(22)  $e_2 \models \xi_2$  by assumption

Contradicts (18).

(23)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

**Case**  $e_1 \models \xi_1, e_2 \not\models_{\text{?}}^{\dagger} \xi_2$ .

(16)  $e_1 \models \xi_1$  by assumption

(17)  $e_1 \not\models_{\text{?}} \xi_1$  by assumption

(18)  $e_2 \not\models \xi_2$  by assumption

(19)  $e_2 \not\models_{\text{?}} \xi_2$  by assumption

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (14i).**

(20)  $e_2 \models \xi_2$  by assumption

Contradicts (18).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (16g).**

(22)  $e_1 \models_{\text{?}} \xi_1$  by assumption

Contradicts (17).

**Case (16h).**

(22)  $e_2 \models_{\text{?}} \xi_2$  by assumption

Contradicts (19).

**Case (16i).**

(22)  $e_1 \models_{\text{?}} \xi_1$  by assumption

Contradicts (17).

- |      |  |                                     |
|------|--|-------------------------------------|
| (23) | $(e_1, e_2) \not\models? (\xi_1, \xi_2)$         | by contradiction                    |
| (24) | $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ | by Lemma 2.0.20 on<br>(21) and (23) |

**Case**  $e_1 \models? \xi_1, e_2 \models \xi_2$ .

- |      |                                      |                                   |
|------|--------------------------------------|-----------------------------------|
| (16) | $e_1 \not\models \xi_1$              | by assumption                     |
| (17) | $e_1 \models? \xi_1$                 | by assumption                     |
| (18) | $e_2 \models \xi_2$                  | by assumption                     |
| (19) | $e_2 \not\models? \xi_2$             | by assumption                     |
| (20) | $(e_1, e_2) \models? (\xi_1, \xi_2)$ | by Rule (16g) on (17)<br>and (18) |

- |      |  |                       |
|------|--|-----------------------|
| (21) | $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ | by Rule (17a) on (20) |
|------|--|-----------------------|

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case** (14j).

- |      |                              |               |
|------|------------------------------|---------------|
| (22) | $(e_1, e_2)$ <b>notintro</b> | by assumption |
|------|------------------------------|---------------|

Contradicts Lemma 4.0.8.

**Case** (14i).

- |      |                     |               |
|------|---------------------|---------------|
| (22) | $e_1 \models \xi_1$ | by assumption |
|------|---------------------|---------------|

Contradicts (16).

- |      |   |                  |
|------|---|------------------|
| (23) | $(e_1, e_2) \not\models (\xi_1, \xi_2)$ | by contradiction |
|------|---|------------------|

**Case**  $e_1 \models? \xi_1, e_2 \models? \xi_2$ .

- |      |                                      |                                   |
|------|--------------------------------------|-----------------------------------|
| (16) | $e_1 \not\models \xi_1$              | by assumption                     |
| (17) | $e_1 \models? \xi_1$                 | by assumption                     |
| (18) | $e_2 \not\models \xi_2$              | by assumption                     |
| (19) | $e_2 \models? \xi_2$                 | by assumption                     |
| (20) | $(e_1, e_2) \models? (\xi_1, \xi_2)$ | by Rule (16i) on (17)<br>and (19) |

- |      |  |                       |
|------|--|-----------------------|
| (21) | $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ | by Rule (17a) on (20) |
|------|--|-----------------------|

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case** (14j).

- |      |                              |               |
|------|------------------------------|---------------|
| (22) | $(e_1, e_2)$ <b>notintro</b> | by assumption |
|------|------------------------------|---------------|

Contradicts Lemma 4.0.8.

**Case** (14i).

- |      |                     |               |
|------|---------------------|---------------|
| (22) | $e_1 \models \xi_1$ | by assumption |
|------|---------------------|---------------|

Contradicts (16).

(23)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

**Case**  $e_1 \models? \xi_1, e_2 \not\models?^\dagger \xi_2$ .

(16)  $e_1 \not\models \xi_1$  by assumption

(17)  $e_1 \models? \xi_1$  by assumption

(18)  $e_2 \not\models \xi_2$  by assumption

(19)  $e_2 \not\models? \xi_2$  by assumption

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case** (14j).

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case** (14i).

(20)  $e_1 \models \xi_1$  by assumption

Contradicts (16).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case** (16b).

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case** (16g).

(22)  $e_2 \models \xi_2$  by assumption

Contradicts (18).

**Case** (16h).

(22)  $e_2 \models? \xi_2$  by assumption

Contradicts (19).

**Case** (16i).

(22)  $e_2 \models? \xi_2$  by assumption

Contradicts (19).

(23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction

(24)  $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models?^\dagger \xi_1, e_2 \models \xi_2$ .

(16)  $e_1 \not\models \xi_1$  by assumption

(17)  $e_1 \not\models? \xi_1$  by assumption

(18)  $e_2 \models \xi_2$  by assumption

(19)  $e_2 \not\models? \xi_2$  by assumption  
 Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.  
**Case (14j).**  
 (20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.  
**Case (14i).**  
 (20)  $e_1 \models \xi_1$  by assumption  
 Contradicts (16).  
  
 (21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction  
 Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, the following cases apply.  
**Case (16b).**  
 (22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.  
**Case (16g).**  
 (22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).  
**Case (16h).**  
 (22)  $e_2 \models? \xi_2$  by assumption  
 Contradicts (19).  
**Case (16i).**  
 (22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).  
  
 (23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction  
 (24)  $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$  by Lemma 2.0.20 on (21) and (23)  
**Case  $e_1 \not\models?^\dagger \xi_1, e_2 \models? \xi_2$ .**  
 (16)  $e_1 \not\models \xi_1$  by assumption  
 (17)  $e_1 \not\models? \xi_1$  by assumption  
 (18)  $e_2 \not\models \xi_2$  by assumption  
 (19)  $e_2 \models? \xi_2$  by assumption  
 Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.  
**Case (14j).**  
 (20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.  
**Case (14i).**

(20)  $e_2 \models \xi_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.

**Case (16g).**

(22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).

**Case (16h).**

(22)  $e_1 \models \xi_1$  by assumption  
 Contradicts (16).

**Case (16i).**

(22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).

(23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction

(24)  $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models?^\dagger \xi_1, e_2 \not\models?^\dagger \xi_2$ .

(16)  $e_1 \not\models \xi_1$  by assumption  
 (17)  $e_1 \not\models? \xi_1$  by assumption  
 (18)  $e_2 \not\models \xi_2$  by assumption  
 (19)  $e_2 \not\models? \xi_2$  by assumption

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (14) on it, only two cases apply.

**Case (14j).**

(20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.

**Case (14i).**

(20)  $e_2 \models \xi_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (16) on it, the following cases apply.

**Case (16b).**

(22) $(e_1, e_2)$ <b>notintro</b>	by assumption
Contradicts Lemma 4.0.8.	
<b>Case (16g).</b>	
(22) $e_1 \models_{\tau} \xi_1$	by assumption
Contradicts (17).	
<b>Case (16h).</b>	
(22) $e_2 \models_{\tau} \xi_2$	by assumption
Contradicts (19).	
<b>Case (16i).</b>	
(22) $e_1 \models_{\tau} \xi_1$	by assumption
Contradicts (17).	
(23) $(e_1, e_2) \not\models_{\tau} (\xi_1, \xi_2)$	by contradiction
(24) $(e_1, e_2) \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$	by Lemma 2.0.20 on (21) and (23)

□

**Definition 2.1.1** (Entailment of Constraints). *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \models \xi_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **val** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies  $e \models \xi_2$*

**Definition 2.1.2** (Potential Entailment of Constraints). *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \models_{\tau}^{\dagger} \xi_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies  $e \models_{\tau}^{\dagger} \xi_2$*

**Corollary 2.1.1.** *Suppose that  $\xi : \tau$  and  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final**. Then  $\top \models_{\tau}^{\dagger} \xi$  implies  $e \models_{\tau}^{\dagger} \xi$*

*Proof.*

(1) $\xi : \tau$	by assumption
(2) $\cdot ; \Gamma \vdash e : \tau$	by assumption
(3) $e$ <b>final</b>	by assumption
(4) $\top \models_{\tau}^{\dagger} \xi$	by assumption
(5) $e_1 \models \top$	by Rule (14a)
(6) $e_1 \models_{\tau}^{\dagger} \top$	by Rule (17b) on (5)
(7) $\top : \tau$	by Rule (8a)
(8) $e_1 \models_{\tau}^{\dagger} \xi_r$	by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

□

### 3 Static Semantics

$$\begin{aligned}
\tau &::= \text{num} \mid (\tau_1 \rightarrow \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e &::= x \mid \underline{n} \\
&\quad \mid (\lambda x : \tau. e) \mid e_1(e_2) \\
&\quad \mid (e_1, e_2) \\
&\quad \mid \text{inl}_\tau(e) \mid \text{inr}_\tau(e) \mid \text{match}(e)\{\hat{r}s\} \\
&\quad \mid \textcolor{violet}{\mathbb{O}}^u \mid \textcolor{violet}{\langle} e \textcolor{violet}{\rangle}^u \\
\hat{r}s &::= (rs \mid r \mid rs) \\
rs &::= \cdot \mid (r \mid rs') \\
r &::= p \Rightarrow e \\
p &::= x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \text{inl}(p) \mid \text{inr}(p) \mid \textcolor{violet}{\mathbb{O}}^w \mid \textcolor{violet}{\langle} p \textcolor{violet}{\rangle}^w \\
\boxed{(\hat{r}s)^\diamond = rs} &\quad rs \text{ can be obtained by erasing pointer from } \hat{r}s
\end{aligned}$$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \quad (18a)$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \quad (18b)$$

$$\boxed{\Gamma; \Delta \vdash e : \tau} \quad e \text{ is of type } \tau$$

$$\frac{\text{TVar}}{\Gamma, x : \tau; \Delta \vdash x : \tau} \quad (19a)$$

$$\frac{\text{TEHole}}{\Gamma; \Delta, u :: \tau \vdash \textcolor{violet}{\mathbb{O}}^u : \tau} \quad (19b)$$

$$\frac{\text{THole} \quad \Gamma; \Delta, u :: \tau \vdash e : \tau'}{\Gamma; \Delta, u :: \tau \vdash \textcolor{violet}{\langle} e \textcolor{violet}{\rangle}^u : \tau} \quad (19c)$$

$$\frac{\text{TNum}}{\Gamma; \Delta \vdash \underline{n} : \text{num}} \quad (19d)$$

$$\frac{\text{TLam} \quad \Gamma, x : \tau_1; \Delta \vdash e : \tau_2}{\Gamma; \Delta \vdash (\lambda x : \tau_1. e) : (\tau_1 \rightarrow \tau_2)} \quad (19e)$$

$$\frac{\text{TAp} \quad \Gamma; \Delta \vdash e_1 : (\tau_2 \rightarrow \tau) \quad \Gamma; \Delta \vdash e_2 : \tau_2}{\Gamma; \Delta \vdash e_1(e_2) : \tau} \quad (19f)$$

$$\frac{\text{TPair} \quad \Gamma; \Delta \vdash e_1 : \tau_1 \quad \Gamma; \Delta \vdash e_2 : \tau_2}{\Gamma; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \quad (19g)$$

$$\frac{\text{TPrl} \quad \Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma; \Delta \vdash \text{prl}(e) : \tau_1} \quad (19h)$$



$$\frac{\text{TPrr}}{\Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)} \quad \Gamma; \Delta \vdash \text{pr}(e) : \tau_2 \quad (19i)$$

$$\frac{\text{TInl}}{\Gamma; \Delta \vdash e : \tau_1} \quad \Gamma; \Delta \vdash \text{inl}_{\tau_2}(e) : (\tau_1 + \tau_2) \quad (19j)$$

$$\frac{\text{TInr}}{\Gamma; \Delta \vdash e : \tau_2} \quad \Gamma; \Delta \vdash \text{inr}_{\tau_1}(e) : (\tau_1 + \tau_2) \quad (19k)$$

$$\frac{\text{TMatchZPre} \quad \Gamma; \Delta \vdash e : \tau \quad \Gamma; \Delta \vdash [\perp]r \mid rs : \tau[\xi] \Rightarrow \tau' \quad \top \models_{\tau}^{\dagger} \xi}{\Gamma; \Delta \vdash \text{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \quad (19l)$$

$$\frac{\text{TMatchNZPre} \quad \Gamma; \Delta \vdash e : \tau \quad e \text{ final} \quad \Gamma; \Delta \vdash [\perp]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \quad \Gamma; \Delta \vdash [\perp \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \quad e \not\models_{\tau}^{\dagger} \xi_{pre} \quad \top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}}{\Gamma; \Delta \vdash \text{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \quad (19m)$$

$\boxed{p : \tau[\xi] \dashv \Gamma; \Delta}$   $p$  is assigned type  $\tau$  and emits constraint  $\xi$

$$\frac{\text{PTVar}}{x : \tau[\top] \dashv \cdot; x : \tau} \quad (20a)$$

$$\frac{\text{PTWild}}{\_ : \tau[\top] \dashv \cdot; \cdot} \quad (20b)$$

$$\frac{\text{PTEHole}}{\langle \rangle^w : \tau[?] \dashv \cdot; w :: \tau} \quad (20c)$$

$$\frac{\text{PTHole} \quad p : \tau[\xi] \dashv \Gamma; \Delta}{\langle p \rangle^w : \tau'[?] \dashv \Gamma; \Delta, w :: \tau'} \quad (20d)$$

$$\frac{\text{PTNum}}{\underline{n} : \text{num}[\underline{n}] \dashv \cdot; \cdot} \quad (20e)$$

$$\frac{\text{PTInl} \quad p : \tau_1[\xi] \dashv \Gamma; \Delta}{\text{inl}(p) : (\tau_1 + \tau_2)[\text{inl}(\xi)] \dashv \Gamma; \Delta} \quad (20f)$$

$$\frac{\text{PTInr} \quad p : \tau_2[\xi] \dashv \Gamma; \Delta}{\text{inr}(p) : (\tau_1 + \tau_2)[\text{inr}(\xi)] \dashv \Gamma; \Delta} \quad (20g)$$

$$\frac{\text{PTPair} \quad p_1 : \tau_1[\xi_1] \dashv \Gamma_1; \Delta_1 \quad p_2 : \tau_2[\xi_2] \dashv \Gamma_2; \Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv \Gamma_1 \uplus \Gamma_2; \Delta_1 \uplus \Delta_2} \quad (20h)$$

$$\boxed{\Gamma ; \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'} \quad \begin{array}{l} r \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\frac{\text{CTRrule} \quad p : \tau[\xi] \dashv \vdash \Gamma_p ; \Delta_p \quad \Gamma \uplus \Gamma_p ; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma ; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \quad (21a)$$

$$\boxed{\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'} \quad \begin{array}{l} rs \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\frac{\text{CTOneRules} \quad \Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \xi_r \not\models \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \quad (22a)$$

$$\frac{\text{CTRrules} \quad \Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \quad \xi_r \not\models \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \quad (22b)$$

**Lemma 3.0.1.** *If  $p : \tau[\xi] \dashv \vdash \Gamma ; \Delta$  then  $\xi : \tau$ .*

*Proof.* By rule induction over Rules (20). □

**Lemma 3.0.2.** *If  $\cdot ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  then  $\xi_r : \tau_1$ .*

*Proof.* By rule induction over Rules (21). □

**Lemma 3.0.3.** *If  $\cdot ; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$  then  $\xi_{rs} : \tau_1$ .*

*Proof.* By rule induction over Rules (22). □

**Lemma 3.0.4.** *If  $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$  and  $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$  and  $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$  then  $\Gamma ; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$*

*Proof.*

- (1)  $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$  by assumption
- (2)  $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$  by assumption
- (3)  $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$  by assumption

By rule induction over Rules (22) on (1).

**Case (22a).**

- (4)  $rs = r' \mid \cdot$  by assumption
- (5)  $\xi_{rs} = \xi'_r$  by assumption
- (6)  $\Gamma ; \Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$  by assumption
- (7)  $\xi'_r \not\models \xi_{pre}$  by assumption
- (8)  $\Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r]r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$  by Rule (22a) on (2) and (3)
- (9)  $\Gamma ; \Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$  by Rule (22b) on (6) and (8) and (7)

$$(10) \quad \Gamma ; \Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau' \\ \text{by Definition 18 on (9)}$$

**Case (22b).**

$$\begin{aligned} (4) \quad rs &= r' \mid rs' && \text{by assumption} \\ (5) \quad \xi_{rs} &= \xi'_r \vee \xi'_{rs} && \text{by assumption} \\ (6) \quad \Gamma ; \Delta \vdash r' : \tau[\xi'_r] &\Rightarrow \tau' && \text{by assumption} \\ (7) \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] &\Rightarrow \tau' && \text{by assumption} \\ (8) \quad \xi'_r &\not\models \xi_{pre} && \text{by assumption} \\ (9) \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^\diamond : \tau[\xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by IH on (7) and (2)} \\ &\text{and (3)} \\ (10) \quad \Gamma ; \Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^\diamond) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by Rule (22b) on (6)} \\ &\text{and (9) and (8)} \\ (11) \quad \Gamma ; \Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by Definition 18 on} \\ &\text{(10)} \end{aligned}$$

□

**Lemma 3.0.5** (Substitution). *If  $\Gamma, x : \tau ; \Delta \vdash e_0 : \tau_0$  and  $\Gamma ; \Delta \vdash e : \tau$  then  $\Gamma ; \Delta \vdash [e/x]e_0 : \tau_0$*

**Lemma 3.0.6** (Simultaneous Substitution). *If  $\Gamma \uplus \Gamma' ; \Delta \vdash e : \tau$  and  $\theta : \Gamma'$  then  $\Gamma ; \Delta \vdash [\theta]e : \tau$*

**Lemma 3.0.7** (Substitution Typing). *If  $e \triangleright p \dashv\vdash \theta$  and  $\cdot ; \Delta_e \vdash e : \tau$  and  $p : \tau[\xi] \dashv\vdash \Gamma ; \Delta$  then  $\theta : \Gamma$*

Proof by induction on the derivation of  $e \triangleright p \dashv\vdash \theta$ .

**Theorem 3.1** (Determinism). *If  $\cdot ; \Delta \vdash e : \tau$  then exactly one of the following holds*

1.  $e \text{ val}$
2.  $e \text{ indet}$
3.  $e \mapsto e'$  for some unique  $e'$

## 4 Dynamic Semantics

$\boxed{e \text{ val}}$      $e$  is a value

$$\frac{\text{VNum}}{\underline{n \text{ val}}} \quad (23a)$$

$$\frac{\text{VLam}}{(\lambda x : \tau. e) \text{ val}} \quad (23b)$$

$$\frac{\text{VPair} \quad e_1 \text{ val} \quad e_2 \text{ val}}{(e_1, e_2) \text{ val}} \quad (23c)$$

$$\frac{\text{VInl} \quad e \text{ val}}{\text{inl}_\tau(e) \text{ val}} \quad (23d)$$

$$\frac{\text{VInr} \quad e \text{ val}}{\text{inr}_\tau(e) \text{ val}} \quad (23e)$$

$\boxed{e \text{ indet}}$      $e$  is indeterminate

$$\frac{\text{IEHole}}{\llbracket \cdot \rrbracket^u \text{ indet}} \quad (24a)$$

$$\frac{\text{IHole} \quad e \text{ final}}{\llbracket e \rrbracket^u \text{ indet}} \quad (24b)$$

$$\frac{\text{IAp} \quad e_1 \text{ indet} \quad e_2 \text{ final}}{e_1(e_2) \text{ indet}} \quad (24c)$$

$$\frac{\text{IPairL} \quad e_1 \text{ indet} \quad e_2 \text{ val}}{(e_1, e_2) \text{ indet}} \quad (24d)$$

$$\frac{\text{IPairR} \quad e_1 \text{ val} \quad e_2 \text{ indet}}{(e_1, e_2) \text{ indet}} \quad (24e)$$

$$\frac{\text{IPair} \quad e_1 \text{ indet} \quad e_2 \text{ indet}}{(e_1, e_2) \text{ indet}} \quad (24f)$$

$$\frac{\text{IPrl} \quad e \text{ indet}}{\text{prl}(e) \text{ indet}} \quad (24g)$$

$$\frac{\text{IPrr} \quad e \text{ indet}}{\text{pr}(e) \text{ indet}} \quad (24h)$$

$$\frac{\text{IInL} \quad e \text{ indet}}{\text{inl}_\tau(e) \text{ indet}} \quad (24i)$$

$$\frac{\text{IInR} \quad e \text{ indet}}{\text{inr}_\tau(e) \text{ indet}} \quad (24j)$$

$$\frac{\text{IMatch} \quad e \text{ final} \quad e ? p_r}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \text{ indet}} \quad (24k)$$

$$\boxed{e \text{ final}} \quad e \text{ is final}$$

$$\frac{\text{FVal} \quad e \text{ val}}{e \text{ final}} \quad (25a)$$

$$\frac{\text{FIndet} \quad e \text{ indet}}{e \text{ final}} \quad (25b)$$

$$\boxed{e \text{ notintro}} \quad e \text{ cannot be a value syntactically}$$

$$\frac{\text{NVEHole}}{\text{⋈}^u \text{ notintro}} \quad (26a)$$

$$\frac{\text{NVHole}}{\text{⋈}(e)^u \text{ notintro}} \quad (26b)$$

$$\frac{\text{NVAp}}{e_1(e_2) \text{ notintro}} \quad (26c)$$

$$\frac{\text{NVMatch}}{\text{match}(e)\{\hat{r}s\} \text{ notintro}} \quad (26d)$$

$$\frac{\text{NVPrI}}{\text{prl}(e) \text{ notintro}} \quad (26e)$$

$$\frac{\text{NVPrR}}{\text{pr}(e) \text{ notintro}} \quad (26f)$$

$$\boxed{e' \in \text{values}(e)} \quad e' \text{ is one of the possible values of } e$$

$$\frac{\text{IVVal} \quad e \text{ val} \quad \cdot; \Delta \vdash e : \tau}{e \in \text{values}(e)} \quad (27a)$$

$$\frac{\text{IVIndet} \quad e \text{ notintro} \quad \cdot; \Delta \vdash e : \tau \quad e' \text{ val} \quad \cdot; \Delta \vdash e' : \tau}{e' \in \text{values}(e)} \quad (27b)$$

$$\frac{\text{IVInl} \quad \text{inl}_{\tau_2}(e_1) \text{ indet} \quad \cdot; \Delta \vdash \text{inl}_{\tau_2}(e_1) : \tau \quad e'_1 \in \text{values}(e_1)}{\text{inl}_{\tau_2}(e'_1) \in \text{values}(\text{inl}_{\tau_2}(e_1))} \quad (27c)$$

$$\frac{\text{IVInr} \quad \text{inr}_{\tau_1}(e_2) \text{ indet} \quad \cdot; \Delta \vdash \text{inr}_{\tau_1}(e_2) : \tau \quad e'_2 \in \text{values}(e_2)}{\text{inr}_{\tau_1}(e'_2) \in \text{values}(\text{inr}_{\tau_1}(e_2))} \quad (27d)$$

$$\frac{\text{IVPair} \quad (e_1, e_2) \text{ indet} \quad \cdot; \Delta \vdash (e_1, e_2) : \tau \quad e'_1 \in \text{values}(e_1) \quad e'_2 \in \text{values}(e_2)}{(e'_1, e'_2) \in \text{values}((e_1, e_2))} \quad (27e)$$

**Lemma 4.0.1.** *If  $e$  indet and  $\cdot; \Delta \vdash e : \tau$  and  $\dot{\xi} : \tau$  and  $e \not\models_{\tau}^{\dagger} \dot{\xi}$  then  $e' \not\models_{\tau}^{\dagger} \dot{\xi}$  for all  $e' \in \text{complete}(e)$ .*

*Proof.*

- (1)  $e$  indet by assumption
- (2)  $\cdot; \Delta \vdash e : \tau$  by assumption
- (3)  $\dot{\xi} : \tau$  by assumption
- (4)  $e \not\models_{\tau}^{\dagger} \dot{\xi}$  by assumption

By rule induction over Rules (8) on (3).

**Case (8a).**

- (5)  $\dot{\xi} = \top$  by assumption
- (6)  $e \models \top$  by Rule (14a)
- (7)  $e \models_{\tau}^{\dagger} \top$  by Rule (17b) on (6)

Contradicts (4).

**Case (1b).**

- (5)  $\dot{\xi} = ?$  by assumption
- (6)  $e \models_{\tau} ?$  by Rule (16a)
- (7)  $e \models_{\tau}^{\dagger} ?$  by Rule (17a) on (6)

Contradicts (4).

**Case (8c).**

- (5)  $\dot{\xi} = \underline{n}$  by assumption

- |  |               |
|--|---------------|
| (6) $\tau = \text{num}$                | by assumption |
| (7) $\underline{n} \text{ refutable?}$ | by Rule (10a) |

By rule induction over Rules (24) on (1).

**Case (24a).**

- |  |                              |
|--|------------------------------|
| (8) $e = \mathbb{O}^u$   | by assumption                |
| (9) $\mathbb{O}^u \text{ notintro}$                            | by Rule (26a)                |
| (10) $\mathbb{O}^u \models_{\text{?}} \underline{n}$           | by Rule (16b) on (9) and (7) |
| (11) $\mathbb{O}^u \models_{\text{?}}^{\dagger} \underline{n}$ | by Rule (17a) on (10)        |

Contradicts (4).

**Case (24b).**

- |   |                              |
|---|------------------------------|
| (8) $e = \langle e_1 \rangle^u$   | by assumption                |
| (9) $\langle e_1 \rangle^u \text{ notintro}$                            | by Rule (26b)                |
| (10) $\langle e_1 \rangle^u \models_{\text{?}} \underline{n}$           | by Rule (16b) on (9) and (7) |
| (11) $\langle e_1 \rangle^u \models_{\text{?}}^{\dagger} \underline{n}$ | by Rule (17a) on (10)        |

Contradicts (4).

**Case (24c).**

- |  |                              |
|--|------------------------------|
| (8) $e = e_1(e_2)$   | by assumption                |
| (9) $e_1(e_2) \text{ notintro}$                            | by Rule (26c)                |
| (10) $e_1(e_2) \models_{\text{?}} \underline{n}$           | by Rule (16b) on (9) and (7) |
| (11) $e_1(e_2) \models_{\text{?}}^{\dagger} \underline{n}$ | by Rule (17a) on (10)        |

Contradicts (4).

**Case (24g).**

- |   |                              |
|---|------------------------------|
| (8) $e = \text{prl}(e_1)$   | by assumption                |
| (9) $\text{prl}(e_1) \text{ notintro}$                            | by Rule (26e)                |
| (10) $\text{prl}(e_1) \models_{\text{?}} \underline{n}$           | by Rule (16b) on (9) and (7) |
| (11) $\text{prl}(e_1) \models_{\text{?}}^{\dagger} \underline{n}$ | by Rule (17a) on (10)        |

Contradicts (4).

**Case (24h).**

- |   |                              |
|---|------------------------------|
| (8) $e = \text{prr}(e_1)$                               | by assumption                |
| (9) $\text{prr}(e_1) \text{ notintro}$                  | by Rule (26f)                |
| (10) $\text{prr}(e_1) \models_{\text{?}} \underline{n}$ | by Rule (16b) on (9) and (7) |

(11)  $\text{pr}(e_1) \models_{\tau}^{\dagger} \underline{n}$  by Rule (17a) on (10)

Contradicts (4).

**Case (24k).**

(8)  $e = \text{match}(e_1)\{\hat{r}s\}$  by assumption

(9)  $\text{match}(e_1)\{\hat{r}s\} \text{ notintro}$  by Rule (26d)

(10)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau} \underline{n}$  by Rule (16b) on (9) and (7)

(11)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau}^{\dagger} \underline{n}$  by Rule (17a) on (10)

Contradicts (4).

**Case (24d), (24e), (24f).**

(8)  $e = (e_1, e_2)$  by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

**Case (24i).**

(8)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

**Case (24j).**

(8)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

**Case (8g).**

(5)  $\dot{\xi} = \text{inl}(\dot{\xi}_1)$  by assumption

(6)  $\tau = (\tau_1 + \tau_2)$  by assumption

(7)  $\dot{\xi}_1 : \tau_1$  by assumption

(8)  $\text{inl}(\dot{\xi}_1) \text{ refutable?}$  by Rule (10b)

By rule induction over Rules (24) on (1).

**Case (24a).**

(9)  $e = \text{inl}^u$  by assumption

(10)  $\text{inl}^u \text{ notintro}$  by Rule (26a)

(11)  $\text{inl}^u \models_{\tau} \text{inl}(\dot{\xi}_1)$  by Rule (16b) on (10) and (8)

(12)  $\text{inl}^u \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24b).**



(9) $e = \langle e_1 \rangle^u$	by assumption
(10) $\langle e_1 \rangle^u \text{ notintro}$	by Rule (26b)
(11) $\langle e_1 \rangle^u \models_{\dot{?}} \text{inl}(\dot{\xi}_1)$	by Rule (16b) on (10) and (8)
(12) $\langle e_1 \rangle^u \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$	by Rule (17a) on (11)
Contradicts (4).	
<b>Case (24c).</b>	
(9) $e = e_1(e_2)$	by assumption
(10) $e_1(e_2) \text{ notintro}$	by Rule (26c)
(11) $e_1(e_2) \models_{\dot{?}} \text{inl}(\dot{\xi}_1)$	by Rule (16b) on (10) and (8)
(12) $e_1(e_2) \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$	by Rule (17a) on (11)
Contradicts (4).	
<b>Case (24g).</b>	
(9) $e = \text{prl}(e_1)$	by assumption
(10) $\text{prl}(e_1) \text{ notintro}$	by Rule (26e)
(11) $\text{prl}(e_1) \models_{\dot{?}} \text{inl}(\dot{\xi}_1)$	by Rule (16b) on (10) and (8)
(12) $\text{prl}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$	by Rule (17a) on (11)
Contradicts (4).	
<b>Case (24h).</b>	
(9) $e = \text{prr}(e_1)$	by assumption
(10) $\text{prr}(e_1) \text{ notintro}$	by Rule (26f)
(11) $\text{prr}(e_1) \models_{\dot{?}} \text{inl}(\dot{\xi}_1)$	by Rule (16b) on (10) and (8)
(12) $\text{prr}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$	by Rule (17a) on (11)
Contradicts (4).	
<b>Case (24k).</b>	
(9) $e = \text{match}(e_1)\{\hat{r}s\}$	by assumption
(10) $\text{match}(e_1)\{\hat{r}s\} \text{ notintro}$	by Rule (26d)
(11) $\text{match}(e_1)\{\hat{r}s\} \models_{\dot{?}} \text{inl}(\dot{\xi}_1)$	by Rule (16b) on (10) and (8)
(12) $\text{match}(e_1)\{\hat{r}s\} \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$	by Rule (17a) on (11)
Contradicts (4).	
<b>Case (24d), (24e), (24f).</b>	
(9) $e = (e_1, e_2)$	by assumption

By rule induction over Rules (24) on (1), no rule applies due to syntactic contradiction.

**Case (24i).**

- (9)  $e = \text{inl}_{\tau'_2}(e_1)$  by assumption
- (10)  $e_1 \text{ indet}$  by assumption

By rule induction over Rules (19) on (2), only one rule applies.

**Case (19j).**

- (11)  $\tau'_2 = \tau_2$  by assumption
- (12)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (13)  $e_1 \not\models_{\tau_1}^{\dagger} \dot{\xi}_1$  by Lemma 2.0.11 on (4)

- (14) if  $e'_1 \in \text{values}(e_1)$  then  $e'_1 \not\models_{\tau_1}^{\dagger} \dot{\xi}_1$  by IH on (10) and (12) and (7) and (13)

To show if  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  then  $e' \not\models_{\tau_2}^{\dagger} \text{inl}(\dot{\xi}_1)$ , we assume  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$ .

- (15)  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  by assumption

By rule induction over Rules (27) on (15).

**Case (27a).**

- (16)  $\text{inl}_{\tau_2}(e_1) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

- (16)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.6

**Case (27c).**

- (16)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption
- (17)  $e'_1 \in \text{values}(e_1)$  by assumption
- (18)  $e'_1 \not\models_{\tau_1}^{\dagger} \dot{\xi}_1$  by (14) on (17)
- (19)  $\text{inl}_{\tau_2}(e'_1) \not\models_{\tau_2}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.11 on (18)

**Case (24j).**

- (9)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

To show if  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  then  $e' \not\models_{\tau_1}^{\dagger} \text{inl}(\dot{\xi}_1)$ , we assume  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$ .

- (10)  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  by assumption

By rule induction over Rules (27) on (10).

**Case (27a).**

- (11)  $\text{inr}_{\tau_1}(e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(11)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7

**Case (27d).**

(11)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption

(12)  $\text{inr}_{\tau_1}(e'_2) \not\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 1.0.19

**Case (8h).**

(5)  $\dot{\xi} = \text{inr}(\dot{\xi}_2)$  by assumption

(6)  $\tau = (\tau_1 + \tau_2)$  by assumption

(7)  $\dot{\xi}_2 : \tau_2$  by assumption

(8)  $\text{inr}(\dot{\xi}_2) \text{ refutable}_{\tau}$  by Rule (10c)

By rule induction over Rules (24) on (1).

**Case (24a).**

(9)  $e = \llbracket \cdot \rrbracket^u$  by assumption

(10)  $\llbracket \cdot \rrbracket^u \text{ notintro}$  by Rule (26a)

(11)  $\llbracket \cdot \rrbracket^u \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (10) and (8)

(12)  $\llbracket \cdot \rrbracket^u \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24b).**

(9)  $e = \llbracket e_1 \rrbracket^u$  by assumption

(10)  $\llbracket e_1 \rrbracket^u \text{ notintro}$  by Rule (26b)

(11)  $\llbracket e_1 \rrbracket^u \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (10) and (8)

(12)  $\llbracket e_1 \rrbracket^u \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24c).**

(9)  $e = e_1(e_2)$  by assumption

(10)  $e_1(e_2) \text{ notintro}$  by Rule (26c)

(11)  $e_1(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (10) and (8)

(12)  $e_1(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24g).**

- (9)  $e = \text{prl}(e_1)$  by assumption
- (10)  $\text{prl}(e_1) \text{ notintro}$  by Rule (26e)
- (11)  $\text{prl}(e_1) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (10) and (8)
- (12)  $\text{prl}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24h).**

- (9)  $e = \text{prr}(e_1)$  by assumption
- (10)  $\text{prr}(e_1) \text{ notintro}$  by Rule (26f)
- (11)  $\text{prr}(e_1) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (10) and (8)
- (12)  $\text{prr}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24k).**

- (9)  $e = \text{match}(e_1)\{\hat{r}s\}$  by assumption
- (10)  $\text{match}(e_1)\{\hat{r}s\} \text{ notintro}$  by Rule (26d)
- (11)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (16b) on (10) and (8)
- (12)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (17a) on (11)

Contradicts (4).

**Case (24d), (24e), (24f).**

- (9)  $e = (e_1, e_2)$  by assumption

By rule induction over Rules (24) on (1), no rule applies due to syntactic contradiction.

**Case (24i).**

- (9)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

To show if  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  then  $e' \not\models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$ , we assume  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$ .

- (10)  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  by assumption

By rule induction over Rules (27) on (10).

**Case (27a).**

- (11)  $\text{inl}_{\tau_2}(e_1) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

- (11)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.6

**Case (27c).**

- (11)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption
- (12)  $\text{inl}_{\tau_2}(e'_1) \not\vdash_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Lemma 1.0.18

**Case (24j).**

- (9)  $e = \text{inr}_{\tau'_1}(e_2)$  by assumption
- (10)  $e_2 \text{ indet}$  by assumption

By rule induction over Rules (19) on (2), only one rule applies.

**Case (19k).**

- (11)  $\tau'_1 = \tau_1$  by assumption
- (12)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption
- (13)  $e_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.11 on (4)

- (14) if  $e'_2 \in \text{values}(e_2)$  then  $e'_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$   
by IH on (10) and (12)  
and (7) and (13)

To show if  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  then  $e' \not\vdash_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$ , we assume  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$ .

- (15)  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  by assumption

By rule induction over Rules (27) on (15).

**Case (27a).**

- (16)  $\text{inr}_{\tau_1}(e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

- (16)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7

**Case (27d).**

- (16)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption
- (17)  $e'_2 \in \text{values}(e_2)$  by assumption
- (18)  $e'_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by (14) on (17)
- (19)  $\text{inr}_{\tau_1}(e'_2) \not\vdash_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Lemma 2.0.12 on (18)

**Case (8i).**

- (5)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption
- (6)  $\tau = (\tau_1 \times \tau_2)$  by assumption
- (7)  $\dot{\xi}_1 : \tau_1$  by assumption
- (8)  $\dot{\xi}_2 : \tau_2$  by assumption

By rule induction over Rules (24) on (1).

**Case** (24a), (24b), (24c), (24g), (24h), (24k).

- (9)  $e = \langle \rangle^u, \langle e_1 \rangle^u, e_1(e_2), \text{prl}(e_1), \text{pr}(e_1), \text{match}(e_1)\{\hat{r}\hat{s}\}$   
by assumption
- (10)  $e$  **notintro** by Rules (26)
- (11)  $\text{prl}(e)$  **notintro** by Rule (26e)
- (12)  $\text{pr}(e)$  **notintro** by Rule (26f)
- (13)  $\text{prl}(e)$  **indet** by Rule (24g) on (1)
- (14)  $\text{pr}(e)$  **indet** by Rule (24h) on (1)
- (15)  $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$  by Rule (19h) on (2)
- (16)  $\cdot; \Delta \vdash \text{pr}(e) : \tau_2$  by Rule (19i) on (2)

By case analysis on the result of  $\text{satisfyormay}(\text{prl}(e), \dot{\xi}_1)$ .

**Case** true.

- (17)  $\text{satisfyormay}(\text{prl}(e), \dot{\xi}_1) = \text{true}$   
by assumption
- (18)  $\text{prl}(e) \models_{\dot{\xi}_1}^\dagger$  by Lemma 1.0.21 on (17)

By case analysis on the result of  $\text{satisfyormay}(\text{pr}(e), \dot{\xi}_2)$ .

**Case** true.

- (19)  $\text{satisfyormay}(\text{pr}(e), \dot{\xi}_2) = \text{true}$   
by assumption
- (20)  $\text{pr}(e) \models_{\dot{\xi}_2}^\dagger$  by Lemma 1.0.21 on (19)

By rule induction over Rules (17) on (18).

**Case** (17b).

- (21)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

By rule induction over Rules (17) on (20).

**Case** (17b).

- (22)  $\text{pr}(e) \models \dot{\xi}_2$  by assumption
  - (23)  $e \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (14j) on (10) and (21) and (22)
  - (24)  $e \models_{\dot{\xi}_1, \dot{\xi}_2}^\dagger$  by Rule (17b) on (23)
- Contradicts (4).

**Case** (17a).

- (22)  $\text{pr}(e) \models_{\dot{\xi}_2}$  by assumption
- (23)  $\dot{\xi}_2$  **refutable?** by Lemma 1.0.14 on (12) and (22)
- (24)  $(\dot{\xi}_1, \dot{\xi}_2)$  **refutable?** by Rule (10e) on (23)
- (25)  $e \models_{\dot{\xi}_1, \dot{\xi}_2}$  by Rule (16b) on (10) and (24)

(26)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (25)

**Case (17a).**

(21)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\dot{\xi}_1 \text{ refutable}_{\dot{?}}$  by Lemma 1.0.14 on (11) and (21)

(23)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by Rule (10d) on (22)

(24)  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (16b) on (10) and (23)

(25)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17a) on (24)

**Case false.**

(19)  $\text{satisfyormay}(\text{prr}(e), \dot{\xi}_2) = \text{false}$

by assumption

(20)  $\text{prr}(e) \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Lemma 1.0.21 on (19)

(21) if  $e'_2 \in \text{values}(\text{prr}(e))$  then  $e'_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$   
by IH on (14) and (16) and (8) and (20)

To show if  $e' \in \text{values}(e)$  then  $e' \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}(e)$ .

(22)  $e' \in \text{values}(e)$  by assumption

By rule induction over Rules (27) on (22), only two rules apply.

**Case (27a).**

(23)  $e \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(23)  $e' \text{ val}$  by assumption

(24)  $\cdot; \Delta \vdash e' : (\tau_1 \times \tau_2)$  by assumption

By rule induction over Rules (23) on (23).

**Case (23a).**

(25)  $e' = \underline{n}$  by assumption

By rule induction over Rules (19) on (24), no rule applies due to syntactic contradiction.

**Case (23b).**

(25)  $e' = (\lambda x : \tau'. e'_1)$  by assumption

By rule induction over Rules (19) on (24), no rule applies due to syntactic contradiction.

**Case (23c).**

(25)  $e' = (e'_1, e'_2)$  by assumption

(26)  $e'_2 \text{ val}$  by assumption

By rule induction over Rules (19) on (24), only one rule applies.

**Case (19g).**

- (27)  $\cdot; \Delta \vdash e'_2 : \tau_2$  by assumption
- (28)  $e'_2 \in \text{values}(\text{pr}(e))$  by Rule (27b) on (12) and (16) and (26) and (27)
- (29)  $e'_2 \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by (21) on (28)
- (30)  $(e'_1, e'_2) \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (27)

**Case (23d).**

- (25)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption

By rule induction over Rules (19) on (24), no rule applies due to syntactic contradiction.

**Case (23e).**

- (25)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption

By rule induction over Rules (19) on (24), no rule applies due to syntactic contradiction.

**Case false.**

- (17)  $\text{satisfyormay}(\text{prl}(e), \dot{\xi}_1) = \text{false}$  by assumption
- (18)  $\text{prl}(e) \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by Lemma 1.0.21 on (17)
- (19) if  $e'_1 \in \text{values}(\text{prl}(e))$  then  $e'_1 \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by IH on (13) and (15) and (7) and (18)

To show if  $e' \in \text{values}(e)$  then  $e' \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}(e)$ .

- (20)  $e' \in \text{values}(e)$  by assumption

By rule induction over Rules (27) on (20), only two rules apply.

**Case (27a).**

- (21)  $e \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

- (21)  $e' \text{ val}$  by assumption
- (22)  $\cdot; \Delta \vdash e' : (\tau_1 \times \tau_2)$  by assumption

By rule induction over Rules (23) on (21).

**Case (23a).**

- (23)  $e' = \underline{n}$  by assumption

By rule induction over Rules (19) on (22), no rule applies due to syntactic contradiction.



**Case (23b).**

(23)  $e' = (\lambda x : \tau'. e'_1)$  by assumption

By rule induction over Rules (19) on (22), no rule applies due to syntactic contradiction.

**Case (23c).**

(23)  $e' = (e'_1, e'_2)$  by assumption

(24)  $e'_1 \text{ val}$  by assumption

By rule induction over Rules (19) on (22), only one rule applies.

**Case (19g).**

(25)  $\cdot; \Delta \vdash e'_1 : \tau_1$  by assumption

(26)  $e'_1 \in \text{values}(\text{prl}(e))$  by Rule (27b) on (11) and (15) and (24) and (25)

(27)  $e'_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  by (19) on (26)

(28)  $(e'_1, e'_2) \not\vdash_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (27)

**Case (23d).**

(23)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption

By rule induction over Rules (19) on (22), no rule applies due to syntactic contradiction.

**Case (23e).**

(23)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption

By rule induction over Rules (19) on (22), no rule applies due to syntactic contradiction.

**Case (24d).**

(9)  $e = (e_1, e_2)$  by assumption

(10)  $e_1 \text{ indet}$  by assumption

(11)  $e_2 \text{ val}$  by assumption

(12)  $e_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  or  $e_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.13 on (4)

By case analysis on the disjunction in (12).

**Case  $e_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$ .**

(13)  $e_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  by assumption

By rule induction over Rules (19) on (2), only one rule applies.

**Case (19g).**

(14)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption

(15) if  $e'_1 \in \text{values}(e_1)$  then  $e'_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  by IH on (10) and (14) and (7) and (13)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}((e_1, e_2))$ .

(16)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (27) on (16).

**Case (27a).**

(17)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(17)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (27e).**

(17)  $e' = (e'_1, e'_2)$  by assumption

(18)  $e'_1 \in \text{values}(e_1)$  by assumption

(19)  $e'_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by (15) on (18)

(20)  $(e'_1, e'_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (19)

**Case  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .**

(13)  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by assumption

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}((e_1, e_2))$ .

(14)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (27) on (14).

**Case (27a).**

(15)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(15)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (27e).**

(15)  $e' = (e'_1, e'_2)$  by assumption

(16)  $e'_2 \in \text{values}(e_2)$  by assumption

By rule induction over Rules (27) on (16).

**Case (27a).**

(17)  $e'_2 = e_2$  by assumption

(18)  $e'_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by (17) and (13)

(19)  $(e'_1, e'_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (18)

**Case (27b).**

(17)  $e_2 \text{ notintro}$  by assumption

Contradicts (11) by Lemma 4.0.11.

**Case (27c), (27d), (27e).**

(17)  $e_2$  **indet** by assumption

Contradicts (11) by Lemma 4.0.10.

**Case (24e).**

(9)  $e = (e_1, e_2)$  by assumption

(10)  $e_1$  **val** by assumption

(11)  $e_2$  **indet** by assumption

(12)  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$  or  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$  by Lemma 2.0.13 on (4)

By case analysis on the disjunction in (12).

**Case  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$ .**

(13)  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

To show that if  $e' \in \mathbf{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \mathbf{values}((e_1, e_2))$ .

(14)  $e' \in \mathbf{values}((e_1, e_2))$  by assumption

By rule induction over Rules (27) on (14).

**Case (27a).**

(15)  $(e_1, e_2)$  **val** by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(15)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (27e).**

(15)  $e' = (e'_1, e'_2)$  by assumption

(16)  $e'_1 \in \mathbf{values}(e_1)$  by assumption

By rule induction over Rules (27) on (16).

**Case (27a).**

(17)  $e'_1 = e_1$  by assumption

(18)  $e'_1 \not\models_{\dot{?}} \dot{\xi}_1$  by (17) and (13)

(19)  $(e'_1, e'_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (18)

**Case (27b).**

(17)  $e_1$  **notintro** by assumption

Contradicts (10) by Lemma 4.0.11.

**Case (27c), (27d), (27e).**

(17)  $e_1$  **indet** by assumption

Contradicts (10) by Lemma 4.0.10.

**Case  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$ .**

(13)  $e_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by assumption

By rule induction over Rules (19) on (2), only one rule applies.

**Case (19g).**

(14)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(15) if  $e'_2 \in \text{values}(e_2)$  then  $e'_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$   
by IH on (11) and (14)  
and (8) and (13)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\vdash_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ ,  
we assume  $e' \in \text{values}((e_1, e_2))$ .

(16)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (27) on (16).

**Case (27a).**

(17)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(17)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (27e).**

(17)  $e' = (e'_1, e'_2)$  by assumption

(18)  $e'_2 \in \text{values}(e_2)$  by assumption

(19)  $e'_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by (15) on (18)

(20)  $(e'_1, e'_2) \not\vdash_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on  
(19)

**Case (24f).**

(9)  $e = (e_1, e_2)$  by assumption

(10)  $e_1 \text{ indet}$  by assumption

(11)  $e_2 \text{ indet}$  by assumption

(12)  $e_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  or  $e_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.13 on  
(4)

By rule induction over Rules (19) on (2), only one rule applies.

**Case (19g).**

(13)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption

(14)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

By case analysis on the disjunction in (12).

**Case  $e_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$ .**

(15)  $e_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  by assumption

(16) if  $e'_1 \in \text{values}(e_1)$  then  $e'_1 \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$   
by IH on (10) and (13)  
and (7) and (15)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}((e_1, e_2))$ .

(17)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (27) on (17).

**Case (27a).**

(18)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(18)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (27e).**

(18)  $e' = (e'_1, e'_2)$  by assumption

(19)  $e'_1 \in \text{values}(e_1)$  by assumption

(20)  $e'_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by (16) on (19)

(21)  $(e'_1, e'_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (20)

**Case  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .**

(15)  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by assumption

(16) if  $e'_2 \in \text{values}(e_2)$  then  $e'_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$   
by IH on (11) and (14)  
and (8) and (15)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}((e_1, e_2))$ .

(17)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (27) on (17).

**Case (27a).**

(18)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.10.

**Case (27b).**

(18)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (27e).**

(18)  $e' = (e'_1, e'_2)$  by assumption

(19)  $e'_2 \in \text{values}(e_2)$  by assumption

(20)  $e'_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by (16) on (19)

(21)  $(e'_1, e'_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (20)

**Case (24i).**

(9)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

**Case (24j).**

(9)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

**Case (8f).**

- (5)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption
- (6)  $\dot{\xi}_1 : \tau_1$  by assumption
- (7)  $\dot{\xi}_2 : \tau_2$  by assumption
- (8)  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption
- (9)  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by Lemma 2.0.10 on (8)
- (10)  $e \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.10 on (8)
- (11) if  $e' \in \text{values}(e)$  then  $e' \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by IH on (1) and (2) and (6) and (9)
- (12) if  $e' \in \text{values}(e)$  then  $e' \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by IH on (1) and (2) and (7) and (10)

To show that if  $e' \in \text{values}(e)$  then  $e' \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ , we assume  $e' \in \text{values}(e)$ .

- (13)  $e' \in \text{values}(e)$  by assumption
- (14)  $e' \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by (11) on (13)
- (15)  $e' \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by (12) on (13)
- (16)  $e' \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Lemma 2.0.10 on (14) and (15)

□

$\theta : \Gamma$      $\theta$  is of type  $\Gamma$

$$\frac{\text{STEmpty}}{\emptyset : \cdot} \quad (28a)$$

$$\frac{\text{STExtend} \quad \theta : \Gamma_{\theta} \quad \Gamma ; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_{\theta}, x : \tau} \quad (28b)$$

$p \text{ refutable?}$      $p$  is refutable

$$\frac{\text{RNum}}{\underline{n} \text{ refutable?}} \quad (29a)$$

$$\frac{\text{REHole}}{\langle \rangle^w \text{ refutable?}} \quad (29b)$$

$$\frac{\text{RHole}}{\langle p \rangle^w \text{ refutable?}} \quad (29c)$$

$$\frac{\text{RInl}}{\text{inl}(p) \text{ refutable?}} \quad (29d)$$

$$\frac{\text{RInr}}{\text{inr}(p) \text{ refutable?}} \quad (29e)$$

$$\frac{\text{RPairL}}{p_1 \text{ refutable?}} \quad (29f)$$

$$\frac{}{(p_1, p_2) \text{ refutable?}}$$

$$\frac{\text{RPairR}}{p_2 \text{ refutable?}} \quad (29g)$$

$$\frac{}{(p_1, p_2) \text{ refutable?}}$$

$e \triangleright p \dashv\!\!\vdash \theta$   $e$  matches  $p$ , emitting  $\theta$

$$\frac{\text{MVar}}{e \triangleright x \dashv\!\!\vdash e/x} \quad (30a)$$

$$\frac{\text{MWild}}{e \triangleright \_ \dashv\!\!\vdash \cdot} \quad (30b)$$

$$\frac{\text{MNum}}{\underline{n} \triangleright \underline{n} \dashv\!\!\vdash \cdot} \quad (30c)$$

$$\frac{\text{MPair}}{e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1 \quad e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2} \quad (30d)$$

$$\frac{}{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2}$$

$$\frac{\text{MInl}}{e \triangleright p \dashv\!\!\vdash \theta} \quad (30e)$$

$$\frac{}{\text{inl}_\tau(e) \triangleright \text{inl}(p) \dashv\!\!\vdash \theta}$$

$$\frac{\text{MInr}}{e \triangleright p \dashv\!\!\vdash \theta} \quad (30f)$$

$$\frac{}{\text{inr}_\tau(e) \triangleright \text{inr}(p) \dashv\!\!\vdash \theta}$$

$$\frac{\text{MNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1 \quad \text{prl}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2} \quad (30g)$$

$e ? p$       $e$  may match  $p$

$$\frac{\text{MMEHole}}{e ? (\textcolor{violet}{\parallel})^w} \quad (31a)$$

$$\frac{\text{MMHole}}{e ? (\textcolor{violet}{p})^w} \quad (31b)$$

$$\frac{\text{MMNotIntro} \quad e \text{ notintro} \quad p \text{ refutable?}}{e ? p} \quad (31c)$$

$$\frac{\text{MMPairL} \quad e_1 ? p_1 \quad e_2 \triangleright p_2 \dashv \parallel \theta_2}{(e_1, e_2) ? (p_1, p_2)} \quad (31d)$$

$$\frac{\text{MMPairR} \quad e_1 \triangleright p_1 \dashv \parallel \theta_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (31e)$$

$$\frac{\text{MMPair} \quad e_1 ? p_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (31f)$$

$$\frac{\text{MMInl} \quad e ? p}{\text{inl}_\tau(e) ? \text{inl}(p)} \quad (31g)$$

$$\frac{\text{MMInr} \quad e ? p}{\text{inr}_\tau(e) ? \text{inr}(p)} \quad (31h)$$

$e \perp p$       $e$  does not match  $p$

$$\frac{\text{NMNum} \quad n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}} \quad (32a)$$

$$\frac{\text{NMPairL} \quad e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)} \quad (32b)$$

$$\frac{\text{NMPairR} \quad e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)} \quad (32c)$$

$$\frac{\text{NMConfL}}{\text{inr}_\tau(e) \perp \text{inl}(p)} \quad (32d)$$

$$\frac{\text{NMConfR}}{\text{inl}_\tau(e) \perp \text{inr}(p)} \quad (32e)$$



$$\frac{\text{NMInl} \quad e \perp p}{\text{inr}_\tau(e) \perp \text{inl}(p)} \quad (32f)$$

$$\frac{\text{NMInr} \quad e \perp p}{\text{inl}_\tau(e) \perp \text{inr}(p)} \quad (32g)$$

$\boxed{e \mapsto e'}$       $e$  takes a step to  $e'$

$$\frac{\text{ITHole} \quad e \mapsto e'}{\langle e \rangle^u \mapsto \langle e' \rangle^u} \quad (33a)$$

$$\frac{\text{ITApFun} \quad e_1 \mapsto e'_1}{e_1(e_2) \mapsto e'_1(e_2)} \quad (33b)$$

$$\frac{\text{ITApArg} \quad e_1 \text{ val} \quad e_2 \mapsto e'_2}{e_1(e_2) \mapsto e_1(e'_2)} \quad (33c)$$

$$\frac{\text{ITAP} \quad e_2 \text{ val}}{(\lambda x : \tau. e_1)(e_2) \mapsto [e_2/x]e_1} \quad (33d)$$

$$\frac{\text{ITPairL} \quad e_1 \mapsto e'_1}{(e_1, e_2) \mapsto (e'_1, e_2)} \quad (33e)$$

$$\frac{\text{ITPairR} \quad e_1 \text{ val} \quad e_2 \mapsto e'_2}{(e_1, e_2) \mapsto (e_1, e'_2)} \quad (33f)$$

$$\frac{\text{ITPrL} \quad (e_1, e_2) \text{ final}}{\text{prl}((e_1, e_2)) \mapsto e_1} \quad (33g)$$

$$\frac{\text{ITPrR} \quad (e_1, e_2) \text{ final}}{\text{prr}((e_1, e_2)) \mapsto e_2} \quad (33h)$$

$$\frac{\text{ITInl} \quad e \mapsto e'}{\text{inl}_\tau(e) \mapsto \text{inl}_\tau(e')} \quad (33i)$$

$$\frac{\text{ITInr} \quad e \mapsto e'}{\text{inr}_\tau(e) \mapsto \text{inr}_\tau(e')} \quad (33j)$$

$$\frac{\text{ITExpMatch} \quad e \mapsto e'}{\text{match}(e)\{\hat{r}s\} \mapsto \text{match}(e')\{\hat{r}s\}} \quad (33k)$$

$$\begin{array}{c}
\text{ITSuccMatch} \\
\frac{e \text{ final} \quad e \triangleright p_r \dashv\!\parallel \theta}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \\
\text{ITFailMatch} \\
\frac{e \text{ final} \quad e \perp p_r}{\text{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \text{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \\
(33\text{m})
\end{array} \tag{33l}$$

**Lemma 4.0.2.** *If  $\text{inl}_{\tau_2}(e_1)$  final then  $e_1$  final.*

*Proof.* By rule induction over Rules (25) on  $\text{inl}_{\tau_2}(e_1)$  final.

**Case (25a).**

(17)  $\text{inl}_{\tau_2}(e_1)$  val by assumption

By rule induction over Rules (23) on (17), only one case applies.

**Case (23d).**

(18)  $e_1$  val by assumption  
(19)  $e_1$  final by Rule (25a) on (18)

**Case (25b).**

(17)  $\text{inl}_{\tau_2}(e_1)$  indet by assumption

By rule induction over Rules (24) on (17), only one case applies.

**Case (24i).**

(18)  $e_1$  indet by assumption  
(19)  $e_1$  final by Rule (25b) on (18)

□

**Lemma 4.0.3.** *If  $\text{inr}_{\tau_1}(e_2)$  final then  $e_2$  final.*

*Proof.* By rule induction over Rules (25) on  $\text{inr}_{\tau_1}(e_2)$  final.

**Case (25a).**

(1)  $\text{inr}_{\tau_1}(e_2)$  val by assumption

By rule induction over Rules (23) on (1), only one case applies.

**Case (23d).**

(2)  $e_2$  val by assumption  
(3)  $e_2$  final by Rule (25a) on (2)

**Case (25b).**

(1)  $\text{inr}_{\tau_1}(e_2) \text{ indet}$  by assumption

By rule induction over Rules (24) on (1), only one case applies.

**Case (24i).**

(2)  $e_2 \text{ indet}$  by assumption  
 (3)  $e_2 \text{ final}$  by Rule (25b) on (2)

□

**Lemma 4.0.4.** *If  $(e_1, e_2) \text{ final}$  then  $e_1 \text{ final}$  and  $e_2 \text{ final}$ .*

*Proof.* By rule induction over Rules (25) on  $(e_1, e_2) \text{ final}$ .

**Case (25a).**

(1)  $(e_1, e_2) \text{ val}$  by assumption

By rule induction over Rules (23) on (1), only one case applies.

**Case (23c).**

(2)  $e_1 \text{ val}$  by assumption  
 (3)  $e_2 \text{ val}$  by assumption  
 (4)  $e_1 \text{ final}$  by Rule (25a) on (2)  
 (5)  $e_2 \text{ final}$  by Rule (25a) on (3)

**Case (25b).**

(1)  $(e_1, e_2) \text{ indet}$  by assumption

By rule induction over Rules (24) on (1), only three cases apply.

**Case (24d).**

(2)  $e_1 \text{ indet}$  by assumption  
 (3)  $e_2 \text{ val}$  by assumption  
 (4)  $e_1 \text{ final}$  by Rule (25b) on (2)  
 (5)  $e_1 \text{ final}$  by Rule (25a) on (3)

**Case (24e).**

(2)  $e_1 \text{ val}$  by assumption  
 (3)  $e_2 \text{ indet}$  by assumption  
 (4)  $e_1 \text{ final}$  by Rule (25a) on (2)  
 (5)  $e_1 \text{ final}$  by Rule (25b) on (3)

**Case (24f).**

(2)  $e_1 \text{ indet}$  by assumption

(3) $e_2$ <b>indet</b>	by assumption
(4) $e_1$ <b>final</b>	by Rule (25b) on (2)
(5) $e_1$ <b>final</b>	by Rule (25b) on (3)

□

**Lemma 4.0.5.** *There doesn't exist  $\underline{n}$  such that  $\underline{n}$  **notintro**.*

*Proof.* By rule induction over Rules (26) on  $\underline{n}$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.6.** *There doesn't exist  $\text{inl}_\tau(e)$  such that  $\text{inl}_\tau(e)$  **notintro**.*

*Proof.* By rule induction over Rules (26) on  $\text{inl}_\tau(e)$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.7.** *There doesn't exist  $\text{inr}_\tau(e)$  such that  $\text{inr}_\tau(e)$  **notintro**.*

*Proof.* By rule induction over Rules (26) on  $\text{inr}_\tau(e)$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.8.** *There doesn't exist  $(e_1, e_2)$  such that  $(e_1, e_2)$  **notintro**.*

*Proof.* By rule induction over Rules (26) on  $(e_1, e_2)$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.9.** *If  $e$  **final** and  $e$  **notintro** then  $e$  **indet**.*

*Proof Sketch.* By rule induction over Rules (26) on  $e$  **notintro**, for each case, by rule induction over Rules (23) on  $e$  **val** and we notice that  $e$  **val** is not derivable. By rule induction over Rules (25) on  $e$  **final**, Rule (25a) result in a contradiction with the fact that  $e$  **val** is not derivable while Rule (25b) tells us  $e$  **indet**. □

**Lemma 4.0.10.** *There doesn't exist such an expression  $e$  such that both  $e$  **val** and  $e$  **indet**.*

**Lemma 4.0.11.** *There doesn't exist such an expression  $e$  such that both  $e$  **val** and  $e$  **notintro**.*

**Lemma 4.0.12** (Finality). *There doesn't exist such an expression  $e$  such that both  $e$  **final** and  $e \mapsto e'$  for some  $e'$*

*Proof.* Assume there exists such an  $e$  such that both  $e$  **final** and  $e \mapsto e'$  for some  $e'$  then proof by contradiction.

By rule induction over Rules (25) and Rules (33), i.e., over Rules (23) and Rules (33) and over Rules (24) and Rules (33) respectively. The proof can be done by straightforward observation of syntactic contradictions. □

**Lemma 4.0.13** (Matching Determinism). *If  $e$  **final** and  $\cdot ; \Delta_e \vdash e : \tau$  and  $p : \tau[\xi] \dashv \vdash \Gamma ; \Delta$  then exactly one of the following holds*

1.  $e \triangleright p \dashv\!\!\vdash \theta$  for some  $\theta$

2.  $e ? p$

3.  $e \perp p$

*Proof.*

- |   |               |
|---|---------------|
| (1) $e \text{ final}$                               | by assumption |
| (2) $\cdot; \Delta_e \vdash e : \tau$               | by assumption |
| (3) $p : \tau[\xi] \dashv\!\!\vdash \Gamma; \Delta$ | by assumption |

By rule induction over Rules (20) on (3), we would show one conclusion is derivable while the other two are not.

**Case (20a).**

- |   |               |
|---|---------------|
| (4) $p = x$                                   | by assumption |
| (5) $e \triangleright x \dashv\!\!\vdash e/x$ | by Rule (30a) |

Assume  $e ? x$ . By rule induction over Rules (31) on it, only one case applies.

**Case (31c).**

- |                            |               |
|----------------------------|---------------|
| (6) $x \text{ refutable?}$ | by assumption |
|----------------------------|---------------|

By rule induction over Rules (29) on (6), no case applies due to syntactic contradiction.

- |                        |                  |
|------------------------|------------------|
| (7) $e ? \overline{x}$ | by contradiction |
|------------------------|------------------|

Assume  $e \perp x$ . By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

- |                            |                  |
|----------------------------|------------------|
| (8) $e \perp \overline{x}$ | by contradiction |
|----------------------------|------------------|

**Case (20b).**

- |  |               |
|--|---------------|
| (4) $p = \_$                                     | by assumption |
| (5) $e \triangleright \_ \dashv\!\!\vdash \cdot$ | by Rule (30b) |

Assume  $e ? \_$ . By rule induction over Rules (31) on it, only one case applies.

**Case (31c).**

- |                             |               |
|-----------------------------|---------------|
| (6) $\_ \text{ refutable?}$ | by assumption |
|-----------------------------|---------------|

By rule induction over Rules (29) on (6), no case applies due to syntactic contradiction.

(7)  $e \not\vdash \_$  by contradiction

Assume  $e \perp \_$ . By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

(8)  $e \not\vdash \_$  by contradiction

**Case (20c).**

(4)  $p = \langle \rangle^w$  by assumption

(5)  $e ? \langle \rangle^w$  by Rule (31a)

Assume  $e \triangleright \langle \rangle^w \dashv \vdash \theta$  for some  $\theta$ . By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(6)  $e \triangleright \langle \rangle^w \dashv \vdash \theta$  by contradiction

Assume  $e \perp \langle \rangle^w$ . By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

(7)  $e \not\vdash \langle \rangle^w$  by contradiction

**Case (20d).**

(4)  $p = \langle p_0 \rangle^w$  by assumption

(5)  $e ? \langle p_0 \rangle^w$  by Rule (31b)

Assume  $e \triangleright \langle p_0 \rangle^w \dashv \vdash \theta$  for some  $\theta$ . By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(6)  $e \triangleright \langle p_0 \rangle^w \dashv \vdash \theta$  by contradiction

Assume  $e \perp \langle p_0 \rangle^w$ . By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

(7)  $e \not\vdash \langle p_0 \rangle^w$  by contradiction

**Case (20e).**

(4)  $p = \underline{n_2}$  by assumption

(5)  $\tau = \mathbf{num}$  by assumption

(6)  $\xi = \underline{n_2}$  by assumption

(7)  $\underline{n_2}$  **refutable?** by Rule (29a)

By rule induction over Rules (19) on (2), the following cases apply.

**Case** (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (8)  $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{rs\}$   
by assumption
- (9)  $e$  **notintro** by Rule  
(26a),(26b),(26c),(26d),(26e),(26f)
- (10)  $e ? \underline{n_2}$  by Rule (16b) on (7)  
and (9)

Assume  $e \triangleright \underline{n_2} \dashv\!\!\!\dashv \theta$  for some  $\theta$ . By rule induction over it, no case applies due to syntactic contradiction.

(11)  $e \triangleright \underline{n_2} \dashv\!\!\!\dashv \theta$  by contradiction

Assume  $e \perp \underline{n_2}$ . By rule induction over it, no case applies due to syntactic contradiction.

(12)  $e \perp \underline{n_2}$  by contradiction

**Case** (19d).

(8)  $e = \underline{n_1}$

Assume  $\underline{n_1} ? \underline{n_2}$ . By rule induction over Rules (31) on it, only two cases apply.

**Case** (31c).

(9)  $\underline{n_1}$  **notintro** by assumption

Contradicts Lemma 4.0.5.

(10)  $\underline{n_1} ? \underline{n_2}$  by contradiction

By case analysis on whether  $n_1 = n_2$ .

**Case**  $n_1 = n_2$ .

(11)  $n_1 = n_2$  by assumption

(12)  $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\!\dashv \cdot$  by Rule (30c)

Assume  $\underline{n_1} \perp \underline{n_2}$ . By rule induction over Rules (32) on it, only one case applies.

**Case** (32a).

(13)  $n_1 \neq n_2$  by assumption

Contradicts (11).

(14)  $\underline{n_1} \perp \underline{n_2}$  by contradiction

**Case**  $n_1 \neq n_2$ .

(11)  $n_1 \neq n_2$  by assumption

- (12)  $\underline{n_1} \perp n_2$  by Rule (32a) on (11)  
 Assume  $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash \theta$  for some  $\theta$ . By rule induction over Rules (30) on it, no case applies due to syntactic contradiction.  
 (13)  $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash \theta$  by contradiction

**Case (20f).**

- (4)  $p = \text{inl}(p_1)$  by assumption  
 (5)  $\tau = (\tau_1 + \tau_2)$  by assumption  
 (6)  $\xi = \text{inl}(\xi_1)$  by assumption  
 (7)  $p_1 : \tau_1[\xi_1] \dashv\!\!\vdash \Gamma ; \Delta$  by assumption  
 (8)  $\text{inl}(p_1) \text{ refutable?}$  by Rule (29d)

By rule induction over Rules (19) on (2), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- (9)  $e = \text{!}^u, \text{!}(e_0)^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
 by assumption  
 (10)  $e \text{ notintro}$  by Rule (26a),(26b),(26c),(26d),(26e),(26f)  
 (11)  $e ? \text{inl}(p_1)$  by Rule (16b) on (8) and (10)

Assume  $e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1$  for some  $\theta_1$ . By rule induction over it, no case applies due to syntactic contradiction.

- (12)  $\underline{e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1}$  by contradiction

Assume  $e \perp \text{inl}(p_1)$ . By rule induction over it, no case applies due to syntactic contradiction.

- (13)  $\underline{e \perp \text{inl}(p_1)}$  by contradiction

**Case (19j).**

- (9)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption  
 (10)  $\cdot ; \Delta_e \vdash e_1 : \tau_1$  by assumption  
 (11)  $e_1 \text{ final}$  by Lemma 4.0.2 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$ ,  $e_1 ? p_1$ , and  $e_1 \perp p_1$  holds.

By case analysis on which one holds.

**Case  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ .**

- (12)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption  
 (13)  $\underline{e_1 ? p_1}$  by assumption  
 (14)  $\underline{e_1 \perp p_1}$  by assumption  
 (15)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1$  by Rule (30e) on (12)



Assume  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ . By rule induction over Rules (31) on it, only two cases apply.

**Case (31c).**

(16)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.6.

**Case (31g).**

(16)  $e_1 ? p_1$  by assumption

Contradicts (13).

(17)  $\overline{\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)}$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32f).**

(18)  $e_1 \perp p_1$  by assumption

Contradicts (14).

(19)  $\overline{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}$  by contradiction

**Case  $e_1 ? p_1$ .**

(12)  $\overline{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}$  by assumption

(13)  $e_1 ? p_1$  by assumption

(14)  $\overline{e_1 \perp p_1}$  by assumption

(15)  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  by Rule (31g) on (13)

Assume  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30e).**

(16)  $e_1 \triangleright p_1 \dashv\!\!\dashv \theta$  by assumption

Contradicts (12).

(17)  $\overline{\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta}$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32f).**

(18)  $e_1 \perp p_1$  by assumption

Contradicts (14).

(19)  $\overline{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}$  by contradiction

**Case  $e_1 \perp p_1$ .**

(12)  $\overline{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}$  by assumption

(13)  $\overline{e_1 ? p_1}$  by assumption

(14)  $e_1 \perp p_1$  by assumption

(15)  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$  by Rule (32f) on (14)

Assume  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30e).**

(16)  $e_1 \triangleright p_1 \dashv\!\!\dashv \theta$  by assumption

Contradicts (12).

(17)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ . By rule induction over Rules (31) on it, only two cases apply.

**Case (31c).**

(18)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.6.

**Case (31g).**

(18)  $e_1 ? p_1$  by assumption

Contradicts (13).

(19)  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  by contradiction

**Case (20g).**

(4)  $p = \text{inr}(p_2)$  by assumption

(5)  $\tau = (\tau_1 + \tau_2)$  by assumption

(6)  $\xi = \text{inr}(\xi_2)$  by assumption

(7)  $p_2 : \tau_2[\xi_2] \dashv\!\!\dashv \Gamma ; \Delta$  by assumption

(8)  $\text{inr}(p_2) \text{ refutable?}$  by Rule (29e)

By rule induction over Rules (19) on (2), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

(9)  $e = \text{inl}^u, \text{inl}^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption

(10)  $e \text{ notintro}$  by Rule  
(26a),(26b),(26c),(26d),(26e),(26f)

(11)  $e ? \text{inr}(p_2)$  by Rule (16b) on (8)  
and (10)

Assume  $e \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta_2$  for some  $\theta_2$ . By rule induction over it, no case applies due to syntactic contradiction.

(12)  $e \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta_2$  by contradiction

Assume  $e \perp \text{inr}(p_2)$ . By rule induction over it, no case applies due to syntactic contradiction.

(13)  $e \perp \overline{\text{inr}(p_2)}$  by contradiction

**Case (19k).**

(9)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

(10)  $\cdot; \Delta_e \vdash e_2 : \tau_2$  by assumption

(11)  $e_2 \text{ final}$  by Lemma 4.0.3 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$  for some  $\theta_2$ ,  $e_2 ? p_2$ , and  $e_2 \perp p_2$  holds.

By case analysis on which one holds.

**Case  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$ .**

(12)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$  by assumption

(13)  $e_2 ? \overline{p_2}$  by assumption

(14)  $e_2 \perp \overline{p_2}$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta_2$  by Rule (30f) on (12)

Assume  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ . By rule induction over Rules (31) on it, only two cases apply.

**Case (31c).**

(16)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7.

**Case (31h).**

(16)  $e_2 ? p_2$  by assumption

Contradicts (13).

(17)  $\overline{\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)}$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32g).**

(18)  $e_2 \perp p_2$  by assumption

Contradicts (14).

(19)  $\overline{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}$  by contradiction

**Case  $e_2 ? p_2$ .**

(12)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$  by assumption

(13)  $e_2 ? p_2$  by assumption

(14)  $e_2 \perp \overline{p_2}$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$  by Rule (31h) on (13)

Assume  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30f).**

(16)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$  by assumption

Contradicts (12).

(17)  $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta}{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta}$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32g).**

(18)  $e_2 \perp p_2$  by assumption

Contradicts (14).

(19)  $\frac{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}$  by contradiction

**Case  $e_2 \perp p_2$ .**

(12)  $\frac{e_2 \triangleright p_2 \dashv\vdash \theta}{e_2 \triangleright p_2 \dashv\vdash \theta}$  by assumption

(13)  $\frac{e_2 \triangleright p_2}{e_2 \triangleright p_2}$  by assumption

(14)  $e_2 \perp p_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$  by Rule (32g) on (14)

Assume  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30f).**

(16)  $e_2 \triangleright p_2 \dashv\vdash \theta$  by assumption

Contradicts (12).

(17)  $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta}{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta}$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)$ . By rule induction over Rules (31) on it, only two cases apply.

**Case (31c).**

(18)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7.

**Case (31h).**

(18)  $e_2 \triangleright p_2$  by assumption

Contradicts (13).

(19)  $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)}{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)}$  by contradiction

**Case (20h).**

(4)  $p = (p_1, p_2)$  by assumption

(5)  $\tau = (\tau_1 \times \tau_2)$  by assumption

(6)  $\xi = (\xi_1, \xi_2)$  by assumption

(7)  $\Gamma = \Gamma_1 \uplus \Gamma_2$  by assumption

(8)  $\Delta = \Delta_1 \uplus \Delta_2$  by assumption

- (9)  $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 ; \Delta_1$  by assumption  
 (10)  $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 ; \Delta_2$  by assumption

By rule induction over Rules (19) on (2), the following cases apply.

**Case** (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (11)  $e = \text{new } u, \text{new } e_0^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
 by assumption  
 (12)  $e$  **notintro** by Rule (26a),(26b),(26c),(26d),(26e),(26f)  
 (13)  $e$  **indet** by Lemma 4.0.9 on (1) and (12)  
 (14)  $\text{prl}(e)$  **indet** by Rule (24g) on (13)  
 (15)  $\text{prl}(e)$  **final** by Rule (25b) on (14)  
 (16)  $\text{prl}(e)$  **indet** by Rule (24h) on (13)  
 (17)  $\text{prl}(e)$  **final** by Rule (25b) on (16)  
 (18)  $\cdot ; \Delta \vdash \text{prl}(e) : \tau_1$  by Rule (19h) on (2)  
 (19)  $\cdot ; \Delta \vdash \text{prl}(e) : \tau_2$  by Rule (19i) on (2)

Assume  $e \perp (p_1, p_2)$ . By rule induction on it, no case applies due to syntactic contradiction.

- (20)  $e \perp \overline{(p_1, p_2)}$  by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$ ,  $\text{prl}(e) ? p_1$ , and  $\text{prl}(e) \perp p_1$  holds.

By inductive hypothesis on (17) and (19) and (10), exactly one of  $\text{prl}(e) \triangleright p_2 \dashv\vdash \theta_2$ ,  $\text{prl}(e) ? p_2$ , and  $\text{prl}(e) \perp p_2$  holds.

By case analysis on which conclusion holds for  $p_1$  and  $p_2$ . Note that we have already shown  $e \perp \overline{(p_1, p_2)}$ .

**Case**  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1, \text{prl}(e) \triangleright p_2 \dashv\vdash \theta_2$ .

- (21)  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  by assumption  
 (22)  $\overline{\text{prl}(e) ? p_1}$  by assumption  
 (23)  $\overline{\text{prl}(e) \perp p_1}$  by assumption  
 (24)  $\text{prl}(e) \triangleright p_2 \dashv\vdash \theta_2$  by assumption  
 (25)  $\overline{\text{prl}(e) ? p_2}$  by assumption  
 (26)  $\overline{\text{prl}(e) \perp p_2}$  by assumption  
 (27)  $e \triangleright (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2$  by Rule (30g) on (12) and (21) and (24)

Assume  $e ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only one case applies.

**Case** (31c).

- (28)  $(p_1, p_2)$  **refutable?** by assumption

By rule induction over Rules (29), only two cases apply.

**Case (29f).**

- |                               |                                   |
|-------------------------------|-----------------------------------|
| (29) $p_1$ refutable?         | by assumption                     |
| (30) $\text{prl}(e)$ notintro | by Rule (26e)                     |
| (31) $\text{prl}(e) ? p_1$    | by Rule (31c) on (29)<br>and (30) |

Contradicts (22).

**Case (29g).**

- |                               |                                   |
|-------------------------------|-----------------------------------|
| (29) $p_2$ refutable?         | by assumption                     |
| (30) $\text{prr}(e)$ notintro | by Rule (26f)                     |
| (31) $\text{prl}(e) ? p_1$    | by Rule (31c) on (29)<br>and (30) |

Contradicts (22).

- |                                  |                  |
|----------------------------------|------------------|
| (32) $e ? (\overline{p_1, p_2})$ | by contradiction |
|----------------------------------|------------------|

**Case  $\text{prl}(e) \triangleright p_1 \dashv\!\!\dashv \theta_1, \text{prr}(e) ? p_2$ .**

- |  |               |
|--|---------------|
| (21) $\text{prl}(e) \triangleright p_1 \dashv\!\!\dashv \theta_1$            | by assumption |
| (22) $\overline{\text{prl}(e) ? p_1}$  | by assumption |
| (23) $\overline{\text{prl}(e) \perp p_1}$                                    | by assumption |
| (24) $\overline{\text{prr}(e) \triangleright p_2 \dashv\!\!\dashv \theta_2}$ | by assumption |
| (25) $\text{prr}(e) ? p_2$   | by assumption |
| (26) $\overline{\text{prr}(e) \perp p_2}$                                    | by assumption |

Assume  $e \triangleright (p_1, p_2) \dashv\!\!\dashv \theta$ . By rule induction over Rules (30), only one case applies.

**Case (30g).**

- |   |               |
|---|---------------|
| (27) $\theta = \theta_1 \uplus \theta_2$                          | by assumption |
| (28) $\text{prr}(e) \triangleright p_2 \dashv\!\!\dashv \theta_2$ | by assumption |

Contradicts (24).

- |   |                  |
|---|------------------|
| (29) $e \triangleright (\overline{p_1, p_2}) \dashv\!\!\dashv \theta$ | by contradiction |
|---|------------------|

By rule induction over Rules (31) on (25), the following cases apply.

**Case (31a),(31b).**

- |   |                                   |
|---|-----------------------------------|
| (30) $p_2 = \langle \rangle^w, \langle p \rangle^w$ | by assumption                     |
| (31) $p_2$ refutable?                               | by Rule (29b) and Rule (29c)      |
| (32) $(p_1, p_2)$ refutable?                        | by Rule (29g) on (31)             |
| (33) $e ? (p_1, p_2)$                               | by Rule (31c) on (12)<br>and (32) |

**Case (31c).**

- |                       |               |
|-----------------------|---------------|
| (30) $p_2$ refutable? | by assumption |
|-----------------------|---------------|

- (31)  $(p_1, p_2) \text{ refutable?}$  by Rule (29g) on (30)  
 (32)  $e ? (p_1, p_2)$  by Rule (31c) on (12) and (31)

**Case**  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1, \text{prr}(e) \perp p_2$ .

- (21)  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption  
 (22)  $\text{prl}(e) ? p_1$  by assumption  
 (23)  $\text{prl}(e) \perp p_1$  by assumption  
 (24)  $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$  by assumption  
 (25)  $\text{prr}(e) ? p_2$  by assumption  
 (26)  $\text{prr}(e) \perp p_2$  by assumption

By rule induction over Rules (32) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case**  $\text{prl}(e) ? p_1, \text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ .

- (21)  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption  
 (22)  $\text{prl}(e) ? p_1$  by assumption  
 (23)  $\text{prl}(e) \perp p_1$  by assumption  
 (24)  $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$  by assumption  
 (25)  $\text{prr}(e) ? p_2$  by assumption  
 (26)  $\text{prr}(e) \perp p_2$  by assumption

Assume  $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ . By rule induction over Rules (30), only one case applies.

**Case** (30g).

- (27)  $\theta = \theta_1 \uplus \theta_2$  by assumption  
 (28)  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption

Contradicts (21).

- (29)  $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  by contradiction

By rule induction over Rules (31) on (22), the following cases apply.

**Case** (31a),(31b).

- (30)  $p_1 = \langle \rangle^w, \langle p \rangle^w$  by assumption  
 (31)  $p_1 \text{ refutable?}$  by Rule (29b) and Rule (29c)  
 (32)  $(p_1, p_2) \text{ refutable?}$  by Rule (29g) on (31)  
 (33)  $e ? (p_1, p_2)$  by Rule (31c) on (12) and (32)

**Case** (31c).

- (30)  $p_1 \text{ refutable?}$  by assumption  
 (31)  $(p_1, p_2) \text{ refutable?}$  by Rule (29g) on (30)

(32)  $e ? (p_1, p_2)$  by Rule (31c) on (12) and (31)

**Case**  $\text{prl}(e) ? p_1, \text{prr}(e) ? p_2$ .

(21)  $\frac{\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1}{\text{prl}(e) ? p_1}$  by assumption  
 (22)  $\text{prl}(e) ? p_1$  by assumption  
 (23)  $\frac{\text{prl}(e) \perp p_1}{\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1}$  by assumption  
 (24)  $\frac{\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2}{\text{prr}(e) ? p_2}$  by assumption  
 (25)  $\text{prr}(e) ? p_2$  by assumption  
 (26)  $\frac{\text{prr}(e) \perp p_2}{\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2}$  by assumption

Assume  $e \triangleright (p_1, p_2) \dashv\vdash \theta$ . By rule induction over Rules (30), only one case applies.

**Case** (30g).

(27)  $\theta = \theta_1 \uplus \theta_2$  by assumption

(28)  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  by assumption

Contradicts (21).

(29)  $\frac{e \triangleright (p_1, p_2) \dashv\vdash \theta}{e \triangleright (p_1, p_2) \dashv\vdash \theta}$  by contradiction

By rule induction over Rules (31) on (22), the following cases apply.

**Case** (31a),(31b).

(30)  $p_1 = \langle \rangle^w, \langle p \rangle^w$  by assumption  
 (31)  $p_1$  refutable? by Rule (29b) and Rule (29c)  
 (32)  $(p_1, p_2)$  refutable? by Rule (29g) on (31)  
 (33)  $e ? (p_1, p_2)$  by Rule (31c) on (12) and (32)

**Case** (31c).

(30)  $p_1$  refutable? by assumption  
 (31)  $(p_1, p_2)$  refutable? by Rule (29g) on (30)  
 (32)  $e ? (p_1, p_2)$  by Rule (31c) on (12) and (31)

**Case**  $\text{prl}(e) ? p_1, \text{prr}(e) \perp p_2$ .

(21)  $\frac{\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1}{\text{prl}(e) ? p_1}$  by assumption  
 (22)  $\text{prl}(e) ? p_1$  by assumption  
 (23)  $\frac{\text{prl}(e) \perp p_1}{\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1}$  by assumption  
 (24)  $\frac{\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2}{\text{prr}(e) ? p_2}$  by assumption  
 (25)  $\text{prr}(e) ? p_2$  by assumption  
 (26)  $\text{prr}(e) \perp p_2$  by assumption

By rule induction over Rules (32) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.



**Case**  $\text{prl}(e) \perp p_1, \text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ .

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \triangleright p_1}$ | by assumption |
| (22) $\frac{\text{prl}(e) \triangleright p_1}{\text{prl}(e) \triangleright p_1}$                           | by assumption |
| (23) $\text{prl}(e) \perp p_1$   | by assumption |
| (24) $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$  | by assumption |
| (25) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \triangleright p_2}$ | by assumption |
| (26) $\frac{\text{prr}(e) \triangleright p_2}{\text{prr}(e) \triangleright p_2}$                           | by assumption |

By rule induction over Rules (32) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case**  $\text{prl}(e) \perp p_1, \text{prr}(e) ? p_2$ .

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \triangleright p_1}$ | by assumption |
| (22) $\frac{\text{prl}(e) \triangleright p_1}{\text{prl}(e) \triangleright p_1}$                           | by assumption |
| (23) $\text{prl}(e) \perp p_1$   | by assumption |
| (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \triangleright p_2}$ | by assumption |
| (25) $\text{prr}(e) ? p_2$   | by assumption |
| (26) $\frac{\text{prr}(e) ? p_2}{\text{prr}(e) ? p_2}$   | by assumption |

By rule induction over Rules (32) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case**  $\text{prl}(e) \perp p_1, \text{prr}(e) \perp p_2$ .

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \triangleright p_1}$ | by assumption |
| (22) $\frac{\text{prl}(e) \triangleright p_1}{\text{prl}(e) \triangleright p_1}$                           | by assumption |
| (23) $\text{prl}(e) \perp p_1$   | by assumption |
| (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \triangleright p_2}$ | by assumption |
| (25) $\text{prr}(e) \perp p_2$   | by assumption |
| (26) $\frac{\text{prr}(e) \perp p_2}{\text{prr}(e) \perp p_2}$   | by assumption |

By rule induction over Rules (32) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case** (19g).

- |  |                       |
|--|-----------------------|
| (11) $e = (e_1, e_2)$                    | by assumption         |
| (12) $\cdot; \Delta \vdash e_1 : \tau_1$ | by assumption         |
| (13) $\cdot; \Delta \vdash e_2 : \tau_2$ | by assumption         |
| (14) $e_1$ <b>final</b>                  | by Lemma 4.0.4 on (1) |
| (15) $e_2$ <b>final</b>                  | by Lemma 4.0.4 on (1) |

By inductive hypothesis on (9) and (12) and (14), exactly one of  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ ,  $e_1 ? p_1$ , and  $e_1 \perp p_1$  holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ ,  $e_2 ? p_2$ , and  $e_2 \perp p_2$  holds.

By case analysis on which conclusion holds for  $p_1$  and  $p_2$ .

**Case**  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1, e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ .

- |      |  |                                |
|------|--|--------------------------------|
| (16) | $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$                               | by assumption                  |
| (17) | $\underline{e_1 ? p_1}$  | by assumption                  |
| (18) | $\underline{e_1 \perp p_1}$  | by assumption                  |
| (19) | $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$                               | by assumption                  |
| (20) | $\underline{e_2 ? p_2}$  | by assumption                  |
| (21) | $\underline{e_2 \perp p_2}$  | by assumption                  |
| (22) | $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2$ | by Rule (30d) on (16) and (19) |

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only four cases apply.

**Case** (31c).

- |      |                               |               |
|------|-------------------------------|---------------|
| (23) | $(e_1, e_2) \text{ notintro}$ | by assumption |
|------|-------------------------------|---------------|

Contradicts Lemma 4.0.8.

**Case** (31d).

- |      |             |               |
|------|-------------|---------------|
| (23) | $e_1 ? p_1$ | by assumption |
|------|-------------|---------------|

Contradicts (17).

**Case** (31e).

- |      |             |               |
|------|-------------|---------------|
| (23) | $e_2 ? p_2$ | by assumption |
|------|-------------|---------------|

Contradicts (20).

**Case** (31f).

- |      |             |               |
|------|-------------|---------------|
| (23) | $e_1 ? p_1$ | by assumption |
|------|-------------|---------------|

Contradicts (17).

- |      |                                       |                  |
|------|---------------------------------------|------------------|
| (24) | $\underline{(e_1, e_2) ? (p_1, p_2)}$ | by contradiction |
|------|---------------------------------------|------------------|

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (32) on it, only two cases apply.

**Case** (32b).

- |      |                 |               |
|------|-----------------|---------------|
| (25) | $e_1 \perp p_1$ | by assumption |
|------|-----------------|---------------|

Contradicts (18).

**Case** (32c).

- |      |                 |               |
|------|-----------------|---------------|
| (25) | $e_2 \perp p_2$ | by assumption |
|------|-----------------|---------------|

Contradicts (21).

- |      |   |                  |
|------|---|------------------|
| (26) | $\underline{(e_1, e_2) \perp (p_1, p_2)}$ | by contradiction |
|------|---|------------------|

**Case**  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1, e_2 ? p_2$ .

- |      |  |               |
|------|--|---------------|
| (16) | $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$             | by assumption |
| (17) | $\underline{e_1 ? p_1}$  | by assumption |
| (18) | $\underline{e_1 \perp p_1}$                                    | by assumption |
| (19) | $\underline{e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2}$ | by assumption |

- (20)  $e_2 ? p_2$  by assumption  
 (21)  $\overline{e_2 \perp p_2}$  by assumption  
 (22)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (31e) on (16) and (20)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30d).**

- (23)  $\theta = \theta_1 \uplus \theta_2$   
 (24)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  by assumption  
 Contradicts (19).

- (25)  $\overline{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta}$  by contradiction

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (32) on it, only two cases apply.

**Case (32b).**

- (26)  $e_1 \perp p_1$  by assumption  
 Contradicts (18).

**Case (32c).**

- (26)  $e_2 \perp p_2$  by assumption  
 Contradicts (21).

- (27)  $\overline{(e_1, e_2) \perp (p_1, p_2)}$  by contradiction

**Case**  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1, e_2 \perp p_2$ .

- (16)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption  
 (17)  $\overline{e_1 ? p_1}$  by assumption  
 (18)  $\overline{e_1 \perp p_1}$  by assumption  
 (19)  $\overline{e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2}$  by assumption  
 (20)  $\overline{e_2 ? p_2}$  by assumption  
 (21)  $e_2 \perp p_2$  by assumption  
 (22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (32c) on (21)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30d).**

- (23)  $\theta = \theta_1 \uplus \theta_2$   
 (24)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  by assumption  
 Contradicts (19).

- (25)  $\overline{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta}$  by contradiction

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only four cases apply.

**Case (31c).**

(26)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (31d).**

(26)  $e_1 ? p_1$  by assumption

Contradicts (17).

**Case (31e).**

(26)  $e_2 ? p_2$  by assumption

Contradicts (20).

**Case (31f).**

(26)  $e_1 ? p_1$  by assumption

Contradicts (17).

(27)  $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

**Case**  $e_1 ? p_1, e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ .

(16)  $\frac{e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{by assumption}}$

(17)  $e_1 ? p_1$  by assumption

(18)  $\frac{e_1 \perp p_1}{\text{by assumption}}$

(19)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  by assumption

(20)  $\frac{e_2 ? p_2}{\text{by assumption}}$

(21)  $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (31d) on (17) and (19)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption

Contradicts (16).

(25)  $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta}{\text{by contradiction}}$

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (32) on it, only two cases apply.

**Case (32b).**

(26)  $e_1 \perp p_1$  by assumption

Contradicts (18).

**Case (32c).**

(26)  $e_2 \perp p_2$  by assumption

Contradicts (21).

(27)  $\frac{(e_1, e_2) \perp (p_1, p_2)}{\text{by contradiction}}$

**Case**  $e_1 ? p_1, e_2 ? p_2$ .

(16)  $\frac{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}{\text{by assumption}}$   
 (17)  $e_1 ? p_1$  by assumption  
 (18)  $\frac{e_1 \perp p_1}{\text{by assumption}}$   
 (19)  $\frac{e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2}{\text{by assumption}}$   
 (20)  $e_2 ? p_2$  by assumption  
 (21)  $\frac{e_2 \perp p_2}{\text{by assumption}}$   
 (22)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (31f) on (17) and (20)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case** (30d).

(23)  $\theta = \theta_1 \uplus \theta_2$   
 (24)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$  by assumption  
 Contradicts (19).

(25)  $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta}{\text{by contradiction}}$

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (32) on it, only two cases apply.

**Case** (32b).

(26)  $e_1 \perp p_1$  by assumption  
 Contradicts (18).

**Case** (32c).

(26)  $e_2 \perp p_2$  by assumption  
 Contradicts (21).

(27)  $\frac{(e_1, e_2) \perp (p_1, p_2)}{\text{by contradiction}}$

**Case**  $e_1 ? p_1, e_2 \perp p_2$ .

(16)  $\frac{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}{\text{by assumption}}$   
 (17)  $e_1 ? p_1$  by assumption  
 (18)  $\frac{e_1 \perp p_1}{\text{by assumption}}$   
 (19)  $\frac{e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2}{\text{by assumption}}$   
 (20)  $\frac{e_2 ? p_2}{\text{by assumption}}$   
 (21)  $e_2 \perp p_2$  by assumption  
 (22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (32c) on (21)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case** (30d).

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption  
 Contradicts (19).

(25)  $(e_1, e_2) \triangleright \overline{(p_1, p_2) \dashv\!\!\parallel \theta}$  by contradiction

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only four cases apply.

**Case (31c).**

(26)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.

**Case (31d).**

(26)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption  
 Contradicts (19).

**Case (31e).**

(26)  $e_2 ? p_2$  by assumption  
 Contradicts (20).

**Case (31f).**

(26)  $e_2 ? p_2$  by assumption  
 Contradicts (20).

(27)  $(e_1, e_2) ? \overline{(p_1, p_2)}$  by contradiction

**Case**  $e_1 \perp p_1, e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ .

(16)  $\overline{e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1}$  by assumption

(17)  $\overline{e_1 ? p_1}$  by assumption

(18)  $e_1 \perp p_1$  by assumption

(19)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption

(20)  $\overline{e_2 ? p_2}$  by assumption

(21)  $\overline{e_2 \perp p_2}$  by assumption

(22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (32b) on (18)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$  by assumption  
 Contradicts (16).

(25)  $(e_1, e_2) \triangleright \overline{(p_1, p_2) \dashv\!\!\parallel \theta}$  by contradiction

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only four cases apply.

**Case (31c).**

(26)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (31d).**

(26)  $e_1 ? p_1$  by assumption

Contradicts (17).

**Case (31e).**

(26)  $e_2 ? p_2$  by assumption

Contradicts (20).

**Case (31f).**

(26)  $e_1 ? p_1$  by assumption

Contradicts (17).

(27)  $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

**Case  $e_1 \perp p_1, e_2 ? p_2$ .**

(16)  $\frac{e_1 \triangleright p_1 \dashv\!\!\!\vdash \theta_1}{\text{by assumption}}$

(17)  $\frac{e_1 ? p_1}{\text{by assumption}}$

(18)  $e_1 \perp p_1$  by assumption

(19)  $\frac{e_2 \triangleright p_2 \dashv\!\!\!\vdash \theta_2}{\text{by assumption}}$

(20)  $e_2 ? p_2$  by assumption

(21)  $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (32b) on (18)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\vdash \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case (30d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_2 \triangleright p_2 \dashv\!\!\!\vdash \theta_2$  by assumption

Contradicts (19).

(25)  $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\vdash \theta}{\text{by contradiction}}$

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only four cases apply.

**Case (31c).**

(26)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (31d).**

(26)  $e_2 \triangleright p_2 \dashv\!\!\!\vdash \theta_2$  by assumption

Contradicts (19).

**Case (31e).**

(26)  $e_1 \triangleright p_1 \dashv\!\!\!\vdash \theta_1$  by assumption

Contradicts (16).

**Case (31f).**

(26)  $e_1 ? p_1$  by assumption  
 Contradicts (17).

(27)  $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

**Case**  $e_1 \perp p_1, e_2 \perp p_2$ .

(16)  $\frac{e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1}{\text{by assumption}}$

(17)  $\frac{e_1 ? p_1}{\text{by assumption}}$

(18)  $e_1 \perp p_1$  by assumption

(19)  $\frac{e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}{\text{by assumption}}$

(20)  $e_2 ? p_2$  by assumption

(21)  $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (32b) on (18)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ . By rule induction over Rules (30) on it, only one case applies.

**Case** (30d).

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption

Contradicts (19).

(25)  $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta}{\text{by contradiction}}$

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (31) on it, only four cases apply.

**Case** (31c).

(26)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case** (31d).

(26)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption

Contradicts (19).

**Case** (31e).

(26)  $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$  by assumption

Contradicts (16).

**Case** (31f).

(26)  $e_1 ? p_1$  by assumption

Contradicts (17).

(27)  $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

□

**Lemma 4.0.14** (Matching Coherence of Constraint). *Suppose that  $\cdot; \Delta_e \vdash e : \tau$  and  $e$  final and  $p : \tau[\xi] \dashv\!\!\parallel \Gamma; \Delta$ . Then we have*



1.  $e \models \xi \text{ iff } e \triangleright p \dashv\!\!\vdash \theta$
2.  $e \models_{?} \xi \text{ iff } e ? p$
3.  $e \not\models_{?}^{\dagger} \xi \text{ iff } e \perp p$

*Proof.*

- (1)  $\cdot; \Delta_e \vdash e : \tau$  by assumption
- (2)  $e \text{ final}$  by assumption
- (3)  $p : \tau[\xi] \dashv\!\!\vdash \Gamma ; \Delta$  by assumption

Given Lemma 3.0.1, Theorem 2.1, and Lemma 4.0.13, it is sufficient to prove

1.  $e \models \xi \text{ iff } e \triangleright p \dashv\!\!\vdash \theta$
2.  $e \models_{?} \xi \text{ iff } e ? p$

By rule induction over Rules (20) on (3).

**Case (20a).**

- (4)  $p = x$  by assumption
- (5)  $\xi = \top$  by assumption

1. Prove  $e \models \top$  implies  $e \triangleright x \dashv\!\!\vdash \theta$  for some  $\theta$ .

- (6)  $e \triangleright x \dashv\!\!\vdash e/x$  by Rule (30a)

2. Prove  $e \triangleright x \dashv\!\!\vdash \theta$  implies  $e \models \top$ .

- (6)  $e \models \top$  by Rule (14a)

3. Prove  $e \models_{?} \top$  implies  $e ? x$ .

- (6)  $e \not\models_{?} \top$  by Lemma 2.0.3

Vacuously true.

4. Prove  $e ? x$  implies  $e \models_{?} \top$ .

By rule induction over Rules (31), we notice that either,  $e ? x$  is in syntactic contradiction with all the cases, or the premise  $x \text{ refutable}_{?}$  is not derivable. Hence,  $e ? x$  are not derivable. And thus vacuously true.

**Case (20b).**

- (4)  $p = \_$  by assumption
- (5)  $\xi = \top$  by assumption

1. Prove  $e \models \top$  implies  $e \triangleright \_ \dashv\!\!\vdash \theta$  for some  $\theta$ .

- (6)  $e \triangleright \_ \dashv\!\!\vdash \cdot$  by Rule (30a)

2. Prove  $e \triangleright \_ \dashv \! \vdash \theta$  implies  $e \models \top$ .

(6)  $e \models \top$  by Rule (14a)

3. Prove  $e \models ? \top$  implies  $e ? \_$ .

(6)  $e \not\models ? \top$  by Lemma 2.0.3

Vacuously true.

4. Prove  $e ? \_$  implies  $e \models ? \xi$ .

By rule induction over Rules (31), we notice that either,  $e ? \_$  is in syntactic contradiction with all the cases, or the premise  $\_ \text{refutable?}$  is not derivable. Hence,  $e ? \_$  are not derivable. And thus vacuously true.

**Case (20c).**

(4)  $p = \langle \rangle^w$  by assumption

(5)  $\xi = ?$  by assumption

(6)  $\bar{\xi} = ?$  by Definition 9

1. Prove  $e \models ?$  implies  $e \triangleright \langle \rangle^w \dashv \! \vdash \theta$  for some  $\theta$ .

(7)  $e \not\models ?$  by Rule (30a)

Vacuously true.

2. Prove  $e \triangleright \langle \rangle^w \dashv \! \vdash \theta$  implies  $e \models ?$ .

By rule induction over Rules (30), we notice that  $e \triangleright \langle \rangle^w \dashv \! \vdash \theta$  is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove  $e \models ?$  implies  $e ? \langle \rangle^w$ .

(7)  $e ? \langle \rangle^w$  by Rule (31a)

4. Prove  $e ? \langle \rangle^w$  implies  $e \models ? ?$ .

(7)  $e \models ? ?$  by Rule (16a)

**Case (20d).**

(4)  $p = \langle p_0 \rangle^w$  by assumption

(5)  $\xi = ?$  by assumption

1. Prove  $e \models ?$  implies  $e \triangleright \langle p_0 \rangle^w \dashv \! \vdash \theta$  for some  $\theta$ .

(6)  $e \not\models ?$  by Rule (30a)

Vacuously true.

2. Prove  $e \triangleright \langle p_0 \rangle^w \dashv\!\!\vdash \theta$  implies  $e \models ?$ .  
By rule induction over Rules (30), we notice that  $e \triangleright \langle p_0 \rangle^w \dashv\!\!\vdash \theta$  is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.
3. Prove  $e \models ?$  implies  $e ? \langle p_0 \rangle^w$ .  
(6)  $e ? \langle p_0 \rangle^w$  by Rule (31b)
4. Prove  $e ? \langle p_0 \rangle^w$  implies  $e \models ?$ .  
(6)  $e \models ?$  by Rule (16a)

**Case (20e).**

- (4)  $p = \underline{n}$  by assumption
- (5)  $\xi = \underline{n}$  by assumption

1. Prove  $e \models \underline{n}$  implies  $e \triangleright \underline{n} \dashv\!\!\vdash \theta$  for some  $\theta$ .  
(6)  $e \models \underline{n}$  by assumption  
By rule induction over Rules (14) on (6), only one case applies.  
**Case (14b).**  
(7)  $e = \underline{n}$  by assumption  
(8)  $\underline{n} \triangleright \underline{n} \dashv\!\!\vdash \cdot$  by Rule (30c)

2. Prove  $e \triangleright \underline{n} \dashv\!\!\vdash \theta$  implies  $e \models \underline{n}$ .  
(6)  $e \triangleright \underline{n} \dashv\!\!\vdash \theta$  by assumption  
By rule induction over Rules (30) on (6), only one case applies.  
**Case (30c).**  
(7)  $e = \underline{n}$  by assumption  
(8)  $\theta = \cdot$  by assumption  
(9)  $\underline{n} \models \underline{n}$  by Rule (14b)

3. Prove  $e \models ? \underline{n}$  implies  $e ? \underline{n}$ .  
(6)  $e \models ? \underline{n}$  by assumption  
By rule induction over Rules (16) on (6), only one case applies.  
**Case (16b).**  
(7)  $e$  **notintro** by assumption  
(8)  $\underline{n}$  **refutable?** by Rule (29a)  
(9)  $e ? \underline{n}$  by Rule (31c) on (7) and (8)

4. Prove  $e ? \underline{n}$  implies  $e \models ? \underline{n}$ .  
(6)  $e ? \underline{n}$  by assumption

By rule induction over Rules (31) on (6), only one case applies.

**Case (31c).**

- |                                       |                                |
|---------------------------------------|--------------------------------|
| (7) $e$ <b>notintro</b>               | by assumption                  |
| (8) $\underline{n}$ <b>refutable?</b> | by Rule (10a)                  |
| (9) $e \models? \underline{n}$        | by Rule (16) on (7)<br>and (8) |

**Case (20f).**

- |  |               |
|--|---------------|
| (4) $p = \text{inl}(p_1)$                              | by assumption |
| (5) $\xi = \text{inl}(\xi_1)$                          | by assumption |
| (6) $\tau = (\tau_1 + \tau_2)$                         | by assumption |
| (7) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma ; \Delta$ | by assumption |

By rule induction over Rules (19) on (1), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- |  |  |
|--|--|
| (8) $e = \text{inl}^u, \text{inl}^u(e_0), e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$ | by assumption                                  |
| (9) $e$ <b>notintro</b>  | by Rule<br>(26a),(26b),(26c),(26d),(26e),(26f) |

1. Prove  $e \models \text{inl}(\xi_1)$  implies  $e \triangleright \text{inl}(p_1) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (14) on  $e \models \text{inl}(\xi_1)$ , no case applies due to syntactic contradiction.  
Therefore, vacuously true.
2. Prove  $e \triangleright \text{inl}(p_1) \dashv\vdash \theta$  implies  $e \models \text{inl}(\xi_1)$ . By rule induction over Rules (30) on  $e \triangleright \text{inl}(p_1) \dashv\vdash \theta$ , no case applies due to syntactic contradiction.  
Therefore, vacuously true.
3. Prove  $e \models? \text{inl}(\xi_1)$  implies  $e ? \text{inl}(p_1)$ .

(10) $\text{inl}(p_1)$ <b>refutable?</b>	by Rule (29d)
(11) $e ? \text{inl}(p_1)$	by Rule (31c) on (9) and (10)
4. Prove  $e ? \text{inl}(p_1)$  implies  $e \models? \text{inl}(\xi_1)$ .

(10) $\text{inl}(\xi_1)$ <b>refutable?</b>	by Rule (10b)
(11) $e \models? \text{inl}(\xi_1)$	by Rule (16b) on (9) and (10)

**Case (19j).**

- |  |                       |
|--|-----------------------|
| (8) $e = \text{inl}_{\tau_2}(e_1)$         | by assumption         |
| (9) $\cdot ; \Delta_e \vdash e_1 : \tau_1$ | by assumption         |
| (10) $e_1$ <b>final</b>                    | by Lemma 4.0.2 on (2) |

By inductive hypothesis on (10) and (9) and (7).

$$(11) \quad e_1 \models \xi_1 \text{ iff } e_1 \triangleright p_1 \dashv\!\!\vdash \theta \text{ for some } \theta$$

$$(12) \quad e_1 \models? \xi_1 \text{ iff } e_1 ? p_1$$

1. Prove  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  implies  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$  for some  $\theta$ .

$$(13) \quad \text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1) \quad \text{by assumption}$$

By rule induction over Rules (14) on (13), only one case applies.

**Case (14g).**

$$(14) \quad e_1 \models \xi_1 \quad \text{by assumption}$$

$$(15) \quad e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1 \text{ for some } \theta_1 \quad \text{by (11) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1 \quad \text{by Rule (30e) on (15)}$$

2. Prove  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$  implies  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ .

$$(13) \quad \text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta \quad \text{by assumption}$$

By rule induction over Rules (30) on (13), only one case applies.

**Case (30e).**

$$(14) \quad e_1 \triangleright p_1 \dashv\!\!\vdash \theta \quad \text{by assumption}$$

$$(15) \quad e_1 \models \xi_1 \quad \text{by (11) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1) \quad \text{by Rule (14g) on (15)}$$

3. Prove  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  implies  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ .

$$(13) \quad \text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1) \quad \text{by assumption}$$

By rule induction over Rules (16) on (13), only two cases apply.

**Case (16b).**

$$(14) \quad \text{inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

Contradicts Lemma 4.0.6.

**Case (16e).**

$$(14) \quad e_1 \models? \xi_1 \quad \text{by assumption}$$

$$(15) \quad e_1 ? p_1 \quad \text{by (12) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1) \quad \text{by Rule (31g) on (15)}$$

4. Prove  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  implies  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ .

$$(13) \quad \text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1) \quad \text{by assumption}$$

By rule induction over Rules (31) on (13), only two cases apply.

**Case (31c).**

$$(14) \quad \text{inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

Contradicts Lemma 4.0.6.

**Case (31g).**

$$(14) \quad e_1 ? p_1 \quad \text{by assumption}$$

$$(15) \quad e_1 \models? \xi_1 \quad \text{by (12) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1) \quad \text{by Rule (16e) on (15)}$$

**Case (20g).**

- (4)  $p = \mathbf{inr}(p_2)$  by assumption
- (5)  $\xi = \mathbf{inr}(\xi_2)$  by assumption
- (6)  $\tau = (\tau_1 + \tau_2)$  by assumption
- (7)  $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma ; \Delta$  by assumption

By rule induction over Rules (19) on (1), the following cases apply.

**Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).**

- (8)  $e = \mathbb{0}^u, \mathbb{1}_{e_0}^u, e_1(e_2), \mathbf{prl}(e_0), \mathbf{prl}(e_0), \mathbf{match}(e_0)\{\hat{r}s\}$   
by assumption
- (9)  $e \mathbf{notintro}$  by Rule  
(26a),(26b),(26c),(26d),(26e),(26f)

1. Prove  $e \models \mathbf{inr}(\xi_2)$  implies  $e \triangleright \mathbf{inr}(p_2) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (14) on  $e \models \mathbf{inr}(\xi_2)$ , no case applies due to syntactic contradiction.  
Therefore, vacuously true.
2. Prove  $e \triangleright \mathbf{inr}(p_2) \dashv\vdash \theta$  implies  $e \models \mathbf{inr}(\xi_2)$ . By rule induction over Rules (30) on  $e \triangleright \mathbf{inr}(p_2) \dashv\vdash \theta$ , no case applies due to syntactic contradiction.  
Therefore, vacuously true.
3. Prove  $e \models_{\text{?}} \mathbf{inr}(\xi_2)$  implies  $e \text{?} \mathbf{inr}(p_2)$ .
  - (10)  $\mathbf{inr}(p_2) \mathbf{refutable?}$  by Rule (29e)
  - (11)  $e \text{?} \mathbf{inr}(p_2)$  by Rule (31c) on (9)  
and (10)
4. Prove  $e \text{?} \mathbf{inr}(p_2)$  implies  $e \models_{\text{?}} \mathbf{inr}(\xi_2)$ .
  - (10)  $\mathbf{inr}(\xi_2) \mathbf{refutable?}$  by Rule (10c)
  - (11)  $e \models_{\text{?}} \mathbf{inr}(\xi_2)$  by Rule (16b) on (9)  
and (10)

**Case (19k).**

- (8)  $e = \mathbf{inr}_{\tau_1}(e_2)$  by assumption
- (9)  $\cdot ; \Delta_e \vdash e_2 : \tau_2$  by assumption
- (10)  $e_2 \mathbf{final}$  by Lemma 4.0.2 on (2)

By inductive hypothesis on (10) and (9) and (7).

- (11)  $e_2 \models \xi_2$  iff  $e_2 \triangleright p_2 \dashv\vdash \theta$  for some  $\theta$
- (12)  $e_2 \models_{\text{?}} \xi_2$  iff  $e_2 \text{?} p_2$

1. Prove  $\mathbf{inr}_{\tau_1}(e_2) \models \mathbf{inr}(\xi_2)$  implies  $\mathbf{inr}_{\tau_1}(e_2) \triangleright \mathbf{inr}(p_2) \dashv\vdash \theta$  for some  $\theta$ .
  - (13)  $\mathbf{inr}_{\tau_1}(e_2) \models \mathbf{inr}(\xi_2)$  by assumption

By rule induction over Rules (14) on (13), only one case applies.

**Case (14g).**

- (14)  $e_2 \models \xi_2$  by assumption
- (15)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_1$  for some  $\theta_1$  by (11) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\parallel \theta_1$  by Rule (30e) on (15)

2. Prove  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\parallel \theta$  implies  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ .

- (13)  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\parallel \theta$  by assumption

By rule induction over Rules (30) on (13), only one case applies.

**Case (30e).**

- (14)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta$  by assumption
- (15)  $e_2 \models \xi_2$  by (11) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by Rule (14g) on (15)

3. Prove  $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$  implies  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ .

- (13)  $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$  by assumption

By rule induction over Rules (16) on (13), only two cases apply.

**Case (16b).**

- (14)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.6.

**Case (16e).**

- (14)  $e_2 \models? \xi_2$  by assumption
- (15)  $e_2 ? p_2$  by (12) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$  by Rule (31g) on (15)

4. Prove  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$  implies  $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$ .

- (13)  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$  by assumption

By rule induction over Rules (31) on (13), only two cases apply.

**Case (31c).**

- (14)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.6.

**Case (31g).**

- (14)  $e_2 ? p_2$  by assumption
- (15)  $e_2 \models? \xi_2$  by (12) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$  by Rule (16e) on (15)

**Case (20h).**

- (4)  $p = (p_1, p_2)$  by assumption
- (5)  $\xi = (\xi_1, \xi_2)$  by assumption
- (6)  $\tau = (\tau_1 \times \tau_2)$  by assumption
- (7)  $\Gamma = \Gamma_1 \uplus \Gamma_2$  by assumption
- (8)  $\Delta = \Delta_1 \uplus \Delta_2$  by assumption

- (9)  $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 ; \Delta_1$  by assumption  
 (10)  $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 ; \Delta_2$  by assumption

By rule induction over Rules (19) on (1), the following cases apply.

**Case** (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (11)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
 by assumption  
 (12)  $e$  **notintro** by Rule (26a),(26b),(26c),(26d),(26e),(26f)  
 (13)  $e$  **indet** by Lemma 4.0.9 on (2) and (12)  
 (14)  $\text{prl}(e)$  **indet** by Rule (24g) on (13)  
 (15)  $\text{prl}(e)$  **final** by Rule (25b) on (14)  
 (16)  $\text{prr}(e)$  **indet** by Rule (24h) on (13)  
 (17)  $\text{prr}(e)$  **final** by Rule (25b) on (16)  
 (18)  $\cdot ; \Delta \vdash \text{prl}(e) : \tau_1$  by Rule (19h) on (1)  
 (19)  $\cdot ; \Delta \vdash \text{prr}(e) : \tau_2$  by Rule (19i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

- (20)  $\text{prl}(e) \models \xi_1$  iff  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  for some  $\theta_1$   
 (21)  $\text{prl}(e) \models? \xi_1$  iff  $\text{prl}(e) ? p_1$   
 (22)  $\text{prr}(e) \models \xi_2$  iff  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  for some  $\theta_2$   
 (23)  $\text{prr}(e) \models? \xi_2$  iff  $\text{prr}(e) ? p_2$

1. Prove  $e \models (\xi_1, \xi_2)$  implies  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  for some  $\theta$ .

- (24)  $e \models (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (14) on (24), only one case applies.

**Case** (14j).

- (25)  $\text{prl}(e) \models \xi_1$  by assumption  
 (26)  $\text{prr}(e) \models \xi_2$  by assumption  
 (27)  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  by (20) on (25)  
 (28)  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  by (22) on (26)  
 (29)  $e \triangleright (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2$  by Rule (30g) on (12) and (27) and (28)

2. Prove  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  implies  $e \models (\xi_1, \xi_2)$ .

- (24)  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  by assumption

By rule induction over Rules (30) on (24), only one case applies.

**Case** (30g).

- (25)  $\theta = \theta_1 \uplus \theta_2$  by assumption  
 (26)  $\text{prl}(e) \triangleright \xi_1 \dashv\vdash \theta_1$  by assumption



- (27)  $\text{prr}(e) \triangleright \xi_2 \dashv\!\!\vdash \theta_2$  by assumption
- (28)  $\text{prl}(e) \models \xi_1$  by (20) on (26)
- (29)  $\text{prr}(e) \models \xi_2$  by (22) on (27)
- (30)  $e \models (\xi_1, \xi_2)$  by Rule (14j) on (12) and (28) and (29)

3. Prove  $e \models_{\text{?}} (\xi_1, \xi_2)$  implies  $e \text{ ? } (p_1, p_2)$ .

- (24)  $e \models_{\text{?}} (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (16) on (24), only one case applies.

**Case (16b).**

- (25)  $(\xi_1, \xi_2) \text{ refutable?}$  by assumption

By rule induction over Rules (10) on (25), only two cases apply.

**Case (10d).**

- (26)  $\xi_1 \text{ refutable?}$  by assumption
- (27)  $\text{prl}(e) \text{ notintro}$  by Rule (26e)
- (28)  $\text{prl}(e) \models_{\text{?}} \xi_1$  by Rule (16b) on (26) and (27)
- (29)  $\text{prl}(e) \text{ ? } p_1$  by (21) on (28)

By rule induction over Rules (31) on (29), only three cases apply.

**Case (31a),(31b).**

- (30)  $p_1 = \langle \rangle^w, \langle p_0 \rangle^w$  by assumption
- (31)  $p_1 \text{ refutable?}$  by Rule (29b) and Rule (29c)
- (32)  $(p_1, p_2) \text{ refutable?}$  by Rule (29f) on (31)
- (33)  $e \text{ ? } (p_1, p_2)$  by Rule (31c) on (12) and (32)

**Case (31c).**

- (30)  $p_1 \text{ refutable?}$  by assumption
- (31)  $(p_1, p_2) \text{ refutable?}$  by Rule (29f) on (30)
- (32)  $e \text{ ? } (p_1, p_2)$  by Rule (31c) on (12) and (31)

**Case (10e).**

- (26)  $\xi_2 \text{ refutable?}$  by assumption
- (27)  $\text{prr}(e) \text{ notintro}$  by Rule (26e)
- (28)  $\text{prr}(e) \models_{\text{?}} \xi_2$  by Rule (16b) on (26) and (27)
- (29)  $\text{prr}(e) \text{ ? } p_2$  by (23) on (28)

By rule induction over Rules (31) on (29), only three cases apply.

**Case (31a),(31b).**

- |   |                                |
|---|--------------------------------|
| (30) $p_2 = \langle \rangle^w, \langle p_0 \rangle^w$ | by assumption                  |
| (31) $p_2$ refutable?                                 | by Rule (29b) and Rule (29c)   |
| (32) $(p_1, p_2)$ refutable?                          | by Rule (29g) on (31)          |
| (33) $e ? (p_1, p_2)$                                 | by Rule (31c) on (12) and (32) |

**Case (31c).**

- |                              |                                |
|------------------------------|--------------------------------|
| (30) $p_2$ refutable?        | by assumption                  |
| (31) $(p_1, p_2)$ refutable? | by Rule (29g) on (30)          |
| (32) $e ? (p_1, p_2)$        | by Rule (31c) on (12) and (31) |

4. Prove  $e ? (p_1, p_2)$  implies  $e \models_{\mathcal{I}} (\xi_1, \xi_2)$ .

- |                       |               |
|-----------------------|---------------|
| (24) $e ? (p_1, p_2)$ | by assumption |
|-----------------------|---------------|

By rule induction over Rules (31) on (24), only one case applies.

**Case (31c).**

- |                              |               |
|------------------------------|---------------|
| (25) $(p_1, p_2)$ refutable? | by assumption |
|------------------------------|---------------|

By rule induction over Rules (29) on (25), only two cases apply.

**Case (29f).**

- |  |                                |
|--|--------------------------------|
| (26) $p_1$ refutable?                            | by assumption                  |
| (27) $\text{prl}(e)$ notintro                    | by Rule (26e)                  |
| (28) $\text{prl}(e) ? p_1$                       | by Rule (31c) on (26) and (27) |
| (29) $\text{prl}(e) \models_{\mathcal{I}} \xi_1$ | by (21) on (28)                |

By rule induction over Rules (16) on (29), only three cases apply.

**Case (16a).**

- |   |                                |
|---|--------------------------------|
| (30) $\xi_1 = ?$                              | by assumption                  |
| (31) $\xi_1$ refutable?                       | by Rule (2b)                   |
| (32) $(\xi_1, \xi_2)$ refutable?              | by Rule (10d) on (31)          |
| (33) $e \models_{\mathcal{I}} (\xi_1, \xi_2)$ | by Rule (16b) on (12) and (32) |

**Case (16b).**

- |   |                                |
|---|--------------------------------|
| (30) $\xi_1$ refutable?                       | by assumption                  |
| (31) $(\xi_1, \xi_2)$ refutable?              | by Rule (10d) on (30)          |
| (32) $e \models_{\mathcal{I}} (\xi_1, \xi_2)$ | by Rule (16b) on (12) and (31) |

**Case (29g).**

- |                               |               |
|-------------------------------|---------------|
| (26) $p_2$ refutable?         | by assumption |
| (27) $\text{prr}(e)$ notintro | by Rule (26e) |

(28)  $\text{pr}(e) \text{ ? } p_2$  by Rule (31c) on (26)  
and (27)

(29)  $\text{pr}(e) \models_{\text{?}} \xi_2$  by (23) on (28)

By rule induction over Rules (16) on (29), only three cases apply.

**Case (16a).**

(30)  $\xi_2 = ?$  by assumption

(31)  $\xi_2 \text{ refutable?}$  by Rule (2b)

(32)  $(\xi_1, \xi_2) \text{ refutable?}$  by Rule (10e) on (31)

(33)  $e \models_{\text{?}} (\xi_1, \xi_2)$  by Rule (16b) on (12)  
and (32)

**Case (16b).**

(30)  $\xi_2 \text{ refutable?}$  by assumption

(31)  $(\xi_1, \xi_2) \text{ refutable?}$  by Rule (10e) on (30)

(32)  $e \models_{\text{?}} (\xi_1, \xi_2)$  by Rule (16b) on (12)  
and (31)

**Case (19g).**

(11)  $e = (e_1, e_2)$  by assumption

(12)  $\cdot; \Delta_e \vdash e_1 : \tau_1$  by assumption

(13)  $\cdot; \Delta_e \vdash e_2 : \tau_2$  by assumption

(14)  $e_1 \text{ final}$  by Lemma 4.0.4 on (2)

(15)  $e_2 \text{ final}$  by Lemma 4.0.4 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16)  $e_1 \models \xi_1$  iff  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$

(17)  $e_1 \models_{\text{?}} \xi_1$  iff  $e_1 \text{ ? } p_1$

(18)  $e_2 \models \xi_2$  iff  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  for some  $\theta_2$

(19)  $e_2 \models_{\text{?}} \xi_2$  iff  $e_2 \text{ ? } p_2$

1. Prove  $(e_1, e_2) \models (\xi_1, \xi_2)$  implies  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  for some  $\theta$ .

(20)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (14) on (20), only two cases apply.

**Case (14i).**

(21)  $e_1 \models \xi_1$  by assumption

(22)  $e_2 \models \xi_2$  by assumption

(23)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$  by (16) on (21)

(24)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  for some  $\theta_2$  by (18) on (22)

(25)  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2$  by Rule (30d) on (23)  
and (24)

**Case (14j).**

(21)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

2. Prove  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  implies  $(e_1, e_2) \models (\xi_1, \xi_2)$ .

(20)  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  by assumption

By rule induction over Rules (30) on (20), only two cases apply.

**Case (30d).**

(21)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$  by assumption

(22)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  for some  $\theta_2$  by assumption

(23)  $e_1 \models \xi_1$  by (16) on (21)

(24)  $e_2 \models \xi_2$  by (18) on (22)

(25)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (14i) on (23) and (24)

**Case (30g).**

(21)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

3. Prove  $(e_1, e_2) \models? (\xi_1, \xi_2)$  implies  $(e_1, e_2) ? (p_1, p_2)$ .

(20)  $(e_1, e_2) \models? (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (16) on (20), only four cases apply.

**Case (16b).**

(21)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.8.

**Case (16g).**

(21)  $e_1 \models? \xi_1$  by assumption

(22)  $e_2 \models \xi_2$  by assumption

(23)  $e_1 ? p_1$  by (17) on (21)

(24)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  by (18) on (22)

(25)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (31d) on (23) and (24)

**Case (16h).**

(21)  $e_1 \models \xi_1$  by assumption

(22)  $e_2 \models? \xi_2$  by assumption

(23)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  by (16) on (21)

(24)  $e_2 ? p_2$  by (19) on (22)

(25)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (31e) on (23) and (24)

**Case (16i).**

(21)  $e_1 \models? \xi_1$  by assumption

(22)  $e_2 \models? \xi_2$  by assumption

(23)  $e_1 ? p_1$  by (17) on (21)

- (24)  $e_2 ? p_2$  by (19) on (22)  
 (25)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (31f) on (23) and (24)
4. Prove  $(e_1, e_2) ? (p_1, p_2)$  implies  $(e_1, e_2) \models? (\xi_1, \xi_2)$ .  
 (20)  $(e_1, e_2) ? (p_1, p_2)$  by assumption  
 By rule induction over Rules (31) on (20), only four cases apply.  
**Case (31c).**  
 (21)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.8.  
**Case (31d).**  
 (21)  $e_1 ? p_1$  by assumption  
 (22)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption  
 (23)  $e_1 \models? \xi_1$  by (17) on (21)  
 (24)  $e_2 \models \xi_2$  by (18) on (22)  
 (25)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (16g) on (23) and (24)  
**Case (31e).**  
 (21)  $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$  by assumption  
 (22)  $e_2 ? p_2$  by assumption  
 (23)  $e_1 \models \xi_1$  by (16) on (21)  
 (24)  $e_2 \models? \xi_2$  by (19) on (22)  
 (25)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (16h) on (23) and (24)  
**Case (31f).**  
 (21)  $e_1 ? p_1$  by assumption  
 (22)  $e_2 ? p_2$  by assumption  
 (23)  $e_1 \models? \xi_1$  by (17) on (21)  
 (24)  $e_2 \models? \xi_2$  by (19) on (22)  
 (25)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (16i) on (23) and (24)

□

## 5 Preservation and Progress

**Theorem 5.1** (Preservation). *If  $\cdot ; \Delta \vdash e : \tau$  and  $e \mapsto e'$  then  $\cdot ; \Delta \vdash e' : \tau$*

*Proof.* By rule induction over Rules (19) on typing judgment of  $e$ . For simplicity, we only consider two cases for match expressions here.

**Case (19l).**

- (1)  $\cdot ; \Delta \vdash \text{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$  by assumption

- (2)  $\text{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$  by assumption
- (3)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (4)  $\cdot; \Delta \vdash [\perp](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$  by assumption
- (5)  $\top \models_{\tau}^{\dagger} \xi$  by assumption

By rule induction over Rules (33) on (2).

**Case (33k).**

- (6)  $e' = \text{match}(e'_1)\{\cdot \mid r \mid rs\}$  by assumption
- (7)  $e_1 \mapsto e'_1$  by assumption
- (8)  $\cdot; \Delta \vdash e'_1 : \tau_1$  by IH on (3) and (7)
- (9)  $\cdot; \Delta \vdash \text{match}(e'_1)\{\cdot \mid r \mid rs\} : \tau$  by Rule (19l) on (8) and (4) and (5)

**Case (33l).**

- (6)  $r = p_r \Rightarrow e_r$  by assumption
- (7)  $e' = [\theta](e_r)$  by assumption
- (8)  $e_1 \triangleright p_r \dashv \theta$  by assumption

By rule induction over Rules (22) on (4).

**Case (22a).**

- (9)  $\xi = \xi_r$  by assumption
- (10)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (11)  $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (10)
- (12)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (21a) on (10)
- (13)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (3) and (11) and (8)
- (14)  $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (12) and (13)

**Case (22b).**

- (9)  $\xi = \xi_r \vee \xi_{rs}$  by assumption
- (10)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (11)  $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (10)
- (12)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (21a) on (10)
- (13)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (3) and (11) and (8)
- (14)  $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (12) and (13)

**Case (33m).**

- (6)  $rs = r' \mid rs'$  by assumption
- (7)  $e' = \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$  by assumption
- (8)  $e_1 \text{ final}$  by assumption
- (9)  $e_1 \perp p_r$  by assumption

By rule induction over Rules (22) on (4).

**Case (22a).** Syntactic contradiction of  $rs$ .

**Case (22b).**

- (10)  $\xi = \xi_r \vee \xi_{rs}$  by assumption
- (11)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (12)  $\cdot; \Delta \vdash [\perp \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$  by assumption
- (13)  $\xi_r \not\equiv \perp$  by assumption
- (14)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (11)
- (15)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (21a) on (11)
- (16)  $\cdot; \Delta \vdash [\perp](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$  by Rule (22a) on (11) and (13)
- (17)  $e_1 \not\equiv_{\tau}^{\dagger} \xi_r$  by Lemma 4.0.14 on (3) and (8) and (14) and (9)
- (18)  $\cdot; \Delta \vdash \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$  by Rule (19m) on (3) and (8) and (16) and (12) and (17) and (5)

**Case (19m).**

- (1)  $rs_{pre} = r_{pre} \mid rs'_{pre}$  by assumption
- (2)  $\cdot; \Delta \vdash \text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$  by assumption
- (3)  $\text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$  by assumption
- (4)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (5)  $e_1 \text{ final}$  by assumption
- (6)  $\cdot; \Delta \vdash [\perp]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$  by assumption
- (7)  $\cdot; \Delta \vdash [\perp \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$  by assumption
- (8)  $e_1 \not\equiv_{\tau}^{\dagger} \xi_{pre}$  by assumption

(9)  $\top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}$  by assumption

By rule induction over Rules (33) on (3).

**Case (33k).**

(10)  $e' = \text{match}(e'_1)\{rs_{pre} \mid r \mid rs_{post}\}$  by assumption

(11)  $e_1 \mapsto e'_1$  by assumption

By Lemma 4.0.12, (11) contradicts (5).

**Case (33l).**

(10)  $r = p_r \Rightarrow e_r$  by assumption

(11)  $e' = [\theta](e_r)$  by assumption

(12)  $e_1 \triangleright p_r \dashv \theta$  by assumption

By rule induction over Rules (22) on (7).

**Case (22a).**

(13)  $\xi_{rest} = \xi_r$  by assumption

(14)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption

(15)  $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (14)

(16)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (21a) on (14)

(17)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (4) and (15) and (12)

(18)  $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (16) and (17)

**Case (22b).**

(13)  $\xi_{rest} = \xi_r \vee \xi_{rs}$  by assumption

(14)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption

(15)  $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$  by assumption

(16)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by assumption

(17)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (4) and (15) and (12)

(18)  $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (16) and (17)

**Case (33m).**

(10)  $r = p_r \Rightarrow e_r$  by assumption

(11)  $rs_{post} = r' \mid rs'$  by assumption

(12)  $e' = \text{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\}$  by assumption

(13)  $e_1 \perp p_r$  by assumption

By rule induction over Rules (22) on (7).



**Case (22a).** Syntactic contradiction of  $rs_{post}$ .

**Case (22b).**

- (14)  $\xi_{rest} = \xi_r \vee \xi_{post}$  by assumption
- (15)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (16)  $\cdot; \Delta \vdash [\perp \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$   
by assumption
- (17)  $\xi_r \not\models \xi_{pre}$  by assumption
- (18)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (15)
- (19)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (21a) on (15)
- (20)  $\xi_r : \tau_1$  by Lemma 3.0.2 on (15)
- (21)  $\xi_{pre} : \tau_1$  by Lemma 3.0.3 on (6)
- (22)  $\xi_r \not\models \perp \vee \xi_{pre}$  by Lemma 2.0.6 on (20) and (21) and (17)
- (23)  $\cdot; \Delta \vdash [\perp](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$   
by Lemma 3.0.4 on (6) and (15) and (22)
- (24)  $e_1 \not\models_{\tau}^\dagger \xi_r$  by Lemma 4.0.14 on (4) and (5) and (18) and (13)
- (25)  $e_1 \not\models_{\tau}^\dagger \xi_{pre} \vee \xi_r$  by Lemma 2.0.7 on (8) and (24)
- (26)  $\cdot; \Delta \vdash \text{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond \mid r' \mid rs'\} : \tau$   
by Rule (19m) on (4) and (5) and (23) and (16) and (25) and (9)

□

**Theorem 5.2** (Progress). *If  $\cdot; \Delta \vdash e : \tau$  then either  $e$  final or  $e \mapsto e'$  for some  $e'$ .*

*Proof.* By rule induction over Rules (19) on typing judgment of  $e$ . For simplicity, we only consider two cases for match expressions here.

**Case (19l).**

- (1)  $\cdot; \Delta \vdash \text{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$  by assumption
- (2)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (3)  $\cdot; \Delta \vdash [\perp](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$  by assumption
- (4)  $\top \models_{\tau}^\dagger \xi$  by assumption

By IH on (2).

**Case** Scrutinee takes a step.

- (5)  $e_1 \mapsto e'_1$  by assumption
- (6)  $\text{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \text{match}(e'_1)\{\cdot \mid r \mid rs\}$  by Rule (33k) on (5)

**Case** Scrutinee is final.

- (5)  $e_1$  **final** by assumption

By rule induction over Rules (22) on (3).

**Case** (22a).

- (6)  $rs = \cdot$  by assumption
- (7)  $\xi = \xi_r$  by assumption
- (8)  $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (9)  $r = p_r \Rightarrow e_r$  by Inversion of Rule (21a) on (8)
- (10)  $p_r : \tau_1[\xi_r] \dashv\!\parallel \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (8)
- (11)  $e_1 \models_{\tau}^{\dagger} \xi_r$  by Corollary 2.1.1 on (5) and (4)

By rule induction over Rules (17) on (11).

**Case** (17a).

- (12)  $e_1 \models_{\tau} \xi_r$  by assumption
- (13)  $e_1 ? p_r$  by Lemma 4.0.14 on (2) and (5) and (10) and (12)
- (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$  **indet** by Rule (24k) on (5) and (13)
- (15)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$  **final** by Rule (25b) on (14)

**Case** (17b).

- (12)  $e_1 \models \xi_r$  by assumption
- (13)  $e_1 \triangleright p_r \dashv\!\parallel \theta$  by Lemma 4.0.14 on (2) and (5) and (10) and (12)
- (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$  by Rule (33l) on (5) and (13)

**Case** (22b).

- (6)  $rs = r' \mid rs'$  by assumption

- (7)  $\xi = \xi_r \vee \xi_{rs}$  by assumption
- (8)  $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (9)  $r = p_r \Rightarrow e_r$  by Inversion of Rule (21a) on (8)
- (10)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (8)

By Lemma 4.0.13 on (2) and (5) and (10).

**Case** Scrutinee matches pattern.

- (11)  $e_1 \triangleright p_r \dashv\vdash \theta$  by assumption
- (12)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$  by Rule (33l) on (5) and (11)

**Case** Scrutinee may matches pattern.

- (11)  $e_1 ? p_r$  by assumption
- (12)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}$  **indet** by Rule (24k) on (5) and (11)
- (13)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}$  **final** by Rule (25b) on (12)

**Case** Scrutinee doesn't matche pattern.

- (11)  $e_1 \perp p_r$  by assumption
- (12)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$   
 $\mapsto \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$  by Rule (33m) on (5) and (11)

**Case** (19m).

- (1)  $rs_{pre} = r_{pre} \mid rs'_{pre}$  by assumption
- (2)  $\cdot; \Delta \vdash \text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$  by assumption
- (3)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (4)  $e_1$  **final** by assumption
- (5)  $\cdot; \Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$  by assumption
- (6)  $e_1 \not\models_{\tau}^{\dagger} \xi_{pre}$  by assumption
- (7)  $\top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}$  by assumption

By rule induction over Rules (22) on (5).

**Case** (22a).

- (5)  $rs_{post} = \cdot$  by assumption

- (6)  $\xi_{rest} = \xi_r$  by assumption
- (7)  $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (8)  $r = p_r \Rightarrow e_r$  by Inversion of Rule (21a) on (7)
- (9)  $p_r : \tau_1[\xi_r] \dashv\!\!\parallel \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (7)
- (10)  $e_1 \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_r$  by Corollary 2.1.1 on (4) and (7)
- (11)  $e_1 \models_{\tau}^{\dagger} \xi_r$  by Lemma 2.0.8 on (10) and (6)

By rule induction over Rules (17) on (11).

**Case (17a).**

- (12)  $e_1 \models_{\tau} \xi_r$  by assumption
- (13)  $e_1 \text{ ? } p_r$  by Lemma 4.0.14 on (3) and (4) and (9) and (12)
- (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{indet}$  by Rule (24k) on (4) and (13)
- (15)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{final}$  by Rule (25b) on (14)

**Case (17b).**

- (12)  $e_1 \models \xi_r$  by assumption
- (13)  $e_1 \triangleright p_r \dashv\!\!\parallel \theta$  by Lemma 4.0.14 on (3) and (4) and (9) and (12)
- (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$  by Rule (33l) on (4) and (13)

**Case (22b).**

- (5)  $rs_{post} = r' \mid rs'_{post}$  by assumption
- (6)  $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (7)  $r = p_r \Rightarrow e_r$  by Inversion of Rule (21a) on (6)
- (8)  $p_r : \tau_1[\xi_r] \dashv\!\!\parallel \Gamma_r ; \Delta_r$  by Inversion of Rule (21a) on (6)

By Lemma 4.0.13 on (3) and (4) and (8).

**Case Scrutinee matches pattern.**

- (9)  $e_1 \triangleright p_r \dashv\!\!\parallel \theta$  by assumption

$$(10) \text{ match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$$

by Rule (33l) on (4)  
and (9)

**Case** Scrutinee may matches pattern.

$$(9) e_1 ? p_r \quad \text{by assumption}$$

$$(10) \text{ match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \text{indet}$$

by Rule (24k) on (4)  
and (9)

$$(11) \text{ match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \text{final}$$

by Rule (25b) on (10)

**Case** Scrutinee doesn't matche pattern.

$$(9) e_1 \perp p_r \quad \text{by assumption}$$

$$(10) \text{ match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\}$$

$$\mapsto \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$$

by Rule (33m) on (4)  
and (9)

□

## 6 Decidability

$\Xi \text{ incon}$     A finite set of constraints,  $\Xi$ , is inconsistent

$$\frac{\text{CINCTruth} \quad \Xi \text{ incon}}{\Xi, \top \text{ incon}} \quad (34a)$$

$$\frac{\text{CINCFalse}}{\Xi, \perp \text{ incon}} \quad (34b)$$

$$\frac{\text{CINCNum} \quad n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2} \text{ incon}} \quad (34c)$$

$$\frac{\text{CINCNotNum}}{\Xi, \underline{n}, \neg \text{incon}} \quad (34d)$$

$$\frac{\text{CINCAnd} \quad \Xi, \xi_1, \xi_2 \text{ incon}}{\Xi, \xi_1 \wedge \xi_2 \text{ incon}} \quad (34e)$$

$$\frac{\text{CINCOr} \quad \Xi, \xi_1 \text{ incon} \quad \Xi, \xi_2 \text{ incon}}{\Xi, \xi_1 \vee \xi_2 \text{ incon}} \quad (34f)$$

$$\frac{\text{CINCI}_{\text{nj}}}{\overline{\Xi, \text{inl}(\xi_1), \text{inr}(\xi_2)} \text{ incon}} \quad (34\text{g})$$

$$\frac{\frac{\text{CINCI}_{\text{nl}}}{\Xi \text{ incon}}}{\text{inl}(\Xi) \text{ incon}} \quad (34\text{h})$$

$$\frac{\frac{\text{CINCI}_{\text{nr}}}{\Xi \text{ incon}}}{\text{inr}(\Xi) \text{ incon}} \quad (34\text{i})$$

$$\frac{\frac{\text{CINCPairL}}{\Xi_1 \text{ incon}}}{(\Xi_1, \Xi_2) \text{ incon}} \quad (34\text{j})$$

$$\frac{\frac{\text{CINCPairR}}{\Xi_2 \text{ incon}}}{(\Xi_1, \Xi_2) \text{ incon}} \quad (34\text{k})$$

**Lemma 6.0.1** (Decidability of Inconsistency). *Suppose  $\dot{\top}(\xi) = \xi$ . It is decidable whether  $\xi \text{ incon}$ .*

**Lemma 6.0.2** (Inconsistency and Entailment of Constraint). *Suppose that  $\dot{\top}(\xi) = \xi$ . Then  $\bar{\xi} \text{ incon}$  iff  $\top \models \xi$*

**Lemma 6.0.3.** *If  $e \models \xi$  then  $e \models \dot{\top}(\xi)$*

*Proof.* By rule induction over Rules (14), it is obvious to see that  $\dot{\top}(\xi) = \xi$ .  $\square$

**Lemma 6.0.4.** *If  $e \models_{\text{?}} \xi$  then  $e \models_{\text{?}}^{\dot{\top}} \dot{\top}(\xi)$ .*

*Proof.*

$$(11) \quad e \models_{\text{?}} \xi \quad \text{by assumption}$$

By Rule Induction over Rules (16) on (11).

**Case (16a).**

$$\begin{aligned} (12) \quad \xi &= ? & \text{by assumption} \\ (13) \quad e &\models \top & \text{by Rule (14a)} \\ (14) \quad e &\models_{\text{?}}^{\dot{\top}} \top & \text{by Rule (17b) on (13)} \end{aligned}$$

**Case (16b).**

$$\begin{aligned} (12) \quad e &\text{ notintro} & \text{by assumption} \\ (13) \quad \xi &\text{ refutable?} & \text{by assumption} \end{aligned}$$

By Lemma 2.0.15 on (12) and (13) and case analysis on its conclusion.  
By rule induction over Rules (10).

**Case  $\dot{\top}(\xi)$  refutable<sub>?</sub>.**

(14)  $\dot{\top}(\xi)$  refutable<sub>?</sub>

by assumption

(15)  $e \models_{\dot{?}} \dot{\top}(\xi)$

by Rule (16b) on (12)  
and (14)

(16)  $e \models_{\dot{?}}^{\dagger} \dot{\top}(\xi)$

by Rule (17b) on (15)

**Case  $e \models \dot{\top}(\xi)$ .**

(14)  $e \models \dot{\top}(\xi)$

by assumption

(15)  $e \models_{\dot{?}}^{\dagger} \top$

by Rule (17b) on (14)

**Case (16c).**

(12)  $\xi = \xi_1 \vee \xi_2$

by assumption

(13)  $e \models_{\dot{?}} \xi_1$

by assumption

(14)  $e \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1)$

by IH on (13)

(15)  $e \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$

by Lemma 2.0.10 on  
(14)

**Case (16d).**

(12)  $\xi = \xi_1 \vee \xi_2$

by assumption

(13)  $e \models_{\dot{?}} \xi_2$

by assumption

(14)  $e \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_2)$

by IH on (13)

(15)  $e \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$

by Lemma 2.0.10 on  
(14)

**Case (16e).**

(12)  $e = \text{inl}_{\tau_2}(e_1)$

by assumption

(13)  $\xi = \text{inl}(\xi_1)$

by assumption

(14)  $e_1 \models_{\dot{?}} \xi_1$

by assumption

(15)  $e_1 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1)$

by IH on (14)

(16)  $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\top}(\xi_1))$

by Lemma 2.0.11 on  
(15)

**Case (16f).**

(12)  $e = \text{inr}_{\tau_1}(e_2)$

by assumption

(13)  $\xi = \text{inr}(\xi_2)$

by assumption

(14)	$e_2 \models_{\tau_1} \xi_2$	by assumption
(15)	$e_2 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_2)$	by IH on (14)
(16)	$\text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\dot{\top}(\xi_2))$	by Lemma 2.0.12 on (15)

**Case (16g).**

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{\tau_1} \xi_1$	by assumption
(15)	$e_2 \models_{\tau_1} \xi_2$	by assumption
(16)	$e_1 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_1)$	by IH on (14)
(17)	$e_2 \models_{\tau_1} \dot{\top}(\xi_2)$	by Lemma 6.0.3 on (15)
(18)	$e_2 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_2)$	by Rule (17b) on (17)
(19)	$(e_1, e_2) \models_{\tau_1}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (16) and (18)

**Case (16h).**

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{\tau_1} \xi_1$	by assumption
(15)	$e_2 \models_{\tau_1} \xi_2$	by assumption
(16)	$e_1 \models_{\tau_1} \dot{\top}(\xi_1)$	by Lemma 6.0.3 on (14)
(17)	$e_1 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_1)$	by Rule (17b) on (16)
(18)	$e_2 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_2)$	by IH on (15)
(19)	$(e_1, e_2) \models_{\tau_1}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (17) and (18)

**Case (16i).**

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{\tau_1} \xi_1$	by assumption
(15)	$e_2 \models_{\tau_1} \xi_2$	by assumption



- |   |   |
|---|---|
| (16) $e_1 \models_{\text{?}}^{\dagger} \dot{\top}(\xi_1)$<br>(17) $e_2 \models_{\text{?}}^{\dagger} \dot{\top}(\xi_2)$<br>(18) $(e_1, e_2) \models_{\text{?}}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by IH on (14)<br>by IH on (15)<br>by Lemma 2.0.13 on<br>(16) and (17) |
|---|---|

□

**Lemma 6.0.5.**  $e \models_{\text{?}}^{\dagger} \xi$  iff  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$

*Proof.* 1. Sufficiency:

- |  |               |
|--|---------------|
| (1) $e \models_{\text{?}}^{\dagger} \xi$ | by assumption |
|--|---------------|

By rule induction over Rules (17) on (1)

**Case (17b).**

- |  |                       |
|--|-----------------------|
| (2) $e \models \xi$                                  | by assumption         |
| (3) $e \models \dot{\top}(\xi)$                      | by Lemma 6.0.3 on (2) |
| (4) $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$ | by Rule (17b) on (3)  |

**Case (17a).**

- |  |                       |
|--|-----------------------|
| (2) $e \models_{\text{?}} \xi$                       | by assumption         |
| (3) $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$ | by Lemma 6.0.4 on (2) |

2. Necessity:

- |  |               |
|--|---------------|
| (1) $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$ | by assumption |
|--|---------------|

By structural induction on  $\xi$ ,

**Case  $\xi = \top, \perp, n, \neg$ .**

- |  |                          |
|--|--------------------------|
| (2) $e \models_{\text{?}}^{\dagger} \xi$ | by (1) and Definition 12 |
|--|--------------------------|

**Case  $\xi = ?$ .**

- |  |                      |
|--|----------------------|
| (2) $e \models_{\text{?}} ?$           | by Rule (16a)        |
| (3) $e \models_{\text{?}}^{\dagger} ?$ | by Rule (17a) on (2) |

**Case  $\xi = \xi_1 \vee \xi_2$ .**

- |   |                  |
|---|------------------|
| (2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Definition 12 |
|---|------------------|

By rule induction over Rules (17) on (1),

**Case (17b).**

$$(3) \quad e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad \text{by assumption}$$

By rule induction over Rules (14) on (3) and two cases apply,

**Case (14e).**

$$\begin{aligned} (4) \quad e &\models \dot{\top}(\xi_1) && \text{by assumption} \\ (5) \quad e &\models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_1) && \text{by Rule (17b) on (4)} \\ (6) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_1 && \text{by IH on (5)} \\ (7) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2 && \text{by Lemma 2.0.10 on (6)} \end{aligned}$$

**Case (14f).**

$$\begin{aligned} (4) \quad e &\models \dot{\top}(\xi_2) && \text{by assumption} \\ (5) \quad e &\models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_2) && \text{by Rule (17b) on (4)} \\ (6) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_2 && \text{by IH on (5)} \\ (7) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2 && \text{by Lemma 2.0.10 on (6)} \end{aligned}$$

**Case (17a).**

$$(3) \quad e \models_{\dot{?}} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad \text{by assumption}$$

By rule induction over Rules (16) on (3) and two cases apply,

**Case (16c).**

$$\begin{aligned} (4) \quad e &\models_{\dot{?}} \dot{\top}(\xi_1) && \text{by assumption} \\ (5) \quad e &\models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_1) && \text{by Rule (17a) on (4)} \\ (6) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_1 && \text{by IH on (5)} \\ (7) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2 && \text{by Lemma 2.0.10 on (6)} \end{aligned}$$

**Case (16d).**

$$\begin{aligned} (4) \quad e &\models_{\dot{?}} \dot{\top}(\xi_2) && \text{by assumption} \\ (5) \quad e &\models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_2) && \text{by Rule (17a) on (4)} \\ (6) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_2 && \text{by IH on (5)} \\ (7) \quad e &\models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2 && \text{by Lemma 2.0.10 on (6)} \end{aligned}$$

**Case  $\xi = \text{inl}(\xi_1)$ .**

$$\begin{aligned} (2) \quad e &= \text{inl}_{r_2}(e_1) && \text{by assumption} \\ (3) \quad \dot{\top}(\xi) &= \text{inl}(\dot{\top}(\xi_1)) && \text{by assumption} \end{aligned}$$

By rule induction over Rules (17) on (1),

**Case (17b).**

(4)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\top}(\xi_1))$  by assumption

By rule induction over Rules (14) and only one case applies,

**Case (14g).**

(5)  $e_1 \models \dot{\top}(\xi_1)$  by assumption

(6)  $e_1 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1)$  by Rule (17b) on (5)

(7)  $e_1 \models_{\dot{?}}^{\dagger} \xi_1$  by IH on (6)

(8)  $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.11 on (7)

**Case (17a).**

(4)  $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}} \text{inl}(\dot{\top}(\xi_1))$  by assumption

By rule induction over Rules (16) and only one case applies,

**Case (16e).**

(5)  $e_1 \models_{\dot{?}} \dot{\top}(\xi_1)$  by assumption

(6)  $e_1 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1)$  by Rule (17a) on (5)

(7)  $e_1 \models_{\dot{?}}^{\dagger} \xi_1$  by IH on (6)

(8)  $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.11 on (7)

**Case  $\xi = \text{inr}(\xi_2)$ .**

(2)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

(3)  $\dot{\top}(\xi) = \text{inr}(\dot{\top}(\xi_2))$  by assumption

By rule induction over Rules (17) on (1),

**Case (17b).**

(4)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\top}(\xi_2))$  by assumption

By rule induction over Rules (14) and only one case applies,

**Case (14h).**

(5)  $e_2 \models \dot{\top}(\xi_2)$  by assumption

(6)  $e_2 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_2)$  by Rule (17b) on (5)

(7)  $e_2 \models_{\dot{?}}^{\dagger} \xi_2$  by IH on (6)

(8)  $\text{inr}_{\tau_1}(e_2) \models_{\dot{?}}^{\dagger} \text{inr}(\xi_2)$  by Lemma 2.0.12 on (7)

**Case (17a).**

(4)  $\text{inr}_{\tau_1}(e_2) \models_{\dot{?}} \text{inr}(\dot{\top}(\xi_2))$  by assumption

By rule induction over Rules (16) and only one case applies,

**Case (16f).**

(5)  $e_2 \models_{\dot{?}} \dot{\top}(\xi_2)$  by assumption

(6)  $e_2 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_2)$  by Rule (17a) on (5)

(7)  $e_2 \models_{\dot{?}}^{\dagger} \xi_2$  by IH on (6)

$$(8) \text{ inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{ inr}(\xi_2) \quad \text{by Lemma 2.0.12 on (7)}$$

**Case**  $\xi = (\xi_1, \xi_2)$ .

$$(2) e = (e_1, e_2) \quad \text{by assumption}$$

$$(3) \dot{\vdash}(\xi) = \dot{\vdash}(\xi_1) \wedge \dot{\vdash}(\xi_2) \quad \text{by Definition 12}$$

By rule induction over Rules (17) on (1),

**Case** (17b).

$$(4) (e_1, e_2) \models (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (14) on (4) and only one case applies,

**Case** (14i).

$$(5) e_1 \models \dot{\vdash}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models \dot{\vdash}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_1) \quad \text{by Rule (17b) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_2) \quad \text{by Rule (17b) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

$$(10) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (8)}$$

$$(11) (e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2) \quad \text{by Lemma 2.0.13 on (9) and (10)}$$

**Case** (17a).

$$(4) (e_1, e_2) \models_{\tau} (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (16) on (4) and three cases apply,

**Case** (16g).

$$(5) e_1 \models_{\tau} \dot{\vdash}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models \dot{\vdash}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_1) \quad \text{by Rule (17a) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_2) \quad \text{by Rule (17b) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

$$(10) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (8)}$$

$$(11) (e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2) \quad \text{by Lemma 2.0.13 on (9) and (10)}$$

**Case** (16h).

$$(5) e_1 \models \dot{\vdash}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models_{\tau} \dot{\vdash}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_1) \quad \text{by Rule (17b) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_2) \quad \text{by Rule (17a) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

- |   |                                 |
|---|---------------------------------|
| (10) $e_2 \models_{\tau}^{\dagger} \xi_2$                 | by IH on (8)                    |
| (11) $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

**Case (16i).**

- |   |                                 |
|---|---------------------------------|
| (5) $e_1 \models_{\tau}^{\dagger} \dot{\tau}(\xi_1)$      | by assumption                   |
| (6) $e_2 \models_{\tau}^{\dagger} \dot{\tau}(\xi_2)$      | by assumption                   |
| (7) $e_1 \models_{\tau}^{\dagger} \dot{\tau}(\xi_1)$      | by Rule (17a) on (5)            |
| (8) $e_2 \models_{\tau}^{\dagger} \dot{\tau}(\xi_2)$      | by Rule (17a) on (6)            |
| (9) $e_1 \models_{\tau}^{\dagger} \xi_1$                  | by IH on (7)                    |
| (10) $e_2 \models_{\tau}^{\dagger} \xi_2$                 | by IH on (8)                    |
| (11) $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

□

**Lemma 6.0.6.** Assume  $\dot{\tau}(\xi) = \xi$ . Then  $\top \models_{\tau}^{\dagger} \xi$  iff  $\top \models \xi$ .

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:
2. Necessity:

□

**Theorem 6.1.**  $\top \models_{\tau}^{\dagger} \xi$  iff  $\top \models \dot{\tau}(\xi)$ .

**Lemma 6.1.1.** Assume that  $e \text{ val}$ . Then  $e \models_{\tau}^{\dagger} \xi$  iff  $e \models \dot{\tau}(\xi)$

*Proof.*

- |                     |               |
|---------------------|---------------|
| (1) $e \text{ val}$ | by assumption |
|---------------------|---------------|

We prove sufficiency and necessity separately.

1. Sufficiency:

- |                                      |               |
|--------------------------------------|---------------|
| (2) $e \models_{\tau}^{\dagger} \xi$ | by assumption |
|--------------------------------------|---------------|

By rule induction over Rules (17) on (2).

**Case (17b).**

- |                                 |                       |
|---------------------------------|-----------------------|
| (3) $e \models \xi$             | by assumption         |
| (4) $e \models \dot{\tau}(\xi)$ | by Lemma 6.0.3 on (3) |

**Case (17a).**

- |                                      |               |
|--------------------------------------|---------------|
| (3) $e \models_{\tau}^{\dagger} \xi$ | by assumption |
|--------------------------------------|---------------|

By rule induction over Rules (16) on (3).

**Case (16a).**

- |                                 |                                    |
|---------------------------------|------------------------------------|
| (4) $\xi = ?$                   | by assumption                      |
| (5) $e \models \dot{\top}(\xi)$ | by Rule (14a) and<br>Definition 12 |

**Case (16b).**

- |                          |               |
|--------------------------|---------------|
| (4) $e \text{ notintro}$ | by assumption |
|--------------------------|---------------|

By rule induction over Rules (26) on (4), for each case, by rule induction over Rules (23) on (1), no case applies due to syntactic contradiction.

**Case (16c).**

- |  |                      |
|--|----------------------|
| (4) $\xi = \xi_1 \vee \xi_2$                                     | by assumption        |
| (5) $e \models? \xi_1$   | by assumption        |
| (6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Equation 12       |
| (7) $e \models?^{\dagger} \xi_1$                                 | by Rule (17a) on (5) |
| (8) $e \models \dot{\top}(\xi_1)$                                | by IH on (7)         |
| (9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$         | by Rule (14e) on (8) |

**Case (16d).**

- |  |                      |
|--|----------------------|
| (4) $\xi = \xi_1 \vee \xi_2$                                     | by assumption        |
| (5) $e \models? \xi_2$   | by assumption        |
| (6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Equation 12       |
| (7) $e \models?^{\dagger} \xi_2$                                 | by Rule (17a) on (5) |
| (8) $e \models \dot{\top}(\xi_2)$                                | by IH on (7)         |
| (9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$         | by Rule (14f) on (8) |

**Case (16e).**

- |   |                      |
|---|----------------------|
| (4) $\xi = \text{inl}(\xi_1)$                         | by assumption        |
| (5) $e \models? \xi_1$                                | by assumption        |
| (6) $\dot{\top}(\xi) = \text{inl}(\dot{\top}(\xi_1))$ | by Equation 12       |
| (7) $e \models?^{\dagger} \xi_1$                      | by Rule (17a) on (5) |
| (8) $e \models \dot{\top}(\xi_1)$                     | by IH on (7)         |
| (9) $e \models \text{inl}(\dot{\top}(\xi_1))$         | by Rule (14g) on (8) |

**Case (16f).**

- |   |                      |
|---|----------------------|
| (4) $\xi = \text{inr}(\xi_2)$                         | by assumption        |
| (5) $e \models? \xi_2$                                | by assumption        |
| (6) $\dot{\top}(\xi) = \text{inr}(\dot{\top}(\xi_2))$ | by Equation 12       |
| (7) $e \models?^{\dagger} \xi_2$                      | by Rule (17a) on (5) |

- |   |                      |
|---|----------------------|
| (8) $e \models \dot{\top}(\xi_2)$               | by IH on (7)         |
| (9) $e \models \mathbf{inr}(\dot{\top}(\xi_2))$ | by Rule (14h) on (8) |

**Case (16g).**

- |  |                                   |
|--|-----------------------------------|
| (4) $e = (e_1, e_2)$   | by assumption                     |
| (5) $\xi = (\xi_1, \xi_2)$                                       | by assumption                     |
| (6) $e_1 \models? \xi_1$   | by assumption                     |
| (7) $e_2 \models \xi_2$  | by assumption                     |
| (8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$   | by Equation 12                    |
| (9) $e_1 \models?^{\dagger} \xi_1$                               | by Rule (17a) on (6)              |
| (10) $e_2 \models?^{\dagger} \xi_2$                              | by Rule (17b) on (7)              |
| (11) $e_1 \models \dot{\top}(\xi_1)$                             | by IH on (9)                      |
| (12) $e_2 \models \dot{\top}(\xi_2)$                             | by IH on (10)                     |
| (13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (14i) on (11)<br>and (12) |

**Case (16h).**

- |  |                                   |
|--|-----------------------------------|
| (4) $e = (e_1, e_2)$   | by assumption                     |
| (5) $\xi = (\xi_1, \xi_2)$                                       | by assumption                     |
| (6) $e_1 \models \xi_1$  | by assumption                     |
| (7) $e_2 \models? \xi_2$   | by assumption                     |
| (8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$   | by Equation 12                    |
| (9) $e_1 \models?^{\dagger} \xi_1$                               | by Rule (17b) on (6)              |
| (10) $e_2 \models?^{\dagger} \xi_2$                              | by Rule (17a) on (7)              |
| (11) $e_1 \models \dot{\top}(\xi_1)$                             | by IH on (9)                      |
| (12) $e_2 \models \dot{\top}(\xi_2)$                             | by IH on (10)                     |
| (13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (14i) on (11)<br>and (12) |

**Case (16i).**

- |  |                                   |
|--|-----------------------------------|
| (4) $e = (e_1, e_2)$   | by assumption                     |
| (5) $\xi = (\xi_1, \xi_2)$                                       | by assumption                     |
| (6) $e_1 \models? \xi_1$   | by assumption                     |
| (7) $e_2 \models? \xi_2$   | by assumption                     |
| (8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$   | by Equation 12                    |
| (9) $e_1 \models?^{\dagger} \xi_1$                               | by Rule (17a) on (6)              |
| (10) $e_2 \models?^{\dagger} \xi_2$                              | by Rule (17a) on (7)              |
| (11) $e_1 \models \dot{\top}(\xi_1)$                             | by IH on (9)                      |
| (12) $e_2 \models \dot{\top}(\xi_2)$                             | by IH on (10)                     |
| (13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (14i) on (11)<br>and (12) |

2. Necessity:

$$(2) \quad e \models \dot{\top}(\xi) \quad \text{by assumption}$$

By structural induction on  $\xi$ .

**Case**  $\xi = \top, \perp, n, \neg$ .

$$(3) \quad \xi = \dot{\top}(\xi) \quad \text{by Equation 12}$$

$$(4) \quad e \models_{\dot{?}}^{\dot{?}} \xi \quad \text{by Rule (17b) on (2)}$$

**Case**  $\xi = ?$ .

$$(3) \quad e \models_{\dot{?}} ? \quad \text{by Rule (16a)}$$

$$(4) \quad e \models_{\dot{?}}^{\dot{?}} ? \quad \text{by Rule (17a) on (3)}$$

**Case**  $\xi = \xi_1 \wedge \xi_2$ .

$$(3) \quad \dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2), only one case applies.

**Case** (14d).

$$(4) \quad e \models \dot{\top}(\xi_1) \quad \text{by assumption}$$

$$(5) \quad e \models \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(6) \quad e \models_{\dot{?}}^{\dot{?}} \xi_1 \quad \text{by IH on (4)}$$

$$(7) \quad e \models_{\dot{?}}^{\dot{?}} \xi_2 \quad \text{by IH on (5)}$$

$$(8) \quad e \models \xi_1 \wedge \xi_2 \quad \text{by Lemma 2.0.9 on (6) and (7)}$$

**Case**  $\xi = \xi_1 \vee \xi_2$ .

$$(3) \quad \dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only two cases apply.

**Case** (14e).

$$(4) \quad e \models \dot{\top}(\xi_1) \quad \text{by assumption}$$

$$(5) \quad e \models_{\dot{?}}^{\dot{?}} \xi_1 \quad \text{by IH on (4)}$$

$$(6) \quad e \models_{\dot{?}}^{\dot{?}} \xi_1 \vee \xi_2 \quad \text{by Lemma 2.0.10 on (5)}$$

**Case** (14f).

$$(4) \quad e \models \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(5) \quad e \models_{\dot{?}}^{\dot{?}} \xi_2 \quad \text{by IH on (4)}$$



$$(6) \quad e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2 \quad \text{by Lemma 2.0.10 on (5)}$$

**Case**  $\xi = \text{inl}(\xi_1)$ .

$$(3) \quad \dot{\vdash}(\xi) = \text{inl}(\dot{\vdash}(\xi_1)) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only one case applies.

**Case** (14g).

$$\begin{aligned} (4) \quad e &= \text{inl}_{\tau_2}(e_1) && \text{by assumption} \\ (5) \quad e_1 &\models \dot{\vdash}(\xi_1) && \text{by assumption} \\ (6) \quad e_1 &\models_{\tau}^{\dagger} \xi_1 && \text{by IH on (5)} \\ (7) \quad \text{inl}_{\tau_2}(e_1) &\models_{\tau}^{\dagger} \text{inl}(\xi_1) && \text{by Lemma 2.0.11 on (6)} \end{aligned}$$

**Case**  $\xi = \text{inr}(\xi_2)$ .

$$(3) \quad \dot{\vdash}(\xi) = \text{inr}(\dot{\vdash}(\xi_2)) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only one case applies.

**Case** (14h).

$$\begin{aligned} (4) \quad e &= \text{inr}_{\tau_1}(e_2) && \text{by assumption} \\ (5) \quad e_2 &\models \dot{\vdash}(\xi_2) && \text{by assumption} \\ (6) \quad e_2 &\models_{\tau}^{\dagger} \xi_2 && \text{by IH on (5)} \\ (7) \quad \text{inr}_{\tau_1}(e_2) &\models_{\tau}^{\dagger} \text{inr}(\xi_2) && \text{by Lemma 2.0.12 on (6)} \end{aligned}$$

**Case**  $\xi = (\xi_1, \xi_2)$ .

$$(3) \quad \dot{\vdash}(\xi) = (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only one case applies.

**Case** (14i).

$$\begin{aligned} (4) \quad e &= (e_1, e_2) && \text{by assumption} \\ (5) \quad e_1 &\models \dot{\vdash}(\xi_1) && \text{by assumption} \\ (6) \quad e_2 &\models \dot{\vdash}(\xi_2) && \text{by assumption} \\ (7) \quad e_1 &\models_{\tau}^{\dagger} \xi_1 && \text{by IH on (5)} \\ (8) \quad e_2 &\models_{\tau}^{\dagger} \xi_2 && \text{by IH on (6)} \\ (9) \quad (e_1, e_2) &\models_{\tau}^{\dagger} (\xi_1, \xi_2) && \text{by Lemma 2.0.13 on (7) and (8)} \end{aligned}$$

□

**Lemma 6.1.2.**  $e \models \xi \text{ iff } e \models \dot{\vdash}(\xi)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

(1)  $e \models \xi$  by assumption

By rule induction over Rules (14) on (1).

**Case (14a).**

(2)  $\xi = \top$  by assumption  
 (3)  $e \models \dot{\perp}(\top)$  by (1) and Definition 13

**Case (14b).**

(2)  $\xi = \underline{n}$  by assumption  
 (3)  $e \models \dot{\perp}(\underline{n})$  by (1) and Definition 13

**Case (14c).**

(2)  $\xi = \underline{\neg}$  by assumption  
 (3)  $e \models \dot{\perp}(\underline{\neg})$  by (1) and Definition 13

**Case (14d).**

(2)  $\xi = \xi_1 \wedge \xi_2$  by assumption  
 (3)  $e \models \xi_1$  by assumption  
 (4)  $e \models \xi_2$  by assumption  
 (5)  $e \models \dot{\perp}(\xi_1)$  by IH on (3)  
 (6)  $e \models \dot{\perp}(\xi_2)$  by IH on (4)  
 (7)  $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$  by Rule (14d) on (5) and (6)  
 (8)  $e \models \dot{\perp}(\xi_1 \wedge \xi_2)$  by (7) and Definition 13

**Case (14e).**

(2)  $\xi = \xi_1 \vee \xi_2$  by assumption  
 (3)  $e \models \xi_1$  by assumption  
 (4)  $e \models \dot{\perp}(\xi_1)$  by IH on (3)  
 (5)  $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$  by Rule (14e) on (4)  
 (6)  $e \models \dot{\perp}(\xi_1 \vee \xi_2)$  by (5) and Definition 13

**Case (14f).**

(2)  $\xi = \xi_1 \vee \xi_2$  by assumption  
 (3)  $e \models \xi_2$  by assumption  
 (4)  $e \models \dot{\perp}(\xi_2)$  by IH on (3)  
 (5)  $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$  by Rule (14f) on (4)

(6)  $e \models \dot{\perp}(\xi_1 \vee \xi_2)$  by (5) and Definition 13

**Case (14g).**

(2)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption  
 (3)  $\xi = \text{inl}(\xi_1)$  by assumption  
 (4)  $e_1 \models \xi_1$  by assumption  
 (5)  $e_1 \models \dot{\perp}(\xi_1)$  by IH on (4)  
 (6)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\perp}(\xi_1))$  by Rule (14g) on (5)  
 (7)  $\text{inl}_{\tau_2}(e_1) \models \dot{\perp}(\text{inl}(\xi_1))$  by (6) and Definition 13

**Case (14h).**

(2)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption  
 (3)  $\xi = \text{inr}(\xi_2)$  by assumption  
 (4)  $e_2 \models \xi_2$  by assumption  
 (5)  $e_2 \models \dot{\perp}(\xi_2)$  by IH on (4)  
 (6)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\perp}(\xi_2))$  by Rule (14h) on (5)  
 (7)  $\text{inr}_{\tau_1}(e_2) \models \dot{\perp}(\text{inr}(\xi_2))$  by (6) and Definition 13

**Case (14i).**

(2)  $e = (e_1, e_2)$  by assumption  
 (3)  $\xi = (\xi_1, \xi_2)$  by assumption  
 (4)  $e_1 \models \xi_1$  by assumption  
 (5)  $e_2 \models \xi_2$  by assumption  
 (6)  $e_1 \models \dot{\perp}(\xi_1)$  by IH on (4)  
 (7)  $e_2 \models \dot{\perp}(\xi_2)$  by IH on (5)  
 (8)  $(e_1, e_2) \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$  by Rule (14i) on (6) and (7)  
 (9)  $(e_1, e_2) \models \dot{\perp}((\xi_1, \xi_2))$  by (8) and Definition 13

2. Necessity:

(1)  $e \models \dot{\perp}(\xi)$  by assumption

By structural induction on  $\xi$ .

**Case  $\xi = \top, \perp, \underline{n}, \underline{\mathcal{N}}$ .**

(2)  $e \models \xi$  by (1) and Definition 13

**Case  $\xi = ?$ .**

(2)  $e \models \perp$  by (1) and Definition 13  
 (3)  $e \not\models \perp$  by Lemma 2.0.1

(3) contradicts (2).

**Case**  $\xi = \xi_1 \wedge \xi_2$ .

(2)  $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$  by (1) and Definition 13

By rule induction over Rules (14) on (2) and only case applies.

**Case** (14d).

(3)  $e \models \dot{\perp}(\xi_1)$  by assumption  
 (4)  $e \models \dot{\perp}(\xi_2)$  by assumption  
 (5)  $e \models \xi_1$  by IH on (3)  
 (6)  $e \models \xi_2$  by IH on (4)  
 (7)  $e \models \xi_1 \wedge \xi_2$  by Rule (14d) on (5) and (6)

**Case**  $\xi = \xi_1 \vee \xi_2$ .

(2)  $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$  by (1) and Definition 13

By rule induction over Rules (14) on (2) and only two cases apply.

**Case** (14e).

(3)  $e \models \dot{\perp}(\xi_1)$  by assumption  
 (4)  $e \models \xi_1$  by IH on (3)  
 (5)  $e \models \xi_1 \vee \xi_2$  by Rule (14e) on (4)

**Case** (14f).

(3)  $e \models \dot{\perp}(\xi_2)$  by assumption  
 (4)  $e \models \xi_2$  by IH on (3)  
 (5)  $e \models \xi_1 \vee \xi_2$  by Rule (14f) on (4)

**Case**  $\xi = \text{inl}(\xi_1)$ .

(2)  $e \models \text{inl}(\dot{\perp}(\xi_1))$  by (1) and Definition 13

By rule induction over Rules (14) on (2) and only one case applies.

**Case** (14g).

(3)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption  
 (4)  $e_1 \models \dot{\perp}(\xi_1)$  by assumption  
 (5)  $e_1 \models \xi_1$  by IH on (4)  
 (6)  $e \models \text{inl}(\xi_1)$  by Rule (14g) on (5)

**Case**  $\xi = \text{inr}(\xi_2)$ .

(2)  $e \models \text{inr}(\dot{\perp}(\xi_2))$  by (1) and Definition 13

By rule induction over Rules (14) on (2) and only one case applies.

**Case (14h).**

(3)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption  
 (4)  $e_2 \models \dot{\perp}(\xi_2)$  by assumption  
 (5)  $e_2 \models \xi_2$  by IH on (4)  
 (6)  $e \models \text{inr}(\xi_2)$  by Rule (14h) on (5)

**Case  $\xi = (\xi_1, \xi_2)$ .**

(2)  $e \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$  by (1) and Definition 13

By rule induction over Rules (14) on (2) and only case applies.

**Case (14i).**

(3)  $e = (e_1, e_2)$  by assumption  
 (4)  $e_1 \models \dot{\perp}(\xi_1)$  by assumption  
 (5)  $e_2 \models \dot{\perp}(\xi_2)$  by assumption  
 (6)  $e_1 \models \xi_1$  by IH on (4)  
 (7)  $e_2 \models \xi_2$  by IH on (5)  
 (8)  $e \models (\xi_1, \xi_2)$  by Rule (14i) on (6) and (7)

□

**Lemma 6.1.3.** Assume  $e \text{ val}$  and  $\dot{\top}(\xi) = \xi$ . Then  $e \not\models \xi$  iff  $e \models \bar{\xi}$ .

**Theorem 6.2.**  $\xi_r \models \xi_{rs}$  iff  $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\perp}(\xi_{rs})$ .