

# 1 Match Constraint Language

$\dot{\xi} ::= \top \mid ? \mid \underline{n} \mid \text{inl}(\dot{\xi}) \mid \text{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$   
 $\boxed{\dot{\xi} : \tau}$      $\dot{\xi}$  constrains final expressions of type  $\tau$

$$\frac{\text{CTTruth}}{\top : \tau} \quad (1a)$$

$$\frac{\text{CTUnknown}}{? : \tau} \quad (1b)$$

$$\frac{\text{CTNum}}{\underline{n} : \text{num}} \quad (1c)$$

$$\frac{\text{CTInl} \quad \dot{\xi}_1 : \tau_1}{\text{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)} \quad (1d)$$

$$\frac{\text{CTInr} \quad \dot{\xi}_2 : \tau_2}{\text{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)} \quad (1e)$$

$$\frac{\text{CTPair} \quad \dot{\xi}_1 : \tau_1 \quad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)} \quad (1f)$$

$$\frac{\text{CTOr} \quad \dot{\xi}_1 : \tau \quad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau} \quad (1g)$$

$\boxed{\dot{\xi} \text{ refutable?}}$      $\dot{\xi}$  is refutable

$$\frac{\text{RXNum}}{\underline{n} \text{ refutable?}} \quad (2a)$$

$$\frac{\text{RXUnknown}}{? \text{ refutable?}} \quad (2b)$$

$$\frac{\text{RXInl}}{\text{inl}(\dot{\xi}) \text{ refutable?}} \quad (2c)$$

$$\frac{\text{RXInr}}{\text{inr}(\dot{\xi}) \text{ refutable?}} \quad (2d)$$

$$\frac{\text{RXPairL} \quad \dot{\xi}_1 \text{ refutable?}}{(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}} \quad (2e)$$

$$\frac{\text{RXPairR} \quad \dot{\xi}_2 \text{ refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_?} \quad (2f)$$

$$\frac{\text{RXOr} \quad \dot{\xi}_1 \text{ refutable}_? \quad \dot{\xi}_2 \text{ refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2 \text{ refutable}_?} \quad (2g)$$

$$\boxed{\text{refutable}_?(\dot{\xi})}$$

$$\text{refutable}_?(T) = \text{false} \quad (3a)$$

$$\text{refutable}_?(\underline{n}) = \text{true} \quad (3b)$$

$$\text{refutable}_?(?) = \text{true} \quad (3c)$$

$$\text{refutable}_?(\text{inl}(\dot{\xi})) = \text{true} \quad (3d)$$

$$\text{refutable}_?(\text{inr}(\dot{\xi})) = \text{true} \quad (3e)$$

$$\text{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{refutable}_?(\dot{\xi}_1) \text{ or } \text{refutable}_?(\dot{\xi}_2) \quad (3f)$$

$$\text{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \text{refutable}_?(\dot{\xi}_1) \text{ and } \text{refutable}_?(\dot{\xi}_2) \quad (3g)$$

**Lemma 1.0.1** (Soundness and Completeness of Refutable Constraints).  $\dot{\xi} \text{ refutable}_?$  iff  $\text{refutable}_?(\dot{\xi}) = \text{true}$ .

$$\boxed{e \models \dot{\xi}} \quad e \text{ satisfies } \dot{\xi}$$

$$\frac{\text{CSTruth}}{e \models T} \quad (4a)$$

$$\frac{\text{CSNum}}{\underline{n} \models \underline{n}} \quad (4b)$$

$$\frac{\text{CSInl} \quad e_1 \models \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)} \quad (4c)$$

$$\frac{\text{CSInr} \quad e_2 \models \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)} \quad (4d)$$

$$\frac{\text{CSPair} \quad e_1 \models \dot{\xi}_1 \quad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)} \quad (4e)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \dot{\xi}_1 \quad \text{prl}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)} \quad (4f)$$

$$\frac{\text{CSOrL} \quad e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \quad (4g)$$

$$\frac{\text{CSOrR} \quad e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \quad (4h)$$

$$\boxed{\text{satisfy}(e, \dot{\xi})}$$

$$\text{satisfy}(e, \top) = \text{true} \quad (5a)$$

$$\text{satisfy}(n_1, n_2) = (n_1 = n_2) \quad (5b)$$

$$\text{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{satisfy}(e, \dot{\xi}_1) \text{ or } \text{satisfy}(e, \dot{\xi}_2) \quad (5c)$$

$$\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{satisfy}(e_1, \dot{\xi}_1) \quad (5d)$$

$$\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = \text{satisfy}(e_2, \dot{\xi}_2) \quad (5e)$$

$$\text{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(e_1, \dot{\xi}_1) \text{ and } \text{satisfy}(e_2, \dot{\xi}_2) \quad (5f)$$

$$\text{satisfy}(\text{p}^u, (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(\text{prl}(\text{p}^u), \dot{\xi}_1) \text{ and } \text{satisfy}(\text{pr}(\text{p}^u), \dot{\xi}_2) \quad (5g)$$

$$\text{satisfy}(\text{p}(e)^u, (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(\text{prl}(\text{p}(e)^u), \dot{\xi}_1) \text{ and } \text{satisfy}(\text{pr}(\text{p}(e)^u), \dot{\xi}_2) \quad (5h)$$

$$\begin{aligned} \text{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(e_1(e_2)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(e_1(e_2)), \dot{\xi}_2) \end{aligned} \quad (5i)$$

$$\begin{aligned} \text{satisfy}(\text{match}(e)\{rs\}, (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{match}(e)\{rs\}), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{match}(e)\{rs\}), \dot{\xi}_2) \end{aligned} \quad (5j)$$

$$\begin{aligned} \text{satisfy}(\text{prl}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{prl}(e)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{prl}(e)), \dot{\xi}_2) \end{aligned} \quad (5k)$$

$$\begin{aligned} \text{satisfy}(\text{pr}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{pr}(e)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{pr}(e)), \dot{\xi}_2) \end{aligned} \quad (5l)$$

$$\text{Otherwise } \text{satisfy}(e, \dot{\xi}) = \text{false} \quad (5m)$$

**Lemma 1.0.2** (Soundness and Completeness of Satisfaction Judgment).  $e \models \dot{\xi}$  iff  $\text{satisfy}(e, \dot{\xi}) = \text{true}$ .

$$\boxed{e \models_{\tau} \dot{\xi}} \quad e \text{ may satisfy } \dot{\xi}$$

$$\frac{\text{CMSUnknown}}{e \models_{\tau} ?} \quad (6a)$$

$$\frac{\text{CMSInl} \quad e_1 \models_{\tau} \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)} \quad (6b)$$

$$\frac{\text{CMSInr} \quad e_2 \models_{\tau} \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)} \quad (6c)$$

$$\frac{\text{CMSPairL} \quad e_1 \models_{\tau} \dot{\xi}_1 \quad e_2 \models_{\tau} \dot{\xi}_2}{(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6d)$$

$$\frac{\text{CMSPairR} \quad e_1 \models_{\tau} \dot{\xi}_1 \quad e_2 \models_{\tau} \dot{\xi}_2}{(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6e)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{\tau} \dot{\xi}_1 \quad e_2 \models_{\tau} \dot{\xi}_2}{(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6f)$$

$$\frac{\text{CMSOrL} \quad e \models_{\tau} \dot{\xi}_1 \quad e \not\models_{\tau} \dot{\xi}_2}{e \models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2} \quad (6g)$$

$$\frac{\text{CMSOrR} \quad e \not\models_{\tau} \dot{\xi}_1 \quad e \models_{\tau} \dot{\xi}_2}{e \models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2} \quad (6h)$$

$$\frac{\text{CMSNotIntro} \quad e \text{ notintro} \quad \dot{\xi} \text{ refutable}_{\tau}}{e \models_{\tau} \dot{\xi}} \quad (6i)$$

$$\boxed{\text{maysatisfy}(e, \dot{\xi})}$$

$$\text{maysatisfy}(e, ?) = \text{true} \quad (7a)$$

$$\text{maysatisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{maysatisfy}(e_1, \dot{\xi}_1) \quad (7b)$$

$$\text{maysatisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = \text{maysatisfy}(e_2, \dot{\xi}_2) \quad (7c)$$

$$\text{maysatisfy}(\text{inl}_{\tau_2}(e_1), \text{inr}(\dot{\xi}_2)) = \text{false} \quad (7d)$$

$$\text{maysatisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\dot{\xi}_1)) = \text{false} \quad (7e)$$

$$\begin{aligned} \text{maysatisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = & \left( \text{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \text{satisfy}(e_2, \dot{\xi}_2) \right) \\ & \text{or } \left( \text{satisfy}(e_1, \dot{\xi}_1) \text{ and } \text{maysatisfy}(e_2, \dot{\xi}_2) \right) \\ & \text{or } \left( \text{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \text{maysatisfy}(e_2, \dot{\xi}_2) \right) \end{aligned} \quad (7f)$$

$$\begin{aligned} \text{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = & \left( \text{maysatisfy}(e, \dot{\xi}_1) \text{ and } \left( \text{not } \text{satisfy}(e, \dot{\xi}_2) \right) \right) \\ & \text{or } \left( \left( \text{not } \text{satisfy}(e, \dot{\xi}_1) \right) \text{ and } \text{maysatisfy}(e, \dot{\xi}_2) \right) \end{aligned} \quad (7g)$$

$$\text{maysatisfy}(e, \dot{\xi}) = \text{notintro}(e) \text{ and } \text{refutable}_{\tau}(\dot{\xi}) \quad (7h)$$

**Lemma 1.0.3** (Soundness and Completeness of Maybe Satisfaction Judgment).  
 $e \models_{\tau} \dot{\xi}$  iff  $\text{maysatisfy}(e, \dot{\xi}) = \text{true}$ .

$$\boxed{e \models_{\tau}^{\dagger} \dot{\xi}} \quad e \text{ satisfies or may satisfy } \dot{\xi} \quad \begin{array}{c} \text{CSMSMay} \\ \frac{e \models_{\tau} \dot{\xi}}{e \models_{\tau}^{\dagger} \dot{\xi}} \end{array} \quad (8a)$$

$$\begin{array}{c} \text{CSMSSat} \\ \frac{e \models \dot{\xi}}{e \models_{\tau}^{\dagger} \dot{\xi}} \end{array} \quad (8b)$$

$$\boxed{\text{satisfyormay}(e, \dot{\xi})} \quad \text{satisfyormay}(e, \dot{\xi}) = \text{satisfy}(e, \dot{\xi}) \text{ or } \text{maysatisfy}(e, \dot{\xi}) \quad (9)$$

**Lemma 1.0.4** (Soundness and Completeness of Satisfaction or Maybe Satisfaction).  $e \models_{\tau}^{\dagger} \dot{\xi}$  iff  $\text{satisfyormay}(e, \dot{\xi})$ .

**Lemma 1.0.5.** If  $\dot{\xi} : \tau$  then there exists  $e$  such that  $e \text{ val}$  and  $\cdot ; \Delta \vdash e : \tau$  and  $e \models_{\tau}^{\dagger} \dot{\xi}$ .

**Lemma 1.0.6.**  $e \not\models_{\tau} \top$

**Lemma 1.0.7.**  $e \not\models_{\tau} ?$

**Lemma 1.0.8.**  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\tau}^{\dagger} \dot{\xi}_2$  iff  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$

**Lemma 1.0.9.** If  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  and  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1$  then  $e \models_{\tau}^{\dagger} \dot{\xi}_2$

**Lemma 1.0.10.** If  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  then  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  and  $e \models_{\tau}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$

**Lemma 1.0.11.**  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  iff  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$

**Lemma 1.0.12.**  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  iff  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$

**Lemma 1.0.13.**  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  and  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  iff  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$

**Lemma 1.0.14.** Assume  $e \text{ notintro}$ . If  $e \models_{\tau} \dot{\xi}$  or  $e \not\models \dot{\xi}$  then  $\dot{\xi} \text{ refutable}_{\tau}$ .

**Lemma 1.0.15.** If  $e \text{ notintro}$  and  $e \models \dot{\xi}$  then  $\dot{\xi} \text{ not refutable}_{\tau}$ .

**Lemma 1.0.16.**  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  is not derivable.

**Lemma 1.0.17.**  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  is not derivable.

**Lemma 1.0.18.**  $e \not\models \dot{\xi}$  and  $e \not\models_{\tau} \dot{\xi}$  iff  $e \not\models_{\tau}^{\dagger} \dot{\xi}$ .

**Theorem 1.1** (Exclusiveness of Satisfaction Judgment). *If  $\dot{\xi} : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e$  **final** then exactly one of the following holds*

1.  $e \models \dot{\xi}$
2.  $e \models_{\tau} \dot{\xi}$
3.  $e \not\models_{\tau} \dot{\xi}$

**Definition 1.1.1** (Entailment of Constraints). *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \models \dot{\xi}_2$  iff for all  $e$  such that  $\cdot; \Delta \vdash e : \tau$  and  $e$  **val** we have  $e \models_{\tau} \dot{\xi}_1$  implies  $e \models \dot{\xi}_2$*

**Definition 1.1.2** (Potential Entailment of Constraints). *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \models^{\dagger} \dot{\xi}_2$  iff for all  $e$  such that  $\cdot; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies  $e \models_{\tau}^{\dagger} \dot{\xi}_2$*

**Corollary 1.1.1.** *Suppose that  $\dot{\xi} : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e$  **final**. Then  $\top \models^{\dagger} \dot{\xi}$  implies  $e \models_{\tau}^{\dagger} \dot{\xi}$*

## 2 Normal Match Constraint Language

$\xi ::= \top \mid \perp \mid \underline{n} \mid \mathcal{H} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \text{inl}(\xi) \mid \text{inr}(\xi) \mid (\xi_1, \xi_2)$   
 $\boxed{\xi : \tau}$      $\xi$  constrains final expressions of type  $\tau$

$$\frac{\text{CTTruth}}{\top : \tau} \quad (10a)$$

$$\frac{\text{CTFalsity}}{\perp : \tau} \quad (10b)$$

$$\frac{\text{CTNum}}{\underline{n} : \text{num}} \quad (10c)$$

$$\frac{\text{CTNotNum}}{\mathcal{H} : \text{num}} \quad (10d)$$

$$\frac{\text{CTAnd} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \quad (10e)$$

$$\frac{\text{CTOr} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \quad (10f)$$

$$\frac{\text{CTInl} \quad \xi_1 : \tau_1}{\text{inl}(\xi_1) : (\tau_1 + \tau_2)} \quad (10g)$$

$$\frac{\text{CTInr} \quad \xi_2 : \tau_2}{\text{inr}(\xi_2) : (\tau_1 + \tau_2)} \quad (10h)$$

$$\frac{\text{CTPair} \quad \xi_1 : \tau_1 \quad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \quad (10i)$$

$$\boxed{\xi_1^- = \xi_2} \quad \text{dual of } \xi_1 \text{ is } \xi_2$$

$$\begin{aligned} \overline{\top} &= \perp \\ \overline{\perp} &= \top \\ \overline{\underline{n}} &= \underline{\not n} \\ \overline{\underline{\not n}} &= \underline{n} \\ \overline{\xi_1 \wedge \xi_2} &= \overline{\xi_1} \vee \overline{\xi_2} \\ \overline{\xi_1 \vee \xi_2} &= \overline{\xi_1} \wedge \overline{\xi_2} \\ \overline{\text{inl}(\xi_1)} &= \text{inl}(\overline{\xi_1}) \vee \text{inr}(\top) \\ \overline{\text{inr}(\xi_2)} &= \text{inr}(\overline{\xi_2}) \vee \text{inl}(\top) \\ \overline{(\xi_1, \xi_2)} &= (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \\ \overline{\overline{\xi}} &= \xi \end{aligned}$$

$$\boxed{e \models \xi} \quad e \text{ satisfies } \xi$$

$$\frac{\text{CSTruth}}{e \models \top} \quad (12a)$$

$$\frac{\text{CSNum}}{\underline{n} \models \underline{n}} \quad (12b)$$

$$\frac{\text{CSNotNum} \quad n_1 \neq n_2}{\underline{n_1} \models \underline{\not n_2}} \quad (12c)$$

$$\frac{\text{CSAnd} \quad e \models \xi_1 \quad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \quad (12d)$$

$$\frac{\text{CSOrL} \quad e \models \xi_1}{e \models \xi_1 \vee \xi_2} \quad (12e)$$

$$\frac{\text{CSOrR} \quad e \models \xi_2}{e \models \xi_1 \vee \xi_2} \quad (12f)$$

$$\text{CSInl} \quad \frac{e_1 \models \xi_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)} \quad (12g)$$

$$\text{CSInr} \quad \frac{e_2 \models \xi_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)} \quad (12h)$$

$$\text{CSPair} \quad \frac{e_1 \models \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \quad (12i)$$

**Lemma 2.0.1.** *Assume  $e \text{ val}$ . Then  $e \not\models \xi$  iff  $e \models \bar{\xi}$ .*

**Theorem 2.1** (Exclusiveness of Satisfaction Judgment). *If  $\xi : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e \text{ val}$  then exactly one of the following holds*

1.  $e \models \xi$
2.  $e \models \bar{\xi}$

**Lemma 2.1.1.**  $e \models \bar{\xi}_1 \vee \xi_2$  iff  $e \models \xi_2$  whenever  $e \models \xi_1$ .

**Definition 2.1.1** (Entailment of Constraints). *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \models \xi_2$  iff for all  $e$  such that  $\cdot; \Delta \vdash e : \tau$  and  $e \text{ val}$  we have  $e \models \xi_1$  implies  $e \models \xi_2$*

**Corollary 2.1.1** (Material Entailment of Complete Constraint).  $\xi_1 \models \xi_2$  iff  $\top \models \bar{\xi}_1 \vee \xi_2$ .

## 2.1 Relationship with Incomplete Constraint Language

**Lemma 2.1.2.** *Assume that  $e \text{ val}$ . Then  $e \models_{\tau}^{\dagger} \xi$  iff  $e \models \dot{\top}(\xi)$ .*

**Lemma 2.1.3.**  $e \models_{\tau}^{\dagger} \xi$  iff  $e \models \dot{\perp}(\xi)$

**Lemma 2.1.4.** *Suppose  $\xi : \tau$ . Then  $e \models_{\tau}^{\dagger} \xi$  for all  $e$  such that  $\cdot; \Delta \vdash e : \tau$  and  $e \text{ final}$  iff  $e \models_{\tau}^{\dagger} \xi$  for all  $e$  such that  $\cdot; \Delta \vdash e : \tau$  and  $e \text{ val}$ .*

**Theorem 2.2.**  $\top \models_{\tau}^{\dagger} \xi$  iff  $\top \models \dot{\top}(\xi)$ .

**Theorem 2.3.**  $\xi_1 \models_{\tau}^{\dagger} \xi_2$  iff  $\dot{\top}(\xi_1) \models \dot{\perp}(\xi_2)$ .



### 3 Static Semantics

$$\begin{aligned}
\tau &::= \mathbf{num} \mid (\tau_1 \rightarrow \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e &::= x \mid \underline{n} \\
&\quad \mid \lambda x : \tau. e \mid e_1(e_2) \\
&\quad \mid (e_1, e_2) \\
&\quad \mid \mathbf{inl}_\tau(e) \mid \mathbf{inr}_\tau(e) \mid \mathbf{match}(e)\{\hat{r}s\} \\
&\quad \mid \mathbb{0}^u \mid \mathbb{[e]}^u \\
\hat{r}s &::= (rs \mid r \mid rs) \\
rs &::= \cdot \mid (r \mid rs') \\
r &::= p \Rightarrow e \\
p &::= x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \mathbf{inl}(p) \mid \mathbf{inr}(p) \mid \mathbb{0}^w \mid \mathbb{[p]}_\tau^w \\
\boxed{(\hat{r}s)^\diamond = rs} &\quad rs \text{ can be obtained by erasing pointer from } \hat{r}s
\end{aligned}$$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \quad (13a)$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \quad (13b)$$

$$\boxed{\Gamma; \Delta \vdash e : \tau} \quad e \text{ is of type } \tau$$

$$\frac{\text{TVar}}{\Gamma, x : \tau; \Delta \vdash x : \tau} \quad (14a)$$

$$\frac{\text{TEHole}}{\Gamma; \Delta, u :: \tau \vdash \mathbb{0}^u : \tau} \quad (14b)$$

$$\frac{\text{THole} \quad \Gamma; \Delta, u :: \tau \vdash e : \tau'}{\Gamma; \Delta, u :: \tau \vdash \mathbb{[e]}^u : \tau} \quad (14c)$$

$$\frac{\text{TNum}}{\Gamma; \Delta \vdash \underline{n} : \mathbf{num}} \quad (14d)$$

$$\frac{\text{TLam} \quad \Gamma, x : \tau_1; \Delta \vdash e : \tau_2}{\Gamma; \Delta \vdash \lambda x : \tau_1. e : (\tau_1 \rightarrow \tau_2)} \quad (14e)$$

$$\frac{\text{TAp} \quad \Gamma; \Delta \vdash e_1 : (\tau_2 \rightarrow \tau) \quad \Gamma; \Delta \vdash e_2 : \tau_2}{\Gamma; \Delta \vdash e_1(e_2) : \tau} \quad (14f)$$

$$\frac{\text{TPair} \quad \Gamma; \Delta \vdash e_1 : \tau_1 \quad \Gamma; \Delta \vdash e_2 : \tau_2}{\Gamma; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \quad (14g)$$

$$\frac{\text{TPrl} \quad \Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma; \Delta \vdash \mathbf{prl}(e) : \tau_1} \quad (14h)$$

$$\frac{\text{TPrr}}{\Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)} \quad \Gamma; \Delta \vdash \text{pr}(e) : \tau_2 \quad (14i)$$

$$\frac{\text{TInl} \quad \Gamma; \Delta \vdash e : \tau_1}{\Gamma; \Delta \vdash \text{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \quad (14j)$$

$$\frac{\text{TInr} \quad \Gamma; \Delta \vdash e : \tau_2}{\Gamma; \Delta \vdash \text{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \quad (14k)$$

$$\frac{\text{TMatchZPre} \quad \Gamma; \Delta \vdash e : \tau \quad \Gamma; \Delta \vdash [\perp]r \mid rs : \tau[\xi] \Rightarrow \tau' \quad \top \models_{\tau}^{\dagger} \xi}{\Gamma; \Delta \vdash \text{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \quad (14l)$$

$$\frac{\text{TMatchNZPre} \quad \Gamma; \Delta \vdash e : \tau \quad \Gamma; \Delta \vdash [\perp]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \quad \Gamma; \Delta \vdash [\perp \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \quad \forall e'. e' \in \text{values}(e) \Rightarrow e' \not\models_{\tau}^{\dagger} \xi_{pre} \quad \top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}}{\Gamma; \Delta \vdash \text{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \quad (14m)$$

$p : \tau[\xi] \dashv \Gamma; \Delta$   $p$  is assigned type  $\tau$  and emits constraint  $\xi$

$$\frac{\text{PTVar}}{x : \tau[\top] \dashv x : \tau; \cdot} \quad (15a)$$

$$\frac{\text{PTWild}}{\_ : \tau[\top] \dashv \cdot; \cdot} \quad (15b)$$

$$\frac{\text{PTEHole}}{\text{⋈}^w : \tau[?] \dashv \cdot; w :: \tau} \quad (15c)$$

$$\frac{\text{PTHole} \quad p : \tau[\xi] \dashv \Gamma; \Delta}{\text{⋈}^w_{\tau} : \tau'[?] \dashv \Gamma; \Delta, w :: \tau'} \quad (15d)$$

$$\frac{\text{PTNum}}{\underline{n} : \text{num}[\underline{n}] \dashv \cdot; \cdot} \quad (15e)$$

$$\frac{\text{PTInl} \quad p : \tau_1[\xi] \dashv \Gamma; \Delta}{\text{inl}(p) : (\tau_1 + \tau_2)[\text{inl}(\xi)] \dashv \Gamma; \Delta} \quad (15f)$$

$$\frac{\text{PTInr} \quad p : \tau_2[\xi] \dashv \Gamma; \Delta}{\text{inr}(p) : (\tau_1 + \tau_2)[\text{inr}(\xi)] \dashv \Gamma; \Delta} \quad (15g)$$

$$\text{PTPair} \quad \frac{p_1 : \tau_1[\xi_1] \dashv \Gamma_1 ; \Delta_1 \quad p_2 : \tau_2[\xi_2] \dashv \Gamma_2 ; \Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv \Gamma_1 \uplus \Gamma_2 ; \Delta_1 \uplus \Delta_2} \quad (15h)$$

$$\boxed{\Gamma ; \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'} \quad \begin{array}{l} r \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\text{CTRrule} \quad \frac{p : \tau[\xi] \dashv \Gamma_p ; \Delta_p \quad \Gamma \uplus \Gamma_p ; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma ; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \quad (16a)$$

$$\boxed{\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'} \quad \begin{array}{l} rs \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\text{CTOneRules} \quad \frac{\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \xi_r \not\dot{\dashv} \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \quad (17a)$$

$$\text{CTRrules} \quad \frac{\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \quad \xi_r \not\dot{\dashv} \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \quad (17b)$$

**Lemma 3.0.1.** *If  $p : \tau[\xi] \dashv \Gamma ; \Delta$  then  $\xi : \tau$ .*

**Lemma 3.0.2.** *If  $\cdot ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  then  $\xi_r : \tau_1$ .*

**Lemma 3.0.3.** *If  $\cdot ; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$  then  $\xi_{rs} : \tau_1$ .*

**Lemma 3.0.4.** *If  $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$  and  $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$  and  $\xi_r \not\dot{\dashv} \xi_{pre} \vee \xi_{rs}$  then  $\Gamma ; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$*

**Lemma 3.0.5** (Substitution). *If  $\Gamma, x : \tau ; \Delta \vdash e_0 : \tau_0$  and  $\Gamma ; \Delta \vdash e : \tau$  then  $\Gamma ; \Delta \vdash [e/x]e_0 : \tau_0$*

**Lemma 3.0.6** (Simultaneous Substitution). *If  $\Gamma \uplus \Gamma' ; \Delta \vdash e : \tau$  and  $\theta : \Gamma'$  then  $\Gamma ; \Delta \vdash [\theta]e : \tau$*

**Lemma 3.0.7** (Substitution Typing). *If  $e \triangleright p \dashv \theta$  and  $\cdot ; \Delta_e \vdash e : \tau$  and  $p : \tau[\xi] \dashv \Gamma ; \Delta$  then  $\theta : \Gamma$*

Proof by induction on the derivation of  $e \triangleright p \dashv \theta$ .

**Theorem 3.1** (Determinism). *If  $\cdot ; \Delta \vdash e : \tau$  then exactly one of the following holds*

1.  $e \text{ val}$
2.  $e \text{ indet}$
3.  $e \mapsto e'$  for some unique  $e'$

## 4 Dynamic Semantics

$\boxed{e \text{ val}}$      $e$  is a value

$$\frac{\text{VNum}}{\underline{n \text{ val}}} \quad (18a)$$

$$\frac{\text{VLam}}{\underline{\lambda x : \tau. e \text{ val}}} \quad (18b)$$

$$\frac{\text{VPair} \quad e_1 \text{ val} \quad e_2 \text{ val}}{(e_1, e_2) \text{ val}} \quad (18c)$$

$$\frac{\text{VInl} \quad e \text{ val}}{\underline{\text{inl}_\tau(e) \text{ val}}} \quad (18d)$$

$$\frac{\text{VInr} \quad e \text{ val}}{\underline{\text{inr}_\tau(e) \text{ val}}} \quad (18e)$$

$\boxed{e \text{ indet}}$      $e$  is indeterminate

$$\frac{\text{IEHole}}{\underline{\text{⌈⌋}^u \text{ indet}}} \quad (19a)$$

$$\frac{\text{IHole} \quad e \text{ final}}{\underline{\text{⌈}e\text{⌋}^u \text{ indet}}} \quad (19b)$$

$$\frac{\text{IAp} \quad e_1 \text{ indet} \quad e_2 \text{ final}}{\underline{e_1(e_2) \text{ indet}}} \quad (19c)$$

$$\frac{\text{IPairL} \quad e_1 \text{ indet} \quad e_2 \text{ val}}{\underline{(e_1, e_2) \text{ indet}}} \quad (19d)$$

$$\frac{\text{IPairR} \quad e_1 \text{ val} \quad e_2 \text{ indet}}{\underline{(e_1, e_2) \text{ indet}}} \quad (19e)$$

$$\frac{\text{IPair} \quad e_1 \text{ indet} \quad e_2 \text{ indet}}{\underline{(e_1, e_2) \text{ indet}}} \quad (19f)$$

$$\frac{\text{IPrl} \quad e \text{ final}}{\underline{\text{prl}(e) \text{ indet}}} \quad (19g)$$

$$\frac{\text{IPrr} \quad e \text{ final}}{\text{pr}(e) \text{ indet}} \quad (19h)$$

$$\frac{\text{IInL} \quad e \text{ indet}}{\text{inl}_\tau(e) \text{ indet}} \quad (19i)$$

$$\frac{\text{IInR} \quad e \text{ indet}}{\text{inr}_\tau(e) \text{ indet}} \quad (19j)$$

$$\frac{\text{IMatch} \quad e \text{ final} \quad e ? p_r}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \text{ indet}} \quad (19k)$$

$$\boxed{e \text{ final}} \quad e \text{ is final}$$

$$\frac{\text{FVal} \quad e \text{ val}}{e \text{ final}} \quad (20a)$$

$$\frac{\text{FIndet} \quad e \text{ indet}}{e \text{ final}} \quad (20b)$$

**Lemma 4.0.1.** *If  $\text{inl}_\tau(e)$  final then  $e$  final.*

**Lemma 4.0.2.** *If  $\text{inr}_\tau(e)$  final then  $e$  final.*

**Lemma 4.0.3.** *If  $(e_1, e_2)$  final then  $e_1$  final and  $e_2$  final.*

$$\boxed{e \text{ notintro}} \quad e \text{ cannot be a value syntactically}$$

$$\frac{\text{NVEHole}}{\text{⋈}^u \text{ notintro}} \quad (21a)$$

$$\frac{\text{NVHole}}{\text{⋈}(e)^u \text{ notintro}} \quad (21b)$$

$$\frac{\text{NVAp}}{e_1(e_2) \text{ notintro}} \quad (21c)$$

$$\frac{\text{NVMatch}}{\text{match}(e)\{\hat{r}s\} \text{ notintro}} \quad (21d)$$

$$\frac{\text{NVPrI}}{\text{prl}(e) \text{ notintro}} \quad (21e)$$

$$\frac{\text{NVPrR}}{\text{pr}(e) \text{ notintro}} \quad (21f)$$

$$\boxed{\text{notintro}(e)}$$

$$\text{notintro}(\mathbb{0}^u) = \text{true} \quad (22a)$$

$$\text{notintro}(\mathbb{1}^u) = \text{true} \quad (22b)$$

$$\text{notintro}(e_1(e_2)) = \text{true} \quad (22c)$$

$$\text{notintro}(\text{match}(e)\{rs\}) = \text{true} \quad (22d)$$

$$\text{notintro}(\text{prl}(e)) = \text{true} \quad (22e)$$

$$\text{notintro}(\text{prr}(e)) = \text{true} \quad (22f)$$

$$\text{Otherwise } \text{notintro}(e) = \text{false} \quad (22g)$$

**Lemma 4.0.4** (Soundness and Completeness of NotIntro Judgment).  $e \text{ notintro}$  iff  $\text{notintro}(e)$ .

$$\boxed{e' \in \text{values}(e)}$$

$e'$  is one of the possible values of  $e$

$$\frac{\text{IVVal} \quad e \text{ val} \quad \cdot; \Delta \vdash e : \tau}{e \in \text{values}(e)} \quad (23a)$$

$$\frac{\text{IVIndet} \quad e \text{ notintro} \quad \cdot; \Delta \vdash e : \tau \quad e' \text{ val} \quad \cdot; \Delta \vdash e' : \tau}{e' \in \text{values}(e)} \quad (23b)$$

$$\frac{\text{IVInl} \quad e' \in \text{values}(e)}{\lambda x : \tau. e' \in \text{values}(\lambda x : \tau. e)} \quad (23c)$$

$$\frac{\text{IVInl} \quad e'_1 \in \text{values}(e_1)}{\text{inl}_{\tau_2}(e'_1) \in \text{values}(\text{inl}_{\tau_2}(e_1))} \quad (23d)$$

$$\frac{\text{IVInr} \quad e'_2 \in \text{values}(e_2)}{\text{inr}_{\tau_1}(e'_2) \in \text{values}(\text{inr}_{\tau_1}(e_2))} \quad (23e)$$

$$\frac{\text{IVPair} \quad e'_1 \in \text{values}(e_1) \quad e'_2 \in \text{values}(e_2)}{(e'_1, e'_2) \in \text{values}((e_1, e_2))} \quad (23f)$$

**Lemma 4.0.5.** If  $e' \in \text{values}(e)$  and  $\cdot; \Delta \vdash e : \tau$  then  $\cdot; \Delta \vdash e' : \tau$ .

**Lemma 4.0.6.** If  $e' \in \text{values}(e)$  then  $e' \text{ val}$ .

**Lemma 4.0.7.** If  $e \text{ indet}$  then there exists  $e'$  such that  $e' \in \text{values}(e)$ .

**Lemma 4.0.8.** Assume  $e \text{ final}$  and  $\cdot; \Delta \vdash e : \tau$  and  $\dot{\xi} : \tau$ . Then  $e \not\dot{\models}_?^{\dot{\xi}} \dot{\xi}$  iff  $\forall e'. e' \in \text{values}(e) \implies e' \not\dot{\models}_?^{\dot{\xi}} \dot{\xi}$ .

**Lemma 4.0.9.** If  $e \text{ indet}$  and  $\cdot; \Delta \vdash e : \tau$  and  $\dot{\xi} : \tau$  and there exists  $e'$  such that  $e' \in \text{values}(e)$  and  $e' \dot{\models}_?^{\dot{\xi}} \dot{\xi}$  then  $e \dot{\models}_?^{\dot{\xi}} \dot{\xi}$ .

$\boxed{\theta : \Gamma}$      $\theta$  is of type  $\Gamma$

$$\frac{\text{STEmpty}}{\emptyset : \cdot} \quad (24a)$$

$$\frac{\text{STExtend} \quad \theta : \Gamma_\theta \quad \Gamma ; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau} \quad (24b)$$

$\boxed{p \text{ refutable?}}$      $p$  is refutable

$$\frac{\text{RNum}}{\underline{n} \text{ refutable?}} \quad (25a)$$

$$\frac{\text{REHole}}{\langle \rangle^w \text{ refutable?}} \quad (25b)$$

$$\frac{\text{RHole}}{\langle p \rangle_\tau^w \text{ refutable?}} \quad (25c)$$

$$\frac{\text{RInl}}{\text{inl}(p) \text{ refutable?}} \quad (25d)$$

$$\frac{\text{RInr}}{\text{inr}(p) \text{ refutable?}} \quad (25e)$$

$$\frac{\text{RPairL} \quad p_1 \text{ refutable?}}{(p_1, p_2) \text{ refutable?}} \quad (25f)$$

$$\frac{\text{RPairR} \quad p_2 \text{ refutable?}}{(p_1, p_2) \text{ refutable?}} \quad (25g)$$

$\boxed{e \triangleright p \dashv\!\!\parallel \theta}$      $e$  matches  $p$ , emitting  $\theta$

$$\frac{\text{MVar}}{e \triangleright x \dashv\!\!\parallel e/x} \quad (26a)$$

$$\frac{\text{MWild}}{e \triangleright \_ \dashv\!\!\parallel \cdot} \quad (26b)$$

$$\frac{\text{MNum}}{\underline{n} \triangleright \underline{n} \dashv\!\!\parallel \cdot} \quad (26c)$$

$$\frac{\text{MPair} \quad e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1 \quad e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta_1 \uplus \theta_2} \quad (26d)$$

$$\frac{\text{MInl} \quad e \triangleright p \dashv\!\!\parallel \theta}{\text{inl}_\tau(e) \triangleright \text{inl}(p) \dashv\!\!\parallel \theta} \quad (26e)$$

$$\frac{\text{MInr} \quad e \triangleright p \dashv\!\!\parallel \theta}{\text{inr}_\tau(e) \triangleright \text{inr}(p) \dashv\!\!\parallel \theta} \quad (26f)$$

$$\frac{\text{MNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \triangleright p_1 \dashv\!\!\parallel \theta_1 \quad \text{prl}(e) \triangleright p_2 \dashv\!\!\parallel \theta_2}{e \triangleright (p_1, p_2) \dashv\!\!\parallel \theta_1 \uplus \theta_2} \quad (26g)$$

$\boxed{e ? p}$      $e$  may match  $p$

$$\frac{\text{MMEHole}}{e ? \langle \rangle^w} \quad (27a)$$

$$\frac{\text{MMHole}}{e ? \langle p \rangle_\tau^w} \quad (27b)$$

$$\frac{\text{MMNotIntro} \quad e \text{ notintro} \quad p \text{ refutable?}}{e ? p} \quad (27c)$$

$$\frac{\text{MMPairL} \quad e_1 ? p_1 \quad e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}{(e_1, e_2) ? (p_1, p_2)} \quad (27d)$$

$$\frac{\text{MMPairR} \quad e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (27e)$$

$$\frac{\text{MMPair} \quad e_1 ? p_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (27f)$$

$$\frac{\text{MMInl} \quad e ? p}{\text{inl}_\tau(e) ? \text{inl}(p)} \quad (27g)$$

$$\frac{\text{MMInr} \quad e ? p}{\text{inr}_\tau(e) ? \text{inr}(p)} \quad (27h)$$

$\boxed{e \perp p}$      $e$  does not match  $p$

$$\frac{\text{NMNum} \quad n_1 \neq n_2}{n_1 \perp n_2} \quad (28a)$$

$$\frac{\text{NMPairL} \quad e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)} \quad (28b)$$



$$\frac{\text{NMPairR} \quad e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)} \quad (28c)$$

$$\frac{\text{NMConfL}}{\mathbf{inr}_\tau(e) \perp \mathbf{inl}(p)} \quad (28d)$$

$$\frac{\text{NMConfR}}{\mathbf{inl}_\tau(e) \perp \mathbf{inr}(p)} \quad (28e)$$

$$\frac{\text{NMInl} \quad e \perp p}{\mathbf{inr}_\tau(e) \perp \mathbf{inl}(p)} \quad (28f)$$

$$\frac{\text{NMInr} \quad e \perp p}{\mathbf{inl}_\tau(e) \perp \mathbf{inr}(p)} \quad (28g)$$

$\boxed{e \mapsto e'}$       $e$  takes a step to  $e'$

$$\frac{\text{ITHole} \quad e \mapsto e'}{\langle\!\langle e \rangle\!\rangle^u \mapsto \langle\!\langle e' \rangle\!\rangle^u} \quad (29a)$$

$$\frac{\text{ITApFun} \quad e_1 \mapsto e'_1}{e_1(e_2) \mapsto e'_1(e_2)} \quad (29b)$$

$$\frac{\text{ITApArg} \quad e_1 \mathbf{val} \quad e_2 \mapsto e'_2}{e_1(e_2) \mapsto e_1(e'_2)} \quad (29c)$$

$$\frac{\text{ITAP} \quad e_2 \mathbf{val}}{\lambda x : \tau. e_1(e_2) \mapsto [e_2/x]e_1} \quad (29d)$$

$$\frac{\text{ITPairL} \quad e_1 \mapsto e'_1}{(e_1, e_2) \mapsto (e'_1, e_2)} \quad (29e)$$

$$\frac{\text{ITPairR} \quad e_1 \mathbf{val} \quad e_2 \mapsto e'_2}{(e_1, e_2) \mapsto (e_1, e'_2)} \quad (29f)$$

$$\frac{\text{ITPrL} \quad (e_1, e_2) \mathbf{final}}{\mathbf{prl}((e_1, e_2)) \mapsto e_1} \quad (29g)$$

$$\frac{\text{ITPrR} \quad (e_1, e_2) \mathbf{final}}{\mathbf{prr}((e_1, e_2)) \mapsto e_2} \quad (29h)$$

$$\frac{\text{ITInl} \quad e \mapsto e'}{\text{inl}_\tau(e) \mapsto \text{inl}_\tau(e')} \quad (29i)$$

$$\frac{\text{ITInr} \quad e \mapsto e'}{\text{inr}_\tau(e) \mapsto \text{inr}_\tau(e')} \quad (29j)$$

$$\frac{\text{ITExpMatch} \quad e \mapsto e'}{\text{match}(e)\{\hat{r}s\} \mapsto \text{match}(e')\{\hat{r}s\}} \quad (29k)$$

$$\frac{\text{ITSuccMatch} \quad e \text{ final} \quad e \triangleright p_r \dashv \parallel \theta}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \quad (29l)$$

$$\frac{\text{ITFailMatch} \quad e \text{ final} \quad e \perp p_r}{\text{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \text{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \quad (29m)$$

**Lemma 4.0.10.** *If  $\text{inl}_{\tau_2}(e_1)$  final then  $e_1$  final.*

**Lemma 4.0.11.** *If  $\text{inr}_{\tau_1}(e_2)$  final then  $e_2$  final.*

**Lemma 4.0.12.** *If  $(e_1, e_2)$  final then  $e_1$  final and  $e_2$  final.*

**Lemma 4.0.13.** *There doesn't exist  $\underline{n}$  such that  $\underline{n}$  notintro.*

**Lemma 4.0.14.** *There doesn't exist  $\text{inl}_\tau(e)$  such that  $\text{inl}_\tau(e)$  notintro.*

**Lemma 4.0.15.** *There doesn't exist  $\text{inr}_\tau(e)$  such that  $\text{inr}_\tau(e)$  notintro.*

**Lemma 4.0.16.** *There doesn't exist  $(e_1, e_2)$  such that  $(e_1, e_2)$  notintro.*

**Lemma 4.0.17.** *If  $e$  final and  $e$  notintro then  $e$  indet.*

**Lemma 4.0.18.** *There doesn't exist such an expression  $e$  such that both  $e$  val and  $e$  indet.*

**Lemma 4.0.19.** *There doesn't exist such an expression  $e$  such that both  $e$  val and  $e$  notintro.*

**Lemma 4.0.20** (Finality). *There doesn't exist such an expression  $e$  such that both  $e$  final and  $e \mapsto e'$  for some  $e'$*

**Lemma 4.0.21** (Matching Determinism). *If  $e$  final and  $\cdot; \Delta_e \vdash e : \tau$  and  $p : \tau[\xi] \dashv \parallel \Gamma; \Delta$  then exactly one of the following holds*

1.  $e \triangleright p \dashv \parallel \theta$  for some  $\theta$
2.  $e ? p$
3.  $e \perp p$

**Lemma 4.0.22** (Matching Coherence of Constraint). *Suppose that  $\cdot; \Delta_e \vdash e : \tau$  and  $e$  **final** and  $p : \tau[\xi] \dashv \vdash \Gamma; \Delta$ . Then we have*

1.  $e \dot{\models}_{\xi} \text{ iff } e \triangleright p \dashv \vdash \theta$
2.  $e \dot{\models}_{\tau} \xi \text{ iff } e ? p$
3.  $e \not\dot{\models}_{\tau} \xi \text{ iff } e \perp p$

## 5 Preservation and Progress

**Theorem 5.1** (Preservation). *If  $\cdot; \Delta \vdash e : \tau$  and  $e \mapsto e'$  then  $\cdot; \Delta \vdash e' : \tau$*

**Theorem 5.2** (Progress). *If  $\cdot; \Delta \vdash e : \tau$  then either  $e$  **final** or  $e \mapsto e'$  for some  $e'$ .*

## 6 Decidability

$$\boxed{\dot{\top}(\xi) = \xi}$$

$$\dot{\top}(\top) = \top \tag{30a}$$

$$\dot{\top}(?) = \top \tag{30b}$$

$$\dot{\top}(\underline{n}) = \underline{n} \tag{30c}$$

$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{30d}$$

$$\dot{\top}(\mathbf{inl}(\xi)) = \mathbf{inl}(\dot{\top}(\xi)) \tag{30e}$$

$$\dot{\top}(\mathbf{inr}(\xi)) = \mathbf{inr}(\dot{\top}(\xi)) \tag{30f}$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{30g}$$

$$\boxed{\dot{\perp}(\xi) = \xi}$$

$$\dot{\perp}(\top) = \top \tag{31a}$$

$$\dot{\perp}(?) = \perp \tag{31b}$$

$$\dot{\perp}(\underline{n}) = \underline{n} \tag{31c}$$

$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \tag{31d}$$

$$\dot{\perp}(\mathbf{inl}(\xi)) = \mathbf{inl}(\dot{\perp}(\xi)) \tag{31e}$$

$$\dot{\perp}(\mathbf{inr}(\xi)) = \mathbf{inr}(\dot{\perp}(\xi)) \tag{31f}$$

$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \tag{31g}$$

$$\boxed{\Xi \text{ incon}}$$

A finite set of constraints,  $\Xi$ , is inconsistent

$$\frac{\text{CINCTruth} \quad \Xi \text{ incon}}{\Xi, \top \text{ incon}} \quad (32a)$$

$$\frac{\text{CINCFalse}}{\Xi, \perp \text{ incon}} \quad (32b)$$

$$\frac{\text{CINCNum} \quad n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2} \text{ incon}} \quad (32c)$$

$$\frac{\text{CINCNotNum}}{\Xi, \underline{n}, \underline{\neg n} \text{ incon}} \quad (32d)$$

$$\frac{\text{CINCAnd} \quad \Xi, \xi_1, \xi_2 \text{ incon}}{\Xi, \xi_1 \wedge \xi_2 \text{ incon}} \quad (32e)$$

$$\frac{\text{CINCOr} \quad \Xi, \xi_1 \text{ incon} \quad \Xi, \xi_2 \text{ incon}}{\Xi, \xi_1 \vee \xi_2 \text{ incon}} \quad (32f)$$

$$\frac{\text{CINCInj}}{\Xi, \text{inl}(\xi_1), \text{inr}(\xi_2) \text{ incon}} \quad (32g)$$

$$\frac{\text{CINCInl} \quad \xi_1, \dots, \xi_n \text{ incon}}{\text{inl}(\xi_1), \dots, \text{inl}(\xi_n) \text{ incon}} \quad (32h)$$

$$\frac{\text{CINCInr} \quad \xi_1, \dots, \xi_n \text{ incon}}{\text{inr}(\xi_1), \dots, \text{inr}(\xi_n) \text{ incon}} \quad (32i)$$

$$\frac{\text{CINCPairL} \quad \xi_{11}, \dots, \xi_{n1} \text{ incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \dots, (\xi_{n1}, \xi_{n2}) \text{ incon}} \quad (32j)$$

$$\frac{\text{CINCPairR} \quad \xi_{12}, \dots, \xi_{n2} \text{ incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \dots, (\xi_{n1}, \xi_{n2}) \text{ incon}} \quad (32k)$$

**Lemma 6.0.1** (Decidability of Inconsistency). *It is decidable whether  $\xi \text{ incon}$ .*

**Lemma 6.0.2** (Inconsistency and Entailment of Constraint).  *$\bar{\xi} \text{ incon}$  iff  $\top \models \xi$*

**Theorem 6.1** (Decidability of Exhaustiveness). *It is decidable whether  $\top \models_{\tau}^{\dagger} \xi$ .*

**Theorem 6.2** (Decidability of Redundancy). *It is decidable whether  $\dot{\xi}_1 \models \dot{\xi}_2$ .*