1 Match Constraint Language

$\xi \ ::= \ \top \mid ? \mid \underline{n} \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

CTTruth
$$\frac{}{\top : \tau} \tag{1a}$$

CTUnknown
$$\frac{}{? : \tau} \tag{1b}$$

CTNum
$$\frac{}{\underline{n} : \texttt{num}} \tag{1c}$$

CTOr
$$\frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{1d}$$

CTInl
$$\frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{1e}$$

CTInr
$$\frac{\xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)} \tag{1f}$$

CTPair
$$\frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \tag{1g}$$

$\boxed{\xi \ \texttt{refutable}}$ ξ is refutable

RXNum
$$\frac{}{\underline{n} \ \texttt{refutable}} \tag{2a}$$

RXUnknown
$$\frac{}{? \ \texttt{refutable}} \tag{2b}$$

RXInl
$$\frac{}{\texttt{inl}(\xi) \ \texttt{refutable}} \tag{2c}$$

RXInr
$$\frac{}{\texttt{inr}(\xi) \ \texttt{refutable}} \tag{2d}$$

RXPairL
$$\frac{\xi_1 \ \texttt{refutable}}{(\xi_1, \xi_2) \ \texttt{refutable}} \tag{2e}$$

$$\frac{\xi_2 \texttt{ refutable}}{(\xi_1, \xi_2) \texttt{ refutable}} \text{ RXPairR} \tag{2f}$$

$$\frac{\xi_1 \texttt{ refutable} \qquad \xi_2 \texttt{ refutable}}{\xi_1 \vee \xi_2 \texttt{ refutable}} \text{ RXOr} \tag{2g}$$

$\boxed{\mathit{refutable}(\xi)}$

$$\mathit{refutable}(\underline{n}) = \text{true} \tag{3a}$$

$$\mathit{refutable}(?) = \text{true} \tag{3b}$$

$$\mathit{refutable}(\texttt{inl}(\xi)) = \mathit{refutable}(\xi) \tag{3c}$$

$$\mathit{refutable}(\texttt{inr}(\xi)) = \mathit{refutable}(\xi) \tag{3d}$$

$$\mathit{refutable}((\xi_1, \xi_2)) = \mathit{refutable}(\xi_1) \text{ or } \mathit{refutable}(\xi_2) \tag{3e}$$

$$\mathit{refutable}(\xi_1 \vee \xi_2) = \mathit{refutable}(\xi_1) \text{ and } \mathit{refutable}(\xi_2) \tag{3f}$$

$$\text{Otherwise} \quad \mathit{refutable}(\xi) = \text{false} \tag{3g}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\frac{}{e \models \top} \text{ CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \models \underline{n}} \text{ CSNum} \tag{4b}$$

$$\frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2} \text{ CSOrL} \tag{4c}$$

$$\frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2} \text{ CSOrR} \tag{4d}$$

$$\frac{e_1 \models \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)} \text{ CSInl} \tag{4e}$$

$$\frac{e_2 \models \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)} \text{ CSInr} \tag{4f}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \text{ CSPair} \tag{4g}$$

$$\frac{e \texttt{ notintro} \qquad \texttt{prl}(e) \models \xi_1 \qquad \texttt{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)} \text{ CSNotIntroPair} \tag{4h}$$

$\boxed{\mathit{satisfy}(e, \xi)}$

$$\mathit{satisfy}(e, \top) = \text{true} \tag{5a}$$

$$\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{5b}$$

$$\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1) \text{ or } \mathit{satisfy}(e, \xi_2) \tag{5c}$$

$$\mathit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\xi_1)) = \mathit{satisfy}(e_1, \xi_1) \tag{5d}$$

$$\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\xi_2)) = \mathit{satisfy}(e_2, \xi_2) \tag{5e}$$

$$\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \mathit{satisfy}(e_1, \xi_1) \text{ and } \mathit{satisfy}(e_2, \xi_2) \tag{5f}$$

$$\mathit{satisfy}(⦇⦈^u, (\xi_1, \xi_2)) = \mathit{satisfy}(\mathtt{prl}(⦇⦈^u), \xi_1) \text{ and } \mathit{satisfy}(\mathtt{prr}(⦇⦈^u), \xi_2) \tag{5g}$$

$$\mathit{satisfy}(⦇e⦈^u, (\xi_1, \xi_2)) = \mathit{satisfy}(\mathtt{prl}(⦇e⦈^u), \xi_1) \text{ and } \mathit{satisfy}(\mathtt{prr}(⦇e⦈^u), \xi_2) \tag{5h}$$

$$\begin{aligned}\mathit{satisfy}(e_1(e_2), (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\mathtt{prl}(e_1(e_2)), \xi_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(e_1(e_2)), \xi_2)\end{aligned} \tag{5i}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{match}(e)\{\hat{rs}\}, (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\mathtt{prl}(\mathtt{match}(e)\{\hat{rs}\}), \xi_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{match}(e)\{\hat{rs}\}), \xi_2)\end{aligned} \tag{5j}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{prl}(e), (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\mathtt{prl}(\mathtt{prl}(e)), \xi_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{prl}(e)), \xi_2)\end{aligned} \tag{5k}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{prr}(e), (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\mathtt{prl}(\mathtt{prr}(e)), \xi_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{prr}(e)), \xi_2)\end{aligned} \tag{5l}$$

$$\text{Otherwise} \quad \mathit{satisfy}(e, \xi) = \text{false} \tag{5m}$$

$\boxed{e \models_? \xi}$ e may satisfy ξ

$$\text{CMSUnknown} \quad \frac{}{e \models_? ?} \tag{6a}$$

$$\text{CMSNotIntro} \quad \frac{e \ \mathtt{notintro} \qquad \xi \ \mathtt{refutable}}{e \models_? \xi} \tag{6b}$$

$$\text{CMSOrL} \quad \frac{e \models_? \xi_1 \qquad e \not\models \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{6c}$$

$$\text{CMSOrR} \quad \frac{e \not\models \xi_1 \qquad e \models_? \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{6d}$$

$$\text{CMSInl} \quad \frac{e_1 \models_? \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)} \tag{6e}$$

$$\frac{e_2 \models_? \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)}\ \text{CMSInr} \tag{6f}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)}\ \text{CMSPairL} \tag{6g}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)}\ \text{CMSPairR} \tag{6h}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)}\ \text{CMSPair} \tag{6i}$$

$\boxed{e \models_?^\dagger \xi}$ e satisfies or may satisfy ξ

$$\frac{e \models_? \xi}{e \models_?^\dagger \xi}\ \text{CSMSMay} \tag{7a}$$

$$\frac{e \models \xi}{e \models_?^\dagger \xi}\ \text{CSMSSat} \tag{7b}$$

Lemma 1.0.1. $e \not\models \bot$

Proof. By rule induction over Rules (14), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. □

Lemma 1.0.2. $e \not\models_? \bot$

Proof. Assume $e \models_? \bot$. By rule induction over Rules (16) on $e \models_? \bot$, only one case applies.

Case (16b).

(1) $\bot$ `refutable` by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. □

Lemma 1.0.3. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (16) on $e \models_? \top$, only one case applies.

Case (16b).

(1) $\top$ `refutable` by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. $\square$

Lemma 1.0.4. $e \not\models ?$

Proof. By rule induction over Rules (14), we notice that $e \models ?$ is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 1.0.5. $e \models_?^\dagger \xi$ *iff* $e \models_?^\dagger \xi \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

   (1) $e \models_?^\dagger \xi$ — by assumption

   By rule induction over Rules (17) on (1).

   Case (17a).

   (2) $e \models_? \xi$ — by assumption
   (3) $e \models_? \xi \vee \bot$ — by Rule (16c) on (2) and Lemma 2.0.1
   (4) $e \models_?^\dagger \xi \vee \bot$ — by Rule (17a) on (3)

   Case (17b).

   (2) $e \models \xi$ — by assumption
   (3) $e \models \xi \vee \bot$ — by Rule (14e) on (2)
   (4) $e \models_?^\dagger \xi \vee \bot$ — by Rule (17b) on (3)

2. Necessity:

   (1) $e \models_?^\dagger \xi \vee \bot$ — by assumption

   By rule induction over Rules (17) on (1).

   Case (17a).

   (2) $e \models_? \xi \vee \bot$ — by assumption

   By rule induction over Rules (16) on (2), only two of them apply.

   Case (16c).

   (3) $e \models_? \xi$ — by assumption
   (4) $e \models_?^\dagger \xi$ — by Rule (17a) on (3)

Case (16d)**.**

(3) $e \models_? \perp$ by assumption

(4) $e \not\models_? \perp$ by Lemma 2.0.2

(3) contradicts (4).

Case (17b)**.**

(2) $e \models \xi \vee \perp$ by assumption

By rule induction over Rules (14) on (2), only two of them apply.

Case (14e)**.**

(3) $e \models \xi$ by assumption

(4) $e \models_?^\dagger \xi$ by Rule (17b) on (3)

Case (14f)**.**

(3) $e \models \perp$ by assumption

(4) $e \not\models \perp$ by Lemma 2.0.1

(3) contradicts (4).

□

Corollary 1.0.1. $\top \models_?^\dagger \xi$ *iff* $\top \models_?^\dagger \xi \vee \perp$

Proof. By Definition 2.1.2 and Lemma 2.0.5. □

Lemma 1.0.6. *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$*. Then* $\xi_1 \not\models \xi_2$ *iff* $\xi_1 \not\models \xi_2 \vee \perp$

Proof.

(1) $\xi_1 : \tau$ by assumption

(2) $\xi_2 : \tau$ by assumption

(3) $\perp : \tau$ by Rule (8b)

(4) $\xi_2 \vee \perp : \tau$ by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

   (5) $\xi_1 \not\models \xi_2$ by assumption

   To prove $\xi_1 \not\models \xi_2 \vee \perp$, assume $\xi_1 \models \xi_2 \vee \perp$.

   (6) $\xi_1 \models \xi_2 \vee \perp$ by assumption

   For all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies

   (7) $e \models \xi_2 \vee \perp$ by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

Case (14e).

(8) $e \models \xi_2$ by assumption

(9) $\xi_1 \models \xi_2$ by Definition 2.1.1 on (8)

(5) contradicts (9).

Case (14f).

(8) $e \models \perp$ by assumption

(9) $e \not\models \perp$ by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2 \vee \perp$

2. Necessity:

(5) $\xi_1 \not\models \xi_2 \vee \perp$ by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

(6) $\xi_1 \models \xi_2$ by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \xi_1$ implies

(7) $e \models \xi_2$ by Definition 2.1.1 on (1) and (2) and (6)

(8) $e \models \xi_2 \vee \perp$ by Rule (14e) on (7)

(9) $\xi_1 \models \xi_2 \vee \perp$ by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2$

$\square$

Lemma 1.0.7. *If* $e \not\models^{\dagger}_{?} \xi_1$ *and* $e \not\models^{\dagger}_{?} \xi_2$ *then* $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

Proof. Assume, for the sake of contradiction, that $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$.

(1) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \xi_1$ by assumption

(3) $e \not\models^{\dagger}_{?} \xi_2$ by assumption

By rule induction over Rules (17) on (1).

Case (17b)**.**

(4) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (14) on (4) and only two of them apply.

Case (14e)**.**

(5) $e \models \xi_1$ by assumption

(6) $e \models^{\dagger}_{?} \xi_1$ by Rule (17b) on (5)

(6) contradicts (2).

Case (14f)**.**

(5) $e \models \xi_2$ by assumption

(6) $e \models^{\dagger}_{?} \xi_2$ by Rule (17b) on (5)

(6) contradicts (3).

Case (17a)**.**

(4) $e \models_{?} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (4) and only two of them apply.

Case (16c)**.**

(5) $e \models_{?} \xi_1$ by assumption

(6) $e \models^{\dagger}_{?} \xi_1$ by Rule (17a) on (5)

(6) contradicts (2).

Case (16d)**.**

(5) $e \models_{?} \xi_2$ by assumption

(6) $e \models^{\dagger}_{?} \xi_2$ by Rule (17a) on (5)

(6) contradicts (3).

The conclusion holds as follows:

1. $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

□

Lemma 1.0.8. *If* $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ *and* $e \not\models^{\dagger}_{?} \xi_1$ *then* $e \models^{\dagger}_{?} \xi_2$

Proof.

(1) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b)**.**

(3) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (14) on (3) and only two of them apply.

Case (14e)**.**

(4) $e \models \xi_1$ by assumption

(5) $e \models^{\dagger}_{?} \xi_1$ by Rule (17b) on (4)

(5) contradicts (2).

Case (14f)**.**

(4) $e \models \xi_2$ by assumption

(5) $e \models^{\dagger}_{?} \xi_2$ by Rule (17b) on (4)

Case (17a)**.**

(3) $e \models_{?} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (3) and only two of them apply.

Case (16c)**.**

(4) $e \models_{?} \xi_1$ by assumption

(5) $e \models^{\dagger}_{?} \xi_1$ by Rule (17a) on (4)

(5) contradicts (2).

Case (16d)**.**

(4) $e \models_{?} \xi_2$ by assumption

(5) $e \models^{\dagger}_{?} \xi_2$ by Rule (17a) on (4)

□

Lemma 1.0.9. *If* $e \models^{\dagger}_{?} \xi_1$ *and* $e \models^{\dagger}_{?} \xi_2$ *then* $e \models^{\dagger}_{?} \xi_1 \wedge \xi_2$

Lemma 1.0.10. *If* $e \models^{\dagger}_{?} \xi_1$ *then* $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ *and* $e \models^{\dagger}_{?} \xi_2 \vee \xi_1$

Proof.

(1) $e \models^{\dagger}_{?} \xi_1$ by assumption ,

By rule induction over Rules (17) on (1),

Case (17b)**.**

(2) $e \models \xi_1$ by assumption

(3) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (2)

(4) $e \models \xi_2 \vee \xi_1$ by Rule (14f) on (2)

(5) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (3)

(6) $e \models_?^\dagger \xi_2 \vee \xi_1$ by Rule (17b) on (4)

Case (17a).

(2) $e \models_? \xi_1$ by assumption

By case analysis on the result of *satisfy*(e, ξ_2).

Case true.

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by assumption

(4) $e \models \xi_2$ by Lemma 2.0.19 on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

(6) $e \models \xi_2 \vee \xi_1$ by Rule (14e) on (4)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (5)

(8) $e \models_?^\dagger \xi_2 \vee \xi_1$ by Rule (17b) on (6)

Case false.

(3) $\mathit{satisfy}(e, \xi_2) = \text{false}$ by assumption

(4) $e \not\models \xi_2$ by Lemma 2.0.19 on (3)

(5) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (2) and (4)

(6) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (5)

□

Lemma 1.0.11. *If* $e_1 \models_?^\dagger \xi_1$ *then* $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$

Proof.

(1) $e_1 \models_?^\dagger \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

(2) $e_1 \models \xi_1$ by assumption

(3) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by Rule (14g) on (2)

(4) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ by Rule (17b) on (3)

Case (17a)**.**

(2) $e_1 \models_? \xi_1$ by assumption

(3) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ by Rule (16e) on (2)

(4) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ by Rule (17a) on (3)

□

Lemma 1.0.12. *If* $e_2 \models_?^\dagger \xi_2$ *then* $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$

Proof.

(1) $e_2 \models_?^\dagger \xi_2$ by assumption

By rule induction over Rules (17) on (1).

Case (17b)**.**

(2) $e_2 \models \xi_2$ by assumption

(3) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (14h) on (2)

(4) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (17b) on (3)

Case (17a)**.**

(2) $e_2 \models_? \xi_2$ by assumption

(3) $\mathtt{inl}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (16f) on (2)

(4) $\mathtt{inl}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (17a) on (3)

□

Lemma 1.0.13. *If* $e_1 \models_?^\dagger \xi_1$ *and* $e_2 \models_?^\dagger \xi_2$ *then* $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$

Lemma 1.0.14 (Soundness and Completeness of Refutable Constraints)**.** ξ `refutable` *iff* $\mathit{refutable}(\xi) = \mathit{true}$.

Lemma 1.0.15. *There does not exist such a constraint* $\xi_1 \wedge \xi_2$ *such that* $\xi_1 \wedge \xi_2$ `refutable`*.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \wedge \xi_2$ `refutable` is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.16. *There does not exist such a constraint* $\xi_1 \vee \xi_2$ *such that* $\xi_1 \vee \xi_2$ `refutable`*.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \vee \xi_2$ `refutable` is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.17. *If* e $\texttt{notintro}$ *and* $e \models \xi$ *then* $\cancel{\xi\ \texttt{refutable}}$.

Proof.

(1) e $\texttt{notintro}$	by assumption
(2) $e \models \xi$	by assumption

By rule induction over Rules (14) on (2).

Case (14a)**.**

(3) $\xi = \top$	by assumption

Assume $\top$ $\texttt{refutable}$. By rule induction over Rules (10), no case applies due to syntactic contradiction.
Therefore, $\cancel{\top\ \texttt{refutable}}$.

Case (14e),(14f)**.**

(3) $\xi = \xi_1 \vee \xi_2$	by assumption
(4) $\cancel{\xi_1 \vee \xi_2\ \texttt{refutable}}$	by Lemma 2.0.17

Case (14d)**.**

(3) $\xi = \xi_1 \wedge \xi_2$	by assumption
(4) $\cancel{\xi_1 \wedge \xi_2\ \texttt{refutable}}$	by Lemma 2.0.16

Case (14j)**.**

(3) $\xi = (\xi_1, \xi_2)$	by assumption
(4) $\texttt{prl}(e) \models \xi_1$	by assumption
(5) $\texttt{prr}(e) \models \xi_2$	by assumption
(6) $\texttt{prl}(e)$ $\texttt{notintro}$	by Rule (26e)
(7) $\texttt{prr}(e)$ $\texttt{notintro}$	by Rule (26f)
(8) $\cancel{\xi_1\ \texttt{refutable}}$	by IH on (6) and (4)
(9) $\cancel{\xi_2\ \texttt{refutable}}$	by IH on (7) and (5)

Assume (ξ_1, ξ_2) $\texttt{refutable}$. By rule induction over Rules (10) on it, only two cases apply.

Case (10d)**.**

(10) ξ_1 $\texttt{refutable}$	by assumption

Contradicts (8).

Case (10e)**.**

(10) ξ_2 $\texttt{refutable}$	by assumption

Contradicts (9).

Therefore, $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}}$.

Otherwise.

(3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

□

Lemma 1.0.18 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \models \xi$ iff satisfy$(e, \xi) = true$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

(1) $e \models \xi$ by assumption

By rule induction over Rules (14) on (1).

Case (14a)**.**

(2) $\xi = \top$ by assumption
(3) $\textit{satisfy}(e, \top) = \text{true}$ by Definition 15a

Case (14b)**.**

(2) $e = \underline{n}$ by assumption
(3) $\xi = \underline{n}$ by assumption
(4) $\textit{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$ by Definition 15b

Case (14c)**.**

(2) $e = \underline{n_1}$ by assumption
(3) $\xi = \cancel{\underline{n_2}}$ by assumption
(4) $n_1 \neq n_2$ by assumption
(5) $\textit{satisfy}(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) = \text{true}$ by Definition 15c on (4)

Case (14d)**.**

(2) $\xi = \xi_1 \wedge \xi_2$ by assumption
(3) $e \models \xi_1$ by assumption
(4) $e \models \xi_2$ by assumption
(5) $\textit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)

(6) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e, \xi_1 \wedge \xi_2) = \mathit{satisfy}(e, \xi_1)$ and $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15d on (5) and (6)

Case (14e).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_1$ by assumption

(4) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)

(5) $\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (4)

Case (14f).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_2$ by assumption

(4) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by IH on (3)

(5) $\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (4)

Case (14g).

(2) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\xi = \mathtt{inl}(\xi_1)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\xi_1)) = \mathit{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 15f on (5)

Case (14h).

(2) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\xi = \mathtt{inl}(\xi_2)$ by assumption

(4) $e_2 \models \xi_2$ by assumption

(5) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\xi_2)) = \mathit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 15g on (5)

Case (14i).

(2) $e = (e_1, e_2)$ by assumption

(3) $\xi = (\xi_1, \xi_2)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $e_2 \models \xi_2$ by assumption

(6) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (5)

(8) $\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$
$\mathit{satisfy}(e_1, \xi_1)$ and $\mathit{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 15h on (6) and (7)

Case (14j)**.**

(2) $\xi = (\xi_1, \xi_2)$ by assumption

(3) e `notintro` by assumption

(4) $\texttt{prl}(e) \models \xi_1$ by assumption

(5) $\texttt{prr}(e) \models \xi_2$ by assumption

(6) $\mathit{satisfy}(\texttt{prl}(e), \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$ by IH on (5)

By rule induction over Rules (26) on (3).

Otherwise.

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(9) $\mathit{satisfy}(e, (\xi_1, \xi_2)) =$
$\mathit{satisfy}(\texttt{prl}(e), \xi_1)$ and $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$
by Definition 15 on (6) and (7)

2. Completeness:

(1) $\mathit{satisfy}(e, \xi) = \text{true}$ by assumption

By structural induction on ξ.

Case $\xi = \top$**.**

(2) $e \models \top$ by Rule (14a)

Case $\xi = \bot, ?$**.**

(2) $\mathit{satisfy}(e, \xi) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$**.**
By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' = n$ by Definition 15b on (1)

(3) $\underline{n'} \models \underline{n}$ by Rule (14b) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \underline{n}) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \cancel{\underline{n}}$**.**

By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' \neq n$ by Definition 15c on (1)

(3) $\underline{n'} \models \cancel{\underline{n}}$ by Rule (14c) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \cancel{\underline{n}}) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by Definition 15d on (1)

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15d on (1)

(4) $e \models \xi_1$ by IH on (2)

(5) $e \models \xi_2$ by IH on (3)

(6) $e \models \xi_1 \wedge \xi_2$ by Rule (14d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $\mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (1)

By case analysis on (2).

Case $\mathit{satisfy}(e, \xi_1) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by assumption

(4) $e \models \xi_1$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (4)

Case $\mathit{satisfy}(e, \xi_2) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by assumption

(4) $e \models \xi_2$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

Case $\xi = \texttt{inl}(\xi_1)$**.**

By structural induction on e.

Case $e = \texttt{inl}_{\tau_2}(e_1)$.

(2)	$\mathit{satisfy}(e_1, \xi_1) = \text{true}$	by Definition 15f on (1)
(3)	$e_1 \models \xi_1$	by IH on (2)
(4)	$\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$	by Rule (14g) on (3)

Otherwise.

(2)	$\mathit{satisfy}(e, \texttt{inl}(\xi_1)) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \texttt{inr}(\xi_2)$.

By structural induction on e.

Case $e = \texttt{inr}_{\tau_1}(e_2)$.

(2)	$\mathit{satisfy}(e_2, \xi_2) = \text{true}$	by Definition 15g on (1)
(3)	$e_2 \models \xi_2$	by IH on (2)
(4)	$\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$	by Rule (14h) on (3)

Otherwise.

(2)	$\mathit{satisfy}(e, \texttt{inr}(\xi_2)) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$.

By structural induction on e.

Case $e = (e_1, e_2)$.

(2)	$\mathit{satisfy}(e_1, \xi_1) = \text{true}$	by Definition 15h on (1)
(3)	$\mathit{satisfy}(e_2, \xi_2) = \text{true}$	by Definition 15h on (1)
(4)	$e_1 \models \xi_1$	by IH on (2)
(5)	$e_2 \models \xi_2$	by IH on (3)
(6)	$(e_1, e_2) \models (\xi_1, \xi_2)$	by Rule (14i) on (4) and (5)

Case $e = \llparenthesis\rrparenthesis^u, \llparenthesis e_0 \rrparenthesis^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$.

(2)	$\mathit{satisfy}(\texttt{prl}(e), \xi_1) = \text{true}$	by Definition 15h on (1)
(3)	$\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$	by Definition 15h on (1)
(4)	$\texttt{prl}(e) \models \xi_1$	by IH on (2)
(5)	$\texttt{prr}(e) \models \xi_2$	by IH on (3)
(6)	$e\ \texttt{notintro}$	by each rule in Rules (26)
(7)	$(e_1, e_2) \models (\xi_1, \xi_2)$	by Rule (14j) on (6) and (4) and (5)

Otherwise.

(2) $\textit{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

$\square$

Lemma 1.0.19. *$e \not\models \xi$ and $e \not\models_? \xi$ iff $e \not\models_?^\dagger \xi$.*

Proof. 1. Sufficiency:

(1) $e \not\models \xi$ by assumption

(2) $e \not\models_? \xi$ by assumption

Assume $e \models_?^\dagger \xi$. By rule induction over Rules (17) on it.

Case (17a)**.**

(3) $e \models \xi$ by assumption

Contradicts (1).

Case (17b)**.**

(3) $e \models_? \xi$ by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \xi$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \xi$ by assumption

Assume $e \models \xi$.

(2) $e \models_?^\dagger \xi$ by Rule (17b) on assumption

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_? \xi$.

(3) $e \models_?^\dagger \xi$ by Rule (17a) on assumption

Contradicts (1). Therefore, $e \not\models_? \xi$.

$\square$

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \models \xi$

2. $e \models_? \xi$

3. $e \not\models^{\dagger}_{?} \xi$

Proof.

(4) $\xi : \tau$	by assumption
(5) $\cdot\,; \Delta \vdash e : \tau$	by assumption
(6) e `final`	by assumption

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

Case (8a)**.**

(7) $\xi = \top$	by assumption
(8) $e \models \top$	by Rule (14a)
(9) $e \not\models_{?} \top$	by Lemma 2.0.3
(10) $e \models^{\dagger}_{?} \top$	by Rule (17b) on (8)

Case (8b)**.**

(7) $\xi = \perp$	by assumption
(8) $e \not\models \perp$	by Lemma 2.0.1
(9) $e \not\models_{?} \perp$	by Lemma 2.0.2
(10) $e \not\models^{\dagger}_{?} \perp$	by Lemma 2.0.20 on (8) and (9)

Case (1b)**.**

(7) $\xi = ?$	by assumption
(8) $e \not\models ?$	by Lemma 2.0.4
(9) $e \models_{?} ?$	by Rule (16a)
(10) $e \models^{\dagger}_{?} ?$	by Rule (17a) on (9)

Case (8c)**.**

(7) $\xi = \underline{n_2}$	by assumption
(8) $\tau = \texttt{num}$	by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on ξ.

(11) $e \not\models \underline{n_2}$ by contradiction

(12) $\underline{n_2}$ $\texttt{refutable}$ by Rule (10a)

(13) $e \models_? \underline{n_2}$ by Rule (16b) on (10) and (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (17a) on (13)

Case (19d)**.**

(9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (16), only one case applies.

Case (16b)**.**

(10) $\underline{n_1}$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.4.

(11) $\underline{n_1} \not\models_? \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$**.**

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 15

(13) $\underline{n_1} \models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (17b) on (13)

Case $n_1 \neq n_2$**.**

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 15

(13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \not\models_?^\dagger \underline{n_2}$ by Lemma 2.0.20 on (11) and (13)

Case (8f)**.**

(7) $\xi = \xi_1 \vee \xi_2$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models_? \xi_1$, and $e \not\models_?^\dagger \xi_1$ holds. The same goes for ξ_2. By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e \models \xi_1, e \models \xi_2$.

(8) $e \models \xi_1$	by assumption
(9) $e \not\models_? \xi_1$	by assumption
(10) $e \models \xi_2$	by assumption
(11) $e \not\models_? \xi_2$	by assumption
(12) $e \models \xi_1 \vee \xi_2$	by Rule (14e) on (8)
(13) $e \models_?^\dagger \xi_1 \vee \xi_2$	by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable`	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \xi_1$	by assumption

Contradicts (9).

Case (16d).

(14) $e \models_? \xi_2$	by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$	by contradiction

Case $e \models \xi_1, e \models_? \xi_2$.

(8) $e \models \xi_1$	by assumption
(9) $e \not\models_? \xi_1$	by assumption
(10) $e \not\models \xi_2$	by assumption
(11) $e \models_? \xi_2$	by assumption
(12) $e \models \xi_1 \vee \xi_2$	by Rule (14e) on (8)
(13) $e \models_?^\dagger \xi_1 \vee \xi_2$	by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable`	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \xi_1$	by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \xi_1$	by assumption

Contradicts (8).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \not\models^\dagger_? \xi_2$**.**

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (8)

(13) $e \models^\dagger_? \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c)**.**

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (16d)**.**

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

(13) $e \models^\dagger_? \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c)**.**

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (16d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models_? \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10).

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (16d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \models_? \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16d) on (11) and (8)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14) on it, only two cases apply.

Case (14e).

(12) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(12) $e \models \xi_2$ by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

(16) $e \not\models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 2.0.20 on (13) and (15)

Case (8g).

(7) $\xi = \texttt{inl}(\xi_1)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inl}(\xi_1)$ by contradiction

By case analysis on the value of *refutable*$(\texttt{inl}(\xi_1))$.

Case *refutable*$(\texttt{inl}(\xi_1))$ = true.

(13) *refutable*$(\texttt{inl}(\xi_1))$ = true by assumption

(14) $\texttt{inl}(\xi_1)$ $\texttt{refutable}$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inl}(\xi_1)$ by Rule (16b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (17a) on (15)

Case *refutable*$(\texttt{inl}(\xi_1))$ = false.

(13) *refutable*$(\texttt{inl}(\xi_1))$ = false by assumption

(14) $\cancel{\texttt{inl}(\xi_1)\ \texttt{refutable}}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(15) $\texttt{inl}(\xi_1)$ $\texttt{refutable}$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (12) and (16)

Case (19j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(11) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(12) e_1 $\texttt{final}$ by Lemma 4.0.1 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \not\models_?^\dagger \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$.

(13) $e_1 \models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ by Rule (14g) on (13)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (17b) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \models_? \xi_1$.

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ by Rule (16e) on (14)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (17a) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

(17) $e_1 \models \xi_1$

Contradicts (13).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \not\models_?^\dagger \xi_1$.

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

(15) $e_1 \models \xi_1$

Contradicts (13).

(16) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(17) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (19k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inl}(\xi_1)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(14) $\texttt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (11) and (13)

Case (8h).

(7) $\xi = \texttt{inr}(\xi_2)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) $e\ \texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inr}(\xi_2)$ by contradiction

By case analysis on the value of *refutable*($\texttt{inr}(\xi_2)$). inr is refutable

Case *refutable*($\texttt{inr}(\xi_2)$) = true.

(13) *refutable*($\texttt{inr}(\xi_2)$) = true by assumption

(14) $\texttt{inr}(\xi_2)\ \texttt{refutable}$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inr}(\xi_2)$ by Rule (16b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inr}(\xi_2)$ by Rule (17a) on (15)

Case *refutable*($\texttt{inr}(\xi_2)$) = false.

(13) *refutable*($\texttt{inr}(\xi_2)$) = false by assumption

(14) $\cancel{\texttt{inr}(\xi_2)\ \texttt{refutable}}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(15) $\texttt{inr}(\xi_2)\ \texttt{refutable}$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inr}(\xi_2)$ by Lemma 2.0.20 on (12) and (16)

Case (19j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ — by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inr}(\xi_2)$ — by contradiction

(14) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inr}(\xi_2)$ — by Lemma 2.0.20 on (11) and (13)

Case (19k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ — by assumption

(11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ — by assumption

(12) $e_2\ \texttt{final}$ — by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \not\models_?^\dagger \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

(13) $e_2 \models \xi_2$ — by assumption

(14) $e_2 \not\models_? \xi_2$ — by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ — by Rule (14g) on (13)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$ — by Rule (17b) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ — by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_? \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inr}(\xi_2)$ — by contradiction

Case $e_2 \models_? \xi_2$.

(13) $e_2 \not\models \xi_2$ — by assumption

(14) $e_2 \models_? \xi_2$ — by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$ — by Rule (16f) on (14)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$ — by Rule (17a) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(17) $e_2 \models \xi_2$

Contradicts (13).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Case $e_2 \not\models^{\dagger}_{?} \xi_2$**.**

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \not\models_{?} \xi_2$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h)**.**

(15) $e_2 \models \xi_2$

Contradicts (13).

(16) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f)**.**

(17) $e_2 \models_{?} \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_{?} \texttt{inr}(\xi_2)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models^{\dagger}_{?} \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (14i)**.**

(7) $\xi = (\xi_1, \xi_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

(10) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

(11) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) $e\ \texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(13) e `indet` by Lemma 4.0.8 on (6) and (12)
(14) $\texttt{prl}(e)$ `indet` by Rule (24g) on (13)
(15) $\texttt{prl}(e)$ `final` by Rule (25b) on (14)
(16) $\texttt{prr}(e)$ `indet` by Rule (24h) on (13)
(17) $\texttt{prr}(e)$ `final` by Rule (25b) on (16)
(18) $\cdot\,;\Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (19h) on (5)
(19) $\cdot\,;\Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\texttt{prl}(e) \models \xi_1$, $\texttt{prl}(e) \models_? \xi_1$, and $\texttt{prl}(e) \not\models_?^\dagger \xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\texttt{prr}(e) \models \xi_2$, $\texttt{prr}(e) \models_? \xi_2$, and $\texttt{prr}(e) \not\models_?^\dagger \xi_2$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models \xi_2$**.**

(20) $\texttt{prl}(e) \models \xi_1$ by assumption
(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption
(22) $\texttt{prr}(e) \models \xi_2$ by assumption
(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption
(24) $e \models (\xi_1, \xi_2)$ by Rule (14j) on (12) and (20) and (22)
(25) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17b) on (24)
(26) $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}}$ by Lemma 2.0.18 on (12) and (24)

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

(27) (ξ_1, ξ_2) `refutable` by assumption

Contradicts (26).

(28) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models_? \xi_2$**.**

(20) $\texttt{prl}(e) \models \xi_1$ by assumption
(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption
(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption
(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 `refutable` by assumption

(28) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 `refutable` by assumption

(28) $\texttt{prr}(e)$ `notintro` by Rule (26f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\texttt{prl}(e) \models_? \xi_1, \texttt{prr}(e) \models \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

assume no "or" and "and" in pair

Case (16b)**.**

(26) ξ_1 $\texttt{refutable}$ by assumption

(27) (ξ_1, ξ_2) $\texttt{refutable}$ by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \models_? \xi_1, \texttt{prr}(e) \models_? \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

assume no "or" and "and" in pair

Case (16b)**.**

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \models_? \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

assume no "or" and "and" in pair

Case (16b)**.**

(26) ξ_1 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \xi_1, \texttt{prr}(e) \models \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 `refutable` by assumption

(28) `prl`(e) `notintro` by Rule (26e)

(29) `prl`$(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 `refutable` by assumption

(28) `prr`(e) `notintro` by Rule (26f)

(29) `prr`$(e) \models_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case `prl`$(e) \not\models_?^\dagger \xi_1$, `prr`$(e) \models_? \xi_2$**.**

(20) `prl`$(e) \not\models \xi_1$ by assumption

(21) `prl`$(e) \not\models_? \xi_1$ by assumption

(22) `prr`$(e) \not\models \xi_2$ by assumption

(23) `prr`$(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) `prl`$(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case `prl`$(e) \not\models_?^\dagger \xi_1$, `prr`$(e) \not\models_?^\dagger \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 `refutable` by assumption

(28) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 `refutable` by assumption

(28) $\texttt{prr}(e)$ `notintro` by Rule (26f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case (19g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(14) e_1 `final` by Lemma 4.0.3 on (6)

(15) e_2 `final` by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \models \overline{\xi_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \models \overline{\xi_2}$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e_1 \models \xi_1, e_2 \models \xi_2$**.**

(16)	$e_1 \models \xi_1$	by assumption
(17)	$e_1 \not\models_? \xi_1$	by assumption
(18)	$e_2 \models \xi_2$	by assumption
(19)	$e_2 \not\models_? \xi_2$	by assumption
(20)	$(e_1, e_2) \models (\xi_1, \xi_2)$	by Rule (14i) on (16) and (18)
(21)	$(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17b) on (20)

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h)**.**

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models_? \xi_2$**.**

(16)	$e_1 \models \xi_1$	by assumption
(17)	$e_1 \not\models_? \xi_1$	by assumption
(18)	$e_2 \not\models \xi_2$	by assumption
(19)	$e_2 \models_? \xi_2$	by assumption
(20)	$(e_1, e_2) \models_? (\xi_1, \xi_2)$	by Rule (16h) on (16) and (19)
(21)	$(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models^{\dagger}_{?} \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \models_? \xi_1, e_2 \models \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (16g) on (17) and (18)

(21) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i)**.**

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \models_? \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (16i) on (17) and (19)

(21) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i)**.**

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \not\models^{\dagger}_? \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models^{\dagger}_? (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models^{\dagger}_? \xi_1, e_2 \models \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h)**.**

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

$\square$

Definition 1.1.1 (Entailment of Constraints)**.** *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$*. Then* $\xi_1 \models \xi_2$ *iff for all* e *such that* $\cdot\,; \Delta \vdash e : \tau$ *and* e `val` *we have* $e \models_?^\dagger \xi_1$ *implies* $e \models \xi_2$

Definition 1.1.2 (Potential Entailment of Constraints)**.** *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$*. Then* $\xi_1 \models_?^\dagger \xi_2$ *iff for all* e *such that* $\cdot\,; \Delta \vdash e : \tau$ *and* e `final` *we have* $e \models_?^\dagger \xi_1$ *implies* $e \models_?^\dagger \xi_2$

Corollary 1.1.1. *Suppose that* $\xi : \tau$ *and* $\cdot;\Delta \vdash e : \tau$ *and* e `final`*. Then* $\top \models_?^\dagger \xi$ *implies* $e \models_?^\dagger \xi$

Proof.

(1) $\xi : \tau$ by assumption

(2) $\cdot\,; \Gamma \vdash e : \tau$ by assumption

(3) e `final` by assumption

(4) $\top \models_?^\dagger \xi$ by assumption

(5) $e_1 \models \top$ by Rule (14a)

(6) $e_1 \models_?^\dagger \top$ by Rule (17b) on (5)

(7) $\top : \tau$ by Rule (8a)

(8) $e_1 \models_?^\dagger \xi_r$ by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

$\square$

2 Match Constraint Language

$\xi ::= \top \mid \bot \mid \underline{n} \mid \cancel{\underline{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\frac{}{\top : \tau}\ \text{CTTruth} \tag{8a}$$

$$\frac{}{\bot : \tau}\ \text{CTFalsity} \tag{8b}$$

$$\frac{}{\underline{n} : \texttt{num}}\ \text{CTNum} \tag{8c}$$

$$\frac{}{\cancel{\underline{n}} : \texttt{num}}\ \text{CTNotNum} \tag{8d}$$

$$\frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau}\ \text{CTAnd} \tag{8e}$$

$$\frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau}\ \text{CTOr} \tag{8f}$$

$$\frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)}\ \text{CTInl} \tag{8g}$$

$$\frac{\xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)}\ \text{CTInr} \tag{8h}$$

$$\frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)}\ \text{CTPair} \tag{8i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\overline{\top} = \bot \tag{9a}$$
$$\overline{\bot} = \top \tag{9b}$$
$$\overline{\underline{n}} = \cancel{\underline{n}} \tag{9c}$$
$$\overline{\cancel{\underline{n}}} = \underline{n} \tag{9d}$$
$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \tag{9e}$$
$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \tag{9f}$$
$$\overline{\texttt{inl}(\xi_1)} = \texttt{inl}(\overline{\xi_1}) \vee \texttt{inr}(\top) \tag{9g}$$
$$\overline{\texttt{inr}(\xi_2)} = \texttt{inr}(\overline{\xi_2}) \vee \texttt{inl}(\top) \tag{9h}$$
$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \tag{9i}$$

$\boxed{\xi\ \texttt{refutable}}$ ξ is refutable

$$\text{RXNum}\quad \frac{}{\underline{n}\ \texttt{refutable}} \tag{10a}$$
$$\text{RXInl}\quad \frac{}{\texttt{inl}(\xi)\ \texttt{refutable}} \tag{10b}$$
$$\text{RXInr}\quad \frac{}{\texttt{inr}(\xi)\ \texttt{refutable}} \tag{10c}$$
$$\text{RXPairL}\quad \frac{\xi_1\ \texttt{refutable}}{(\xi_1, \xi_2)\ \texttt{refutable}} \tag{10d}$$
$$\text{RXPairR}\quad \frac{\xi_2\ \texttt{refutable}}{(\xi_1, \xi_2)\ \texttt{refutable}} \tag{10e}$$
$$\text{RXOr}\quad \frac{\xi_1\ \texttt{refutable} \qquad \xi_2\ \texttt{refutable}}{\xi_1 \vee \xi_2\ \texttt{refutable}} \tag{10f}$$

$\boxed{\mathit{refutable}(\xi)}$

$$\mathit{refutable}(\underline{n}) = \text{true} \tag{11a}$$
$$\mathit{refutable}(\cancel{\underline{n}}) = \text{true} \tag{11b}$$
$$\mathit{refutable}(?) = \text{true} \tag{11c}$$
$$\mathit{refutable}(\texttt{inl}(\xi)) = \mathit{refutable}(\xi) \tag{11d}$$
$$\mathit{refutable}(\texttt{inr}(\xi)) = \mathit{refutable}(\xi) \tag{11e}$$
$$\mathit{refutable}((\xi_1, \xi_2)) = \mathit{refutable}(\xi_1) \text{ or } \mathit{refutable}(\xi_2) \tag{11f}$$
$$\mathit{refutable}(\xi_1 \vee \xi_2) = \mathit{refutable}(\xi_1) \text{ and } \mathit{refutable}(\xi_2) \tag{11g}$$
$$\text{Otherwise}\quad \mathit{refutable}(\xi) = \text{false} \tag{11h}$$

$\boxed{\dot{\top}(\xi_1) = \xi_2}$

$$\dot{\top}(\top) = \top \tag{12a}$$

$$\dot{\top}(\bot) = \bot \tag{12b}$$

$$\dot{\top}(?) = \top \tag{12c}$$

$$\dot{\top}(\underline{n}) = \underline{n} \tag{12d}$$

$$\dot{\top}(\cancel{\underline{n}}) = \cancel{\underline{n}} \tag{12e}$$

$$\dot{\top}(\xi_1 \land \xi_2) = \dot{\top}(\xi_1) \land \dot{\top}(\xi_2) \tag{12f}$$

$$\dot{\top}(\xi_1 \lor \xi_2) = \dot{\top}(\xi_1) \lor \dot{\top}(\xi_2) \tag{12g}$$

$$\dot{\top}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\top}(\xi)) \tag{12h}$$

$$\dot{\top}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\top}(\xi)) \tag{12i}$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{12j}$$

$\boxed{\dot{\bot}(\xi_1) = \xi_2}$

$$\dot{\bot}(\top) = \top \tag{13a}$$

$$\dot{\bot}(\bot) = \bot \tag{13b}$$

$$\dot{\bot}(?) = \bot \tag{13c}$$

$$\dot{\bot}(\underline{n}) = \underline{n} \tag{13d}$$

$$\dot{\bot}(\cancel{\underline{n}}) = \cancel{\underline{n}} \tag{13e}$$

$$\dot{\bot}(\xi_1 \land \xi_2) = \dot{\bot}(\xi_1) \land \dot{\bot}(\xi_2) \tag{13f}$$

$$\dot{\bot}(\xi_1 \lor \xi_2) = \dot{\bot}(\xi_1) \lor \dot{\bot}(\xi_2) \tag{13g}$$

$$\dot{\bot}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\bot}(\xi)) \tag{13h}$$

$$\dot{\bot}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\bot}(\xi)) \tag{13i}$$

$$\dot{\bot}((\xi_1, \xi_2)) = (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \tag{13j}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\begin{array}{c}\text{CSTruth}\\ \dfrac{}{e \models \top}\end{array} \tag{14a}$$

$$\begin{array}{c}\text{CSNum}\\ \dfrac{}{\underline{n} \models \underline{n}}\end{array} \tag{14b}$$

$$\begin{array}{c}\text{CSNotNum}\\ \dfrac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}}\end{array} \tag{14c}$$

$$\frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2}\ \text{CSAnd} \tag{14d}$$

$$\frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2}\ \text{CSOrL} \tag{14e}$$

$$\frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2}\ \text{CSOrR} \tag{14f}$$

$$\frac{e_1 \models \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)}\ \text{CSInl} \tag{14g}$$

$$\frac{e_2 \models \xi_2}{\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)}\ \text{CSInr} \tag{14h}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)}\ \text{CSPair} \tag{14i}$$

$$\frac{e\ \mathtt{notintro} \qquad \mathtt{prl}(e) \models \xi_1 \qquad \mathtt{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)}\ \text{CSNotIntroPair} \tag{14j}$$

$\boxed{\mathit{satisfy}(e, \xi)}$

$$\mathit{satisfy}(e, \top) = \text{true} \tag{15a}$$

$$\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{15b}$$

$$\mathit{satisfy}(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) \tag{15c}$$

$$\mathit{satisfy}(e, \xi_1 \wedge \xi_2) = \mathit{satisfy}(e, \xi_1) \text{ and } \mathit{satisfy}(e, \xi_2) \tag{15d}$$

$$\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1) \text{ or } \mathit{satisfy}(e, \xi_2) \tag{15e}$$

$$\mathit{satisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\xi_1)) = \mathit{satisfy}(e_1, \xi_1) \tag{15f}$$

$$\mathit{satisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\xi_2)) = \mathit{satisfy}(e_2, \xi_2) \tag{15g}$$

$$\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \mathit{satisfy}(e_1, \xi_1) \text{ and } \mathit{satisfy}(e_2, \xi_2) \tag{15h}$$

$$\mathit{satisfy}(\llparenthesis\rrparenthesis^u, (\xi_1, \xi_2)) = \mathit{satisfy}(\texttt{prl}(\llparenthesis\rrparenthesis^u), \xi_1) \text{ and } \mathit{satisfy}(\texttt{prr}(\llparenthesis\rrparenthesis^u), \xi_2) \tag{15i}$$

$$\mathit{satisfy}(\llparenthesis e\rrparenthesis^u, (\xi_1, \xi_2)) = \mathit{satisfy}(\texttt{prl}(\llparenthesis e\rrparenthesis^u), \xi_1) \text{ and } \mathit{satisfy}(\texttt{prr}(\llparenthesis e\rrparenthesis^u), \xi_2) \tag{15j}$$

$$\begin{aligned}\mathit{satisfy}(e_1(e_2), (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\texttt{prl}(e_1(e_2)), \xi_1) \\ & \text{and } \mathit{satisfy}(\texttt{prr}(e_1(e_2)), \xi_2)\end{aligned} \tag{15k}$$

$$\begin{aligned}\mathit{satisfy}(\texttt{match}(e)\{\hat{rs}\}, (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\texttt{prl}(\texttt{match}(e)\{\hat{rs}\}), \xi_1) \\ & \text{and } \mathit{satisfy}(\texttt{prr}(\texttt{match}(e)\{\hat{rs}\}), \xi_2)\end{aligned} \tag{15l}$$

$$\begin{aligned}\mathit{satisfy}(\texttt{prl}(e), (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\texttt{prl}(\texttt{prl}(e)), \xi_1) \\ & \text{and } \mathit{satisfy}(\texttt{prr}(\texttt{prl}(e)), \xi_2)\end{aligned} \tag{15m}$$

$$\begin{aligned}\mathit{satisfy}(\texttt{prr}(e), (\xi_1, \xi_2)) = {} & \mathit{satisfy}(\texttt{prl}(\texttt{prr}(e)), \xi_1) \\ & \text{and } \mathit{satisfy}(\texttt{prr}(\texttt{prr}(e)), \xi_2)\end{aligned} \tag{15n}$$

$$\text{Otherwise} \quad \mathit{satisfy}(e, \xi) = \text{false} \tag{15o}$$

$\boxed{e \models_? \xi}$ e may satisfy ξ

CMSUnknown

$$\frac{}{e \models_? ?} \tag{16a}$$

CMSNotIntro

$$\frac{e\ \texttt{notintro} \qquad \xi\ \texttt{refutable}}{e \models_? \xi} \tag{16b}$$

CMSOrL

$$\frac{e \models_? \xi_1 \qquad e \not\models \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{16c}$$

CMSOrR

$$\frac{e \not\models \xi_1 \qquad e \models_? \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{16d}$$

CMSInl

$$\frac{e_1 \models_? \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)} \tag{16e}$$

$$\frac{e_2 \models_? \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)} \text{ CMSInr} \tag{16f}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \text{ CMSPairL} \tag{16g}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \text{ CMSPairR} \tag{16h}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \text{ CMSPair} \tag{16i}$$

$\boxed{e \models_?^\dagger \xi}$ e satisfies or may satisfy ξ

$$\frac{e \models_? \xi}{e \models_?^\dagger \xi} \text{ CSMSMay} \tag{17a}$$

$$\frac{e \models \xi}{e \models_?^\dagger \xi} \text{ CSMSSat} \tag{17b}$$

Lemma 2.0.1. $e \not\models \bot$

Proof. By rule induction over Rules (14), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. $\square$

Lemma 2.0.2. $e \not\models_? \bot$

Proof. Assume $e \models_? \bot$. By rule induction over Rules (16) on $e \models_? \bot$, only one case applies.

Case (16b).

(1) $\bot$ `refutable` by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. $\square$

Lemma 2.0.3. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (16) on $e \models_? \top$, only one case applies.

Case (16b).

(1) $\top$ `refutable` by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. □

Lemma 2.0.4. $e \not\models ?$

Proof. By rule induction over Rules (14), we notice that $e \models ?$ is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 2.0.5. $e \models_?^\dagger \xi$ *iff* $e \models_?^\dagger \xi \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

   (1) $e \models_?^\dagger \xi$ — by assumption

   By rule induction over Rules (17) on (1).

   Case (17a)**.**

   (2) $e \models_? \xi$ — by assumption
   (3) $e \models_? \xi \vee \bot$ — by Rule (16c) on (2) and Lemma 2.0.1
   (4) $e \models_?^\dagger \xi \vee \bot$ — by Rule (17a) on (3)

   Case (17b)**.**

   (2) $e \models \xi$ — by assumption
   (3) $e \models \xi \vee \bot$ — by Rule (14e) on (2)
   (4) $e \models_?^\dagger \xi \vee \bot$ — by Rule (17b) on (3)

2. Necessity:

   (1) $e \models_?^\dagger \xi \vee \bot$ — by assumption

   By rule induction over Rules (17) on (1).

   Case (17a)**.**

   (2) $e \models_? \xi \vee \bot$ — by assumption

   By rule induction over Rules (16) on (2), only two of them apply.

   Case (16c)**.**

   (3) $e \models_? \xi$ — by assumption
   (4) $e \models_?^\dagger \xi$ — by Rule (17a) on (3)

Case (16d).

(3) $e \models_? \bot$ by assumption

(4) $e \not\models_? \bot$ by Lemma 2.0.2

(3) contradicts (4).

Case (17b).

(2) $e \models \xi \vee \bot$ by assumption

By rule induction over Rules (14) on (2), only two of them apply.

Case (14e).

(3) $e \models \xi$ by assumption

(4) $e \models_?^\dagger \xi$ by Rule (17b) on (3)

Case (14f).

(3) $e \models \bot$ by assumption

(4) $e \not\models \bot$ by Lemma 2.0.1

(3) contradicts (4).

□

Corollary 2.0.1. $\top \models_?^\dagger \xi$ *iff* $\top \models_?^\dagger \xi \vee \bot$

Proof. By Definition 2.1.2 and Lemma 2.0.5. □

Lemma 2.0.6. *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$*. Then* $\xi_1 \not\models \xi_2$ *iff* $\xi_1 \not\models \xi_2 \vee \bot$

Proof.

(1) $\xi_1 : \tau$ by assumption

(2) $\xi_2 : \tau$ by assumption

(3) $\bot : \tau$ by Rule (8b)

(4) $\xi_2 \vee \bot : \tau$ by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5) $\xi_1 \not\models \xi_2$ by assumption

To prove $\xi_1 \not\models \xi_2 \vee \bot$, assume $\xi_1 \models \xi_2 \vee \bot$.

(6) $\xi_1 \models \xi_2 \vee \bot$ by assumption

For all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies

(7) $e \models \xi_2 \vee \bot$ by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

Case (14e)**.**

(8) $e \models \xi_2$ by assumption

(9) $\xi_1 \models \xi_2$ by Definition 2.1.1 on (8)

(5) contradicts (9).

Case (14f)**.**

(8) $e \models \bot$ by assumption

(9) $e \not\models \bot$ by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2 \vee \bot$

2. Necessity:

(5) $\xi_1 \not\models \xi_2 \vee \bot$ by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

(6) $\xi_1 \models \xi_2$ by assumption

For all e such that $\cdot\,;\Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \xi_1$ implies

(7) $e \models \xi_2$ by Definition 2.1.1 on (1) and (2) and (6)

(8) $e \models \xi_2 \vee \bot$ by Rule (14e) on (7)

(9) $\xi_1 \models \xi_2 \vee \bot$ by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2$

$\square$

Lemma 2.0.7. *If* $e \not\models^{\dagger}_{?} \xi_1$ *and* $e \not\models^{\dagger}_{?} \xi_2$ *then* $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

Proof. Assume, for the sake of contradiction, that $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$.

(1) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \xi_1$ by assumption

(3) $e \not\models^{\dagger}_{?} \xi_2$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

(4) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (14) on (4) and only two of them apply.

Case (14e).

(5) $e \models \xi_1$ by assumption

(6) $e \models_?^\dagger \xi_1$ by Rule (17b) on (5)

(6) contradicts (2).

Case (14f).

(5) $e \models \xi_2$ by assumption

(6) $e \models_?^\dagger \xi_2$ by Rule (17b) on (5)

(6) contradicts (3).

Case (17a).

(4) $e \models_? \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (4) and only two of them apply.

Case (16c).

(5) $e \models_? \xi_1$ by assumption

(6) $e \models_?^\dagger \xi_1$ by Rule (17a) on (5)

(6) contradicts (2).

Case (16d).

(5) $e \models_? \xi_2$ by assumption

(6) $e \models_?^\dagger \xi_2$ by Rule (17a) on (5)

(6) contradicts (3).

The conclusion holds as follows:

1. $e \not\models_?^\dagger \xi_1 \vee \xi_2$

□

Lemma 2.0.8. *If* $e \models_?^\dagger \xi_1 \vee \xi_2$ *and* $e \not\models_?^\dagger \xi_1$ *then* $e \models_?^\dagger \xi_2$

Proof.

(1) $e \models_?^\dagger \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models_?^\dagger \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

(3) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (14) on (3) and only two of them apply.

Case (14e).

(4) $e \models \xi_1$ by assumption

(5) $e \models^{\dagger}_{?} \xi_1$ by Rule (17b) on (4)

(5) contradicts (2).

Case (14f).

(4) $e \models \xi_2$ by assumption

(5) $e \models^{\dagger}_{?} \xi_2$ by Rule (17b) on (4)

Case (17a).

(3) $e \models_{?} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (3) and only two of them apply.

Case (16c).

(4) $e \models_{?} \xi_1$ by assumption

(5) $e \models^{\dagger}_{?} \xi_1$ by Rule (17a) on (4)

(5) contradicts (2).

Case (16d).

(4) $e \models_{?} \xi_2$ by assumption

(5) $e \models^{\dagger}_{?} \xi_2$ by Rule (17a) on (4)

□

Lemma 2.0.9. *If* $e \models^{\dagger}_{?} \xi_1$ *and* $e \models^{\dagger}_{?} \xi_2$ *then* $e \models^{\dagger}_{?} \xi_1 \wedge \xi_2$

Lemma 2.0.10. *If* $e \models^{\dagger}_{?} \xi_1$ *then* $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ *and* $e \models^{\dagger}_{?} \xi_2 \vee \xi_1$

Proof.

(1) $e \models^{\dagger}_{?} \xi_1$ by assumption ,

By rule induction over Rules (17) on (1),

Case (17b).

(2) $e \models \xi_1$ by assumption

(3) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (2)

(4) $e \models \xi_2 \vee \xi_1$ by Rule (14f) on (2)

(5) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (17b) on (3)

(6) $e \models^{\dagger}_{?} \xi_2 \vee \xi_1$ by Rule (17b) on (4)

Case (17a)**.**

(2) $e \models_{?} \xi_1$ by assumption

By case analysis on the result of $\mathit{satisfy}(e, \xi_2)$.

Case true**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by assumption

(4) $e \models \xi_2$ by Lemma 2.0.19 on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

(6) $e \models \xi_2 \vee \xi_1$ by Rule (14e) on (4)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (17b) on (5)

(8) $e \models^{\dagger}_{?} \xi_2 \vee \xi_1$ by Rule (17b) on (6)

Case false**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{false}$ by assumption

(4) $e \not\models \xi_2$ by Lemma 2.0.19 on (3)

(5) $e \models_{?} \xi_1 \vee \xi_2$ by Rule (16c) on (2) and (4)

(6) $e \models^{\dagger}_{?} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Rule (17a) on (5)

□

Lemma 2.0.11. *If* $e_1 \models^{\dagger}_{?} \xi_1$ *then* $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\xi_1)$

Proof.

(1) $e_1 \models^{\dagger}_{?} \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b)**.**

(2) $e_1 \models \xi_1$ by assumption

(3) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by Rule (14g) on (2)

(4) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\xi_1)$ by Rule (17b) on (3)

Case (17a).

(2) $e_1 \models_? \xi_1$	by assumption
(3) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$	by Rule (16e) on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$	by Rule (17a) on (3)

$\square$

Lemma 2.0.12. *If* $e_2 \models_?^\dagger \xi_2$ *then* $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$

Proof.

(1) $e_2 \models_?^\dagger \xi_2$	by assumption

By rule induction over Rules (17) on (1).

Case (17b).

(2) $e_2 \models \xi_2$	by assumption
(3) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$	by Rule (14h) on (2)
(4) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$	by Rule (17b) on (3)

Case (17a).

(2) $e_2 \models_? \xi_2$	by assumption
(3) $\texttt{inl}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$	by Rule (16f) on (2)
(4) $\texttt{inl}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$	by Rule (17a) on (3)

$\square$

Lemma 2.0.13. *If* $e_1 \models_?^\dagger \xi_1$ *and* $e_2 \models_?^\dagger \xi_2$ *then* $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$

Lemma 2.0.14 (Soundness and Completeness of Refutable Constraints)**.** ξ `refutable` *iff refutable*$(\xi) = $ *true.*

Lemma 2.0.15. *If* e `notintro` *and* ξ `refutable` *then either* $\dot{\top}(\xi)$ `refutable` *or* $e \models \dot{\top}(\xi)$.

Proof. By structural induction on ξ. $\square$

Lemma 2.0.16. *There does not exist such a constraint* $\xi_1 \wedge \xi_2$ *such that* $\xi_1 \wedge \xi_2$ `refutable`*.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \wedge \xi_2$ `refutable` is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 2.0.17. *There does not exist such a constraint* $\xi_1 \vee \xi_2$ *such that* $\xi_1 \vee \xi_2$ `refutable`*.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \vee \xi_2$ `refutable` is in syntactic contradiction with all the cases, hence not derivable. ☐

Lemma 2.0.18. *If* e `notintro` *and* $e \models \xi$ *then* $\cancel{\xi\ \texttt{refutable}}$.

Proof.

(1)	e `notintro`	by assumption
(2)	$e \models \xi$	by assumption

By rule induction over Rules (14) on (2).

Case (14a)**.**

(3)	$\xi = \top$	by assumption

Assume $\top$ `refutable`. By rule induction over Rules (10), no case applies due to syntactic contradiction.
Therefore, $\cancel{\top\ \texttt{refutable}}$.

Case (14e),(14f)**.**

(3)	$\xi = \xi_1 \vee \xi_2$	by assumption
(4)	$\cancel{\xi_1 \vee \xi_2\ \texttt{refutable}}$	by Lemma 2.0.17

Case (14d)**.**

(3)	$\xi = \xi_1 \wedge \xi_2$	by assumption
(4)	$\cancel{\xi_1 \wedge \xi_2\ \texttt{refutable}}$	by Lemma 2.0.16

Case (14j)**.**

(3)	$\xi = (\xi_1, \xi_2)$	by assumption
(4)	$\texttt{prl}(e) \models \xi_1$	by assumption
(5)	$\texttt{prr}(e) \models \xi_2$	by assumption
(6)	$\texttt{prl}(e)$ `notintro`	by Rule (26e)
(7)	$\texttt{prr}(e)$ `notintro`	by Rule (26f)
(8)	$\cancel{\xi_1\ \texttt{refutable}}$	by IH on (6) and (4)
(9)	$\cancel{\xi_2\ \texttt{refutable}}$	by IH on (7) and (5)

Assume (ξ_1, ξ_2) `refutable`. By rule induction over Rules (10) on it, only two cases apply.

Case (10d)**.**

(10)	ξ_1 `refutable`	by assumption

Contradicts (8).

Case (10e)**.**

(10) ξ_2 `refutable` by assumption

Contradicts (9).

Therefore, $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}}$.

Otherwise.

(3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

$\square$

Lemma 2.0.19 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \models \xi$ iff satisfy$(e, \xi) =$ true.*

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models \xi$ by assumption

   By rule induction over Rules (14) on (1).

   Case (14a)**.**

   (2) $\xi = \top$ by assumption
   (3) $\mathit{satisfy}(e, \top) = \text{true}$ by Definition 15a

   Case (14b)**.**

   (2) $e = \underline{n}$ by assumption
   (3) $\xi = \underline{n}$ by assumption
   (4) $\mathit{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$ by Definition 15b

   Case (14c)**.**

   (2) $e = \underline{n_1}$ by assumption
   (3) $\xi = \cancel{\underline{n_2}}$ by assumption
   (4) $n_1 \neq n_2$ by assumption
   (5) $\mathit{satisfy}(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) = \text{true}$ by Definition 15c on (4)

   Case (14d)**.**

   (2) $\xi = \xi_1 \wedge \xi_2$ by assumption
   (3) $e \models \xi_1$ by assumption

(4) $e \models \xi_2$ by assumption

(5) $\textit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)

(6) $\textit{satisfy}(e, \xi_2) = \text{true}$ by IH on (4)

(7) $\textit{satisfy}(e, \xi_1 \wedge \xi_2) = \textit{satisfy}(e, \xi_1)$ and $\textit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15d on (5) and (6)

Case (14e).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_1$ by assumption

(4) $\textit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)

(5) $\textit{satisfy}(e, \xi_1 \vee \xi_2) = \textit{satisfy}(e, \xi_1)$ or $\textit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (4)

Case (14f).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_2$ by assumption

(4) $\textit{satisfy}(e, \xi_2) = \text{true}$ by IH on (3)

(5) $\textit{satisfy}(e, \xi_1 \vee \xi_2) = \textit{satisfy}(e, \xi_1)$ or $\textit{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (4)

Case (14g).

(2) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\xi = \mathtt{inl}(\xi_1)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $\textit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(6) $\textit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\xi_1)) = \textit{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 15f on (5)

Case (14h).

(2) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\xi = \mathtt{inl}(\xi_2)$ by assumption

(4) $e_2 \models \xi_2$ by assumption

(5) $\textit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (4)

(6) $\textit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\xi_2)) = \textit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 15g on (5)

Case (14i).

(2) $e = (e_1, e_2)$ by assumption

(3) $\xi = (\xi_1, \xi_2)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $e_2 \models \xi_2$ by assumption

(6) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (5)

(8) $\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$
$\mathit{satisfy}(e_1, \xi_1)$ and $\mathit{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 15h on (6) and (7)

Case (14j)**.**

(2) $\xi = (\xi_1, \xi_2)$ by assumption

(3) e `notintro` by assumption

(4) $\texttt{prl}(e) \models \xi_1$ by assumption

(5) $\texttt{prr}(e) \models \xi_2$ by assumption

(6) $\mathit{satisfy}(\texttt{prl}(e), \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$ by IH on (5)

By rule induction over Rules (26) on (3).

Otherwise.

(8) $e = ⦇⦈^u, ⦇e_0⦈^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(9) $\mathit{satisfy}(e, (\xi_1, \xi_2)) =$
$\mathit{satisfy}(\texttt{prl}(e), \xi_1)$ and $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$
by Definition 15 on (6) and (7)

2. Completeness:

(1) $\mathit{satisfy}(e, \xi) = \text{true}$ by assumption

By structural induction on ξ.

Case $\xi = \top$**.**

(2) $e \models \top$ by Rule (14a)

Case $\xi = \bot, ?$**.**

(2) $\mathit{satisfy}(e, \xi) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$**.**

By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' = n$ — by Definition 15b on (1)

(3) $\underline{n'} \models \underline{n}$ — by Rule (14b) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \underline{n}) = \text{false}$ — by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \cancel{\underline{n}}$**.**

By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' \neq n$ — by Definition 15c on (1)

(3) $\underline{n'} \models \cancel{\underline{n}}$ — by Rule (14c) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \cancel{\underline{n}}) = \text{false}$ — by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $\mathit{satisfy}(e, \xi_1) = \text{true}$ — by Definition 15d on (1)

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ — by Definition 15d on (1)

(4) $e \models \xi_1$ — by IH on (2)

(5) $e \models \xi_2$ — by IH on (3)

(6) $e \models \xi_1 \wedge \xi_2$ — by Rule (14d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $\mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$ — by Definition 15e on (1)

By case analysis on (2).

Case $\mathit{satisfy}(e, \xi_1) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_1) = \text{true}$ — by assumption

(4) $e \models \xi_1$ — by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ — by Rule (14e) on (4)

Case $\mathit{satisfy}(e, \xi_2) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ — by assumption

(4) $e \models \xi_2$ — by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ — by Rule (14f) on (4)

Case $\xi = \mathtt{inl}(\xi_1)$.

By structural induction on e.

Case $e = \mathtt{inl}_{\tau_2}(e_1)$.

(2)	$\mathit{satisfy}(e_1, \xi_1) = \text{true}$	by Definition 15f on (1)
(3)	$e_1 \models \xi_1$	by IH on (2)
(4)	$\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$	by Rule (14g) on (3)

Otherwise.

(2)	$\mathit{satisfy}(e, \mathtt{inl}(\xi_1)) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \mathtt{inr}(\xi_2)$.

By structural induction on e.

Case $e = \mathtt{inr}_{\tau_1}(e_2)$.

(2)	$\mathit{satisfy}(e_2, \xi_2) = \text{true}$	by Definition 15g on (1)
(3)	$e_2 \models \xi_2$	by IH on (2)
(4)	$\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$	by Rule (14h) on (3)

Otherwise.

(2)	$\mathit{satisfy}(e, \mathtt{inr}(\xi_2)) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$.

By structural induction on e.

Case $e = (e_1, e_2)$.

(2)	$\mathit{satisfy}(e_1, \xi_1) = \text{true}$	by Definition 15h on (1)
(3)	$\mathit{satisfy}(e_2, \xi_2) = \text{true}$	by Definition 15h on (1)
(4)	$e_1 \models \xi_1$	by IH on (2)
(5)	$e_2 \models \xi_2$	by IH on (3)
(6)	$(e_1, e_2) \models (\xi_1, \xi_2)$	by Rule (14i) on (4) and (5)

Case $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$.

(2)	$\mathit{satisfy}(\mathtt{prl}(e), \xi_1) = \text{true}$	by Definition 15h on (1)
(3)	$\mathit{satisfy}(\mathtt{prr}(e), \xi_2) = \text{true}$	by Definition 15h on (1)
(4)	$\mathtt{prl}(e) \models \xi_1$	by IH on (2)
(5)	$\mathtt{prr}(e) \models \xi_2$	by IH on (3)
(6)	$e\ \mathtt{notintro}$	by each rule in Rules (26)

(7) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14j) on (6) and (4) and (5)

Otherwise.

(2) $\mathit{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

□

Lemma 2.0.20. *$e \not\models \xi$ and $e \not\models_? \xi$ iff $e \not\models_?^\dagger \xi$.*

Proof. 1. Sufficiency:

(1) $e \not\models \xi$ by assumption

(2) $e \not\models_? \xi$ by assumption

Assume $e \models_?^\dagger \xi$. By rule induction over Rules (17) on it.

Case (17a).

(3) $e \models \xi$ by assumption

Contradicts (1).

Case (17b).

(3) $e \models_? \xi$ by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \xi$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \xi$ by assumption

Assume $e \models \xi$.

(2) $e \models_?^\dagger \xi$ by Rule (17b) on assumption

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_? \xi$.

(3) $e \models_?^\dagger \xi$ by Rule (17a) on assumption

Contradicts (1). Therefore, $e \not\models_? \xi$.

□

Theorem 2.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \models \xi$

2. $e \models_? \xi$

3. $e \not\models^\dagger_? \xi$

Proof.

(4) $\xi : \tau$ — by assumption

(5) $\cdot\,;\Delta \vdash e : \tau$ — by assumption

(6) $e\ \texttt{final}$ — by assumption

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

Case (8a)**.**

(7) $\xi = \top$ — by assumption

(8) $e \models \top$ — by Rule (14a)

(9) $e \not\models_? \top$ — by Lemma 2.0.3

(10) $e \models^\dagger_? \top$ — by Rule (17b) on (8)

Case (8b)**.**

(7) $\xi = \bot$ — by assumption

(8) $e \not\models \bot$ — by Lemma 2.0.1

(9) $e \not\models_? \bot$ — by Lemma 2.0.2

(10) $e \not\models^\dagger_? \bot$ — by Lemma 2.0.20 on (8) and (9)

Case (1b)**.**

(7) $\xi = ?$ — by assumption

(8) $e \not\models\ ?$ — by Lemma 2.0.4

(9) $e \models_?\ ?$ — by Rule (16a)

(10) $e \models^\dagger_?\ ?$ — by Rule (17a) on (9)

Case (8c)**.**

(7) $\xi = \underline{n_2}$ — by assumption

(8) $\tau = \texttt{num}$ — by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on ξ.

(11) $e \not\models \underline{n_2}$ by contradiction

(12) $\underline{n_2}$ $\texttt{refutable}$ by Rule (10a)

(13) $e \models_? \underline{n_2}$ by Rule (16b) on (10) and (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (17a) on (13)

Case (19d).

(9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (16), only one case applies.

Case (16b).

(10) $\underline{n_1}$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.4.

(11) $\underline{n_1} \not\models_? \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 15

(13) $\underline{n_1} \models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (17b) on (13)

Case $n_1 \neq n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 15

(13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \not\models_?^\dagger \underline{n_2}$ by Lemma 2.0.20 on (11) and (13)

Case (8f).

(7) $\xi = \xi_1 \vee \xi_2$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models_? \xi_1$, and $e \not\models_?^\dagger \xi_1$ holds. The same goes for ξ_2. By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e \models \xi_1, e \models \xi_2$**.**

(8)	$e \models \xi_1$	by assumption
(9)	$e \not\models_? \xi_1$	by assumption
(10)	$e \models \xi_2$	by assumption
(11)	$e \not\models_? \xi_2$	by assumption
(12)	$e \models \xi_1 \vee \xi_2$	by Rule (14e) on (8)
(13)	$e \models_?^\dagger \xi_1 \vee \xi_2$	by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c)**.**

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (16d)**.**

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \models_? \xi_2$**.**

(8)	$e \models \xi_1$	by assumption
(9)	$e \not\models_? \xi_1$	by assumption
(10)	$e \not\models \xi_2$	by assumption
(11)	$e \models_? \xi_2$	by assumption
(12)	$e \models \xi_1 \vee \xi_2$	by Rule (14e) on (8)
(13)	$e \models_?^\dagger \xi_1 \vee \xi_2$	by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c)**.**

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \not\models^\dagger_? \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (8)

(13) $e \models^\dagger_? \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

(13) $e \models^\dagger_? \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (16d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models_? \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10).

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{?} \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_{?} \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_{?} \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (16d).

(14) $e \models_{?} \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_{?} \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \xi_1, e \models_{?} \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{?} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_{?} \xi_2$ by assumption

(12) $e \models_{?} \xi_1 \vee \xi_2$ by Rule (16d) on (11) and (8)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14) on it, only two cases apply.

Case (14e).

(12) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(12) $e \models \xi_2$ by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ `refutable` by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

(16) $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$ — by Lemma 2.0.20 on (13) and (15)

Case (8g).

(7) $\xi = \texttt{inl}(\xi_1)$ — by assumption

(8) $\tau = (\tau_1 + \tau_2)$ — by assumption

(9) $\xi_1 : \tau_1$ — by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ — by assumption

(11) e $\texttt{notintro}$ — by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inl}(\xi_1)$ — by contradiction

By case analysis on the value of *refutable*$(\texttt{inl}(\xi_1))$.

Case *refutable*$(\texttt{inl}(\xi_1)) = \text{true}$**.**

(13) *refutable*$(\texttt{inl}(\xi_1)) = \text{true}$ — by assumption

(14) $\texttt{inl}(\xi_1)$ $\texttt{refutable}$ — by Lemma 2.0.14 on (13)

(15) $e \models_{?} \texttt{inl}(\xi_1)$ — by Rule (16b) on (11) and (14)

(16) $e \models^{\dagger}_{?} \texttt{inl}(\xi_1)$ — by Rule (17a) on (15)

Case *refutable*$(\texttt{inl}(\xi_1)) = \text{false}$**.**

(13) *refutable*$(\texttt{inl}(\xi_1)) = \text{false}$ — by assumption

(14) $\cancel{\texttt{inl}(\xi_1)\ \texttt{refutable}}$ — by Lemma 2.0.14 on (13)

Assume $e \models_{?} \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(15) $\texttt{inl}(\xi_1)$ $\texttt{refutable}$ — by assumption

Contradicts (14).

(16) $e \not\models_{?} \texttt{inl}(\xi_1)$ — by contradiction

(17) $e \not\models^{\dagger}_{?} \texttt{inl}(\xi_1)$ — by Lemma 2.0.20 on (12) and (16)

Case (19j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(11) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(12) e_1 $\texttt{final}$ by Lemma 4.0.1 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \not\models_?^\dagger \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$.

(13) $e_1 \models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ by Rule (14g) on (13)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (17b) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \models_? \xi_1$.

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ by Rule (16e) on (14)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (17a) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

(17) $e_1 \models \xi_1$

Contradicts (13).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \not\models_?^\dagger \xi_1$.

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

Assume $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

(15) $e_1 \models \xi_1$

Contradicts (13).

(16) $\mathtt{inl}_{\tau_2}(e_1) \not\models \mathtt{inl}(\xi_1)$ by contradiction

Assume $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(17) $\mathtt{inl}_{\tau_2}(e_1)$ $\mathtt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inl}(\xi_1)$ by contradiction

(19) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (19k).

(10) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inl}(\xi_1)$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inl}(\xi_1)$ by contradiction

(14) $\mathtt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \mathtt{inl}(\xi_1)$ by Lemma 2.0.20 on (11) and (13)

Case (8h).

(7) $\xi = \mathtt{inr}(\xi_2)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(10) $e = \llparenthesis\rrparenthesis^u, \llparenthesis e_0 \rrparenthesis^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inr}(\xi_2)$ by contradiction

By case analysis on the value of *refutable*$(\texttt{inr}(\xi_2))$. inr is refutable

Case *refutable*$(\texttt{inr}(\xi_2)) =$ true.

(13) *refutable*$(\texttt{inr}(\xi_2)) =$ true by assumption

(14) $\texttt{inr}(\xi_2)$ $\texttt{refutable}$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inr}(\xi_2)$ by Rule (16b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inr}(\xi_2)$ by Rule (17a) on (15)

Case *refutable*$(\texttt{inr}(\xi_2)) =$ false.

(13) *refutable*$(\texttt{inr}(\xi_2)) =$ false by assumption

(14) $\cancel{\texttt{inr}(\xi_2)\ \texttt{refutable}}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(15) $\texttt{inr}(\xi_2)$ $\texttt{refutable}$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inr}(\xi_2)$ by Lemma 2.0.20 on (12) and (16)

Case (19j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(14) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inr}(\xi_2)$ by Lemma 2.0.20 on (11) and (13)

Case (19k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(12) $e_2\ \texttt{final}$ by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \not\models_?^\dagger \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

(13) $e_2 \models \xi_2$ by assumption

(14) $e_2 \not\models_? \xi_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ by Rule (14g) on (13)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$ by Rule (17b) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_? \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inr}(\xi_2)$ by contradiction

Case $e_2 \models_? \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \models_? \xi_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$ by Rule (16f) on (14)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$ by Rule (17a) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(17) $e_2 \models \xi_2$

Contradicts (13).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Case $e_2 \not\models_?^\dagger \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \not\models_? \xi_2$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(15) $e_2 \models \xi_2$

Contradicts (13).

(16) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(17) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_? \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (14i).

(7) $\xi = (\xi_1, \xi_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

(10) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(11) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(13) e $\texttt{indet}$ by Lemma 4.0.8 on (6) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (24g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (25b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (24h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (25b) on (16)

(18) $\cdot\,; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (19h) on (5)

(19) $\cdot\,; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\texttt{prl}(e) \models \xi_1$, $\texttt{prl}(e) \models_? \xi_1$, and $\texttt{prl}(e) \not\models_?^\dagger \xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\texttt{prr}(e) \models \xi_2$, $\texttt{prr}(e) \models_? \xi_2$, and $\texttt{prr}(e) \not\models_?^\dagger \xi_2$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

(24) $e \models (\xi_1, \xi_2)$ by Rule (14j) on (12) and (20) and (22)

(25) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17b) on (24)

(26) $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}}$ by Lemma 2.0.18 on (12) and (24)

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(27) (ξ_1, ξ_2) $\texttt{refutable}$ by assumption

Contradicts (26).

(28) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models_? \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^{\dagger} (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \not\models_?^{\dagger} \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 `refutable` by assumption

(28) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 `refutable` by assumption

(28) $\texttt{prr}(e)$ `notintro` by Rule (26f)

(29) $\texttt{prr}(e) \vDash_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \nvDash_? (\xi_1, \xi_2)$ by contradiction

(31) $e \nvDash_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\texttt{prl}(e) \vDash_? \xi_1, \texttt{prr}(e) \vDash \xi_2$.

(20) $\texttt{prl}(e) \nvDash \xi_1$ by assumption

(21) $\texttt{prl}(e) \vDash_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \vDash \xi_2$ by assumption

(23) $\texttt{prr}(e) \nvDash_? \xi_2$ by assumption

Assume $e \vDash (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\texttt{prl}(e) \vDash \xi_1$ by assumption

Contradicts (20).

(25) $e \nvDash (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_1 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \vDash_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \vDash_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \vDash_? \xi_1, \texttt{prr}(e) \vDash_? \xi_2$.

(20) $\texttt{prl}(e) \nvDash \xi_1$ by assumption

(21) $\texttt{prl}(e) \vDash_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \nvDash \xi_2$ by assumption

(23) $\texttt{prr}(e) \vDash_? \xi_2$ by assumption

Assume $e \vDash (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\texttt{prl}(e) \vDash \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \models_? \xi_1, \mathtt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\mathtt{prl}(e) \not\models \xi_1$ by assumption

(21) $\mathtt{prl}(e) \models_? \xi_1$ by assumption

(22) $\mathtt{prr}(e) \not\models \xi_2$ by assumption

(23) $\mathtt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\mathtt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_1 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \not\models_?^\dagger \xi_1, \mathtt{prr}(e) \models \xi_2$.

(20) $\mathtt{prl}(e) \not\models \xi_1$ by assumption

(21) $\mathtt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\mathtt{prr}(e) \models \xi_2$ by assumption

(23) $\mathtt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\mathtt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 `refutable` by assumption

(28) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 `refutable` by assumption

(28) $\texttt{prr}(e)$ `notintro` by Rule (26f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\texttt{prl}(e) \not\models_?^\dagger \xi_1, \texttt{prr}(e) \models_? \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

assume no "or" and "and" in pair

Case (16b).

(26) ξ_2 `refutable` by assumption

(27) (ξ_1, ξ_2) `refutable` by Rule (10e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$**.**

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j)**.**

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

(26) (ξ_1, ξ_2) $\texttt{refutable}$ by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d)**.**

(27) ξ_1 $\texttt{refutable}$ by assumption

(28) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (26e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e)**.**

(27) ξ_2 $\texttt{refutable}$ by assumption

(28) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (26f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case (19g)**.**

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\, ; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(14) e_1 `final` by Lemma 4.0.3 on (6)

(15) e_2 `final` by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \models \overline{\xi_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \models \overline{\xi_2}$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e_1 \models \xi_1, e_2 \models \xi_2$**.**

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (16) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17b) on (20)

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h)**.**

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models_? \xi_2$**.**

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (16h) on (16) and (19)

(21) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models^{\dagger}_{?} \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_{?} \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_{?} \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_{?} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_{?} \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_{?} \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_{?} \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \models_? \xi_1, e_2 \models \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (16g) on (17) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i)**.**

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \models_? \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (16i) on (17) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i)**.**

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h)**.**

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

Case (16i)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \not\models_?^\dagger \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (14i)**.**

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

$\square$

Definition 2.1.1 (Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e* `val` *we have $e \models_?^\dagger \xi_1$ implies $e \models \xi_2$*

Definition 2.1.2 (Potential Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models_?^\dagger \xi_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e* `final` *we have $e \models_?^\dagger \xi_1$ implies $e \models_?^\dagger \xi_2$*

Corollary 2.1.1. *Suppose that $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e* `final`*. Then $\top \models_?^\dagger \xi$ implies $e \models_?^\dagger \xi$*

Proof.

(1) $\xi : \tau$ by assumption

(2) $\cdot\,; \Gamma \vdash e : \tau$ by assumption

(3) e `final` by assumption

(4) $\top \models_?^\dagger \xi$ by assumption

(5) $e_1 \models \top$ by Rule (14a)

(6) $e_1 \models_?^\dagger \top$ by Rule (17b) on (5)

(7) $\top : \tau$ by Rule (8a)

(8) $e_1 \models_?^\dagger \xi_r$ by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

$\square$

3 Static Semantics

$$
\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & (\lambda x : \tau . e) \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
 & \mid & \llparenthesis\rrparenthesis^u \mid \llparenthesis e \rrparenthesis^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid \llparenthesis\rrparenthesis^w \mid \llparenthesis p \rrparenthesis^w
\end{array}
$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{18a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{18b}$$

$\boxed{\Gamma \,;\, \Delta \vdash e : \tau}$ e is of type τ

$$\text{TVar} \quad \frac{}{\Gamma, x : \tau \,;\, \Delta \vdash x : \tau} \tag{19a}$$

$$\text{TEHole} \quad \frac{}{\Gamma \,;\, \Delta, u :: \tau \vdash \llparenthesis\rrparenthesis^u : \tau} \tag{19b}$$

$$\text{THole} \quad \frac{\Gamma \,;\, \Delta, u :: \tau \vdash e : \tau'}{\Gamma \,;\, \Delta, u :: \tau \vdash \llparenthesis e \rrparenthesis^u : \tau} \tag{19c}$$

$$\text{TNum} \quad \frac{}{\Gamma \,;\, \Delta \vdash \underline{n} : \texttt{num}} \tag{19d}$$

$$\text{TLam} \quad \frac{\Gamma, x : \tau_1 \,;\, \Delta \vdash e : \tau_2}{\Gamma \,;\, \Delta \vdash (\lambda x : \tau_1 . e) : (\tau_1 \to \tau_2)} \tag{19e}$$

$$\text{TAp} \quad \frac{\Gamma \,;\, \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma \,;\, \Delta \vdash e_2 : \tau_2}{\Gamma \,;\, \Delta \vdash e_1(e_2) : \tau} \tag{19f}$$

$$\text{TPair} \quad \frac{\Gamma \,;\, \Delta \vdash e_1 : \tau_1 \qquad \Gamma \,;\, \Delta \vdash e_2 : \tau_2}{\Gamma \,;\, \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \tag{19g}$$

$$\text{TPrl} \quad \frac{\Gamma \,;\, \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma \,;\, \Delta \vdash \texttt{prl}(e) : \tau_1} \tag{19h}$$

$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \texttt{prr}(e) : \tau_2}\ \text{TPrr} \tag{19i}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau_1}{\Gamma\,;\Delta \vdash \texttt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)}\ \text{TInl} \tag{19j}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash \texttt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)}\ \text{TInr} \tag{19k}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad \Gamma\,;\Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \models^{\dagger}_{?} \xi}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{\cdot \mid r \mid rs\} : \tau'}\ \text{TMatchZPre} \tag{19l}$$

$$\frac{\begin{array}{c}\Gamma\,;\Delta \vdash e : \tau \qquad e\ \texttt{final} \qquad \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \\ \Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \qquad e \not\models^{\dagger}_{?} \xi_{pre} \qquad \top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}\end{array}}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'}\ \text{TMatchNZPre} \tag{19m}$$

$\boxed{p : \tau[\xi] \dashv\vdash \Gamma\,;\Delta}$ p is assigned type τ and emits constraint ξ

$$\frac{}{x : \tau[\top] \dashv\vdash \cdot\,; x : \tau}\ \text{PTVar} \tag{20a}$$

$$\frac{}{\_ : \tau[\top] \dashv\vdash \cdot\,;\cdot}\ \text{PTWild} \tag{20b}$$

$$\frac{}{(\!|\,|\!)^w : \tau[?] \dashv\vdash \cdot\,; w :: \tau}\ \text{PTEHole} \tag{20c}$$

$$\frac{p : \tau[\xi] \dashv\vdash \Gamma\,;\Delta}{(\!|p|\!)^w : \tau'[?] \dashv\vdash \Gamma\,;\Delta, w :: \tau'}\ \text{PTHole} \tag{20d}$$

$$\frac{}{\underline{n} : \texttt{num}[\underline{n}] \dashv\vdash \cdot\,;\cdot}\ \text{PTNum} \tag{20e}$$

$$\frac{p : \tau_1[\xi] \dashv\vdash \Gamma\,;\Delta}{\texttt{inl}(p) : (\tau_1 + \tau_2)[\texttt{inl}(\xi)] \dashv\vdash \Gamma\,;\Delta}\ \text{PTInl} \tag{20f}$$

$$\frac{p : \tau_2[\xi] \dashv\vdash \Gamma\,;\Delta}{\texttt{inr}(p) : (\tau_1 + \tau_2)[\texttt{inr}(\xi)] \dashv\vdash \Gamma\,;\Delta}\ \text{PTInr} \tag{20g}$$

$$\frac{p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1\,;\Delta_1 \qquad p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2\,;\Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\vdash \Gamma_1 \uplus \Gamma_2\,;\Delta_1 \uplus \Delta_2}\ \text{PTPair} \tag{20h}$$

$\boxed{\Gamma\,;\Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

$$\frac{p : \tau[\xi] \dashv\!\vdash \Gamma_p\,;\Delta_p \qquad \Gamma \uplus \Gamma_p\,;\Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma\,;\Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'}\ \text{CTRule} \tag{21a}$$

$\boxed{\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

$$\frac{\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,;\Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'}\ \text{CTOneRules} \tag{22a}$$

$$\frac{\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,;\Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'}\ \text{CTRules} \tag{22b}$$

Lemma 3.0.1. *If* $p : \tau[\xi] \dashv\!\vdash \Gamma\,;\Delta$ *then* $\xi : \tau$.

Proof. By rule induction over Rules (20). ☐

Lemma 3.0.2. *If* $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Proof. By rule induction over Rules (21). ☐

Lemma 3.0.3. *If* $\cdot\,;\Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Proof. By rule induction over Rules (22). ☐

Lemma 3.0.4. *If* $\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma\,;\Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Proof.

(1) $\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ by assumption

(2) $\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ by assumption

(3) $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ by assumption

By rule induction over Rules (22) on (1).

Case (22a).

(4) $rs = r' \mid \cdot$ by assumption

(5) $\xi_{rs} = \xi'_r$ by assumption

(6) $\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$ by assumption

(7) $\xi'_r \not\models \xi_{pre}$ by assumption

(8) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$ by Rule (22a) on (2) and (3)

(9) $\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$ by Rule (22b) on (6) and (8) and (7)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$
by Definition 18 on (9)

Case (22b)**.**

(4) $rs = r' \mid rs'$ — by assumption

(5) $\xi_{rs} = \xi'_r \vee \xi'_{rs}$ — by assumption

(6) $\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$ — by assumption

(7) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] \Rightarrow \tau'$ — by assumption

(8) $\xi'_r \not\models \xi_{pre}$ — by assumption

(9) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^{\diamond} : \tau[\xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by IH on (7) and (2) and (3)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^{\diamond}) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by Rule (22b) on (6) and (9) and (8)

(11) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by Definition 18 on (10)

$\square$

Lemma 3.0.5 (Substitution)**.** *If* $\Gamma, x : \tau\,;\Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta \vdash e : \tau$ *then* $\Gamma\,;\Delta \vdash [e/x]e_0 : \tau_0$

Lemma 3.0.6 (Simultaneous Substitution)**.** *If* $\Gamma \uplus \Gamma'\,;\Delta \vdash e : \tau$ *and* $\theta : \Gamma'$ *then* $\Gamma\,;\Delta \vdash [\theta]e : \tau$

Lemma 3.0.7 (Substitution Typing)**.** *If* $e \triangleright p \dashv\!\vert\, \theta$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\vert\, \Gamma\,;\Delta$ *then* $\theta : \Gamma$

Proof by induction on the derivation of $e \triangleright p \dashv\!\vert\, \theta$.

Theorem 3.1 (Determinism)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

4 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\frac{}{\underline{n}\ \texttt{val}}\ \text{VNum} \tag{23a}$$

$$\frac{}{(\lambda x : \tau . e)\ \texttt{val}}\ \text{VLam} \tag{23b}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}}\ \text{VPair} \tag{23c}$$

$$\frac{e\ \texttt{val}}{\texttt{inl}_\tau(e)\ \texttt{val}}\ \text{VInl} \tag{23d}$$

$$\frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}}\ \text{VInr} \tag{23e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{}{(\!|\!)^u\ \texttt{indet}}\ \text{IEHole} \tag{24a}$$

$$\frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}}\ \text{IHole} \tag{24b}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}}\ \text{IAp} \tag{24c}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairL} \tag{24d}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairR} \tag{24e}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPair} \tag{24f}$$

$$\frac{e\ \texttt{indet}}{\texttt{prl}(e)\ \texttt{indet}}\ \text{IPrl} \tag{24g}$$

$$\frac{e\ \texttt{indet}}{\texttt{prr}(e)\ \texttt{indet}}\ \text{IPrr} \tag{24h}$$

$$\frac{e\ \texttt{indet}}{\texttt{inl}_\tau(e)\ \texttt{indet}}\ \text{IInL} \tag{24i}$$

$$\frac{e\ \texttt{indet}}{\texttt{inr}_\tau(e)\ \texttt{indet}}\ \text{IInR} \tag{24j}$$

$$\frac{e\ \texttt{final} \qquad e\ ?\ p_r}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \texttt{indet}}\ \text{IMatch} \tag{24k}$$

$\boxed{e\ \texttt{final}}$ e is final

$$\frac{e\ \texttt{val}}{e\ \texttt{final}}\ \text{FVal} \tag{25a}$$

$$\frac{e\ \texttt{indet}}{e\ \texttt{final}}\ \text{FIndet} \tag{25b}$$

$\boxed{e\ \texttt{notintro}}$ e cannot be a value syntactically

$$\frac{}{(\!|\!)^u\ \texttt{notintro}}\ \text{NVEHole} \tag{26a}$$

$$\frac{}{(\!|e|\!)^u\ \texttt{notintro}}\ \text{NVHole} \tag{26b}$$

$$\frac{}{e_1(e_2)\ \texttt{notintro}}\ \text{NVAp} \tag{26c}$$

$$\frac{}{\texttt{match}(e)\{\hat{rs}\}\ \texttt{notintro}}\ \text{NVMatch} \tag{26d}$$

$$\frac{}{\texttt{prl}(e)\ \texttt{notintro}}\ \text{NVPrl} \tag{26e}$$

$$\frac{}{\texttt{prr}(e)\ \texttt{notintro}}\ \text{NVPrr} \tag{26f}$$

$\boxed{\theta : \Gamma}$ θ is of type Γ

$$\frac{}{\emptyset : \cdot}\ \text{STEmpty} \tag{27a}$$

$$\frac{\theta : \Gamma_\theta \qquad \Gamma ; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau} \text{ STExtend} \tag{27b}$$

$\boxed{p\ \texttt{refutable}}$ p is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}} \text{ RNum} \tag{28a}$$

$$\frac{}{(\!|\!)^w\ \texttt{refutable}} \text{ REHole} \tag{28b}$$

$$\frac{}{(\!|p|\!)^w\ \texttt{refutable}} \text{ RHole} \tag{28c}$$

$$\frac{}{\texttt{inl}(p)\ \texttt{refutable}} \text{ RInl} \tag{28d}$$

$$\frac{}{\texttt{inr}(p)\ \texttt{refutable}} \text{ RInr} \tag{28e}$$

$$\frac{p_1\ \texttt{refutable}}{(p_1, p_2)\ \texttt{refutable}} \text{ RPairL} \tag{28f}$$

$$\frac{p_2\ \texttt{refutable}}{(p_1, p_2)\ \texttt{refutable}} \text{ RPairR} \tag{28g}$$

$\boxed{e \rhd p \dashv\!\vdash \theta}$ e matches p, emitting θ

$$\frac{}{e \rhd x \dashv\!\vdash e/x} \text{ MVar} \tag{29a}$$

$$\frac{}{e \rhd \_ \dashv\!\vdash \cdot} \text{ MWild} \tag{29b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashv\!\vdash \cdot} \text{ MNum} \tag{29c}$$

$$\frac{e_1 \rhd p_1 \dashv\!\vdash \theta_1 \qquad e_2 \rhd p_2 \dashv\!\vdash \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2} \text{ MPair} \tag{29d}$$

$$\frac{e \rhd p \dashv\!\vdash \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\vdash \theta} \text{ MInl} \tag{29e}$$

$$\frac{e \rhd p \dashv\!\vdash \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\vdash \theta} \text{ MInr} \tag{29f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{prl}(e) \rhd p_1 \dashv\!\mid \theta_1 \qquad \texttt{prr}(e) \rhd p_2 \dashv\!\mid \theta_2}{e \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{29g}$$

$\boxed{e \mathrel{?} p}$ e may match p

$$\frac{}{e \mathrel{?} (\!|\!)^w}\ \text{MMEHole} \tag{30a}$$

$$\frac{}{e \mathrel{?} (\!|p|\!)^w}\ \text{MMHole} \tag{30b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}}{e \mathrel{?} p}\ \text{MMNotIntro} \tag{30c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\!\mid \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{30d}$$

$$\frac{e_1 \rhd p_1 \dashv\!\mid \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{30e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{30f}$$

$$\frac{e \mathrel{?} p}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)}\ \text{MMInl} \tag{30g}$$

$$\frac{e \mathrel{?} p}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)}\ \text{MMInr} \tag{30h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{31a}$$

$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairL} \tag{31b}$$

$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairR} \tag{31c}$$

$$\frac{}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)}\ \text{NMConfL} \tag{31d}$$

NMConfR
$$\frac{}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{31e}$$

NMInl
$$\frac{e \perp p}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)} \tag{31f}$$

NMInr
$$\frac{e \perp p}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{31g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

ITHole
$$\frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{32a}$$

ITApFun
$$\frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \tag{32b}$$

ITApArg
$$\frac{e_1 \ \texttt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')} \tag{32c}$$

ITAP
$$\frac{e_2 \ \texttt{val}}{(\lambda x : \tau . e_1)(e_2) \mapsto [e_2/x]e_1} \tag{32d}$$

ITPairL
$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \tag{32e}$$

ITPairR
$$\frac{e_1 \ \texttt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \tag{32f}$$

ITPrl
$$\frac{(e_1, e_2) \ \texttt{final}}{\texttt{prl}((e_1, e_2)) \mapsto e_1} \tag{32g}$$

ITPrr
$$\frac{(e_1, e_2) \ \texttt{final}}{\texttt{prr}((e_1, e_2)) \mapsto e_2} \tag{32h}$$

ITInl
$$\frac{e \mapsto e'}{\texttt{inl}_\tau(e) \mapsto \texttt{inl}_\tau(e')} \tag{32i}$$

ITInr
$$\frac{e \mapsto e'}{\texttt{inr}_\tau(e) \mapsto \texttt{inr}_\tau(e')} \tag{32j}$$

$$\frac{e \mapsto e'}{\texttt{match}(e)\{\hat{rs}\} \mapsto \texttt{match}(e')\{\hat{rs}\}} \text{ ITExpMatch} \tag{32k}$$

$$\frac{e\ \texttt{final} \qquad e \rhd p_r \dashv\vdash \theta}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \text{ ITSuccMatch} \tag{32l}$$

$$\frac{e\ \texttt{final} \qquad e \perp p_r}{\texttt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \text{ ITFailMatch} \tag{32m}$$

Lemma 4.0.1. *If* $\texttt{inl}_{\tau_2}(e_1)\ \texttt{final}$ *then* $e_1\ \texttt{final}$.

Proof. By rule induction over Rules (25) on $\texttt{inl}_{\tau_2}(e_1)\ \texttt{final}$.

Case (25a).

(1) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{val}$ — by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23d).

(2) $e_1\ \texttt{val}$ — by assumption

(3) $e_1\ \texttt{final}$ — by Rule (25a) on (2)

Case (25b).

(1) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{indet}$ — by assumption

By rule induction over Rules (24) on (1), only one case applies.

Case (24i).

(2) $e_1\ \texttt{indet}$ — by assumption

(3) $e_1\ \texttt{final}$ — by Rule (25b) on (2)

□

Lemma 4.0.2. *If* $\texttt{inr}_{\tau_1}(e_2)\ \texttt{final}$ *then* $e_2\ \texttt{final}$.

Proof. By rule induction over Rules (25) on $\texttt{inr}_{\tau_1}(e_2)\ \texttt{final}$.

Case (25a).

(1) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{val}$ — by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23d).

(2)	e_2 `val`	by assumption
(3)	e_2 `final`	by Rule (25a) on (2)

Case (25b)**.**

(1)	$\texttt{inr}_{\tau_1}(e_2)$ `indet`	by assumption

By rule induction over Rules (24) on (1), only one case applies.

Case (24i)**.**

(2)	e_2 `indet`	by assumption
(3)	e_2 `final`	by Rule (25b) on (2)

□

Lemma 4.0.3. *If* (e_1, e_2) `final` *then* e_1 `final` *and* e_2 `final`*.*

Proof. By rule induction over Rules (25) on (e_1, e_2) `final`.

Case (25a)**.**

(1)	(e_1, e_2) `val`	by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23c)**.**

(2)	e_1 `val`	by assumption
(3)	e_2 `val`	by assumption
(4)	e_1 `final`	by Rule (25a) on (2)
(5)	e_2 `final`	by Rule (25a) on (3)

Case (25b)**.**

(1)	(e_1, e_2) `indet`	by assumption

By rule induction over Rules (24) on (1), only three cases apply.

Case (24d)**.**

(2)	e_1 `indet`	by assumption
(3)	e_2 `val`	by assumption
(4)	e_1 `final`	by Rule (25b) on (2)
(5)	e_1 `final`	by Rule (25a) on (3)

Case (24e)**.**

(2)	e_1 `val`	by assumption
(3)	e_2 `indet`	by assumption
(4)	e_1 `final`	by Rule (25a) on (2)

(5) e_1 `final` by Rule (25b) on (3)

Case (24f).

(2) e_1 `indet` by assumption

(3) e_2 `indet` by assumption

(4) e_1 `final` by Rule (25b) on (2)

(5) e_1 `final` by Rule (25b) on (3)

□

Lemma 4.0.4. *There doesn't exist $\underline{n}$ such that $\underline{n}$* `notintro`*.*

Proof. By rule induction over Rules (26) on $\underline{n}$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 4.0.5. *There doesn't exist $\mathtt{inl}_\tau(e)$ such that $\mathtt{inl}_\tau(e)$* `notintro`*.*

Proof. By rule induction over Rules (26) on $\mathtt{inl}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 4.0.6. *There doesn't exist $\mathtt{inr}_\tau(e)$ such that $\mathtt{inr}_\tau(e)$* `notintro`*.*

Proof. By rule induction over Rules (26) on $\mathtt{inr}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 4.0.7. *There doesn't exist (e_1, e_2) such that (e_1, e_2)* `notintro`*.*

Proof. By rule induction over Rules (26) on (e_1, e_2) `notintro`, no case applies due to syntactic contradiction. □

Lemma 4.0.8. *If e* `final` *and e* `notintro` *then e* `indet`*.*

Proof Sketch. By rule induction over Rules (26) on e `notintro`, for each case, by rule induction over Rules (23) on e `val` and we notice that e `val` is not derivable. By rule induction over Rules (25) on e `final`, Rule (25a) result in a contradiction with the fact that e `val` is not derivable while Rule (25b) tells us e `indet`. □

Lemma 4.0.9 (Finality)**.** *There doesn't exist such an expression e such that both e* `final` *and $e \mapsto e'$ for some e'*

Proof. Assume there exists such an e such that both e `final` and $e \mapsto e'$ for some e' then proof by contradiction.

By rule induction over Rules (25) and Rules (32), *i.e.*, over Rules (23) and Rules (32) and over Rules (24) and Rules (32) respectively. The proof can be done by straightforward observation of syntactic contradictions. □

Lemma 4.0.10 (Matching Determinism)**.** *If e* `final` *and $\cdot\,;\Delta_e \vdash e : \tau$ and $p : \tau[\xi] \dashv\!\vdash \Gamma\,;\Delta$ then exactly one of the following holds*

1. $e \rhd p \dashv\!\vert\, \theta$ *for some* θ

2. $e \mathrel{?} p$

3. $e \perp p$

Proof.

(1) e `final` by assumption

(2) $\cdot\,; \Delta_e \vdash e : \tau$ by assumption

(3) $p : \tau[\xi] \dashv\!\vert\, \Gamma\,; \Delta$ by assumption

By rule induction over Rules (20) on (3), we would show one conclusion is derivable while the other two are not.

Case (20a).

(4) $p = x$ by assumption

(5) $e \rhd x \dashv\!\vert\, e/x$ by Rule (29a)

Assume $e \mathrel{?} x$. By rule induction over Rules (30) on it, only one case applies.

Case (30c).

(6) x `refutable` by assumption

By rule induction over Rules (28) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \mathrel{?} x}$ by contradiction

Assume $e \perp x$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp x}$ by contradiction

Case (20b).

(4) $p = \_$ by assumption

(5) $e \rhd \_ \dashv\!\vert\, \cdot$ by Rule (29b)

Assume $e \mathrel{?} \_$. By rule induction over Rules (30) on it, only one case applies.

Case (30c).

(6) $\_$ `refutable` by assumption

By rule induction over Rules (28) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \mathrel{?} \_}$ by contradiction

Assume $e \perp \_$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp \_}$ by contradiction

Case (20c)**.**

(4) $p = \llparenthesis\rrparenthesis^w$ by assumption

(5) $e \mathrel{?} \llparenthesis\rrparenthesis^w$ by Rule (30a)

Assume $e \rhd \llparenthesis\rrparenthesis^w \dashv \theta$ for some θ. By rule induction over Rules (30) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd \llparenthesis\rrparenthesis^w \dashv \theta}$ by contradiction

Assume $e \perp \llparenthesis\rrparenthesis^w$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp \llparenthesis\rrparenthesis^w}$ by contradiction

Case (20d)**.**

(4) $p = \llparenthesis p_0 \rrparenthesis^w$ by assumption

(5) $e \mathrel{?} \llparenthesis p_0 \rrparenthesis^w$ by Rule (30b)

Assume $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv \theta$ for some θ. By rule induction over Rules (30) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv \theta}$ by contradiction

Assume $e \perp \llparenthesis p_0 \rrparenthesis^w$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp \llparenthesis p_0 \rrparenthesis^w}$ by contradiction

Case (20e)**.**

(4) $p = \underline{n_2}$ by assumption

(5) $\tau = \texttt{num}$ by assumption

(6) $\xi = \underline{n_2}$ by assumption

(7) $\underline{n_2}$ $\texttt{refutable}$ by Rule (28a)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(10) $e\ ?\ \underline{n_2}$ by Rule (16b) on (7) and (9)

Assume $e \rhd \underline{n_2} \dashv\!\mid \theta$ for some θ. By rule induction over it, no case applies due to syntactic contradiction.

(11) $\cancel{e \rhd \underline{n_2} \dashv\!\mid \theta}$ by contradiction

Assume $e \perp \underline{n_2}$. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \perp \underline{n_2}}$ by contradiction

Case (19d).

(8) $e = \underline{n_1}$

Assume $\underline{n_1}\ ?\ \underline{n_2}$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(9) $\underline{n_1}$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.4.

(10) $\cancel{\underline{n_1}\ ?\ \underline{n_2}}$ by contradiction

By case analysis on whether $n_1 = n_2$.

Case $n_1 = n_2$.

(11) $n_1 = n_2$ by assumption

(12) $\underline{n_1} \rhd \underline{n_2} \dashv\!\mid \cdot$ by Rule (29c)

Assume $\underline{n_1} \perp \underline{n_2}$. By rule induction over Rules (31) on it, only one case applies.

Case (31a).

(13) $n_1 \neq n_2$ by assumption

Contradicts (11).

(14) $\cancel{\underline{n_1} \perp \underline{n_2}}$ by contradiction

Case $n_1 \neq n_2$.

(11) $n_1 \neq n_2$ by assumption

(12) $\underline{n_1} \perp \underline{n_2}$ by Rule (31a) on (11)

Assume $\underline{n_1} \rhd \underline{n_2} \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (29) on it, no case applies due to syntactic contradiction.

(13) $\cancel{\underline{n_1} \rhd \underline{n_2} \dashv\!\vert\, \theta}$ by contradiction

Case (20f).

(4) $p = \texttt{inl}(p_1)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \texttt{inl}(\xi_1)$ by assumption

(7) $p_1 : \tau_1[\xi_1] \dashv\!\vert\, \Gamma \,;\, \Delta$ by assumption

(8) $\texttt{inl}(p_1)$ $\texttt{refutable}$ by Rule (28d)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(11) $e \mathbin{?} \texttt{inl}(p_1)$ by Rule (16b) on (8) and (10)

Assume $e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1$ for some θ_1. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1}$ by contradiction

Assume $e \perp \texttt{inl}(p_1)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inl}(p_1)}$ by contradiction

Case (19j).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(10) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$ by assumption

(11) e_1 $\texttt{final}$ by Lemma 4.0.1 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1, $e_1 \mathbin{?} p_1$, and $e_1 \perp p_1$ holds.
By case analysis on which one holds.

Case $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$.

(12) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

(13) $\cancel{e_1 \mathbin{?} p_1}$ by assumption

(14) $\cancel{e_1 \perp p_1}$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1$ by Rule (29e) on (12)

Assume $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(16) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ — by assumption

Contradicts Lemma 4.0.5.

Case (30g).

(16) $e_1 \,?\, p_1$ — by assumption

Contradicts (13).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)}$ — by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (31) on it, only one case applies.

Case (31f).

(18) $e_1 \perp p_1$ — by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ — by contradiction

Case $e_1 \,?\, p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ — by assumption

(13) $e_1 \,?\, p_1$ — by assumption

(14) $\cancel{e_1 \perp p_1}$ — by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$ — by Rule (30g) on (13)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (29) on it, only one case applies.

Case (29e).

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta$ — by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta}$ — by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (31) on it, only one case applies.

Case (31f).

(18) $e_1 \perp p_1$ — by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ — by contradiction

Case $e_1 \perp p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ — by assumption

(13) $\cancel{e_1 \,?\, p_1}$ — by assumption

(14) $e_1 \perp p_1$ — by assumption

(15) $\mathtt{inl}_{\tau_2}(e_1) \perp \mathtt{inl}(p_1)$ by Rule (31f) on (14)

Assume $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\mid \theta$ for some θ. By rule induction over Rules (29) on it, only one case applies.

Case (29e).

(16) $e_1 \rhd p_1 \dashv\!\mid \theta$ by assumption

Contradicts (12).

(17) $\cancel{\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\mid \theta}$ by contradiction

Assume $\mathtt{inl}_{\tau_2}(e_1) \mathbin{?} \mathtt{inl}(p_1)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(18) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (30g).

(18) $e_1 \mathbin{?} p_1$ by assumption

Contradicts (13).

(19) $\cancel{\mathtt{inl}_{\tau_2}(e_1) \mathbin{?} \mathtt{inl}(p_1)}$ by contradiction

Case (20g).

(4) $p = \mathtt{inr}(p_2)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \mathtt{inr}(\xi_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\mid \Gamma \,;\, \Delta$ by assumption

(8) $\mathtt{inr}(p_2)\ \mathtt{refutable}$ by Rule (28e)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) $e\ \mathtt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(11) $e \mathbin{?} \mathtt{inr}(p_2)$ by Rule (16b) on (8) and (10)

Assume $e \rhd \mathtt{inr}(p_2) \dashv\!\mid \theta_2$ for some θ_2. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \mathtt{inr}(p_2) \dashv\!\mid \theta_2}$ by contradiction

Assume $e \perp \mathtt{inr}(p_2)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inr}(p_2)}$ by contradiction

Case (19k).

(9) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(10) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ by assumption

(11) e_2 $\texttt{final}$ by Lemma 4.0.2 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2, $e_2 \,?\, p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which one holds.

Case $e_2 \rhd p_2 \dashv\!\mid \theta_2$.

(12) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(13) $\cancel{e_2 \,?\, p_2}$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\mid \theta_2$ by Rule (29f) on (12)

Assume $\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(16) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.6.

Case (30h).

(16) $e_2 \,?\, p_2$ by assumption

Contradicts (13).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$. By rule induction over Rules (31) on it, only one case applies.

Case (31g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)}$ by contradiction

Case $e_2 \,?\, p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\mid \theta}$ by assumption

(13) $e_2 \,?\, p_2$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)$ by Rule (30h) on (13)

Assume $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\mid \theta$ for some θ. By rule induction over Rules (29) on it, only one case applies.

Case (29f).

(16) $e_2 \rhd p_2 \dashv\!\mid \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\vert\, \theta}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$. By rule induction over Rules (31) on it, only one case applies.

Case (31g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)}$ by contradiction

Case $e_2 \perp p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta}$ by assumption

(13) $\cancel{e_2 \,?\, p_2}$ by assumption

(14) $e_2 \perp p_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$ by Rule (31g) on (14)

Assume $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (29) on it, only one case applies.

Case (29f).

(16) $e_2 \rhd p_2 \dashv\!\vert\, \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\vert\, \theta}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(18) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.6.

Case (30h).

(18) $e_2 \,?\, p_2$ by assumption

Contradicts (13).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)}$ by contradiction

Case (20h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\tau = (\tau_1 \times \tau_2)$ by assumption

(6) $\xi = (\xi_1, \xi_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\vert \Gamma_1 \,;\, \Delta_1$ — by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\vert \Gamma_2 \,;\, \Delta_2$ — by assumption

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

(11) $e = \llparenthesis\rrparenthesis^u, \llparenthesis e_0 \rrparenthesis^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ — by assumption

(12) $e\ \texttt{notintro}$ — by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(13) $e\ \texttt{indet}$ — by Lemma 4.0.8 on (1) and (12)

(14) $\texttt{prl}(e)\ \texttt{indet}$ — by Rule (24g) on (13)

(15) $\texttt{prl}(e)\ \texttt{final}$ — by Rule (25b) on (14)

(16) $\texttt{prr}(e)\ \texttt{indet}$ — by Rule (24h) on (13)

(17) $\texttt{prr}(e)\ \texttt{final}$ — by Rule (25b) on (16)

(18) $\cdot\,;\Delta \vdash \texttt{prl}(e) : \tau_1$ — by Rule (19h) on (2)

(19) $\cdot\,;\Delta \vdash \texttt{prr}(e) : \tau_2$ — by Rule (19i) on (2)

Assume $e \perp (p_1, p_2)$. By rule induction on it, no case applies due to syntactic contradiction.

(20) $\cancel{e \perp (p_1, p_2)}$ — by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of $\texttt{prl}(e) \rhd p_1 \dashv\vert \theta_1$, $\texttt{prl}(e)\,?\,p_1$, and $\texttt{prl}(e) \perp p_1$ holds.
By inductive hypothesis on (17) and (19) and (10), exactly one of $\texttt{prr}(e) \rhd p_2 \dashv\vert \theta_2$, $\texttt{prr}(e)\,?\,p_2$, and $\texttt{prr}(e) \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2. Note that we have already shown $\cancel{e \perp (p_1, p_2)}$.

Case $\texttt{prl}(e) \rhd p_1 \dashv\vert \theta_1, \texttt{prr}(e) \rhd p_2 \dashv\vert \theta_2$**.**

(21) $\texttt{prl}(e) \rhd p_1 \dashv\vert \theta_1$ — by assumption

(22) $\cancel{\texttt{prl}(e)\,?\,p_1}$ — by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ — by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashv\vert \theta_2$ — by assumption

(25) $\cancel{\texttt{prr}(e)\,?\,p_2}$ — by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ — by assumption

(27) $e \rhd (p_1, p_2) \dashv\vert \theta_1 \uplus \theta_2$ — by Rule (29g) on (12) and (21) and (24)

Assume $e\,?\,(p_1, p_2)$. By rule induction over Rules (30) on it, only one case applies.

Case (30c)**.**

(28) $(p_1, p_2)\ \texttt{refutable}$ — by assumption

By rule induction over Rules (28), only two cases apply.

Case (28f).

(29) p_1 `refutable` by assumption

(30) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(31) $\texttt{prl}(e) \mathrel{?} p_1$ by Rule (30c) on (29) and (30)

Contradicts (22).

Case (28g).

(29) p_2 `refutable` by assumption

(30) $\texttt{prr}(e)$ `notintro` by Rule (26f)

(31) $\texttt{prl}(e) \mathrel{?} p_1$ by Rule (30c) on (29) and (30)

Contradicts (22).

(32) $\cancel{e \mathrel{?} (p_1, p_2)}$ by contradiction

Case $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1, \texttt{prr}(e) \mathrel{?} p_2$.

(21) $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (29), only one case applies.

Case (29g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

Contradicts (24).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

By rule induction over Rules (30) on (25), the following cases apply.

Case (30a),(30b).

(30) $p_2 = (\!|\,|\!)^w, (\!|p|\!)^w$ by assumption

(31) p_2 `refutable` by Rule (28b) and Rule (28c)

(32) (p_1, p_2) `refutable` by Rule (28g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c).

(30) p_2 `refutable` by assumption

(31) (p_1, p_2) `refutable` by Rule (28g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case $\texttt{prl}(e) \rhd p_1 \dashV \theta_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\texttt{prl}(e) \rhd p_1 \dashV \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption

(26) $\texttt{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (31) on (26), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \rhd p_2 \dashV \theta_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$ by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (29), only one case applies.

Case (29g)**.**

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashV \theta_1$ by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashV \theta}$ by contradiction

By rule induction over Rules (30) on (22), the following cases apply.

Case (30a),(30b)**.**

(30) $p_1 = \llparenthesis\rrparenthesis^w, \llparenthesis p\rrparenthesis^w$ by assumption

(31) p_1 `refutable` by Rule (28b) and Rule (28c)

(32) (p_1, p_2) `refutable` by Rule (28g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c)**.**

(30) p_1 `refutable` by assumption

(31) (p_1, p_2) `refutable` by Rule (28g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \mathrel{?} p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\vdash \theta$. By rule induction over Rules (29), only one case applies.

Case (29g)**.**

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1$ by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\vdash \theta}$ by contradiction

By rule induction over Rules (30) on (22), the following cases apply.

Case (30a),(30b)**.**

(30) $p_1 = (\!|\,|\!)^w, (\!|p|\!)^w$ by assumption

(31) p_1 $\texttt{refutable}$ by Rule (28b) and Rule (28c)

(32) (p_1, p_2) $\texttt{refutable}$ by Rule (28g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c)**.**

(30) p_1 $\texttt{refutable}$ by assumption

(31) (p_1, p_2) $\texttt{refutable}$ by Rule (28g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2}$ by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption

(26) $\texttt{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (31) on (26), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \rhd p_2 \dashv\!\!\dashv \theta_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\!\dashv \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\texttt{prr}(e) \rhd p_2 \dashv\!\!\dashv \theta_2$ by assumption
(25) $\cancel{\texttt{prr}(e) \,?\, p_2}$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (31) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \,?\, p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\!\dashv \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\!\dashv \theta_2}$ by assumption
(25) $\texttt{prr}(e) \,?\, p_2$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (31) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \perp p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\!\dashv \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\!\dashv \theta_2}$ by assumption
(25) $\texttt{prr}(e) \,?\, p_2$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (31) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case (19g).

(11) $e = (e_1, e_2)$ by assumption
(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption
(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption
(14) $e_1\ \texttt{final}$ by Lemma 4.0.3 on (1)
(15) $e_2\ \texttt{final}$ by Lemma 4.0.3 on (1)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \rhd p_1 \dashv\!\!\dashv \theta_1$, $e_1 \,?\, p_1$, and $e_1 \perp p_1$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \rhd p_2 \dashv\!\!\dashv \theta_2$, $e_2 \,?\, p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2.

Case $e_1 \rhd p_1 \dashv\!\!\mid \theta_1, e_2 \rhd p_2 \dashv\!\!\mid \theta_2$.

(16) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption
(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption
(18) $\cancel{e_1 \perp p_1}$ by assumption
(19) $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption
(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption
(21) $\cancel{e_2 \perp p_2}$ by assumption
(22) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta_1 \uplus \theta_2$ by Rule (29d) on (16) and (19)

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(23) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (30e).

(23) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (30f).

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(24) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

(25) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (31c).

(25) $e_2 \perp p_2$ by assumption

Contradicts (21).

(26) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\!\mid \theta_1, e_2 \mathrel{?} p_2$.

(16) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption
(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption
(18) $\cancel{e_1 \perp p_1}$ by assumption
(19) $\cancel{e_2 \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (30e) on (16) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\dashv \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\dashv \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\dashv \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (31c).

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\dashv \theta_1, e_2 \perp p_2$.

(16) $e_1 \rhd p_1 \dashv\!\dashv \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\dashv \theta_2}$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\dashv \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\dashv \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\dashv \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (30e).

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (30f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \rhd p_2 \dashv\!\mid \theta_2$.

(16) $\cancel{e_1 \rhd p_1 \dashv\!\mid \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (30d) on (17) and (19)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (31c).

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \mathrel{?} p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (30f) on (17) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashV \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (31c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

Contradicts (19).

Case (30e).

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (30f).

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \rhd p_2 \dashv\!\mid \theta_2$.

(16) $\cancel{e_1 \rhd p_1 \dashv\!\mid \theta_1}$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (30e).

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (30f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \mathrel{?} p_2$.

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2)} \dashV \theta$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

Case (30e).

(26) $e_1 \rhd p_1 \dashV \theta_1$ by assumption

Contradicts (16).

Case (30f).

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashV \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c)**.**

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d)**.**

(26) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

Case (30e)**.**

(26) $e_1 \rhd p_1 \dashV \theta_1$ by assumption

Contradicts (16).

Case (30f)**.**

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

□

Lemma 4.0.11 (Matching Coherence of Constraint)**.** *Suppose that* $\cdot ; \Delta_e \vdash e : \tau$ *and* e `final` *and* $p : \tau[\xi] \dashV \Gamma ; \Delta$*. Then we have*

1. $e \models \xi$ *iff* $e \rhd p \dashv\vdash \theta$

2. $e \models_? \xi$ *iff* $e \,?\, p$

3. $e \not\models_?^\dagger \xi$ *iff* $e \perp p$

Proof.

(1) $\cdot\,; \Delta_e \vdash e : \tau$ by assumption

(2) e `final` by assumption

(3) $p : \tau[\xi] \dashv\vdash \Gamma\,; \Delta$ by assumption

Given Lemma 3.0.1, Theorem 2.1, and Lemma 4.0.10, it is sufficient to prove

1. $e \models \xi$ iff $e \rhd p \dashv\vdash \theta$
2. $e \models_? \xi$ iff $e \,?\, p$

By rule induction over Rules (20) on (3).

Case (20a)**.**

(4) $p = x$ by assumption

(5) $\xi = \top$ by assumption

1. Prove $e \models \top$ implies $e \rhd x \dashv\vdash \theta$ for some θ.

   (6) $e \rhd x \dashv\vdash e/x$ by Rule (29a)

2. Prove $e \rhd x \dashv\vdash \theta$ implies $e \models \top$.

   (6) $e \models \top$ by Rule (14a)

3. Prove $e \models_? \top$ implies $e \,?\, x$.

   (6) $e \not\models_? \top$ by Lemma 2.0.3

   Vacuously true.

4. Prove $e \,?\, x$ implies $e \models_? \top$.

   By rule induction over Rules (30), we notice that either, $e \,?\, x$ is in syntactic contradiction with all the cases, or the premise x `refutable` is not derivable. Hence, $e \,?\, x$ are not derivable. And thus vacuously true.

Case (20b)**.**

(4) $p = \_$ by assumption

(5) $\xi = \top$ by assumption

1. Prove $e \models \top$ implies $e \rhd \_ \dashv\vdash \theta$ for some θ.

   (6) $e \rhd \_ \dashv\vdash \cdot$ by Rule (29a)

2. Prove $e \rhd \_ \dashv\vert\, \theta$ implies $e \models \top$.

   (6) $e \models \top$ — by Rule (14a)

3. Prove $e \models_? \top$ implies $e\,?\,\_$.

   (6) $e \not\models_? \top$ — by Lemma 2.0.3

   Vacuously true.

4. Prove $e\,?\,\_$ implies $e \models_? \xi$.

   By rule induction over Rules (30), we notice that either, $e?\_$ is in syntactic contradiction with all the cases, or the premise $\_$ `refutable` is not derivable. Hence, $e\,?\,\_$ are not derivable. And thus vacuously true.

Case (20c)**.**

(4) $p = \llparenthesis\rrparenthesis^w$ — by assumption

(5) $\xi = ?$ — by assumption

(6) $\overline{\xi} = ?$ — by Definition 9

1. Prove $e \models ?$ implies $e \rhd \llparenthesis\rrparenthesis^w \dashv\vert\, \theta$ for some θ.

   (7) $e \not\models ?$ — by Rule (29a)

   Vacuously true.

2. Prove $e \rhd \llparenthesis\rrparenthesis^w \dashv\vert\, \theta$ implies $e \models ?$.

   By rule induction over Rules (29), we notice that $e \rhd \llparenthesis\rrparenthesis^w \dashv\vert\, \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? ?$ implies $e\,?\,\llparenthesis\rrparenthesis^w$.

   (7) $e\,?\,\llparenthesis\rrparenthesis^w$ — by Rule (30a)

4. Prove $e\,?\,\llparenthesis\rrparenthesis^w$ implies $e \models_? ?$.

   (7) $e \models_? ?$ — by Rule (16a)

Case (20d)**.**

(4) $p = \llparenthesis p_0 \rrparenthesis^w$ — by assumption

(5) $\xi = ?$ — by assumption

1. Prove $e \models ?$ implies $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\vert\, \theta$ for some θ.

   (6) $e \not\models ?$ — by Rule (29a)

   Vacuously true.

2. Prove $e \rhd (\!|p_0|\!)^w \dashv\!\vdash \theta$ implies $e \models\, ?$.
   By rule induction over Rules (29), we notice that $e \rhd (\!|p_0|\!)^w \dashv\!\vdash \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? \, ?$ implies $e \mathrel{?} (\!|p_0|\!)^w$.

   (6) $e \mathrel{?} (\!|p_0|\!)^w$ — by Rule (30b)

4. Prove $e \mathrel{?} (\!|p_0|\!)^w$ implies $e \models_? \, ?$.

   (6) $e \models_? \, ?$ — by Rule (16a)

Case (20e)**.**

(4) $p = \underline{n}$ — by assumption

(5) $\xi = \underline{n}$ — by assumption

1. Prove $e \models \underline{n}$ implies $e \rhd \underline{n} \dashv\!\vdash \theta$ for some θ.

   (6) $e \models \underline{n}$ — by assumption

   By rule induction over Rules (14) on (6), only one case applies.

   Case (14b)**.**

   (7) $e = \underline{n}$ — by assumption

   (8) $\underline{n} \rhd \underline{n} \dashv\!\vdash \cdot$ — by Rule (29c)

2. Prove $e \rhd \underline{n} \dashv\!\vdash \theta$ implies $e \models \underline{n}$.

   (6) $e \rhd \underline{n} \dashv\!\vdash \theta$ — by assumption

   By rule induction over Rules (29) on (6), only one case applies.

   Case (29c)**.**

   (7) $e = \underline{n}$ — by assumption

   (8) $\theta = \cdot$ — by assumption

   (9) $\underline{n} \models \underline{n}$ — by Rule (14b)

3. Prove $e \models_? \underline{n}$ implies $e \mathrel{?} \underline{n}$.

   (6) $e \models_? \underline{n}$ — by assumption

   By rule induction over Rules (16) on (6), only one case applies.

   Case (16b)**.**

   (7) e `notintro` — by assumption

   (8) $\underline{n}$ `refutable` — by Rule (28a)

   (9) $e \mathrel{?} \underline{n}$ — by Rule (30c) on (7) and (8)

4. Prove $e \mathrel{?} \underline{n}$ implies $e \models_? \underline{n}$.

   (6) $e \mathrel{?} \underline{n}$ — by assumption

By rule induction over Rules (30) on (6), only one case applies.

Case (30c).

(7) e `notintro` by assumption

(8) $\underline{n}$ `refutable` by Rule (10a)

(9) $e \models_? \underline{n}$ by Rule (16) on (7) and (8)

Case (20f).

(4) $p = \texttt{inl}(p_1)$ by assumption

(5) $\xi = \texttt{inl}(\xi_1)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $p_1 : \tau_1[\xi_1] \dashv\!\!| \, \Gamma ; \Delta$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e `notintro` by Rule (26a),(26b),(26c),(26d),(26e),(26f)

1. Prove $e \models \texttt{inl}(\xi_1)$ implies $e \rhd \texttt{inl}(p_1) \dashv\!\!| \, \theta$ for some θ. By rule induction over Rules (14) on $e \models \texttt{inl}(\xi_1)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \texttt{inl}(p_1) \dashv\!\!| \, \theta$ implies $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (29) on $e \rhd \texttt{inl}(p_1) \dashv\!\!| \, \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \texttt{inl}(\xi_1)$ implies $e \,?\, \texttt{inl}(p_1)$.

   (10) $\texttt{inl}(p_1)$ `refutable` by Rule (28d)

   (11) $e \,?\, \texttt{inl}(p_1)$ by Rule (30c) on (9) and (10)
4. Prove $e \,?\, \texttt{inl}(p_1)$ implies $e \models_? \texttt{inl}(\xi_1)$.

   (10) $\texttt{inl}(\xi_1)$ `refutable` by Rule (10b)

   (11) $e \models_? \texttt{inl}(\xi_1)$ by Rule (16b) on (9) and (10)

Case (19j).

(8) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(9) $\cdot\, ; \Delta_e \vdash e_1 : \tau_1$ by assumption

(10) e_1 `final` by Lemma 4.0.1 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\!\mid \theta$ for some θ

(12) $e_1 \models_? \xi_1$ iff $e_1 \,?\, p_1$

1. Prove $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ implies $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\!\mid \theta$ for some θ.

   (13) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ — by assumption

   By rule induction over Rules (14) on (13), only one case applies.

   Case (14g).

   (14) $e_1 \models \xi_1$ — by assumption

   (15) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ for some θ_1 — by (11) on (14)

   (16) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\!\mid \theta_1$ — by Rule (29e) on (15)

2. Prove $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\!\mid \theta$ implies $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$.

   (13) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\!\mid \theta$ — by assumption

   By rule induction over Rules (29) on (13), only one case applies.

   Case (29e).

   (14) $e_1 \rhd p_1 \dashv\!\!\mid \theta$ — by assumption

   (15) $e_1 \models \xi_1$ — by (11) on (14)

   (16) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ — by Rule (14g) on (15)

3. Prove $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ implies $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$.

   (13) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ — by assumption

   By rule induction over Rules (16) on (13), only two cases apply.

   Case (16b).

   (14) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ — by assumption

   Contradicts Lemma 4.0.5.

   Case (16e).

   (14) $e_1 \models_? \xi_1$ — by assumption

   (15) $e_1 \,?\, p_1$ — by (12) on (14)

   (16) $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$ — by Rule (30g) on (15)

4. Prove $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$ implies $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$.

   (13) $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$ — by assumption

   By rule induction over Rules (30) on (13), only two cases apply.

   Case (30c).

   (14) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ — by assumption

   Contradicts Lemma 4.0.5.

   Case (30g).

   (14) $e_1 \,?\, p_1$ — by assumption

   (15) $e_1 \models_? \xi_1$ — by (12) on (14)

   (16) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ — by Rule (16e) on (15)

Case (20g).

(4) $p = \mathtt{inr}(p_2)$ by assumption

(5) $\xi = \mathtt{inr}(\xi_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!| \Gamma\,;\Delta$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e $\mathtt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

1. Prove $e \models \mathtt{inr}(\xi_2)$ implies $e \rhd \mathtt{inr}(p_2) \dashv\!| \theta$ for some θ. By rule induction over Rules (14) on $e \models \mathtt{inr}(\xi_2)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \mathtt{inr}(p_2) \dashv\!| \theta$ implies $e \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (29) on $e \rhd \mathtt{inr}(p_2) \dashv\!| \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \mathtt{inr}(\xi_2)$ implies $e\,?\,\mathtt{inr}(p_2)$.

   (10) $\mathtt{inr}(p_2)$ $\mathtt{refutable}$ by Rule (28e)

   (11) $e\,?\,\mathtt{inr}(p_2)$ by Rule (30c) on (9) and (10)
4. Prove $e\,?\,\mathtt{inr}(p_2)$ implies $e \models_? \mathtt{inr}(\xi_2)$.

   (10) $\mathtt{inr}(\xi_2)$ $\mathtt{refutable}$ by Rule (10c)

   (11) $e \models_? \mathtt{inr}(\xi_2)$ by Rule (16b) on (9) and (10)

Case (19k).

(8) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(9) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ by assumption

(10) e_2 $\mathtt{final}$ by Lemma 4.0.1 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!| \theta$ for some θ

(12) $e_2 \models_? \xi_2$ iff $e_2\,?\,p_2$

1. Prove $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!| \theta$ for some θ.

   (13) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by assumption

By rule induction over Rules (14) on (13), only one case applies.

Case (14g).

(14) $e_2 \models \xi_2$ by assumption

(15) $e_2 \rhd p_2 \dashV \theta_1$ for some θ_1 by (11) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashV \theta_1$ by Rule (29e) on (15)

2. Prove $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashV \theta$ implies $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashV \theta$ by assumption

By rule induction over Rules (29) on (13), only one case applies.

Case (29e).

(14) $e_2 \rhd p_2 \dashV \theta$ by assumption

(15) $e_2 \models \xi_2$ by (11) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (14g) on (15)

3. Prove $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by assumption

By rule induction over Rules (16) on (13), only two cases apply.

Case (16b).

(14) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (16e).

(14) $e_2 \models_? \xi_2$ by assumption

(15) $e_2 \,?\, p_2$ by (12) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ by Rule (30g) on (15)

4. Prove $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ by assumption

By rule induction over Rules (30) on (13), only two cases apply.

Case (30c).

(14) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (30g).

(14) $e_2 \,?\, p_2$ by assumption

(15) $e_2 \models_? \xi_2$ by (12) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (16e) on (15)

Case (20h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\xi = (\xi_1, \xi_2)$ by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\!\vert\, \Gamma_1 ; \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\!\vert\, \Gamma_2 ; \Delta_2$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(13) e $\texttt{indet}$ by Lemma 4.0.8 on (2) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (24g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (25b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (24h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (25b) on (16)

(18) $\cdot ; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (19h) on (1)

(19) $\cdot ; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (19i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

(20) $\texttt{prl}(e) \models \xi_1$ iff $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1

(21) $\texttt{prl}(e) \models_? \xi_1$ iff $\texttt{prl}(e)\ ?\ p_1$

(22) $\texttt{prr}(e) \models \xi_2$ iff $\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2$ for some θ_2

(23) $\texttt{prr}(e) \models_? \xi_2$ iff $\texttt{prr}(e)\ ?\ p_2$

1. Prove $e \models (\xi_1, \xi_2)$ implies $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ for some θ.

   (24) $e \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (14) on (24), only one case applies.

   Case (14j).

   (25) $\texttt{prl}(e) \models \xi_1$ by assumption

   (26) $\texttt{prr}(e) \models \xi_2$ by assumption

   (27) $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ by (20) on (25)

   (28) $\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2$ by (22) on (26)

   (29) $e \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2$ by Rule (29g) on (12) and (27) and (28)

2. Prove $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ implies $e \models (\xi_1, \xi_2)$.

   (24) $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ by assumption

   By rule induction over Rules (29) on (24), only one case applies.

   Case (29g).

   (25) $\theta = \theta_1 \uplus \theta_2$ by assumption

   (26) $\texttt{prl}(e) \rhd \xi_1 \dashv\!\vert\, \theta_1$ by assumption

(27) $\texttt{prr}(e) \rhd \xi_2 \dashv\!| \theta_2$ by assumption

(28) $\texttt{prl}(e) \models \xi_1$ by (20) on (26)

(29) $\texttt{prr}(e) \models \xi_2$ by (22) on (27)

(30) $e \models (\xi_1, \xi_2)$ by Rule (14j) on (12) and (28) and (29)

3. Prove $e \models_? (\xi_1, \xi_2)$ implies $e \mathrel{?} (p_1, p_2)$.

(24) $e \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (16) on (24), only one case applies.

Case (16b)**.**

(25) (ξ_1, ξ_2) `refutable` by assumption

By rule induction over Rules (10) on (25), only two cases apply.

Case (10d)**.**

(26) ξ_1 `refutable` by assumption

(27) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(28) $\texttt{prl}(e) \models_? \xi_1$ by Rule (16b) on (26) and (27)

(29) $\texttt{prl}(e) \mathrel{?} p_1$ by (21) on (28)

By rule induction over Rules (30) on (29), only three cases apply.

Case (30a),(30b)**.**

(30) $p_1 = (\!|\,|\!)^w, (\!|p_0|\!)^w$ by assumption

(31) p_1 `refutable` by Rule (28b) and Rule (28c)

(32) (p_1, p_2) `refutable` by Rule (28f) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c)**.**

(30) p_1 `refutable` by assumption

(31) (p_1, p_2) `refutable` by Rule (28f) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case (10e)**.**

(26) ξ_2 `refutable` by assumption

(27) $\texttt{prr}(e)$ `notintro` by Rule (26e)

(28) $\texttt{prr}(e) \models_? \xi_2$ by Rule (16b) on (26) and (27)

(29) $\texttt{prr}(e) \mathrel{?} p_2$ by (23) on (28)

By rule induction over Rules (30) on (29), only three cases apply.

Case (30a),(30b)**.**

(30) $p_2 = \llparenthesis\rrparenthesis^w, \llparenthesis p_0\rrparenthesis^w$ by assumption

(31) p_2 `refutable` by Rule (28b) and Rule (28c)

(32) (p_1, p_2) `refutable` by Rule (28g) on (31)

(33) $e\,?\,(p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c)**.**

(30) p_2 `refutable` by assumption

(31) (p_1, p_2) `refutable` by Rule (28g) on (30)

(32) $e\,?\,(p_1, p_2)$ by Rule (30c) on (12) and (31)

4. Prove $e\,?\,(p_1, p_2)$ implies $e \models_? (\xi_1, \xi_2)$.

(24) $e\,?\,(p_1, p_2)$ by assumption

By rule induction over Rules (30) on (24), only one case applies.

Case (30c)**.**

(25) (p_1, p_2) `refutable` by assumption

By rule induction over Rules (28) on (25), only two cases apply.

Case (28f)**.**

(26) p_1 `refutable` by assumption

(27) $\texttt{prl}(e)$ `notintro` by Rule (26e)

(28) $\texttt{prl}(e)\,?\,p_1$ by Rule (30c) on (26) and (27)

(29) $\texttt{prl}(e) \models_? \xi_1$ by (21) on (28)

By rule induction over Rules (16) on (29), only three cases apply.

Case (16a)**.**

(30) $\xi_1 = ?$ by assumption

(31) ξ_1 `refutable` by Rule (2b)

(32) (ξ_1, ξ_2) `refutable` by Rule (10d) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (32)

Case (16b)**.**

(30) ξ_1 `refutable` by assumption

(31) (ξ_1, ξ_2) `refutable` by Rule (10d) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (31)

Case (28g)**.**

(26) p_2 `refutable` by assumption

(27) $\texttt{prr}(e)$ `notintro` by Rule (26e)

(28) $\texttt{prr}(e) \mathrel{?} p_2$ by Rule (30c) on (26) and (27)

(29) $\texttt{prr}(e) \models_? \xi_2$ by (23) on (28)

By rule induction over Rules (16) on (29), only three cases apply.

Case (16a)**.**

(30) $\xi_2 = ?$ by assumption

(31) $\xi_2\ \texttt{refutable}$ by Rule (2b)

(32) $(\xi_1, \xi_2)\ \texttt{refutable}$ by Rule (10e) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (32)

Case (16b)**.**

(30) $\xi_2\ \texttt{refutable}$ by assumption

(31) $(\xi_1, \xi_2)\ \texttt{refutable}$ by Rule (10e) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (31)

Case (19g)**.**

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot ; \Delta_e \vdash e_1 : \tau_1$ by assumption

(13) $\cdot ; \Delta_e \vdash e_2 : \tau_2$ by assumption

(14) $e_1\ \texttt{final}$ by Lemma 4.0.3 on (2)

(15) $e_2\ \texttt{final}$ by Lemma 4.0.3 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1

(17) $e_1 \models_? \xi_1$ iff $e_1 \mathrel{?} p_1$

(18) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2

(19) $e_2 \models_? \xi_2$ iff $e_2 \mathrel{?} p_2$

1. Prove $(e_1, e_2) \models (\xi_1, \xi_2)$ implies $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$ for some θ.

   (20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (14) on (20), only two cases apply.

   Case (14i)**.**

   (21) $e_1 \models \xi_1$ by assumption

   (22) $e_2 \models \xi_2$ by assumption

   (23) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1 by (16) on (21)

   (24) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2 by (18) on (22)

   (25) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2$ by Rule (29d) on (23) and (24)

Case (14j).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

2. Prove $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\dashv \theta$ implies $(e_1, e_2) \models (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\dashv \theta$ by assumption

By rule induction over Rules (29) on (20), only two cases apply.

Case (29d).

(21) $e_1 \rhd p_1 \dashv\!\!\dashv \theta_1$ for some θ_1 by assumption

(22) $e_2 \rhd p_2 \dashv\!\!\dashv \theta_2$ for some θ_2 by assumption

(23) $e_1 \models \xi_1$ by (16) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (23) and (24)

Case (29g).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

3. Prove $(e_1, e_2) \models_? (\xi_1, \xi_2)$ implies $(e_1, e_2)\, ?\, (p_1, p_2)$.

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (16) on (20), only four cases apply.

Case (16b).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models \xi_2$ by assumption

(23) $e_1\, ?\, p_1$ by (17) on (21)

(24) $e_2 \rhd p_2 \dashv\!\!\dashv \theta_2$ by (18) on (22)

(25) $(e_1, e_2)\, ?\, (p_1, p_2)$ by Rule (30d) on (23) and (24)

Case (16h).

(21) $e_1 \models \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1 \rhd p_1 \dashv\!\!\dashv \theta_1$ by (16) on (21)

(24) $e_2\, ?\, p_2$ by (19) on (22)

(25) $(e_1, e_2)\, ?\, (p_1, p_2)$ by Rule (30e) on (23) and (24)

Case (16i).

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1\, ?\, p_1$ by (17) on (21)

(24) $e_2 \,?\, p_2$ by (19) on (22)

(25) $(e_1, e_2) \,?\, (p_1, p_2)$ by Rule (30f) on (23) and (24)

4. Prove $(e_1, e_2) \,?\, (p_1, p_2)$ implies $(e_1, e_2) \models_? (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \,?\, (p_1, p_2)$ by assumption

By rule induction over Rules (30) on (20), only four cases apply.

Case (30c).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(21) $e_1 \,?\, p_1$ by assumption

(22) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(23) $e_1 \models_? \xi_1$ by (17) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) \,?\, (p_1, p_2)$ by Rule (16g) on (23) and (24)

Case (30e).

(21) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ by assumption

(22) $e_2 \,?\, p_2$ by assumption

(23) $e_1 \models \xi_1$ by (16) on (21)

(24) $e_2 \models_? \xi_2$ by (19) on (22)

(25) $(e_1, e_2) \,?\, (p_1, p_2)$ by Rule (16h) on (23) and (24)

Case (30f).

(21) $e_1 \,?\, p_1$ by assumption

(22) $e_2 \,?\, p_2$ by assumption

(23) $e_1 \models_? \xi_1$ by (17) on (21)

(24) $e_2 \models_? \xi_2$ by (19) on (22)

(25) $(e_1, e_2) \,?\, (p_1, p_2)$ by Rule (16i) on (23) and (24)

□

5 Preservation and Progress

Theorem 5.1 (Preservation). *If* $\cdot\,; \Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,; \Delta \vdash e' : \tau$

Proof. By rule induction over Rules (19) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (19l).

(1) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

(2) $\texttt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$ by assumption

(3) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(4) $\cdot\,;\Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ by assumption

(5) $\top \models^{\dagger}_{?} \xi$ by assumption

By rule induction over Rules (32) on (2).

Case (32k).

(6) $e' = \texttt{match}(e_1')\{\cdot \mid r \mid rs\}$ by assumption

(7) $e_1 \mapsto e_1'$ by assumption

(8) $\cdot\,;\Delta \vdash e_1' : \tau_1$ by IH on (3) and (7)

(9) $\cdot\,;\Delta \vdash \texttt{match}(e_1')\{\cdot \mid r \mid rs\} : \tau$ by Rule (19l) on (8) and (4) and (5)

Case (32l).

(6) $r = p_r \Rightarrow e_r$ by assumption

(7) $e' = [\theta](e_r)$ by assumption

(8) $e_1 \rhd p_r \dashv\!\!\mid \theta$ by assumption

By rule induction over Rules (22) on (4).

Case (22a).

(9) $\xi = \xi_r$ by assumption

(10) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(11) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (10)

(12) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (10)

(13) $\theta : \Gamma_r$ by Lemma 3.0.7 on (3) and (11) and (8)

(14) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (12) and (13)

Case (22b).

(9) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(10) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(11) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (10)

(12) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (10)

(13) $\theta : \Gamma_r$ by Lemma 3.0.7 on (3) and (11) and (8)

(14) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (12) and (13)

Case (32m).

(6) $rs = r' \mid rs'$ by assumption

(7) $e' = \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by assumption

(8) $e_1\ \texttt{final}$ by assumption

(9) $e_1 \perp p_r$ by assumption

By rule induction over Rules (22) on (4).

Case (22a). Syntactic contradiction of rs.

Case (22b).

(10) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(11) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(12) $\cdot\,;\Delta \vdash [\perp \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$ by assumption

(13) $\xi_r \not\models \perp$ by assumption

(14) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (11)

(15) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (11)

(16) $\cdot\,;\Delta \vdash [\perp](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$ by Rule (22a) on (11) and (13)

(17) $e_1 \not\models^{\dagger}_{?} \xi_r$ by Lemma 4.0.11 on (3) and (8) and (14) and (9)

(18) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$ by Rule (19m) on (3) and (8) and (16) and (12) and (17) and (5)

Case (19m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$ by assumption

(4) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(5) $e_1\ \texttt{final}$ by assumption

(6) $\cdot\,;\Delta \vdash [\perp]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$ by assumption

(7) $\cdot\,;\Delta \vdash [\perp \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(8) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(9) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (32) on (3).

Case (32k)**.**

(10) $e' = \texttt{match}(e'_1)\{rs_{pre} \mid r \mid rs_{post}\}$ by assumption
(11) $e_1 \mapsto e'_1$ by assumption

By Lemma 4.0.9, (11) contradicts (5).

Case (32l)**.**

(10) $r = p_r \Rightarrow e_r$ by assumption
(11) $e' = [\theta](e_r)$ by assumption
(12) $e_1 \rhd p_r \dashv\vdash \theta$ by assumption

By rule induction over Rules (22) on (7).

Case (22a)**.**

(13) $\xi_{rest} = \xi_r$ by assumption
(14) $\cdot \,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
(15) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$ by Inversion of Rule (21a) on (14)
(16) $\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (14)
(17) $\theta : \Gamma_r$ by Lemma 3.0.7 on (4) and (15) and (12)
(18) $\cdot \,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (16) and (17)

Case (22b)**.**

(13) $\xi_{rest} = \xi_r \vee \xi_{rs}$ by assumption
(14) $\cdot \,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
(15) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$ by assumption
(16) $\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by assumption
(17) $\theta : \Gamma_r$ by Lemma 3.0.7 on (4) and (15) and (12)
(18) $\cdot \,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (16) and (17)

Case (32m)**.**

(10) $r = p_r \Rightarrow e_r$ by assumption
(11) $rs_{post} = r' \mid rs'$ by assumption
(12) $e' = \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\}$ by assumption
(13) $e_1 \perp p_r$ by assumption

By rule induction over Rules (22) on (7).

Case (22a)**.** Syntactic contradiction of rs_{post}.

Case (22b)**.**

(14)	$\xi_{rest} = \xi_r \vee \xi_{post}$	by assumption
(15)	$\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$	by assumption
(16)	$\cdot\,; \Delta \vdash [\bot \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$	by assumption
(17)	$\xi_r \not\models \xi_{pre}$	by assumption
(18)	$p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$	by Inversion of Rule (21a) on (15)
(19)	$\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$	by Inversion of Rule (21a) on (15)
(20)	$\xi_r : \tau_1$	by Lemma 3.0.2 on (15)
(21)	$\xi_{pre} : \tau_1$	by Lemma 3.0.3 on (6)
(22)	$\xi_r \not\models \bot \vee \xi_{pre}$	by Lemma 2.0.6 on (20) and (21) and (17)
(23)	$\cdot\,; \Delta \vdash [\bot](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$	by Lemma 3.0.4 on (6) and (15) and (22)
(24)	$e_1 \not\models^{\dagger}_{?} \xi_r$	by Lemma 4.0.11 on (4) and (5) and (18) and (13)
(25)	$e_1 \not\models^{\dagger}_{?} \xi_{pre} \vee \xi_r$	by Lemma 2.0.7 on (8) and (24)
(26)	$\cdot\,; \Delta \vdash \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\} : \tau$	by Rule (19m) on (4) and (5) and (23) and (16) and (25) and (9)

□

Theorem 5.2 (Progress)**.** *If* $\cdot\,; \Delta \vdash e : \tau$ *then either* e $\texttt{final}$ *or* $e \mapsto e'$ *for some* e'*.*

Proof. By rule induction over Rules (19) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (19l)**.**

(1)	$\cdot\,; \Delta \vdash \texttt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$	by assumption
(2)	$\cdot\,; \Delta \vdash e_1 : \tau_1$	by assumption
(3)	$\cdot\,; \Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$	by assumption
(4)	$\top \models^{\dagger}_{?} \xi$	by assumption

By IH on (2).

Case Scrutinee takes a step.

(5) $e_1 \mapsto e_1'$ by assumption

(6) $\texttt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \texttt{match}(e_1')\{\cdot \mid r \mid rs\}$ by Rule (32k) on (5)

Case Scrutinee is final.

(5) e_1 $\texttt{final}$ by assumption

By rule induction over Rules (22) on (3).

Case (22a).

(6) $rs = \cdot$ by assumption

(7) $\xi = \xi_r$ by assumption

(8) $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (8)

(11) $e_1 \models_?^\dagger \xi_r$ by Corollary 2.1.1 on (5) and (4)

By rule induction over Rules (17) on (11).

Case (17a).

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \,?\, p_r$ by Lemma 4.0.11 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{indet}$ by Rule (24k) on (5) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{final}$ by Rule (25b) on (14)

Case (17b).

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\vert\, \theta$ by Lemma 4.0.11 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (32l) on (5) and (13)

Case (22b).

(6) $rs = r' \mid rs'$ by assumption

(7) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(8) $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (8)

By Lemma 4.0.10 on (2) and (5) and (10).

Case Scrutinee matches pattern.

(11) $e_1 \rhd p_r \dashv\vert\, \theta$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$ by Rule (32l) on (5) and (11)

Case Scrutinee may matches pattern.

(11) $e_1\,?\,p_r$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}\ \texttt{indet}$ by Rule (24k) on (5) and (11)

(13) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}\ \texttt{final}$ by Rule (25b) on (12)

Case Scrutinee doesn't matche pattern.

(11) $e_1 \perp p_r$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$
$\mapsto \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by Rule (32m) on (5) and (11)

Case (19m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(4) $e_1\ \texttt{final}$ by assumption

(5) $\cdot\,;\Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(6) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(7) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (22) on (5).

Case (22a).

(5) $rs_{post} = \cdot$ by assumption

(6) $\xi_{rest} = \xi_r$ by assumption

(7) $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(8) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (7)

(9) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (7)

(10) $e_1 \models^{\dagger}_{?} \xi_{pre} \vee \xi_r$ by Corollary 2.1.1 on (4) and (7)

(11) $e_1 \models^{\dagger}_{?} \xi_r$ by Lemma 2.0.8 on (10) and (6)

By rule induction over Rules (17) on (11).

Case (17a)**.**

(12) $e_1 \models_{?} \xi_r$ by assumption

(13) $e_1 \,?\, p_r$ by Lemma 4.0.11 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{indet}$ by Rule (24k) on (4) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{final}$ by Rule (25b) on (14)

Case (17b)**.**

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\vert\, \theta$ by Lemma 4.0.11 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (32l) on (4) and (13)

Case (22b)**.**

(5) $rs_{post} = r' \mid rs'_{post}$ by assumption

(6) $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(7) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (6)

(8) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r\,;\Delta_r$ by Inversion of Rule (21a) on (6)

By Lemma 4.0.10 on (3) and (4) and (8).

Case Scrutinee matches pattern**.**

(9) $e_1 \rhd p_r \dashv\vert\, \theta$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$ by Rule (32l) on (4) and (9)

Case Scrutinee may matches pattern.

(9) $e_1 \;?\; p_r$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}\ \mathtt{indet}$ by Rule (24k) on (4) and (9)

(11) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}\ \mathtt{final}$ by Rule (25b) on (10)

Case Scrutinee doesn't matche pattern.

(9) $e_1 \perp p_r$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\}$
$\mapsto \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$ by Rule (32m) on (4) and (9)

$\square$

6 Decidability

$\boxed{\Xi\ \mathtt{incon}}$ A finite set of constraints, Ξ, is inconsistent

$$\text{CINCTruth}\quad \frac{\Xi\ \mathtt{incon}}{\Xi, \top\ \mathtt{incon}} \tag{33a}$$

$$\text{CINCFalsity}\quad \frac{}{\Xi, \bot\ \mathtt{incon}} \tag{33b}$$

$$\text{CINCNum}\quad \frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \mathtt{incon}} \tag{33c}$$

$$\text{CINCNotNum}\quad \frac{}{\Xi, \underline{n}, \cancel{\underline{n}}\ \mathtt{incon}} \tag{33d}$$

$$\text{CINCAnd}\quad \frac{\Xi, \xi_1, \xi_2\ \mathtt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \mathtt{incon}} \tag{33e}$$

$$\text{CINCOr}\quad \frac{\Xi, \xi_1\ \mathtt{incon} \qquad \Xi, \xi_2\ \mathtt{incon}}{\Xi, \xi_1 \vee \xi_2\ \mathtt{incon}} \tag{33f}$$

CINCInj

$$\frac{}{\Xi, \mathtt{inl}(\xi_1), \mathtt{inr}(\xi_2)\ \mathtt{incon}} \tag{33g}$$

CINCInl

$$\frac{\Xi\ \mathtt{incon}}{\mathtt{inl}(\Xi)\ \mathtt{incon}} \tag{33h}$$

CINCInr

$$\frac{\Xi\ \mathtt{incon}}{\mathtt{inr}(\Xi)\ \mathtt{incon}} \tag{33i}$$

CINCPairL

$$\frac{\Xi_1\ \mathtt{incon}}{(\Xi_1, \Xi_2)\ \mathtt{incon}} \tag{33j}$$

CINCPairR

$$\frac{\Xi_2\ \mathtt{incon}}{(\Xi_1, \Xi_2)\ \mathtt{incon}} \tag{33k}$$

Lemma 6.0.1 (Decidability of Inconsistency)**.** *Suppose* $\dot{\top}(\xi) = \xi$. *It is decidable whether* ξ $\mathtt{incon}$.

Lemma 6.0.2 (Inconsistency and Entailment of Constraint)**.** *Suppose that* $\dot{\top}(\xi) = \xi$. *Then* $\overline{\xi}$ $\mathtt{incon}$ *iff* $\top \models \xi$

Lemma 6.0.3. *If* $e \models \xi$ *then* $e \models \dot{\top}(\xi)$

Proof. By rule induction over Rules (14), it is obvious to see that $\dot{\top}(\xi) = \xi$. $\square$

Lemma 6.0.4. *If* $e \models_? \xi$ *then* $e \models_?^{\dagger} \dot{\top}(\xi)$.

Proof.

(11) $e \models_? \xi$ by assumption

By Rule Induction over Rules (16) on (11).

Case (16a)**.**

(12) $\xi = ?$ by assumption

(13) $e \models \top$ by Rule (14a)

(14) $e \models_?^{\dagger} \top$ by Rule (17b) on (13)

Case (16b)**.**

(12) e $\mathtt{notintro}$ by assumption

(13) ξ $\mathtt{refutable}$ by assumption

By Lemma 2.0.15 on (12) and (13) and case analysis on its conclusion. By rule induction over Rules (10).

Case $\dot{\top}(\xi)$ `refutable`.

(14)	$\dot{\top}(\xi)$ `refutable`	by assumption
(15)	$e \models_? \dot{\top}(\xi)$	by Rule (16b) on (12) and (14)
(16)	$e \models_?^\dagger \dot{\top}(\xi)$	by Rule (17b) on (15)

Case $e \models \dot{\top}(\xi)$.

(14)	$e \models \dot{\top}(\xi)$	by assumption
(15)	$e \models_?^\dagger \top$	by Rule (17b) on (14)

Case (16c).

(12)	$\xi = \xi_1 \vee \xi_2$	by assumption
(13)	$e \models_? \xi_1$	by assumption
(14)	$e \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (13)
(15)	$e \models_?^\dagger \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 2.0.10 on (14)

Case (16d).

(12)	$\xi = \xi_1 \vee \xi_2$	by assumption
(13)	$e \models_? \xi_2$	by assumption
(14)	$e \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (13)
(15)	$e \models_?^\dagger \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 2.0.10 on (14)

Case (16e).

(12)	$e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(13)	$\xi = \mathtt{inl}(\xi_1)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (14)
(16)	$\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\dot{\top}(\xi_1))$	by Lemma 2.0.11 on (15)

Case (16f).

(12)	$e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(13)	$\xi = \mathtt{inr}(\xi_2)$	by assumption

(14) $e_2 \models_? \xi_2$	by assumption
(15) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (14)
(16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\dot{\top}(\xi_2))$	by Lemma 2.0.12 on (15)

Case (16g).

(12) $e = (e_1, e_2)$	by assumption
(13) $\xi = (\xi_1, \xi_2)$	by assumption
(14) $e_1 \models_? \xi_1$	by assumption
(15) $e_2 \models \xi_2$	by assumption
(16) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (14)
(17) $e_2 \models \dot{\top}(\xi_2)$	by Lemma 6.0.3 on (15)
(18) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by Rule (17b) on (17)
(19) $(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (16) and (18)

Case (16h).

(12) $e = (e_1, e_2)$	by assumption
(13) $\xi = (\xi_1, \xi_2)$	by assumption
(14) $e_1 \models \xi_1$	by assumption
(15) $e_2 \models_? \xi_2$	by assumption
(16) $e_1 \models \dot{\top}(\xi_1)$	by Lemma 6.0.3 on (14)
(17) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by Rule (17b) on (16)
(18) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (15)
(19) $(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (17) and (18)

Case (16i).

(12) $e = (e_1, e_2)$	by assumption
(13) $\xi = (\xi_1, \xi_2)$	by assumption
(14) $e_1 \models_? \xi_1$	by assumption
(15) $e_2 \models_? \xi_2$	by assumption

(16) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ — by IH on (14)

(17) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ — by IH on (15)

(18) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ — by Lemma 2.0.13 on (16) and (17)

$\square$

Lemma 6.0.5. $e \models^{\dagger}_{?} \xi$ *iff* $e \models^{\dagger}_{?} \dot{\top}(\xi)$

Proof. 1. Sufficiency:

(1) $e \models^{\dagger}_{?} \xi$ — by assumption

By rule induction over Rules (17) on (1)

Case (17b)**.**

(2) $e \models \xi$ — by assumption

(3) $e \models \dot{\top}(\xi)$ — by Lemma 6.0.3 on (2)

(4) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ — by Rule (17b) on (3)

Case (17a)**.**

(2) $e \models_{?} \xi$ — by assumption

(3) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ — by Lemma 6.0.4 on (2)

2. Necessity:

(1) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ — by assumption

By structural induction on ξ,

Case $\xi = \top, \bot, \underline{n}, \cancel{\underline{n}}$**.**

(2) $e \models^{\dagger}_{?} \xi$ — by (1) and Definition 12

Case $\xi = ?$**.**

(2) $e \models_{?} ?$ — by Rule (16a)

(3) $e \models^{\dagger}_{?} ?$ — by Rule (17a) on (2)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ — by Definition 12

By rule induction over Rules (17) on (1),

Case (17b).

(3) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ — by assumption

By rule induction over Rules (14) on (3) and two cases apply,

Case (14e).

(4) $e \models \dot{\top}(\xi_1)$ — by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_1)$ — by Rule (17b) on (4)

(6) $e \models^{\dagger}_{?} \xi_1$ — by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ — by Lemma 2.0.10 on (6)

Case (14f).

(4) $e \models \dot{\top}(\xi_2)$ — by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_2)$ — by Rule (17b) on (4)

(6) $e \models^{\dagger}_{?} \xi_2$ — by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ — by Lemma 2.0.10 on (6)

Case (17a).

(3) $e \models_{?} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ — by assumption

By rule induction over Rules (16) on (3) and two cases apply,

Case (16c).

(4) $e \models_{?} \dot{\top}(\xi_1)$ — by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_1)$ — by Rule (17a) on (4)

(6) $e \models^{\dagger}_{?} \xi_1$ — by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ — by Lemma 2.0.10 on (6)

Case (16d).

(4) $e \models_{?} \dot{\top}(\xi_2)$ — by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_2)$ — by Rule (17a) on (4)

(6) $e \models^{\dagger}_{?} \xi_2$ — by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ — by Lemma 2.0.10 on (6)

Case $\xi = \texttt{inl}(\xi_1)$.

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ — by assumption

(3) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ — by assumption

By rule induction over Rules (17) on (1),

Case (17b).

(4) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (14) and only one case applies,

Case (14g).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (17b) on (5)

(7) $e_1 \models_?^\dagger \xi_1$ by IH on (6)

(8) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.11 on (7)

Case (17a).

(4) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (16) and only one case applies,

Case (16e).

(5) $e_1 \models_? \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (17a) on (5)

(7) $e_1 \models_?^\dagger \xi_1$ by IH on (6)

(8) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.11 on (7)

Case $\xi = \texttt{inr}(\xi_2)$.

(2) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (17) on (1),

Case (17b).

(4) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (14) and only one case applies,

Case (14h).

(5) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (17b) on (5)

(7) $e_2 \models_?^\dagger \xi_2$ by IH on (6)

(8) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$ by Lemma 2.0.12 on (7)

Case (17a).

(4) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (16) and only one case applies,

Case (16f).

(5) $e_2 \models_? \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (17a) on (5)

(7) $e_2 \models_?^\dagger \xi_2$ by IH on (6)

(8) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ — by Lemma 2.0.12 on (7)

Case $\xi = (\xi_1, \xi_2)$**.**

(2) $e = (e_1, e_2)$ — by assumption

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ — by Definition 12

By rule induction over Rules (17) on (1),

Case (17b)**.**

(4) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ — by assumption

By rule induction over Rules (14) on (4) and only one case applies,

Case (14i)**.**

(5) $e_1 \models \dot{\top}(\xi_1)$ — by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ — by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ — by Rule (17b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ — by Rule (17b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ — by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ — by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ — by Lemma 2.0.13 on (9) and (10)

Case (17a)**.**

(4) $(e_1, e_2) \models_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ — by assumption

By rule induction over Rules (16) on (4) and three cases apply,

Case (16g)**.**

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ — by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ — by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ — by Rule (17a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ — by Rule (17b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ — by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ — by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ — by Lemma 2.0.13 on (9) and (10)

Case (16h)**.**

(5) $e_1 \models \dot{\top}(\xi_1)$ — by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ — by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ — by Rule (17b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ — by Rule (17a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ — by IH on (7)

(10) $e_2 \models^\dagger_? \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^\dagger_? (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

Case (16i)**.**

(5) $e_1 \models_? \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_? \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^\dagger_? \dot{\top}(\xi_1)$ by Rule (17a) on (5)

(8) $e_2 \models^\dagger_? \dot{\top}(\xi_2)$ by Rule (17a) on (6)

(9) $e_1 \models^\dagger_? \xi_1$ by IH on (7)

(10) $e_2 \models^\dagger_? \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^\dagger_? (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

□

Lemma 6.0.6. *Assume* $\dot{\top}(\xi) = \xi$. *Then* $\top \models^\dagger_? \xi$ *iff* $\top \models \xi$.

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

2. Necessity:

□

Theorem 6.1. $\top \models^\dagger_? \xi$ *iff* $\top \models \dot{\top}(\xi)$.

Lemma 6.1.1. *Assume that* e `val`. *Then* $e \models^\dagger_? \xi$ *iff* $e \models \dot{\top}(\xi)$

Proof.

(1) e `val` by assumption

We prove sufficiency and necessity separately.

1. Sufficiency:

   (2) $e \models^\dagger_? \xi$ by assumption

   By rule induction over Rules (17) on (2).

   Case (17b)**.**

   (3) $e \models \xi$ by assumption

   (4) $e \models \dot{\top}(\xi)$ by Lemma 6.0.3 on (3)

   Case (17a)**.**

   (3) $e \models_? \xi$ by assumption

By rule induction over Rules (16) on (3).

Case (16a)**.**

(4) $\xi = ?$ by assumption

(5) $e \models \dot{\top}(\xi)$ by Rule (14a) and Definition 12

Case (16b)**.**

(4) e `notintro` by assumption

By rule induction over Rules (26) on (4), for each case, by rule induction over Rules (23) on (1), no case applies due to syntactic contradiction.

Case (16c)**.**

(4) $\xi = \xi_1 \vee \xi_2$ by assumption

(5) $e \models_? \xi_1$ by assumption

(6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Equation 12

(7) $e \models_?^\dagger \xi_1$ by Rule (17a) on (5)

(8) $e \models \dot{\top}(\xi_1)$ by IH on (7)

(9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Rule (14e) on (8)

Case (16d)**.**

(4) $\xi = \xi_1 \vee \xi_2$ by assumption

(5) $e \models_? \xi_2$ by assumption

(6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Equation 12

(7) $e \models_?^\dagger \xi_2$ by Rule (17a) on (5)

(8) $e \models \dot{\top}(\xi_2)$ by IH on (7)

(9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Rule (14f) on (8)

Case (16e)**.**

(4) $\xi = \texttt{inl}(\xi_1)$ by assumption

(5) $e \models_? \xi_1$ by assumption

(6) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ by Equation 12

(7) $e \models_?^\dagger \xi_1$ by Rule (17a) on (5)

(8) $e \models \dot{\top}(\xi_1)$ by IH on (7)

(9) $e \models \texttt{inl}(\dot{\top}(\xi_1))$ by Rule (14g) on (8)

Case (16f)**.**

(4) $\xi = \texttt{inr}(\xi_2)$ by assumption

(5) $e \models_? \xi_2$ by assumption

(6) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$ by Equation 12

(7) $e \models_?^\dagger \xi_2$ by Rule (17a) on (5)

(8) $e \models \dot{\top}(\xi_2)$	by IH on (7)
(9) $e \models \texttt{inr}(\dot{\top}(\xi_2))$	by Rule (14h) on (8)

Case (16g)**.**

(4) $e = (e_1, e_2)$	by assumption
(5) $\xi = (\xi_1, \xi_2)$	by assumption
(6) $e_1 \models_? \xi_1$	by assumption
(7) $e_2 \models \xi_2$	by assumption
(8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12
(9) $e_1 \models_?^{\dagger} \xi_1$	by Rule (17a) on (6)
(10) $e_2 \models_?^{\dagger} \xi_2$	by Rule (17b) on (7)
(11) $e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12) $e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (14i) on (11) and (12)

Case (16h)**.**

(4) $e = (e_1, e_2)$	by assumption
(5) $\xi = (\xi_1, \xi_2)$	by assumption
(6) $e_1 \models \xi_1$	by assumption
(7) $e_2 \models_? \xi_2$	by assumption
(8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12
(9) $e_1 \models_?^{\dagger} \xi_1$	by Rule (17b) on (6)
(10) $e_2 \models_?^{\dagger} \xi_2$	by Rule (17a) on (7)
(11) $e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12) $e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (14i) on (11) and (12)

Case (16i)**.**

(4) $e = (e_1, e_2)$	by assumption
(5) $\xi = (\xi_1, \xi_2)$	by assumption
(6) $e_1 \models_? \xi_1$	by assumption
(7) $e_2 \models_? \xi_2$	by assumption
(8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12
(9) $e_1 \models_?^{\dagger} \xi_1$	by Rule (17a) on (6)
(10) $e_2 \models_?^{\dagger} \xi_2$	by Rule (17a) on (7)
(11) $e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12) $e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (14i) on (11) and (12)

2. Necessity:

(2) $e \models \dot{\top}(\xi)$ by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\cancel{n}}$**.**

(3) $\xi = \dot{\top}(\xi)$ by Equation 12
(4) $e \models^{\dagger}_{?} \xi$ by Rule (17b) on (2)

Case $\xi = ?$**.**

(3) $e \models_{?} ?$ by Rule (16a)
(4) $e \models^{\dagger}_{?} ?$ by Rule (17a) on (3)

Case $\xi = \xi_1 \wedge \xi_2$**.**

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by Equation 12

By rule induction over Rules (14) on (2), only one case applies.

Case (14d)**.**

(4) $e \models \dot{\top}(\xi_1)$ by assumption
(5) $e \models \dot{\top}(\xi_2)$ by assumption
(6) $e \models^{\dagger}_{?} \xi_1$ by IH on (4)
(7) $e \models^{\dagger}_{?} \xi_2$ by IH on (5)
(8) $e \models \xi_1 \wedge \xi_2$ by Lemma 2.0.9 on (6) and (7)

Case $\xi = \xi_1 \vee \xi_2$**.**

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Equation 12

By rule induction over Rules (14) on (2) and only two cases apply.

Case (14e)**.**

(4) $e \models \dot{\top}(\xi_1)$ by assumption
(5) $e \models^{\dagger}_{?} \xi_1$ by IH on (4)
(6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (5)

Case (14f)**.**

(4) $e \models \dot{\top}(\xi_2)$ by assumption
(5) $e \models^{\dagger}_{?} \xi_2$ by IH on (4)

(6) $e \models^{\dagger}_{?} \xi_1 \lor \xi_2$	by Lemma 2.0.10 on (5)

Case $\xi = \mathtt{inl}(\xi_1)$.

(3) $\dot{\top}(\xi) = \mathtt{inl}(\dot{\top}(\xi_1))$	by Equation 12

By rule induction over Rules (14) on (2) and only one case applies.

Case (14g).

(4) $e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(5) $e_1 \models \dot{\top}(\xi_1)$	by assumption
(6) $e_1 \models^{\dagger}_{?} \xi_1$	by IH on (5)
(7) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\xi_1)$	by Lemma 2.0.11 on (6)

Case $\xi = \mathtt{inr}(\xi_2)$.

(3) $\dot{\top}(\xi) = \mathtt{inr}(\dot{\top}(\xi_2))$	by Equation 12

By rule induction over Rules (14) on (2) and only one case applies.

Case (14h).

(4) $e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(5) $e_2 \models \dot{\top}(\xi_2)$	by assumption
(6) $e_2 \models^{\dagger}_{?} \xi_2$	by IH on (5)
(7) $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inr}(\xi_2)$	by Lemma 2.0.12 on (6)

Case $\xi = (\xi_1, \xi_2)$.

(3) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12

By rule induction over Rules (14) on (2) and only one case applies.

Case (14i).

(4) $e = (e_1, e_2)$	by assumption
(5) $e_1 \models \dot{\bot}(\xi_1)$	by assumption
(6) $e_2 \models \dot{\bot}(\xi_2)$	by assumption
(7) $e_1 \models^{\dagger}_{?} \xi_1$	by IH on (5)
(8) $e_2 \models^{\dagger}_{?} \xi_2$	by IH on (6)
(9) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$	by Lemma 2.0.13 on (7) and (8)

□

Lemma 6.1.2. *$e \models \xi$ iff $e \models \dot{\bot}(\xi)$*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models \xi$	by assumption

By rule induction over Rules (14) on (1).

Case (14a).

(2) $\xi = \top$	by assumption
(3) $e \models \dot{\bot}(\top)$	by (1) and Definition 13

Case (14b).

(2) $\xi = \underline{n}$	by assumption
(3) $e \models \dot{\bot}(\underline{n})$	by (1) and Definition 13

Case (14c).

(2) $\xi = \cancel{\underline{n}}$	by assumption
(3) $e \models \dot{\bot}(\cancel{\underline{n}})$	by (1) and Definition 13

Case (14d).

(2) $\xi = \xi_1 \wedge \xi_2$	by assumption
(3) $e \models \xi_1$	by assumption
(4) $e \models \xi_2$	by assumption
(5) $e \models \dot{\bot}(\xi_1)$	by IH on (3)
(6) $e \models \dot{\bot}(\xi_2)$	by IH on (4)
(7) $e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$	by Rule (14d) on (5) and (6)
(8) $e \models \dot{\bot}(\xi_1 \wedge \xi_2)$	by (7) and Definition 13

Case (14e).

(2) $\xi = \xi_1 \vee \xi_2$	by assumption
(3) $e \models \xi_1$	by assumption
(4) $e \models \dot{\bot}(\xi_1)$	by IH on (3)
(5) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (14e) on (4)
(6) $e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 13

Case (14f).

(2) $\xi = \xi_1 \vee \xi_2$	by assumption
(3) $e \models \xi_2$	by assumption
(4) $e \models \dot{\bot}(\xi_2)$	by IH on (3)
(5) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (14f) on (4)

(6) $e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 13

Case (14g).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$	by assumption
(3) $\xi = \texttt{inl}(\xi_1)$	by assumption
(4) $e_1 \models \xi_1$	by assumption
(5) $e_1 \models \dot{\bot}(\xi_1)$	by IH on (4)
(6) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\bot}(\xi_1))$	by Rule (14g) on (5)
(7) $\texttt{inl}_{\tau_2}(e_1) \models \dot{\bot}(\texttt{inl}(\xi_1))$	by (6) and Definition 13

Case (14h).

(2) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption
(3) $\xi = \texttt{inr}(\xi_2)$	by assumption
(4) $e_2 \models \xi_2$	by assumption
(5) $e_2 \models \dot{\bot}(\xi_2)$	by IH on (4)
(6) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\bot}(\xi_2))$	by Rule (14h) on (5)
(7) $\texttt{inr}_{\tau_1}(e_2) \models \dot{\bot}(\texttt{inr}(\xi_2))$	by (6) and Definition 13

Case (14i).

(2) $e = (e_1, e_2)$	by assumption
(3) $\xi = (\xi_1, \xi_2)$	by assumption
(4) $e_1 \models \xi_1$	by assumption
(5) $e_2 \models \xi_2$	by assumption
(6) $e_1 \models \dot{\bot}(\xi_1)$	by IH on (4)
(7) $e_2 \models \dot{\bot}(\xi_2)$	by IH on (5)
(8) $(e_1, e_2) \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$	by Rule (14i) on (6) and (7)
(9) $(e_1, e_2) \models \dot{\bot}((\xi_1, \xi_2))$	by (8) and Definition 13

2. Necessity:

(1) $e \models \dot{\bot}(\xi)$	by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\not n}$**.**

(2) $e \models \xi$	by (1) and Definition 13

Case $\xi = ?$**.**

(2) $e \models \bot$ by (1) and Definition 13

(3) $e \not\models \bot$ by Lemma 2.0.1

(3) contradicts (2).

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only case applies.

Case (14d)**.**

(3) $e \models \dot{\bot}(\xi_1)$ by assumption

(4) $e \models \dot{\bot}(\xi_2)$ by assumption

(5) $e \models \xi_1$ by IH on (3)

(6) $e \models \xi_2$ by IH on (4)

(7) $e \models \xi_1 \wedge \xi_2$ by Rule (14d) on (5) and (6)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only two cases apply.

Case (14e)**.**

(3) $e \models \dot{\bot}(\xi_1)$ by assumption

(4) $e \models \xi_1$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (4)

Case (14f)**.**

(3) $e \models \dot{\bot}(\xi_2)$ by assumption

(4) $e \models \xi_2$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

Case $\xi = \texttt{inl}(\xi_1)$**.**

(2) $e \models \texttt{inl}(\dot{\bot}(\xi_1))$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only one case applies.

Case (14g)**.**

(3) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(4) $e_1 \models \dot{\bot}(\xi_1)$ by assumption

(5) $e_1 \models \xi_1$ by IH on (4)

(6) $e \models \texttt{inl}(\xi_1)$ by Rule (14g) on (5)

Case $\xi = \texttt{inr}(\xi_2)$**.**

(2) $e \models \texttt{inr}(\dot{\bot}(\xi_2))$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only one case applies.

Case (14h)**.**

(3) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption
(4) $e_2 \models \dot{\bot}(\xi_2)$ by assumption
(5) $e_2 \models \xi_2$ by IH on (4)
(6) $e \models \texttt{inr}(\xi_2)$ by Rule (14h) on (5)

Case $\xi = (\xi_1, \xi_2)$**.**

(2) $e \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only case applies.

Case (14i)**.**

(3) $e = (e_1, e_2)$ by assumption
(4) $e_1 \models \dot{\bot}(\xi_1)$ by assumption
(5) $e_2 \models \dot{\bot}(\xi_2)$ by assumption
(6) $e_1 \models \xi_1$ by IH on (4)
(7) $e_2 \models \xi_2$ by IH on (5)
(8) $e \models (\xi_1, \xi_2)$ by Rule (14i) on (6) and (7)

$\square$

Lemma 6.1.3. *Assume* e `val` *and* $\dot{\top}(\xi) = \xi$. *Then* $e \not\models \xi$ *iff* $e \models \overline{\xi}$.

Theorem 6.2. $\xi_r \models \xi_{rs}$ *iff* $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\bot}(\xi_{rs})$.