1 Match Constraint Language

$$\dot{\xi} \;::=\; \top \mid \bot \mid ? \mid \underline{n} \mid \mathtt{inl}(\dot{\xi}) \mid \mathtt{inr}(\dot{\xi}) \mid (\dot{\xi},\dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$$

$\boxed{\dot{\xi} : \tau}$ $\dot{\xi}$ constrains final expressions of type τ

$$\frac{\text{CTTruth}}{\top : \tau} \tag{1a}$$

$$\frac{\text{CTUnknown}}{? : \tau} \tag{1b}$$

$$\frac{\text{CTNum}}{\underline{n} : \mathtt{num}} \tag{1c}$$

$$\frac{\text{CTInl} \quad \dot{\xi}_1 : \tau_1}{\mathtt{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)} \tag{1d}$$

$$\frac{\text{CTInr} \quad \dot{\xi}_2 : \tau_2}{\mathtt{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)} \tag{1e}$$

$$\frac{\text{CTPair} \quad \dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)} \tag{1f}$$

$$\frac{\text{CTOr} \quad \dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau} \tag{1g}$$

$\boxed{\dot{\xi} \;\mathtt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\frac{\text{RXNum}}{\underline{n} \;\mathtt{refutable}_?} \tag{2a}$$

$$\frac{\text{RXUnknown}}{? \;\mathtt{refutable}_?} \tag{2b}$$

$$\frac{\text{RXInl}}{\mathtt{inl}(\dot{\xi}) \;\mathtt{refutable}_?} \tag{2c}$$

$$\frac{\text{RXInr}}{\mathtt{inr}(\dot{\xi}) \;\mathtt{refutable}_?} \tag{2d}$$

$$\frac{\text{RXPairL} \quad \dot{\xi}_1 \;\mathtt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \;\mathtt{refutable}_?} \tag{2e}$$

$$\frac{\text{RXPairR}}{(\dot{\xi}_1, \dot{\xi}_2) \texttt{ refutable}_?} \tag{2f}$$

$$\frac{\text{RXOr} \quad \dot{\xi}_1 \texttt{ refutable}_? \qquad \dot{\xi}_2 \texttt{ refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2 \texttt{ refutable}_?} \tag{2g}$$

$\boxed{refutable_?(\dot{\xi})}$

$$refutable_?(\underline{n}) = \text{true} \tag{3a}$$

$$refutable_?(?) = \text{true} \tag{3b}$$

$$refutable_?(\texttt{inl}(\dot{\xi})) = refutable_?(\dot{\xi}) \tag{3c}$$

$$refutable_?(\texttt{inr}(\dot{\xi})) = refutable_?(\dot{\xi}) \tag{3d}$$

$$refutable_?((\dot{\xi}_1, \dot{\xi}_2)) = refutable_?(\dot{\xi}_1) \text{ or } refutable_?(\dot{\xi}_2) \tag{3e}$$

$$refutable_?(\dot{\xi}_1 \vee \dot{\xi}_2) = refutable_?(\dot{\xi}_1) \text{ and } refutable_?(\dot{\xi}_2) \tag{3f}$$

$$\text{Otherwise} \quad refutable_?(\dot{\xi}) = \text{false} \tag{3g}$$

$\boxed{e \models \dot{\xi}}$ $\quad e$ satisfies $\dot{\xi}$

$$\frac{\text{CSTruth}}{e \models \top} \tag{4a}$$

$$\frac{\text{CSNum}}{\underline{n} \models \underline{n}} \tag{4b}$$

$$\frac{\text{CSInl} \quad e_1 \models \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)} \tag{4c}$$

$$\frac{\text{CSInr} \quad e_2 \models \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)} \tag{4d}$$

$$\frac{\text{CSPair} \quad e_1 \models \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)} \tag{4e}$$

$$\frac{\text{CSNotIntroPair} \quad e \texttt{ notintro} \qquad \texttt{prl}(e) \models \dot{\xi}_1 \qquad \texttt{prr}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)} \tag{4f}$$

$$\frac{\text{CSOrL} \quad e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \tag{4g}$$

$$\text{CSOrR} \quad \frac{e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \tag{4h}$$

$\boxed{satisfy(e, \dot{\xi})}$

$$satisfy(e, \top) = \text{true} \tag{5a}$$

$$satisfy(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{5b}$$

$$satisfy(e, \dot{\xi}_1 \vee \dot{\xi}_2) = satisfy(e, \dot{\xi}_1) \text{ or } satisfy(e, \dot{\xi}_2) \tag{5c}$$

$$satisfy(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = satisfy(e_1, \dot{\xi}_1) \tag{5d}$$

$$satisfy(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = satisfy(e_2, \dot{\xi}_2) \tag{5e}$$

$$satisfy((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(e_1, \dot{\xi}_1) \text{ and } satisfy(e_2, \dot{\xi}_2) \tag{5f}$$

$$satisfy(\langle\!\langle\rangle\!\rangle^u, (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(\text{prl}(\langle\!\langle\rangle\!\rangle^u), \dot{\xi}_1) \text{ and } satisfy(\text{prr}(\langle\!\langle\rangle\!\rangle^u), \dot{\xi}_2) \tag{5g}$$

$$satisfy(\langle\!\langle e\rangle\!\rangle^u, (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(\text{prl}(\langle\!\langle e\rangle\!\rangle^u), \dot{\xi}_1) \text{ and } satisfy(\text{prr}(\langle\!\langle e\rangle\!\rangle^u), \dot{\xi}_2) \tag{5h}$$

$$satisfy(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(\text{prl}(e_1(e_2)), \dot{\xi}_1)$$
$$\text{and } satisfy(\text{prr}(e_1(e_2)), \dot{\xi}_2) \tag{5i}$$

$$satisfy(\text{match}(e)\{\hat{rs}\}, (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(\text{prl}(\text{match}(e)\{\hat{rs}\}), \dot{\xi}_1)$$
$$\text{and } satisfy(\text{prr}(\text{match}(e)\{\hat{rs}\}), \dot{\xi}_2) \tag{5j}$$

$$satisfy(\text{prl}(e), (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(\text{prl}(\text{prl}(e)), \dot{\xi}_1)$$
$$\text{and } satisfy(\text{prr}(\text{prl}(e)), \dot{\xi}_2) \tag{5k}$$

$$satisfy(\text{prr}(e), (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(\text{prl}(\text{prr}(e)), \dot{\xi}_1)$$
$$\text{and } satisfy(\text{prr}(\text{prr}(e)), \dot{\xi}_2) \tag{5l}$$

$$\text{Otherwise} \quad satisfy(e, \dot{\xi}) = \text{false} \tag{5m}$$

$\boxed{e \models_? \dot{\xi}}$ e may satisfy $\dot{\xi}$

$$\text{CMSUnknown} \quad \frac{}{e \models_? \, ?} \tag{6a}$$

$$\text{CMSInl} \quad \frac{e_1 \models_? \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models_? \text{inl}(\dot{\xi}_1)} \tag{6b}$$

$$\text{CMSInr} \quad \frac{e_2 \models_? \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models_? \text{inr}(\dot{\xi}_2)} \tag{6c}$$

$$\text{CMSPairL} \quad \frac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)} \tag{6d}$$

CMSPairR
$$\frac{e_1 \models \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)} \tag{6e}$$

CMSPair
$$\frac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)} \tag{6f}$$

CMSOrL
$$\frac{e \models_? \dot{\xi}_1 \qquad e \not\models \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2} \tag{6g}$$

CMSOrR
$$\frac{e \not\models \dot{\xi}_1 \qquad e \models_? \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2} \tag{6h}$$

CMSNotIntro
$$\frac{e \;\texttt{notintro} \qquad \dot{\xi} \;\texttt{refutable}_?}{e \models_? \dot{\xi}} \tag{6i}$$

$\boxed{e \models_?^\dagger \dot{\xi}}$ $\qquad$ e satisfies or may satisfy $\dot{\xi}$

CSMSMay
$$\frac{e \models_? \dot{\xi}}{e \models_?^\dagger \dot{\xi}} \tag{7a}$$

CSMSSat
$$\frac{e \models \dot{\xi}}{e \models_?^\dagger \dot{\xi}} \tag{7b}$$

Lemma 1.0.1. $e \not\models \bot$

Proof. By rule induction over Rules (14), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. □

Lemma 1.0.2. $e \not\models_? \bot$

Proof. Assume $e \models_? \bot$. By rule induction over Rules (16) on $e \models_? \bot$, only one case applies.

Case (16b).

(1) $\bot \;\texttt{refutable}_?$ $\qquad\qquad\qquad\qquad$ by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. □

Lemma 1.0.3. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (16) on $e \models_? \top$, only one case applies.

Case (16b)**.**

    (1) $\top\ \texttt{refutable}_?$ by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. $\square$

Lemma 1.0.4. $e \not\models\ ?$

Proof. By rule induction over Rules (14), we notice that $e \models\ ?$ is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 1.0.5. $e \models_?^\dagger \dot\xi$ *iff* $e \models_?^\dagger \dot\xi \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

    (1) $e \models_?^\dagger \dot\xi$ by assumption

By rule induction over Rules (17) on (1).

Case (17a)**.**

    (2) $e \models_? \dot\xi$ by assumption
    (3) $e \models_? \dot\xi \vee \bot$ by Rule (16c) on (2)
                                      and Lemma 2.0.1
    (4) $e \models_?^\dagger \dot\xi \vee \bot$ by Rule (17a) on (3)

Case (17b)**.**

    (2) $e \models \dot\xi$ by assumption
    (3) $e \models \dot\xi \vee \bot$ by Rule (14e) on (2)
    (4) $e \models_?^\dagger \dot\xi \vee \bot$ by Rule (17b) on (3)

2. Necessity:

    (1) $e \models_?^\dagger \dot\xi \vee \bot$ by assumption

By rule induction over Rules (17) on (1).

Case (17a)**.**

    (2) $e \models_? \dot\xi \vee \bot$ by assumption

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By rule induction over Rules (16) on (2), only two of them apply.

Case (16c).

(3)	$e \models_? \dot{\xi}$	by assumption
(4)	$e \models_?^\dagger \dot{\xi}$	by Rule (17a) on (3)

Case (16d).

(3)	$e \models_? \bot$	by assumption
(4)	$e \not\models_? \bot$	by Lemma 2.0.2

    (3) contradicts (4).

Case (17b).

(2)	$e \models \dot{\xi} \vee \bot$	by assumption

By rule induction over Rules (14) on (2), only two of them apply.

Case (14e).

(3)	$e \models \dot{\xi}$	by assumption
(4)	$e \models_?^\dagger \dot{\xi}$	by Rule (17b) on (3)

Case (14f).

(3)	$e \models \bot$	by assumption
(4)	$e \not\models \bot$	by Lemma 2.0.1

    (3) contradicts (4).

$\square$

Corollary 1.0.1. $\top \models_?^\dagger \dot{\xi}$ *iff* $\top \models_?^\dagger \dot{\xi} \vee \bot$

Proof. Follows directly from Definition 2.1.2 and Lemma 2.0.5. $\square$

Lemma 1.0.6. *Suppose that* $\dot{\xi}_1 : \tau$ *and* $\dot{\xi}_2 : \tau$. *Then* $\dot{\xi}_1 \not\models \dot{\xi}_2$ *iff* $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \bot$

Proof.

(1)	$\dot{\xi}_1 : \tau$	by assumption
(2)	$\dot{\xi}_2 : \tau$	by assumption
(3)	$\bot : \tau$	by Rule (8b)
(4)	$\dot{\xi}_2 \vee \bot : \tau$	by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5)	$\dot{\xi}_1 \not\models \dot{\xi}_2$	by assumption

    To prove $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \bot$, assume $\dot{\xi}_1 \models \dot{\xi}_2 \vee \bot$.

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(6) $\dot\xi_1 \models \dot\xi_2 \vee \bot$ by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \dot\xi_1$ implies

(7) $e \models \dot\xi_2 \vee \bot$ by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

Case (14e)**.**

    (8) $e \models \dot\xi_2$ by assumption

    (9) $\dot\xi_1 \models \dot\xi_2$ by Definition 2.1.1 on (8)

    (5) contradicts (9).

Case (14f)**.**

    (8) $e \models \bot$ by assumption

    (9) $e \not\models \bot$ by Lemma 2.0.1

    (8) contradicts (9).

The conclusion holds as follows:

(a) $\dot\xi_1 \not\models \dot\xi_2 \vee \bot$

2. Necessity:

    (5) $\dot\xi_1 \not\models \dot\xi_2 \vee \bot$ by assumption

To prove $\dot\xi_1 \not\models \dot\xi_2$, assume $\dot\xi_1 \models \dot\xi_2$.

    (6) $\dot\xi_1 \models \dot\xi_2$ by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \dot\xi_1$ implies

    (7) $e \models \dot\xi_2$ by Definition 2.1.1 on (1) and (2) and (6)

    (8) $e \models \dot\xi_2 \vee \bot$ by Rule (14e) on (7)

    (9) $\dot\xi_1 \models \dot\xi_2 \vee \bot$ by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

(a) $\dot\xi_1 \not\models \dot\xi_2$

                                           $\square$

Lemma 1.0.7. $e \not\models_?^\dagger \dot\xi_1$ and $e \not\models_?^\dagger \dot\xi_2$ iff $e \not\models_?^\dagger \dot\xi_1 \vee \dot\xi_2$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency: to show $e \not\models_?^\dagger \dot\xi_1 \vee \dot\xi_2$, we assume $e \models_?^\dagger \dot\xi_1 \vee \dot\xi_2$.

(1) $e \models_?^\dagger \dot\xi_1 \vee \dot\xi_2$	by assumption
(2) $e \not\models_?^\dagger \dot\xi_1$	by assumption
(3) $e \not\models_?^\dagger \dot\xi_2$	by assumption

   By rule induction over Rules (17) on (1).

   Case (17b).

(4) $e \models \dot\xi_1 \vee \dot\xi_2$	by assumption

   By rule induction over Rules (14) on (4) and only two of them apply.

   Case (14e).

(5) $e \models \dot\xi_1$	by assumption
(6) $e \models_?^\dagger \dot\xi_1$	by Rule (17b) on (5)

   (6) contradicts (2).

   Case (14f).

(5) $e \models \dot\xi_2$	by assumption
(6) $e \models_?^\dagger \dot\xi_2$	by Rule (17b) on (5)

   (6) contradicts (3).

   Case (17a).

(4) $e \models_? \dot\xi_1 \vee \dot\xi_2$	by assumption

   By rule induction over Rules (16) on (4) and only two of them apply.

   Case (16c).

(5) $e \models_? \dot\xi_1$	by assumption
(6) $e \models_?^\dagger \dot\xi_1$	by Rule (17a) on (5)

   (6) contradicts (2).

   Case (16d).

(5) $e \models_? \dot\xi_2$	by assumption
(6) $e \models_?^\dagger \dot\xi_2$	by Rule (17a) on (5)

   (6) contradicts (3).

   The conclusion holds as follows:

   (a) $e \not\models_?^\dagger \dot\xi_1 \vee \dot\xi_2$

2. Necessity:

(1) $e \not\models_?^\dagger \dot\xi_1 \vee \dot\xi_2$ by assumption

We show $e \not\models_?^\dagger \dot\xi_1$ and $e \not\models_?^\dagger \dot\xi_2$ separately.

(a) To show $e \not\models_?^\dagger \dot\xi_1$, we assume $e \models_?^\dagger \dot\xi_1$.

    (2) $e \models_?^\dagger \dot\xi_1$ by assumption

    (3) $e \models_?^\dagger \dot\xi_1 \vee \dot\xi_2$ by Lemma 2.0.10 on (2)

    Contradicts (1).

(b) To show $e \not\models_?^\dagger \dot\xi_2$, we assume $e \models_?^\dagger \dot\xi_2$.

    (2) $e \models_?^\dagger \dot\xi_2$ by assumption

    (3) $e \models_?^\dagger \dot\xi_1 \vee \dot\xi_2$ by Lemma 2.0.10 on (2)

    Contradicts (1).

In conclusion, $e \not\models_?^\dagger \dot\xi_1$ and $e \not\models_?^\dagger \dot\xi_2$.

$\square$

Lemma 1.0.8. *If $e \models_?^\dagger \dot\xi_1 \vee \dot\xi_2$ and $e \not\models_?^\dagger \dot\xi_1$ then $e \models_?^\dagger \dot\xi_2$*

Proof.

(4) $e \models_?^\dagger \dot\xi_1 \vee \dot\xi_2$ by assumption

(5) $e \not\models_?^\dagger \dot\xi_1$ by assumption

By rule induction over Rules (17) on (4).

Case (17b).

    (6) $e \models \dot\xi_1 \vee \dot\xi_2$ by assumption

By rule induction over Rules (14) on (6) and only two of them apply.

Case (14e).

    (7) $e \models \dot\xi_1$ by assumption

    (8) $e \models_?^\dagger \dot\xi_1$ by Rule (17b) on (7)

    (8) contradicts (5).

Case (14f).

    (7) $e \models \dot\xi_2$ by assumption

    (8) $e \models_?^\dagger \dot\xi_2$ by Rule (17b) on (7)

Case (17a).

(6) $e \models_? \dot\xi_1 \lor \dot\xi_2$ by assumption

By rule induction over Rules (16) on (6) and only two of them apply.

Case (16c).

    (7) $e \models_? \dot\xi_1$ by assumption

    (8) $e \models_?^\dagger \dot\xi_1$ by Rule (17a) on (7)

   (8) contradicts (5).

Case (16d).

    (7) $e \models_? \dot\xi_2$ by assumption

    (8) $e \models_?^\dagger \dot\xi_2$ by Rule (17a) on (7)

$\square$

Lemma 1.0.9. *If* $e \models_?^\dagger \dot\xi_1$ *then* $e \models_?^\dagger \dot\xi_1 \lor \dot\xi_2$ *and* $e \models_?^\dagger \dot\xi_2 \lor \dot\xi_1$

Proof.

   (1) $e \models_?^\dagger \dot\xi_1$ by assumption   ,

By rule induction over Rules (17) on (1),

Case (17b).

    (2) $e \models \dot\xi_1$ by assumption

    (3) $e \models \dot\xi_1 \lor \dot\xi_2$ by Rule (14e) on (2)

    (4) $e \models \dot\xi_2 \lor \dot\xi_1$ by Rule (14f) on (2)

    (5) $e \models_?^\dagger \dot\xi_1 \lor \dot\xi_2$ by Rule (17b) on (3)

    (6) $e \models_?^\dagger \dot\xi_2 \lor \dot\xi_1$ by Rule (17b) on (4)

Case (17a).

    (2) $e \models_? \dot\xi_1$ by assumption

By case analysis on the result of $satisfy(e, \dot\xi_2)$.

Case true.

    (3) $satisfy(e, \dot\xi_2) = \text{true}$ by assumption

    (4) $e \models \dot\xi_2$ by Lemma 2.0.19 on (3)

    (5) $e \models \dot\xi_1 \lor \dot\xi_2$ by Rule (14f) on (4)

    (6) $e \models \dot\xi_2 \lor \dot\xi_1$ by Rule (14e) on (4)

| (7) $e \models^\dagger_? \dot\xi_1 \vee \dot\xi_2$ | by Rule (17b) on (5) |
| (8) $e \models^\dagger_? \dot\xi_2 \vee \dot\xi_1$ | by Rule (17b) on (6) |

Case false**.**

(3) $satisfy(e, \dot\xi_2) = $ false	by assumption
(4) $e \not\models \dot\xi_2$	by Lemma 2.0.19 on (3)
(5) $e \models_? \dot\xi_1 \vee \dot\xi_2$	by Rule (16c) on (2) and (4)
(6) $e \models^\dagger_? \dot\xi_1 \vee \dot\xi_2$	by Rule (17a) on (5)

$\square$

Lemma 1.0.10. $e_1 \models^\dagger_? \dot\xi_1$ *iff* $\mathtt{inl}_{\tau_2}(e_1) \models^\dagger_? \mathtt{inl}(\dot\xi_1)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

| (1) $e_1 \models^\dagger_? \dot\xi_1$ | by assumption |

By rule induction over Rules (17) on (1).

Case (17b)**.**

(2) $e_1 \models \dot\xi_1$	by assumption
(3) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot\xi_1)$	by Rule (14g) on (2)
(4) $\mathtt{inl}_{\tau_2}(e_1) \models^\dagger_? \mathtt{inl}(\dot\xi_1)$	by Rule (17b) on (3)

Case (17a)**.**

(2) $e_1 \models_? \dot\xi_1$	by assumption
(3) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot\xi_1)$	by Rule (16e) on (2)
(4) $\mathtt{inl}_{\tau_2}(e_1) \models^\dagger_? \mathtt{inl}(\dot\xi_1)$	by Rule (17a) on (3)

2. Necessity:

| (1) $\mathtt{inl}_{\tau_2}(e_1) \models^\dagger_? \mathtt{inl}(\dot\xi_1)$ | by assumption |

By rule induction over Rules (17) on (1).

Case (17b)**.**

| (2) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot\xi_1)$ | by assumption |

By rule induction over Rules (14) on (2), only one rule applies.

Case (14g).

$$\text{(3) } e_1 \models \dot{\xi}_1 \qquad\qquad\qquad\qquad \text{by assumption}$$
$$\text{(4) } e_1 \models_?^\dagger \dot{\xi}_1 \qquad\qquad\qquad\qquad \text{by Rule (17b) on (3)}$$

Case (17a).

$$\text{(2) } \texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1) \qquad\qquad \text{by assumption}$$

By rule induction over Rules (16) on (2), only two rules apply.

Case (16e).

$$\text{(3) } e_1 \models_? \dot{\xi}_1 \qquad\qquad\qquad\qquad \text{by assumption}$$
$$\text{(4) } e_1 \models_?^\dagger \dot{\xi}_1 \qquad\qquad\qquad\qquad \text{by Rule (17a) on (3)}$$

Case (16b).

$$\text{(3) } \texttt{inl}_{\tau_2}(e_1) \ \texttt{notintro} \qquad\qquad \text{by assumption}$$

By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

$\square$

Lemma 1.0.11. $e_2 \models_?^\dagger \dot{\xi}_2$ *iff* $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

$$\text{(1) } e_2 \models_?^\dagger \dot{\xi}_2 \qquad\qquad\qquad\qquad \text{by assumption}$$

By rule induction over Rules (17) on (1).

Case (17b).

$$\text{(2) } e_2 \models \dot{\xi}_2 \qquad\qquad\qquad\qquad \text{by assumption}$$
$$\text{(3) } \texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2) \qquad\qquad \text{by Rule (14h) on (2)}$$
$$\text{(4) } \texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2) \qquad\qquad \text{by Rule (17b) on (3)}$$

Case (17a).

$$\text{(2) } e_2 \models_? \dot{\xi}_2 \qquad\qquad\qquad\qquad \text{by assumption}$$
$$\text{(3) } \texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\xi}_2) \qquad\qquad \text{by Rule (16f) on (2)}$$
$$\text{(4) } \texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2) \qquad\qquad \text{by Rule (17a) on (3)}$$

2. Necessity:

$$\text{(1) } \texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2) \qquad\qquad \text{by assumption}$$

By rule induction over Rules (17) on (1).

Case (17b).

$\qquad$ (2) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$ $\qquad$ by assumption

By rule induction over Rules (14) on (2), only one rule applies.

Case (14h).
$\qquad$ (3) $e_2 \models \dot{\xi}_2$ $\qquad$ by assumption
$\qquad$ (4) $e_2 \models_?^\dagger \dot{\xi}_2$ $\qquad$ by Rule (17b) on (3)

Case (17a).
$\qquad$ (2) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$ $\qquad$ by assumption

By rule induction over Rules (16) on (2), only two rules apply.

Case (16f).
$\qquad$ (3) $e_2 \models_? \dot{\xi}_2$ $\qquad$ by assumption
$\qquad$ (4) $e_2 \models_?^\dagger \dot{\xi}_2$ $\qquad$ by Rule (17a) on (3)

Case (16b).
$\qquad$ (3) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ $\qquad$ by assumption
$\qquad$ By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 1.0.12. $e_1 \models_?^\dagger \dot{\xi}_1$ *and* $e_2 \models_?^\dagger \dot{\xi}_2$ *iff* $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:
$\qquad$ (1) $e_1 \models_?^\dagger \dot{\xi}_1$ $\qquad$ by assumption
$\qquad$ (2) $e_2 \models_?^\dagger \dot{\xi}_2$ $\qquad$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).
$\qquad$ (3) $e_1 \models \dot{\xi}_1$ $\qquad$ by assumption

By rule induction over Rules (17) on (2).

Case (17b).
$\qquad$ (4) $e_2 \models \dot{\xi}_2$ $\qquad$ by assumption
$\qquad$ (5) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by Rule (14i) on (3) and (4)
$\qquad$ (6) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by Rule (17b) on (5)

Case (17a).
$\qquad$ (4) $e_2 \models_? \dot{\xi}_2$ $\qquad$ by assumption
$\qquad$ (5) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by Rule (16h) on (3) and (4)

$\quad\quad\quad$ (6) $(e_1, e_2) \models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad\quad$ by Rule (17a) on (5)

Case (17a).

$\quad\quad\quad$ (4) $e_1 \models_? \dot{\xi}_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

By rule induction over Rules (17) on (2).

Case (17b).

$\quad\quad\quad$ (5) $e_2 \models \dot{\xi}_2$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad$ by Rule (16g) on (4)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ and (5)

$\quad\quad\quad$ (7) $(e_1, e_2) \models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad$ by Rule (17a) on (6)

Case (17a).

$\quad\quad\quad$ (5) $e_2 \models_? \dot{\xi}_2$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad$ by Rule (16h) on (4)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ and (5)

$\quad\quad\quad$ (7) $(e_1, e_2) \models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad$ by Rule (17a) on (6)

2. Necessity:

$\quad\quad$ (1) $(e_1, e_2) \models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad\quad$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

$\quad\quad\quad$ (2) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad\quad$ by assumption

By rule induction over Rules (14) on (2), only one rule applies.

Case (14i).

$\quad\quad\quad$ (3) $e_1 \models \dot{\xi}_1$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (4) $e_2 \models \dot{\xi}_2$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (5) $e_1 \models^\dagger_? \dot{\xi}_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by Rule (17b) on (3)

$\quad\quad\quad$ (6) $e_2 \models^\dagger_? \dot{\xi}_2$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by Rule (17b) on (4)

Case (17a).

$\quad\quad\quad$ (2) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ $\quad\quad\quad\quad\quad\quad$ by assumption

By rule induction over Rules (16) on (2), only three rules apply.

Case (16g).

$\quad\quad\quad$ (3) $e_1 \models_? \dot{\xi}_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (4) $e_2 \models \dot{\xi}_2$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (5) $e_1 \models^\dagger_? \dot{\xi}_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by Rule (17a) on (3)

$\quad\quad\quad$ (6) $e_2 \models^\dagger_? \dot{\xi}_2$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by Rule (17b) on (4)

Case (16h).

(3)	$e_1 \models \dot{\xi}_1$	by assumption
(4)	$e_2 \models_? \dot{\xi}_2$	by assumption
(5)	$e_1 \models_?^\dagger \dot{\xi}_1$	by Rule (17b) on (3)
(6)	$e_2 \models_?^\dagger \dot{\xi}_2$	by Rule (17a) on (4)

Case (16i).

(3)	$e_1 \models_? \dot{\xi}_1$	by assumption
(4)	$e_2 \models_? \dot{\xi}_2$	by assumption
(5)	$e_1 \models_?^\dagger \dot{\xi}_1$	by Rule (17a) on (3)
(6)	$e_2 \models_?^\dagger \dot{\xi}_2$	by Rule (17a) on (4)

$\square$

Lemma 1.0.13 (Soundness and Completeness of Refutable Constraints). $\dot{\xi}$ `refutable`? *iff* $refutable_?(\dot{\xi}) = true$.

Lemma 1.0.14. *There does not exist such a constraint $\dot{\xi}_1 \wedge \dot{\xi}_2$ such that $\dot{\xi}_1 \wedge \dot{\xi}_2$* `refutable`?.

Proof. By rule induction over Rules (10), we notice that $\dot{\xi}_1 \wedge \dot{\xi}_2$ `refutable`? is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 1.0.15. *There does not exist such a constraint $\dot{\xi}_1 \vee \dot{\xi}_2$ such that $\dot{\xi}_1 \vee \dot{\xi}_2$* `refutable`?.

Proof. By rule induction over Rules (10), we notice that $\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`? is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 1.0.16. *If e* `notintro` *and $e \models \dot{\xi}$ then $\dot{\xi}$* ~~`refutable`?~~.

Proof.

(1)	e `notintro`	by assumption
(2)	$e \models \dot{\xi}$	by assumption

By rule induction over Rules (14) on (2).

Case (14a).

(3)	$\dot{\xi} = \top$	by assumption

Assume $\top$ `refutable`?. By rule induction over Rules (10), no case applies due to syntactic contradiction.
Therefore, $\top$ ~~`refutable`?~~.

Case (14e),(14f).

(3)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(4)	$\dot{\xi}_1 \vee \dot{\xi}_2$ ~~`refutable`?~~	by Lemma 2.0.17

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Case (14d).

(3) $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$ by assumption

(4) $\dot{\xi}_1 \wedge \dot{\xi}_2 \text{ } \widecancel{\texttt{refutable}_?}$ by Lemma 2.0.16

Case (14j).

(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(4) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(5) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(6) $\texttt{prl}(e) \texttt{ notintro}$ by Rule (26e)

(7) $\texttt{prr}(e) \texttt{ notintro}$ by Rule (26f)

(8) $\dot{\xi}_1 \text{ } \widecancel{\texttt{refutable}_?}$ by IH on (6) and (4)

(9) $\dot{\xi}_2 \text{ } \widecancel{\texttt{refutable}_?}$ by IH on (7) and (5)

Assume $(\dot{\xi}_1, \dot{\xi}_2) \texttt{ refutable}_?$. By rule induction over Rules (10) on it, only two cases apply.

Case (10d).

(10) $\dot{\xi}_1 \texttt{ refutable}_?$ by assumption

Contradicts (8).

Case (10e).

(10) $\dot{\xi}_2 \texttt{ refutable}_?$ by assumption

Contradicts (9).

Therefore, $(\dot{\xi}_1, \dot{\xi}_2) \text{ } \widecancel{\texttt{refutable}_?}$.

Otherwise.

(3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

$\square$

Lemma 1.0.17. $\texttt{inl}_{\tau_2}(e_1) \models_?^{\dagger} \texttt{inr}(\dot{\xi}_2)$ *is not derivable.*

Proof. We prove by assuming $\texttt{inl}_{\tau_2}(e_1) \models_?^{\dagger} \texttt{inr}(\dot{\xi}_2)$ and obtaining a contradiction.

(1) $\texttt{inl}_{\tau_2}(e_1) \models_?^{\dagger} \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (17) on (1).

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Case (17b)**.**

$$(2)\ \text{inl}_{\tau_2}(e_1) \models \text{inr}(\dot{\xi}_2) \qquad \text{by assumption}$$

By rule induction over Rules (14) on (2), no rule applies due to syntactic contradiction.

Case (17a)**.**

$$(2)\ \text{inl}_{\tau_2}(e_1) \models_? \text{inr}(\dot{\xi}_2) \qquad \text{by assumption}$$

By rule induction over Rules (16) on (2), only one rule applies.

> **Case** (16b)**.**
>
> $$(3)\ \text{inl}_{\tau_2}(e_1)\ \texttt{notintro} \qquad \text{by assumption}$$
>
> By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

$\square$

Lemma 1.0.18. $\text{inr}_{\tau_1}(e_2) \models_?^{\dagger} \text{inl}(\dot{\xi}_1)$ *is not derivable.*

Proof. We prove by assuming $\text{inr}_{\tau_1}(e_2) \models_?^{\dagger} \text{inl}(\dot{\xi}_1)$ and obtaining a contradiction.

$$(1)\ \text{inr}_{\tau_1}(e_2) \models_?^{\dagger} \text{inl}(\dot{\xi}_1) \qquad \text{by assumption}$$

By rule induction over Rules (17) on (1).

Case (17b)**.**

$$(2)\ \text{inr}_{\tau_1}(e_2) \models \text{inl}(\dot{\xi}_1) \qquad \text{by assumption}$$

By rule induction over Rules (14) on (2), no rule applies due to syntactic contradiction.

Case (17a)**.**

$$(2)\ \text{inr}_{\tau_1}(e_2) \models_? \text{inl}(\dot{\xi}_1) \qquad \text{by assumption}$$

By rule induction over Rules (16) on (2), only one rule applies.

> **Case** (16b)**.**
>
> $$(3)\ \text{inr}_{\tau_1}(e_2)\ \texttt{notintro} \qquad \text{by assumption}$$
>
> By rule induction over Rules (26) on (3), no rule applies due to syntactic contradiction.

$\square$

Lemma 1.0.19 (Soundness and Completeness of Satisfaction Judgment). $e \models \dot{\xi}$ iff $satisfy(e, \dot{\xi}) = true$.

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models \dot{\xi}$ by assumption

   By rule induction over Rules (14) on (1).

   Case (14a).

        (2) $\dot{\xi} = \top$ by assumption
        (3) $satisfy(e, \top) = true$ by Definition 15a

   Case (14b).

        (2) $e = \underline{n}$ by assumption
        (3) $\dot{\xi} = \underline{n}$ by assumption
        (4) $satisfy(\underline{n}, \underline{n}) = (n = n) = true$ by Definition 15b

   Case (14c).

        (2) $e = \underline{n_1}$ by assumption
        (3) $\dot{\xi} = \underline{\not{n_2}}$ by assumption
        (4) $n_1 \neq n_2$ by assumption
        (5) $satisfy(\underline{n_1}, \underline{\not{n_2}}) = (n_1 \neq n_2) = true$ by Definition 15c on (4)

   Case (14d).

        (2) $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$ by assumption
        (3) $e \models \dot{\xi}_1$ by assumption
        (4) $e \models \dot{\xi}_2$ by assumption
        (5) $satisfy(e, \dot{\xi}_1) = true$ by IH on (3)
        (6) $satisfy(e, \dot{\xi}_2) = true$ by IH on (4)
        (7) $satisfy(e, \dot{\xi}_1 \wedge \dot{\xi}_2) = satisfy(e, \dot{\xi}_1)$ and $satisfy(e, \dot{\xi}_2) = true$
                                        by Definition 15d on
                                        (5) and (6)

   Case (14e).

        (2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption
        (3) $e \models \dot{\xi}_1$ by assumption
        (4) $satisfy(e, \dot{\xi}_1) = true$ by IH on (3)

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(5) $satisfy(e, \dot{\xi}_1 \vee \dot{\xi}_2) = satisfy(e, \dot{\xi}_1)$ or $satisfy(e, \dot{\xi}_2) = \text{true}$

by Definition 15e on (4)

Case (14f).

(2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption
(3) $e \models \dot{\xi}_2$ by assumption
(4) $satisfy(e, \dot{\xi}_2) = \text{true}$ by IH on (3)
(5) $satisfy(e, \dot{\xi}_1 \vee \dot{\xi}_2) = satisfy(e, \dot{\xi}_1)$ or $satisfy(e, \dot{\xi}_2) = \text{true}$

by Definition 15e on (4)

Case (14g).

(2) $e = \text{inl}_{\tau_2}(e_1)$ by assumption
(3) $\dot{\xi} = \text{inl}(\dot{\xi}_1)$ by assumption
(4) $e_1 \models \dot{\xi}_1$ by assumption
(5) $satisfy(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)
(6) $satisfy(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = satisfy(e_1, \dot{\xi}_1) = \text{true}$

by Definition 15f on (5)

Case (14h).

(2) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
(3) $\dot{\xi} = \text{inl}(\dot{\xi}_2)$ by assumption
(4) $e_2 \models \dot{\xi}_2$ by assumption
(5) $satisfy(e_2, \dot{\xi}_2) = \text{true}$ by IH on (4)
(6) $satisfy(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = satisfy(e_2, \dot{\xi}_2) = \text{true}$

by Definition 15g on (5)

Case (14i).

(2) $e = (e_1, e_2)$ by assumption
(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption
(4) $e_1 \models \dot{\xi}_1$ by assumption
(5) $e_2 \models \dot{\xi}_2$ by assumption
(6) $satisfy(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)
(7) $satisfy(e_2, \dot{\xi}_2) = \text{true}$ by IH on (5)
(8) $satisfy((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = satisfy(e_1, \dot{\xi}_1)$ and $satisfy(e_2, \dot{\xi}_2) = \text{true}$

by Definition 15h on (6) and (7)

Case (14j).

$\quad$ (2) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (3) e `notintro` $\qquad\qquad\qquad\qquad\quad$ by assumption

$\quad$ (4) $\mathtt{prl}(e) \models \dot{\xi}_1$ $\qquad\qquad\qquad\quad\;$ by assumption

$\quad$ (5) $\mathtt{prr}(e) \models \dot{\xi}_2$ $\qquad\qquad\qquad\quad\;$ by assumption

$\quad$ (6) $satisfy(\mathtt{prl}(e), \dot{\xi}_1) = \text{true}$ $\qquad\quad$ by IH on (4)

$\quad$ (7) $satisfy(\mathtt{prr}(e), \dot{\xi}_2) = \text{true}$ $\qquad\quad$ by IH on (5)

By rule induction over Rules (26) on (3).

Otherwise.

$\quad$ (8) $e = (\!(\,)\!)^u, (\!(e_0)\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}s\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (9) $satisfy(e, (\dot{\xi}_1, \dot{\xi}_2)) =$

$\qquad satisfy(\mathtt{prl}(e), \dot{\xi}_1) \text{ and } satisfy(\mathtt{prr}(e), \dot{\xi}_2) = \text{true}$

$\qquad\qquad\qquad\qquad\qquad\qquad$ by Definition 15 on (6)

$\qquad\qquad\qquad\qquad\qquad\qquad$ and (7)

2. Completeness:

$\quad$ (1) $satisfy(e, \dot{\xi}) = \text{true}$ $\qquad\qquad\qquad$ by assumption

By structural induction on $\dot{\xi}$.

Case $\dot{\xi} = \top$.

$\quad$ (2) $e \models \top$ $\qquad\qquad\qquad\qquad\qquad$ by Rule (14a)

Case $\dot{\xi} = \bot, ?$.

$\quad$ (2) $satisfy(e, \dot{\xi}) = \text{false}$ $\qquad\qquad$ by Definition 15o

$\quad$ (2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \underline{n}$.

$\quad$ By structural induction on e.

$\quad$ **Case** $e = \underline{n'}$.

$\qquad$ (2) $n' = n$ $\qquad\qquad\qquad\qquad$ by Definition 15b on

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (1)

$\qquad$ (3) $\underline{n'} \models \underline{n}$ $\qquad\qquad\qquad\qquad$ by Rule (14b) on (2)

$\quad$ **Otherwise.**

$\qquad$ (2) $satisfy(e, \underline{n}) = \text{false}$ $\qquad\quad$ by Definition 15o

$\qquad$ (2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \not{\underline{x}}$.

$\quad$ By structural induction on e.

Case $e = \underline{n'}$.

   (2) $n' \neq n$ by Definition 15c on (1)

   (3) $\underline{n'} \models \cancel{\varkappa}$ by Rule (14c) on (2)

Otherwise.

   (2) $satisfy(e,\cancel{\varkappa}) = \text{false}$ by Definition 15o

  (2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$.

   (2) $satisfy(e, \dot{\xi}_1) = \text{true}$ by Definition 15d on (1)

   (3) $satisfy(e, \dot{\xi}_2) = \text{true}$ by Definition 15d on (1)

   (4) $e \models \dot{\xi}_1$ by IH on (2)

   (5) $e \models \dot{\xi}_2$ by IH on (3)

   (6) $e \models \dot{\xi}_1 \wedge \dot{\xi}_2$ by Rule (14d) on (4) and (5)

Case $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$.

   (2) $satisfy(e, \dot{\xi}_1) \text{ or } satisfy(e, \dot{\xi}_2) = \text{true}$

                  by Definition 15e on (1)

  By case analysis on (2).

  Case $satisfy(e, \dot{\xi}_1) = \text{true}$.

    (3) $satisfy(e, \dot{\xi}_1) = \text{true}$ by assumption

    (4) $e \models \dot{\xi}_1$ by IH on (3)

    (5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (14e) on (4)

  Case $satisfy(e, \dot{\xi}_2) = \text{true}$.

    (3) $satisfy(e, \dot{\xi}_2) = \text{true}$ by assumption

    (4) $e \models \dot{\xi}_2$ by IH on (3)

    (5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (14f) on (4)

Case $\dot{\xi} = \mathtt{inl}(\dot{\xi}_1)$.

  By structural induction on e.

  Case $e = \mathtt{inl}_{\tau_2}(e_1)$.

    (2) $satisfy(e_1, \dot{\xi}_1) = \text{true}$ by Definition 15f on (1)

    (3) $e_1 \models \dot{\xi}_1$ by IH on (2)

    (4) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\xi}_1)$ by Rule (14g) on (3)

  Otherwise.

(2) $satisfy(e, \mathtt{inl}(\dot{\xi_1})) = $ false   by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \mathtt{inr}(\dot{\xi_2})$.

By structural induction on e.

Case $e = \mathtt{inr}_{\tau_1}(e_2)$.

(2) $satisfy(e_2, \dot{\xi_2}) = $ true   by Definition 15g on (1)

(3) $e_2 \models \dot{\xi_2}$ by IH on (2)

(4) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi_2})$ by Rule (14h) on (3)

Otherwise.

(2) $satisfy(e, \mathtt{inr}(\dot{\xi_2})) = $ false   by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = (\dot{\xi_1}, \dot{\xi_2})$.

By structural induction on e.

Case $e = (e_1, e_2)$.

(2) $satisfy(e_1, \dot{\xi_1}) = $ true   by Definition 15h on (1)

(3) $satisfy(e_2, \dot{\xi_2}) = $ true   by Definition 15h on (1)

(4) $e_1 \models \dot{\xi_1}$ by IH on (2)

(5) $e_2 \models \dot{\xi_2}$ by IH on (3)

(6) $(e_1, e_2) \models (\dot{\xi_1}, \dot{\xi_2})$ by Rule (14i) on (4) and (5)

Case $e = (\![\|]\!)^u, (\![e_0]\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}s\}$.

(2) $satisfy(\mathtt{prl}(e), \dot{\xi_1}) = $ true   by Definition 15h on (1)

(3) $satisfy(\mathtt{prr}(e), \dot{\xi_2}) = $ true   by Definition 15h on (1)

(4) $\mathtt{prl}(e) \models \dot{\xi_1}$ by IH on (2)

(5) $\mathtt{prr}(e) \models \dot{\xi_2}$ by IH on (3)

(6) e $\mathtt{notintro}$ by each rule in Rules (26)

(7) $(e_1, e_2) \models (\dot{\xi_1}, \dot{\xi_2})$ by Rule (14j) on (6) and (4) and (5)

Otherwise.

(2) $satisfy(e, (\dot{\xi_1}, \dot{\xi_2})) = $ false   by Definition 15o

(2) contradicts (1) and thus vacuously true.

$\square$

Lemma 1.0.20. ────────────────────────────────────

Lemma 1.0.21. $e \not\models \dot\xi$ *and* $e \not\models_? \dot\xi$ *iff* $e \not\models_?^\dagger \dot\xi$.

Proof.   1. Sufficiency:

(1) $e \not\models \dot\xi$ by assumption

(2) $e \not\models_? \dot\xi$ by assumption

Assume $e \models_?^\dagger \dot\xi$. By rule induction over Rules (17) on it.

Case (17a)**.**

(3) $e \models \dot\xi$ by assumption

Contradicts (1).

Case (17b)**.**

(3) $e \models_? \dot\xi$ by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \dot\xi$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \dot\xi$ by assumption

Assume $e \models \dot\xi$.

(2) $e \models_?^\dagger \dot\xi$ by Rule (17b) on assumption

Contradicts (1). Therefore, $e \not\models \dot\xi$. Assume $e \models_? \dot\xi$.

(3) $e \models_?^\dagger \dot\xi$ by Rule (17a) on assumption

Contradicts (1). Therefore, $e \not\models_? \dot\xi$.

□

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If* $\dot\xi : \tau$ *and* $\cdot; \Delta \vdash e : \tau$ *and* e `final` *then exactly one of the following holds*

1. $e \models \dot\xi$

2. $e \models_? \dot\xi$

3. $e \not\models_?^\dagger \dot\xi$

Proof.

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(4) $\dot{\xi} : \tau$		by assumption
(5) $\cdot\,; \Delta \vdash e : \tau$		by assumption
(6) e final		by assumption

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

Case (8a).

(7) $\dot{\xi} = \top$		by assumption
(8) $e \models \top$		by Rule (14a)
(9) $e \not\models_? \top$		by Lemma 2.0.3
(10) $e \models^\dagger_? \top$		by Rule (17b) on (8)

Case (8b).

(7) $\dot{\xi} = \bot$		by assumption
(8) $e \not\models \bot$		by Lemma 2.0.1
(9) $e \not\models_? \bot$		by Lemma 2.0.2
(10) $e \not\models^\dagger_? \bot$		by Lemma 2.0.20 on (8) and (9)

Case (1b).

(7) $\dot{\xi} = ?$		by assumption
(8) $e \not\models ?$		by Lemma 2.0.4
(9) $e \models_? ?$		by Rule (16a)
(10) $e \models^\dagger_? ?$		by Rule (17a) on (9)

Case (8c).

(7) $\dot{\xi} = \underline{n_2}$		by assumption
(8) $\tau = $ num		by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = (\!|\,|\!)^u, (\!| e_0 |\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$
by assumption

$\quad$ (10) e `notintro` $\hspace{4.5cm}$ by Rule
$\hspace{7.5cm}$ (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on $\dot{\xi}$.

$\quad$ (11) $e \not\models \underline{n_2}$ $\hspace{5.5cm}$ by contradiction
$\quad$ (12) $\underline{n_2}$ `refutable`$_?$ $\hspace{4.3cm}$ by Rule (10a)
$\quad$ (13) $e \models_? \underline{n_2}$ $\hspace{4.8cm}$ by Rule (16b) on (10)
$\hspace{9cm}$ and (12)

$\quad$ (14) $e \models_?^{\dagger} \underline{n_2}$ $\hspace{4.6cm}$ by Rule (17a) on (13)

Case (19d)**.**

$\quad$ (9) $e = \underline{n_1}$ $\hspace{5.4cm}$ by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (16), only one case applies.

$\quad$ **Case** (16b)**.**

$\quad\quad$ (10) $\underline{n_1}$ `notintro` $\hspace{3.7cm}$ by assumption
$\quad\quad$ Contradicts Lemma 4.0.5.

$\quad\quad$ (11) $\underline{n_1} \not\models_? \underline{n_2}$ $\hspace{4cm}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

$\quad$ **Case** $n_1 = n_2$**.**
$\quad\quad$ (12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ $\hspace{1.7cm}$ by Definition 15
$\quad\quad$ (13) $\underline{n_1} \models \underline{n_2}$ $\hspace{3.9cm}$ by Lemma 2.0.19 on
$\hspace{9cm}$ (12)

$\quad\quad$ (14) $e \models_?^{\dagger} \underline{n_2}$ $\hspace{3.9cm}$ by Rule (17b) on (13)
$\quad$ **Case** $n_1 \neq n_2$**.**
$\quad\quad$ (12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ $\hspace{1.5cm}$ by Definition 15
$\quad\quad$ (13) $\underline{n_1} \not\models \underline{n_2}$ $\hspace{3.9cm}$ by Lemma 2.0.19 on
$\hspace{9cm}$ (12)

$\quad\quad$ (14) $e \not\models_?^{\dagger} \underline{n_2}$ $\hspace{3.8cm}$ by Lemma 2.0.20 on
$\hspace{9cm}$ (11) and (13)

Case (8f)**.**

$\quad$ (7) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ $\hspace{4.6cm}$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \dot{\xi}_1$, $e \models_? \dot{\xi}_1$, and $e \not\models_?^{\dagger} \dot{\xi}_1$ holds. The same goes for $\dot{\xi}_2$. By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $e \models \dot{\xi}_1, e \models \dot{\xi}_2$.

(8)	$e \models \dot{\xi}_1$	by assumption
(9)	$e \not\models_? \dot{\xi}_1$	by assumption
(10)	$e \models \dot{\xi}_2$	by assumption
(11)	$e \not\models_? \dot{\xi}_2$	by assumption
(12)	$e \models \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (14e) on (8)
(13)	$e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (17b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14)	$\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_?$	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14)	$e \models_? \dot{\xi}_1$	by assumption

Contradicts (9).

Case (16d).

(14)	$e \models_? \dot{\xi}_2$	by assumption

Contradicts (11).

(15)	$e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$	by contradiction

Case $e \models \dot{\xi}_1, e \models_? \dot{\xi}_2$.

(8)	$e \models \dot{\xi}_1$	by assumption
(9)	$e \not\models_? \dot{\xi}_1$	by assumption
(10)	$e \not\models \dot{\xi}_2$	by assumption
(11)	$e \models_? \dot{\xi}_2$	by assumption
(12)	$e \models \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (14e) on (8)
(13)	$e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (17b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14)	$\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_?$	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14)	$e \models_? \dot{\xi}_1$	by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \dot{\xi}_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models \dot{\xi}_1, e \not\models_?^\dagger \dot{\xi}_2$.

(8) $e \models \dot{\xi}_1$ by assumption
(9) $e \not\models_? \dot{\xi}_1$ by assumption
(10) $e \not\models \dot{\xi}_2$ by assumption
(11) $e \not\models_? \dot{\xi}_2$ by assumption
(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (14e) on (8)
(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (17b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\dot{\xi}_1 \vee \dot{\xi}_2 \; \texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \dot{\xi}_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \dot{\xi}_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models_? \dot{\xi}_1, e \models \dot{\xi}_2$.

(8) $e \not\models \dot{\xi}_1$ by assumption
(9) $e \models_? \dot{\xi}_1$ by assumption
(10) $e \models \dot{\xi}_2$ by assumption
(11) $e \not\models_? \dot{\xi}_2$ by assumption
(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (14f) on (10)
(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (17b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\dot{\xi}_1 \vee \dot{\xi}_2 \; \texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

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Case (16c)**.**

  (14) $e \not\models \dot{\xi}_2$ by assumption

  Contradicts (10).

Case (16d)**.**

  (14) $e \models_? \dot{\xi}_2$ by assumption

  Contradicts (11).

  (15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models_? \dot{\xi}_1, e \models_? \dot{\xi}_2$**.**

  (8) $e \not\models \dot{\xi}_1$ by assumption

  (9) $e \models_? \dot{\xi}_1$ by assumption

  (10) $e \not\models \dot{\xi}_2$ by assumption

  (11) $e \models_? \dot{\xi}_2$ by assumption

  (12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16c) on (9)

                  and (10)

  (13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (17a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (14), only two cases apply.

Case (14e)**.**

  (14) $e \models \dot{\xi}_1$ by assumption

  Contradicts (8)

Case (14f)**.**

  (14) $e \models \dot{\xi}_2$ by assumption

  Contradicts (10)

  (15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models_? \dot{\xi}_1, e \not\models_?^\dagger \dot{\xi}_2$**.**

  (8) $e \not\models \dot{\xi}_1$ by assumption

  (9) $e \models_? \dot{\xi}_1$ by assumption

  (10) $e \not\models \dot{\xi}_2$ by assumption

  (11) $e \not\models_? \dot{\xi}_2$ by assumption

  (12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16c) on (9)

                  and (10)

  (13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (17a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

      (14) $e \models \dot{\xi}_1$ by assumption

   Contradicts (8).

Case (14f).

      (14) $e \models \dot{\xi}_2$ by assumption

   Contradicts (10).

    (15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models_?^\dagger \dot{\xi}_1, e \models \dot{\xi}_2$.

    (8) $e \not\models \dot{\xi}_1$ by assumption

    (9) $e \not\models_? \dot{\xi}_1$ by assumption

   (10) $e \models \dot{\xi}_2$ by assumption

   (11) $e \not\models_? \dot{\xi}_2$ by assumption

   (12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (14f) on (10)

   (13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (17b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

      (14) $\dot{\xi}_1 \vee \dot{\xi}_2 \ \texttt{refutable}_?$ by assumption

   Contradicts Lemma 2.0.17.

Case (16c).

      (14) $e \not\models \dot{\xi}_2$ by assumption

   Contradicts (10).

Case (16d).

      (14) $e \models_? \dot{\xi}_2$ by assumption

   Contradicts (11).

    (15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models_?^\dagger \dot{\xi}_1, e \models_? \dot{\xi}_2$.

    (8) $e \not\models \dot{\xi}_1$ by assumption

    (9) $e \not\models_? \dot{\xi}_1$ by assumption

   (10) $e \not\models \dot{\xi}_2$ by assumption

   (11) $e \models_? \dot{\xi}_2$ by assumption

   (12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16d) on (11) and (8)

   (13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (17a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

    (14) $e \models \dot{\xi}_1$ by assumption

    Contradicts (8)

Case (14f).

    (14) $e \models \dot{\xi}_2$ by assumption

    Contradicts (10)

    (15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models_?^\dagger \dot{\xi}_1, e \not\models_?^\dagger \dot{\xi}_2$.

    (8) $e \not\models \dot{\xi}_1$ by assumption

    (9) $e \not\models_? \dot{\xi}_1$ by assumption

    (10) $e \not\models \dot{\xi}_2$ by assumption

    (11) $e \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (14) on it, only two cases apply.

Case (14e).

    (12) $e \models \dot{\xi}_1$ by assumption

    Contradicts (8).

Case (14f).

    (12) $e \models \dot{\xi}_2$ by assumption

    Contradicts (10).

    (13) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

    (14) $\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_?$ by assumption

    Contradicts Lemma 2.0.17.

Case (16c).

    (14) $e \models_? \dot{\xi}_1$ by assumption

    Contradicts (9).

Case (16d).

    (14) $e \models_? \dot{\xi}_2$ by assumption

    Contradicts (11).

$$(15) \quad e \not\models_? \dot\xi_1 \vee \dot\xi_2 \qquad \text{by contradiction}$$

$$(16) \quad e \not\models_?^\dagger \dot\xi_1 \vee \dot\xi_2 \qquad \text{by Lemma 2.0.20 on}$$
(13) and (15)

Case (8g)**.**

$$(7) \quad \dot\xi = \mathtt{inl}(\dot\xi_1) \qquad \text{by assumption}$$

$$(8) \quad \tau = (\tau_1 + \tau_2) \qquad \text{by assumption}$$

$$(9) \quad \dot\xi_1 : \tau_1 \qquad \text{by assumption}$$

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

$$(10) \quad e = (\![\,]\!)^u, (\![e_0]\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$$
by assumption

$$(11) \quad e \; \mathtt{notintro} \qquad \text{by Rule}$$
(26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \mathtt{inl}(\dot\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

$$(12) \quad e \not\models \mathtt{inl}(\dot\xi_1) \qquad \text{by contradiction}$$

By case analysis on the value of $\textit{refutable}_?(\mathtt{inl}(\dot\xi_1))$.

Case $\textit{refutable}_?(\mathtt{inl}(\dot\xi_1)) = \text{true}$**.**

$$(13) \quad \textit{refutable}_?(\mathtt{inl}(\dot\xi_1)) = \text{true} \qquad \text{by assumption}$$

$$(14) \quad \mathtt{inl}(\dot\xi_1) \; \mathtt{refutable}_? \qquad \text{by Lemma 2.0.14 on}$$
(13)

$$(15) \quad e \models_? \mathtt{inl}(\dot\xi_1) \qquad \text{by Rule (16b) on (11)}$$
and (14)

$$(16) \quad e \models_?^\dagger \mathtt{inl}(\dot\xi_1) \qquad \text{by Rule (17a) on (15)}$$

Case $\textit{refutable}_?(\mathtt{inl}(\dot\xi_1)) = \text{false}$**.**

$$(13) \quad \textit{refutable}_?(\mathtt{inl}(\dot\xi_1)) = \text{false} \qquad \text{by assumption}$$

$$(14) \quad \cancel{\mathtt{inl}(\dot\xi_1) \; \mathtt{refutable}_?} \qquad \text{by Lemma 2.0.14 on}$$
(13)

Assume $e \models_? \mathtt{inl}(\dot\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

$$(15) \quad \mathtt{inl}(\dot\xi_1) \; \mathtt{refutable}_? \qquad \text{by assumption}$$
Contradicts (14).

$$(16) \quad e \not\models_? \mathtt{inl}(\dot\xi_1) \qquad \text{by contradiction}$$

$$(17) \quad e \not\models^\dagger_? \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by Lemma 2.0.20 on}$$
$$\text{(12) and (16)}$$

Case (19j).

$$(10) \quad e = \mathtt{inl}_{\tau_2}(e_1) \qquad\qquad \text{by assumption}$$
$$(11) \quad \cdot\,;\Delta \vdash e_1 : \tau_1 \qquad\qquad \text{by assumption}$$
$$(12) \quad e_1 \ \mathtt{final} \qquad\qquad \text{by Lemma 4.0.2 on (6)}$$

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \dot\xi_1$, $e_1 \models_? \dot\xi_1$, and $e_1 \not\models^\dagger_? \dot\xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \dot\xi_1$.

$$(13) \quad e_1 \models \dot\xi_1 \qquad\qquad \text{by assumption}$$
$$(14) \quad e_1 \not\models_? \dot\xi_1 \qquad\qquad \text{by assumption}$$
$$(15) \quad \mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by Rule (14g) on (13)}$$
$$(16) \quad \mathtt{inl}_{\tau_2}(e_1) \models^\dagger_? \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by Rule (17b) on (15)}$$

Assume $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot\xi_1)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

$$(17) \quad \mathtt{inl}_{\tau_2}(e_1) \ \mathtt{notintro} \qquad \text{by assumption}$$

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

$$(17) \quad e_1 \models_? \dot\xi_1$$

Contradicts (14).

$$(18) \quad \mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by contradiction}$$

Case $e_1 \models_? \dot\xi_1$.

$$(13) \quad e_1 \not\models \dot\xi_1 \qquad\qquad \text{by assumption}$$
$$(14) \quad e_1 \models_? \dot\xi_1 \qquad\qquad \text{by assumption}$$
$$(15) \quad \mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by Rule (16e) on (14)}$$
$$(16) \quad \mathtt{inl}_{\tau_2}(e_1) \models^\dagger_? \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by Rule (17a) on (15)}$$

Assume $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

$$(17) \quad e_1 \models \dot\xi_1$$

Contradicts (13).

$$(18) \quad \mathtt{inl}_{\tau_2}(e_1) \not\models \mathtt{inl}(\dot\xi_1) \qquad\qquad \text{by contradiction}$$

Case $e_1 \not\models_?^\dagger \dot{\xi}_1$**.**

    (13) $e_1 \not\models \dot{\xi}_1$ by assumption
    (14) $e_1 \not\models_? \dot{\xi}_1$ by assumption

Assume $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\xi}_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g)**.**

    (15) $e_1 \models \dot{\xi}_1$
   Contradicts (13).

    (16) $\mathtt{inl}_{\tau_2}(e_1) \not\models \mathtt{inl}(\dot{\xi}_1)$ by contradiction

Assume $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

    (17) $\mathtt{inl}_{\tau_2}(e_1) \ \mathtt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e)**.**

    (17) $e_1 \models_? \dot{\xi}_1$
   Contradicts (14).

    (18) $\mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inl}(\dot{\xi}_1)$ by contradiction
    (19) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on
                                                  (16) and (18)

Case (19k)**.**

    (10) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inl}(\dot{\xi}_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

    (11) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inl}(\dot{\xi}_1)$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inl}(\dot{\xi}_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

    (12) $\mathtt{inr}_{\tau_1}(e_2) \ \mathtt{notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

    (13) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inl}(\dot{\xi}_1)$ by contradiction
    (14) $\mathtt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on
                                                 (11) and (13)

Case (8h)**.**

  (7) $\dot{\xi} = \mathtt{inr}(\dot{\xi}_2)$ by assumption

  (8) $\tau = (\tau_1 + \tau_2)$ by assumption

  (9) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

  (10) $e = (\!|\rangle\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}\hat{s}\}$

                by assumption

  (11) $e \; \mathtt{notintro}$ by Rule

               (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

  (12) $e \not\models \mathtt{inr}(\dot{\xi}_2)$ by contradiction

By case analysis on the value of $refutable_?(\mathtt{inr}(\dot{\xi}_2))$. ⟵ inr is refutable

 Case $refutable_?(\mathtt{inr}(\dot{\xi}_2)) = \mathrm{true}$**.**

   (13) $refutable_?(\mathtt{inr}(\dot{\xi}_2)) = \mathrm{true}$ by assumption

   (14) $\mathtt{inr}(\dot{\xi}_2) \; \mathtt{refutable}_?$ by Lemma 2.0.14 on (13)

   (15) $e \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (16b) on (11) and (14)

   (16) $e \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ by Rule (17a) on (15)

 Case $refutable_?(\mathtt{inr}(\dot{\xi}_2)) = \mathrm{false}$**.**

   (13) $refutable_?(\mathtt{inr}(\dot{\xi}_2)) = \mathrm{false}$ by assumption

   (14) $\cancel{\mathtt{inr}(\dot{\xi}_2) \; \mathtt{refutable}_?}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

  Case (16b)**.**

    (15) $\mathtt{inr}(\dot{\xi}_2) \; \mathtt{refutable}_?$ by assumption

   Contradicts (14).

   (16) $e \not\models_? \mathtt{inr}(\dot{\xi}_2)$ by contradiction

   (17) $e \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ by Lemma 2.0.20 on (12) and (16)

Case (19j)**.**

  (10) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

   (11) $\mathtt{inl}_{\tau_2}(e_1) \not\models \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Assume $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

   (12) $\mathtt{inl}_{\tau_2}(e_1) \, \mathtt{notintro}$ by assumption

   By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inr}(\dot{\xi}_2)$ by contradiction

   (14) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ by Lemma 2.0.20 on (11) and (13)

Case (19k).

   (10) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

   (11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

   (12) $e_2 \, \mathtt{final}$ by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \dot{\xi}_2$, $e_2 \models_? \dot{\xi}_2$, and $e_2 \not\models_?^\dagger \dot{\xi}_2$ holds. By case analysis on which one holds.

Case $e_2 \models \dot{\xi}_2$.

   (13) $e_2 \models \dot{\xi}_2$ by assumption

   (14) $e_2 \not\models_? \dot{\xi}_2$ by assumption

   (15) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$ by Rule (14g) on (13)

   (16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ by Rule (17b) on (15)

   Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

   Case (16b).

   (17) $\mathtt{inr}_{\tau_1}(e_2) \, \mathtt{notintro}$ by assumption

   By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

   Case (16f).

   (17) $e_2 \models_? \dot{\xi}_2$

   Contradicts (14).

   (18) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Case $e_2 \models_? \dot{\xi}_2$.

   (13) $e_2 \not\models \dot{\xi}_2$ by assumption

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(14) $e_2 \models_? \dot{\xi}_2$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (16f) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ by Rule (17a) on (15)

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(17) $e_2 \models \dot{\xi}_2$

Contradicts (13).

(18) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Case $e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

(13) $e_2 \not\models \dot{\xi}_2$ by assumption

(14) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(15) $e_2 \models \dot{\xi}_2$

Contradicts (13).

(16) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(17) $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_? \dot{\xi}_2$

Contradicts (14).

(18) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inr}(\dot{\xi}_2)$ by contradiction

(19) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on (16) and (18)

Case (14i).

(7) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\dot{\xi}_1 : \tau_1$ by assumption

(10) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

(11)	$e = (\!|\!|)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$	
		by assumption
(12)	e $\mathtt{notintro}$	by Rule (26a),(26b),(26c),(26d),(26e),(26f)
(13)	e $\mathtt{indet}$	by Lemma 4.0.9 on (6) and (12)
(14)	$\mathtt{prl}(e)$ $\mathtt{indet}$	by Rule (24g) on (13)
(15)	$\mathtt{prl}(e)$ $\mathtt{final}$	by Rule (25b) on (14)
(16)	$\mathtt{prr}(e)$ $\mathtt{indet}$	by Rule (24h) on (13)
(17)	$\mathtt{prr}(e)$ $\mathtt{final}$	by Rule (25b) on (16)
(18)	$\cdot\,; \Delta \vdash \mathtt{prl}(e) : \tau_1$	by Rule (19h) on (5)
(19)	$\cdot\,; \Delta \vdash \mathtt{prr}(e) : \tau_2$	by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\mathtt{prl}(e) \models \dot\xi_1$, $\mathtt{prl}(e) \models_? \dot\xi_1$, and $\mathtt{prl}(e) \not\models_?^\dagger \dot\xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\mathtt{prr}(e) \models \dot\xi_2$, $\mathtt{prr}(e) \models_? \dot\xi_2$, and $\mathtt{prr}(e) \not\models_?^\dagger \dot\xi_2$ holds.
By case analysis on which conclusion holds for $\dot\xi_1$ and $\dot\xi_2$.

Case $\mathtt{prl}(e) \models \dot\xi_1, \mathtt{prr}(e) \models \dot\xi_2$**.**

(20)	$\mathtt{prl}(e) \models \dot\xi_1$	by assumption
(21)	$\mathtt{prl}(e) \not\models_? \dot\xi_1$	by assumption
(22)	$\mathtt{prr}(e) \models \dot\xi_2$	by assumption
(23)	$\mathtt{prr}(e) \not\models_? \dot\xi_2$	by assumption
(24)	$e \models (\dot\xi_1, \dot\xi_2)$	by Rule (14j) on (12) and (20) and (22)
(25)	$e \models_?^\dagger (\dot\xi_1, \dot\xi_2)$	by Rule (17b) on (24)
(26)	$\cancel{(\dot\xi_1, \dot\xi_2)\ \mathtt{refutable}_?}$	by Lemma 2.0.18 on (12) and (24)

Assume $e \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

(27)	$(\dot\xi_1, \dot\xi_2)$ $\mathtt{refutable}_?$	by assumption

Contradicts (26).

(28)	$e \not\models_? (\dot\xi_1, \dot\xi_2)$	by contradiction

Case $\mathtt{prl}(e) \models \dot\xi_1, \mathtt{prr}(e) \models_? \dot\xi_2$**.**

(20)	$\mathtt{prl}(e) \models \dot\xi_1$	by assumption

(21) $\mathtt{prl}(e) \not\models_? \dot\xi_1$	by assumption
(22) $\mathtt{prr}(e) \not\models \dot\xi_2$	by assumption
(23) $\mathtt{prr}(e) \models_? \dot\xi_2$	by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\mathtt{prr}(e) \models \dot\xi_2$	by assumption

Contradicts (22)

(25) $e \not\models (\dot\xi_1, \dot\xi_2)$	by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

assume no "or" and "and" in pair

(26) $\dot\xi_2$ $\mathtt{refutable}_?$	by assumption
(27) $(\dot\xi_1, \dot\xi_2)$ $\mathtt{refutable}_?$	by Rule (10e) on (26)
(28) $e \models_? (\dot\xi_1, \dot\xi_2)$	by Rule (16b) on (12) and (27)
(29) $e \models_?^\dagger (\dot\xi_1, \dot\xi_2)$	by Rule (17a) on (28)

Case $\mathtt{prl}(e) \models \dot\xi_1, \mathtt{prr}(e) \not\models_?^\dagger \dot\xi_2$.

(20) $\mathtt{prl}(e) \models \dot\xi_1$	by assumption
(21) $\mathtt{prl}(e) \not\models_? \dot\xi_1$	by assumption
(22) $\mathtt{prr}(e) \not\models \dot\xi_2$	by assumption
(23) $\mathtt{prr}(e) \not\models_? \dot\xi_2$	by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\mathtt{prr}(e) \models \dot\xi_2$	by assumption

Contradicts (22).

(25) $e \not\models (\dot\xi_1, \dot\xi_2)$	by contradiction

Assume $e \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) $(\dot\xi_1, \dot\xi_2)$ $\mathtt{refutable}_?$	by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) $\dot\xi_1$ $\mathtt{refutable}_?$	by assumption
(28) $\mathtt{prl}(e)$ $\mathtt{notintro}$	by Rule (26e)

(29) $\mathtt{prl}(e) \models_? \dot\xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) $\dot\xi_2\ \mathtt{refutable}_?$ by assumption

(28) $\mathtt{prr}(e)\ \mathtt{notintro}$ by Rule (26f)

(29) $\mathtt{prr}(e) \models_? \dot\xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\dot\xi_1, \dot\xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\dot\xi_1, \dot\xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\mathtt{prl}(e) \models_? \dot\xi_1, \mathtt{prr}(e) \models \dot\xi_2$**.**

(20) $\mathtt{prl}(e) \not\models \dot\xi_1$ by assumption

(21) $\mathtt{prl}(e) \models_? \dot\xi_1$ by assumption

(22) $\mathtt{prr}(e) \models \dot\xi_2$ by assumption

(23) $\mathtt{prr}(e) \not\models_? \dot\xi_2$ by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\mathtt{prl}(e) \models \dot\xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

Case (16b).

(26) $\dot\xi_1\ \mathtt{refutable}_?$ by assumption

(27) $(\dot\xi_1, \dot\xi_2)\ \mathtt{refutable}_?$ by Rule (10e) on (26)

(28) $e \models_? (\dot\xi_1, \dot\xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\dot\xi_1, \dot\xi_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \models_? \dot\xi_1, \mathtt{prr}(e) \models_? \dot\xi_2$**.**

(20) $\mathtt{prl}(e) \not\models \dot\xi_1$ by assumption

(21) $\mathtt{prl}(e) \models_? \dot\xi_1$ by assumption

(22) $\mathtt{prr}(e) \not\models \dot\xi_2$ by assumption

(23) $\mathtt{prr}(e) \models_? \dot\xi_2$ by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14), only one case applies.

assume no "or" and "and" in pair

Case (14j)**.**

    (24) $\mathtt{prl}(e) \models \dot{\xi}_1$ by assumption

  Contradicts (20).

  (25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b)**.**

    (26) $\dot{\xi}_2 \; \mathtt{refutable}_?$ by assumption

    (27) $(\dot{\xi}_1, \dot{\xi}_2) \; \mathtt{refutable}_?$ by Rule (10e) on (26)

    (28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16b) on (12)

                              and (27)

    (29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \models_? \dot{\xi}_1, \mathtt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$**.**

  (20) $\mathtt{prl}(e) \not\models \dot{\xi}_1$ by assumption

  (21) $\mathtt{prl}(e) \models_? \dot{\xi}_1$ by assumption

  (22) $\mathtt{prr}(e) \not\models \dot{\xi}_2$ by assumption

  (23) $\mathtt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

  Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14), only one case applies.

  Case (14j)**.**

    (24) $\mathtt{prl}(e) \models \dot{\xi}_1$ by assumption

    Contradicts (20)

  (25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

Case (16b)**.**

    (26) $\dot{\xi}_1 \; \mathtt{refutable}_?$ by assumption

    (27) $(\dot{\xi}_1, \dot{\xi}_2) \; \mathtt{refutable}_?$ by Rule (10e) on (26)

    (28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16b) on (12)

                              and (27)

    (29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \not\models_?^\dagger \dot{\xi}_1, \mathtt{prr}(e) \models \dot{\xi}_2$**.**

  (20) $\mathtt{prl}(e) \not\models \dot{\xi}_1$ by assumption

  (21) $\mathtt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

  (22) $\mathtt{prr}(e) \models \dot{\xi}_2$ by assumption

  (23) $\mathtt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

Note (margin, upper): assume no "or" and "and" in pair

Note (margin, lower): assume no "or" and "and" in pair

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j)**.**

    (24) $\mathtt{prl}(e) \models \dot\xi_1$ by assumption

  Contradicts (20)

  (25) $e \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

Assume $e \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

    (26) $(\dot\xi_1, \dot\xi_2)$ $\mathtt{refutable}_?$ by assumption

  By rule induction over Rules (10) on (26), only two cases apply.

  Case (10d)**.**

     (27) $\dot\xi_1$ $\mathtt{refutable}_?$ by assumption

     (28) $\mathtt{prl}(e)$ $\mathtt{notintro}$ by Rule (26e)

     (29) $\mathtt{prl}(e) \models_? \dot\xi_1$ by Rule (16b) on (28) and (27)

   Contradicts (21).

  Case (10e)**.**

     (27) $\dot\xi_2$ $\mathtt{refutable}_?$ by assumption

     (28) $\mathtt{prr}(e)$ $\mathtt{notintro}$ by Rule (26f)

     (29) $\mathtt{prr}(e) \models_? \dot\xi_2$ by Rule (16b) on (28) and (27)

   Contradicts (23).

  (30) $e \not\models_? (\dot\xi_1, \dot\xi_2)$ by contradiction

  (31) $e \not\models_?^\dagger (\dot\xi_1, \dot\xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\mathtt{prl}(e) \not\models_?^\dagger \dot\xi_1, \mathtt{prr}(e) \models_? \dot\xi_2$**.**

  (20) $\mathtt{prl}(e) \not\models \dot\xi_1$ by assumption

  (21) $\mathtt{prl}(e) \not\models_? \dot\xi_1$ by assumption

  (22) $\mathtt{prr}(e) \not\models \dot\xi_2$ by assumption

  (23) $\mathtt{prr}(e) \models_? \dot\xi_2$ by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

    (24) $\mathtt{prl}(e) \models \dot\xi_1$ by assumption

  Contradicts (20).

(25) $e \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

    (26) $\dot\xi_2\ \texttt{refutable}_?$ by assumption

    (27) $(\dot\xi_1, \dot\xi_2)\ \texttt{refutable}_?$ by Rule (10e) on (26)

    (28) $e \models_? (\dot\xi_1, \dot\xi_2)$ by Rule (16b) on (12)
           and (27)

    (29) $e \models_?^\dagger (\dot\xi_1, \dot\xi_2)$ by Rule (17a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \dot\xi_1, \texttt{prr}(e) \not\models_?^\dagger \dot\xi_2$.

    (20) $\texttt{prl}(e) \not\models \dot\xi_1$ by assumption

    (21) $\texttt{prl}(e) \not\models_? \dot\xi_1$ by assumption

    (22) $\texttt{prr}(e) \not\models \dot\xi_2$ by assumption

    (23) $\texttt{prr}(e) \not\models_? \dot\xi_2$ by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

    (24) $\texttt{prl}(e) \models \dot\xi_1$ by assumption

  Contradicts (20)

    (25) $e \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

Assume $e \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

    (26) $(\dot\xi_1, \dot\xi_2)\ \texttt{refutable}_?$ by assumption

  By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

    (27) $\dot\xi_1\ \texttt{refutable}_?$ by assumption

    (28) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (26e)

    (29) $\texttt{prl}(e) \models_? \dot\xi_1$ by Rule (16b) on (28)
           and (27)

  Contradicts (21).

Case (10e).

    (27) $\dot\xi_2\ \texttt{refutable}_?$ by assumption

    (28) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (26f)

    (29) $\texttt{prr}(e) \models_? \dot\xi_2$ by Rule (16b) on (28)
           and (27)

  Contradicts (23).

assume no "or" and "and" in pair

(30)	$e \not\models_? (\dot\xi_1, \dot\xi_2)$	by contradiction
(31)	$e \not\models_?^\dagger (\dot\xi_1, \dot\xi_2)$	by Lemma 2.0.20 on (25) and (30)

Case (19g).

(11)	$e = (e_1, e_2)$	by assumption
(12)	$\cdot\,; \Delta \vdash e_1 : \tau_1$	by assumption
(13)	$\cdot\,; \Delta \vdash e_2 : \tau_2$	by assumption
(14)	e_1 `final`	by Lemma 4.0.4 on (6)
(15)	e_2 `final`	by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \dot\xi_1$, $e_1 \models_? \dot\xi_1$, and $e_1 \models \overline{\dot\xi_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \dot\xi_2$, $e_2 \models_? \dot\xi_2$, and $e_2 \models \overline{\dot\xi_2}$ holds.
By case analysis on which conclusion holds for $\dot\xi_1$ and $\dot\xi_2$.

Case $e_1 \models \dot\xi_1, e_2 \models \dot\xi_2$**.**

(16)	$e_1 \models \dot\xi_1$	by assumption
(17)	$e_1 \not\models_? \dot\xi_1$	by assumption
(18)	$e_2 \models \dot\xi_2$	by assumption
(19)	$e_2 \not\models_? \dot\xi_2$	by assumption
(20)	$(e_1, e_2) \models (\dot\xi_1, \dot\xi_2)$	by Rule (14i) on (16) and (18)
(21)	$(e_1, e_2) \models_?^\dagger (\dot\xi_1, \dot\xi_2)$	by Rule (17b) on (20)

Assume $(e_1, e_2) \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (16g).

(22)	$e_1 \models_? \dot\xi_1$	by assumption

Contradicts (17).

Case (16h).

(22)	$e_2 \models_? \dot\xi_2$	by assumption

Contradicts (19).

Case (16i).

(22)	$e_1 \models_? \dot\xi_1$	by assumption

Contradicts (17).

(23)	$(e_1, e_2) \not\models_? (\dot\xi_1, \dot\xi_2)$	by contradiction

Case $e_1 \models \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$**.**

(16)	$e_1 \models \dot{\xi}_1$	by assumption
(17)	$e_1 \not\models_? \dot{\xi}_1$	by assumption
(18)	$e_2 \not\models \dot{\xi}_2$	by assumption
(19)	$e_2 \models_? \dot{\xi}_2$	by assumption
(20)	$(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (16h) on (16) and (19)
(21)	$(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (14i)**.**

(22)	$e_2 \models \dot{\xi}_2$	by assumption

Contradicts (18).

(23)	$(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$	by contradiction

Case $e_1 \models \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

(16)	$e_1 \models \dot{\xi}_1$	by assumption
(17)	$e_1 \not\models_? \dot{\xi}_1$	by assumption
(18)	$e_2 \not\models \dot{\xi}_2$	by assumption
(19)	$e_2 \not\models_? \dot{\xi}_2$	by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(20)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (14i)**.**

(20)	$e_2 \models \dot{\xi}_2$	by assumption

Contradicts (18).

(21)	$(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$	by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

(22)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (16g)**.**

    (22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (16h)**.**

    (22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (16i)**.**

    (22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

    (23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

    (24) $(e_1, e_2) \not\models_?^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \models_? \dot{\xi}_1, e_2 \models \dot{\xi}_2$**.**

    (16) $e_1 \not\models \dot{\xi}_1$ by assumption

    (17) $e_1 \models_? \dot{\xi}_1$ by assumption

    (18) $e_2 \models \dot{\xi}_2$ by assumption

    (19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

    (20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16g) on (17) and (18)

    (21) $(e_1, e_2) \models_?^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

    (22) (e_1, e_2) `notintro`   by assumption

Contradicts Lemma 4.0.8.

Case (14i)**.**

    (22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

    (23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$**.**

    (16) $e_1 \not\models \dot{\xi}_1$ by assumption

    (17) $e_1 \models_? \dot{\xi}_1$ by assumption

    (18) $e_2 \not\models \dot{\xi}_2$ by assumption

    (19) $e_2 \models_? \dot{\xi}_2$ by assumption

(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16i) on (17) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

    (22) (e_1, e_2) `notintro`   by assumption

  Contradicts Lemma 4.0.8.

Case (14i).

    (22) $e_1 \models \dot{\xi}_1$ by assumption

  Contradicts (16).

    (23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

    (16) $e_1 \not\models \dot{\xi}_1$ by assumption

    (17) $e_1 \models_? \dot{\xi}_1$ by assumption

    (18) $e_2 \not\models \dot{\xi}_2$ by assumption

    (19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

    (20) (e_1, e_2) `notintro`   by assumption

  Contradicts Lemma 4.0.8.

Case (14i).

    (20) $e_1 \models \dot{\xi}_1$ by assumption

  Contradicts (16).

    (21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

    (22) (e_1, e_2) `notintro`   by assumption

  Contradicts Lemma 4.0.8.

Case (16g).

    (22) $e_2 \models \dot{\xi}_2$ by assumption

  Contradicts (18).

Case (16h).

    (22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (16i).

$\quad$ (22) $e_2 \models_? \dot{\xi}_2$ $\qquad\qquad$ by assumption

$\quad$ Contradicts (19).

$\quad$ (23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by contradiction

$\quad$ (24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by Lemma 2.0.20 on

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (21) and (23)

Case $e_1 \not\models_?^\dagger \dot{\xi}_1, e_2 \models \dot{\xi}_2$**.**

$\quad$ (16) $e_1 \not\models \dot{\xi}_1$ $\qquad\qquad\qquad$ by assumption

$\quad$ (17) $e_1 \not\models_? \dot{\xi}_1$ $\qquad\qquad\quad$ by assumption

$\quad$ (18) $e_2 \models \dot{\xi}_2$ $\qquad\qquad\qquad$ by assumption

$\quad$ (19) $e_2 \not\models_? \dot{\xi}_2$ $\qquad\qquad\quad$ by assumption

$\quad$ Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

$\quad$ (20) (e_1, e_2) `notintro` $\qquad$ by assumption

$\quad$ Contradicts Lemma 4.0.8.

Case (14i).

$\quad$ (20) $e_1 \models \dot{\xi}_1$ $\qquad\qquad\qquad$ by assumption

$\quad$ Contradicts (16).

$\quad$ (21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by contradiction

$\quad$ Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

$\quad$ (22) (e_1, e_2) `notintro` $\qquad$ by assumption

$\quad$ Contradicts Lemma 4.0.8.

Case (16g).

$\quad$ (22) $e_1 \models_? \dot{\xi}_1$ $\qquad\qquad\quad$ by assumption

$\quad$ Contradicts (17).

Case (16h).

$\quad$ (22) $e_2 \models_? \dot{\xi}_2$ $\qquad\qquad\quad$ by assumption

$\quad$ Contradicts (19).

Case (16i).

$\quad$ (22) $e_1 \models_? \dot{\xi}_1$ $\qquad\qquad\quad$ by assumption

$\quad$ Contradicts (17).

$\quad$ (23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ $\qquad$ by contradiction

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(24) $(e_1, e_2) \not\models_?^\dagger (\dot\xi_1, \dot\xi_2)$ by Lemma 2.0.20 on
(21) and (23)

Case $e_1 \not\models_?^\dagger \dot\xi_1, e_2 \models_? \dot\xi_2$.

    (16) $e_1 \not\models \dot\xi_1$ by assumption
    (17) $e_1 \not\models_? \dot\xi_1$ by assumption
    (18) $e_2 \not\models \dot\xi_2$ by assumption
    (19) $e_2 \models_? \dot\xi_2$ by assumption

Assume $(e_1, e_2) \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

  Case (14j).

      (20) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

  Case (14i).

      (20) $e_2 \models \dot\xi_2$ by assumption

    Contradicts (18).

    (21) $(e_1, e_2) \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

  Case (16b).

      (22) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

  Case (16g).

      (22) $e_1 \models_? \dot\xi_1$ by assumption

    Contradicts (17).

  Case (16h).

      (22) $e_1 \models \dot\xi_1$ by assumption

    Contradicts (16).

  Case (16i).

      (22) $e_1 \models_? \dot\xi_1$ by assumption

    Contradicts (17).

    (23) $(e_1, e_2) \not\models_? (\dot\xi_1, \dot\xi_2)$ by contradiction
    (24) $(e_1, e_2) \not\models_?^\dagger (\dot\xi_1, \dot\xi_2)$ by Lemma 2.0.20 on
(21) and (23)

Case $e_1 \not\models_?^\dagger \dot\xi_1, e_2 \not\models_?^\dagger \dot\xi_2$.

    (16) $e_1 \not\models \dot\xi_1$ by assumption
    (17) $e_1 \not\models_? \dot\xi_1$ by assumption
    (18) $e_2 \not\models \dot\xi_2$ by assumption

$\qquad$ (19) $\ e_2 \not\models_? \dot\xi_2 \qquad\qquad\qquad$ by assumption

Assume $(e_1, e_2) \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

$\qquad$ (20) $\ (e_1, e_2) \ \mathtt{notintro} \qquad\qquad$ by assumption

Contradicts Lemma 4.0.8.

Case (14i)**.**

$\qquad$ (20) $\ e_2 \models \dot\xi_2 \qquad\qquad\qquad\quad$ by assumption

Contradicts (18).

$\qquad$ (21) $\ (e_1, e_2) \not\models (\dot\xi_1, \dot\xi_2) \qquad\qquad$ by contradiction

Assume $(e_1, e_2) \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

$\qquad$ (22) $\ (e_1, e_2) \ \mathtt{notintro} \qquad\qquad$ by assumption

Contradicts Lemma 4.0.8.

Case (16g)**.**

$\qquad$ (22) $\ e_1 \models_? \dot\xi_1 \qquad\qquad\qquad$ by assumption

Contradicts (17).

Case (16h)**.**

$\qquad$ (22) $\ e_2 \models_? \dot\xi_2 \qquad\qquad\qquad$ by assumption

Contradicts (19).

Case (16i)**.**

$\qquad$ (22) $\ e_1 \models_? \dot\xi_1 \qquad\qquad\qquad$ by assumption

Contradicts (17).

$\qquad$ (23) $\ (e_1, e_2) \not\models_? (\dot\xi_1, \dot\xi_2) \qquad\qquad$ by contradiction

$\qquad$ (24) $\ (e_1, e_2) \not\models_?^\dagger (\dot\xi_1, \dot\xi_2) \qquad\qquad$ by Lemma 2.0.20 on
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ $ (21) and (23)

$\square$

Definition 1.1.1 (Entailment of Constraints)**.** *Suppose that $\dot\xi_1 : \tau$ and $\dot\xi_2 : \tau$. Then $\dot\xi_1 \models \dot\xi_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e $\mathtt{val}$ we have $e \models_?^\dagger \dot\xi_1$ implies $e \models \dot\xi_2$*

Definition 1.1.2 (Potential Entailment of Constraints)**.** *Suppose that $\dot\xi_1 : \tau$ and $\dot\xi_2 : \tau$. Then $\dot\xi_1 \models_?^\dagger \dot\xi_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e $\mathtt{final}$ we have $e \models_?^\dagger \dot\xi_1$ implies $e \models_?^\dagger \dot\xi_2$*

Corollary 1.1.1. *Suppose that $\dot\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e $\mathtt{final}$. Then $\top \models_?^\dagger \dot\xi$ implies $e \models_?^\dagger \dot\xi$*

Proof.

(1)	$\dot{\xi} : \tau$	by assumption
(2)	$\cdot\,; \Gamma \vdash e : \tau$	by assumption
(3)	$e\ \texttt{final}$	by assumption
(4)	$\top \models^{\dagger}_{?} \dot{\xi}$	by assumption
(5)	$e_1 \models \top$	by Rule (14a)
(6)	$e_1 \models^{\dagger}_{?} \top$	by Rule (17b) on (5)
(7)	$\top : \tau$	by Rule (8a)
(8)	$e_1 \models^{\dagger}_{?} \dot{\xi}_r$	by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

$\square$

2   Match Constraint Language

$$\xi \ ::= \ \top \mid \bot \mid \underline{n} \mid \underline{\cancel{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\frac{\text{CTTruth}}{\top : \tau} \tag{8a}$$

$$\frac{\text{CTFalsity}}{\bot : \tau} \tag{8b}$$

$$\frac{\text{CTNum}}{\underline{n} : \texttt{num}} \tag{8c}$$

$$\frac{\text{CTNotNum}}{\underline{\cancel{n}} : \texttt{num}} \tag{8d}$$

$$\frac{\text{CTAnd} \quad \xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \tag{8e}$$

$$\frac{\text{CTOr} \quad \xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{8f}$$

$$\frac{\text{CTInl} \quad \xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{8g}$$

$$\frac{\text{CTInr} \quad \xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)} \tag{8h}$$

$$\frac{\text{CTPair}}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \qquad \xi_2 : \tau_2 \qquad (8\text{i})$$

$$\frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \tag{8i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\overline{\top} = \bot \tag{9a}$$

$$\overline{\bot} = \top \tag{9b}$$

$$\overline{\underline{n}} = \underline{\cancel{n}} \tag{9c}$$

$$\overline{\underline{\cancel{n}}} = \underline{n} \tag{9d}$$

$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \tag{9e}$$

$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \tag{9f}$$

$$\overline{\texttt{inl}(\xi_1)} = \texttt{inl}(\overline{\xi_1}) \vee \texttt{inr}(\top) \tag{9g}$$

$$\overline{\texttt{inr}(\xi_2)} = \texttt{inr}(\overline{\xi_2}) \vee \texttt{inl}(\top) \tag{9h}$$

$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \tag{9i}$$

$\boxed{\xi \ \texttt{refutable}_?}$ ξ is refutable

$$\frac{\text{RXNum}}{\underline{n} \ \texttt{refutable}_?} \tag{10a}$$

$$\frac{\text{RXInl}}{\texttt{inl}(\xi) \ \texttt{refutable}_?} \tag{10b}$$

$$\frac{\text{RXInr}}{\texttt{inr}(\xi) \ \texttt{refutable}_?} \tag{10c}$$

$$\frac{\text{RXPairL} \quad \xi_1 \ \texttt{refutable}_?}{(\xi_1, \xi_2) \ \texttt{refutable}_?} \tag{10d}$$

$$\frac{\text{RXPairR} \quad \xi_2 \ \texttt{refutable}_?}{(\xi_1, \xi_2) \ \texttt{refutable}_?} \tag{10e}$$

$$\frac{\text{RXOr} \quad \xi_1 \ \texttt{refutable}_? \qquad \xi_2 \ \texttt{refutable}_?}{\xi_1 \vee \xi_2 \ \texttt{refutable}_?} \tag{10f}$$

$\boxed{refutable_?(\xi)}$

$$refutable_?(\underline{n}) = \text{true} \tag{11a}$$

$$refutable_?(\underline{\varkappa}) = \text{true} \tag{11b}$$

$$refutable_?(?) = \text{true} \tag{11c}$$

$$refutable_?(\mathtt{inl}(\xi)) = refutable_?(\xi) \tag{11d}$$

$$refutable_?(\mathtt{inr}(\xi)) = refutable_?(\xi) \tag{11e}$$

$$refutable_?((\xi_1, \xi_2)) = refutable_?(\xi_1) \text{ or } refutable_?(\xi_2) \tag{11f}$$

$$refutable_?(\xi_1 \vee \xi_2) = refutable_?(\xi_1) \text{ and } refutable_?(\xi_2) \tag{11g}$$

$$\text{Otherwise} \quad refutable_?(\xi) = \text{false} \tag{11h}$$

$\boxed{\dot{\top}(\xi_1) = \xi_2}$

$$\dot{\top}(\top) = \top \tag{12a}$$

$$\dot{\top}(\bot) = \bot \tag{12b}$$

$$\dot{\top}(?) = \top \tag{12c}$$

$$\dot{\top}(\underline{n}) = \underline{n} \tag{12d}$$

$$\dot{\top}(\underline{\varkappa}) = \underline{\varkappa} \tag{12e}$$

$$\dot{\top}(\xi_1 \wedge \xi_2) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \tag{12f}$$

$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{12g}$$

$$\dot{\top}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\top}(\xi)) \tag{12h}$$

$$\dot{\top}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\top}(\xi)) \tag{12i}$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{12j}$$

$\boxed{\dot{\bot}(\xi_1) = \xi_2}$

$$\dot{\bot}(\top) = \top \tag{13a}$$

$$\dot{\bot}(\bot) = \bot \tag{13b}$$

$$\dot{\bot}(?) = \bot \tag{13c}$$

$$\dot{\bot}(\underline{n}) = \underline{n} \tag{13d}$$

$$\dot{\bot}(\underline{\varkappa}) = \underline{\varkappa} \tag{13e}$$

$$\dot{\bot}(\xi_1 \wedge \xi_2) = \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2) \tag{13f}$$

$$\dot{\bot}(\xi_1 \vee \xi_2) = \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2) \tag{13g}$$

$$\dot{\bot}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\bot}(\xi)) \tag{13h}$$

$$\dot{\bot}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\bot}(\xi)) \tag{13i}$$

$$\dot{\bot}((\xi_1, \xi_2)) = (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \tag{13j}$$

$\boxed{e \models \xi}$ e satisfies ξ

CSTruth
$$\frac{}{e \models \top} \tag{14a}$$

CSNum
$$\frac{}{\underline{n} \models \underline{n}} \tag{14b}$$

CSNotNum
$$\frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}} \tag{14c}$$

CSAnd
$$\frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \tag{14d}$$

CSOrL
$$\frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2} \tag{14e}$$

CSOrR
$$\frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2} \tag{14f}$$

CSInl
$$\frac{e_1 \models \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)} \tag{14g}$$

CSInr
$$\frac{e_2 \models \xi_2}{\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)} \tag{14h}$$

CSPair
$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \tag{14i}$$

CSNotIntroPair
$$\frac{e \ \mathtt{notintro} \qquad \mathtt{prl}(e) \models \xi_1 \qquad \mathtt{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)} \tag{14j}$$

$\boxed{satisfy(e, \xi)}$

$$satisfy(e, \top) = \text{true} \tag{15a}$$

$$satisfy(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{15b}$$

$$satisfy(\underline{n_1}, \underline{\cancel{n_2}}) = (n_1 \neq n_2) \tag{15c}$$

$$satisfy(e, \xi_1 \wedge \xi_2) = satisfy(e, \xi_1) \text{ and } satisfy(e, \xi_2) \tag{15d}$$

$$satisfy(e, \xi_1 \vee \xi_2) = satisfy(e, \xi_1) \text{ or } satisfy(e, \xi_2) \tag{15e}$$

$$satisfy(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\xi_1)) = satisfy(e_1, \xi_1) \tag{15f}$$

$$satisfy(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\xi_2)) = satisfy(e_2, \xi_2) \tag{15g}$$

$$satisfy((e_1, e_2), (\xi_1, \xi_2)) = satisfy(e_1, \xi_1) \text{ and } satisfy(e_2, \xi_2) \tag{15h}$$

$$satisfy(\llparenthesis\rrparenthesis^u, (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(\llparenthesis\rrparenthesis^u), \xi_1) \text{ and } satisfy(\mathtt{prr}(\llparenthesis\rrparenthesis^u), \xi_2) \tag{15i}$$

$$satisfy(\llparenthesis e\rrparenthesis^u, (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(\llparenthesis e\rrparenthesis^u), \xi_1) \text{ and } satisfy(\mathtt{prr}(\llparenthesis e\rrparenthesis^u), \xi_2) \tag{15j}$$

$$satisfy(e_1(e_2), (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(e_1(e_2)), \xi_1) \\ \text{and } satisfy(\mathtt{prr}(e_1(e_2)), \xi_2) \tag{15k}$$

$$satisfy(\mathtt{match}(e)\{\hat{rs}\}, (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(\mathtt{match}(e)\{\hat{rs}\}), \xi_1) \\ \text{and } satisfy(\mathtt{prr}(\mathtt{match}(e)\{\hat{rs}\}), \xi_2) \tag{15l}$$

$$satisfy(\mathtt{prl}(e), (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(\mathtt{prl}(e)), \xi_1) \\ \text{and } satisfy(\mathtt{prr}(\mathtt{prl}(e)), \xi_2) \tag{15m}$$

$$satisfy(\mathtt{prr}(e), (\xi_1, \xi_2)) = satisfy(\mathtt{prl}(\mathtt{prr}(e)), \xi_1) \\ \text{and } satisfy(\mathtt{prr}(\mathtt{prr}(e)), \xi_2) \tag{15n}$$

$$\text{Otherwise} \quad satisfy(e, \xi) = \text{false} \tag{15o}$$

$\boxed{e \models_? \xi}$ $\quad$ e may satisfy ξ

$$\frac{\text{CMSUnknown}}{e \models_? \,?} \tag{16a}$$

$$\frac{\text{CMSNotIntro}}{e \; \mathtt{notintro} \quad \xi \; \mathtt{refutable}_?}{e \models_? \xi} \tag{16b}$$

$$\frac{\text{CMSOrL}}{e \models_? \xi_1 \quad e \not\models \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{16c}$$

$$\frac{\text{CMSOrR}}{e \not\models \xi_1 \quad e \models_? \xi_2}{e \models_? \xi_1 \vee \xi_2} \tag{16d}$$

$$\frac{\text{CMSInl}}{e_1 \models_? \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)} \tag{16e}$$

CMSInr
$$\frac{e_2 \models_? \xi_2}{\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)} \tag{16f}$$

CMSPairL
$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \tag{16g}$$

CMSPairR
$$\frac{e_1 \models \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \tag{16h}$$

CMSPair
$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \tag{16i}$$

$\boxed{e \models_?^\dagger \xi}$ $\qquad$ e satisfies or may satisfy ξ

CSMSMay
$$\frac{e \models_? \xi}{e \models_?^\dagger \xi} \tag{17a}$$

CSMSSat
$$\frac{e \models \xi}{e \models_?^\dagger \xi} \tag{17b}$$

Lemma 2.0.1. $e \not\models \bot$

Proof. By rule induction over Rules (14), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. $\qquad\square$

Lemma 2.0.2. $e \not\models_? \bot$

Proof. Assume $e \models_? \bot$. By rule induction over Rules (16) on $e \models_? \bot$, only one case applies.

Case (16b).

(1) $\bot$ `refutable`$_?$ $\qquad\qquad\qquad\qquad$ by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. $\qquad\square$

Lemma 2.0.3. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (16) on $e \models_? \top$, only one case applies.

Case (16b).

(1) $\top$ `refutable`$_?$ $\qquad\qquad\qquad\qquad$ by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. $\qquad\square$

Lemma 2.0.4. $e \not\models\, ?$

Proof. By rule induction over Rules (14), we notice that $e \models\, ?$ is in syntactic contradiction with all the cases, hence not derivable. $\qquad\square$

Lemma 2.0.5. $e \models_?^\dagger \xi$ *iff* $e \models_?^\dagger \xi \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

   (1) $e \models_?^\dagger \xi$ by assumption

   By rule induction over Rules (17) on (1).

   Case (17a)**.**

   (2) $e \models_? \xi$ by assumption
   (3) $e \models_? \xi \vee \bot$ by Rule (16c) on (2)
                                                            and Lemma 2.0.1
   (4) $e \models_?^\dagger \xi \vee \bot$ by Rule (17a) on (3)

   Case (17b)**.**

   (2) $e \models \xi$ by assumption
   (3) $e \models \xi \vee \bot$ by Rule (14e) on (2)
   (4) $e \models_?^\dagger \xi \vee \bot$ by Rule (17b) on (3)

2. Necessity:

   (1) $e \models_?^\dagger \xi \vee \bot$ by assumption

   By rule induction over Rules (17) on (1).

   Case (17a)**.**

   (2) $e \models_? \xi \vee \bot$ by assumption

   By rule induction over Rules (16) on (2), only two of them apply.

   Case (16c)**.**

   (3) $e \models_? \xi$ by assumption
   (4) $e \models_?^\dagger \xi$ by Rule (17a) on (3)

Case (16d)**.**

  (3) $e \models_? \bot$ by assumption

  (4) $e \not\models_? \bot$ by Lemma 2.0.2

 (3) contradicts (4).

Case (17b)**.**

  (2) $e \models \xi \vee \bot$ by assumption

By rule induction over Rules (14) on (2), only two of them apply.

Case (14e)**.**

  (3) $e \models \xi$ by assumption

  (4) $e \models_?^\dagger \xi$ by Rule (17b) on (3)

Case (14f)**.**

  (3) $e \models \bot$ by assumption

  (4) $e \not\models \bot$ by Lemma 2.0.1

 (3) contradicts (4).

                       □

Corollary 2.0.1. $\top \models_?^\dagger \xi$ *iff* $\top \models_?^\dagger \xi \vee \bot$

Proof. By Definition 2.1.2 and Lemma 2.0.5.   □

Lemma 2.0.6. *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$. *Then* $\xi_1 \not\models \xi_2$ *iff* $\xi_1 \not\models \xi_2 \vee \bot$

Proof.

 (1) $\xi_1 : \tau$ by assumption

 (2) $\xi_2 : \tau$ by assumption

 (3) $\bot : \tau$ by Rule (8b)

 (4) $\xi_2 \vee \bot : \tau$ by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

 1. Sufficiency:

  (5) $\xi_1 \not\models \xi_2$ by assumption

 To prove $\xi_1 \not\models \xi_2 \vee \bot$, assume $\xi_1 \models \xi_2 \vee \bot$.

  (6) $\xi_1 \models \xi_2 \vee \bot$ by assumption

 For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies

  (7) $e \models \xi_2 \vee \bot$ by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

Case (14e)**.**

  (8) $e \models \xi_2$ by assumption

  (9) $\xi_1 \models \xi_2$ by Definition 2.1.1 on (8)

  (5) contradicts (9).

Case (14f)**.**

  (8) $e \models \bot$ by assumption

  (9) $e \not\models \bot$ by Lemma 2.0.1

  (8) contradicts (9).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2 \vee \bot$

2. Necessity:

  (5) $\xi_1 \not\models \xi_2 \vee \bot$ by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

  (6) $\xi_1 \models \xi_2$ by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^{\dagger} \xi_1$ implies

  (7) $e \models \xi_2$ by Definition 2.1.1 on (1) and (2) and (6)

  (8) $e \models \xi_2 \vee \bot$ by Rule (14e) on (7)

  (9) $\xi_1 \models \xi_2 \vee \bot$ by Definition 2.1.1 on (8)

  (9) contradicts (5).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2$

$\square$

Lemma 2.0.7. *If $e \not\models_?^{\dagger} \xi_1$ and $e \not\models_?^{\dagger} \xi_2$ then $e \not\models_?^{\dagger} \xi_1 \vee \xi_2$*

Proof. Assume, for the sake of contradiction, that $e \models_?^{\dagger} \xi_1 \vee \xi_2$.

  (1) $e \models_?^{\dagger} \xi_1 \vee \xi_2$ by assumption

  (2) $e \not\models_?^{\dagger} \xi_1$ by assumption

  (3) $e \not\models_?^{\dagger} \xi_2$ by assumption

By rule induction over Rules (17) on (1).

Case (17b)**.**

$\qquad$ (4) $\;e \models \xi_1 \vee \xi_2$ $\hfill$ by assumption

$\quad$ By rule induction over Rules (14) on (4) and only two of them apply.

$\quad$ **Case** (14e)**.**

$\qquad\qquad$ (5) $\;e \models \xi_1$ $\hfill$ by assumption

$\qquad\qquad$ (6) $\;e \models_?^\dagger \xi_1$ $\hfill$ by Rule (17b) on (5)

$\qquad$ (6) contradicts (2).

$\quad$ **Case** (14f)**.**

$\qquad\qquad$ (5) $\;e \models \xi_2$ $\hfill$ by assumption

$\qquad\qquad$ (6) $\;e \models_?^\dagger \xi_2$ $\hfill$ by Rule (17b) on (5)

$\qquad$ (6) contradicts (3).

Case (17a)**.**

$\qquad$ (4) $\;e \models_? \xi_1 \vee \xi_2$ $\hfill$ by assumption

$\quad$ By rule induction over Rules (16) on (4) and only two of them apply.

$\quad$ **Case** (16c)**.**

$\qquad\qquad$ (5) $\;e \models_? \xi_1$ $\hfill$ by assumption

$\qquad\qquad$ (6) $\;e \models_?^\dagger \xi_1$ $\hfill$ by Rule (17a) on (5)

$\qquad$ (6) contradicts (2).

$\quad$ **Case** (16d)**.**

$\qquad\qquad$ (5) $\;e \models_? \xi_2$ $\hfill$ by assumption

$\qquad\qquad$ (6) $\;e \models_?^\dagger \xi_2$ $\hfill$ by Rule (17a) on (5)

$\qquad$ (6) contradicts (3).

$\quad$ The conclusion holds as follows:

1. $e \not\models_?^\dagger \xi_1 \vee \xi_2$

$\hfill \square$

Lemma 2.0.8. *If* $e \models_?^\dagger \xi_1 \vee \xi_2$ *and* $e \not\models_?^\dagger \xi_1$ *then* $e \models_?^\dagger \xi_2$

Proof.

$\qquad$ (1) $\;e \models_?^\dagger \xi_1 \vee \xi_2$ $\hfill$ by assumption

$\qquad$ (2) $\;e \not\models_?^\dagger \xi_1$ $\hfill$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

$$(3)\ \ e \models \xi_1 \vee \xi_2 \qquad\qquad\qquad \text{by assumption}$$

By rule induction over Rules (14) on (3) and only two of them apply.

Case (14e).

$$(4)\ \ e \models \xi_1 \qquad\qquad\qquad \text{by assumption}$$
$$(5)\ \ e \models_?^\dagger \xi_1 \qquad\qquad\qquad \text{by Rule (17b) on (4)}$$

(5) contradicts (2).

Case (14f).

$$(4)\ \ e \models \xi_2 \qquad\qquad\qquad \text{by assumption}$$
$$(5)\ \ e \models_?^\dagger \xi_2 \qquad\qquad\qquad \text{by Rule (17b) on (4)}$$

Case (17a).

$$(3)\ \ e \models_? \xi_1 \vee \xi_2 \qquad\qquad\qquad \text{by assumption}$$

By rule induction over Rules (16) on (3) and only two of them apply.

Case (16c).

$$(4)\ \ e \models_? \xi_1 \qquad\qquad\qquad \text{by assumption}$$
$$(5)\ \ e \models_?^\dagger \xi_1 \qquad\qquad\qquad \text{by Rule (17a) on (4)}$$

(5) contradicts (2).

Case (16d).

$$(4)\ \ e \models_? \xi_2 \qquad\qquad\qquad \text{by assumption}$$
$$(5)\ \ e \models_?^\dagger \xi_2 \qquad\qquad\qquad \text{by Rule (17a) on (4)}$$

$\square$

Lemma 2.0.9. *If $e \models_?^\dagger \xi_1$ and $e \models_?^\dagger \xi_2$ then $e \models_?^\dagger \xi_1 \wedge \xi_2$*

Lemma 2.0.10. *If $e \models_?^\dagger \xi_1$ then $e \models_?^\dagger \xi_1 \vee \xi_2$ and $e \models_?^\dagger \xi_2 \vee \xi_1$*

Proof.

$$(1)\ \ e \models_?^\dagger \xi_1 \qquad\qquad\qquad \text{by assumption} \qquad\qquad ,$$

By rule induction over Rules (17) on (1),

Case (17b).

$$(2)\ \ e \models \xi_1 \qquad\qquad\qquad \text{by assumption}$$
$$(3)\ \ e \models \xi_1 \vee \xi_2 \qquad\qquad\qquad \text{by Rule (14e) on (2)}$$
$$(4)\ \ e \models \xi_2 \vee \xi_1 \qquad\qquad\qquad \text{by Rule (14f) on (2)}$$

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(5) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (3)

(6) $e \models_?^\dagger \xi_2 \vee \xi_1$ by Rule (17b) on (4)

Case (17a).

     (2) $e \models_? \xi_1$ by assumption

By case analysis on the result of $satisfy(e, \xi_2)$.

Case true.

     (3) $satisfy(e, \xi_2) = \text{true}$ by assumption

     (4) $e \models \xi_2$ by Lemma 2.0.19 on (3)

     (5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

     (6) $e \models \xi_2 \vee \xi_1$ by Rule (14e) on (4)

     (7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (5)

     (8) $e \models_?^\dagger \xi_2 \vee \xi_1$ by Rule (17b) on (6)

Case false.

     (3) $satisfy(e, \xi_2) = \text{false}$ by assumption

     (4) $e \not\models \xi_2$ by Lemma 2.0.19 on (3)

     (5) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (2) and (4)

     (6) $e \models_?^\dagger \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Rule (17a) on (5)

$\square$

Lemma 2.0.11. *If* $e_1 \models_?^\dagger \xi_1$ *then* $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$

Proof.

     (1) $e_1 \models_?^\dagger \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

     (2) $e_1 \models \xi_1$ by assumption

     (3) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by Rule (14g) on (2)

     (4) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ by Rule (17b) on (3)

Case (17a).

$$
\begin{array}{lll}
(2) & e_1 \models_? \xi_1 & \text{by assumption} \\
(3) & \mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1) & \text{by Rule (16e) on (2)} \\
(4) & \mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1) & \text{by Rule (17a) on (3)}
\end{array}
$$

$\square$

Lemma 2.0.12. *If $e_2 \models_?^\dagger \xi_2$ then $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$*

Proof.

$$
\begin{array}{lll}
(1) & e_2 \models_?^\dagger \xi_2 & \text{by assumption}
\end{array}
$$

By rule induction over Rules (17) on (1).

Case (17b).

$$
\begin{array}{lll}
(2) & e_2 \models \xi_2 & \text{by assumption} \\
(3) & \mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2) & \text{by Rule (14h) on (2)} \\
(4) & \mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2) & \text{by Rule (17b) on (3)}
\end{array}
$$

Case (17a).

$$
\begin{array}{lll}
(2) & e_2 \models_? \xi_2 & \text{by assumption} \\
(3) & \mathtt{inl}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2) & \text{by Rule (16f) on (2)} \\
(4) & \mathtt{inl}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2) & \text{by Rule (17a) on (3)}
\end{array}
$$

$\square$

Lemma 2.0.13. *If $e_1 \models_?^\dagger \xi_1$ and $e_2 \models_?^\dagger \xi_2$ then $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$*

Lemma 2.0.14 (Soundness and Completeness of Refutable Constraints)**.** *ξ $\mathtt{refutable}_?$ iff $refutable_?(\xi) = true$.*

Lemma 2.0.15. *If e $\mathtt{notintro}$ and ξ $\mathtt{refutable}_?$ then either $\dot\top(\xi)$ $\mathtt{refutable}_?$ or $e \models \dot\top(\xi)$.*

Proof. By structural induction on ξ. $\square$

Lemma 2.0.16. *There does not exist such a constraint $\xi_1 \wedge \xi_2$ such that $\xi_1 \wedge \xi_2$ $\mathtt{refutable}_?$.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \wedge \xi_2$ $\mathtt{refutable}_?$ is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 2.0.17. *There does not exist such a constraint $\xi_1 \vee \xi_2$ such that $\xi_1 \vee \xi_2$ $\mathtt{refutable}_?$.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \vee \xi_2$ `refutable`$_?$ is in syntactic contradiction with all the cases, hence not derivable. $\qquad\square$

Lemma 2.0.18. *If e `notintro` and $e \models \xi$ then ξ ~~refutable$_?$~~.*

Proof.

    (1) e `notintro`   by assumption

    (2) $e \models \xi$ by assumption

By rule induction over Rules (14) on (2).

Case (14a)**.**

        (3) $\xi = \top$ by assumption

    Assume $\top$ `refutable`$_?$. By rule induction over Rules (10), no case applies due to syntactic contradiction.
    Therefore, $\top$ ~~refutable$_?$~~.

Case (14e),(14f)**.**

        (3) $\xi = \xi_1 \vee \xi_2$ by assumption

        (4) $\xi_1 \vee \xi_2$ ~~refutable$_?$~~   by Lemma 2.0.17

Case (14d)**.**

        (3) $\xi = \xi_1 \wedge \xi_2$ by assumption

        (4) $\xi_1 \wedge \xi_2$ ~~refutable$_?$~~   by Lemma 2.0.16

Case (14j)**.**

        (3) $\xi = (\xi_1, \xi_2)$ by assumption

        (4) $\mathtt{prl}(e) \models \xi_1$ by assumption

        (5) $\mathtt{prr}(e) \models \xi_2$ by assumption

        (6) $\mathtt{prl}(e)$ `notintro`   by Rule (26e)

        (7) $\mathtt{prr}(e)$ `notintro`   by Rule (26f)

        (8) ξ_1 ~~refutable$_?$~~   by IH on (6) and (4)

        (9) ξ_2 ~~refutable$_?$~~   by IH on (7) and (5)

    Assume (ξ_1, ξ_2) `refutable`$_?$. By rule induction over Rules (10) on it, only two cases apply.

    Case (10d)**.**

           (10) ξ_1 `refutable`$_?$ by assumption

        Contradicts (8).

Case (10e).

       (10) ξ_2 `refutable`$_?$ by assumption

     Contradicts (9).

Therefore, $\cancel{(\xi_1, \xi_2) \text{ refutable}_?}$.

Otherwise.

       (3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

$\square$

Lemma 2.0.19 (Soundness and Completeness of Satisfaction Judgment). $e \models \xi$ *iff satisfy*$(e, \xi) = true$.

Proof. We prove soundness and completeness separately.

1. Soundness:

       (1) $e \models \xi$ by assumption

By rule induction over Rules (14) on (1).

Case (14a).

       (2) $\xi = \top$ by assumption
       (3) *satisfy*$(e, \top) = \text{true}$ by Definition 15a

Case (14b).

       (2) $e = \underline{n}$ by assumption
       (3) $\xi = \underline{n}$ by assumption
       (4) *satisfy*$(\underline{n}, \underline{n}) = (n = n) = \text{true}$ by Definition 15b

Case (14c).

       (2) $e = \underline{n_1}$ by assumption
       (3) $\xi = \cancel{\underline{n_2}}$ by assumption
       (4) $n_1 \neq n_2$ by assumption
       (5) *satisfy*$(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) = \text{true}$ by Definition 15c on (4)

Case (14d).

       (2) $\xi = \xi_1 \wedge \xi_2$ by assumption

(3) $e \models \xi_1$ by assumption

(4) $e \models \xi_2$ by assumption

(5) $satisfy(e, \xi_1) = \text{true}$ by IH on (3)

(6) $satisfy(e, \xi_2) = \text{true}$ by IH on (4)

(7) $satisfy(e, \xi_1 \wedge \xi_2) = satisfy(e, \xi_1)$ and $satisfy(e, \xi_2) = \text{true}$
     by Definition 15d on (5) and (6)

Case (14e).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_1$ by assumption

(4) $satisfy(e, \xi_1) = \text{true}$ by IH on (3)

(5) $satisfy(e, \xi_1 \vee \xi_2) = satisfy(e, \xi_1)$ or $satisfy(e, \xi_2) = \text{true}$
     by Definition 15e on (4)

Case (14f).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption

(3) $e \models \xi_2$ by assumption

(4) $satisfy(e, \xi_2) = \text{true}$ by IH on (3)

(5) $satisfy(e, \xi_1 \vee \xi_2) = satisfy(e, \xi_1)$ or $satisfy(e, \xi_2) = \text{true}$
     by Definition 15e on (4)

Case (14g).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\xi = \texttt{inl}(\xi_1)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $satisfy(e_1, \xi_1) = \text{true}$ by IH on (4)

(6) $satisfy(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\xi_1)) = satisfy(e_1, \xi_1) = \text{true}$
     by Definition 15f on (5)

Case (14h).

(2) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\xi = \texttt{inl}(\xi_2)$ by assumption

(4) $e_2 \models \xi_2$ by assumption

(5) $satisfy(e_2, \xi_2) = \text{true}$ by IH on (4)

(6) $satisfy(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\xi_2)) = satisfy(e_2, \xi_2) = \text{true}$
     by Definition 15g on (5)

Case (14i).

(2)	$e = (e_1, e_2)$	by assumption
(3)	$\xi = (\xi_1, \xi_2)$	by assumption
(4)	$e_1 \models \xi_1$	by assumption
(5)	$e_2 \models \xi_2$	by assumption
(6)	$satisfy(e_1, \xi_1) = \text{true}$	by IH on (4)
(7)	$satisfy(e_2, \xi_2) = \text{true}$	by IH on (5)

(8) $satisfy((e_1, e_2), (\xi_1, \xi_2)) =$
$satisfy(e_1, \xi_1)$ and $satisfy(e_2, \xi_2) = \text{true}$

                                      by Definition 15h on
(6) and (7)

Case (14j).

(2)	$\xi = (\xi_1, \xi_2)$	by assumption
(3)	$e\ \mathtt{notintro}$	by assumption
(4)	$\mathtt{prl}(e) \models \xi_1$	by assumption
(5)	$\mathtt{prr}(e) \models \xi_2$	by assumption
(6)	$satisfy(\mathtt{prl}(e), \xi_1) = \text{true}$	by IH on (4)
(7)	$satisfy(\mathtt{prr}(e), \xi_2) = \text{true}$	by IH on (5)

By rule induction over Rules (26) on (3).

Otherwise.

(8) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$
                                       by assumption

(9) $satisfy(e, (\xi_1, \xi_2)) =$
$satisfy(\mathtt{prl}(e), \xi_1)$ and $satisfy(\mathtt{prr}(e), \xi_2) = \text{true}$

                                        by Definition 15 on (6)
and (7)

2. Completeness:

(1)	$satisfy(e, \xi) = \text{true}$	by assumption

By structural induction on ξ.

Case $\xi = \top$.

(2)	$e \models \top$	by Rule (14a)

Case $\xi = \bot, ?$.

(2)	$satisfy(e, \xi) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$.

By structural induction on e.

Case $e = \underline{n'}$.

    (2) $n' = n$ by Definition 15b on (1)

    (3) $\underline{n'} \models \underline{n}$ by Rule (14b) on (2)

Otherwise.

    (2) $satisfy(e, \underline{n}) = \text{false}$ by Definition 15o

    (2) contradicts (1) and thus vacuously true.

Case $\xi = \cancel{\mathcal{H}}$.

  By structural induction on e.

    Case $e = \underline{n'}$.

      (2) $n' \neq n$ by Definition 15c on (1)

      (3) $\underline{n'} \models \cancel{\mathcal{H}}$ by Rule (14c) on (2)

    Otherwise.

      (2) $satisfy(e, \cancel{\mathcal{H}}) = \text{false}$ by Definition 15o

      (2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$.

    (2) $satisfy(e, \xi_1) = \text{true}$ by Definition 15d on (1)

    (3) $satisfy(e, \xi_2) = \text{true}$ by Definition 15d on (1)

    (4) $e \models \xi_1$ by IH on (2)

    (5) $e \models \xi_2$ by IH on (3)

    (6) $e \models \xi_1 \wedge \xi_2$ by Rule (14d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$.

    (2) $satisfy(e, \xi_1)$ or $satisfy(e, \xi_2) = \text{true}$
                        by Definition 15e on (1)

  By case analysis on (2).

    Case $satisfy(e, \xi_1) = \text{true}$.

      (3) $satisfy(e, \xi_1) = \text{true}$ by assumption

      (4) $e \models \xi_1$ by IH on (3)

      (5) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (4)

    Case $satisfy(e, \xi_2) = \text{true}$.

      (3) $satisfy(e, \xi_2) = \text{true}$ by assumption

      (4) $e \models \xi_2$ by IH on (3)

      (5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

Case $\xi = \mathtt{inl}(\xi_1)$.

By structural induction on e.

Case $e = \mathtt{inl}_{\tau_2}(e_1)$.

(2)	$satisfy(e_1, \xi_1) = \text{true}$	by Definition 15f on (1)
(3)	$e_1 \models \xi_1$	by IH on (2)
(4)	$\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$	by Rule (14g) on (3)

Otherwise.

(2)	$satisfy(e, \mathtt{inl}(\xi_1)) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \mathtt{inr}(\xi_2)$.

By structural induction on e.

Case $e = \mathtt{inr}_{\tau_1}(e_2)$.

(2)	$satisfy(e_2, \xi_2) = \text{true}$	by Definition 15g on (1)
(3)	$e_2 \models \xi_2$	by IH on (2)
(4)	$\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$	by Rule (14h) on (3)

Otherwise.

(2)	$satisfy(e, \mathtt{inr}(\xi_2)) = \text{false}$	by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$.

By structural induction on e.

Case $e = (e_1, e_2)$.

(2)	$satisfy(e_1, \xi_1) = \text{true}$	by Definition 15h on (1)
(3)	$satisfy(e_2, \xi_2) = \text{true}$	by Definition 15h on (1)
(4)	$e_1 \models \xi_1$	by IH on (2)
(5)	$e_2 \models \xi_2$	by IH on (3)
(6)	$(e_1, e_2) \models (\xi_1, \xi_2)$	by Rule (14i) on (4) and (5)

Case $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$.

(2)	$satisfy(\mathtt{prl}(e), \xi_1) = \text{true}$	by Definition 15h on (1)
(3)	$satisfy(\mathtt{prr}(e), \xi_2) = \text{true}$	by Definition 15h on (1)
(4)	$\mathtt{prl}(e) \models \xi_1$	by IH on (2)
(5)	$\mathtt{prr}(e) \models \xi_2$	by IH on (3)
(6)	e $\mathtt{notintro}$	by each rule in Rules (26)

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$$\text{(7)} \quad (e_1, e_2) \models (\xi_1, \xi_2) \qquad\qquad \text{by Rule (14j) on (6)}$$
$$\text{and (4) and (5)}$$

Otherwise.

$$\text{(2)} \quad satisfy(e, (\xi_1, \xi_2)) = \text{false} \qquad \text{by Definition 15o}$$

(2) contradicts (1) and thus vacuously true.

$\square$

Lemma 2.0.20. $e \not\models \xi$ and $e \not\models_? \xi$ iff $e \not\models_?^\dagger \xi$.

Proof.   1. Sufficiency:

$$\text{(1)} \quad e \not\models \xi \qquad\qquad\qquad\qquad\qquad \text{by assumption}$$
$$\text{(2)} \quad e \not\models_? \xi \qquad\qquad\qquad\qquad\qquad \text{by assumption}$$

Assume $e \models_?^\dagger \xi$. By rule induction over Rules (17) on it.

Case (17a)**.**

$$\text{(3)} \quad e \models \xi \qquad\qquad\qquad\qquad\qquad \text{by assumption}$$

Contradicts (1).

Case (17b)**.**

$$\text{(3)} \quad e \models_? \xi \qquad\qquad\qquad\qquad\qquad \text{by assumption}$$

Contradicts (2).

Therefore, $e \models_?^\dagger \xi$ is not derivable.

2. Necessity:

$$\text{(1)} \quad e \not\models_?^\dagger \xi \qquad\qquad\qquad\qquad\qquad \text{by assumption}$$

Assume $e \models \xi$.

$$\text{(2)} \quad e \models_?^\dagger \xi \qquad\qquad\qquad\qquad\qquad \text{by Rule (17b) on}$$
$$\text{assumption}$$

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_? \xi$.

$$\text{(3)} \quad e \models_?^\dagger \xi \qquad\qquad\qquad\qquad\qquad \text{by Rule (17a) on}$$
$$\text{assumption}$$

Contradicts (1). Therefore, $e \not\models_? \xi$.

$\square$

Theorem 2.1 (Exclusiveness of Satisfaction Judgment)**.** *If* $\xi : \tau$ *and* $\cdot; \Delta \vdash e : \tau$ *and* e `final` *then exactly one of the following holds*

1. $e \models \xi$

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2. $e \models_? \xi$

3. $e \not\models_?^\dagger \xi$

Proof.

(4) $\xi : \tau$ by assumption

(5) $\cdot\,; \Delta \vdash e : \tau$ by assumption

(6) $e\ \mathtt{final}$ by assumption

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

Case (8a)**.**

(7) $\xi = \top$ by assumption

(8) $e \models \top$ by Rule (14a)

(9) $e \not\models_? \top$ by Lemma 2.0.3

(10) $e \models_?^\dagger \top$ by Rule (17b) on (8)

Case (8b)**.**

(7) $\xi = \bot$ by assumption

(8) $e \not\models \bot$ by Lemma 2.0.1

(9) $e \not\models_? \bot$ by Lemma 2.0.2

(10) $e \not\models_?^\dagger \bot$ by Lemma 2.0.20 on (8) and (9)

Case (1b)**.**

(7) $\xi = {?}$ by assumption

(8) $e \not\models {?}$ by Lemma 2.0.4

(9) $e \models_? {?}$ by Rule (16a)

(10) $e \models_?^\dagger {?}$ by Rule (17a) on (9)

Case (8c)**.**

(7) $\xi = \underline{n_2}$ by assumption

(8) $\tau = \mathtt{num}$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

    (9) $e = (\!| |\!)^u, (\!| e_0 |\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$

                                       by assumption

   (10) e $\mathtt{notintro}$ by Rule
                                       (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on ξ.

   (11) $e \not\models \underline{n_2}$ by contradiction

   (12) $\underline{n_2}$ $\mathtt{refutable}_?$ by Rule (10a)

   (13) $e \models_? \underline{n_2}$ by Rule (16b) on (10)
                                       and (12)

   (14) $e \models_?^\dagger \underline{n_2}$ by Rule (17a) on (13)

Case (19d)**.**

    (9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (16), only one case applies.

Case (16b)**.**

    (10) $\underline{n_1}$ $\mathtt{notintro}$ by assumption

    Contradicts Lemma 4.0.5.

   (11) $\underline{n_1} \not\models_? \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$**.**

    (12) $satisfy(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 15

    (13) $\underline{n_1} \models \underline{n_2}$ by Lemma 2.0.19 on
                                   (12)

    (14) $e \models_?^\dagger \underline{n_2}$ by Rule (17b) on (13)

Case $n_1 \neq n_2$**.**

    (12) $satisfy(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 15

    (13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 2.0.19 on
                                     (12)

    (14) $e \not\models_?^\dagger \underline{n_2}$ by Lemma 2.0.20 on
                                   (11) and (13)

Case (8f)**.**

    (7) $\xi = \xi_1 \vee \xi_2$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models_? \xi_1$, and $e \not\models_?^\dagger \xi_1$ holds. The same goes for ξ_2. By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e \models \xi_1, e \models \xi_2$**.**

(8)	$e \models \xi_1$	by assumption
(9)	$e \not\models_? \xi_1$	by assumption
(10)	$e \models \xi_2$	by assumption
(11)	$e \not\models_? \xi_2$	by assumption
(12)	$e \models \xi_1 \vee \xi_2$	by Rule (14e) on (8)
(13)	$e \models_?^\dagger \xi_1 \vee \xi_2$	by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14)	$\xi_1 \vee \xi_2$ `refutable`$_?$	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14)	$e \models_? \xi_1$	by assumption

Contradicts (9).

Case (16d).

(14)	$e \models_? \xi_2$	by assumption

Contradicts (11).

(15)	$e \not\models_? \xi_1 \vee \xi_2$	by contradiction

Case $e \models \xi_1, e \models_? \xi_2$**.**

(8)	$e \models \xi_1$	by assumption
(9)	$e \not\models_? \xi_1$	by assumption
(10)	$e \not\models \xi_2$	by assumption
(11)	$e \models_? \xi_2$	by assumption
(12)	$e \models \xi_1 \vee \xi_2$	by Rule (14e) on (8)
(13)	$e \models_?^\dagger \xi_1 \vee \xi_2$	by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14)	$\xi_1 \vee \xi_2$ `refutable`$_?$	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14)	$e \models_? \xi_1$	by assumption

Contradicts (9).

    Case (16d).

        (14) $e \not\models \xi_1$ by assumption

    Contradicts (8).

    (15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \not\models_?^\dagger \xi_2$**.**

    (8) $e \models \xi_1$ by assumption

    (9) $e \not\models_? \xi_1$ by assumption

    (10) $e \not\models \xi_2$ by assumption

    (11) $e \not\models_? \xi_2$ by assumption

    (12) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (8)

    (13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

    Case (16b).

        (14) $\xi_1 \vee \xi_2$ `refutable`$_?$ by assumption

    Contradicts Lemma 2.0.17.

    Case (16c).

        (14) $e \models_? \xi_1$ by assumption

    Contradicts (9).

    Case (16d).

        (14) $e \not\models \xi_1$ by assumption

    Contradicts (8).

    (15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models \xi_2$**.**

    (8) $e \not\models \xi_1$ by assumption

    (9) $e \models_? \xi_1$ by assumption

    (10) $e \models \xi_2$ by assumption

    (11) $e \not\models_? \xi_2$ by assumption

    (12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

    (13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

    Case (16b).

        (14) $\xi_1 \vee \xi_2$ `refutable`$_?$ by assumption

Contradicts Lemma 2.0.17.

Case (16c).

  (14) $e \not\models \xi_2$ by assumption

 Contradicts (10).

Case (16d).

  (14) $e \models_? \xi_2$ by assumption

 Contradicts (11).

 (15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \models_? \xi_2$**.**

  (8) $e \not\models \xi_1$ by assumption

  (9) $e \models_? \xi_1$ by assumption

  (10) $e \not\models \xi_2$ by assumption

  (11) $e \models_? \xi_2$ by assumption

  (12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (9)
                  and (10)

  (13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

  (14) $e \models \xi_1$ by assumption

 Contradicts (8)

Case (14f).

  (14) $e \models \xi_2$ by assumption

 Contradicts (10)

 (15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \models_? \xi_1, e \not\models_?^\dagger \xi_2$**.**

  (8) $e \not\models \xi_1$ by assumption

  (9) $e \models_? \xi_1$ by assumption

  (10) $e \not\models \xi_2$ by assumption

  (11) $e \not\models_? \xi_2$ by assumption

  (12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16c) on (9)
                  and (10)

  (13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

    (14) $e \models \xi_1$ by assumption

  Contradicts (8).

Case (14f).

    (14) $e \models \xi_2$ by assumption

  Contradicts (10).

    (15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \models \xi_2$**.**

    (8) $e \not\models \xi_1$ by assumption

    (9) $e \not\models_? \xi_1$ by assumption

    (10) $e \models \xi_2$ by assumption

    (11) $e \not\models_? \xi_2$ by assumption

    (12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

    (13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

    (14) $\xi_1 \vee \xi_2$ `refutable`$_?$ by assumption

  Contradicts Lemma 2.0.17.

Case (16c).

    (14) $e \not\models \xi_2$ by assumption

  Contradicts (10).

Case (16d).

    (14) $e \models_? \xi_2$ by assumption

  Contradicts (11).

    (15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \models_? \xi_2$**.**

    (8) $e \not\models \xi_1$ by assumption

    (9) $e \not\models_? \xi_1$ by assumption

    (10) $e \not\models \xi_2$ by assumption

    (11) $e \models_? \xi_2$ by assumption

    (12) $e \models_? \xi_1 \vee \xi_2$ by Rule (16d) on (11) and (8)

    (13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \lor \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$	by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$	by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \lor \xi_2$	by contradiction

Case $e \not\models_?^\dagger \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \not\models \xi_1$	by assumption
(9) $e \not\models_? \xi_1$	by assumption
(10) $e \not\models \xi_2$	by assumption
(11) $e \not\models_? \xi_2$	by assumption

Assume $e \models \xi_1 \lor \xi_2$. By rule induction over Rules (14) on it, only two cases apply.

Case (14e).

(12) $e \models \xi_1$	by assumption

Contradicts (8).

Case (14f).

(12) $e \models \xi_2$	by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \lor \xi_2$	by contradiction

Assume $e \models_? \xi_1 \lor \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \lor \xi_2$ `refutable`$_?$	by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_? \xi_1$	by assumption

Contradicts (9).

Case (16d).

(14) $e \models_? \xi_2$	by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \lor \xi_2$	by contradiction

(16) $e \not\models^\dagger_? \xi_1 \vee \xi_2$ by Lemma 2.0.20 on (13) and (15)

Case (8g).

(7) $\xi = \mathtt{inl}(\xi_1)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(10) $e = (\!| \cdot |\!)^u, (\!| e_0 |\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) e $\mathtt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \mathtt{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \mathtt{inl}(\xi_1)$ by contradiction

By case analysis on the value of $\mathit{refutable}_?(\mathtt{inl}(\xi_1))$.

Case $\mathit{refutable}_?(\mathtt{inl}(\xi_1)) = \text{true}$.

(13) $\mathit{refutable}_?(\mathtt{inl}(\xi_1)) = \text{true}$ by assumption

(14) $\mathtt{inl}(\xi_1)$ $\mathtt{refutable}_?$ by Lemma 2.0.14 on (13)

(15) $e \models_? \mathtt{inl}(\xi_1)$ by Rule (16b) on (11) and (14)

(16) $e \models^\dagger_? \mathtt{inl}(\xi_1)$ by Rule (17a) on (15)

Case $\mathit{refutable}_?(\mathtt{inl}(\xi_1)) = \text{false}$.

(13) $\mathit{refutable}_?(\mathtt{inl}(\xi_1)) = \text{false}$ by assumption

(14) $\mathtt{inl}(\xi_1)$ $\cancel{\mathtt{refutable}_?}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \mathtt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(15) $\mathtt{inl}(\xi_1)$ $\mathtt{refutable}_?$ by assumption

Contradicts (14).

(16) $e \not\models_? \mathtt{inl}(\xi_1)$ by contradiction

(17) $e \not\models^\dagger_? \mathtt{inl}(\xi_1)$ by Lemma 2.0.20 on (12) and (16)

Case (19j).

$\quad$ (10) $\ e = \mathtt{inl}_{\tau_2}(e_1)$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (11) $\ \cdot\,; \Delta \vdash e_1 : \tau_1$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (12) $\ e_1$ `final` $\qquad\qquad\qquad\qquad\qquad$ by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \not\models_?^\dagger \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$.

$\quad$ (13) $\ e_1 \models \xi_1$ $\qquad\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (14) $\ e_1 \not\models_? \xi_1$ $\qquad\qquad\qquad\qquad\quad$ by assumption

$\quad$ (15) $\ \mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ $\qquad\qquad$ by Rule (14g) on (13)

$\quad$ (16) $\ \mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ $\qquad\quad$ by Rule (17b) on (15)

Assume $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

$\quad$ (17) $\ \mathtt{inl}_{\tau_2}(e_1)$ `notintro` $\qquad$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

$\quad$ (17) $\ e_1 \models_? \xi_1$

Contradicts (14).

$\quad$ (18) $\ \mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inl}(\xi_1)$ $\qquad\qquad$ by contradiction

Case $e_1 \models_? \xi_1$.

$\quad$ (13) $\ e_1 \not\models \xi_1$ $\qquad\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (14) $\ e_1 \models_? \xi_1$ $\qquad\qquad\qquad\qquad\quad$ by assumption

$\quad$ (15) $\ \mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ $\qquad\quad$ by Rule (16e) on (14)

$\quad$ (16) $\ \mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ $\qquad\quad$ by Rule (17a) on (15)

Assume $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

$\quad$ (17) $\ e_1 \models \xi_1$

Contradicts (13).

$\quad$ (18) $\ \mathtt{inl}_{\tau_2}(e_1) \not\models \mathtt{inl}(\xi_1)$ $\qquad\qquad$ by contradiction

Case $e_1 \not\models_?^\dagger \xi_1$.

$\quad$ (13) $\ e_1 \not\models \xi_1$ $\qquad\qquad\qquad\qquad\qquad$ by assumption

$\qquad$(14) $e_1 \not\models_? \xi_1$ $\qquad$ by assumption

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g)**.**

$\qquad$(15) $e_1 \models \xi_1$

Contradicts (13).

$\qquad$(16) $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$ $\qquad$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \models_? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

$\qquad$(17) $\text{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ $\qquad$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e)**.**

$\qquad$(17) $e_1 \models_? \xi_1$

Contradicts (14).

$\qquad$(18) $\text{inl}_{\tau_2}(e_1) \not\models_? \text{inl}(\xi_1)$ $\qquad$ by contradiction

$\qquad$(19) $\text{inl}_{\tau_2}(e_1) \not\models_?^{\dagger} \text{inl}(\xi_1)$ $\qquad$ by Lemma 2.0.20 on (16) and (18)

Case (19k)**.**

$\qquad$(10) $e = \text{inr}_{\tau_1}(e_2)$ $\qquad$ by assumption

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

$\qquad$(11) $\text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\xi_1)$ $\qquad$ by contradiction

Assume $\text{inr}_{\tau_1}(e_2) \models_? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

$\qquad$(12) $\text{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ $\qquad$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

$\qquad$(13) $\text{inr}_{\tau_1}(e_2) \not\models_? \text{inl}(\xi_1)$ $\qquad$ by contradiction

$\qquad$(14) $\text{inr}_{\tau_1}(e_2) \not\models_?^{\dagger} \text{inl}(\xi_1)$ $\qquad$ by Lemma 2.0.20 on (11) and (13)

Case (8h)**.**

$\qquad$(7) $\xi = \text{inr}(\xi_2)$ $\qquad$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

    (10) $e = (\!|)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}s\}$
                                            by assumption

    (11) e $\mathtt{notintro}$ by Rule
                                             (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

    (12) $e \not\models \mathtt{inr}(\xi_2)$ by contradiction

By case analysis on the value of $\mathit{refutable}_?(\mathtt{inr}(\xi_2))$. _____ [inr is refutable]

    Case $\mathit{refutable}_?(\mathtt{inr}(\xi_2)) = \mathrm{true}$**.**

        (13) $\mathit{refutable}_?(\mathtt{inr}(\xi_2)) = \mathrm{true}$ by assumption

        (14) $\mathtt{inr}(\xi_2)$ $\mathtt{refutable}_?$ by Lemma 2.0.14 on
                                              (13)

        (15) $e \models_? \mathtt{inr}(\xi_2)$ by Rule (16b) on (11)
                                              and (14)

        (16) $e \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (17a) on (15)

    Case $\mathit{refutable}_?(\mathtt{inr}(\xi_2)) = \mathrm{false}$**.**

        (13) $\mathit{refutable}_?(\mathtt{inr}(\xi_2)) = \mathrm{false}$ by assumption

        (14) ~~$\mathtt{inr}(\xi_2)$ $\mathtt{refutable}_?$~~   by Lemma 2.0.14 on
                                              (13)

Assume $e \models_? \mathtt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

    Case (16b)**.**

        (15) $\mathtt{inr}(\xi_2)$ $\mathtt{refutable}_?$ by assumption
       Contradicts (14).

        (16) $e \not\models_? \mathtt{inr}(\xi_2)$ by contradiction

        (17) $e \not\models_?^\dagger \mathtt{inr}(\xi_2)$ by Lemma 2.0.20 on
                                              (12) and (16)

Case (19j)**.**

    (10) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

    (11) $\mathtt{inl}_{\tau_2}(e_1) \not\models \mathtt{inr}(\xi_2)$ by contradiction

Assume $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

    (12) $\mathtt{inl}_{\tau_2}(e_1)$ `notintro`   by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

    (13) $\mathtt{inl}_{\tau_2}(e_1) \not\models_? \mathtt{inr}(\xi_2)$ by contradiction

    (14) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inr}(\xi_2)$ by Lemma 2.0.20 on (11) and (13)

Case (19k).

    (10) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

    (11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

    (12) e_2 `final`   by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \not\models_?^\dagger \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

    (13) $e_2 \models \xi_2$ by assumption

    (14) $e_2 \not\models_? \xi_2$ by assumption

    (15) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (14g) on (13)

    (16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (17b) on (15)

Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

    (17) $\mathtt{inr}_{\tau_1}(e_2)$ `notintro`   by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

    (17) $e_2 \models_? \xi_2$

Contradicts (14).

    (18) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inr}(\xi_2)$ by contradiction

Case $e_2 \models_? \xi_2$.

    (13) $e_2 \not\models \xi_2$ by assumption

    (14) $e_2 \models_? \xi_2$ by assumption

    (15) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (16f) on (14)

    (16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Rule (17a) on (15)

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h)**.**

  (17) $e_2 \models \xi_2$

  Contradicts (13).

  (18) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inr}(\xi_2)$ by contradiction

Case $e_2 \not\models_?^\dagger \xi_2$**.**

  (13) $e_2 \not\models \xi_2$ by assumption

  (14) $e_2 \not\models_? \xi_2$ by assumption

 Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h)**.**

  (15) $e_2 \models \xi_2$

  Contradicts (13).

  (16) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inr}(\xi_2)$ by contradiction

 Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

  (17) $\mathtt{inr}_{\tau_1}(e_2) \, \mathtt{notintro}$ by assumption

  By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f)**.**

  (17) $e_2 \models_? \xi_2$

  Contradicts (14).

  (18) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inr}(\xi_2)$ by contradiction

  (19) $\mathtt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \mathtt{inl}(\xi_1)$ by Lemma 2.0.20 on
                      (16) and (18)

Case (14i)**.**

  (7) $\xi = (\xi_1, \xi_2)$ by assumption

  (8) $\tau = (\tau_1 \times \tau_2)$ by assumption

  (9) $\xi_1 : \tau_1$ by assumption

  (10) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

(11) $e = (\!|)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$
        by assumption

(12) e $\mathtt{notintro}$ by Rule
                      (26a),(26b),(26c),(26d),(26e),(26f)

(13) e $\mathtt{indet}$ by Lemma 4.0.9 on (6)
                      and (12)

(14) $\mathtt{prl}(e)$ $\mathtt{indet}$ by Rule (24g) on (13)

(15) $\mathtt{prl}(e)$ $\mathtt{final}$ by Rule (25b) on (14)

(16) $\mathtt{prr}(e)$ $\mathtt{indet}$ by Rule (24h) on (13)

(17) $\mathtt{prr}(e)$ $\mathtt{final}$ by Rule (25b) on (16)

(18) $\cdot\,; \Delta \vdash \mathtt{prl}(e) : \tau_1$ by Rule (19h) on (5)

(19) $\cdot\,; \Delta \vdash \mathtt{prr}(e) : \tau_2$ by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of
$\mathtt{prl}(e) \models \xi_1$, $\mathtt{prl}(e) \models_? \xi_1$, and $\mathtt{prl}(e) \not\models_?^\dagger \xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of
$\mathtt{prr}(e) \models \xi_2$, $\mathtt{prr}(e) \models_? \xi_2$, and $\mathtt{prr}(e) \not\models_?^\dagger \xi_2$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $\mathtt{prl}(e) \models \xi_1, \mathtt{prr}(e) \models \xi_2$.

    (20) $\mathtt{prl}(e) \models \xi_1$ by assumption

    (21) $\mathtt{prl}(e) \not\models_? \xi_1$ by assumption

    (22) $\mathtt{prr}(e) \models \xi_2$ by assumption

    (23) $\mathtt{prr}(e) \not\models_? \xi_2$ by assumption

    (24) $e \models (\xi_1, \xi_2)$ by Rule (14j) on (12)
                          and (20) and (22)

    (25) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17b) on (24)

    (26) $\cancel{(\xi_1, \xi_2) \; \mathtt{refutable}_?}$ by Lemma 2.0.18 on
                          (12) and (24)

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it,
only one case applies.

Case (16b).

    (27) $(\xi_1, \xi_2) \; \mathtt{refutable}_?$ by assumption
    Contradicts (26).

    (28) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $\mathtt{prl}(e) \models \xi_1, \mathtt{prr}(e) \models_? \xi_2$.

    (20) $\mathtt{prl}(e) \models \xi_1$ by assumption

    (21) $\mathtt{prl}(e) \not\models_? \xi_1$ by assumption

    (22) $\mathtt{prr}(e) \not\models \xi_2$ by assumption

    (23) $\mathtt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

    (24) $\mathtt{prr}(e) \models \xi_2$ by assumption

  Contradicts (22)

  (25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

    (26) ξ_2 $\mathtt{refutable}_?$ by assumption

    (27) (ξ_1, ξ_2) $\mathtt{refutable}_?$ by Rule (10e) on (26)

    (28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

    (29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \models \xi_1, \mathtt{prr}(e) \not\models_?^\dagger \xi_2$**.**

  (20) $\mathtt{prl}(e) \models \xi_1$ by assumption

  (21) $\mathtt{prl}(e) \not\models_? \xi_1$ by assumption

  (22) $\mathtt{prr}(e) \not\models \xi_2$ by assumption

  (23) $\mathtt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

    (24) $\mathtt{prr}(e) \models \xi_2$ by assumption

  Contradicts (22).

  (25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

    (26) (ξ_1, ξ_2) $\mathtt{refutable}_?$ by assumption

  By rule induction over Rules (10) on (26), only two cases apply.

  Case (10d).

    (27) ξ_1 $\mathtt{refutable}_?$ by assumption

    (28) $\mathtt{prl}(e)$ $\mathtt{notintro}$ by Rule (26e)

    (29) $\mathtt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

  Contradicts (21).

  Case (10e).

assume no "or" and "and" in pair

(27) ξ_2 `refutable`$_?$	by assumption
(28) `prr`(e) `notintro`	by Rule (26f)
(29) `prr`$(e) \models_? \xi_2$	by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$	by contradiction
(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$	by Lemma 2.0.20 on (25) and (30)

Case `prl`$(e) \models_? \xi_1$, `prr`$(e) \models \xi_2$**.**

(20) `prl`$(e) \not\models \xi_1$	by assumption
(21) `prl`$(e) \models_? \xi_1$	by assumption
(22) `prr`$(e) \models \xi_2$	by assumption
(23) `prr`$(e) \not\models_? \xi_2$	by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

(24) `prl`$(e) \models \xi_1$	by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$	by contradiction

By rule induction over Rules (16) on (21), only one case applies.

assume no "or" and "and" in pair

Case (16b)**.**

(26) ξ_1 `refutable`$_?$	by assumption
(27) (ξ_1, ξ_2) `refutable`$_?$	by Rule (10e) on (26)
(28) $e \models_? (\xi_1, \xi_2)$	by Rule (16b) on (12) and (27)
(29) $e \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (28)

Case `prl`$(e) \models_? \xi_1$, `prr`$(e) \models_? \xi_2$**.**

(20) `prl`$(e) \not\models \xi_1$	by assumption
(21) `prl`$(e) \models_? \xi_1$	by assumption
(22) `prr`$(e) \not\models \xi_2$	by assumption
(23) `prr`$(e) \models_? \xi_2$	by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

(24) `prl`$(e) \models \xi_1$	by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

<div style="border:1px solid orange;background:orange;color:white;float:right">assume no "or" and "and" in pair</div>

Case (16b).

    (26) ξ_2 `refutable`$_?$ by assumption

    (27) (ξ_1, ξ_2) `refutable`$_?$ by Rule (10e) on (26)

    (28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

    (29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \models_? \xi_1, \mathtt{prr}(e) \not\models_?^\dagger \xi_2$.

    (20) $\mathtt{prl}(e) \not\models \xi_1$ by assumption

    (21) $\mathtt{prl}(e) \models_? \xi_1$ by assumption

    (22) $\mathtt{prr}(e) \not\models \xi_2$ by assumption

    (23) $\mathtt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

    (24) $\mathtt{prl}(e) \models \xi_1$ by assumption

  Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (21), only one case applies.

<div style="border:1px solid orange;background:orange;color:white;float:right">assume no "or" and "and" in pair</div>

Case (16b).

    (26) ξ_1 `refutable`$_?$ by assumption

    (27) (ξ_1, ξ_2) `refutable`$_?$ by Rule (10e) on (26)

    (28) $e \models_? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

    (29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\mathtt{prl}(e) \not\models_?^\dagger \xi_1, \mathtt{prr}(e) \models_? \xi_2$.

    (20) $\mathtt{prl}(e) \not\models \xi_1$ by assumption

    (21) $\mathtt{prl}(e) \not\models_? \xi_1$ by assumption

    (22) $\mathtt{prr}(e) \models \xi_2$ by assumption

    (23) $\mathtt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

    (24) $\mathtt{prl}(e) \models \xi_1$ by assumption

  Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b)**.**

    (26) $(\xi_1, \xi_2) \; \mathtt{refutable}_?$ by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d)**.**

    (27) $\xi_1 \; \mathtt{refutable}_?$ by assumption

    (28) $\mathtt{prl}(e) \; \mathtt{notintro}$ by Rule (26e)

    (29) $\mathtt{prl}(e) \models_? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e)**.**

    (27) $\xi_2 \; \mathtt{refutable}_?$ by assumption

    (28) $\mathtt{prr}(e) \; \mathtt{notintro}$ by Rule (26f)

    (29) $\mathtt{prr}(e) \models_? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\mathtt{prl}(e) \not\models_?^\dagger \xi_1, \mathtt{prr}(e) \models_? \xi_2$**.**

    (20) $\mathtt{prl}(e) \not\models \xi_1$ by assumption

    (21) $\mathtt{prl}(e) \not\models_? \xi_1$ by assumption

    (22) $\mathtt{prr}(e) \not\models \xi_2$ by assumption

    (23) $\mathtt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j)**.**

    (24) $\mathtt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b)**.**

    (26) $\xi_2 \; \mathtt{refutable}_?$ by assumption

    (27) $(\xi_1, \xi_2) \; \mathtt{refutable}_?$ by Rule (10e) on (26)

> assume no "or" and "and" in pair

(28) $e \models_? (\xi_1, \xi_2)$ — by Rule (16b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ — by Rule (17a) on (28)

Case $\mathtt{prl}(e) \not\models_?^\dagger \xi_1, \mathtt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\mathtt{prl}(e) \not\models \xi_1$ — by assumption

(21) $\mathtt{prl}(e) \not\models_? \xi_1$ — by assumption

(22) $\mathtt{prr}(e) \not\models \xi_2$ — by assumption

(23) $\mathtt{prr}(e) \not\models_? \xi_2$ — by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\mathtt{prl}(e) \models \xi_1$ — by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ — by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) $\mathtt{refutable}_?$ — by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 $\mathtt{refutable}_?$ — by assumption

(28) $\mathtt{prl}(e)$ $\mathtt{notintro}$ — by Rule (26e)

(29) $\mathtt{prl}(e) \models_? \xi_1$ — by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 $\mathtt{refutable}_?$ — by assumption

(28) $\mathtt{prr}(e)$ $\mathtt{notintro}$ — by Rule (26f)

(29) $\mathtt{prr}(e) \models_? \xi_2$ — by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ — by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ — by Lemma 2.0.20 on (25) and (30)

Case (19g).

(11) $e = (e_1, e_2)$ — by assumption

(12) $\cdot; \Delta \vdash e_1 : \tau_1$ — by assumption

(13) $\cdot\,;\Delta \vdash e_2 : \tau_2$ by assumption

(14) e_1 `final`   by Lemma 4.0.4 on (6)

(15) e_2 `final`   by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \models \overline{\xi_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \models \overline{\xi_2}$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e_1 \models \xi_1, e_2 \models \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (16) and (18)

(21) $(e_1, e_2) \models_?^{\dagger} (\xi_1, \xi_2)$ by Rule (17b) on (20)

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro`   by assumption

Contradicts Lemma 4.0.8.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (16h) on (16) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

    (22) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (14i)**.**

    (22) $e_2 \models \xi_2$ by assumption

    Contradicts (18).

    (23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models_?^\dagger \xi_2$**.**

    (16) $e_1 \models \xi_1$ by assumption

    (17) $e_1 \not\models_? \xi_1$ by assumption

    (18) $e_2 \not\models \xi_2$ by assumption

    (19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

    (20) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (14i)**.**

    (20) $e_2 \models \xi_2$ by assumption

    Contradicts (18).

    (21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b)**.**

    (22) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (16g)**.**

    (22) $e_1 \models_? \xi_1$ by assumption

    Contradicts (17).

Case (16h)**.**

    (22) $e_2 \models_? \xi_2$ by assumption

    Contradicts (19).

Case (16i)**.**

    (22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23)	$(e_1, e_2) \not\models_? (\xi_1, \xi_2)$	by contradiction
(24)	$(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$	by Lemma 2.0.20 on (21) and (23)

Case $e_1 \models_? \xi_1, e_2 \models \xi_2$**.**

(16)	$e_1 \not\models \xi_1$	by assumption
(17)	$e_1 \models_? \xi_1$	by assumption
(18)	$e_2 \models \xi_2$	by assumption
(19)	$e_2 \not\models_? \xi_2$	by assumption
(20)	$(e_1, e_2) \models_? (\xi_1, \xi_2)$	by Rule (16g) on (17) and (18)
(21)	$(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (14i)**.**

(22)	$e_1 \models \xi_1$	by assumption

Contradicts (16).

(23)	$(e_1, e_2) \not\models (\xi_1, \xi_2)$	by contradiction

Case $e_1 \models_? \xi_1, e_2 \models_? \xi_2$**.**

(16)	$e_1 \not\models \xi_1$	by assumption
(17)	$e_1 \models_? \xi_1$	by assumption
(18)	$e_2 \not\models \xi_2$	by assumption
(19)	$e_2 \models_? \xi_2$	by assumption
(20)	$(e_1, e_2) \models_? (\xi_1, \xi_2)$	by Rule (16i) on (17) and (19)
(21)	$(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j)**.**

(22)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (14i)**.**

(22)	$e_1 \models \xi_1$	by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \not\models_?^\dagger \xi_2$.

    (16) $e_1 \not\models \xi_1$ by assumption
    (17) $e_1 \models_? \xi_1$ by assumption
    (18) $e_2 \not\models \xi_2$ by assumption
    (19) $e_2 \not\models_? \xi_2$ by assumption

  Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

  Case (14j).

      (20) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

  Case (14i).

      (20) $e_1 \models \xi_1$ by assumption

    Contradicts (16).

    (21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

  Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

  Case (16b).

      (22) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

  Case (16g).

      (22) $e_2 \models \xi_2$ by assumption

    Contradicts (18).

  Case (16h).

      (22) $e_2 \models_? \xi_2$ by assumption

    Contradicts (19).

  Case (16i).

      (22) $e_2 \models_? \xi_2$ by assumption

    Contradicts (19).

    (23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction
    (24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models \xi_2$.

    (16) $e_1 \not\models \xi_1$ by assumption
    (17) $e_1 \not\models_? \xi_1$ by assumption
    (18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

    (20) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (14i).

    (20) $e_1 \models \xi_1$ by assumption

    Contradicts (16).

    (21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

    (22) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (16g).

    (22) $e_1 \models_? \xi_1$ by assumption

    Contradicts (17).

Case (16h).

    (22) $e_2 \models_? \xi_2$ by assumption

    Contradicts (19).

Case (16i).

    (22) $e_1 \models_? \xi_1$ by assumption

    Contradicts (17).

    (23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

    (24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \models_? \xi_2$.

    (16) $e_1 \not\models \xi_1$ by assumption

    (17) $e_1 \not\models_? \xi_1$ by assumption

    (18) $e_2 \not\models \xi_2$ by assumption

    (19) $e_2 \models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

    (20) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) `notintro`   by assumption

Contradicts Lemma 4.0.8.

Case (16g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

Case (16i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) `notintro`   by assumption

Contradicts Lemma 4.0.8.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

$\qquad$ (22) (e_1, e_2) `notintro` $\qquad$ by assumption

$\qquad$ Contradicts Lemma 4.0.8.

$\quad$ **Case** (16g).

$\qquad$ (22) $e_1 \models_? \xi_1$ $\qquad$ by assumption

$\qquad$ Contradicts (17).

$\quad$ **Case** (16h).

$\qquad$ (22) $e_2 \models_? \xi_2$ $\qquad$ by assumption

$\qquad$ Contradicts (19).

$\quad$ **Case** (16i).

$\qquad$ (22) $e_1 \models_? \xi_1$ $\qquad$ by assumption

$\qquad$ Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ $\qquad$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ $\qquad$ by Lemma 2.0.20 on (21) and (23)

$\hfill\square$

Definition 2.1.1 (Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `val` we have $e \models_?^\dagger \xi_1$ implies $e \models \xi_2$*

Definition 2.1.2 (Potential Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models_?^\dagger \xi_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies $e \models_?^\dagger \xi_2$*

Corollary 2.1.1. *Suppose that $\xi : \tau$ and $\cdot\,; \Delta \vdash e : \tau$ and e `final`. Then $\top \models_?^\dagger \xi$ implies $e \models_?^\dagger \xi$*

Proof.

(1) $\xi : \tau$ $\qquad$ by assumption

(2) $\cdot\,; \Gamma \vdash e : \tau$ $\qquad$ by assumption

(3) e `final` $\qquad$ by assumption

(4) $\top \models_?^\dagger \xi$ $\qquad$ by assumption

(5) $e_1 \models \top$ $\qquad$ by Rule (14a)

(6) $e_1 \models_?^\dagger \top$ $\qquad$ by Rule (17b) on (5)

(7) $\top : \tau$ $\qquad$ by Rule (8a)

(8) $e_1 \models_?^\dagger \xi_r$ $\qquad$ by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

$\hfill\square$

3   Static Semantics

$$
\begin{aligned}
\tau \quad &::= \quad \mathtt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e \quad &::= \quad x \mid \underline{n} \\
&\mid \quad (\lambda x : \tau.e) \mid e_1(e_2) \\
&\mid \quad (e_1, e_2) \\
&\mid \quad \mathtt{inl}_\tau(e) \mid \mathtt{inr}_\tau(e) \mid \mathtt{match}(e)\{\hat{rs}\} \\
&\mid \quad (\!|\,|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} \quad &::= \quad (rs \mid r \mid rs) \\
rs \quad &::= \quad \cdot \mid (r \mid rs') \\
r \quad &::= \quad p \Rightarrow e \\
p \quad &::= \quad x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \mathtt{inl}(p) \mid \mathtt{inr}(p) \mid (\!|\,|\!)^w \mid (\!|p|\!)^w
\end{aligned}
$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{18a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{18b}$$

$\boxed{\Gamma\,;\Delta \vdash e : \tau}$ e is of type τ

TVar
$$\frac{}{\Gamma, x : \tau\,;\Delta \vdash x : \tau} \tag{19a}$$

TEHole
$$\frac{}{\Gamma\,;\Delta, u :: \tau \vdash (\!|\,|\!)^u : \tau} \tag{19b}$$

THole
$$\frac{\Gamma\,;\Delta, u :: \tau \vdash e : \tau'}{\Gamma\,;\Delta, u :: \tau \vdash (\!|e|\!)^u : \tau} \tag{19c}$$

TNum
$$\frac{}{\Gamma\,;\Delta \vdash \underline{n} : \mathtt{num}} \tag{19d}$$

TLam
$$\frac{\Gamma, x : \tau_1\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash (\lambda x : \tau_1.e) : (\tau_1 \to \tau_2)} \tag{19e}$$

TAp
$$\frac{\Gamma\,;\Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma\,;\Delta \vdash e_2 : \tau_2}{\Gamma\,;\Delta \vdash e_1(e_2) : \tau} \tag{19f}$$

TPair
$$\frac{\Gamma\,;\Delta \vdash e_1 : \tau_1 \qquad \Gamma\,;\Delta \vdash e_2 : \tau_2}{\Gamma\,;\Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \tag{19g}$$

TPrl
$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \mathtt{prl}(e) : \tau_1} \tag{19h}$$

$$\text{TPrr}$$
$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \mathtt{prr}(e) : \tau_2} \tag{19i}$$

$$\text{TInl}$$
$$\frac{\Gamma\,;\Delta \vdash e : \tau_1}{\Gamma\,;\Delta \vdash \mathtt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \tag{19j}$$

$$\text{TInr}$$
$$\frac{\Gamma\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash \mathtt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \tag{19k}$$

$$\text{TMatchZPre}$$
$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad \Gamma\,;\Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \models_?^\dagger \xi}{\Gamma\,;\Delta \vdash \mathtt{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \tag{19l}$$

$$\text{TMatchNZPre}$$
$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad e\ \mathtt{final} \qquad \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau'}{\Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \qquad e \not\models_?^\dagger \xi_{pre} \qquad \top \models_?^\dagger \xi_{pre} \vee \xi_{rest}}{\Gamma\,;\Delta \vdash \mathtt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \tag{19m}$$

$$\boxed{p : \tau[\xi] \dashv\vdash \Gamma\,;\Delta} \qquad p \text{ is assigned type } \tau \text{ and emits constraint } \xi$$

$$\text{PTVar}$$
$$\frac{}{x : \tau[\top] \dashv\vdash \cdot\,;x : \tau} \tag{20a}$$

$$\text{PTWild}$$
$$\frac{}{\_ : \tau[\top] \dashv\vdash \cdot\,;\cdot} \tag{20b}$$

$$\text{PTEHole}$$
$$\frac{}{(\!|\,|\!)^w : \tau[?] \dashv\vdash \cdot\,;w :: \tau} \tag{20c}$$

$$\text{PTHole}$$
$$\frac{p : \tau[\xi] \dashv\vdash \Gamma\,;\Delta}{(\!|p|\!)^w : \tau'[?] \dashv\vdash \Gamma\,;\Delta, w :: \tau'} \tag{20d}$$

$$\text{PTNum}$$
$$\frac{}{\underline{n} : \mathtt{num}[\underline{n}] \dashv\vdash \cdot\,;\cdot} \tag{20e}$$

$$\text{PTInl}$$
$$\frac{p : \tau_1[\xi] \dashv\vdash \Gamma\,;\Delta}{\mathtt{inl}(p) : (\tau_1 + \tau_2)[\mathtt{inl}(\xi)] \dashv\vdash \Gamma\,;\Delta} \tag{20f}$$

$$\text{PTInr}$$
$$\frac{p : \tau_2[\xi] \dashv\vdash \Gamma\,;\Delta}{\mathtt{inr}(p) : (\tau_1 + \tau_2)[\mathtt{inr}(\xi)] \dashv\vdash \Gamma\,;\Delta} \tag{20g}$$

$$\text{PTPair}$$
$$\frac{p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1\,;\Delta_1 \qquad p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2\,;\Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\vdash \Gamma_1 \uplus \Gamma_2\,;\Delta_1 \uplus \Delta_2} \tag{20h}$$

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$$\boxed{\Gamma\,;\Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'} \qquad \begin{array}{l} r \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

CTRule
$$\dfrac{p : \tau[\xi] \dashv\vdash \Gamma_p\,; \Delta_p \qquad \Gamma \uplus \Gamma_p\,; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma\,; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \tag{21a}$$

$$\boxed{\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'} \qquad \begin{array}{l} rs \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

CTOneRules
$$\dfrac{\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \tag{22a}$$

CTRules
$$\dfrac{\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\,; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'}$$
$$\tag{22b}$$

Lemma 3.0.1. *If* $p : \tau[\xi] \dashv\vdash \Gamma\,; \Delta$ *then* $\xi : \tau$.

Proof. By rule induction over Rules (20). $\qquad\square$

Lemma 3.0.2. *If* $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Proof. By rule induction over Rules (21). $\qquad\square$

Lemma 3.0.3. *If* $\cdot\,; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Proof. By rule induction over Rules (22). $\qquad\square$

Lemma 3.0.4. *If* $\Gamma\,; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma\,; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Proof.

(1)	$\Gamma\,; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$	by assumption
(2)	$\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$	by assumption
(3)	$\xi_r \not\models \xi_{pre} \vee \xi_{rs}$	by assumption

By rule induction over Rules (22) on (1).

Case (22a)**.**

(4)	$rs = r' \mid \cdot$	by assumption
(5)	$\xi_{rs} = \xi_r'$	by assumption
(6)	$\Gamma\,; \Delta \vdash r' : \tau[\xi_r'] \Rightarrow \tau'$	by assumption
(7)	$\xi_r' \not\models \xi_{pre}$	by assumption
(8)	$\Gamma\,; \Delta \vdash [\xi_{pre} \vee \xi_r']r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$	by Rule (22a) on (2) and (3)
(9)	$\Gamma\,; \Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi_r' \vee \xi_r] \Rightarrow \tau'$	by Rule (22b) on (6) and (8) and (7)

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(10) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^{\diamond} : \tau[\xi_r' \vee \xi_r] \Rightarrow \tau'$

by Definition 18 on (9)

Case (22b).

(4) $rs = r' \mid rs'$ — by assumption

(5) $\xi_{rs} = \xi_r' \vee \xi_{rs}'$ — by assumption

(6) $\Gamma\,;\Delta \vdash r' : \tau[\xi_r'] \Rightarrow \tau'$ — by assumption

(7) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi_r']rs' : \tau[\xi_{rs}'] \Rightarrow \tau'$ — by assumption

(8) $\xi_r' \not\models \xi_{pre}$ — by assumption

(9) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi_r'](rs' \mid r \mid \cdot)^{\diamond} : \tau[\xi_{rs}' \vee \xi_r] \Rightarrow \tau'$

by IH on (7) and (2) and (3)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^{\diamond}) : \tau[\xi_r' \vee \xi_{rs}' \vee \xi_r] \Rightarrow \tau'$

by Rule (22b) on (6) and (9) and (8)

(11) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^{\diamond} : \tau[\xi_r' \vee \xi_{rs}' \vee \xi_r] \Rightarrow \tau'$

by Definition 18 on (10)

$\square$

Lemma 3.0.5 (Substitution). *If* $\Gamma, x : \tau\,;\Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta \vdash e : \tau$ *then* $\Gamma\,;\Delta \vdash [e/x]e_0 : \tau_0$

Lemma 3.0.6 (Simultaneous Substitution). *If* $\Gamma \uplus \Gamma'\,;\Delta \vdash e : \tau$ *and* $\theta : \Gamma'$ *then* $\Gamma\,;\Delta \vdash [\theta]e : \tau$

Lemma 3.0.7 (Substitution Typing). *If* $e \triangleright p \dashv\!\mid \theta$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\mid \Gamma\,;\Delta$ *then* $\theta : \Gamma$

Proof by induction on the derivation of $e \triangleright p \dashv\!\mid \theta$.

Theorem 3.1 (Determinism). *If* $\cdot\,;\Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`

2. e `indet`

3. $e \mapsto e'$ for some unique e'

4   Dynamic Semantics

$\boxed{e \; \mathtt{val}}$ e is a value

$$\frac{\text{VNum}}{\underline{n} \; \mathtt{val}} \tag{23a}$$

$$\frac{\text{VLam}}{(\lambda x : \tau.e) \; \mathtt{val}} \tag{23b}$$

$$\frac{\text{VPair} \quad e_1 \; \mathtt{val} \qquad e_2 \; \mathtt{val}}{(e_1, e_2) \; \mathtt{val}} \tag{23c}$$

$$\frac{\text{VInl} \quad e \; \mathtt{val}}{\mathtt{inl}_\tau(e) \; \mathtt{val}} \tag{23d}$$

$$\frac{\text{VInr} \quad e \; \mathtt{val}}{\mathtt{inr}_\tau(e) \; \mathtt{val}} \tag{23e}$$

$\boxed{e \; \mathtt{indet}}$ e is indeterminate

$$\frac{\text{IEHole}}{(\!|\;|\!)^u \; \mathtt{indet}} \tag{24a}$$

$$\frac{\text{IHole} \quad e \; \mathtt{final}}{(\!|e|\!)^u \; \mathtt{indet}} \tag{24b}$$

$$\frac{\text{IAp} \quad e_1 \; \mathtt{indet} \qquad e_2 \; \mathtt{final}}{e_1(e_2) \; \mathtt{indet}} \tag{24c}$$

$$\frac{\text{IPairL} \quad e_1 \; \mathtt{indet} \qquad e_2 \; \mathtt{val}}{(e_1, e_2) \; \mathtt{indet}} \tag{24d}$$

$$\frac{\text{IPairR} \quad e_1 \; \mathtt{val} \qquad e_2 \; \mathtt{indet}}{(e_1, e_2) \; \mathtt{indet}} \tag{24e}$$

$$\frac{\text{IPair} \quad e_1 \; \mathtt{indet} \qquad e_2 \; \mathtt{indet}}{(e_1, e_2) \; \mathtt{indet}} \tag{24f}$$

$$\frac{\text{IPrl} \quad e \; \mathtt{indet}}{\mathtt{prl}(e) \; \mathtt{indet}} \tag{24g}$$

$$\frac{\text{IPrr}}{\text{prr}(e) \text{ indet}}$$ (24h)

$$\frac{\text{IInL}}{\text{inl}_\tau(e) \text{ indet}}$$ (24i)

$$\frac{\text{IInR}}{\text{inr}_\tau(e) \text{ indet}}$$ (24j)

$$\frac{\text{IMatch} \qquad e \text{ final} \qquad e \mathbin{?} p_r}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \text{ indet}}$$ (24k)

$\boxed{e \text{ final}}$ e is final

$$\frac{\text{FVal} \qquad e \text{ val}}{e \text{ final}}$$ (25a)

$$\frac{\text{FIndet} \qquad e \text{ indet}}{e \text{ final}}$$ (25b)

$\boxed{e \text{ notintro}}$ e cannot be a value syntactically

$$\frac{\text{NVEHole}}{(\!|\,|\!)^u \text{ notintro}}$$ (26a)

$$\frac{\text{NVHole}}{(\!|e|\!)^u \text{ notintro}}$$ (26b)

$$\frac{\text{NVAp}}{e_1(e_2) \text{ notintro}}$$ (26c)

$$\frac{\text{NVMatch}}{\text{match}(e)\{\hat{rs}\} \text{ notintro}}$$ (26d)

$$\frac{\text{NVPrl}}{\text{prl}(e) \text{ notintro}}$$ (26e)

$$\frac{\text{NVPrr}}{\text{prr}(e) \text{ notintro}}$$ (26f)

$\boxed{\text{complete}(e)}$ for e final and $\cdot\,; \Delta \vdash e : \tau$

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$$\text{complete}(e) = \{e\} \quad \text{if } e \text{ val} \tag{27a}$$

$$\text{complete}(e) = \{e' \mid e : \tau \text{ and } e \text{ val}\} \quad \text{if } e \text{ notintro} \tag{27b}$$

$$\text{complete}(\text{inl}_{\tau_2}(e_1)) = \{\text{inl}_{\tau_2}(e_1') \mid e_1' \in \text{complete}(e_1)\} \tag{27c}$$

$$\text{complete}(\text{inr}_{\tau_1}(e_2)) = \{\text{inr}_{\tau_1}(e_2') \mid e_2' \in \text{complete}(e_2)\} \tag{27d}$$

$$\text{complete}((e_1, e_2)) = \{(e_1', e_2') \mid e_1' \in \text{complete}(e_1) \text{ and } e_2' \in \text{complete}(e_2)\} \tag{27e}$$

Lemma 4.0.1. *If e indet and $\cdot\,; \Delta \vdash e : \tau$ and $\dot\xi : \tau$ and $e \not\models_?^\dagger \dot\xi$ then $e' \not\models_?^\dagger \dot\xi$ for all $e' \in \text{complete}(e)$.*

Proof.

(1)	e indet	by assumption
(2)	$\cdot\,; \Delta \vdash e : \tau$	by assumption
(3)	$\dot\xi : \tau$	by assumption
(4)	$e \not\models_?^\dagger \dot\xi$	by assumption

By rule induction over Rules (8) on (3).

Case (8a).

(5)	$\dot\xi = \top$	by assumption
(6)	$e \models \top$	by Rule (14a)
(7)	$e \models_?^\dagger \top$	by Rule (17b) on (6)

Contradicts (4).

Case (1b).

(5)	$\dot\xi = {?}$	by assumption
(6)	$e \models_? {?}$	by Rule (16a)
(7)	$e \models_?^\dagger {?}$	by Rule (17a) on (6)

Contradicts (4).

Case (8c).

(5)	$\dot\xi = \underline{n}$	by assumption
(6)	$\tau = \text{num}$	by assumption
(7)	$\underline{n}$ refutable$_?$	by Rule (10a)

By rule induction over Rules (24) on (1).

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Case (24a).

(8)	$e = (\!\|\!)^u$	by assumption
(9)	$(\!\|\!)^u$ notintro	by Rule (26a)
(10)	$(\!\|\!)^u \models_? \underline{n}$	by Rule (16b) on (9) and (7)
(11)	$(\!\|\!)^u \models_?^\dagger \underline{n}$	by Rule (17a) on (10)

Case (24b).

(8)	$e = (\!|e_1|\!)^u$	by assumption
(9)	$(\!|e_1|\!)^u$ notintro	by Rule (26b)
(10)	$(\!|e_1|\!)^u \models_? \underline{n}$	by Rule (16b) on (9) and (7)
(11)	$(\!|e_1|\!)^u \models_?^\dagger \underline{n}$	by Rule (17a) on (10)

Case (24c).

(8)	$e = e_1(e_2)$	by assumption
(9)	$e_1(e_2)$ notintro	by Rule (26c)
(10)	$e_1(e_2) \models_? \underline{n}$	by Rule (16b) on (9) and (7)
(11)	$e_1(e_2) \models_?^\dagger \underline{n}$	by Rule (17a) on (10)

Case (24g).

(8)	$e = \mathtt{prl}(e_1)$	by assumption
(9)	$\mathtt{prl}(e_1)$ notintro	by Rule (26e)
(10)	$\mathtt{prl}(e_1) \models_? \underline{n}$	by Rule (16b) on (9) and (7)
(11)	$\mathtt{prl}(e_1) \models_?^\dagger \underline{n}$	by Rule (17a) on (10)

Case (24h).

(8)	$e = \mathtt{prr}(e_1)$	by assumption
(9)	$\mathtt{prr}(e_1)$ notintro	by Rule (26f)
(10)	$\mathtt{prr}(e_1) \models_? \underline{n}$	by Rule (16b) on (9) and (7)
(11)	$\mathtt{prr}(e_1) \models_?^\dagger \underline{n}$	by Rule (17a) on (10)

Case (24k).

(8)	$e = \mathtt{match}(e_1)\{\hat{rs}\}$	by assumption
(9)	$\mathtt{match}(e_1)\{\hat{rs}\}$ notintro	by Rule (26d)
(10)	$\mathtt{match}(e_1)\{\hat{rs}\} \models_? \underline{n}$	by Rule (16b) on (9) and (7)
(11)	$\mathtt{match}(e_1)\{\hat{rs}\} \models_?^\dagger \underline{n}$	by Rule (17a) on (10)

Case (24d), (24e), (24f).

$\qquad$ (8) $\;e = (e_1, e_2)$ $\qquad\qquad\qquad$ by assumption

$\qquad$ By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

Case (24i).

$\qquad$ (8) $\;e = \mathtt{inl}_{\tau_2}(e_1)$ $\qquad\qquad\qquad$ by assumption

$\qquad$ By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

Case (24j).

$\qquad$ (8) $\;e = \mathtt{inr}_{\tau_1}(e_2)$ $\qquad\qquad\qquad$ by assumption

$\qquad$ By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

Case (8g).

$\qquad$ (5) $\;\dot\xi = \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (6) $\;\tau = (\tau_1 + \tau_2)$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (7) $\;\dot\xi_1 : \tau_1$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (8) $\;\mathtt{inl}(\dot\xi_1)\ \mathtt{refutable}_?$ $\qquad\qquad\qquad$ by Rule (10b)

By rule induction over Rules (24) on (1).

Case (24a).

$\qquad$ (9) $\;e = (\!|\,|\!)^u$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (10) $\;(\!|\,|\!)^u\ \mathtt{notintro}$ $\qquad\qquad\qquad$ by Rule (26a)

$\qquad$ (11) $\;(\!|\,|\!)^u \models_? \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by Rule (16b) on (10) and (8)

$\qquad$ (12) $\;(\!|\,|\!)^u \models_?^\dagger \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by Rule (17a) on (11)

Case (24b).

$\qquad$ (9) $\;e = (\!|e_1|\!)^u$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (10) $\;(\!|e_1|\!)^u\ \mathtt{notintro}$ $\qquad\qquad\qquad$ by Rule (26b)

$\qquad$ (11) $\;(\!|e_1|\!)^u \models_? \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by Rule (16b) on (10) and (8)

$\qquad$ (12) $\;(\!|e_1|\!)^u \models_?^\dagger \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by Rule (17a) on (11)

Case (24c).

$\qquad$ (9) $\;e = e_1(e_2)$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (10) $\;e_1(e_2)\ \mathtt{notintro}$ $\qquad\qquad\qquad$ by Rule (26c)

$\qquad$ (11) $\;e_1(e_2) \models_? \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by Rule (16b) on (10) and (8)

$\qquad$ (12) $\;e_1(e_2) \models_?^\dagger \mathtt{inl}(\dot\xi_1)$ $\qquad\qquad\qquad$ by Rule (17a) on (11)

Case (24g).

$\quad$ (9) $\ e = \mathtt{prl}(e_1)$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (10) $\ \mathtt{prl}(e_1)\ \mathtt{notintro}$ $\qquad\qquad\qquad$ by Rule (26e)

$\quad$ (11) $\ \mathtt{prl}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$ $\qquad\qquad$ by Rule (16b) on (10)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ and (8)

$\quad$ (12) $\ \mathtt{prl}(e_1) \models_?^{\dagger} \mathtt{inl}(\dot{\xi}_1)$ $\qquad\qquad$ by Rule (17a) on (11)

Case (24h).

$\quad$ (9) $\ e = \mathtt{prr}(e_1)$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (10) $\ \mathtt{prr}(e_1)\ \mathtt{notintro}$ $\qquad\qquad\qquad$ by Rule (26f)

$\quad$ (11) $\ \mathtt{prr}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$ $\qquad\qquad$ by Rule (16b) on (10)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ and (8)

$\quad$ (12) $\ \mathtt{prr}(e_1) \models_?^{\dagger} \mathtt{inl}(\dot{\xi}_1)$ $\qquad\qquad$ by Rule (17a) on (11)

Case (24k).

$\quad$ (9) $\ e = \mathtt{match}(e_1)\{\hat{rs}\}$ $\qquad\qquad\ $ by assumption

$\quad$ (10) $\ \mathtt{match}(e_1)\{\hat{rs}\}\ \mathtt{notintro}$ $\quad\ $ by Rule (26d)

$\quad$ (11) $\ \mathtt{match}(e_1)\{\hat{rs}\} \models_? \mathtt{inl}(\dot{\xi}_1)$ $\ $ by Rule (16b) on (10)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ and (8)

$\quad$ (12) $\ \mathtt{match}(e_1)\{\hat{rs}\} \models_?^{\dagger} \mathtt{inl}(\dot{\xi}_1)$ $\ $ by Rule (17a) on (11)

Case (24d), (24e), (24f).

$\quad$ (9) $\ e = (e_1, e_2)$ $\qquad\qquad\qquad\qquad\ $ by assumption

By rule induction over Rules (24) on (1), no rule applies due to syntactic contradiction.

Case (24i).

$\quad$ (9) $\ e = \mathtt{inl}_{\tau_2'}(e_1)$ $\qquad\qquad\qquad\ $ by assumption

$\quad$ (10) $\ e_1\ \mathtt{indet}$ $\qquad\qquad\qquad\qquad\ $ by assumption

By rule induction over Rules (19) on (2), only one rule applies.

$\quad$ **Case (19j).**

$\qquad$ (11) $\ \tau_2' = \tau_2$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (12) $\ \cdot\,; \Delta \vdash e_1 : \tau_1$ $\qquad\qquad\quad$ by assumption

$\qquad$ (13) $\ e_1 \not\models_?^{\dagger} \dot{\xi}_1$ $\qquad\qquad\qquad$ by Lemma 2.0.11 on
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ $ (4)

$\qquad$ (14) $\ e_1' \not\models_?^{\dagger} \dot{\xi}_1$ for any $e_1' \in \mathtt{complete}(e_1)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ by IH on (10) and (12)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ and (7) and (13)

$\qquad$ (15) $\ e' \not\models_?^{\dagger} \mathtt{inl}(\dot{\xi}_1)$ for any
$\qquad\qquad\quad e' \in \{\mathtt{inl}_{\tau_2}(e_1') \mid e_1' \in \mathtt{complete}(e_1)\}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ by Lemma 2.0.11 on
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (14)

(16) $e' \not\models_?^\dagger \texttt{inl}(\dot\xi_1)$ for any $e' \in \texttt{complete}(\texttt{inl}_{\tau_2}(e_1))$
by Equation 27 on (15)

Case (24j).

(9) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(10) $e' \not\models_?^\dagger \texttt{inl}(\dot\xi_1)$ for any $e' \in \{\texttt{inr}_{\tau_1}(e_2')\}$
by Lemma 1.0.18

(11) $e' \not\models_?^\dagger \texttt{inl}(\dot\xi_1)$ for any $e' \in \{\texttt{inr}_{\tau_1}(e_2') \mid e_2' \in \texttt{complete}(e_2)\}$
by Weakening on (10)

(12) $e' \not\models_?^\dagger \texttt{inl}(\dot\xi_1)$ for any $e' \in \texttt{complete}(\texttt{inr}_{\tau_1}(e_2))$
by Equation 27 on (11)

Case (8h).

(5) $\dot\xi = \texttt{inr}(\dot\xi_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $\dot\xi_2 : \tau_2$ by assumption

(8) $\texttt{inr}(\dot\xi_2)$ $\texttt{refutable}_?$ by Rule (10c)

By rule induction over Rules (24) on (1).

Case (24a).

(9) $e = (\!|\!|)^u$ by assumption

(10) $(\!|\!|)^u$ $\texttt{notintro}$ by Rule (26a)

(11) $(\!|\!|)^u \models_? \texttt{inr}(\dot\xi_2)$ by Rule (16b) on (10) and (8)

(12) $(\!|\!|)^u \models_?^\dagger \texttt{inr}(\dot\xi_2)$ by Rule (17a) on (11)

Case (24b).

(9) $e = (\!| e_1 |\!)^u$ by assumption

(10) $(\!| e_1 |\!)^u$ $\texttt{notintro}$ by Rule (26b)

(11) $(\!| e_1 |\!)^u \models_? \texttt{inr}(\dot\xi_2)$ by Rule (16b) on (10) and (8)

(12) $(\!| e_1 |\!)^u \models_?^\dagger \texttt{inr}(\dot\xi_2)$ by Rule (17a) on (11)

Case (24c).

(9) $e = e_1(e_2)$ by assumption

(10) $e_1(e_2)$ $\texttt{notintro}$ by Rule (26c)

(11) $e_1(e_2) \models_? \texttt{inr}(\dot\xi_2)$ by Rule (16b) on (10) and (8)

(12) $e_1(e_2) \models_?^\dagger \texttt{inr}(\dot\xi_2)$ by Rule (17a) on (11)

Case (24g).

(9) $e = \mathtt{prl}(e_1)$	by assumption
(10) $\mathtt{prl}(e_1)$ notintro	by Rule (26e)
(11) $\mathtt{prl}(e_1) \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (16b) on (10) and (8)
(12) $\mathtt{prl}(e_1) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (17a) on (11)

Case (24h).

(9) $e = \mathtt{prr}(e_1)$	by assumption
(10) $\mathtt{prr}(e_1)$ notintro	by Rule (26f)
(11) $\mathtt{prr}(e_1) \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (16b) on (10) and (8)
(12) $\mathtt{prr}(e_1) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (17a) on (11)

Case (24k).

(9) $e = \mathtt{match}(e_1)\{\hat{rs}\}$	by assumption
(10) $\mathtt{match}(e_1)\{\hat{rs}\}$ notintro	by Rule (26d)
(11) $\mathtt{match}(e_1)\{\hat{rs}\} \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (16b) on (10) and (8)
(12) $\mathtt{match}(e_1)\{\hat{rs}\} \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (17a) on (11)

Case (24d), (24e), (24f).

| (9) $e = (e_1, e_2)$ | by assumption |

By rule induction over Rules (24) on (1), no rule applies due to syntactic contradiction.

Case (24i).

(9) $e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(10) $e' \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ for any $e' \in \{\mathtt{inl}_{\tau_2}(e_1')\}$	
	by Lemma 1.0.17
(11) $e' \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ for any $e' \in \{\mathtt{inl}_{\tau_2}(e_1') \mid e_1' \in \mathtt{complete}(e_1)\}$	
	by Weakening on (10)
(12) $e' \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ for any $e' \in \mathtt{complete}(\mathtt{inl}_{\tau_2}(e_1))$	
	by Equation 27 on (11)

Case (24j).

| (9) $e = \mathtt{inr}_{\tau_1'}(e_2)$ | by assumption |
| (10) e_2 indet | by assumption |

By rule induction over Rules (19) on (2), only one rule applies.

Case (19k).

| (11) $\tau_1' = \tau_1$ | by assumption |
| (12) $\cdot\,; \Delta \vdash e_2 : \tau_2$ | by assumption |

(13) $e_2 \not\models_?^\dagger \dot{\xi}_2$ 　　　　　　　　　 by Lemma 2.0.11 on (4)

(14) $e_2' \not\models_?^\dagger \dot{\xi}_2$ for any $e_2' \in \mathtt{complete}(e_2)$
　　　　　　　　　 by IH on (10) and (12) and (7) and (13)

(15) $e' \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ for any
$e' \in \{\mathtt{inr}_{\tau_1}(e_2') \mid e_2' \in \mathtt{complete}(e_2)\}$
　　　　　　　　　 by Lemma 2.0.12 on (14)

(16) $e' \not\models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ for any $e' \in \mathtt{complete}(\mathtt{inr}_{\tau_1}(e_2))$
　　　　　　　　　 by Equation 27 on (15)

Case (8i).

(5) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ 　　　　　　　　　 by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ 　　　　　　　　　 by assumption

(7) $\dot{\xi}_1 : \tau_1$ 　　　　　　　　　 by assumption

(8) $\dot{\xi}_2 : \tau_2$ 　　　　　　　　　 by assumption

By rule induction over Rules (24) on (1).

Case (24a), (24b), (24c), (24g), (24h), (24k).

(9) $e = (\!| \,)^u, (\!|e_1|\!)^u, e_1(e_2), \mathtt{prl}(e_1), \mathtt{prr}(e_1), \mathtt{match}(e_1)\{\hat{rs}\}$
　　　　　　　　　 by assumption

(10) e notintro 　　　　　　　 by Rules (26)

(11) $\mathtt{complete}(e) = \{e' \mid e' \text{ val and } e' : (\tau_1 \times \tau_2)\}$
　　　　　　　　　 by Equation 27 on (10)

(12) $\mathtt{prl}(e)$ indet 　　　　　　 by Rule (24g) on (1)

(13) $\mathtt{prr}(e)$ indet 　　　　　　 by Rule (24h) on (1)

(14) $\cdot\,; \Delta \vdash \mathtt{prl}(e) : \tau_1$ 　　　　 by Rule (19h) on (2)

(15) $\cdot\,; \Delta \vdash \mathtt{prr}(e) : \tau_2$ 　　　　 by Rule (19i) on (2)

By case analysis on the result of $\mathit{satisfyormay}(\mathtt{prl}(e), \dot{\xi}_1)$.

Case true.

(16) $\mathit{satisfyormay}(\mathtt{prl}(e), \dot{\xi}_1) = \text{true}$
　　　　　　　　　 by assumption

(17) $\mathtt{prl}(e) \models_?^\dagger \dot{\xi}_1$ 　　　　 by Lemma 1.0.20 on (16)

Case false.

(16) $\mathit{satisfyormay}(\mathtt{prl}(e), \dot{\xi}_1) = \text{false}$
　　　　　　　　　 by assumption

(17) $\mathtt{prl}(e) \not\models_?^\dagger \dot{\xi}_1$ 　　　 by Lemma 1.0.20 on (16)

(18) $e'_1 \not\models^\dagger_? \dot\xi_1$ for any $e'_1 \in \texttt{complete}(\texttt{prl}(e))$
                      by IH on (12) and (14)
                      and (7) and (17)

(19) $e'_1 \not\models^\dagger_? \dot\xi_1$ for any $e'_1 \in \{e'_1 \mid e'_1 \ \texttt{val} \text{ and } e'_1 : \tau_1\}$
                      by Equation 27 on (18)

Then for any $e' \in \{e' \mid e' \ \texttt{val} \text{ and } e' : (\tau_1 \times \tau_2)\}$,

(20) $e' \ \texttt{val}$ by assumption

(21) $\cdot\,; \Delta \vdash e' : (\tau_1 \times \tau_2)$ by assumption

By rule induction over Rules (23) on (20).

Case (23a).

(22) $e' = \underline{n}$ by assumption

By rule induction over Rules (19) on (21), no rule applies due to syntactic contradiction.

Case (23b).

(22) $e' = (\lambda x : \tau'.e'_1)$ by assumption

By rule induction over Rules (19) on (21), no rule applies due to syntactic contradiction.

Case (23c).

(22) $e' = (e'_1, e'_2)$ by assumption

(23) $e'_1 \ \texttt{val}$ by assumption

By rule induction over Rules (19) on (21), only one rule applies.

Case (19g).

(24) $\cdot\,; \Delta \vdash e'_1 : \tau_1$ by assumption

(25) $e'_1 \not\models^\dagger_? \dot\xi_1$ by (19) on (23) and
                    (24)

(26) $(e'_1, e'_2) \not\models^\dagger_? (\dot\xi_1, \dot\xi_2)$ by Lemma 2.0.13 on
                    (25)

Case (23d).

(22) $e' = \texttt{inl}_{\tau_2}(e'_1)$ by assumption

By rule induction over Rules (19) on (21), no rule applies due to syntactic contradiction.

Case (23e).

(22) $e' = \texttt{inr}_{\tau_1}(e'_2)$ by assumption

By rule induction over Rules (19) on (21), no rule applies due to syntactic contradiction.

To conclude, $(e'_1, e'_2) \not\models^\dagger_? (\dot\xi_1, \dot\xi_2)$ for any $e' \in \texttt{complete}(e)$.

Case (24d).

(9) $e = (e_1, e_2)$ by assumption

(10) $e_1 \ \texttt{indet}$ by assumption

(11) e_2 `val`   by assumption

(12) $e_1 \not\models_?^\dagger \dot{\xi}_1$ or $e_2 \not\models_?^\dagger \dot{\xi}_2$ by Lemma 2.0.13 on
(4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models_?^\dagger \dot{\xi}_1$.

    (13) $e_1 \not\models_?^\dagger \dot{\xi}_1$ by assumption

    By rule induction over Rules (19) on (2), only one rule applies.

    Case (19g).

        (14) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

        (15) $e_1' \not\models_?^\dagger \dot{\xi}_1$ for any $e_1' \in$ `complete`(e_1)
                           by IH on (10) and (14)
                           and (7) and (13)

        (16) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ for any $e' \in \{(e_1', e_2') \mid e_1' \in$
           `complete`(e_1) and $e_2' \in$ `complete`$(e_2)\}$
                           by Lemma 2.0.13 on
(15)

Case $e_2 \not\models_?^\dagger \dot{\xi}_2$.

    (13) $e_2' \not\models_?^\dagger \dot{\xi}_2$ by assumption

    (14) $e_2' \not\models_?^\dagger \dot{\xi}_2$ for any $e_2' \in$ `complete`(e_2)
                           by Equation 27 and
(13)

    (15) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ for any $e' \in \{(e_1', e_2') \mid e_1' \in$
       `complete`(e_1) and $e_2' \in$ `complete`$(e_2)\}$
                           by Lemma 2.0.13 on
(14)

Case (24e).

    (9) $e = (e_1, e_2)$ by assumption

    (10) e_1 `val`   by assumption

    (11) e_2 `indet`   by assumption

    (12) $e_1 \not\models_?^\dagger \dot{\xi}_1$ or $e_2 \not\models_?^\dagger \dot{\xi}_2$ by Lemma 2.0.13 on
(4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models_?^\dagger \dot{\xi}_1$.

    (13) $e_1' \not\models_?^\dagger \dot{\xi}_1$ by assumption

    (14) $e_1' \not\models_?^\dagger \dot{\xi}_1$ for any $e_1' \in$ `complete`(e_1)
                           by Equation 27 and
(13)

    (15) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ for any $e' \in \{(e_1', e_2') \mid e_1' \in$
       `complete`(e_1) and $e_2' \in$ `complete`$(e_2)\}$
                           by Lemma 2.0.13 on
(14)

Case $e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

    (13) $e_2 \not\models_?^\dagger \dot{\xi}_2$ by assumption

By rule induction over Rules (19) on (2), only one rule applies.

Case (19g)**.**

        (14) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

        (15) $e_2' \not\models_?^\dagger \dot{\xi}_2$ for any $e_2' \in \mathtt{complete}(e_2)$

                         by IH on (11) and (14)
                         and (8) and (13)

        (16) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ for any $e' \in \{(e_1', e_2') \mid e_1' \in$
             $\mathtt{complete}(e_1)$ and $e_2' \in \mathtt{complete}(e_2)\}$

                         by Lemma 2.0.13 on
                         (15)

Case (24f)**.**

    (9) $e = (e_1, e_2)$ by assumption

    (10) e_1 `indet`   by assumption

    (11) e_2 `indet`   by assumption

    (12) $e_1 \not\models_?^\dagger \dot{\xi}_1$ or $e_2 \not\models_?^\dagger \dot{\xi}_2$ by Lemma 2.0.13 on
                                      (4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models_?^\dagger \dot{\xi}_1$**.**

    (13) $e_1 \not\models_?^\dagger \dot{\xi}_1$ by assumption

By rule induction over Rules (19) on (2), only one rule applies.

Case (19g)**.**

        (14) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

        (15) $e_1' \not\models_?^\dagger \dot{\xi}_1$ for any $e_1' \in \mathtt{complete}(e_1)$

                         by IH on (10) and (14)
                         and (7) and (13)

        (16) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ for any $e' \in \{(e_1', e_2') \mid e_1' \in$
             $\mathtt{complete}(e_1)$ and $e_2' \in \mathtt{complete}(e_2)\}$

                         by Lemma 2.0.13 on
                         (15)

Case $e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

    (13) $e_2 \not\models_?^\dagger \dot{\xi}_2$ by assumption

By rule induction over Rules (19) on (2), only one rule applies.

Case (19g)**.**

        (14) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

        (15) $e_2' \not\models_?^\dagger \dot{\xi}_2$ for any $e_2' \in \mathtt{complete}(e_2)$

                         by IH on (11) and (14)
                         and (8) and (13)

(16) $(e_1, e_2) \not\models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ for any $e' \in \{(e'_1, e'_2) \mid e'_1 \in$ complete(e_1) and $e'_2 \in$ complete$(e_2)\}$
$\qquad\qquad\qquad\qquad\qquad\qquad$ by Lemma 2.0.13 on (15)

Case $(24i)$.

$\qquad (9)$ $e = \mathtt{inl}_{\tau_2}(e_1)$ $\qquad\qquad\qquad$ by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

Case $(24j)$.

$\qquad (9)$ $e = \mathtt{inr}_{\tau'_1}(e_2)$ $\qquad\qquad\qquad$ by assumption

By rule induction over Rules (19) on (2), no rule applies due to syntactic contradiction.

Case $(8f)$.

$\qquad (5)$ $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad (6)$ $\dot{\xi}_1 : \tau_1$ $\qquad\qquad\qquad\qquad\qquad$ by assumption

$\qquad (7)$ $\dot{\xi}_2 : \tau_2$ $\qquad\qquad\qquad\qquad\qquad$ by assumption

$\qquad (8)$ $e \not\models^\dagger_? \dot{\xi}_1 \vee \dot{\xi}_2$ $\qquad\qquad\qquad$ by assumption

$\qquad (9)$ $e \not\models^\dagger_? \dot{\xi}_1$ $\qquad\qquad\qquad\qquad$ by Lemma 2.0.10 on (8)

$\qquad (10)$ $e \not\models^\dagger_? \dot{\xi}_2$ $\qquad\qquad\qquad\quad$ by Lemma 2.0.10 on (8)

$\qquad (11)$ $e' \not\models^\dagger_? \dot{\xi}_1$ for any $e' \in$ complete(e) $\qquad$ by IH on (1) and (2) and (6) and (9)

$\qquad (12)$ $e' \not\models^\dagger_? \dot{\xi}_2$ for any $e' \in$ complete(e) $\qquad$ by IH on (1) and (2) and (7) and (10)

$\qquad (13)$ $e' \not\models^\dagger_? \dot{\xi}_1 \vee \dot{\xi}_2$ for any $e' \in$ complete(e) $\qquad$ by Lemma 2.0.10 on (11) and (12)

$\square$

$\boxed{\theta : \Gamma}$ $\qquad \theta$ is of type Γ

$$\begin{array}{c} \text{STEmpty} \\ \hline \emptyset : \cdot \end{array} \qquad (28a)$$

$$\begin{array}{c} \text{STExtend} \\ \dfrac{\theta : \Gamma_\theta \qquad \Gamma \,; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau} \end{array} \qquad (28b)$$

$\boxed{p \ \mathtt{refutable}_?}$ $\qquad p$ is refutable

$$\frac{\text{RNum}}{\underline{n} \text{ refutable}_?} \tag{29a}$$

$$\frac{\text{REHole}}{(\!|\!|)^w \text{ refutable}_?} \tag{29b}$$

$$\frac{\text{RHole}}{(\!|p|\!)^w \text{ refutable}_?} \tag{29c}$$

$$\frac{\text{RInl}}{\texttt{inl}(p) \text{ refutable}_?} \tag{29d}$$

$$\frac{\text{RInr}}{\texttt{inr}(p) \text{ refutable}_?} \tag{29e}$$

$$\frac{\text{RPairL} \qquad p_1 \text{ refutable}_?}{(p_1, p_2) \text{ refutable}_?} \tag{29f}$$

$$\frac{\text{RPairR} \qquad p_2 \text{ refutable}_?}{(p_1, p_2) \text{ refutable}_?} \tag{29g}$$

$\boxed{e \rhd p \dashv\!\vdash \theta}$ $\qquad$ e matches p, emitting θ

$$\frac{\text{MVar}}{e \rhd x \dashv\!\vdash e/x} \tag{30a}$$

$$\frac{\text{MWild}}{e \rhd \_ \dashv\!\vdash \cdot} \tag{30b}$$

$$\frac{\text{MNum}}{\underline{n} \rhd \underline{n} \dashv\!\vdash \cdot} \tag{30c}$$

$$\frac{\text{MPair} \qquad e_1 \rhd p_1 \dashv\!\vdash \theta_1 \qquad e_2 \rhd p_2 \dashv\!\vdash \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2} \tag{30d}$$

$$\frac{\text{MInl} \qquad e \rhd p \dashv\!\vdash \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\vdash \theta} \tag{30e}$$

$$\frac{\text{MInr} \qquad e \rhd p \dashv\!\vdash \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\vdash \theta} \tag{30f}$$

$$\frac{\text{MNotIntroPair} \qquad e \text{ notintro} \qquad \texttt{prl}(e) \rhd p_1 \dashv\!\vdash \theta_1 \qquad \texttt{prr}(e) \rhd p_2 \dashv\!\vdash \theta_2}{e \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2} \tag{30g}$$

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$\boxed{e \mathbin{?} p}$ e may match p

$$\frac{\text{MMEHole}}{e \mathbin{?} (\!|\,|\!)^w} \tag{31a}$$

$$\frac{\text{MMHole}}{e \mathbin{?} (\!|p|\!)^w} \tag{31b}$$

$$\frac{\text{MMNotIntro} \quad e \text{ notintro} \qquad p \text{ refutable}_?}{e \mathbin{?} p} \tag{31c}$$

$$\frac{\text{MMPairL} \quad e_1 \mathbin{?} p_1 \qquad e_2 \rhd p_2 \dashv\vdash \theta_2}{(e_1, e_2) \mathbin{?} (p_1, p_2)} \tag{31d}$$

$$\frac{\text{MMPairR} \quad e_1 \rhd p_1 \dashv\vdash \theta_1 \qquad e_2 \mathbin{?} p_2}{(e_1, e_2) \mathbin{?} (p_1, p_2)} \tag{31e}$$

$$\frac{\text{MMPair} \quad e_1 \mathbin{?} p_1 \qquad e_2 \mathbin{?} p_2}{(e_1, e_2) \mathbin{?} (p_1, p_2)} \tag{31f}$$

$$\frac{\text{MMInl} \quad e \mathbin{?} p}{\text{inl}_\tau(e) \mathbin{?} \text{inl}(p)} \tag{31g}$$

$$\frac{\text{MMInr} \quad e \mathbin{?} p}{\text{inr}_\tau(e) \mathbin{?} \text{inr}(p)} \tag{31h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{\text{NMNum} \quad n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}} \tag{32a}$$

$$\frac{\text{NMPairL} \quad e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)} \tag{32b}$$

$$\frac{\text{NMPairR} \quad e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)} \tag{32c}$$

$$\frac{\text{NMConfL}}{\text{inr}_\tau(e) \perp \text{inl}(p)} \tag{32d}$$

$$\frac{\text{NMConfR}}{\text{inl}_\tau(e) \perp \text{inr}(p)} \tag{32e}$$

NMInl
$$\frac{e \perp p}{\mathtt{inr}_\tau(e) \perp \mathtt{inl}(p)} \tag{32f}$$

NMInr
$$\frac{e \perp p}{\mathtt{inl}_\tau(e) \perp \mathtt{inr}(p)} \tag{32g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

ITHole
$$\frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{33a}$$

ITApFun
$$\frac{e_1 \mapsto e'_1}{e_1(e_2) \mapsto e'_1(e_2)} \tag{33b}$$

ITApArg
$$\frac{e_1 \; \mathtt{val} \qquad e_2 \mapsto e'_2}{e_1(e_2) \mapsto e_1(e'_2)} \tag{33c}$$

ITAP
$$\frac{e_2 \; \mathtt{val}}{(\lambda x : \tau.e_1)(e_2) \mapsto [e_2/x]e_1} \tag{33d}$$

ITPairL
$$\frac{e_1 \mapsto e'_1}{(e_1, e_2) \mapsto (e'_1, e_2)} \tag{33e}$$

ITPairR
$$\frac{e_1 \; \mathtt{val} \qquad e_2 \mapsto e'_2}{(e_1, e_2) \mapsto (e_1, e'_2)} \tag{33f}$$

ITPrl
$$\frac{(e_1, e_2) \; \mathtt{final}}{\mathtt{prl}((e_1, e_2)) \mapsto e_1} \tag{33g}$$

ITPrr
$$\frac{(e_1, e_2) \; \mathtt{final}}{\mathtt{prr}((e_1, e_2)) \mapsto e_2} \tag{33h}$$

ITInl
$$\frac{e \mapsto e'}{\mathtt{inl}_\tau(e) \mapsto \mathtt{inl}_\tau(e')} \tag{33i}$$

ITInr
$$\frac{e \mapsto e'}{\mathtt{inr}_\tau(e) \mapsto \mathtt{inr}_\tau(e')} \tag{33j}$$

ITExpMatch
$$\frac{e \mapsto e'}{\mathtt{match}(e)\{\hat{rs}\} \mapsto \mathtt{match}(e')\{\hat{rs}\}} \tag{33k}$$

ITSuccMatch
$$\frac{e \text{ final} \qquad e \rhd p_r \dashv\vdash \theta}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{33l}$$

ITFailMatch
$$\frac{e \text{ final} \qquad e \perp p_r}{\texttt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \tag{33m}$$

Lemma 4.0.2. *If* $\texttt{inl}_{\tau_2}(e_1)$ final *then* e_1 final.

Proof. By rule induction over Rules (25) on $\texttt{inl}_{\tau_2}(e_1)$ final.

Case (25a).

(14)	$\texttt{inl}_{\tau_2}(e_1)$ val	by assumption

By rule induction over Rules (23) on (14), only one case applies.

Case (23d).

(15)	e_1 val	by assumption
(16)	e_1 final	by Rule (25a) on (15)

Case (25b).

(14)	$\texttt{inl}_{\tau_2}(e_1)$ indet	by assumption

By rule induction over Rules (24) on (14), only one case applies.

Case (24i).

(15)	e_1 indet	by assumption
(16)	e_1 final	by Rule (25b) on (15)

$\square$

Lemma 4.0.3. *If* $\texttt{inr}_{\tau_1}(e_2)$ final *then* e_2 final.

Proof. By rule induction over Rules (25) on $\texttt{inr}_{\tau_1}(e_2)$ final.

Case (25a).

(1)	$\texttt{inr}_{\tau_1}(e_2)$ val	by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23d).

(2)	e_2 val	by assumption
(3)	e_2 final	by Rule (25a) on (2)

Case (25b).

(1) $\text{inr}_{\tau_1}(e_2)$ indet   by assumption

By rule induction over Rules (24) on (1), only one case applies.

Case (24i).

(2) e_2 indet   by assumption
(3) e_2 final   by Rule (25b) on (2)

$\square$

Lemma 4.0.4. *If* (e_1, e_2) final *then* e_1 final *and* e_2 final.

Proof. By rule induction over Rules (25) on (e_1, e_2) final.

Case (25a).

(1) (e_1, e_2) val   by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23c).

(2) e_1 val   by assumption
(3) e_2 val   by assumption
(4) e_1 final   by Rule (25a) on (2)
(5) e_2 final   by Rule (25a) on (3)

Case (25b).

(1) (e_1, e_2) indet   by assumption

By rule induction over Rules (24) on (1), only three cases apply.

Case (24d).

(2) e_1 indet   by assumption
(3) e_2 val   by assumption
(4) e_1 final   by Rule (25b) on (2)
(5) e_1 final   by Rule (25a) on (3)

Case (24e).

(2) e_1 val   by assumption
(3) e_2 indet   by assumption
(4) e_1 final   by Rule (25a) on (2)
(5) e_1 final   by Rule (25b) on (3)

Case (24f).

(2) e_1 indet   by assumption

(3)	e_2 `indet`	by assumption
(4)	e_1 `final`	by Rule (25b) on (2)
(5)	e_1 `final`	by Rule (25b) on (3)

$\square$

Lemma 4.0.5. *There doesn't exist $\underline{n}$ such that $\underline{n}$ `notintro`.*

Proof. By rule induction over Rules (26) on $\underline{n}$ `notintro`, no case applies due to syntactic contradiction. $\square$

Lemma 4.0.6. *There doesn't exist $\mathtt{inl}_\tau(e)$ such that $\mathtt{inl}_\tau(e)$ `notintro`.*

Proof. By rule induction over Rules (26) on $\mathtt{inl}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. $\square$

Lemma 4.0.7. *There doesn't exist $\mathtt{inr}_\tau(e)$ such that $\mathtt{inr}_\tau(e)$ `notintro`.*

Proof. By rule induction over Rules (26) on $\mathtt{inr}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. $\square$

Lemma 4.0.8. *There doesn't exist (e_1, e_2) such that (e_1, e_2) `notintro`.*

Proof. By rule induction over Rules (26) on (e_1, e_2) `notintro`, no case applies due to syntactic contradiction. $\square$

Lemma 4.0.9. *If e `final` and e `notintro` then e `indet`.*

Proof Sketch. By rule induction over Rules (26) on e `notintro`, for each case, by rule induction over Rules (23) on e `val` and we notice that e `val` is not derivable. By rule induction over Rules (25) on e `final`, Rule (25a) result in a contradiction with the fact that e `val` is not derivable while Rule (25b) tells us e `indet`. $\square$

Lemma 4.0.10 (Finality). *There doesn't exist such an expression e such that both e `final` and $e \mapsto e'$ for some e'*

Proof. Assume there exists such an e such that both e `final` and $e \mapsto e'$ for some e' then proof by contradiction.

By rule induction over Rules (25) and Rules (33), *i.e.*, over Rules (23) and Rules (33) and over Rules (24) and Rules (33) respectively. The proof can be done by straightforward observation of syntactic contradictions. $\square$

Lemma 4.0.11 (Matching Determinism). *If e `final` and $\cdot \,; \Delta_e \vdash e : \tau$ and $p : \tau[\xi] \dashv\vdash \Gamma \,; \Delta$ then exactly one of the following holds*

1. *$e \triangleright p \dashv\vdash \theta$ for some θ*

2. *$e \,?\, p$*

3. *$e \perp p$*

Proof.

$$(1) \ e \ \texttt{final} \qquad\qquad\qquad\qquad\qquad\qquad \text{by assumption}$$

$$(2) \ \cdot\,;\Delta_e \vdash e : \tau \qquad\qquad\qquad\qquad\qquad \text{by assumption}$$

$$(3) \ p : \tau[\xi] \dashv\mid \Gamma\,;\Delta \qquad\qquad\qquad\qquad \text{by assumption}$$

By rule induction over Rules (20) on (3), we would show one conclusion is derivable while the other two are not.

Case (20a).

$$(4) \ p = x \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{by assumption}$$

$$(5) \ e \rhd x \dashv\mid e/x \qquad\qquad\qquad\qquad\qquad \text{by Rule (30a)}$$

Assume $e\,?\,x$. By rule induction over Rules (31) on it, only one case applies.

Case (31c).

$$(6) \ x \ \texttt{refutable}_? \qquad\qquad\qquad\qquad \text{by assumption}$$

By rule induction over Rules (29) on (6), no case applies due to syntactic contradiction.

$$(7) \ \cancel{e\,?\,x} \qquad\qquad\qquad\qquad\qquad\qquad \text{by contradiction}$$

Assume $e \perp x$. By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

$$(8) \ \cancel{e \perp x} \qquad\qquad\qquad\qquad\qquad\qquad \text{by contradiction}$$

Case (20b).

$$(4) \ p = \_ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{by assumption}$$

$$(5) \ e \rhd \_ \dashv\mid \cdot \qquad\qquad\qquad\qquad\qquad\quad \text{by Rule (30b)}$$

Assume $e\,?\,\_$. By rule induction over Rules (31) on it, only one case applies.

Case (31c).

$$(6) \ \_ \ \texttt{refutable}_? \qquad\qquad\qquad\qquad \text{by assumption}$$

By rule induction over Rules (29) on (6), no case applies due to syntactic contradiction.

$$(7) \ \cancel{e\,?\,\_} \qquad\qquad\qquad\qquad\qquad\qquad \text{by contradiction}$$

Assume $e \perp \_$. By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp \_}$ by contradiction

Case (20c)**.**

    (4) $p = (\!|\!)^w$ by assumption

    (5) $e \,?\, (\!|\!)^w$ by Rule (31a)

Assume $e \rhd (\!|\!)^w \dashv\!\| \theta$ for some θ. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

    (6) $\cancel{e \rhd (\!|\!)^w \dashv\!\| \theta}$ by contradiction

Assume $e \perp (\!|\!)^w$. By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

    (7) $\cancel{e \perp (\!|\!)^w}$ by contradiction

Case (20d)**.**

    (4) $p = (\!|p_0|\!)^w$ by assumption

    (5) $e \,?\, (\!|p_0|\!)^w$ by Rule (31b)

Assume $e \rhd (\!|p_0|\!)^w \dashv\!\| \theta$ for some θ. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

    (6) $\cancel{e \rhd (\!|p_0|\!)^w \dashv\!\| \theta}$ by contradiction

Assume $e \perp (\!|p_0|\!)^w$. By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

    (7) $\cancel{e \perp (\!|p_0|\!)^w}$ by contradiction

Case (20e)**.**

    (4) $p = \underline{n_2}$ by assumption

    (5) $\tau = \texttt{num}$ by assumption

    (6) $\xi = \underline{n_2}$ by assumption

    (7) $\underline{n_2}\ \texttt{refutable}_?$ by Rule (29a)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

    (8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
                                           by assumption

(9) e `notintro` — by Rule $(26a),(26b),(26c),(26d),(26e),(26f)$

(10) $e\,?\,\underline{n_2}$ — by Rule $(16b)$ on (7) and (9)

Assume $e \rhd \underline{n_2} \dashv\!\mid \theta$ for some θ. By rule induction over it, no case applies due to syntactic contradiction.

(11) $\cancel{e \rhd \underline{n_2} \dashv\!\mid \theta}$ — by contradiction

Assume $e \perp \underline{n_2}$. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \perp \underline{n_2}}$ — by contradiction

Case $(19d)$.

(8) $e = \underline{n_1}$

Assume $\underline{n_1}\,?\,\underline{n_2}$. By rule induction over Rules (31) on it, only two cases apply.

Case $(31c)$.

(9) $\underline{n_1}$ `notintro` — by assumption

Contradicts Lemma 4.0.5.

(10) $\cancel{\underline{n_1}\,?\,\underline{n_2}}$ — by contradiction

By case analysis on whether $n_1 = n_2$.

Case $n_1 = n_2$.

(11) $n_1 = n_2$ — by assumption

(12) $\underline{n_1} \rhd \underline{n_2} \dashv\!\mid \cdot$ — by Rule $(30c)$

Assume $\underline{n_1} \perp \underline{n_2}$. By rule induction over Rules (32) on it, only one case applies.

Case $(32a)$.

(13) $n_1 \neq n_2$ — by assumption

Contradicts (11).

(14) $\cancel{\underline{n_1} \perp \underline{n_2}}$ — by contradiction

Case $n_1 \neq n_2$.

(11) $n_1 \neq n_2$ — by assumption

(12) $\underline{n_1} \perp \underline{n_2}$ — by Rule $(32a)$ on (11)

Assume $\underline{n_1} \rhd \underline{n_2} \dashv\!\mid \theta$ for some θ. By rule induction over Rules (30) on it, no case applies due to syntactic contradiction.

(13) $\cancel{\underline{n_1} \rhd \underline{n_2} \dashv\!\mid \theta}$ — by contradiction

Case $(20f)$.

(4) $p = \mathtt{inl}(p_1)$		by assumption
(5) $\tau = (\tau_1 + \tau_2)$		by assumption
(6) $\xi = \mathtt{inl}(\xi_1)$		by assumption
(7) $p_1 : \tau_1[\xi_1] \dashv\mathbin{\|} \Gamma \, ; \Delta$		by assumption
(8) $\mathtt{inl}(p_1) \; \mathtt{refutable}_?$		by Rule (29d)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}s\}$
   by assumption

(10) $e \; \mathtt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(11) $e \, ? \, \mathtt{inl}(p_1)$ by Rule (16b) on (8) and (10)

Assume $e \triangleright \mathtt{inl}(p_1) \dashv\mathbin{\|} \theta_1$ for some θ_1. By rule induction over it, no case applies due to syntactic contradiction.

(12) ~~$e \triangleright \mathtt{inl}(p_1) \dashv\mathbin{\|} \theta_1$~~ by contradiction

Assume $e \perp \mathtt{inl}(p_1)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) ~~$e \perp \mathtt{inl}(p_1)$~~ by contradiction

Case (19j).

(9) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

(10) $\cdot \, ; \Delta_e \vdash e_1 : \tau_1$ by assumption

(11) $e_1 \; \mathtt{final}$ by Lemma 4.0.2 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_1 \triangleright p_1 \dashv\mathbin{\|} \theta_1$ for some θ_1, $e_1 \, ? \, p_1$, and $e_1 \perp p_1$ holds. By case analysis on which one holds.

Case $e_1 \triangleright p_1 \dashv\mathbin{\|} \theta_1$.

(12) $e_1 \triangleright p_1 \dashv\mathbin{\|} \theta_1$ by assumption

(13) ~~$e_1 \, ? \, p_1$~~ by assumption

(14) ~~$e_1 \perp p_1$~~ by assumption

(15) $\mathtt{inl}_{\tau_2}(e_1) \triangleright \mathtt{inl}(p_1) \dashv\mathbin{\|} \theta_1$ by Rule (30e) on (12)

Assume $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31c).

(16) $\mathtt{inl}_{\tau_2}(e_1) \; \mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.6.

Case (31g).

$\quad\quad\quad$ (16) $e_1 \mathbin{?} p_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad$ Contradicts (13).

$\quad\quad\quad$ (17) $\underline{\mathtt{inl}_{\tau_2}(e_1) \mathbin{\cancel{?}} \overline{\mathtt{inl}(p_1)}}$ $\quad\quad\quad\quad$ by contradiction

$\quad$ Assume $\overline{\mathtt{inl}_{\tau_2}(e_1) \perp \mathtt{inl}(p_1)}$. By rule induction over Rules (32)
on it, only one case applies.

$\quad$ **Case** (32f)**.**

$\quad\quad\quad$ (18) $e_1 \perp p_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad$ Contradicts (14).

$\quad\quad\quad$ (19) $\underline{\mathtt{inl}_{\tau_2}(e_1) \cancel{\perp} \overline{\mathtt{inl}(p_1)}}$ $\quad\quad\quad\quad$ by contradiction

Case $e_1 \mathbin{?} p_1$**.**

$\quad\quad\quad$ (12) $\underline{e_1 \mathbin{\triangleright} p_1 \dashv\!\vert \cancel{\theta_1}}$ $\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (13) $e_1 \mathbin{?} p_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (14) $\underline{e_1 \cancel{\perp} \overline{p_1}}$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (15) $\mathtt{inl}_{\tau_2}(e_1) \mathbin{?} \mathtt{inl}(p_1)$ $\quad\quad\quad\quad$ by Rule (31g) on (13)

$\quad$ Assume $\mathtt{inl}_{\tau_2}(e_1) \mathbin{\triangleright} \mathtt{inl}(p_1) \dashv\!\vert \theta$ for some θ. By rule induction
over Rules (30) on it, only one case applies.

$\quad$ **Case** (30e)**.**

$\quad\quad\quad$ (16) $e_1 \mathbin{\triangleright} p_1 \dashv\!\vert \theta$ $\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad$ Contradicts (12).

$\quad\quad\quad$ (17) $\underline{\mathtt{inl}_{\tau_2}(e_1) \mathbin{\cancel{\triangleright}} \overline{\mathtt{inl}(p_1)} \dashv\!\vert \theta}$ $\quad\quad\quad$ by contradiction

$\quad$ Assume $\overline{\mathtt{inl}_{\tau_2}(e_1) \perp \mathtt{inl}(p_1)}$. By rule induction over Rules (32)
on it, only one case applies.

$\quad$ **Case** (32f)**.**

$\quad\quad\quad$ (18) $e_1 \perp p_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad$ Contradicts (14).

$\quad\quad\quad$ (19) $\underline{\mathtt{inl}_{\tau_2}(e_1) \cancel{\perp} \overline{\mathtt{inl}(p_1)}}$ $\quad\quad\quad\quad$ by contradiction

Case $e_1 \perp p_1$**.**

$\quad\quad\quad$ (12) $\underline{e_1 \mathbin{\triangleright} p_1 \dashv\!\vert \cancel{\theta_1}}$ $\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (13) $\underline{e_1 \mathbin{\cancel{?}} \overline{p_1}}$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (14) $e_1 \perp p_1$ $\quad\quad\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad\quad$ (15) $\mathtt{inl}_{\tau_2}(e_1) \perp \mathtt{inl}(p_1)$ $\quad\quad\quad\quad$ by Rule (32f) on (14)

$\quad$ Assume $\mathtt{inl}_{\tau_2}(e_1) \mathbin{\triangleright} \mathtt{inl}(p_1) \dashv\!\vert \theta$ for some θ. By rule induction
over Rules (30) on it, only one case applies.

$\quad$ **Case** (30e)**.**

$\quad\quad\quad$ (16) $e_1 \mathbin{\triangleright} p_1 \dashv\!\vert \theta$ $\quad\quad\quad\quad\quad\quad$ by assumption

$\quad\quad$ Contradicts (12).

(17) $\text{inl}_{\tau_2}(e_1) \not\triangleright \text{inl}(p_1) \dashv\!\mid \theta$ 	by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \mathbin{?} \text{inl}(p_1)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31c).

(18) $\text{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ 	by assumption

Contradicts Lemma 4.0.6.

Case (31g).

(18) $e_1 \mathbin{?} p_1$ 	by assumption

Contradicts (13).

(19) $\text{inl}_{\tau_2}(e_1) \mathbin{\not?} \text{inl}(p_1)$ 	by contradiction

Case (20g).

(4) $p = \text{inr}(p_2)$ 	by assumption

(5) $\tau = (\tau_1 + \tau_2)$ 	by assumption

(6) $\xi = \text{inr}(\xi_2)$ 	by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\mid \Gamma \mathbin{;} \Delta$ 	by assumption

(8) $\text{inr}(p_2)\ \texttt{refutable}_?$ 	by Rule (29e)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = (\!|\!)^u, (\!| e_0 |\!)^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$ 	by assumption

(10) $e\ \texttt{notintro}$ 	by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(11) $e \mathbin{?} \text{inr}(p_2)$ 	by Rule (16b) on (8) and (10)

Assume $e \triangleright \text{inr}(p_2) \dashv\!\mid \theta_2$ for some θ_2. By rule induction over it, no case applies due to syntactic contradiction.

(12) $e \not\triangleright \text{inr}(p_2) \dashv\!\mid \theta_2$ 	by contradiction

Assume $e \perp \text{inr}(p_2)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $e \not\perp \text{inr}(p_2)$ 	by contradiction

Case (19k).

(9) $e = \text{inr}_{\tau_1}(e_2)$ 	by assumption

(10) $\cdot \mathbin{;} \Delta_e \vdash e_2 : \tau_2$ 	by assumption

(11) $e_2\ \texttt{final}$ 	by Lemma 4.0.3 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_2 \triangleright p_2 \dashv\vdash \theta_2$ for some θ_2, $e_2 \,?\, p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which one holds.

Case $e_2 \triangleright p_2 \dashv\vdash \theta_2$**.**

(12)	$e_2 \triangleright p_2 \dashv\vdash \theta_2$	by assumption
(13)	$\cancel{e_2 \,?\, p_2}$	by assumption
(14)	$\cancel{e_2 \perp p_2}$	by assumption
(15)	$\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\vdash \theta_2$	by Rule (30f) on (12)

Assume $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31c)**.**

(16)	$\mathtt{inr}_{\tau_1}(e_2) \ \mathtt{notintro}$	by assumption

Contradicts Lemma 4.0.7.

Case (31h)**.**

(16)	$e_2 \,?\, p_2$	by assumption

Contradicts (13).

(17)	$\cancel{\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)}$	by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)$. By rule induction over Rules (32) on it, only one case applies.

Case (32g)**.**

(18)	$e_2 \perp p_2$	by assumption

Contradicts (14).

(19)	$\cancel{\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)}$	by contradiction

Case $e_2 \,?\, p_2$**.**

(12)	$\cancel{e_2 \triangleright p_2 \dashv\vdash \theta}$	by assumption
(13)	$e_2 \,?\, p_2$	by assumption
(14)	$\cancel{e_2 \perp p_2}$	by assumption
(15)	$\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$	by Rule (31h) on (13)

Assume $\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\vdash \theta$ for some θ. By rule induction over Rules (30) on it, only one case applies.

Case (30f)**.**

(16)	$e_2 \triangleright p_2 \dashv\vdash \theta$	by assumption

Contradicts (12).

(17)	$\cancel{\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\vdash \theta}$	by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)$. By rule induction over Rules (32) on it, only one case applies.

Case (32g)**.**

$\quad$ (18) $e_2 \perp p_2$ $\qquad$ by assumption

$\qquad$ Contradicts (14).

$\quad$ (19) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)}$ $\qquad$ by contradiction

Case $e_2 \perp p_2$**.**

$\quad$ (12) $\cancel{e_2 \triangleright p_2 \dashv\!\vdash \theta}$ $\qquad$ by assumption

$\quad$ (13) $\cancel{e_2 ? p_2}$ $\qquad$ by assumption

$\quad$ (14) $e_2 \perp p_2$ $\qquad$ by assumption

$\quad$ (15) $\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)$ $\qquad$ by Rule (32g) on (14)

$\quad$ Assume $\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\!\vdash \theta$ for some θ. By rule induction over Rules (30) on it, only one case applies.

$\quad$ **Case** (30f)**.**

$\qquad$ (16) $e_2 \triangleright p_2 \dashv\!\vdash \theta$ $\qquad$ by assumption

$\qquad$ Contradicts (12).

$\quad$ (17) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\!\vdash \theta}$ $\qquad$ by contradiction

$\quad$ Assume $\overline{\mathtt{inr}_{\tau_1}(e_2) ? \mathtt{inr}(p_2)}$. By rule induction over Rules (31) on it, only two cases apply.

$\quad$ **Case** (31c)**.**

$\qquad$ (18) $\mathtt{inr}_{\tau_1}(e_2) \mathtt{\ notintro}$ $\qquad$ by assumption

$\qquad$ Contradicts Lemma 4.0.7.

$\quad$ **Case** (31h)**.**

$\qquad$ (18) $e_2 ? p_2$ $\qquad$ by assumption

$\qquad$ Contradicts (13).

$\quad$ (19) $\cancel{\mathtt{inr}_{\tau_1}(e_2) ? \mathtt{inr}(p_2)}$ $\qquad$ by contradiction

Case (20h)**.**

$\quad$ (4) $p = (p_1, p_2)$ $\qquad$ by assumption

$\quad$ (5) $\tau = (\tau_1 \times \tau_2)$ $\qquad$ by assumption

$\quad$ (6) $\xi = (\xi_1, \xi_2)$ $\qquad$ by assumption

$\quad$ (7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ $\qquad$ by assumption

$\quad$ (8) $\Delta = \Delta_1 \uplus \Delta_2$ $\qquad$ by assumption

$\quad$ (9) $p_1 : \tau_1[\xi_1] \dashv\!\vdash \Gamma_1 ; \Delta_1$ $\qquad$ by assumption

$\quad$ (10) $p_2 : \tau_2[\xi_2] \dashv\!\vdash \Gamma_2 ; \Delta_2$ $\qquad$ by assumption

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

(11) $e = (\!|\!|)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$
                 by assumption

(12) e $\mathtt{notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(13) e $\mathtt{indet}$ by Lemma 4.0.9 on (1) and (12)

(14) $\mathtt{prl}(e)$ $\mathtt{indet}$ by Rule (24g) on (13)

(15) $\mathtt{prl}(e)$ $\mathtt{final}$ by Rule (25b) on (14)

(16) $\mathtt{prr}(e)$ $\mathtt{indet}$ by Rule (24h) on (13)

(17) $\mathtt{prr}(e)$ $\mathtt{final}$ by Rule (25b) on (16)

(18) $\cdot\,; \Delta \vdash \mathtt{prl}(e) : \tau_1$ by Rule (19h) on (2)

(19) $\cdot\,; \Delta \vdash \mathtt{prr}(e) : \tau_2$ by Rule (19i) on (2)

Assume $e \perp (p_1, p_2)$. By rule induction on it, no case applies due to syntactic contradiction.

(20) $\cancel{e \perp (p_1, p_2)}$ by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of $\mathtt{prl}(e) \triangleright p_1 \dashv\!\vdash \theta_1$, $\mathtt{prl}(e) \,?\, p_1$, and $\mathtt{prl}(e) \perp p_1$ holds.
By inductive hypothesis on (17) and (19) and (10), exactly one of $\mathtt{prr}(e) \triangleright p_2 \dashv\!\vdash \theta_2$, $\mathtt{prr}(e) \,?\, p_2$, and $\mathtt{prr}(e) \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2. Note that we have already shown $\cancel{e \perp (p_1, p_2)}$.

Case $\mathtt{prl}(e) \triangleright p_1 \dashv\!\vdash \theta_1, \mathtt{prr}(e) \triangleright p_2 \dashv\!\vdash \theta_2$**.**

    (21) $\mathtt{prl}(e) \triangleright p_1 \dashv\!\vdash \theta_1$ by assumption

    (22) $\cancel{\mathtt{prl}(e) \,?\, p_1}$ by assumption

    (23) $\cancel{\mathtt{prl}(e) \perp p_1}$ by assumption

    (24) $\mathtt{prr}(e) \triangleright p_2 \dashv\!\vdash \theta_2$ by assumption

    (25) $\cancel{\mathtt{prr}(e) \,?\, p_2}$ by assumption

    (26) $\cancel{\mathtt{prr}(e) \perp p_2}$ by assumption

    (27) $e \triangleright (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2$ by Rule (30g) on (12) and (21) and (24)

Assume $e \,?\, (p_1, p_2)$. By rule induction over Rules (31) on it, only one case applies.

Case (31c)**.**

    (28) (p_1, p_2) $\mathtt{refutable}_?$ by assumption

By rule induction over Rules (29), only two cases apply.

Case (29f)**.**

    (29) p_1 $\mathtt{refutable}_?$ by assumption

    (30) $\mathtt{prl}(e)$ $\mathtt{notintro}$ by Rule (26e)

    (31) $\mathtt{prl}(e) \,?\, p_1$ by Rule (31c) on (29) and (30)

Contradicts (22).

Case (29g).

(29) p_2 `refutable`$_?$	by assumption
(30) $\text{prr}(e)$ `notintro`	by Rule (26f)
(31) $\text{prl}(e) \mathrel{?} p_1$	by Rule (31c) on (29) and (30)

Contradicts (22).

(32) $e \mathrel{?} \cancel{(p_1, p_2)}$	by contradiction

Case $\text{prl}(e) \rhd p_1 \dashv\mid \theta_1, \text{prr}(e) \mathrel{?} p_2$.

(21) $\text{prl}(e) \rhd p_1 \dashv\mid \theta_1$	by assumption
(22) $\cancel{\text{prl}(e) \mathrel{?} p_1}$	by assumption
(23) $\cancel{\text{prl}(e) \perp p_1}$	by assumption
(24) $\cancel{\text{prr}(e) \rhd p_2 \dashv\mid \theta_2}$	by assumption
(25) $\text{prr}(e) \mathrel{?} p_2$	by assumption
(26) $\cancel{\text{prr}(e) \perp p_2}$	by assumption

Assume $e \rhd (p_1, p_2) \dashv\mid \theta$. By rule induction over Rules (30), only one case applies.

Case (30g).

(27) $\theta = \theta_1 \uplus \theta_2$	by assumption
(28) $\text{prr}(e) \rhd p_2 \dashv\mid \theta_2$	by assumption

Contradicts (24).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\mid \theta}$	by contradiction

By rule induction over Rules (31) on (25), the following cases apply.

Case (31a),(31b).

(30) $p_2 = (\!\mid\!)^w, (\!\mid\! p \!\mid\!)^w$	by assumption
(31) p_2 `refutable`$_?$	by Rule (29b) and Rule (29c)
(32) (p_1, p_2) `refutable`$_?$	by Rule (29g) on (31)
(33) $e \mathrel{?} (p_1, p_2)$	by Rule (31c) on (12) and (32)

Case (31c).

(30) p_2 `refutable`$_?$	by assumption
(31) (p_1, p_2) `refutable`$_?$	by Rule (29g) on (30)
(32) $e \mathrel{?} (p_1, p_2)$	by Rule (31c) on (12) and (31)

Case $\text{prl}(e) \rhd p_1 \dashv\mid \theta_1, \text{prr}(e) \perp p_2$.

(21) $\text{prl}(e) \rhd p_1 \dashv\mid \theta_1$	by assumption
(22) $\cancel{\text{prl}(e) \mathrel{?} p_1}$	by assumption

(23) ~~prl(e) ⊥ p₁~~   by assumption

(24) ~~prr(e) ▷ p₂ ⊣∥θ₂~~   by assumption

(25) ~~prr(e) ? p₂~~   by assumption

(26) prr(e) ⊥ p₂   by assumption

By rule induction over Rules (32) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case prl(e) ? p₁, prr(e) ▷ p₂ ⊣∥θ₂**.**

(21) ~~prl(e) ▷ p₁ ⊣∥θ₁~~   by assumption

(22) prl(e) ? p₁   by assumption

(23) ~~prl(e) ⊥ p₁~~   by assumption

(24) prr(e) ▷ p₂ ⊣∥θ₂   by assumption

(25) ~~prr(e) ? p₂~~   by assumption

(26) ~~prr(e) ⊥ p₂~~   by assumption

Assume $e ▷ (p_1, p_2) ⊣∥θ$. By rule induction over Rules (30), only one case applies.

Case (30g)**.**

(27) $θ = θ_1 ⊎ θ_2$ by assumption

(28) prl(e) ▷ p₁ ⊣∥θ₁   by assumption

Contradicts (21).

(29) ~~$e ▷ (p_1, p_2) ⊣∥θ$~~   by contradiction

By rule induction over Rules (31) on (22), the following cases apply.

Case (31a),(31b)**.**

(30) $p_1 = (\!|\!|)^w, (\!|p|\!)^w$ by assumption

(31) p_1 refutable?   by Rule (29b) and Rule (29c)

(32) (p_1, p_2) refutable?   by Rule (29g) on (31)

(33) $e ? (p_1, p_2)$ by Rule (31c) on (12) and (32)

Case (31c)**.**

(30) p_1 refutable?   by assumption

(31) (p_1, p_2) refutable?   by Rule (29g) on (30)

(32) $e ? (p_1, p_2)$ by Rule (31c) on (12) and (31)

Case prl(e) ? p₁, prr(e) ? p₂**.**

(21) ~~prl(e) ▷ p₁ ⊣∥θ₁~~   by assumption

(22) prl(e) ? p₁   by assumption

(23) ~~prl(e) ⊥ p₁~~   by assumption

(24) $\mathrm{prr}(e) \rhd p_2 \dashv\mid \theta_2$		by assumption
(25) $\mathrm{prr}(e) \mathbin{?} p_2$		by assumption
(26) $\mathrm{prr}(e) \perp p_2$		by assumption

Assume $e \rhd (p_1, p_2) \dashv\mid \theta$. By rule induction over Rules (30), only one case applies.

Case (30g).

(27) $\theta = \theta_1 \uplus \theta_2$		by assumption
(28) $\mathrm{prl}(e) \rhd p_1 \dashv\mid \theta_1$		by assumption

Contradicts (21).

(29) $e \rhd (p_1, p_2) \dashv\mid \theta$		by contradiction

By rule induction over Rules (31) on (22), the following cases apply.

Case (31a),(31b).

(30) $p_1 = (\!\|\!)^w, (\!\|p\!\|)^w$		by assumption
(31) $p_1 \texttt{ refutable}_?$		by Rule (29b) and Rule (29c)
(32) $(p_1, p_2) \texttt{ refutable}_?$		by Rule (29g) on (31)
(33) $e \mathbin{?} (p_1, p_2)$		by Rule (31c) on (12) and (32)

Case (31c).

(30) $p_1 \texttt{ refutable}_?$		by assumption
(31) $(p_1, p_2) \texttt{ refutable}_?$		by Rule (29g) on (30)
(32) $e \mathbin{?} (p_1, p_2)$		by Rule (31c) on (12) and (31)

Case $\mathrm{prl}(e) \mathbin{?} p_1, \mathrm{prr}(e) \perp p_2$.

(21) $\mathrm{prl}(e) \rhd p_1 \dashv\mid \theta_1$		by assumption
(22) $\mathrm{prl}(e) \mathbin{?} p_1$		by assumption
(23) $\mathrm{prl}(e) \perp p_1$		by assumption
(24) $\mathrm{prr}(e) \rhd p_2 \dashv\mid \theta_2$		by assumption
(25) $\mathrm{prr}(e) \mathbin{?} p_2$		by assumption
(26) $\mathrm{prr}(e) \perp p_2$		by assumption

By rule induction over Rules (32) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\mathrm{prl}(e) \perp p_1, \mathrm{prr}(e) \rhd p_2 \dashv\mid \theta_2$.

(21) $\mathrm{prl}(e) \rhd p_1 \dashv\mid \theta_1$		by assumption
(22) $\mathrm{prl}(e) \mathbin{?} p_1$		by assumption
(23) $\mathrm{prl}(e) \perp p_1$		by assumption
(24) $\mathrm{prr}(e) \rhd p_2 \dashv\mid \theta_2$		by assumption

(25) $\cancel{\text{prr}(e)~?~p_2}$		by assumption
(26) $\cancel{\text{prr}(e)~\bot~p_2}$		by assumption

By rule induction over Rules (32) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\text{prl}(e) \bot p_1, \text{prr}(e)~?~p_2$**.**

(21) $\cancel{\text{prl}(e) \rhd p_1 \dashv\!\!\vdash \theta_1}$		by assumption
(22) $\cancel{\text{prl}(e)~?~p_1}$		by assumption
(23) $\text{prl}(e) \bot p_1$		by assumption
(24) $\cancel{\text{prr}(e) \rhd p_2 \dashv\!\!\vdash \theta_2}$		by assumption
(25) $\text{prr}(e)~?~p_2$		by assumption
(26) $\cancel{\text{prr}(e) \bot p_2}$		by assumption

By rule induction over Rules (32) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\text{prl}(e) \bot p_1, \text{prr}(e) \bot p_2$**.**

(21) $\cancel{\text{prl}(e) \rhd p_1 \dashv\!\!\vdash \theta_1}$		by assumption
(22) $\cancel{\text{prl}(e)~?~p_1}$		by assumption
(23) $\text{prl}(e) \bot p_1$		by assumption
(24) $\cancel{\text{prr}(e) \rhd p_2 \dashv\!\!\vdash \theta_2}$		by assumption
(25) $\text{prr}(e)~?~p_2$		by assumption
(26) $\cancel{\text{prr}(e) \bot p_2}$		by assumption

By rule induction over Rules (32) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case (19g)**.**

(11) $e = (e_1, e_2)$		by assumption
(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$		by assumption
(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$		by assumption
(14) e_1 `final`		by Lemma 4.0.4 on (1)
(15) e_2 `final`		by Lemma 4.0.4 on (1)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \rhd p_1 \dashv\!\!\vdash \theta_1$, $e_1~?~p_1$, and $e_1 \bot p_1$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \rhd p_2 \dashv\!\!\vdash \theta_2$, $e_2~?~p_2$, and $e_2 \bot p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2.

Case $e_1 \rhd p_1 \dashv\!\!\vdash \theta_1, e_2 \rhd p_2 \dashv\!\!\vdash \theta_2$**.**

(16) $e_1 \rhd p_1 \dashv\!\!\vdash \theta_1$		by assumption
(17) $\cancel{e_1~?~p_1}$		by assumption
(18) $\cancel{e_1 \bot p_1}$		by assumption

(19)	$e_2 \rhd p_2 \dashv\!\vdash \theta_2$	by assumption
(20)	~~$e_2 \mathbin{?} p_2$~~	by assumption
(21)	~~$e_2 \perp p_2$~~	by assumption
(22)	$(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2$	by Rule (30d) on (16) and (19)

Assume $(e_1, e_2) \mathbin{?} (p_1, p_2)$. By rule induction over Rules (31) on it, only four cases apply.

Case (31c).

(23)	$(e_1, e_2) \; \mathtt{notintro}$	by assumption

Contradicts Lemma 4.0.8.

Case (31d).

(23)	$e_1 \mathbin{?} p_1$	by assumption

Contradicts (17).

Case (31e).

(23)	$e_2 \mathbin{?} p_2$	by assumption

Contradicts (20).

Case (31f).

(23)	$e_1 \mathbin{?} p_1$	by assumption

Contradicts (17).

(24)	~~$(e_1, e_2) \mathbin{?} (p_1, p_2)$~~	by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (32) on it, only two cases apply.

Case (32b).

(25)	$e_1 \perp p_1$	by assumption

Contradicts (18).

Case (32c).

(25)	$e_2 \perp p_2$	by assumption

Contradicts (21).

(26)	~~$(e_1, e_2) \perp (p_1, p_2)$~~	by contradiction

Case $e_1 \rhd p_1 \dashv\!\vdash \theta_1, e_2 \mathbin{?} p_2$**.**

(16)	$e_1 \rhd p_1 \dashv\!\vdash \theta_1$	by assumption
(17)	~~$e_1 \mathbin{?} p_1$~~	by assumption
(18)	~~$e_1 \perp p_1$~~	by assumption
(19)	~~$e_2 \rhd p_2 \dashv\!\vdash \theta_2$~~	by assumption
(20)	$e_2 \mathbin{?} p_2$	by assumption
(21)	~~$e_2 \perp p_2$~~	by assumption
(22)	$(e_1, e_2) \mathbin{?} (p_1, p_2)$	by Rule (31e) on (16) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\! \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d)**.**

    (23) $\theta = \theta_1 \uplus \theta_2$

    (24) $e_2 \rhd p_2 \dashv\! \theta_2$ by assumption

   Contradicts (19).

  (25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\! \theta}$ by contradiction

Assume $\cancel{(e_1, e_2) \perp (p_1, p_2)}$. By rule induction over Rules (32) on it, only two cases apply.

Case (32b)**.**

    (26) $e_1 \perp p_1$ by assumption

   Contradicts (18).

Case (32c)**.**

    (26) $e_2 \perp p_2$ by assumption

   Contradicts (21).

  (27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\! \theta_1, e_2 \perp p_2$**.**

    (16) $e_1 \rhd p_1 \dashv\! \theta_1$ by assumption

    (17) $\cancel{e_1 \mathbin{?} p_1}$ by assumption

    (18) $\cancel{e_1 \perp p_1}$ by assumption

    (19) $\cancel{e_2 \rhd p_2 \dashv\! \theta_2}$ by assumption

    (20) $\cancel{e_2 \mathbin{?} p_2}$ by assumption

    (21) $e_2 \perp p_2$ by assumption

    (22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (32c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\! \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d)**.**

    (23) $\theta = \theta_1 \uplus \theta_2$

    (24) $e_2 \rhd p_2 \dashv\! \theta_2$ by assumption

   Contradicts (19).

  (25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\! \theta}$ by contradiction

Assume $\cancel{(e_1, e_2) \mathbin{?} (p_1, p_2)}$. By rule induction over Rules (31) on it, only four cases apply.

Case (31c)**.**

    (26) $(e_1, e_2) \; \texttt{notintro}$ by assumption

   Contradicts Lemma 4.0.8.

Case (31d)**.**

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

Case (31e)**.**

(26) $e_2 ? p_2$ by assumption

Contradicts (20).

Case (31f)**.**

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) ? (p_1, p_2)}$ by contradiction

Case $e_1 ? p_1, e_2 \rhd p_2 \dashv\! \theta_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\! \theta_1}$ by assumption

(17) $e_1 ? p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\! \theta_2$ by assumption

(20) $\cancel{e_2 ? p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) ? (p_1, p_2)$ by Rule (31d) on (17) and (19)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\! \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\! \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\! \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (32) on it, only two cases apply.

Case (32b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (32c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 ? p_1, e_2 ? p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\! \theta_1}$ by assumption

(17) $e_1 ? p_1$ by assumption

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(18) $\cancel{e_1 \perp p_1}$	by assumption
(19) $\cancel{e_2 \rhd p_2 \dashv\! \vdash \theta_2}$	by assumption
(20) $e_2 \mathbin{?} p_2$	by assumption
(21) $\cancel{e_2 \perp p_2}$	by assumption
(22) $(e_1, e_2) \mathbin{?} (p_1, p_2)$	by Rule (31f) on (17) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\! \vdash \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d).

(23) $\theta = \theta_1 \uplus \theta_2$	
(24) $e_2 \rhd p_2 \dashv\! \vdash \theta_2$	by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\! \vdash \theta}$	by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (32) on it, only two cases apply.

Case (32b).

(26) $e_1 \perp p_1$	by assumption

Contradicts (18).

Case (32c).

(26) $e_2 \perp p_2$	by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$	by contradiction

Case $e_1 \mathbin{?} p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\! \vdash \theta_1}$	by assumption
(17) $e_1 \mathbin{?} p_1$	by assumption
(18) $\cancel{e_1 \perp p_1}$	by assumption
(19) $\cancel{e_2 \rhd p_2 \dashv\! \vdash \theta_2}$	by assumption
(20) $\cancel{e_2 \mathbin{?} p_2}$	by assumption
(21) $e_2 \perp p_2$	by assumption
(22) $(e_1, e_2) \perp (p_1, p_2)$	by Rule (32c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\! \vdash \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d).

(23) $\theta = \theta_1 \uplus \theta_2$	
(24) $e_2 \rhd p_2 \dashv\! \vdash \theta_2$	by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\! \vdash \theta}$	by contradiction

Assume $(e_1, e_2) \mathbin{?} (p_1, p_2)$. By rule induction over Rules (31) on it, only four cases apply.

Case (31c).

    (26) $(e_1, e_2)\ \texttt{notintro}$ by assumption

  Contradicts Lemma 4.0.8.

Case (31d).

    (26) $e_2 \rhd p_2 \dashv\! \parallel \theta_2$ by assumption

  Contradicts (19).

Case (31e).

    (26) $e_2 \mathbin{?} p_2$ by assumption

  Contradicts (20).

Case (31f).

    (26) $e_2 \mathbin{?} p_2$ by assumption

  Contradicts (20).

    (27) $\cancel{(e_1, e_2) \mathbin{?} (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \rhd p_2 \dashv\! \parallel \theta_2$.

    (16) $\cancel{e_1 \rhd p_1 \dashv\! \parallel \theta_1}$ by assumption

    (17) $\cancel{e_1 \mathbin{?} p_1}$ by assumption

    (18) $e_1 \perp p_1$ by assumption

    (19) $e_2 \rhd p_2 \dashv\! \parallel \theta_2$ by assumption

    (20) $\cancel{e_2 \mathbin{?} p_2}$ by assumption

    (21) $\cancel{e_2 \perp p_2}$ by assumption

    (22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (32b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\! \parallel \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d).

    (23) $\theta = \theta_1 \uplus \theta_2$

    (24) $e_1 \rhd p_1 \dashv\! \parallel \theta_1$ by assumption

  Contradicts (16).

    (25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\! \parallel \theta}$ by contradiction

Assume $(e_1, e_2) \mathbin{?} (p_1, p_2)$. By rule induction over Rules (31) on it, only four cases apply.

Case (31c).

    (26) $(e_1, e_2)\ \texttt{notintro}$ by assumption

  Contradicts Lemma 4.0.8.

Case (31d).

    (26) $e_1 \mathbin{?} p_1$ by assumption

  Contradicts (17).

Case (31e)**.**

    (26) $e_2 ? p_2$ by assumption

    Contradicts (20).

Case (31f)**.**

    (26) $e_1 ? p_1$ by assumption

    Contradicts (17).

  (27) $\;\cancel{(e_1, e_2) ? (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 ? p_2$**.**

    (16) $\cancel{e_1 \rhd p_1 \dashv\!\vdash \theta_1}$ by assumption

    (17) $\cancel{e_1 ? p_1}$ by assumption

    (18) $e_1 \perp p_1$ by assumption

    (19) $\cancel{e_2 \rhd p_2 \dashv\!\vdash \theta_2}$ by assumption

    (20) $e_2 ? p_2$ by assumption

    (21) $\cancel{e_2 \perp p_2}$ by assumption

    (22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (32b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d)**.**

    (23) $\theta = \theta_1 \uplus \theta_2$

    (24) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

    Contradicts (19).

  (25) $\;\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta}$ by contradiction

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (31) on it, only four cases apply.

Case (31c)**.**

    (26) (e_1, e_2) `notintro`   by assumption

    Contradicts Lemma 4.0.8.

Case (31d)**.**

    (26) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

    Contradicts (19).

Case (31e)**.**

    (26) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ by assumption

    Contradicts (16).

Case (31f)**.**

    (26) $e_1 ? p_1$ by assumption

    Contradicts (17).

  (27) $\;\cancel{(e_1, e_2) ? (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \perp p_2$**.**

(16)	$\cancel{e_1 \rhd p_1 \dashv\!\vdash \theta_1}$	by assumption
(17)	$\cancel{e_1 ? p_1}$	by assumption
(18)	$e_1 \perp p_1$	by assumption
(19)	$\cancel{e_2 \rhd p_2 \dashv\!\vdash \theta_2}$	by assumption
(20)	$e_2 ? p_2$	by assumption
(21)	$\cancel{e_2 \perp p_2}$	by assumption
(22)	$(e_1, e_2) \perp (p_1, p_2)$	by Rule (32b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta$. By rule induction over Rules (30) on it, only one case applies.

Case (30d)**.**

(23)	$\theta = \theta_1 \uplus \theta_2$	
(24)	$e_2 \rhd p_2 \dashv\!\vdash \theta_2$	by assumption

Contradicts (19).

(25)	$\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta}$	by contradiction

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (31) on it, only four cases apply.

Case (31c)**.**

(26)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

Case (31d)**.**

(26)	$e_2 \rhd p_2 \dashv\!\vdash \theta_2$	by assumption

Contradicts (19).

Case (31e)**.**

(26)	$e_1 \rhd p_1 \dashv\!\vdash \theta_1$	by assumption

Contradicts (16).

Case (31f)**.**

(26)	$e_1 ? p_1$	by assumption

Contradicts (17).

(27)	$\cancel{(e_1, e_2) ? (p_1, p_2)}$	by contradiction

$\square$

Lemma 4.0.12 (Matching Coherence of Constraint)**.** *Suppose that* $\cdot\,; \Delta_e \vdash e : \tau$ *and* e `final` *and* $p : \tau[\xi] \dashv\!\vdash \Gamma\,; \Delta$*. Then we have*

1. $e \models \xi$ *iff* $e \rhd p \dashv\!\vdash \theta$

2. $e \models_? \xi$ *iff* $e ? p$

3. $e \not\models_?^{\dagger} \xi$ *iff* $e \perp p$

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Proof.

(1) $\cdot\,; \Delta_e \vdash e : \tau$ by assumption

(2) e `final`   by assumption

(3) $p : \tau[\xi] \dashv\vdash \Gamma\,; \Delta$ by assumption

Given Lemma 3.0.1, Theorem 2.1, and Lemma 4.0.11, it is sufficient to prove

1. $e \models \xi$ iff $e \triangleright p \dashv\vdash \theta$

2. $e \models_? \xi$ iff $e\ ?\ p$

By rule induction over Rules (20) on (3).

Case (20a).

(4) $p = x$ by assumption

(5) $\xi = \top$ by assumption

1. Prove $e \models \top$ implies $e \triangleright x \dashv\vdash \theta$ for some θ.

    (6) $e \triangleright x \dashv\vdash e/x$ by Rule (30a)

2. Prove $e \triangleright x \dashv\vdash \theta$ implies $e \models \top$.

    (6) $e \models \top$ by Rule (14a)

3. Prove $e \models_? \top$ implies $e\ ?\ x$.

    (6) $e \not\models_? \top$ by Lemma 2.0.3

   Vacuously true.

4. Prove $e\ ?\ x$ implies $e \models_? \top$.

By rule induction over Rules (31), we notice that either, $e?x$ is in syntactic contradiction with all the cases, or the premise x `refutable`$_?$ is not derivable. Hence, $e\ ?\ x$ are not derivable. And thus vacuously true.

Case (20b).

(4) $p = \_$ by assumption

(5) $\xi = \top$ by assumption

1. Prove $e \models \top$ implies $e \triangleright \_ \dashv\vdash \theta$ for some θ.

    (6) $e \triangleright \_ \dashv\vdash \cdot$ by Rule (30a)

2. Prove $e \triangleright \_ \dashv\vdash \theta$ implies $e \models \top$.

    (6) $e \models \top$ by Rule (14a)

3. Prove $e \models_? \top$ implies $e\ ?\ \_$.

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(6) $e \not\models_? \top$ by Lemma 2.0.3

Vacuously true.

4. Prove $e \,?\, \_$ implies $e \models_? \xi$.

By rule induction over Rules (31), we notice that either, $e\,?\,\_$ is in syntactic contradiction with all the cases, or the premise $\_$ `refutable`$_?$ is not derivable. Hence, $e\,?\,\_$ are not derivable. And thus vacuously true.

Case (20c).

(4) $p = (\!|\!)^w$ by assumption

(5) $\xi = \,?$ by assumption

(6) $\overline{\xi} = \,?$ by Definition 9

1. Prove $e \models \,?$ implies $e \rhd (\!|\!)^w \dashv\vdash \theta$ for some θ.

(7) $e \not\models \,?$ by Rule (30a)

Vacuously true.

2. Prove $e \rhd (\!|\!)^w \dashv\vdash \theta$ implies $e \models \,?$.

By rule induction over Rules (30), we notice that $e \rhd (\!|\!)^w \dashv\vdash \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? \,?$ implies $e \,?\, (\!|\!)^w$.

(7) $e \,?\, (\!|\!)^w$ by Rule (31a)

4. Prove $e \,?\, (\!|\!)^w$ implies $e \models_? \,?$.

(7) $e \models_? \,?$ by Rule (16a)

Case (20d).

(4) $p = (\!|p_0|\!)^w$ by assumption

(5) $\xi = \,?$ by assumption

1. Prove $e \models \,?$ implies $e \rhd (\!|p_0|\!)^w \dashv\vdash \theta$ for some θ.

(6) $e \not\models \,?$ by Rule (30a)

Vacuously true.

2. Prove $e \rhd (\!|p_0|\!)^w \dashv\vdash \theta$ implies $e \models \,?$.

By rule induction over Rules (30), we notice that $e \rhd (\!|p_0|\!)^w \dashv\vdash \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? \,?$ implies $e \,?\, (\!|p_0|\!)^w$.

(6) $e \,?\, (\!|p_0|\!)^w$ by Rule (31b)

4. Prove e ? $(\!|p_0|\!)^w$ implies $e \models_? ?$.

    (6) $e \models_? ?$ by Rule (16a)

Case (20e).

    (4) $p = \underline{n}$ by assumption

    (5) $\xi = \underline{n}$ by assumption

1. Prove $e \models \underline{n}$ implies $e \triangleright \underline{n} \dashv\!\mid \theta$ for some θ.

    (6) $e \models \underline{n}$ by assumption

    By rule induction over Rules (14) on (6), only one case applies.

    Case (14b).

        (7) $e = \underline{n}$ by assumption

        (8) $\underline{n} \triangleright \underline{n} \dashv\!\mid \cdot$ by Rule (30c)

2. Prove $e \triangleright \underline{n} \dashv\!\mid \theta$ implies $e \models \underline{n}$.

    (6) $e \triangleright \underline{n} \dashv\!\mid \theta$ by assumption

    By rule induction over Rules (30) on (6), only one case applies.

    Case (30c).

        (7) $e = \underline{n}$ by assumption

        (8) $\theta = \cdot$ by assumption

        (9) $\underline{n} \models \underline{n}$ by Rule (14b)

3. Prove $e \models_? \underline{n}$ implies e ? $\underline{n}$.

    (6) $e \models_? \underline{n}$ by assumption

    By rule induction over Rules (16) on (6), only one case applies.

    Case (16b).

        (7) e `notintro`   by assumption

        (8) $\underline{n}$ `refutable`$_?$ by Rule (29a)

        (9) e ? $\underline{n}$ by Rule (31c) on (7)
                                                      and (8)

4. Prove e ? $\underline{n}$ implies $e \models_? \underline{n}$.

    (6) e ? $\underline{n}$ by assumption

    By rule induction over Rules (31) on (6), only one case applies.

    Case (31c).

        (7) e `notintro`   by assumption

        (8) $\underline{n}$ `refutable`$_?$ by Rule (10a)

        (9) $e \models_? \underline{n}$ by Rule (16) on (7)
                                                      and (8)

Case (20f)**.**

$\quad$ (4) $p = \mathtt{inl}(p_1)$ $\hfill$ by assumption

$\quad$ (5) $\xi = \mathtt{inl}(\xi_1)$ $\hfill$ by assumption

$\quad$ (6) $\tau = (\tau_1 + \tau_2)$ $\hfill$ by assumption

$\quad$ (7) $p_1 : \tau_1[\xi_1] \dashv\mid \Gamma \,;\, \Delta$ $\hfill$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m)**.**

$\quad$ (8) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r s}\}$
$\hfill$ by assumption

$\quad$ (9) e $\mathtt{notintro}$ $\hfill$ by Rule

$\hfill$ (26a),(26b),(26c),(26d),(26e),(26f)

1. Prove $e \models \mathtt{inl}(\xi_1)$ implies $e \vartriangleright \mathtt{inl}(p_1) \dashv\mid \theta$ for some θ. By rule induction over Rules (14) on $e \models \mathtt{inl}(\xi_1)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.

2. Prove $e \vartriangleright \mathtt{inl}(p_1) \dashv\mid \theta$ implies $e \models \mathtt{inl}(\xi_1)$. By rule induction over Rules (30) on $e \vartriangleright \mathtt{inl}(p_1) \dashv\mid \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.

3. Prove $e \models_? \mathtt{inl}(\xi_1)$ implies $e \mathbin{?} \mathtt{inl}(p_1)$.

   $\quad$ (10) $\mathtt{inl}(p_1)$ $\mathtt{refutable}_?$ $\hfill$ by Rule (29d)

   $\quad$ (11) $e \mathbin{?} \mathtt{inl}(p_1)$ $\hfill$ by Rule (31c) on (9)
   $\hfill$ and (10)

4. Prove $e \mathbin{?} \mathtt{inl}(p_1)$ implies $e \models_? \mathtt{inl}(\xi_1)$.

   $\quad$ (10) $\mathtt{inl}(\xi_1)$ $\mathtt{refutable}_?$ $\hfill$ by Rule (10b)

   $\quad$ (11) $e \models_? \mathtt{inl}(\xi_1)$ $\hfill$ by Rule (16b) on (9)
   $\hfill$ and (10)

Case (19j)**.**

$\quad$ (8) $e = \mathtt{inl}_{\tau_2}(e_1)$ $\hfill$ by assumption

$\quad$ (9) $\cdot\,;\, \Delta_e \vdash e_1 : \tau_1$ $\hfill$ by assumption

$\quad$ (10) e_1 $\mathtt{final}$ $\hfill$ by Lemma 4.0.2 on (2)

By inductive hypothesis on (10) and (9) and (7).

$\quad$ (11) $e_1 \models \xi_1$ iff $e_1 \vartriangleright p_1 \dashv\mid \theta$ for some θ

$\quad$ (12) $e_1 \models_? \xi_1$ iff $e_1 \mathbin{?} p_1$

1. Prove $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \vartriangleright \mathtt{inl}(p_1) \dashv\mid \theta$ for some θ.

   $\quad$ (13) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ $\hfill$ by assumption

By rule induction over Rules (14) on (13), only one case applies.

Case (14g).

$\qquad$ (14) $e_1 \models \xi_1$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (15) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ for some θ_1 $\qquad$ by (11) on (14)

$\qquad$ (16) $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta_1$ $\qquad$ by Rule (30e) on (15)

2. Prove $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta$ implies $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$.

$\qquad$ (13) $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\!\mid \theta$ $\qquad$ by assumption

By rule induction over Rules (30) on (13), only one case applies.

Case (30e).

$\qquad$ (14) $e_1 \rhd p_1 \dashv\!\!\mid \theta$ $\qquad\qquad\qquad$ by assumption

$\qquad$ (15) $e_1 \models \xi_1$ $\qquad\qquad\qquad\qquad$ by (11) on (14)

$\qquad$ (16) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ $\qquad\qquad$ by Rule (14g) on (15)

3. Prove $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$.

$\qquad$ (13) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ $\qquad\qquad$ by assumption

By rule induction over Rules (16) on (13), only two cases apply.

Case (16b).

$\qquad$ (14) $\mathtt{inl}_{\tau_2}(e_1) \, \mathtt{notintro}$ $\qquad$ by assumption

$\qquad$ Contradicts Lemma 4.0.6.

Case (16e).

$\qquad$ (14) $e_1 \models_? \xi_1$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (15) $e_1 \, ? \, p_1$ $\qquad\qquad\qquad\qquad$ by (12) on (14)

$\qquad$ (16) $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$ $\qquad\qquad$ by Rule (31g) on (15)

4. Prove $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$.

$\qquad$ (13) $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$ $\qquad\qquad$ by assumption

By rule induction over Rules (31) on (13), only two cases apply.

Case (31c).

$\qquad$ (14) $\mathtt{inl}_{\tau_2}(e_1) \, \mathtt{notintro}$ $\qquad$ by assumption

$\qquad$ Contradicts Lemma 4.0.6.

Case (31g).

$\qquad$ (14) $e_1 \, ? \, p_1$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (15) $e_1 \models_? \xi_1$ $\qquad\qquad\qquad\qquad$ by (12) on (14)

$\qquad$ (16) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ $\qquad\qquad$ by Rule (16e) on (15)

Case (20g).

$\qquad$ (4) $p = \mathtt{inr}(p_2)$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (5) $\xi = \mathtt{inr}(\xi_2)$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (6) $\tau = (\tau_1 + \tau_2)$ $\qquad\qquad\qquad\qquad$ by assumption

$\qquad$ (7) $p_2 : \tau_2[\xi_2] \dashv\!\!\mid \Gamma \, ; \Delta$ $\qquad\qquad\qquad$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}s\}$
         by assumption

(9) e `notintro`
         by Rule (26a),(26b),(26c),(26d),(26e),(26f)

1. Prove $e \models \mathtt{inr}(\xi_2)$ implies $e \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta$ for some θ. By rule induction over Rules (14) on $e \models \mathtt{inr}(\xi_2)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.

2. Prove $e \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta$ implies $e \models \mathtt{inr}(\xi_2)$. By rule induction over Rules (30) on $e \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.

3. Prove $e \models_? \mathtt{inr}(\xi_2)$ implies $e\, ?\, \mathtt{inr}(p_2)$.
   (10) $\mathtt{inr}(p_2)$ `refutable`$_?$
          by Rule (29e)
   (11) $e\, ?\, \mathtt{inr}(p_2)$
          by Rule (31c) on (9) and (10)

4. Prove $e\, ?\, \mathtt{inr}(p_2)$ implies $e \models_? \mathtt{inr}(\xi_2)$.
   (10) $\mathtt{inr}(\xi_2)$ `refutable`$_?$
          by Rule (10c)
   (11) $e \models_? \mathtt{inr}(\xi_2)$
          by Rule (16b) on (9) and (10)

Case (19k).

(8) $e = \mathtt{inr}_{\tau_1}(e_2)$
       by assumption

(9) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$
       by assumption

(10) e_2 `final`
       by Lemma 4.0.2 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_2 \models \xi_2$ iff $e_2 \triangleright p_2 \dashv\!\mid \theta$ for some θ

(12) $e_2 \models_? \xi_2$ iff $e_2\, ?\, p_2$

1. Prove $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta$ for some θ.
   (13) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$
          by assumption
   By rule induction over Rules (14) on (13), only one case applies.
   Case (14g).
   (14) $e_2 \models \xi_2$
          by assumption
   (15) $e_2 \triangleright p_2 \dashv\!\mid \theta_1$ for some θ_1
          by (11) on (14)
   (16) $\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta_1$
          by Rule (30e) on (15)

2. Prove $\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta$ implies $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$.
   (13) $\mathtt{inr}_{\tau_1}(e_2) \triangleright \mathtt{inr}(p_2) \dashv\!\mid \theta$
          by assumption

By rule induction over Rules (30) on (13), only one case applies.

Case (30e).

      (14) $e_2 \rhd p_2 \dashv\vdash \theta$ by assumption

      (15) $e_2 \models \xi_2$ by (11) on (14)

      (16) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (14g) on (15)

3. Prove $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$.

    (13) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by assumption

By rule induction over Rules (16) on (13), only two cases apply.

Case (16b).

      (14) $\mathtt{inr}_{\tau_1}(e_2)$ `notintro`   by assumption

    Contradicts Lemma 4.0.6.

Case (16e).

      (14) $e_2 \models_? \xi_2$ by assumption

      (15) $e_2 \,?\, p_2$ by (12) on (14)

      (16) $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ by Rule (31g) on (15)

4. Prove $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$.

    (13) $\mathtt{inr}_{\tau_1}(e_2) \,?\, \mathtt{inr}(p_2)$ by assumption

By rule induction over Rules (31) on (13), only two cases apply.

Case (31c).

      (14) $\mathtt{inr}_{\tau_1}(e_2)$ `notintro`   by assumption

    Contradicts Lemma 4.0.6.

Case (31g).

      (14) $e_2 \,?\, p_2$ by assumption

      (15) $e_2 \models_? \xi_2$ by (12) on (14)

      (16) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (16e) on (15)

Case (20h).

    (4) $p = (p_1, p_2)$ by assumption

    (5) $\xi = (\xi_1, \xi_2)$ by assumption

    (6) $\tau = (\tau_1 \times \tau_2)$ by assumption

    (7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

    (8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

    (9) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 \,;\, \Delta_1$ by assumption

   (10) $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 \,;\, \Delta_2$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

    (11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{r}\hat{s}\}$

                                            by assumption

(12)	e `notintro`	by Rule (26a),(26b),(26c),(26d),(26e),(26f)
(13)	e `indet`	by Lemma 4.0.9 on (2) and (12)
(14)	`prl`(e) `indet`	by Rule (24g) on (13)
(15)	`prl`(e) `final`	by Rule (25b) on (14)
(16)	`prr`(e) `indet`	by Rule (24h) on (13)
(17)	`prr`(e) `final`	by Rule (25b) on (16)
(18)	$\cdot\,; \Delta \vdash$ `prl`$(e) : \tau_1$	by Rule (19h) on (1)
(19)	$\cdot\,; \Delta \vdash$ `prr`$(e) : \tau_2$	by Rule (19i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

(20)	`prl`$(e) \models \xi_1$ iff `prl`$(e) \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1
(21)	`prl`$(e) \models_? \xi_1$ iff `prl`$(e) \,?\, p_1$
(22)	`prr`$(e) \models \xi_2$ iff `prr`$(e) \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2
(23)	`prr`$(e) \models_? \xi_2$ iff `prr`$(e) \,?\, p_2$

1. Prove $e \models (\xi_1, \xi_2)$ implies $e \rhd (p_1, p_2) \dashv\!\mid \theta$ for some θ.

(24)	$e \models (\xi_1, \xi_2)$	by assumption

   By rule induction over Rules (14) on (24), only one case applies.

   Case (14j).

(25)	`prl`$(e) \models \xi_1$	by assumption
(26)	`prr`$(e) \models \xi_2$	by assumption
(27)	`prl`$(e) \rhd p_1 \dashv\!\mid \theta_1$	by (20) on (25)
(28)	`prr`$(e) \rhd p_2 \dashv\!\mid \theta_2$	by (22) on (26)
(29)	$e \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2$	by Rule (30g) on (12) and (27) and (28)

2. Prove $e \rhd (p_1, p_2) \dashv\!\mid \theta$ implies $e \models (\xi_1, \xi_2)$.

(24)	$e \rhd (p_1, p_2) \dashv\!\mid \theta$	by assumption

   By rule induction over Rules (30) on (24), only one case applies.

   Case (30g).

(25)	$\theta = \theta_1 \uplus \theta_2$	by assumption
(26)	`prl`$(e) \rhd \xi_1 \dashv\!\mid \theta_1$	by assumption
(27)	`prr`$(e) \rhd \xi_2 \dashv\!\mid \theta_2$	by assumption
(28)	`prl`$(e) \models \xi_1$	by (20) on (26)
(29)	`prr`$(e) \models \xi_2$	by (22) on (27)
(30)	$e \models (\xi_1, \xi_2)$	by Rule (14j) on (12) and (28) and (29)

3. Prove $e \models_? (\xi_1, \xi_2)$ implies $e \,?\, (p_1, p_2)$.

(24)	$e \models_? (\xi_1, \xi_2)$	by assumption

By rule induction over Rules (16) on (24), only one case applies.

Case (16b)**.**

  (25) (ξ_1, ξ_2) `refutable`$_?$ by assumption

By rule induction over Rules (10) on (25), only two cases apply.

 Case (10d)**.**

   (26) ξ_1 `refutable`$_?$ by assumption

   (27) $\mathtt{prl}(e)$ `notintro`   by Rule (26e)

   (28) $\mathtt{prl}(e) \models_? \xi_1$ by Rule (16b) on (26) and (27)

   (29) $\mathtt{prl}(e) \, ? \, p_1$ by (21) on (28)

 By rule induction over Rules (31) on (29), only three cases apply.

  Case (31a),(31b)**.**

    (30) $p_1 = (\!|\,|\!)^w, (\!|p_0|\!)^w$ by assumption

    (31) p_1 `refutable`$_?$ by Rule (29b) and Rule (29c)

    (32) (p_1, p_2) `refutable`$_?$ by Rule (29f) on (31)

    (33) $e \, ? \, (p_1, p_2)$ by Rule (31c) on (12) and (32)

  Case (31c)**.**

    (30) p_1 `refutable`$_?$ by assumption

    (31) (p_1, p_2) `refutable`$_?$ by Rule (29f) on (30)

    (32) $e \, ? \, (p_1, p_2)$ by Rule (31c) on (12) and (31)

 Case (10e)**.**

   (26) ξ_2 `refutable`$_?$ by assumption

   (27) $\mathtt{prr}(e)$ `notintro`   by Rule (26e)

   (28) $\mathtt{prr}(e) \models_? \xi_2$ by Rule (16b) on (26) and (27)

   (29) $\mathtt{prr}(e) \, ? \, p_2$ by (23) on (28)

 By rule induction over Rules (31) on (29), only three cases apply.

  Case (31a),(31b)**.**

    (30) $p_2 = (\!|\,|\!)^w, (\!|p_0|\!)^w$ by assumption

    (31) p_2 `refutable`$_?$ by Rule (29b) and Rule (29c)

    (32) (p_1, p_2) `refutable`$_?$ by Rule (29g) on (31)

    (33) $e \, ? \, (p_1, p_2)$ by Rule (31c) on (12) and (32)

  Case (31c)**.**

$$(30)\ p_2\ \texttt{refutable}_?\qquad \text{by assumption}$$
$$(31)\ (p_1, p_2)\ \texttt{refutable}_?\qquad \text{by Rule (29g) on (30)}$$
$$(32)\ e\ ?\ (p_1, p_2)\qquad \text{by Rule (31c) on (12)}$$
$$\text{and (31)}$$

4. Prove $e\ ?\ (p_1, p_2)$ implies $e \models_? (\xi_1, \xi_2)$.

$$(24)\ e\ ?\ (p_1, p_2)\qquad \text{by assumption}$$

By rule induction over Rules (31) on (24), only one case applies.

Case (31c)**.**

$$(25)\ (p_1, p_2)\ \texttt{refutable}_?\qquad \text{by assumption}$$

By rule induction over Rules (29) on (25), only two cases apply.

Case (29f)**.**

$$(26)\ p_1\ \texttt{refutable}_?\qquad \text{by assumption}$$
$$(27)\ \texttt{prl}(e)\ \texttt{notintro}\qquad \text{by Rule (26e)}$$
$$(28)\ \texttt{prl}(e)\ ?\ p_1\qquad \text{by Rule (31c) on (26)}$$
$$\text{and (27)}$$
$$(29)\ \texttt{prl}(e) \models_? \xi_1\qquad \text{by (21) on (28)}$$

By rule induction over Rules (16) on (29), only three cases apply.

Case (16a)**.**

$$(30)\ \xi_1 = ?\qquad \text{by assumption}$$
$$(31)\ \xi_1\ \texttt{refutable}_?\qquad \text{by Rule (2b)}$$
$$(32)\ (\xi_1, \xi_2)\ \texttt{refutable}_?\qquad \text{by Rule (10d) on (31)}$$
$$(33)\ e \models_? (\xi_1, \xi_2)\qquad \text{by Rule (16b) on (12)}$$
$$\text{and (32)}$$

Case (16b)**.**

$$(30)\ \xi_1\ \texttt{refutable}_?\qquad \text{by assumption}$$
$$(31)\ (\xi_1, \xi_2)\ \texttt{refutable}_?\qquad \text{by Rule (10d) on (30)}$$
$$(32)\ e \models_? (\xi_1, \xi_2)\qquad \text{by Rule (16b) on (12)}$$
$$\text{and (31)}$$

Case (29g)**.**

$$(26)\ p_2\ \texttt{refutable}_?\qquad \text{by assumption}$$
$$(27)\ \texttt{prr}(e)\ \texttt{notintro}\qquad \text{by Rule (26e)}$$
$$(28)\ \texttt{prr}(e)\ ?\ p_2\qquad \text{by Rule (31c) on (26)}$$
$$\text{and (27)}$$
$$(29)\ \texttt{prr}(e) \models_? \xi_2\qquad \text{by (23) on (28)}$$

By rule induction over Rules (16) on (29), only three cases apply.

Case (16a)**.**

$$(30)\ \xi_2 = ?\qquad \text{by assumption}$$
$$(31)\ \xi_2\ \texttt{refutable}_?\qquad \text{by Rule (2b)}$$

(32) (ξ_1, ξ_2) `refutable`$_?$	by Rule (10e) on (31)
(33) $e \models_? (\xi_1, \xi_2)$	by Rule (16b) on (12) and (32)

Case (16b).

(30) ξ_2 `refutable`$_?$	by assumption
(31) (ξ_1, ξ_2) `refutable`$_?$	by Rule (10e) on (30)
(32) $e \models_? (\xi_1, \xi_2)$	by Rule (16b) on (12) and (31)

Case (19g).

(11) $e = (e_1, e_2)$	by assumption
(12) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$	by assumption
(13) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$	by assumption
(14) e_1 `final`	by Lemma 4.0.4 on (2)
(15) e_2 `final`	by Lemma 4.0.4 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16) $e_1 \models \xi_1$ iff $e_1 \triangleright p_1 \dashv\Vdash \theta_1$ for some θ_1

(17) $e_1 \models_? \xi_1$ iff $e_1 \,?\, p_1$

(18) $e_2 \models \xi_2$ iff $e_2 \triangleright p_2 \dashv\Vdash \theta_2$ for some θ_2

(19) $e_2 \models_? \xi_2$ iff $e_2 \,?\, p_2$

1. Prove $(e_1, e_2) \models (\xi_1, \xi_2)$ implies $(e_1, e_2) \triangleright (p_1, p_2) \dashv\Vdash \theta$ for some θ.

(20) $(e_1, e_2) \models (\xi_1, \xi_2)$	by assumption

By rule induction over Rules (14) on (20), only two cases apply.

Case (14i).

(21) $e_1 \models \xi_1$	by assumption
(22) $e_2 \models \xi_2$	by assumption
(23) $e_1 \triangleright p_1 \dashv\Vdash \theta_1$ for some θ_1	by (16) on (21)
(24) $e_2 \triangleright p_2 \dashv\Vdash \theta_2$ for some θ_2	by (18) on (22)
(25) $(e_1, e_2) \triangleright (p_1, p_2) \dashv\Vdash \theta_1 \uplus \theta_2$	by Rule (30d) on (23) and (24)

Case (14j).

(21) (e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.8.

2. Prove $(e_1, e_2) \triangleright (p_1, p_2) \dashv\Vdash \theta$ implies $(e_1, e_2) \models (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \triangleright (p_1, p_2) \dashv\Vdash \theta$	by assumption

By rule induction over Rules (30) on (20), only two cases apply.

Case (30d).

(21) $e_1 \triangleright p_1 \dashv\Vdash \theta_1$ for some θ_1	by assumption

$\quad$ (22) $e_2 \rhd p_2 \dashv\vdash \theta_2$ for some θ_2 $\qquad$ by assumption

$\quad$ (23) $e_1 \models \xi_1$ $\qquad$ by (16) on (21)

$\quad$ (24) $e_2 \models \xi_2$ $\qquad$ by (18) on (22)

$\quad$ (25) $(e_1, e_2) \models (\xi_1, \xi_2)$ $\qquad$ by Rule (14i) on (23) and (24)

Case (30g).

$\quad$ (21) (e_1, e_2) `notintro` $\qquad$ by assumption

$\quad$ Contradicts Lemma 4.0.8.

3. Prove $(e_1, e_2) \models_? (\xi_1, \xi_2)$ implies $(e_1, e_2) \,?\, (p_1, p_2)$.

$\quad$ (20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ $\qquad$ by assumption

By rule induction over Rules (16) on (20), only four cases apply.

Case (16b).

$\quad$ (21) (e_1, e_2) `notintro` $\qquad$ by assumption

$\quad$ Contradicts Lemma 4.0.8.

Case (16g).

$\quad$ (21) $e_1 \models_? \xi_1$ $\qquad$ by assumption

$\quad$ (22) $e_2 \models \xi_2$ $\qquad$ by assumption

$\quad$ (23) $e_1 \,?\, p_1$ $\qquad$ by (17) on (21)

$\quad$ (24) $e_2 \rhd p_2 \dashv\vdash \theta_2$ $\qquad$ by (18) on (22)

$\quad$ (25) $(e_1, e_2) \,?\, (p_1, p_2)$ $\qquad$ by Rule (31d) on (23) and (24)

Case (16h).

$\quad$ (21) $e_1 \models \xi_1$ $\qquad$ by assumption

$\quad$ (22) $e_2 \models_? \xi_2$ $\qquad$ by assumption

$\quad$ (23) $e_1 \rhd p_1 \dashv\vdash \theta_1$ $\qquad$ by (16) on (21)

$\quad$ (24) $e_2 \,?\, p_2$ $\qquad$ by (19) on (22)

$\quad$ (25) $(e_1, e_2) \,?\, (p_1, p_2)$ $\qquad$ by Rule (31e) on (23) and (24)

Case (16i).

$\quad$ (21) $e_1 \models_? \xi_1$ $\qquad$ by assumption

$\quad$ (22) $e_2 \models_? \xi_2$ $\qquad$ by assumption

$\quad$ (23) $e_1 \,?\, p_1$ $\qquad$ by (17) on (21)

$\quad$ (24) $e_2 \,?\, p_2$ $\qquad$ by (19) on (22)

$\quad$ (25) $(e_1, e_2) \,?\, (p_1, p_2)$ $\qquad$ by Rule (31f) on (23) and (24)

4. Prove $(e_1, e_2) \,?\, (p_1, p_2)$ implies $(e_1, e_2) \models_? (\xi_1, \xi_2)$.

$\quad$ (20) $(e_1, e_2) \,?\, (p_1, p_2)$ $\qquad$ by assumption

By rule induction over Rules (31) on (20), only four cases apply.

Case (31c).

$\quad$ (21) (e_1, e_2) `notintro` $\qquad$ by assumption

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Contradicts Lemma 4.0.8.

Case (31d).

(21)	$e_1 \,?\, p_1$	by assumption
(22)	$e_2 \rhd p_2 \dashv\!\mid \theta_2$	by assumption
(23)	$e_1 \models_? \xi_1$	by (17) on (21)
(24)	$e_2 \models \xi_2$	by (18) on (22)
(25)	$(e_1, e_2) \,?\, (p_1, p_2)$	by Rule (16g) on (23) and (24)

Case (31e).

(21)	$e_1 \rhd p_1 \dashv\!\mid \theta_1$	by assumption
(22)	$e_2 \,?\, p_2$	by assumption
(23)	$e_1 \models \xi_1$	by (16) on (21)
(24)	$e_2 \models_? \xi_2$	by (19) on (22)
(25)	$(e_1, e_2) \,?\, (p_1, p_2)$	by Rule (16h) on (23) and (24)

Case (31f).

(21)	$e_1 \,?\, p_1$	by assumption
(22)	$e_2 \,?\, p_2$	by assumption
(23)	$e_1 \models_? \xi_1$	by (17) on (21)
(24)	$e_2 \models_? \xi_2$	by (19) on (22)
(25)	$(e_1, e_2) \,?\, (p_1, p_2)$	by Rule (16i) on (23) and (24)

$\square$

5 Preservation and Progress

Theorem 5.1 (Preservation). *If* $\cdot\,; \Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,; \Delta \vdash e' : \tau$

Proof. By rule induction over Rules (19) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (19l).

(1)	$\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$	by assumption
(2)	$\mathtt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$	by assumption
(3)	$\cdot\,; \Delta \vdash e_1 : \tau_1$	by assumption
(4)	$\cdot\,; \Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$	by assumption
(5)	$\top \models_?^\dagger \xi$	by assumption

By rule induction over Rules (33) on (2).

Case (33k).

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$$(6) \quad e' = \texttt{match}(e_1')\{\cdot \mid r \mid rs\} \qquad \text{by assumption}$$
$$(7) \quad e_1 \mapsto e_1' \qquad \text{by assumption}$$
$$(8) \quad \cdot\, ; \Delta \vdash e_1' : \tau_1 \qquad \text{by IH on (3) and (7)}$$
$$(9) \quad \cdot\, ; \Delta \vdash \texttt{match}(e_1')\{\cdot \mid r \mid rs\} : \tau \qquad \text{by Rule (19l) on (8)}$$
$$\text{and (4) and (5)}$$

Case (33l).

$$(6) \quad r = p_r \Rightarrow e_r \qquad \text{by assumption}$$
$$(7) \quad e' = [\theta](e_r) \qquad \text{by assumption}$$
$$(8) \quad e_1 \rhd p_r \dashv\vdash \theta \qquad \text{by assumption}$$

By rule induction over Rules (22) on (4).

Case (22a).

$$(9) \quad \xi = \xi_r \qquad \text{by assumption}$$
$$(10) \quad \cdot\, ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau \qquad \text{by assumption}$$
$$(11) \quad p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \, ; \Delta_r \qquad \text{by Inversion of Rule}$$
$$\text{(21a) on (10)}$$
$$(12) \quad \Gamma_r \, ; \Delta \uplus \Delta_r \vdash e_r : \tau \qquad \text{by Inversion of Rule}$$
$$\text{(21a) on (10)}$$
$$(13) \quad \theta : \Gamma_r \qquad \text{by Lemma 3.0.7 on (3)}$$
$$\text{and (11) and (8)}$$
$$(14) \quad \cdot\, ; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau \qquad \text{by Lemma 3.0.6 on}$$
$$\text{(12) and (13)}$$

Case (22b).

$$(9) \quad \xi = \xi_r \vee \xi_{rs} \qquad \text{by assumption}$$
$$(10) \quad \cdot\, ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau \qquad \text{by assumption}$$
$$(11) \quad p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \, ; \Delta_r \qquad \text{by Inversion of Rule}$$
$$\text{(21a) on (10)}$$
$$(12) \quad \Gamma_r \, ; \Delta \uplus \Delta_r \vdash e_r : \tau \qquad \text{by Inversion of Rule}$$
$$\text{(21a) on (10)}$$
$$(13) \quad \theta : \Gamma_r \qquad \text{by Lemma 3.0.7 on (3)}$$
$$\text{and (11) and (8)}$$
$$(14) \quad \cdot\, ; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau \qquad \text{by Lemma 3.0.6 on}$$
$$\text{(12) and (13)}$$

Case (33m).

$$(6) \quad rs = r' \mid rs' \qquad \text{by assumption}$$
$$(7) \quad e' = \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$$
$$\text{by assumption}$$
$$(8) \quad e_1 \ \texttt{final} \qquad \text{by assumption}$$
$$(9) \quad e_1 \perp p_r \qquad \text{by assumption}$$

By rule induction over Rules (22) on (4).

Case (22a). Syntactic contradiction of rs.

Case (22b).

(10) $\xi = \xi_r \vee \xi_{rs}$	by assumption
(11) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$	by assumption
(12) $\cdot\,; \Delta \vdash [\bot \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$	by assumption
(13) $\xi_r \not\models \bot$	by assumption
(14) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$	by Inversion of Rule (21a) on (11)
(15) $\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$	by Inversion of Rule (21a) on (11)
(16) $\cdot\,; \Delta \vdash [\bot](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$	by Rule (22a) on (11) and (13)
(17) $e_1 \not\models^{\dagger}_{?} \xi_r$	by Lemma 4.0.12 on (3) and (8) and (14) and (9)
(18) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$	by Rule (19m) on (3) and (8) and (16) and (12) and (17) and (5)

Case (19m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$	by assumption
(2) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$	by assumption
(3) $\mathtt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$	by assumption
(4) $\cdot\,; \Delta \vdash e_1 : \tau_1$	by assumption
(5) e_1 $\mathtt{final}$	by assumption
(6) $\cdot\,; \Delta \vdash [\bot]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$	by assumption
(7) $\cdot\,; \Delta \vdash [\bot \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$	by assumption
(8) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$	by assumption
(9) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$	by assumption

By rule induction over Rules (33) on (3).

Case (33k).

(10) $e' = \mathtt{match}(e'_1)\{rs_{pre} \mid r \mid rs_{post}\}$	by assumption
(11) $e_1 \mapsto e'_1$	by assumption

By Lemma 4.0.10, (11) contradicts (5).

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Case (33l)**.**

$\quad$ (10) $\;r = p_r \Rightarrow e_r$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (11) $\;e' = [\theta](e_r)$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (12) $\;e_1 \triangleright p_r \dashv\vdash \theta$ $\qquad\qquad\qquad\quad$ by assumption

By rule induction over Rules (22) on (7).

Case (22a)**.**

$\quad$ (13) $\;\xi_{rest} = \xi_r$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (14) $\;\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ $\quad$ by assumption

$\quad$ (15) $\;p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$ $\qquad$ by Inversion of Rule

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ (21a) on (14)

$\quad$ (16) $\;\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$ $\qquad$ by Inversion of Rule

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ (21a) on (14)

$\quad$ (17) $\;\theta : \Gamma_r$ $\qquad\qquad\qquad\qquad\quad$ by Lemma 3.0.7 on (4)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ and (15) and (12)

$\quad$ (18) $\;\cdot\,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ $\quad$ by Lemma 3.0.6 on

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ (16) and (17)

Case (22b)**.**

$\quad$ (13) $\;\xi_{rest} = \xi_r \vee \xi_{rs}$ $\qquad\qquad\quad$ by assumption

$\quad$ (14) $\;\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ $\quad$ by assumption

$\quad$ (15) $\;p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$ $\qquad$ by assumption

$\quad$ (16) $\;\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$ $\qquad$ by assumption

$\quad$ (17) $\;\theta : \Gamma_r$ $\qquad\qquad\qquad\qquad\quad$ by Lemma 3.0.7 on (4)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ and (15) and (12)

$\quad$ (18) $\;\cdot\,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ $\quad$ by Lemma 3.0.6 on

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ (16) and (17)

Case (33m)**.**

$\quad$ (10) $\;r = p_r \Rightarrow e_r$ $\qquad\qquad\qquad\qquad$ by assumption

$\quad$ (11) $\;rs_{post} = r' \mid rs'$ $\qquad\qquad\qquad$ by assumption

$\quad$ (12) $\;e' = \mathtt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond \mid r' \mid rs'\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ by assumption

$\quad$ (13) $\;e_1 \perp p_r$ $\qquad\qquad\qquad\qquad\quad$ by assumption

By rule induction over Rules (22) on (7).

Case (22a)**.** Syntactic contradiction of rs_{post}.

Case (22b)**.**

$\quad$ (14) $\;\xi_{rest} = \xi_r \vee \xi_{post}$ $\qquad\qquad\quad$ by assumption

$\quad$ (15) $\;\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ $\quad$ by assumption

$\quad$ (16) $\;\cdot\,; \Delta \vdash [\perp \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ by assumption

$\quad$ (17) $\;\xi_r \not\models \xi_{pre}$ $\qquad\qquad\qquad\qquad$ by assumption

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$$(18) \quad p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,;\, \Delta_r \qquad\qquad \text{by Inversion of Rule}$$
$$\text{(21a) on (15)}$$
$$(19) \quad \Gamma_r \,;\, \Delta \uplus \Delta_r \vdash e_r : \tau \qquad\qquad \text{by Inversion of Rule}$$
$$\text{(21a) on (15)}$$
$$(20) \quad \xi_r : \tau_1 \qquad\qquad\qquad \text{by Lemma 3.0.2 on}$$
$$\text{(15)}$$
$$(21) \quad \xi_{pre} : \tau_1 \qquad\qquad\qquad \text{by Lemma 3.0.3 on (6)}$$
$$(22) \quad \xi_r \not\models \bot \vee \xi_{pre} \qquad\qquad \text{by Lemma 2.0.6 on}$$
$$\text{(20) and (21) and (17)}$$
$$(23) \quad \cdot\,;\, \Delta \vdash [\bot](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$$
$$\text{by Lemma 3.0.4 on (6)}$$
$$\text{and (15) and (22)}$$
$$(24) \quad e_1 \not\models_?^{\dagger} \xi_r \qquad\qquad\qquad \text{by Lemma 4.0.12 on}$$
$$\text{(4) and (5) and (18)}$$
$$\text{and (13)}$$
$$(25) \quad e_1 \not\models_?^{\dagger} \xi_{pre} \vee \xi_r \qquad\qquad \text{by Lemma 2.0.7 on (8)}$$
$$\text{and (24)}$$
$$(26) \quad \cdot\,;\, \Delta \vdash \mathtt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\} : \tau$$
$$\text{by Rule (19m) on (4)}$$
$$\text{and (5) and (23) and}$$
$$\text{(16) and (25) and (9)}$$

□

Theorem 5.2 (Progress). *If* $\cdot\,;\, \Delta \vdash e : \tau$ *then either* e `final` *or* $e \mapsto e'$ *for some* e'.

Proof. By rule induction over Rules (19) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (19l).

$$(1) \quad \cdot\,;\, \Delta \vdash \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau \qquad \text{by assumption}$$
$$(2) \quad \cdot\,;\, \Delta \vdash e_1 : \tau_1 \qquad\qquad\qquad \text{by assumption}$$
$$(3) \quad \cdot\,;\, \Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau \qquad \text{by assumption}$$
$$(4) \quad \top \models_?^{\dagger} \xi \qquad\qquad\qquad\qquad \text{by assumption}$$

By IH on (2).

Case Scrutinee takes a step.

$$(5) \quad e_1 \mapsto e_1' \qquad\qquad\qquad\qquad \text{by assumption}$$
$$(6) \quad \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \mathtt{match}(e_1')\{\cdot \mid r \mid rs\}$$
$$\text{by Rule (33k) on (5)}$$

Case Scrutinee is final.

(5) e_1 `final`   by assumption

By rule induction over Rules (22) on (3).

Case (22a)**.**

  (6) $rs = \cdot$ by assumption

  (7) $\xi = \xi_r$ by assumption

  (8) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

  (9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (8)

 (10) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (21a) on (8)

 (11) $e_1 \models^\dagger_? \xi_r$ by Corollary 2.1.1 on (5) and (4)

By rule induction over Rules (17) on (11).

Case (17a)**.**

  (12) $e_1 \models_? \xi_r$ by assumption

  (13) $e_1 \mathbin{?} p_r$ by Lemma 4.0.12 on (2) and (5) and (10) and (12)

  (14) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ `indet`

         by Rule (24k) on (5) and (13)

  (15) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ `final`

         by Rule (25b) on (14)

Case (17b)**.**

  (12) $e_1 \models \xi_r$ by assumption

  (13) $e_1 \triangleright p_r \dashv\vdash \theta$ by Lemma 4.0.12 on (2) and (5) and (10) and (12)

  (14) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$

         by Rule (33l) on (5) and (13)

Case (22b)**.**

  (6) $rs = r' \mid rs'$ by assumption

  (7) $\xi = \xi_r \vee \xi_{rs}$ by assumption

  (8) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

  (9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (8)

 (10) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (21a) on (8)

By Lemma 4.0.11 on (2) and (5) and (10).

Case Scrutinee matches pattern.

 (11) $e_1 \triangleright p_r \dashv\vdash \theta$ by assumption

 (12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$
           by Rule (33l) on (5)
           and (11)

Case Scrutinee may matches pattern.

 (11) $e_1 \; ? \; p_r$ by assumption

 (12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \; \texttt{indet}$
           by Rule (24k) on (5)
           and (11)

 (13) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \; \texttt{final}$
           by Rule (25b) on (12)

Case Scrutinee doesn't matche pattern.

 (11) $e_1 \perp p_r$ by assumption

 (12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$
     $\mapsto \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$
           by Rule (33m) on (5)
           and (11)

Case (19m).

 (1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

 (2) $\cdot\,; \Delta \vdash \texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

 (3) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

 (4) $e_1 \; \texttt{final}$ by assumption

 (5) $\cdot\,; \Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

 (6) $e_1 \not\models^\dagger_? \xi_{pre}$ by assumption

 (7) $\top \models^\dagger_? \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (22) on (5).

Case (22a).

 (5) $rs_{post} = \cdot$ by assumption

 (6) $\xi_{rest} = \xi_r$ by assumption

 (7) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

 (8) $r = p_r \Rightarrow e_r$ by Inversion of Rule
               (21a) on (7)

 (9) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$ by Inversion of Rule
               (21a) on (7)

 (10) $e_1 \models^\dagger_? \xi_{pre} \vee \xi_r$ by Corollary 2.1.1 on
               (4) and (7)

(11) $e_1 \models^{\dagger}_? \xi_r$ by Lemma 2.0.8 on (10) and (6)

By rule induction over Rules (17) on (11).

Case (17a).

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \mathbin{?} p_r$ by Lemma 4.0.12 on (3) and (4) and (9) and (12)

(14) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mathtt{\,indet}$ by Rule (24k) on (4) and (13)

(15) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mathtt{\,final}$ by Rule (25b) on (14)

Case (17b).

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \triangleright p_r \dashv\vdash \theta$ by Lemma 4.0.12 on (3) and (4) and (9) and (12)

(14) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (33l) on (4) and (13)

Case (22b).

(5) $rs_{post} = r' \mid rs'_{post}$ by assumption

(6) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(7) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (6)

(8) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r \,; \Delta_r$ by Inversion of Rule (21a) on (6)

By Lemma 4.0.11 on (3) and (4) and (8).

Case Scrutinee matches pattern.

(9) $e_1 \triangleright p_r \dashv\vdash \theta$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$ by Rule (33l) on (4) and (9)

Case Scrutinee may matches pattern.

(9) $e_1 \mathbin{?} p_r$ by assumption

(10) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mathtt{\,indet}$ by Rule (24k) on (4) and (9)

(11) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}$ $\texttt{final}$
$$\text{by Rule (25b) on (10)}$$

Case Scrutinee doesn't matche pattern.

(9) $e_1 \perp p_r$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\}$
$\mapsto \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$
$$\text{by Rule (33m) on (4)}$$
$$\text{and (9)}$$

$\square$

6   Decidability

$\boxed{\Xi \; \texttt{incon}}$ A finite set of constraints, Ξ, is inconsistent

CINCTruth
$$\frac{\Xi \; \texttt{incon}}{\Xi, \top \; \texttt{incon}} \tag{34a}$$

CINCFalsity
$$\frac{}{\Xi, \bot \; \texttt{incon}} \tag{34b}$$

CINCNum
$$\frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2} \; \texttt{incon}} \tag{34c}$$

CINCNotNum
$$\frac{}{\Xi, \underline{n}, \underline{\cancel{\varkappa}} \; \texttt{incon}} \tag{34d}$$

CINCAnd
$$\frac{\Xi, \xi_1, \xi_2 \; \texttt{incon}}{\Xi, \xi_1 \wedge \xi_2 \; \texttt{incon}} \tag{34e}$$

CINCOr
$$\frac{\Xi, \xi_1 \; \texttt{incon} \qquad \Xi, \xi_2 \; \texttt{incon}}{\Xi, \xi_1 \vee \xi_2 \; \texttt{incon}} \tag{34f}$$

CINCInj
$$\frac{}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2) \; \texttt{incon}} \tag{34g}$$

CINCInl
$$\frac{\Xi \; \texttt{incon}}{\texttt{inl}(\Xi) \; \texttt{incon}} \tag{34h}$$

CINCInr
$$\frac{\Xi \; \texttt{incon}}{\texttt{inr}(\Xi) \; \texttt{incon}} \tag{34i}$$

$$\frac{\text{CINCPairL}}{(\Xi_1, \Xi_2) \text{ incon}} \quad \Xi_1 \text{ incon} \tag{34j}$$

$$\frac{\text{CINCPairR}}{(\Xi_1, \Xi_2) \text{ incon}} \quad \Xi_2 \text{ incon} \tag{34k}$$

Lemma 6.0.1 (Decidability of Inconsistency). *Suppose* $\dot{\top}(\xi) = \xi$. *It is decidable whether* ξ `incon`.

Lemma 6.0.2 (Inconsistency and Entailment of Constraint). *Suppose that* $\dot{\top}(\xi) = \xi$. *Then* $\bar{\bar{\xi}}$ `incon` *iff* $\top \models \xi$

Lemma 6.0.3. *If* $e \models \xi$ *then* $e \models \dot{\top}(\xi)$

Proof. By rule induction over Rules (14), it is obvious to see that $\dot{\top}(\xi) = \xi$. $\square$

Lemma 6.0.4. *If* $e \models_? \xi$ *then* $e \models_?^\dagger \dot{\top}(\xi)$.

Proof.

(11)	$e \models_? \xi$	by assumption

By Rule Induction over Rules (16) on (11).

Case (16a).

(12)	$\xi = ?$	by assumption
(13)	$e \models \top$	by Rule (14a)
(14)	$e \models_?^\dagger \top$	by Rule (17b) on (13)

Case (16b).

(12)	e `notintro`	by assumption
(13)	ξ `refutable`$_?$	by assumption

By Lemma 2.0.15 on (12) and (13) and case analysis on its conclusion. By rule induction over Rules (10).

Case $\dot{\top}(\xi)$ `refutable`$_?$.

(14)	$\dot{\top}(\xi)$ `refutable`$_?$	by assumption
(15)	$e \models_? \dot{\top}(\xi)$	by Rule (16b) on (12) and (14)
(16)	$e \models_?^\dagger \dot{\top}(\xi)$	by Rule (17b) on (15)

Case $e \models \dot{\top}(\xi)$.

(14) $e \models \dot{\top}(\xi)$	by assumption
(15) $e \models_?^{\dagger} \top$	by Rule (17b) on (14)

Case (16c).

(12) $\xi = \xi_1 \vee \xi_2$	by assumption
(13) $e \models_? \xi_1$	by assumption
(14) $e \models_?^{\dagger} \dot{\top}(\xi_1)$	by IH on (13)
(15) $e \models_?^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 2.0.10 on (14)

Case (16d).

(12) $\xi = \xi_1 \vee \xi_2$	by assumption
(13) $e \models_? \xi_2$	by assumption
(14) $e \models_?^{\dagger} \dot{\top}(\xi_2)$	by IH on (13)
(15) $e \models_?^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 2.0.10 on (14)

Case (16e).

(12) $e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(13) $\xi = \mathtt{inl}(\xi_1)$	by assumption
(14) $e_1 \models_? \xi_1$	by assumption
(15) $e_1 \models_?^{\dagger} \dot{\top}(\xi_1)$	by IH on (14)
(16) $\mathtt{inl}_{\tau_2}(e_1) \models_?^{\dagger} \mathtt{inl}(\dot{\top}(\xi_1))$	by Lemma 2.0.11 on (15)

Case (16f).

(12) $e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(13) $\xi = \mathtt{inr}(\xi_2)$	by assumption
(14) $e_2 \models_? \xi_2$	by assumption
(15) $e_2 \models_?^{\dagger} \dot{\top}(\xi_2)$	by IH on (14)
(16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^{\dagger} \mathtt{inr}(\dot{\top}(\xi_2))$	by Lemma 2.0.12 on (15)

Case (16g).

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(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_2 \models \xi_2$	by assumption
(16)	$e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (14)
(17)	$e_2 \models \dot{\top}(\xi_2)$	by Lemma 6.0.3 on (15)
(18)	$e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by Rule (17b) on (17)
(19)	$(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (16) and (18)

Case (16h).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models \xi_1$	by assumption
(15)	$e_2 \models_? \xi_2$	by assumption
(16)	$e_1 \models \dot{\top}(\xi_1)$	by Lemma 6.0.3 on (14)
(17)	$e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by Rule (17b) on (16)
(18)	$e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (15)
(19)	$(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (17) and (18)

Case (16i).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_2 \models_? \xi_2$	by assumption
(16)	$e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (14)
(17)	$e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (15)
(18)	$(e_1, e_2) \models_?^\dagger (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (16) and (17)

$\square$

Lemma 6.0.5. $e \models^{\dagger}_{?} \xi$ *iff* $e \models^{\dagger}_{?} \dot{\top}(\xi)$

Proof.   1. Sufficiency:

    (1) $e \models^{\dagger}_{?} \xi$ by assumption

By rule induction over Rules (17) on (1)

Case (17b).

    (2) $e \models \xi$ by assumption
    (3) $e \models \dot{\top}(\xi)$ by Lemma 6.0.3 on (2)
    (4) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by Rule (17b) on (3)

Case (17a).

    (2) $e \models_{?} \xi$ by assumption
    (3) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by Lemma 6.0.4 on (2)

2. Necessity:

    (1) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by assumption

By structural induction on ξ,

Case $\xi = \top, \bot, \underline{n}, \underline{\varkappa}$.

    (2) $e \models^{\dagger}_{?} \xi$ by (1) and Definition 12

Case $\xi = ?$.

    (2) $e \models_{?} ?$ by Rule (16a)
    (3) $e \models^{\dagger}_{?} ?$ by Rule (17a) on (2)

Case $\xi = \xi_1 \vee \xi_2$.

    (2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Definition 12

By rule induction over Rules (17) on (1),

    Case (17b).

        (3) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

        By rule induction over Rules (14) on (3) and two cases apply,

        Case ($14e$).

            (4) $e \models \dot{\top}(\xi_1)$ by assumption
            (5) $e \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (17b) on (4)

$$(6) \quad e \models^\dagger_? \xi_1 \qquad\qquad \text{by IH on (5)}$$

$$(7) \quad e \models^\dagger_? \xi_1 \vee \xi_2 \qquad\qquad \text{by Lemma 2.0.10 on}$$
$$(6)$$

Case (14f).

$$(4) \quad e \models \dot{\top}(\xi_2) \qquad\qquad \text{by assumption}$$

$$(5) \quad e \models^\dagger_? \dot{\top}(\xi_2) \qquad\qquad \text{by Rule (17b) on (4)}$$

$$(6) \quad e \models^\dagger_? \xi_2 \qquad\qquad \text{by IH on (5)}$$

$$(7) \quad e \models^\dagger_? \xi_1 \vee \xi_2 \qquad\qquad \text{by Lemma 2.0.10 on}$$
$$(6)$$

Case (17a).

$$(3) \quad e \models_? \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \qquad\qquad \text{by assumption}$$

By rule induction over Rules (16) on (3) and two cases apply,

Case (16c).

$$(4) \quad e \models_? \dot{\top}(\xi_1) \qquad\qquad \text{by assumption}$$

$$(5) \quad e \models^\dagger_? \dot{\top}(\xi_1) \qquad\qquad \text{by Rule (17a) on (4)}$$

$$(6) \quad e \models^\dagger_? \xi_1 \qquad\qquad \text{by IH on (5)}$$

$$(7) \quad e \models^\dagger_? \xi_1 \vee \xi_2 \qquad\qquad \text{by Lemma 2.0.10 on}$$
$$(6)$$

Case (16d).

$$(4) \quad e \models_? \dot{\top}(\xi_2) \qquad\qquad \text{by assumption}$$

$$(5) \quad e \models^\dagger_? \dot{\top}(\xi_2) \qquad\qquad \text{by Rule (17a) on (4)}$$

$$(6) \quad e \models^\dagger_? \xi_2 \qquad\qquad \text{by IH on (5)}$$

$$(7) \quad e \models^\dagger_? \xi_1 \vee \xi_2 \qquad\qquad \text{by Lemma 2.0.10 on}$$
$$(6)$$

Case $\xi = \mathtt{inl}(\xi_1)$.

$$(2) \quad e = \mathtt{inl}_{\tau_2}(e_1) \qquad\qquad \text{by assumption}$$

$$(3) \quad \dot{\top}(\xi) = \mathtt{inl}(\dot{\top}(\xi_1)) \qquad\qquad \text{by assumption}$$

By rule induction over Rules (17) on (1),

Case (17b).

$$(4) \quad \mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\top}(\xi_1)) \qquad\qquad \text{by assumption}$$

By rule induction over Rules (14) and only one case applies,

Case (14g).

$$(5) \quad e_1 \models \dot{\top}(\xi_1) \qquad\qquad \text{by assumption}$$

$$(6) \quad e_1 \models^\dagger_? \dot{\top}(\xi_1) \qquad\qquad \text{by Rule (17b) on (5)}$$

$$(7) \quad e_1 \models^\dagger_? \xi_1 \qquad\qquad \text{by IH on (6)}$$

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$\quad$ (8) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ $\qquad$ by Lemma 2.0.11 on (7)

Case (17a).

$\qquad$ (4) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\top}(\xi_1))$ $\qquad$ by assumption

$\quad$ By rule induction over Rules (16) and only one case applies,

$\quad$ **Case (16e).**

$\qquad\quad$ (5) $e_1 \models_? \dot{\top}(\xi_1)$ $\qquad$ by assumption
$\qquad\quad$ (6) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$ $\qquad$ by Rule (17a) on (5)
$\qquad\quad$ (7) $e_1 \models_?^\dagger \xi_1$ $\qquad$ by IH on (6)
$\qquad\quad$ (8) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ $\qquad$ by Lemma 2.0.11 on (7)

Case $\xi = \mathtt{inr}(\xi_2)$.

$\quad$ (2) $e = \mathtt{inr}_{\tau_1}(e_2)$ $\qquad$ by assumption
$\quad$ (3) $\dot{\top}(\xi) = \mathtt{inr}(\dot{\top}(\xi_2))$ $\qquad$ by assumption

By rule induction over Rules (17) on (1),

Case (17b).

$\qquad$ (4) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\top}(\xi_2))$ $\qquad$ by assumption

$\quad$ By rule induction over Rules (14) and only one case applies,

$\quad$ **Case (14h).**

$\qquad\quad$ (5) $e_2 \models \dot{\top}(\xi_2)$ $\qquad$ by assumption
$\qquad\quad$ (6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$ $\qquad$ by Rule (17b) on (5)
$\qquad\quad$ (7) $e_2 \models_?^\dagger \xi_2$ $\qquad$ by IH on (6)
$\qquad\quad$ (8) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ $\qquad$ by Lemma 2.0.12 on (7)

Case (17a).

$\qquad$ (4) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\top}(\xi_2))$ $\qquad$ by assumption

$\quad$ By rule induction over Rules (16) and only one case applies,

$\quad$ **Case (16f).**

$\qquad\quad$ (5) $e_2 \models_? \dot{\top}(\xi_2)$ $\qquad$ by assumption
$\qquad\quad$ (6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$ $\qquad$ by Rule (17a) on (5)
$\qquad\quad$ (7) $e_2 \models_?^\dagger \xi_2$ $\qquad$ by IH on (6)
$\qquad\quad$ (8) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ $\qquad$ by Lemma 2.0.12 on (7)

Case $\xi = (\xi_1, \xi_2)$.

$\quad$ (2) $e = (e_1, e_2)$ $\qquad$ by assumption
$\quad$ (3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ $\qquad$ by Definition 12

By rule induction over Rules (17) on (1),

Case (17b).

      (4) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (14) on (4) and only one case applies,

  Case (14i).

      (5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

      (6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

      (7) $e_1 \models_?^{\dagger} \dot{\top}(\xi_1)$ by Rule (17b) on (5)

      (8) $e_2 \models_?^{\dagger} \dot{\top}(\xi_2)$ by Rule (17b) on (6)

      (9) $e_1 \models_?^{\dagger} \xi_1$ by IH on (7)

    (10) $e_2 \models_?^{\dagger} \xi_2$ by IH on (8)

    (11) $(e_1, e_2) \models_?^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

Case (17a).

      (4) $(e_1, e_2) \models_? (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (16) on (4) and three cases apply,

  Case (16g).

      (5) $e_1 \models_? \dot{\top}(\xi_1)$ by assumption

      (6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

      (7) $e_1 \models_?^{\dagger} \dot{\top}(\xi_1)$ by Rule (17a) on (5)

      (8) $e_2 \models_?^{\dagger} \dot{\top}(\xi_2)$ by Rule (17b) on (6)

      (9) $e_1 \models_?^{\dagger} \xi_1$ by IH on (7)

    (10) $e_2 \models_?^{\dagger} \xi_2$ by IH on (8)

    (11) $(e_1, e_2) \models_?^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

  Case (16h).

      (5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

      (6) $e_2 \models_? \dot{\top}(\xi_2)$ by assumption

      (7) $e_1 \models_?^{\dagger} \dot{\top}(\xi_1)$ by Rule (17b) on (5)

      (8) $e_2 \models_?^{\dagger} \dot{\top}(\xi_2)$ by Rule (17a) on (6)

      (9) $e_1 \models_?^{\dagger} \xi_1$ by IH on (7)

    (10) $e_2 \models_?^{\dagger} \xi_2$ by IH on (8)

    (11) $(e_1, e_2) \models_?^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

  Case (16i).

      (5) $e_1 \models_? \dot{\top}(\xi_1)$ by assumption

      (6) $e_2 \models_? \dot{\top}(\xi_2)$ by assumption

$$\begin{array}{lll}
\text{(7)} & e_1 \models_?^\dagger \dot{\top}(\xi_1) & \text{by Rule (17a) on (5)} \\
\text{(8)} & e_2 \models_?^\dagger \dot{\top}(\xi_2) & \text{by Rule (17a) on (6)} \\
\text{(9)} & e_1 \models_?^\dagger \xi_1 & \text{by IH on (7)} \\
\text{(10)} & e_2 \models_?^\dagger \xi_2 & \text{by IH on (8)} \\
\text{(11)} & (e_1, e_2) \models_?^\dagger (\xi_1, \xi_2) & \text{by Lemma 2.0.13 on} \\
& & \text{(9) and (10)}
\end{array}$$

□

Lemma 6.0.6. *Assume* $\dot{\top}(\xi) = \xi$. *Then* $\top \models_?^\dagger \xi$ *iff* $\top \models \xi$.

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

2. Necessity:

□

Theorem 6.1. $\top \models_?^\dagger \xi$ *iff* $\top \models \dot{\top}(\xi)$.

Lemma 6.1.1. *Assume that* e `val`. *Then* $e \models_?^\dagger \xi$ *iff* $e \models \dot{\top}(\xi)$

Proof.

$$\begin{array}{lll}
\text{(1)} & e \text{ val} & \text{by assumption}
\end{array}$$

We prove sufficiency and necessity separately.

1. Sufficiency:

$$\begin{array}{lll}
\text{(2)} & e \models_?^\dagger \xi & \text{by assumption}
\end{array}$$

By rule induction over Rules (17) on (2).

Case (17b).

$$\begin{array}{lll}
\text{(3)} & e \models \xi & \text{by assumption} \\
\text{(4)} & e \models \dot{\top}(\xi) & \text{by Lemma 6.0.3 on (3)}
\end{array}$$

Case (17a).

$$\begin{array}{lll}
\text{(3)} & e \models_? \xi & \text{by assumption}
\end{array}$$

By rule induction over Rules (16) on (3).

Case (16a).

$$\begin{array}{lll}
\text{(4)} & \xi = ? & \text{by assumption} \\
\text{(5)} & e \models \dot{\top}(\xi) & \text{by Rule (14a) and} \\
& & \text{Definition 12}
\end{array}$$

Case (16b).

$\qquad$ (4) e `notintro` $\qquad$ by assumption

By rule induction over Rules (26) on (4), for each case, by rule induction over Rules (23) on (1), no case applies due to syntactic contradiction.

Case (16c).

$\qquad$ (4) $\xi = \xi_1 \vee \xi_2$ $\qquad$ by assumption
$\qquad$ (5) $e \models_? \xi_1$ $\qquad$ by assumption
$\qquad$ (6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ $\qquad$ by Equation 12
$\qquad$ (7) $e \models_?^{\dagger} \xi_1$ $\qquad$ by Rule (17a) on (5)
$\qquad$ (8) $e \models \dot{\top}(\xi_1)$ $\qquad$ by IH on (7)
$\qquad$ (9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ $\qquad$ by Rule (14e) on (8)

Case (16d).

$\qquad$ (4) $\xi = \xi_1 \vee \xi_2$ $\qquad$ by assumption
$\qquad$ (5) $e \models_? \xi_2$ $\qquad$ by assumption
$\qquad$ (6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ $\qquad$ by Equation 12
$\qquad$ (7) $e \models_?^{\dagger} \xi_2$ $\qquad$ by Rule (17a) on (5)
$\qquad$ (8) $e \models \dot{\top}(\xi_2)$ $\qquad$ by IH on (7)
$\qquad$ (9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ $\qquad$ by Rule (14f) on (8)

Case (16e).

$\qquad$ (4) $\xi = \mathtt{inl}(\xi_1)$ $\qquad$ by assumption
$\qquad$ (5) $e \models_? \xi_1$ $\qquad$ by assumption
$\qquad$ (6) $\dot{\top}(\xi) = \mathtt{inl}(\dot{\top}(\xi_1))$ $\qquad$ by Equation 12
$\qquad$ (7) $e \models_?^{\dagger} \xi_1$ $\qquad$ by Rule (17a) on (5)
$\qquad$ (8) $e \models \dot{\top}(\xi_1)$ $\qquad$ by IH on (7)
$\qquad$ (9) $e \models \mathtt{inl}(\dot{\top}(\xi_1))$ $\qquad$ by Rule (14g) on (8)

Case (16f).

$\qquad$ (4) $\xi = \mathtt{inr}(\xi_2)$ $\qquad$ by assumption
$\qquad$ (5) $e \models_? \xi_2$ $\qquad$ by assumption
$\qquad$ (6) $\dot{\top}(\xi) = \mathtt{inr}(\dot{\top}(\xi_2))$ $\qquad$ by Equation 12
$\qquad$ (7) $e \models_?^{\dagger} \xi_2$ $\qquad$ by Rule (17a) on (5)
$\qquad$ (8) $e \models \dot{\top}(\xi_2)$ $\qquad$ by IH on (7)
$\qquad$ (9) $e \models \mathtt{inr}(\dot{\top}(\xi_2))$ $\qquad$ by Rule (14h) on (8)

Case (16g).

$\qquad$ (4) $e = (e_1, e_2)$ $\qquad$ by assumption
$\qquad$ (5) $\xi = (\xi_1, \xi_2)$ $\qquad$ by assumption

(6)	$e_1 \models_? \xi_1$	by assumption
(7)	$e_2 \models \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (17a) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (17b) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (14i) on (11) and (12)

Case (16h).

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models \xi_1$	by assumption
(7)	$e_2 \models_? \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (17b) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (17a) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (14i) on (11) and (12)

Case (16i).

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models_? \xi_1$	by assumption
(7)	$e_2 \models_? \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (17a) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (17a) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (14i) on (11) and (12)

2. Necessity:

(2)	$e \models \dot{\top}(\xi)$	by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\mathscr{K}}$.

    (3) $\xi = \dot{\top}(\xi)$ by Equation 12

    (4) $e \models^{\dagger}_{?} \xi$ by Rule (17b) on (2)

Case $\xi = \,?$.

    (3) $e \models_{?} \,?$ by Rule (16a)

    (4) $e \models^{\dagger}_{?} \,?$ by Rule (17a) on (3)

Case $\xi = \xi_1 \wedge \xi_2$.

    (3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by Equation 12

By rule induction over Rules (14) on (2), only one case applies.

Case (14d).

    (4) $e \models \dot{\top}(\xi_1)$ by assumption

    (5) $e \models \dot{\top}(\xi_2)$ by assumption

    (6) $e \models^{\dagger}_{?} \xi_1$ by IH on (4)

    (7) $e \models^{\dagger}_{?} \xi_2$ by IH on (5)

    (8) $e \models \xi_1 \wedge \xi_2$ by Lemma 2.0.9 on (6) and (7)

Case $\xi = \xi_1 \vee \xi_2$.

    (3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Equation 12

By rule induction over Rules (14) on (2) and only two cases apply.

Case (14e).

    (4) $e \models \dot{\top}(\xi_1)$ by assumption

    (5) $e \models^{\dagger}_{?} \xi_1$ by IH on (4)

    (6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (5)

Case (14f).

    (4) $e \models \dot{\top}(\xi_2)$ by assumption

    (5) $e \models^{\dagger}_{?} \xi_2$ by IH on (4)

    (6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (5)

Case $\xi = \mathtt{inl}(\xi_1)$.

    (3) $\dot{\top}(\xi) = \mathtt{inl}(\dot{\top}(\xi_1))$ by Equation 12

By rule induction over Rules (14) on (2) and only one case applies.

Case (14g).

(4) $e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(5) $e_1 \models \dot{\top}(\xi_1)$	by assumption
(6) $e_1 \models^{\dagger}_? \xi_1$	by IH on (5)
(7) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_? \mathtt{inl}(\xi_1)$	by Lemma 2.0.11 on (6)

Case $\xi = \mathtt{inr}(\xi_2)$.

(3) $\dot{\top}(\xi) = \mathtt{inr}(\dot{\top}(\xi_2))$	by Equation 12

By rule induction over Rules (14) on (2) and only one case applies.

Case (14h).

(4) $e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(5) $e_2 \models \dot{\top}(\xi_2)$	by assumption
(6) $e_2 \models^{\dagger}_? \xi_2$	by IH on (5)
(7) $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_? \mathtt{inr}(\xi_2)$	by Lemma 2.0.12 on (6)

Case $\xi = (\xi_1, \xi_2)$.

(3) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Equation 12

By rule induction over Rules (14) on (2) and only one case applies.

Case (14i).

(4) $e = (e_1, e_2)$	by assumption
(5) $e_1 \models \dot{\bot}(\xi_1)$	by assumption
(6) $e_2 \models \dot{\bot}(\xi_2)$	by assumption
(7) $e_1 \models^{\dagger}_? \xi_1$	by IH on (5)
(8) $e_2 \models^{\dagger}_? \xi_2$	by IH on (6)
(9) $(e_1, e_2) \models^{\dagger}_? (\xi_1, \xi_2)$	by Lemma 2.0.13 on (7) and (8)

□

Lemma 6.1.2. $e \models \xi$ *iff* $e \models \dot{\bot}(\xi)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models \xi$	by assumption

By rule induction over Rules (14) on (1).

Case (14a).

(2) $\xi = \top$	by assumption
(3) $e \models \dot{\bot}(\top)$	by (1) and Definition 13

Case (14b).

(2)	$\xi = \underline{n}$	by assumption
(3)	$e \models \dot{\bot}(\underline{n})$	by (1) and Definition 13

Case (14c).

(2)	$\xi = \underline{\varkappa}$	by assumption
(3)	$e \models \dot{\bot}(\underline{\varkappa})$	by (1) and Definition 13

Case (14d).

(2)	$\xi = \xi_1 \wedge \xi_2$	by assumption
(3)	$e \models \xi_1$	by assumption
(4)	$e \models \xi_2$	by assumption
(5)	$e \models \dot{\bot}(\xi_1)$	by IH on (3)
(6)	$e \models \dot{\bot}(\xi_2)$	by IH on (4)
(7)	$e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$	by Rule (14d) on (5) and (6)
(8)	$e \models \dot{\bot}(\xi_1 \wedge \xi_2)$	by (7) and Definition 13

Case (14e).

(2)	$\xi = \xi_1 \vee \xi_2$	by assumption
(3)	$e \models \xi_1$	by assumption
(4)	$e \models \dot{\bot}(\xi_1)$	by IH on (3)
(5)	$e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (14e) on (4)
(6)	$e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 13

Case (14f).

(2)	$\xi = \xi_1 \vee \xi_2$	by assumption
(3)	$e \models \xi_2$	by assumption
(4)	$e \models \dot{\bot}(\xi_2)$	by IH on (3)
(5)	$e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (14f) on (4)
(6)	$e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 13

Case (14g).

(2)	$e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(3)	$\xi = \mathtt{inl}(\xi_1)$	by assumption
(4)	$e_1 \models \xi_1$	by assumption

$$(5) \quad e_1 \models \dot{\bot}(\xi_1) \qquad\qquad \text{by IH on (4)}$$
$$(6) \quad \mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\bot}(\xi_1)) \qquad \text{by Rule (14g) on (5)}$$
$$(7) \quad \mathtt{inl}_{\tau_2}(e_1) \models \dot{\bot}(\mathtt{inl}(\xi_1)) \qquad \text{by (6) and Definition 13}$$

Case (14h).

$$(2) \quad e = \mathtt{inr}_{\tau_1}(e_2) \qquad\qquad \text{by assumption}$$
$$(3) \quad \xi = \mathtt{inr}(\xi_2) \qquad\qquad\qquad \text{by assumption}$$
$$(4) \quad e_2 \models \xi_2 \qquad\qquad\qquad\quad \text{by assumption}$$
$$(5) \quad e_2 \models \dot{\bot}(\xi_2) \qquad\qquad\quad \text{by IH on (4)}$$
$$(6) \quad \mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\bot}(\xi_2)) \qquad \text{by Rule (14h) on (5)}$$
$$(7) \quad \mathtt{inr}_{\tau_1}(e_2) \models \dot{\bot}(\mathtt{inr}(\xi_2)) \qquad \text{by (6) and Definition 13}$$

Case (14i).

$$(2) \quad e = (e_1, e_2) \qquad\qquad\qquad \text{by assumption}$$
$$(3) \quad \xi = (\xi_1, \xi_2) \qquad\qquad\qquad \text{by assumption}$$
$$(4) \quad e_1 \models \xi_1 \qquad\qquad\qquad\quad \text{by assumption}$$
$$(5) \quad e_2 \models \xi_2 \qquad\qquad\qquad\quad \text{by assumption}$$
$$(6) \quad e_1 \models \dot{\bot}(\xi_1) \qquad\qquad\quad \text{by IH on (4)}$$
$$(7) \quad e_2 \models \dot{\bot}(\xi_2) \qquad\qquad\quad \text{by IH on (5)}$$
$$(8) \quad (e_1, e_2) \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \qquad \text{by Rule (14i) on (6) and (7)}$$
$$(9) \quad (e_1, e_2) \models \dot{\bot}((\xi_1, \xi_2)) \qquad \text{by (8) and Definition 13}$$

2. Necessity:

$$(1) \quad e \models \dot{\bot}(\xi) \qquad\qquad\qquad \text{by assumption}$$

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\varkappa}$.

$$(2) \quad e \models \xi \qquad\qquad\qquad\qquad \text{by (1) and Definition 13}$$

Case $\xi = ?$.

$$(2) \quad e \models \bot \qquad\qquad\qquad\qquad \text{by (1) and Definition 13}$$
$$(3) \quad e \not\models \bot \qquad\qquad\qquad\qquad \text{by Lemma 2.0.1}$$

(3) contradicts (2).

Case $\xi = \xi_1 \wedge \xi_2$.

$$(2) \quad e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2) \qquad \text{by (1) and Definition 13}$$

By rule induction over Rules (14) on (2) and only case applies.

Case (14d).

$$
\begin{array}{lll}
(3) & e \models \dot{\perp}(\xi_1) & \text{by assumption} \\
(4) & e \models \dot{\perp}(\xi_2) & \text{by assumption} \\
(5) & e \models \xi_1 & \text{by IH on (3)} \\
(6) & e \models \xi_2 & \text{by IH on (4)} \\
(7) & e \models \xi_1 \wedge \xi_2 & \text{by Rule (14d) on (5)} \\
& & \text{and (6)}
\end{array}
$$

Case $\xi = \xi_1 \vee \xi_2$.

$$
\begin{array}{lll}
(2) & e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) & \text{by (1) and Definition} \\
& & 13
\end{array}
$$

By rule induction over Rules (14) on (2) and only two cases apply.

Case (14e).

$$
\begin{array}{lll}
(3) & e \models \dot{\perp}(\xi_1) & \text{by assumption} \\
(4) & e \models \xi_1 & \text{by IH on (3)} \\
(5) & e \models \xi_1 \vee \xi_2 & \text{by Rule (14e) on (4)}
\end{array}
$$

Case (14f).

$$
\begin{array}{lll}
(3) & e \models \dot{\perp}(\xi_2) & \text{by assumption} \\
(4) & e \models \xi_2 & \text{by IH on (3)} \\
(5) & e \models \xi_1 \vee \xi_2 & \text{by Rule (14f) on (4)}
\end{array}
$$

Case $\xi = \mathtt{inl}(\xi_1)$.

$$
\begin{array}{lll}
(2) & e \models \mathtt{inl}(\dot{\perp}(\xi_1)) & \text{by (1) and Definition} \\
& & 13
\end{array}
$$

By rule induction over Rules (14) on (2) and only one case applies.

Case (14g).

$$
\begin{array}{lll}
(3) & e = \mathtt{inl}_{\tau_2}(e_1) & \text{by assumption} \\
(4) & e_1 \models \dot{\perp}(\xi_1) & \text{by assumption} \\
(5) & e_1 \models \xi_1 & \text{by IH on (4)} \\
(6) & e \models \mathtt{inl}(\xi_1) & \text{by Rule (14g) on (5)}
\end{array}
$$

Case $\xi = \mathtt{inr}(\xi_2)$.

$$
\begin{array}{lll}
(2) & e \models \mathtt{inr}(\dot{\perp}(\xi_2)) & \text{by (1) and Definition} \\
& & 13
\end{array}
$$

By rule induction over Rules (14) on (2) and only one case applies.

Case (14h).

$$
\begin{array}{lll}
(3) & e = \mathtt{inr}_{\tau_1}(e_2) & \text{by assumption} \\
(4) & e_2 \models \dot{\perp}(\xi_2) & \text{by assumption}
\end{array}
$$

$$(5) \quad e_2 \models \xi_2 \qquad\qquad\qquad \text{by IH on (4)}$$
$$(6) \quad e \models \text{inr}(\xi_2) \qquad\qquad \text{by Rule (14h) on (5)}$$

Case $\xi = (\xi_1, \xi_2)$.

$$(2) \quad e \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \qquad\qquad \text{by (1) and Definition 13}$$

By rule induction over Rules (14) on (2) and only case applies.

Case (14i).

$$(3) \quad e = (e_1, e_2) \qquad\qquad\qquad \text{by assumption}$$
$$(4) \quad e_1 \models \dot{\bot}(\xi_1) \qquad\qquad\qquad \text{by assumption}$$
$$(5) \quad e_2 \models \dot{\bot}(\xi_2) \qquad\qquad\qquad \text{by assumption}$$
$$(6) \quad e_1 \models \xi_1 \qquad\qquad\qquad\qquad \text{by IH on (4)}$$
$$(7) \quad e_2 \models \xi_2 \qquad\qquad\qquad\qquad \text{by IH on (5)}$$
$$(8) \quad e \models (\xi_1, \xi_2) \qquad\qquad\qquad \text{by Rule (14i) on (6) and (7)}$$

$\square$

Lemma 6.1.3. *Assume e val and $\dot{\top}(\xi) = \xi$. Then $e \not\models \xi$ iff $e \models \overline{\xi}$.*

Theorem 6.2. $\xi_r \models \xi_{rs}$ *iff* $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\bot}(\xi_{rs})$.