

1 Match Constraint Language

$\xi ::= \top \mid ? \mid \underline{n} \mid \xi_1 \vee \xi_2 \mid \text{inl}(\xi) \mid \text{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\frac{\text{CTTruth}}{\overline{\top : \tau}} \quad (1a)$$

$$\frac{\text{CTUnknown}}{\overline{? : \tau}} \quad (1b)$$

$$\frac{\text{CTNum}}{\overline{\underline{n} : \text{num}}} \quad (1c)$$

$$\frac{\text{CTOr} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \quad (1d)$$

$$\frac{\text{CTInl} \quad \xi_1 : \tau_1}{\text{inl}(\xi_1) : (\tau_1 + \tau_2)} \quad (1e)$$

$$\frac{\text{CTInr} \quad \xi_2 : \tau_2}{\text{inr}(\xi_2) : (\tau_1 + \tau_2)} \quad (1f)$$

$$\frac{\text{CTPair} \quad \xi_1 : \tau_1 \quad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \quad (1g)$$

$\boxed{\xi \text{ refutable}}$ ξ is refutable

$$\frac{\text{RXNum}}{\overline{\underline{n} \text{ refutable}}} \quad (2a)$$

$$\frac{\text{RXUnknown}}{\overline{? \text{ refutable}}} \quad (2b)$$

$$\frac{\text{RXInl}}{\overline{\text{inl}(\xi) \text{ refutable}}} \quad (2c)$$

$$\frac{\text{RXInr}}{\overline{\text{inr}(\xi) \text{ refutable}}} \quad (2d)$$

$$\frac{\text{RXPairL} \quad \xi_1 \text{ refutable}}{\overline{(\xi_1, \xi_2) \text{ refutable}}} \quad (2e)$$

$$\frac{\text{RXPairR} \quad \xi_2 \text{ refutable}}{(\xi_1, \xi_2) \text{ refutable}} \quad (2f)$$

$$\frac{\text{RXOr} \quad \xi_1 \text{ refutable} \quad \xi_2 \text{ refutable}}{\xi_1 \vee \xi_2 \text{ refutable}} \quad (2g)$$

$$\boxed{\text{refutable}(\xi)}$$

$$\text{refutable}(\underline{n}) = \text{true} \quad (3a)$$

$$\text{refutable}(\text{?}) = \text{true} \quad (3b)$$

$$\text{refutable}(\text{inl}(\xi)) = \text{refutable}(\xi) \quad (3c)$$

$$\text{refutable}(\text{inr}(\xi)) = \text{refutable}(\xi) \quad (3d)$$

$$\text{refutable}((\xi_1, \xi_2)) = \text{refutable}(\xi_1) \text{ or } \text{refutable}(\xi_2) \quad (3e)$$

$$\text{refutable}(\xi_1 \vee \xi_2) = \text{refutable}(\xi_1) \text{ and } \text{refutable}(\xi_2) \quad (3f)$$

$$\text{Otherwise } \text{refutable}(\xi) = \text{false} \quad (3g)$$

$$\boxed{e \models \xi} \quad e \text{ satisfies } \xi$$

$$\frac{\text{CSTruth}}{e \models \top} \quad (4a)$$

$$\frac{\text{CNum}}{\underline{n} \models n} \quad (4b)$$

$$\frac{\text{CSOrL} \quad e \models \xi_1}{e \models \xi_1 \vee \xi_2} \quad (4c)$$

$$\frac{\text{CSOrR} \quad e \models \xi_2}{e \models \xi_1 \vee \xi_2} \quad (4d)$$

$$\frac{\text{CSInl} \quad e_1 \models \xi_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)} \quad (4e)$$

$$\frac{\text{CSInr} \quad e_2 \models \xi_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)} \quad (4f)$$

$$\frac{\text{CSPair} \quad e_1 \models \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \quad (4g)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \xi_1 \quad \text{prl}(e) \models \xi_2}{e \models (\xi_1, \xi_2)} \quad (4h)$$

$$\boxed{\text{ satisfy}(e, \xi)}$$

$$\text{ satisfy}(e, \top) = \text{ true} \quad (5a)$$

$$\text{ satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \quad (5b)$$

$$\text{ satisfy}(e, \xi_1 \vee \xi_2) = \text{ satisfy}(e, \xi_1) \text{ or } \text{ satisfy}(e, \xi_2) \quad (5c)$$

$$\text{ satisfy}(\text{ inl}_{\tau_2}(e_1), \text{ inl}(\xi_1)) = \text{ satisfy}(e_1, \xi_1) \quad (5d)$$

$$\text{ satisfy}(\text{ inr}_{\tau_1}(e_2), \text{ inr}(\xi_2)) = \text{ satisfy}(e_2, \xi_2) \quad (5e)$$

$$\text{ satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \text{ satisfy}(e_1, \xi_1) \text{ and } \text{ satisfy}(e_2, \xi_2) \quad (5f)$$

$$\text{ satisfy}(\llbracket \cdot \rrbracket^u, (\xi_1, \xi_2)) = \text{ satisfy}(\text{ prl}(\llbracket \cdot \rrbracket^u), \xi_1) \text{ and } \text{ satisfy}(\text{ prr}(\llbracket \cdot \rrbracket^u), \xi_2) \quad (5g)$$

$$\text{ satisfy}(\llbracket e \rrbracket^u, (\xi_1, \xi_2)) = \text{ satisfy}(\text{ prl}(\llbracket e \rrbracket^u), \xi_1) \text{ and } \text{ satisfy}(\text{ prr}(\llbracket e \rrbracket^u), \xi_2) \quad (5h)$$

$$\begin{aligned} \text{ satisfy}(e_1(e_2), (\xi_1, \xi_2)) &= \text{ satisfy}(\text{ prl}(e_1(e_2)), \xi_1) \\ &\quad \text{ and } \text{ satisfy}(\text{ prr}(e_1(e_2)), \xi_2) \end{aligned} \quad (5i)$$

$$\begin{aligned} \text{ satisfy}(\text{ match}(e)\{\hat{r}s\}, (\xi_1, \xi_2)) &= \text{ satisfy}(\text{ prl}(\text{ match}(e)\{\hat{r}s\}), \xi_1) \\ &\quad \text{ and } \text{ satisfy}(\text{ prr}(\text{ match}(e)\{\hat{r}s\}), \xi_2) \end{aligned} \quad (5j)$$

$$\begin{aligned} \text{ satisfy}(\text{ prl}(e), (\xi_1, \xi_2)) &= \text{ satisfy}(\text{ prl}(\text{ prl}(e)), \xi_1) \\ &\quad \text{ and } \text{ satisfy}(\text{ prr}(\text{ prl}(e)), \xi_2) \end{aligned} \quad (5k)$$

$$\begin{aligned} \text{ satisfy}(\text{ prr}(e), (\xi_1, \xi_2)) &= \text{ satisfy}(\text{ prl}(\text{ prr}(e)), \xi_1) \\ &\quad \text{ and } \text{ satisfy}(\text{ prr}(\text{ prr}(e)), \xi_2) \end{aligned} \quad (5l)$$

$$\text{ Otherwise } \text{ satisfy}(e, \xi) = \text{ false} \quad (5m)$$

$$\boxed{e \models? \xi} \quad e \text{ may satisfy } \xi$$

$$\begin{array}{c} \text{CMSUnknown} \\ \hline e \models? ? \end{array} \quad (6a)$$

$$\begin{array}{c} \text{CMSNotIntro} \\ e \text{ notintro} \quad \xi \text{ refutable} \\ \hline e \models? \xi \end{array} \quad (6b)$$

$$\begin{array}{c} \text{CMSOrL} \\ e \models? \xi_1 \quad e \not\models \xi_2 \\ \hline e \models? \xi_1 \vee \xi_2 \end{array} \quad (6c)$$

$$\begin{array}{c} \text{CMSOrR} \\ e \not\models \xi_1 \quad e \models? \xi_2 \\ \hline e \models? \xi_1 \vee \xi_2 \end{array} \quad (6d)$$

$$\begin{array}{c} \text{CMSInl} \\ e_1 \models? \xi_1 \\ \hline \text{ inl}_{\tau_2}(e_1) \models? \text{ inl}(\xi_1) \end{array} \quad (6e)$$

$$\frac{\text{CMSInr} \quad e_2 \models_{\tau} \xi_2}{\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)} \quad (6f)$$

$$\frac{\text{CMSPairL} \quad e_1 \models_{\tau} \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models_{\tau} (\xi_1, \xi_2)} \quad (6g)$$

$$\frac{\text{CMSPairR} \quad e_1 \models \xi_1 \quad e_2 \models_{\tau} \xi_2}{(e_1, e_2) \models_{\tau} (\xi_1, \xi_2)} \quad (6h)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{\tau} \xi_1 \quad e_2 \models_{\tau} \xi_2}{(e_1, e_2) \models_{\tau} (\xi_1, \xi_2)} \quad (6i)$$

$$\boxed{e \models_{\tau}^{\dagger} \xi} \quad e \text{ satisfies or may satisfy } \xi$$

$$\frac{\text{CSMSMay} \quad e \models_{\tau} \xi}{e \models_{\tau}^{\dagger} \xi} \quad (7a)$$

$$\frac{\text{CSMSSat} \quad e \models \xi}{e \models_{\tau}^{\dagger} \xi} \quad (7b)$$

Lemma 1.0.1. $e \not\models \perp$

Proof. By rule induction over Rules (14), we notice that $e \models \perp$ is in syntactic contradiction with all rules, hence not derivable. \square

Lemma 1.0.2. $e \not\models_{\tau} \perp$

Proof. Assume $e \models_{\tau} \perp$. By rule induction over Rules (16) on $e \models_{\tau} \perp$, only one case applies.

Case (16b).

(1) \perp **refutable** by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_{\tau} \perp$ is not derivable. \square

Lemma 1.0.3. $e \not\models_{\tau} \top$

Proof. Assume $e \models_{\tau} \top$. By rule induction over Rules (16) on $e \models_{\tau} \top$, only one case applies.

Case (16b).

(1) \top **refutable** by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_{\text{?}} \top$ is not derivable. \square

Lemma 1.0.4. $e \not\models_{\text{?}}$

Proof. By rule induction over Rules (14), we notice that $e \models_{\text{?}}$ is in syntactic contradiction with all the cases, hence not derivable. \square

Lemma 1.0.5. $e \models_{\text{?}}^{\dagger} \xi$ iff $e \models_{\text{?}}^{\dagger} \xi \vee \perp$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models_{\text{?}}^{\dagger} \xi$ by assumption

By rule induction over Rules (17) on (1).

Case (17a).

(2) $e \models_{\text{?}} \xi$ by assumption
(3) $e \models_{\text{?}} \xi \vee \perp$ by Rule (16c) on (2)
and Lemma 2.0.1
(4) $e \models_{\text{?}}^{\dagger} \xi \vee \perp$ by Rule (17a) on (3)

Case (17b).

(2) $e \models \xi$ by assumption
(3) $e \models \xi \vee \perp$ by Rule (14e) on (2)
(4) $e \models_{\text{?}}^{\dagger} \xi \vee \perp$ by Rule (17b) on (3)

2. Necessity:

(1) $e \models_{\text{?}}^{\dagger} \xi \vee \perp$ by assumption

By rule induction over Rules (17) on (1).

Case (17a).

(2) $e \models_{\text{?}} \xi \vee \perp$ by assumption

By rule induction over Rules (16) on (2), only two of them apply.

Case (16c).

(3) $e \models_{\text{?}} \xi$ by assumption
(4) $e \models_{\text{?}}^{\dagger} \xi$ by Rule (17a) on (3)

Case (16d).

(3) $e \models_{\tau} \perp$

by assumption

(4) $e \not\models_{\tau} \perp$

by Lemma 2.0.2

(3) contradicts (4).

Case (17b).

(2) $e \models \xi \vee \perp$

by assumption

By rule induction over Rules (14) on (2), only two of them apply.

Case (14e).

(3) $e \models \xi$

by assumption

(4) $e \models_{\tau}^{\dagger} \xi$

by Rule (17b) on (3)

Case (14f).

(3) $e \models \perp$

by assumption

(4) $e \not\models \perp$

by Lemma 2.0.1

(3) contradicts (4).

□

Corollary 1.0.1. $\top \models_{\tau}^{\dagger} \xi \text{ iff } \top \models_{\tau}^{\dagger} \xi \vee \perp$

Proof. By Definition 2.1.2 and Lemma 2.0.5.

□

Lemma 1.0.6. *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \not\models \xi_2$ iff $\xi_1 \not\models \xi_2 \vee \perp$*

Proof.

(1) $\xi_1 : \tau$

by assumption

(2) $\xi_2 : \tau$

by assumption

(3) $\perp : \tau$

by Rule (8b)

(4) $\xi_2 \vee \perp : \tau$

by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5) $\xi_1 \not\models \xi_2$

by assumption

To prove $\xi_1 \not\models \xi_2 \vee \perp$, assume $\xi_1 \models \xi_2 \vee \perp$.

(6) $\xi_1 \models \xi_2 \vee \perp$

by assumption

For all e such that $\cdot ; \Delta \vdash e : \tau$ and e **final** we have $e \models_{\tau}^{\dagger} \xi_1$ implies

(7) $e \models \xi_2 \vee \perp$

by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

Case (14e).

- (8) $e \models \xi_2$ by assumption
- (9) $\xi_1 \models \xi_2$ by Definition 2.1.1 on (8)

(5) contradicts (9).

Case (14f).

- (8) $e \models \perp$ by assumption
- (9) $e \not\models \perp$ by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

- (a) $\xi_1 \not\models \xi_2 \vee \perp$

2. Necessity:

- (5) $\xi_1 \not\models \xi_2 \vee \perp$ by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

- (6) $\xi_1 \models \xi_2$ by assumption

For all e such that $\cdot ; \Delta \vdash e : \tau$ and e **final** we have $e \models_{\tau}^{\dagger} \xi_1$ implies

- (7) $e \models \xi_2$ by Definition 2.1.1 on (1) and (2) and (6)
- (8) $e \models \xi_2 \vee \perp$ by Rule (14e) on (7)
- (9) $\xi_1 \models \xi_2 \vee \perp$ by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

- (a) $\xi_1 \not\models \xi_2$

□

Lemma 1.0.7. *If $e \not\models_{\tau}^{\dagger} \xi_1$ and $e \not\models_{\tau}^{\dagger} \xi_2$ then $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$*

Proof. Assume, for the sake of contradiction, that $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$.

- (1) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by assumption
- (2) $e \not\models_{\tau}^{\dagger} \xi_1$ by assumption
- (3) $e \not\models_{\tau}^{\dagger} \xi_2$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

$$(4) \quad e \models \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (14) on (4) and only two of them apply.

Case (14e).

$$(5) \quad e \models \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (17b) on (5)}$$

(6) contradicts (2).

Case (14f).

$$(5) \quad e \models \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (17b) on (5)}$$

(6) contradicts (3).

Case (17a).

$$(4) \quad e \models_{\vdash} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (16) on (4) and only two of them apply.

Case (16c).

$$(5) \quad e \models_{\vdash} \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (17a) on (5)}$$

(6) contradicts (2).

Case (16d).

$$(5) \quad e \models_{\vdash} \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (17a) on (5)}$$

(6) contradicts (3).

The conclusion holds as follows:

$$1. \quad e \not\models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$$

□

Lemma 1.0.8. *If $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$ and $e \not\models_{\vdash}^{\dagger} \xi_1$ then $e \models_{\vdash}^{\dagger} \xi_2$*

Proof.

$$(1) \quad e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

$$(2) \quad e \not\models_{\vdash}^{\dagger} \xi_1 \quad \text{by assumption}$$

By rule induction over Rules (17) on (1).

Case (17b).

(3) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (14) on (3) and only two of them apply.

Case (14e).

(4) $e \models \xi_1$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_1$ by Rule (17b) on (4)

(5) contradicts (2).

Case (14f).

(4) $e \models \xi_2$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_2$ by Rule (17b) on (4)

Case (17a).

(3) $e \models_{\neg} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (3) and only two of them apply.

Case (16c).

(4) $e \models_{\neg} \xi_1$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_1$ by Rule (17a) on (4)

(5) contradicts (2).

Case (16d).

(4) $e \models_{\neg} \xi_2$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_2$ by Rule (17a) on (4)

□

Lemma 1.0.9. *If $e \models_{\neg}^{\dagger} \xi_1$ and $e \models_{\neg}^{\dagger} \xi_2$ then $e \models_{\neg}^{\dagger} \xi_1 \wedge \xi_2$*

Lemma 1.0.10. *If $e \models_{\neg}^{\dagger} \xi_1$ then $e \models_{\neg}^{\dagger} \xi_1 \vee \xi_2$ and $e \models_{\neg}^{\dagger} \xi_2 \vee \xi_1$*

Proof.

(1) $e \models_{\neg}^{\dagger} \xi_1$ by assumption ,

By rule induction over Rules (17) on (1),

Case (17b).

(2) $e \models \xi_1$ by assumption

(3) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (2)

(4) $e \models \xi_2 \vee \xi_1$ by Rule (14f) on (2)

- (5) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17b) on (3)
- (6) $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$ by Rule (17b) on (4)

Case (17a).

- (2) $e \models_{\tau} \xi_1$ by assumption

By case analysis on the result of $\text{satisfy}(e, \xi_2)$.

Case true.

- (3) $\text{satisfy}(e, \xi_2) = \text{true}$ by assumption
- (4) $e \models \xi_2$ by Lemma 2.0.19 on (3)
- (5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)
- (6) $e \models \xi_2 \vee \xi_1$ by Rule (14e) on (4)
- (7) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17b) on (5)
- (8) $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$ by Rule (17b) on (6)

Case false.

- (3) $\text{satisfy}(e, \xi_2) = \text{false}$ by assumption
- (4) $e \not\models \xi_2$ by Lemma 2.0.19 on (3)
- (5) $e \models_{\tau} \xi_1 \vee \xi_2$ by Rule (16c) on (2) and (4)
- (6) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17a) on (5)

□

Lemma 1.0.11. *If $e_1 \models_{\tau}^{\dagger} \xi_1$ then $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$*

Proof.

- (1) $e_1 \models_{\tau}^{\dagger} \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

- (2) $e_1 \models \xi_1$ by assumption
- (3) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ by Rule (14g) on (2)
- (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Rule (17b) on (3)

Case (17a).

- | | |
|-----------------------------------------------------------------------------|----------------------|
| (2) $e_1 \models_{\tau_1} \xi_1$ | by assumption |
| (3) $\text{inl}_{\tau_2}(e_1) \models_{\tau_2} \text{inl}(\xi_1)$ | by Rule (16e) on (2) |
| (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau_2}^{\dagger} \text{inl}(\xi_1)$ | by Rule (17a) on (3) |

□

Lemma 1.0.12. *If $e_2 \models_{\tau_2}^{\dagger} \xi_2$ then $\text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\xi_2)$*

Proof.

- | | |
|--------------------------------------------|---------------|
| (1) $e_2 \models_{\tau_2}^{\dagger} \xi_2$ | by assumption |
|--------------------------------------------|---------------|

By rule induction over Rules (17) on (1).

Case (17b).

- | | |
|-----------------------------------------------------------------------------|----------------------|
| (2) $e_2 \models \xi_2$ | by assumption |
| (3) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ | by Rule (14h) on (2) |
| (4) $\text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\xi_2)$ | by Rule (17b) on (3) |

Case (17a).

- | | |
|-----------------------------------------------------------------------------|----------------------|
| (2) $e_2 \models_{\tau_2} \xi_2$ | by assumption |
| (3) $\text{inl}_{\tau_1}(e_2) \models_{\tau_1} \text{inl}(\xi_2)$ | by Rule (16f) on (2) |
| (4) $\text{inl}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inl}(\xi_2)$ | by Rule (17a) on (3) |

□

Lemma 1.0.13. *If $e_1 \models_{\tau_1}^{\dagger} \xi_1$ and $e_2 \models_{\tau_2}^{\dagger} \xi_2$ then $(e_1, e_2) \models_{\tau_1 \times \tau_2}^{\dagger} (\xi_1, \xi_2)$*

Lemma 1.0.14 (Soundness and Completeness of Refutable Constraints). ξ **refutable** iff $\text{refutable}(\xi) = \text{true}$.

Lemma 1.0.15. *There does not exist such a constraint $\xi_1 \wedge \xi_2$ such that $\xi_1 \wedge \xi_2$ **refutable**.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \wedge \xi_2$ **refutable** is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.16. *There does not exist such a constraint $\xi_1 \vee \xi_2$ such that $\xi_1 \vee \xi_2$ **refutable**.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \vee \xi_2$ **refutable** is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.17. *If e notintro and $e \models \xi$ then ξ ~~refutable~~.*

Proof.

- (1) e notintro by assumption
- (2) $e \models \xi$ by assumption

By rule induction over Rules (14) on (2).

Case (14a).

- (3) $\xi = \top$ by assumption

Assume \top ~~refutable~~. By rule induction over Rules (10), no case applies due to syntactic contradiction.

Therefore, \top ~~refutable~~.

Case (14e),(14f).

- (3) $\xi = \xi_1 \vee \xi_2$ by assumption
- (4) $\xi_1 \vee \xi_2$ ~~refutable~~ by Lemma 2.0.17

Case (14d).

- (3) $\xi = \xi_1 \wedge \xi_2$ by assumption
- (4) $\xi_1 \wedge \xi_2$ ~~refutable~~ by Lemma 2.0.16

Case (14j).

- (3) $\xi = (\xi_1, \xi_2)$ by assumption
- (4) $\text{prl}(e) \models \xi_1$ by assumption
- (5) $\text{prr}(e) \models \xi_2$ by assumption
- (6) $\text{prl}(e)$ notintro by Rule (26e)
- (7) $\text{prr}(e)$ notintro by Rule (26f)
- (8) ξ_1 ~~refutable~~ by IH on (6) and (4)
- (9) ξ_2 ~~refutable~~ by IH on (7) and (5)

Assume (ξ_1, ξ_2) ~~refutable~~. By rule induction over Rules (10) on it, only two cases apply.

Case (10d).

- (10) ξ_1 ~~refutable~~ by assumption

Contradicts (8).

Case (10e).

- (10) ξ_2 ~~refutable~~ by assumption

Contradicts (9).

Therefore, (ξ_1, ξ_2) ~~refutable~~.

Otherwise.

$$(3) \quad e = \underline{n}, \text{inl}_{\tau_2}(e_1), \text{inr}_{\tau_1}(e_2), (e_1, e_2) \quad \text{by assumption}$$

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

□

Lemma 1.0.18 (Soundness and Completeness of Satisfaction Judgment). $e \models \xi$ iff $\text{satisfy}(e, \xi) = \text{true}$.

Proof. We prove soundness and completeness separately.

1. Soundness:

$$(1) \quad e \models \xi \quad \text{by assumption}$$

By rule induction over Rules (14) on (1).

Case (14a).

$$\begin{aligned} (2) \quad \xi &= \top && \text{by assumption} \\ (3) \quad \text{satisfy}(e, \top) &= \text{true} && \text{by Definition 15a} \end{aligned}$$

Case (14b).

$$\begin{aligned} (2) \quad e &= \underline{n} && \text{by assumption} \\ (3) \quad \xi &= \underline{n} && \text{by assumption} \\ (4) \quad \text{satisfy}(\underline{n}, \underline{n}) &= (n = n) = \text{true} && \text{by Definition 15b} \end{aligned}$$

Case (14c).

$$\begin{aligned} (2) \quad e &= \underline{n_1} && \text{by assumption} \\ (3) \quad \xi &= \underline{n_2} && \text{by assumption} \\ (4) \quad n_1 &\neq n_2 && \text{by assumption} \\ (5) \quad \text{satisfy}(\underline{n_1}, \underline{n_2}) &= (n_1 \neq n_2) = \text{true} && \text{by Definition 15c on (4)} \end{aligned}$$

Case (14d).

$$\begin{aligned} (2) \quad \xi &= \xi_1 \wedge \xi_2 && \text{by assumption} \\ (3) \quad e &\models \xi_1 && \text{by assumption} \\ (4) \quad e &\models \xi_2 && \text{by assumption} \\ (5) \quad \text{satisfy}(e, \xi_1) &= \text{true} && \text{by IH on (3)} \end{aligned}$$

- (6) $\text{satisfy}(e, \xi_2) = \text{true}$ by IH on (4)
- (7) $\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1)$ and $\text{satisfy}(e, \xi_2) = \text{true}$
by Definition 15d on (5) and (6)

Case (14e).

- (2) $\xi = \xi_1 \vee \xi_2$ by assumption
- (3) $e \models \xi_1$ by assumption
- (4) $\text{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)
- (5) $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$ or $\text{satisfy}(e, \xi_2) = \text{true}$
by Definition 15e on (4)

Case (14f).

- (2) $\xi = \xi_1 \vee \xi_2$ by assumption
- (3) $e \models \xi_2$ by assumption
- (4) $\text{satisfy}(e, \xi_2) = \text{true}$ by IH on (3)
- (5) $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$ or $\text{satisfy}(e, \xi_2) = \text{true}$
by Definition 15e on (4)

Case (14g).

- (2) $e = \text{inl}_{\tau_2}(e_1)$ by assumption
- (3) $\xi = \text{inl}(\xi_1)$ by assumption
- (4) $e_1 \models \xi_1$ by assumption
- (5) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)
- (6) $\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) = \text{true}$
by Definition 15f on (5)

Case (14h).

- (2) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
- (3) $\xi = \text{inl}(\xi_2)$ by assumption
- (4) $e_2 \models \xi_2$ by assumption
- (5) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (4)
- (6) $\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\xi_2)) = \text{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 15g on (5)

Case (14i).

- (2) $e = (e_1, e_2)$ by assumption
- (3) $\xi = (\xi_1, \xi_2)$ by assumption

- (4) $e_1 \models \xi_1$ by assumption
- (5) $e_2 \models \xi_2$ by assumption
- (6) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)
- (7) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (5)
- (8) $\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$
 $\text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 15h on (6) and (7)

Case (14j).

- (2) $\xi = (\xi_1, \xi_2)$ by assumption
- (3) $e \text{ notintro}$ by assumption
- (4) $\text{prl}(e) \models \xi_1$ by assumption
- (5) $\text{prr}(e) \models \xi_2$ by assumption
- (6) $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$ by IH on (4)
- (7) $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$ by IH on (5)

By rule induction over Rules (26) on (3).

Otherwise.

- (8) $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}\hat{s}\}$
by assumption
- (9) $\text{satisfy}(e, (\xi_1, \xi_2)) =$
 $\text{satisfy}(\text{prl}(e), \xi_1) \text{ and } \text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$
by Definition 15 on (6) and (7)

2. Completeness:

- (1) $\text{satisfy}(e, \xi) = \text{true}$ by assumption

By structural induction on ξ .

Case $\xi = \top$.

- (2) $e \models \top$ by Rule (14a)

Case $\xi = \perp, ?$.

- (2) $\text{satisfy}(e, \xi) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$.

By structural induction on e .

Case $e = \underline{n'}$.

- (2) $n' = n$ by Definition 15b on (1)

(3) $\underline{n}' \models \underline{n}$ by Rule (14b) on (2)

Otherwise.

(2) $\text{satisfy}(e, \underline{n}) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \perp$.

By structural induction on e .

Case $e = \underline{n}'$.

(2) $\underline{n}' \neq \underline{n}$ by Definition 15c on (1)

(3) $\underline{n}' \models \perp$ by Rule (14c) on (2)

Otherwise.

(2) $\text{satisfy}(e, \perp) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$.

(2) $\text{satisfy}(e, \xi_1) = \text{true}$ by Definition 15d on (1)

(3) $\text{satisfy}(e, \xi_2) = \text{true}$ by Definition 15d on (1)

(4) $e \models \xi_1$ by IH on (2)

(5) $e \models \xi_2$ by IH on (3)

(6) $e \models \xi_1 \wedge \xi_2$ by Rule (14d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$.

(2) $\text{satisfy}(e, \xi_1) \text{ or } \text{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (1)

By case analysis on (2).

Case $\text{satisfy}(e, \xi_1) = \text{true}$.

(3) $\text{satisfy}(e, \xi_1) = \text{true}$ by assumption

(4) $e \models \xi_1$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (4)

Case $\text{satisfy}(e, \xi_2) = \text{true}$.

(3) $\text{satisfy}(e, \xi_2) = \text{true}$ by assumption

(4) $e \models \xi_2$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

Case $\xi = \text{inl}(\xi_1)$.

By structural induction on e .

Case $e = \text{inl}_{\tau_2}(e_1)$.

- (2) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 15f on (1)
- (3) $e_1 \models \xi_1$ by IH on (2)
- (4) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ by Rule (14g) on (3)

Otherwise.

- (2) $\text{satisfy}(e, \text{inl}(\xi_1)) = \text{false}$ by Definition 15o
- (2) contradicts (1) and thus vacuously true.

Case $\xi = \text{inr}(\xi_2)$.

By structural induction on e .

Case $e = \text{inr}_{\tau_1}(e_2)$.

- (2) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 15g on (1)
- (3) $e_2 \models \xi_2$ by IH on (2)
- (4) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ by Rule (14h) on (3)

Otherwise.

- (2) $\text{satisfy}(e, \text{inr}(\xi_2)) = \text{false}$ by Definition 15o
- (2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$.

By structural induction on e .

Case $e = (e_1, e_2)$.

- (2) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 15h on (1)
- (3) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 15h on (1)
- (4) $e_1 \models \xi_1$ by IH on (2)
- (5) $e_2 \models \xi_2$ by IH on (3)
- (6) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (4) and (5)

Case $e = (\llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\})$.

- (2) $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$ by Definition 15h on (1)
- (3) $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$ by Definition 15h on (1)
- (4) $\text{prl}(e) \models \xi_1$ by IH on (2)
- (5) $\text{prr}(e) \models \xi_2$ by IH on (3)
- (6) $e \text{ notintro}$ by each rule in Rules (26)
- (7) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14j) on (6) and (4) and (5)

Otherwise.

- (2) $satisfy(e, (\xi_1, \xi_2)) = \text{false}$ by Definition 15o
 (2) contradicts (1) and thus vacuously true.

□

Lemma 1.0.19. $e \not\models \xi$ and $e \not\models_{\text{?}} \xi$ iff $e \not\models_{\text{?}}^{\dagger} \xi$.

Proof. 1. Sufficiency:

- (1) $e \not\models \xi$ by assumption
 (2) $e \not\models_{\text{?}} \xi$ by assumption

Assume $e \models_{\text{?}}^{\dagger} \xi$. By rule induction over Rules (17) on it.

Case (17a).

- (3) $e \models \xi$ by assumption

Contradicts (1).

Case (17b).

- (3) $e \models_{\text{?}} \xi$ by assumption

Contradicts (2).

Therefore, $e \models_{\text{?}}^{\dagger} \xi$ is not derivable.

2. Necessity:

- (1) $e \not\models_{\text{?}}^{\dagger} \xi$ by assumption

Assume $e \models \xi$.

- (2) $e \models_{\text{?}}^{\dagger} \xi$ by Rule (17b) on assumption

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_{\text{?}} \xi$.

- (3) $e \models_{\text{?}}^{\dagger} \xi$ by Rule (17a) on assumption

Contradicts (1). Therefore, $e \not\models_{\text{?}} \xi$.

□

Theorem 1.1 (Exclusiveness of Satisfaction Judgment). *If $\xi : \tau$ and $\Delta \vdash e : \tau$ and e final then exactly one of the following holds*

1. $e \models \xi$
2. $e \models_{\text{?}} \xi$

3. $e \not\models_{\tau}^{\dagger} \xi$

Proof.

- | | |
|-------------------------------------|---------------|
| (4) $\xi : \tau$ | by assumption |
| (5) $\cdot; \Delta \vdash e : \tau$ | by assumption |
| (6) $e \text{ final}$ | by assumption |

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

Case (8a).

- | | |
|----------------------------------------|----------------------|
| (7) $\xi = \top$ | by assumption |
| (8) $e \models \top$ | by Rule (14a) |
| (9) $e \not\models_{\tau} \top$ | by Lemma 2.0.3 |
| (10) $e \models_{\tau}^{\dagger} \top$ | by Rule (17b) on (8) |

Case (8b).

- | | |
|---------------------------------------------|--------------------------------|
| (7) $\xi = \perp$ | by assumption |
| (8) $e \not\models \perp$ | by Lemma 2.0.1 |
| (9) $e \not\models_{\tau} \perp$ | by Lemma 2.0.2 |
| (10) $e \not\models_{\tau}^{\dagger} \perp$ | by Lemma 2.0.20 on (8) and (9) |

Case (1b).

- | | |
|-------------------------------------|----------------------|
| (7) $\xi = ?$ | by assumption |
| (8) $e \not\models ?$ | by Lemma 2.0.4 |
| (9) $e \models_{\tau} ?$ | by Rule (16a) |
| (10) $e \models_{\tau}^{\dagger} ?$ | by Rule (17a) on (9) |

Case (8c).

- | | |
|-----------------------------|---------------|
| (7) $\xi = \underline{n_2}$ | by assumption |
| (8) $\tau = \text{num}$ | by assumption |

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (9) $e = \mathbb{0}^u, (\mathbb{e}_0)^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$
by assumption
- (10) e **notintro**
by Rule
(26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models n_2$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on ξ .

- (11) $e \not\models n_2$
by contradiction
- (12) $\underline{n_2}$ **refutable**
by Rule (10a)
- (13) $e \models_{\text{?}} \underline{n_2}$
by Rule (16b) on (10)
and (12)
- (14) $e \models_{\text{?}}^{\dagger} \underline{n_2}$
by Rule (17a) on (13)

Case (19d).

- (9) $e = \underline{n_1}$
by assumption

Assume $\underline{n_1} \models_{\text{?}} \underline{n_2}$. By rule induction over Rules (16), only one case applies.

Case (16b).

- (10) $\underline{n_1}$ **notintro**
by assumption
- Contradicts Lemma 4.0.4.

- (11) $\underline{n_1} \not\models_{\text{?}} \underline{n_2}$
by contradiction

By case analysis on whether n_1 is equal to n_2 .

Case $n_1 = n_2$.

- (12) $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$
by Definition 15
- (13) $\underline{n_1} \models \underline{n_2}$
by Lemma 2.0.19 on
(12)
- (14) $e \models_{\text{?}}^{\dagger} \underline{n_2}$
by Rule (17b) on (13)

Case $n_1 \neq n_2$.

- (12) $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$
by Definition 15
- (13) $\underline{n_1} \not\models \underline{n_2}$
by Lemma 2.0.19 on
(12)
- (14) $e \not\models_{\text{?}}^{\dagger} \underline{n_2}$
by Lemma 2.0.20 on
(11) and (13)

Case (8f).

- (7) $\xi = \xi_1 \vee \xi_2$
by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models_{\text{?}} \xi_1$, and $e \not\models_{\text{?}}^{\dagger} \xi_1$ holds. The same goes for ξ_2 . By case analysis on which conclusion holds for ξ_1 and ξ_2 .

Case $e \models \xi_1, e \models \xi_2$.

- | | |
|--------------------------------------------|-----------------------|
| (8) $e \models \xi_1$ | by assumption |
| (9) $e \not\models? \xi_1$ | by assumption |
| (10) $e \models \xi_2$ | by assumption |
| (11) $e \not\models? \xi_2$ | by assumption |
| (12) $e \models \xi_1 \vee \xi_2$ | by Rule (14e) on (8) |
| (13) $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (17b) on (12) |

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- | | |
|------------------------------------------|---------------|
| (14) $\xi_1 \vee \xi_2$ refutable | by assumption |
|------------------------------------------|---------------|

Contradicts Lemma 2.0.17.

Case (16c).

- | | |
|-------------------------|---------------|
| (14) $e \models? \xi_1$ | by assumption |
|-------------------------|---------------|

Contradicts (9).

Case (16d).

- | | |
|-------------------------|---------------|
| (14) $e \models? \xi_2$ | by assumption |
|-------------------------|---------------|

Contradicts (11).

- | | |
|----------------------------------------|------------------|
| (15) $e \not\models? \xi_1 \vee \xi_2$ | by contradiction |
|----------------------------------------|------------------|

Case $e \models \xi_1, e \models? \xi_2$.

- | | |
|--------------------------------------------|-----------------------|
| (8) $e \models \xi_1$ | by assumption |
| (9) $e \not\models? \xi_1$ | by assumption |
| (10) $e \not\models \xi_2$ | by assumption |
| (11) $e \models? \xi_2$ | by assumption |
| (12) $e \models \xi_1 \vee \xi_2$ | by Rule (14e) on (8) |
| (13) $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (17b) on (12) |

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- | | |
|------------------------------------------|---------------|
| (14) $\xi_1 \vee \xi_2$ refutable | by assumption |
|------------------------------------------|---------------|

Contradicts Lemma 2.0.17.

Case (16c).

- | | |
|-------------------------|---------------|
| (14) $e \models? \xi_1$ | by assumption |
|-------------------------|---------------|

Contradicts (9).

Case (16d).

- | | |
|----------------------------|---------------|
| (14) $e \not\models \xi_1$ | by assumption |
|----------------------------|---------------|

Contradicts (8).

(15) $e \not\models? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \not\models?^\dagger \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (8)

(13) $e \models?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable** by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models? \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models? \xi_1 \vee \xi_2$ by contradiction

Case $e \models? \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (10)

(13) $e \models?^\dagger \xi_1 \vee \xi_2$ by Rule (17b) on (12)

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable** by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (16d).

(14) $e \models_{\text{?}} \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_{\text{?}} \xi_1 \vee \xi_2$ by contradiction

Case $e \models_{\text{?}} \xi_1, e \models_{\text{?}} \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_{\text{?}} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_{\text{?}} \xi_2$ by assumption

(12) $e \models_{\text{?}} \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \models_{\text{?}} \xi_1, e \not\models_{\text{?}}^{\dagger} \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_{\text{?}} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_{\text{?}} \xi_2$ by assumption

(12) $e \models_{\text{?}} \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(14) $e \models \xi_2$

by assumption

Contradicts (10).

(15) $e \not\models \xi_1 \vee \xi_2$

by contradiction

Case $e \not\models_{\text{?}}^{\dagger} \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$

by assumption

(9) $e \not\models_{\text{?}} \xi_1$

by assumption

(10) $e \models \xi_2$

by assumption

(11) $e \not\models_{\text{?}} \xi_2$

by assumption

(12) $e \models \xi_1 \vee \xi_2$

by Rule (14f) on (10)

(13) $e \models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$

by Rule (17b) on (12)

Assume $e \models_{\text{?}} \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable**

by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$

by assumption

Contradicts (10).

Case (16d).

(14) $e \models_{\text{?}} \xi_2$

by assumption

Contradicts (11).

(15) $e \not\models_{\text{?}} \xi_1 \vee \xi_2$

by contradiction

Case $e \not\models_{\text{?}}^{\dagger} \xi_1, e \models_{\text{?}} \xi_2$.

(8) $e \not\models \xi_1$

by assumption

(9) $e \not\models_{\text{?}} \xi_1$

by assumption

(10) $e \not\models \xi_2$

by assumption

(11) $e \models_{\text{?}} \xi_2$

by assumption

(12) $e \models_{\text{?}} \xi_1 \vee \xi_2$

by Rule (16d) on (11)
and (8)

(13) $e \models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$

by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_{\text{?}}^{\dagger} \xi_1, e \not\models_{\text{?}}^{\dagger} \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{\text{?}} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_{\text{?}} \xi_2$ by assumption

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14) on it, only two cases apply.

Case (14e).

(12) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(12) $e \models \xi_2$ by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Assume $e \models_{\text{?}} \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable** by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_{\text{?}} \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \models_{\text{?}} \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_{\text{?}} \xi_1 \vee \xi_2$ by contradiction

(16) $e \not\models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$ by Lemma 2.0.20 on (13) and (15)

Case (8g).

- (7) $\xi = \text{inl}(\xi_1)$ by assumption
- (8) $\tau = (\tau_1 + \tau_2)$ by assumption
- (9) $\xi_1 : \tau_1$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (10) $e = \emptyset^u, (e_0)^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$
by assumption
- (11) $e \text{ notintro}$ by Rule
(26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (12) $e \not\models \text{inl}(\xi_1)$ by contradiction

By case analysis on the value of $\text{refutable}(\text{inl}(\xi_1))$.

Case $\text{refutable}(\text{inl}(\xi_1)) = \text{true}$.

- (13) $\text{refutable}(\text{inl}(\xi_1)) = \text{true}$ by assumption
- (14) $\text{inl}(\xi_1) \text{ refutable}$ by Lemma 2.0.14 on
(13)
- (15) $e \models? \text{inl}(\xi_1)$ by Rule (16b) on (11)
and (14)
- (16) $e \models?^\dagger \text{inl}(\xi_1)$ by Rule (17a) on (15)

Case $\text{refutable}(\text{inl}(\xi_1)) = \text{false}$.

- (13) $\text{refutable}(\text{inl}(\xi_1)) = \text{false}$ by assumption
- (14) ~~$\text{inl}(\xi_1) \text{ refutable}$~~ by Lemma 2.0.14 on
(13)

Assume $e \models? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

- (15) $\text{inl}(\xi_1) \text{ refutable}$ by assumption
- Contradicts (14).

- (16) $e \not\models? \text{inl}(\xi_1)$ by contradiction
- (17) $e \not\models?^\dagger \text{inl}(\xi_1)$ by Lemma 2.0.20 on
(12) and (16)

Case (19j).

- (10) $e = \text{inl}_{\tau_2}(e_1)$ by assumption

- (11) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
 (12) e_1 **final** by Lemma 4.0.1 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models? \xi_1$, and $e_1 \not\models?^\dagger \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$.

- (13) $e_1 \models \xi_1$ by assumption
 (14) $e_1 \not\models? \xi_1$ by assumption
 (15) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ by Rule (14g) on (13)
 (16) $\text{inl}_{\tau_2}(e_1) \models?^\dagger \text{inl}(\xi_1)$ by Rule (17b) on (15)

Assume $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

- (17) $\text{inl}_{\tau_2}(e_1)$ **notintro** by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

- (17) $e_1 \models? \xi_1$
 Contradicts (14).

- (18) $\text{inl}_{\tau_2}(e_1) \not\models? \text{inl}(\xi_1)$ by contradiction

Case $e_1 \models? \xi_1$.

- (13) $e_1 \not\models \xi_1$ by assumption
 (14) $e_1 \models? \xi_1$ by assumption
 (15) $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ by Rule (16e) on (14)
 (16) $\text{inl}_{\tau_2}(e_1) \models?^\dagger \text{inl}(\xi_1)$ by Rule (17a) on (15)

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

- (17) $e_1 \models \xi_1$
 Contradicts (13).

- (18) $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$ by contradiction

Case $e_1 \not\models?^\dagger \xi_1$.

- (13) $e_1 \not\models \xi_1$ by assumption
 (14) $e_1 \not\models? \xi_1$ by assumption

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

$$(15) \quad e_1 \models \xi_1$$

Contradicts (13).

$$(16) \quad \text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1) \quad \text{by contradiction}$$

Assume $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

$$(17) \quad \text{inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

$$(17) \quad e_1 \models? \xi_1$$

Contradicts (14).

$$(18) \quad \text{inl}_{\tau_2}(e_1) \not\models? \text{inl}(\xi_1) \quad \text{by contradiction}$$

$$(19) \quad \text{inl}_{\tau_2}(e_1) \not\models?^{\dagger} \text{inl}(\xi_1) \quad \text{by Lemma 2.0.20 on (16) and (18)}$$

Case (19k).

$$(10) \quad e = \text{inr}_{\tau_1}(e_2) \quad \text{by assumption}$$

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

$$(11) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\xi_1) \quad \text{by contradiction}$$

Assume $\text{inr}_{\tau_1}(e_2) \models? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

$$(12) \quad \text{inr}_{\tau_1}(e_2) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

$$(13) \quad \text{inr}_{\tau_1}(e_2) \not\models? \text{inl}(\xi_1) \quad \text{by contradiction}$$

$$(14) \quad \text{inr}_{\tau_1}(e_2) \not\models?^{\dagger} \text{inl}(\xi_1) \quad \text{by Lemma 2.0.20 on (11) and (13)}$$

Case (8h).

$$(7) \quad \xi = \text{inr}(\xi_2) \quad \text{by assumption}$$

$$(8) \quad \tau = (\tau_1 + \tau_2) \quad \text{by assumption}$$

$$(9) \quad \xi_2 : \tau_2 \quad \text{by assumption}$$

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (10) $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$
by assumption
- (11) $e \text{ notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (12) $e \not\models \text{inr}(\xi_2)$ by contradiction

By case analysis on the value of $\text{refutable}(\text{inr}(\xi_2))$.

inr is
refutable

Case $\text{refutable}(\text{inr}(\xi_2)) = \text{true}$.

- (13) $\text{refutable}(\text{inr}(\xi_2)) = \text{true}$ by assumption
- (14) $\text{inr}(\xi_2) \text{ refutable}$ by Lemma 2.0.14 on (13)
- (15) $e \models_{\text{?}} \text{inr}(\xi_2)$ by Rule (16b) on (11) and (14)
- (16) $e \models_{\text{?}}^{\dagger} \text{inr}(\xi_2)$ by Rule (17a) on (15)

Case $\text{refutable}(\text{inr}(\xi_2)) = \text{false}$.

- (13) $\text{refutable}(\text{inr}(\xi_2)) = \text{false}$ by assumption
- (14) ~~$\text{inr}(\xi_2) \text{ refutable}$~~ by Lemma 2.0.14 on (13)

Assume $e \models_{\text{?}} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

- (15) $\text{inr}(\xi_2) \text{ refutable}$ by assumption
- Contradicts (14).

- (16) $e \not\models_{\text{?}} \text{inr}(\xi_2)$ by contradiction
- (17) $e \not\models_{\text{?}}^{\dagger} \text{inr}(\xi_2)$ by Lemma 2.0.20 on (12) and (16)

Case (19j).

- (10) $e = \text{inl}_{\tau_2}(e_1)$ by assumption

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (11) $\text{inl}_{\tau_2}(e_1) \not\models \text{inr}(\xi_2)$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \models_{\text{?}} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\text{inl}_{\tau_2}(e_1) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inr}(\xi_2)$ by contradiction

(14) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inr}(\xi_2)$ by Lemma 2.0.20 on (11) and (13)

Case (19k).

(10) $e = \text{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot; \Delta \vdash e_2 : \tau_2$ by assumption

(12) $e_2 \text{ final}$ by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_{\tau} \xi_2$, and $e_2 \not\models_{\tau}^{\dagger} \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

(13) $e_2 \models \xi_2$ by assumption

(14) $e_2 \not\models_{\tau} \xi_2$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ by Rule (14g) on (13)

(16) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ by Rule (17b) on (15)

Assume $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

(17) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_{\tau} \xi_2$

Contradicts (14).

(18) $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\xi_2)$ by contradiction

Case $e_2 \models_{\tau} \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \models_{\tau} \xi_2$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$ by Rule (16f) on (14)

(16) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ by Rule (17a) on (15)

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(17) $e_2 \models \xi_2$
 Contradicts (13).

(18) $\text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2)$ by contradiction

Case $e_2 \not\models_{\tau}^{\dagger} \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \not\models_{\tau} \xi_2$ by assumption

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

(15) $e_2 \models \xi_2$

Contradicts (13).

(16) $\text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2)$ by contradiction

Assume $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(17) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_{\tau} \xi_2$

Contradicts (14).

(18) $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\xi_2)$ by contradiction

(19) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (14i).

(7) $\xi = (\xi_1, \xi_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

(10) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(11) $e = \text{⋈}^u, \text{⋈}_{e_0}^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$
 by assumption

(12) $e \text{ notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

(13) e indet	by Lemma 4.0.8 on (6) and (12)
(14) $\text{prl}(e)$ indet	by Rule (24g) on (13)
(15) $\text{prl}(e)$ final	by Rule (25b) on (14)
(16) $\text{prr}(e)$ indet	by Rule (24h) on (13)
(17) $\text{prr}(e)$ final	by Rule (25b) on (16)
(18) $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$	by Rule (19h) on (5)
(19) $\cdot; \Delta \vdash \text{prr}(e) : \tau_2$	by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\text{prl}(e) \models \xi_1$, $\text{prl}(e) \models? \xi_1$, and $\text{prl}(e) \not\models?^\dagger \xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\text{prr}(e) \models \xi_2$, $\text{prr}(e) \models? \xi_2$, and $\text{prr}(e) \not\models?^\dagger \xi_2$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2 .

Case $\text{prl}(e) \models \xi_1, \text{prr}(e) \models \xi_2$.

(20) $\text{prl}(e) \models \xi_1$	by assumption
(21) $\text{prl}(e) \not\models? \xi_1$	by assumption
(22) $\text{prr}(e) \models \xi_2$	by assumption
(23) $\text{prr}(e) \not\models? \xi_2$	by assumption
(24) $e \models (\xi_1, \xi_2)$	by Rule (14j) on (12) and (20) and (22)
(25) $e \models?^\dagger (\xi_1, \xi_2)$	by Rule (17b) on (24)
(26) (ξ_1, ξ_2) refutable	by Lemma 2.0.18 on (12) and (24)

Assume $e \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(27) (ξ_1, ξ_2) refutable	by assumption
Contradicts (26).	

(28) $e \not\models? (\xi_1, \xi_2)$	by contradiction
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Case $\text{prl}(e) \models \xi_1, \text{prr}(e) \models? \xi_2$.

(20) $\text{prl}(e) \models \xi_1$	by assumption
(21) $\text{prl}(e) \not\models? \xi_1$	by assumption
(22) $\text{prr}(e) \not\models \xi_2$	by assumption
(23) $\text{prr}(e) \models? \xi_2$	by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{prr}(e) \models \xi_2$	by assumption
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Contradicts (22)

(25) $e \not\models (\xi_1, \xi_2)$

by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

(26) ξ_2 **refutable**

by assumption

(27) (ξ_1, ξ_2) **refutable**

by Rule (10e) on (26)

(28) $e \models? (\xi_1, \xi_2)$

by Rule (16b) on (12)
and (27)

(29) $e \models?^\dagger (\xi_1, \xi_2)$

by Rule (17a) on (28)

Case $\text{prl}(e) \models \xi_1, \text{pr}(e) \not\models?^\dagger \xi_2$.

(20) $\text{prl}(e) \models \xi_1$

by assumption

(21) $\text{prl}(e) \not\models? \xi_1$

by assumption

(22) $\text{pr}(e) \not\models \xi_2$

by assumption

(23) $\text{pr}(e) \not\models? \xi_2$

by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it,
only one case applies.

Case (14j).

(24) $\text{pr}(e) \models \xi_2$

by assumption

Contradicts (22).

(25) $e \not\models (\xi_1, \xi_2)$

by contradiction

Assume $e \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it,
only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) **refutable**

by assumption

By rule induction over Rules (10) on (26), only two cases
apply.

Case (10d).

(27) ξ_1 **refutable**

by assumption

(28) $\text{prl}(e)$ **notintro**

by Rule (26e)

(29) $\text{prl}(e) \models? \xi_1$

by Rule (16b) on (28)
and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 **refutable**

by assumption

(28) $\text{pr}(e)$ **notintro**

by Rule (26f)

(29) $\text{pr}(e) \models? \xi_2$

by Rule (16b) on (28)
and (27)

assume no
"or" and
"and" in
pair

Contradicts (23).

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|----------------------------------------------------------|-------------------------------------|
| (30) $e \not\models_{\text{?}} (\xi_1, \xi_2)$ | by contradiction |
| (31) $e \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.20 on
(25) and (30) |

Case $\text{prl}(e) \models_{\text{?}} \xi_1, \text{prr}(e) \models \xi_2$.

- | | |
|---------------------------------------------------|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$ | by assumption |
| (21) $\text{prl}(e) \models_{\text{?}} \xi_1$ | by assumption |
| (22) $\text{prr}(e) \models \xi_2$ | by assumption |
| (23) $\text{prr}(e) \not\models_{\text{?}} \xi_2$ | by assumption |

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

- | | |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
|------------------------------------|---------------|
- Contradicts (20).

- | | |
|-------------------------------------|------------------|
| (25) $e \not\models (\xi_1, \xi_2)$ | by contradiction |
|-------------------------------------|------------------|

By rule induction over Rules (16) on (21), only one case applies.

Case (16b).

- | | |
|------------------------------------------------------|-----------------------------------|
| (26) ξ_1 refutable | by assumption |
| (27) (ξ_1, ξ_2) refutable | by Rule (10e) on (26) |
| (28) $e \models_{\text{?}} (\xi_1, \xi_2)$ | by Rule (16b) on (12)
and (27) |
| (29) $e \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ | by Rule (17a) on (28) |

Case $\text{prl}(e) \models_{\text{?}} \xi_1, \text{prr}(e) \models_{\text{?}} \xi_2$.

- | | |
|-----------------------------------------------|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$ | by assumption |
| (21) $\text{prl}(e) \models_{\text{?}} \xi_1$ | by assumption |
| (22) $\text{prr}(e) \not\models \xi_2$ | by assumption |
| (23) $\text{prr}(e) \models_{\text{?}} \xi_2$ | by assumption |

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

- | | |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
|------------------------------------|---------------|
- Contradicts (20).

- | | |
|-------------------------------------|------------------|
| (25) $e \not\models (\xi_1, \xi_2)$ | by contradiction |
|-------------------------------------|------------------|

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

assume no
"or" and
"and" in
pair

assume no
"or" and
"and" in
pair

- | | |
|------------------------------------------------------|--------------------------------|
| (26) ξ_2 refutable | by assumption |
| (27) (ξ_1, ξ_2) refutable | by Rule (10e) on (26) |
| (28) $e \models_{\text{?}} (\xi_1, \xi_2)$ | by Rule (16b) on (12) and (27) |
| (29) $e \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ | by Rule (17a) on (28) |

Case $\text{prl}(e) \models_{\text{?}} \xi_1, \text{prr}(e) \not\models_{\text{?}}^{\dagger} \xi_2$.

- | | |
|---------------------------------------------------|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$ | by assumption |
| (21) $\text{prl}(e) \models_{\text{?}} \xi_1$ | by assumption |
| (22) $\text{prr}(e) \not\models \xi_2$ | by assumption |
| (23) $\text{prr}(e) \not\models_{\text{?}} \xi_2$ | by assumption |

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

- | | |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
| Contradicts (20) | |

- | | |
|-------------------------------------|------------------|
| (25) $e \not\models (\xi_1, \xi_2)$ | by contradiction |
|-------------------------------------|------------------|

By rule induction over Rules (16) on (21), only one case applies.

Case (16b).

- | | |
|------------------------------------------------------|--------------------------------|
| (26) ξ_1 refutable | by assumption |
| (27) (ξ_1, ξ_2) refutable | by Rule (10e) on (26) |
| (28) $e \models_{\text{?}} (\xi_1, \xi_2)$ | by Rule (16b) on (12) and (27) |
| (29) $e \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ | by Rule (17a) on (28) |

Case $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \xi_1, \text{prr}(e) \models \xi_2$.

- | | |
|---------------------------------------------------|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$ | by assumption |
| (21) $\text{prl}(e) \not\models_{\text{?}} \xi_1$ | by assumption |
| (22) $\text{prr}(e) \models \xi_2$ | by assumption |
| (23) $\text{prr}(e) \not\models_{\text{?}} \xi_2$ | by assumption |

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

- | | |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
| Contradicts (20) | |

- | | |
|-------------------------------------|------------------|
| (25) $e \not\models (\xi_1, \xi_2)$ | by contradiction |
|-------------------------------------|------------------|

Assume $e \models_{\text{?}} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

assume no
"or" and
"and" in
pair

(26) (ξ_1, ξ_2) **refutable** by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 **refutable** by assumption

(28) $\text{prl}(e)$ **notintro** by Rule (26e)

(29) $\text{prl}(e) \models_{\text{?}} \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 **refutable** by assumption

(28) $\text{prr}(e)$ **notintro** by Rule (26f)

(29) $\text{prr}(e) \models_{\text{?}} \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_{\text{?}} (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \xi_1, \text{prr}(e) \models_{\text{?}} \xi_2$.

(20) $\text{prl}(e) \not\models \xi_1$ by assumption

(21) $\text{prl}(e) \not\models_{\text{?}} \xi_1$ by assumption

(22) $\text{prr}(e) \not\models \xi_2$ by assumption

(23) $\text{prr}(e) \models_{\text{?}} \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

(26) ξ_2 **refutable** by assumption

(27) (ξ_1, ξ_2) **refutable** by Rule (10e) on (26)

(28) $e \models_{\text{?}} (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \xi_1, \text{prr}(e) \not\models_{\text{?}}^{\dagger} \xi_2$.

assume no
"or" and
"and" in
pair

- (20) $\text{prl}(e) \not\models \xi_1$ by assumption
- (21) $\text{prl}(e) \not\models? \xi_1$ by assumption
- (22) $\text{prr}(e) \not\models \xi_2$ by assumption
- (23) $\text{prr}(e) \not\models? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

- (24) $\text{prl}(e) \models \xi_1$ by assumption
- Contradicts (20)

- (25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

- (26) (ξ_1, ξ_2) **refutable** by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

- (27) ξ_1 **refutable** by assumption
- (28) $\text{prl}(e)$ **notintro** by Rule (26e)
- (29) $\text{prl}(e) \models? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

- (27) ξ_2 **refutable** by assumption
- (28) $\text{prr}(e)$ **notintro** by Rule (26f)
- (29) $\text{prr}(e) \models? \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

- (30) $e \not\models? (\xi_1, \xi_2)$ by contradiction
- (31) $e \not\models?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case (19g).

- (11) $e = (e_1, e_2)$ by assumption
- (12) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
- (13) $\cdot; \Delta \vdash e_2 : \tau_2$ by assumption
- (14) e_1 **final** by Lemma 4.0.3 on (6)
- (15) e_2 **final** by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models? \xi_1$, and $e_1 \models \bar{\xi}_1$ holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models? \xi_2$, and $e_2 \models \bar{\xi}_2$ holds.

By case analysis on which conclusion holds for ξ_1 and ξ_2 .

Case $e_1 \models \xi_1, e_2 \models \xi_2$.

- (16) $e_1 \models \xi_1$ by assumption
- (17) $e_1 \not\models? \xi_1$ by assumption
- (18) $e_2 \models \xi_2$ by assumption
- (19) $e_2 \not\models? \xi_2$ by assumption
- (20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (16) and (18)
- (21) $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ by Rule (17b) on (20)

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- (22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

- (22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

Case (16h).

- (22) $e_2 \models? \xi_2$ by assumption

Contradicts (19).

Case (16i).

- (22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

- (23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models? \xi_2$.

- (16) $e_1 \models \xi_1$ by assumption
- (17) $e_1 \not\models? \xi_1$ by assumption
- (18) $e_2 \not\models \xi_2$ by assumption
- (19) $e_2 \models? \xi_2$ by assumption
- (20) $(e_1, e_2) \models? (\xi_1, \xi_2)$ by Rule (16h) on (16) and (19)
- (21) $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models? \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction

(24)	$(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$	by Lemma 2.0.20 on (21) and (23)
Case $e_1 \models_? \xi_1, e_2 \models \xi_2$.		
(16)	$e_1 \not\models \xi_1$	by assumption
(17)	$e_1 \models_? \xi_1$	by assumption
(18)	$e_2 \models \xi_2$	by assumption
(19)	$e_2 \not\models_? \xi_2$	by assumption
(20)	$(e_1, e_2) \models_? (\xi_1, \xi_2)$	by Rule (16g) on (17) and (18)
(21)	$(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (20)
Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.		
Case (14j).		
(22)	(e_1, e_2) notintro	by assumption
Contradicts Lemma 4.0.7.		
Case (14i).		
(22)	$e_1 \models \xi_1$	by assumption
Contradicts (16).		
(23)	$(e_1, e_2) \not\models (\xi_1, \xi_2)$	by contradiction
Case $e_1 \models_? \xi_1, e_2 \models_? \xi_2$.		
(16)	$e_1 \not\models \xi_1$	by assumption
(17)	$e_1 \models_? \xi_1$	by assumption
(18)	$e_2 \not\models \xi_2$	by assumption
(19)	$e_2 \models_? \xi_2$	by assumption
(20)	$(e_1, e_2) \models_? (\xi_1, \xi_2)$	by Rule (16i) on (17) and (19)
(21)	$(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (20)
Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.		
Case (14j).		
(22)	(e_1, e_2) notintro	by assumption
Contradicts Lemma 4.0.7.		
Case (14i).		
(22)	$e_1 \models \xi_1$	by assumption
Contradicts (16).		
(23)	$(e_1, e_2) \not\models (\xi_1, \xi_2)$	by contradiction

Case $e_1 \models_{\text{?}} \xi_1, e_2 \not\models_{\text{?}}^{\dagger} \xi_2$.

- (16) $e_1 \not\models \xi_1$ by assumption
- (17) $e_1 \models_{\text{?}} \xi_1$ by assumption
- (18) $e_2 \not\models \xi_2$ by assumption
- (19) $e_2 \not\models_{\text{?}} \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

- (20) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (14i).

- (20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

- (21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- (22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

- (22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

Case (16h).

- (22) $e_2 \models_{\text{?}} \xi_2$ by assumption

Contradicts (19).

Case (16i).

- (22) $e_2 \models_{\text{?}} \xi_2$ by assumption

Contradicts (19).

- (23) $(e_1, e_2) \not\models_{\text{?}} (\xi_1, \xi_2)$ by contradiction

- (24) $(e_1, e_2) \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_{\text{?}}^{\dagger} \xi_1, e_2 \models \xi_2$.

- (16) $e_1 \not\models \xi_1$ by assumption
- (17) $e_1 \not\models_{\text{?}} \xi_1$ by assumption
- (18) $e_2 \models \xi_2$ by assumption
- (19) $e_2 \not\models_{\text{?}} \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models?^\dagger \xi_1, e_2 \models? \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction
 Assume $(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).
 (22) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.

Case (16g).
 (22) $e_1 \models_{\text{?}} \xi_1$ by assumption
 Contradicts (17).

Case (16h).
 (22) $e_1 \models \xi_1$ by assumption
 Contradicts (16).

Case (16i).
 (22) $e_1 \models_{\text{?}} \xi_1$ by assumption
 Contradicts (17).

(23) $(e_1, e_2) \not\models_{\text{?}} (\xi_1, \xi_2)$ by contradiction
 (24) $(e_1, e_2) \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_{\text{?}}^{\dagger} \xi_1, e_2 \not\models_{\text{?}}^{\dagger} \xi_2$.
 (16) $e_1 \not\models \xi_1$ by assumption
 (17) $e_1 \not\models_{\text{?}} \xi_1$ by assumption
 (18) $e_2 \not\models \xi_2$ by assumption
 (19) $e_2 \not\models_{\text{?}} \xi_2$ by assumption
 Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).
 (20) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.

Case (14i).
 (20) $e_2 \models \xi_2$ by assumption
 Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction
 Assume $(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).
 (22) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_{\tau} \xi_1$	by assumption
Contradicts (17).	
Case (16h).	
(22) $e_2 \models_{\tau} \xi_2$	by assumption
Contradicts (19).	
Case (16i).	
(22) $e_1 \models_{\tau} \xi_1$	by assumption
Contradicts (17).	
(23) $(e_1, e_2) \not\models_{\tau} (\xi_1, \xi_2)$	by contradiction
(24) $(e_1, e_2) \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$	by Lemma 2.0.20 on (21) and (23)

□

Definition 1.1.1 (Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e **val** we have $e \models_{\tau}^{\dagger} \xi_1$ implies $e \models \xi_2$*

Definition 1.1.2 (Potential Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models_{\tau}^{\dagger} \xi_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e **final** we have $e \models_{\tau}^{\dagger} \xi_1$ implies $e \models_{\tau}^{\dagger} \xi_2$*

Corollary 1.1.1. *Suppose that $\xi : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e **final**. Then $\top \models_{\tau}^{\dagger} \xi$ implies $e \models_{\tau}^{\dagger} \xi$*

Proof.

(1) $\xi : \tau$	by assumption
(2) $\cdot ; \Gamma \vdash e : \tau$	by assumption
(3) e final	by assumption
(4) $\top \models_{\tau}^{\dagger} \xi$	by assumption
(5) $e_1 \models \top$	by Rule (14a)
(6) $e_1 \models_{\tau}^{\dagger} \top$	by Rule (17b) on (5)
(7) $\top : \tau$	by Rule (8a)
(8) $e_1 \models_{\tau}^{\dagger} \xi_r$	by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

□

2 Match Constraint Language

$\xi ::= \top \mid \perp \mid \underline{n} \mid \underline{\mathcal{N}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \text{inl}(\xi) \mid \text{inr}(\xi) \mid (\xi_1, \xi_2)$
 $\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\frac{\text{CTTruth}}{\overline{\top : \tau}} \quad (8a)$$

$$\frac{\text{CTFalsity}}{\overline{\perp : \tau}} \quad (8b)$$

$$\frac{\text{CTNum}}{\overline{\underline{n} : \text{num}}} \quad (8c)$$

$$\frac{\text{CTNotNum}}{\overline{\underline{\mathcal{N}} : \text{num}}} \quad (8d)$$

$$\frac{\text{CTAnd} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \quad (8e)$$

$$\frac{\text{CTOr} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \quad (8f)$$

$$\frac{\text{CTInl} \quad \xi_1 : \tau_1}{\text{inl}(\xi_1) : (\tau_1 + \tau_2)} \quad (8g)$$

$$\frac{\text{CTInr} \quad \xi_2 : \tau_2}{\text{inr}(\xi_2) : (\tau_1 + \tau_2)} \quad (8h)$$

$$\frac{\text{CTPair} \quad \xi_1 : \tau_1 \quad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \quad (8i)$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\overline{\top} = \perp \quad (9a)$$

$$\overline{\perp} = \top \quad (9b)$$

$$\overline{n} = \neg \quad (9c)$$

$$\overline{\neg} = n \quad (9d)$$

$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \quad (9e)$$

$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \quad (9f)$$

$$\overline{\text{inl}(\xi_1)} = \text{inl}(\overline{\xi_1}) \vee \text{inr}(\top) \quad (9g)$$

$$\overline{\text{inr}(\xi_2)} = \text{inr}(\overline{\xi_2}) \vee \text{inl}(\top) \quad (9h)$$

$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \quad (9i)$$

ξ **refutable**

ξ is refutable

$$\frac{\text{RXNum}}{n \text{ **refutable**}} \quad (10a)$$

$$\frac{\text{RXInl}}{\text{inl}(\xi) \text{ **refutable**}} \quad (10b)$$

$$\frac{\text{RXInr}}{\text{inr}(\xi) \text{ **refutable**}} \quad (10c)$$

$$\frac{\text{RXPairL} \quad \xi_1 \text{ **refutable**}}{(\xi_1, \xi_2) \text{ **refutable**}} \quad (10d)$$

$$\frac{\text{RXPairR} \quad \xi_2 \text{ **refutable**}}{(\xi_1, \xi_2) \text{ **refutable**}} \quad (10e)$$

$$\frac{\text{RXOr} \quad \xi_1 \text{ **refutable**} \quad \xi_2 \text{ **refutable**}}{\xi_1 \vee \xi_2 \text{ **refutable**}} \quad (10f)$$

$\text{refutable}(\xi)$

$$\text{refutable}(n) = \text{true} \quad (11a)$$

$$\text{refutable}(\neg) = \text{true} \quad (11b)$$

$$\text{refutable}(\top) = \text{true} \quad (11c)$$

$$\text{refutable}(\text{inl}(\xi)) = \text{refutable}(\xi) \quad (11d)$$

$$\text{refutable}(\text{inr}(\xi)) = \text{refutable}(\xi) \quad (11e)$$

$$\text{refutable}((\xi_1, \xi_2)) = \text{refutable}(\xi_1) \text{ or } \text{refutable}(\xi_2) \quad (11f)$$

$$\text{refutable}(\xi_1 \vee \xi_2) = \text{refutable}(\xi_1) \text{ and } \text{refutable}(\xi_2) \quad (11g)$$

$$\text{Otherwise } \text{refutable}(\xi) = \text{false} \quad (11h)$$

$$\boxed{\dot{\top}(\xi_1) = \xi_2}$$

$$\dot{\top}(\top) = \top \quad (12a)$$

$$\dot{\top}(\perp) = \perp \quad (12b)$$

$$\dot{\top}(?) = \top \quad (12c)$$

$$\dot{\top}(\underline{n}) = \underline{n} \quad (12d)$$

$$\dot{\top}(\underline{\mathscr{N}}) = \underline{\mathscr{N}} \quad (12e)$$

$$\dot{\top}(\xi_1 \wedge \xi_2) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \quad (12f)$$

$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad (12g)$$

$$\dot{\top}(\mathbf{inl}(\xi)) = \mathbf{inl}(\dot{\top}(\xi)) \quad (12h)$$

$$\dot{\top}(\mathbf{inr}(\xi)) = \mathbf{inr}(\dot{\top}(\xi)) \quad (12i)$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \quad (12j)$$

$$\boxed{\dot{\perp}(\xi_1) = \xi_2}$$

$$\dot{\perp}(\top) = \top \quad (13a)$$

$$\dot{\perp}(\perp) = \perp \quad (13b)$$

$$\dot{\perp}(?) = \perp \quad (13c)$$

$$\dot{\perp}(\underline{n}) = \underline{n} \quad (13d)$$

$$\dot{\perp}(\underline{\mathscr{N}}) = \underline{\mathscr{N}} \quad (13e)$$

$$\dot{\perp}(\xi_1 \wedge \xi_2) = \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2) \quad (13f)$$

$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \quad (13g)$$

$$\dot{\perp}(\mathbf{inl}(\xi)) = \mathbf{inl}(\dot{\perp}(\xi)) \quad (13h)$$

$$\dot{\perp}(\mathbf{inr}(\xi)) = \mathbf{inr}(\dot{\perp}(\xi)) \quad (13i)$$

$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \quad (13j)$$

$$\boxed{e \models \xi} \quad e \text{ satisfies } \xi$$

$$\frac{\text{CSTruth}}{e \models \top} \quad (14a)$$

$$\frac{\text{CSNum}}{\underline{n} \models \underline{n}} \quad (14b)$$

$$\frac{\text{CSNotNum}}{\underline{n_1} \models \underline{\mathscr{N}_2}} \quad (14c)$$

$$\frac{\text{CSAnd} \quad e \models \xi_1 \quad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \quad (14d)$$

$$\frac{\text{CSOrL} \quad e \models \xi_1}{e \models \xi_1 \vee \xi_2} \quad (14e)$$

$$\frac{\text{CSOrR} \quad e \models \xi_2}{e \models \xi_1 \vee \xi_2} \quad (14f)$$

$$\frac{\text{CSInl} \quad e_1 \models \xi_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)} \quad (14g)$$

$$\frac{\text{CSInr} \quad e_2 \models \xi_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)} \quad (14h)$$

$$\frac{\text{CSPair} \quad e_1 \models \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \quad (14i)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \xi_1 \quad \text{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)} \quad (14j)$$

$satisfy(e, \xi)$

$$\text{satisfy}(e, \top) = \text{true} \quad (15a)$$

$$\text{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \quad (15b)$$

$$\text{satisfy}(\underline{n_1}, \underline{\neg n_2}) = (n_1 \neq n_2) \quad (15c)$$

$$\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1) \text{ and } \text{satisfy}(e, \xi_2) \quad (15d)$$

$$\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1) \text{ or } \text{satisfy}(e, \xi_2) \quad (15e)$$

$$\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) \quad (15f)$$

$$\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\xi_2)) = \text{satisfy}(e_2, \xi_2) \quad (15g)$$

$$\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) \quad (15h)$$

$$\text{satisfy}(\llbracket \cdot \rrbracket^u, (\xi_1, \xi_2)) = \text{satisfy}(\text{prl}(\llbracket \cdot \rrbracket^u), \xi_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket \cdot \rrbracket^u), \xi_2) \quad (15i)$$

$$\text{satisfy}(\llbracket e \rrbracket^u, (\xi_1, \xi_2)) = \text{satisfy}(\text{prl}(\llbracket e \rrbracket^u), \xi_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket e \rrbracket^u), \xi_2) \quad (15j)$$

$$\begin{aligned} \text{satisfy}(e_1(e_2), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(e_1(e_2)), \xi_1) \\ &\text{and } \text{satisfy}(\text{pr}(e_1(e_2)), \xi_2) \end{aligned} \quad (15k)$$

$$\begin{aligned} \text{satisfy}(\text{match}(e)\{\hat{r}s\}, (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{match}(e)\{\hat{r}s\}), \xi_1) \\ &\text{and } \text{satisfy}(\text{pr}(\text{match}(e)\{\hat{r}s\}), \xi_2) \end{aligned} \quad (15l)$$

$$\begin{aligned} \text{satisfy}(\text{prl}(e), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{prl}(e)), \xi_1) \\ &\text{and } \text{satisfy}(\text{pr}(\text{prl}(e)), \xi_2) \end{aligned} \quad (15m)$$

$$\begin{aligned} \text{satisfy}(\text{pr}(e), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{pr}(e)), \xi_1) \\ &\text{and } \text{satisfy}(\text{pr}(\text{pr}(e)), \xi_2) \end{aligned} \quad (15n)$$

$$\text{Otherwise } \text{satisfy}(e, \xi) = \text{false} \quad (15o)$$

$$\boxed{e \models? \xi} \quad e \text{ may satisfy } \xi$$

$$\frac{\text{CMSUnknown}}{e \models? ?} \quad (16a)$$

$$\frac{\text{CMSNotIntro} \quad e \text{ notintro} \quad \xi \text{ refutable}}{e \models? \xi} \quad (16b)$$

$$\frac{\text{CMSOrL} \quad e \models? \xi_1 \quad e \not\models \xi_2}{e \models? \xi_1 \vee \xi_2} \quad (16c)$$

$$\frac{\text{CMSOrR} \quad e \not\models \xi_1 \quad e \models? \xi_2}{e \models? \xi_1 \vee \xi_2} \quad (16d)$$

$$\frac{\text{CMSInl} \quad e_1 \models? \xi_1}{\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)} \quad (16e)$$

$$\frac{\text{CMSInr} \quad e_2 \models_{\text{?}} \xi_2}{\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\xi_2)} \quad (16f)$$

$$\frac{\text{CMSPairL} \quad e_1 \models_{\text{?}} \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (16g)$$

$$\frac{\text{CMSPairR} \quad e_1 \models \xi_1 \quad e_2 \models_{\text{?}} \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (16h)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{\text{?}} \xi_1 \quad e_2 \models_{\text{?}} \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (16i)$$

$$\boxed{e \models_{\text{?}}^{\dagger} \xi} \quad e \text{ satisfies or may satisfy } \xi$$

$$\frac{\text{CSMSMay} \quad e \models_{\text{?}} \xi}{e \models_{\text{?}}^{\dagger} \xi} \quad (17a)$$

$$\frac{\text{CSMSSat} \quad e \models \xi}{e \models_{\text{?}}^{\dagger} \xi} \quad (17b)$$

Lemma 2.0.1. $e \not\models \perp$

Proof. By rule induction over Rules (14), we notice that $e \models \perp$ is in syntactic contradiction with all rules, hence not derivable. \square

Lemma 2.0.2. $e \not\models_{\text{?}} \perp$

Proof. Assume $e \models_{\text{?}} \perp$. By rule induction over Rules (16) on $e \models_{\text{?}} \perp$, only one case applies.

Case (16b).

(1) \perp **refutable** by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_{\text{?}} \perp$ is not derivable. \square

Lemma 2.0.3. $e \not\models_{\text{?}} \top$

Proof. Assume $e \models_{\text{?}} \top$. By rule induction over Rules (16) on $e \models_{\text{?}} \top$, only one case applies.

Case (16b).

(1) \top **refutable** by assumption

By rule induction over Rules (10) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_{\text{?}} \top$ is not derivable. \square

Lemma 2.0.4. $e \not\models_{\text{?}} ?$

Proof. By rule induction over Rules (14), we notice that $e \models_{\text{?}} ?$ is in syntactic contradiction with all the cases, hence not derivable. \square

Lemma 2.0.5. $e \models_{\text{?}}^{\dagger} \xi$ iff $e \models_{\text{?}}^{\dagger} \xi \vee \perp$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models_{\text{?}}^{\dagger} \xi$ by assumption

By rule induction over Rules (17) on (1).

Case (17a).

(2) $e \models_{\text{?}} \xi$ by assumption
(3) $e \models_{\text{?}} \xi \vee \perp$ by Rule (16c) on (2)
and Lemma 2.0.1
(4) $e \models_{\text{?}}^{\dagger} \xi \vee \perp$ by Rule (17a) on (3)

Case (17b).

(2) $e \models \xi$ by assumption
(3) $e \models \xi \vee \perp$ by Rule (14e) on (2)
(4) $e \models_{\text{?}}^{\dagger} \xi \vee \perp$ by Rule (17b) on (3)

2. Necessity:

(1) $e \models_{\text{?}}^{\dagger} \xi \vee \perp$ by assumption

By rule induction over Rules (17) on (1).

Case (17a).

(2) $e \models_{\text{?}} \xi \vee \perp$ by assumption

By rule induction over Rules (16) on (2), only two of them apply.

Case (16c).

(3) $e \models_{\text{?}} \xi$ by assumption
(4) $e \models_{\text{?}}^{\dagger} \xi$ by Rule (17a) on (3)

Case (16d).

(3) $e \models_{\tau} \perp$

by assumption

(4) $e \not\models_{\tau} \perp$

by Lemma 2.0.2

(3) contradicts (4).

Case (17b).

(2) $e \models \xi \vee \perp$

by assumption

By rule induction over Rules (14) on (2), only two of them apply.

Case (14e).

(3) $e \models \xi$

by assumption

(4) $e \models_{\tau}^{\dagger} \xi$

by Rule (17b) on (3)

Case (14f).

(3) $e \models \perp$

by assumption

(4) $e \not\models \perp$

by Lemma 2.0.1

(3) contradicts (4).

□

Corollary 2.0.1. $\top \models_{\tau}^{\dagger} \xi \text{ iff } \top \models_{\tau}^{\dagger} \xi \vee \perp$

Proof. By Definition 2.1.2 and Lemma 2.0.5.

□

Lemma 2.0.6. *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \not\models \xi_2$ iff $\xi_1 \not\models \xi_2 \vee \perp$*

Proof.

(1) $\xi_1 : \tau$

by assumption

(2) $\xi_2 : \tau$

by assumption

(3) $\perp : \tau$

by Rule (8b)

(4) $\xi_2 \vee \perp : \tau$

by Rule (8f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5) $\xi_1 \not\models \xi_2$

by assumption

To prove $\xi_1 \not\models \xi_2 \vee \perp$, assume $\xi_1 \models \xi_2 \vee \perp$.

(6) $\xi_1 \models \xi_2 \vee \perp$

by assumption

For all e such that $\cdot ; \Delta \vdash e : \tau$ and e **final** we have $e \models_{\tau}^{\dagger} \xi_1$ implies

(7) $e \models \xi_2 \vee \perp$

by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (14) on (7).

Case (14e).

- (8) $e \models \xi_2$ by assumption
- (9) $\xi_1 \models \xi_2$ by Definition 2.1.1 on (8)

(5) contradicts (9).

Case (14f).

- (8) $e \models \perp$ by assumption
- (9) $e \not\models \perp$ by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

- (a) $\xi_1 \not\models \xi_2 \vee \perp$

2. Necessity:

- (5) $\xi_1 \not\models \xi_2 \vee \perp$ by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

- (6) $\xi_1 \models \xi_2$ by assumption

For all e such that $\cdot ; \Delta \vdash e : \tau$ and e **final** we have $e \models_{\tau}^{\dagger} \xi_1$ implies

- (7) $e \models \xi_2$ by Definition 2.1.1 on (1) and (2) and (6)
- (8) $e \models \xi_2 \vee \perp$ by Rule (14e) on (7)
- (9) $\xi_1 \models \xi_2 \vee \perp$ by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

- (a) $\xi_1 \not\models \xi_2$

□

Lemma 2.0.7. *If $e \not\models_{\tau}^{\dagger} \xi_1$ and $e \not\models_{\tau}^{\dagger} \xi_2$ then $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$*

Proof. Assume, for the sake of contradiction, that $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$.

- (1) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by assumption
- (2) $e \not\models_{\tau}^{\dagger} \xi_1$ by assumption
- (3) $e \not\models_{\tau}^{\dagger} \xi_2$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

$$(4) \quad e \models \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (14) on (4) and only two of them apply.

Case (14e).

$$(5) \quad e \models \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (17b) on (5)}$$

(6) contradicts (2).

Case (14f).

$$(5) \quad e \models \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (17b) on (5)}$$

(6) contradicts (3).

Case (17a).

$$(4) \quad e \models_{\vdash} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (16) on (4) and only two of them apply.

Case (16c).

$$(5) \quad e \models_{\vdash} \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (17a) on (5)}$$

(6) contradicts (2).

Case (16d).

$$(5) \quad e \models_{\vdash} \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (17a) on (5)}$$

(6) contradicts (3).

The conclusion holds as follows:

$$1. \quad e \not\models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$$

□

Lemma 2.0.8. *If $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$ and $e \not\models_{\vdash}^{\dagger} \xi_1$ then $e \models_{\vdash}^{\dagger} \xi_2$*

Proof.

$$(1) \quad e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

$$(2) \quad e \not\models_{\vdash}^{\dagger} \xi_1 \quad \text{by assumption}$$

By rule induction over Rules (17) on (1).

Case (17b).

(3) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (14) on (3) and only two of them apply.

Case (14e).

(4) $e \models \xi_1$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_1$ by Rule (17b) on (4)

(5) contradicts (2).

Case (14f).

(4) $e \models \xi_2$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_2$ by Rule (17b) on (4)

Case (17a).

(3) $e \models_{\neg} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (3) and only two of them apply.

Case (16c).

(4) $e \models_{\neg} \xi_1$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_1$ by Rule (17a) on (4)

(5) contradicts (2).

Case (16d).

(4) $e \models_{\neg} \xi_2$ by assumption

(5) $e \models_{\neg}^{\dagger} \xi_2$ by Rule (17a) on (4)

□

Lemma 2.0.9. *If $e \models_{\neg}^{\dagger} \xi_1$ and $e \models_{\neg}^{\dagger} \xi_2$ then $e \models_{\neg}^{\dagger} \xi_1 \wedge \xi_2$*

Lemma 2.0.10. *If $e \models_{\neg}^{\dagger} \xi_1$ then $e \models_{\neg}^{\dagger} \xi_1 \vee \xi_2$ and $e \models_{\neg}^{\dagger} \xi_2 \vee \xi_1$*

Proof.

(1) $e \models_{\neg}^{\dagger} \xi_1$ by assumption ,

By rule induction over Rules (17) on (1),

Case (17b).

(2) $e \models \xi_1$ by assumption

(3) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (2)

(4) $e \models \xi_2 \vee \xi_1$ by Rule (14f) on (2)

- (5) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17b) on (3)
- (6) $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$ by Rule (17b) on (4)

Case (17a).

- (2) $e \models_{\tau} \xi_1$ by assumption

By case analysis on the result of $\text{satisfy}(e, \xi_2)$.

Case true.

- (3) $\text{satisfy}(e, \xi_2) = \text{true}$ by assumption
- (4) $e \models \xi_2$ by Lemma 2.0.19 on (3)
- (5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)
- (6) $e \models \xi_2 \vee \xi_1$ by Rule (14e) on (4)
- (7) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by Rule (17b) on (5)
- (8) $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$ by Rule (17b) on (6)

Case false.

- (3) $\text{satisfy}(e, \xi_2) = \text{false}$ by assumption
- (4) $e \not\models \xi_2$ by Lemma 2.0.19 on (3)
- (5) $e \models_{\tau} \xi_1 \vee \xi_2$ by Rule (16c) on (2) and (4)
- (6) $e \models_{\tau}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Rule (17a) on (5)

□

Lemma 2.0.11. *If $e_1 \models_{\tau}^{\dagger} \xi_1$ then $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$*

Proof.

- (1) $e_1 \models_{\tau}^{\dagger} \xi_1$ by assumption

By rule induction over Rules (17) on (1).

Case (17b).

- (2) $e_1 \models \xi_1$ by assumption
- (3) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ by Rule (14g) on (2)
- (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Rule (17b) on (3)

Case (17a).

- | | |
|---------------------------------------------------------------------------|----------------------|
| (2) $e_1 \models_{\tau} \xi_1$ | by assumption |
| (3) $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\xi_1)$ | by Rule (16e) on (2) |
| (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ | by Rule (17a) on (3) |

□

Lemma 2.0.12. *If $e_2 \models_{\tau}^{\dagger} \xi_2$ then $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$*

Proof.

- | | |
|------------------------------------------|---------------|
| (1) $e_2 \models_{\tau}^{\dagger} \xi_2$ | by assumption |
|------------------------------------------|---------------|

By rule induction over Rules (17) on (1).

Case (17b).

- | | |
|---------------------------------------------------------------------------|----------------------|
| (2) $e_2 \models \xi_2$ | by assumption |
| (3) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ | by Rule (14h) on (2) |
| (4) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ | by Rule (17b) on (3) |

Case (17a).

- | | |
|---------------------------------------------------------------------------|----------------------|
| (2) $e_2 \models_{\tau} \xi_2$ | by assumption |
| (3) $\text{inl}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$ | by Rule (16f) on (2) |
| (4) $\text{inl}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ | by Rule (17a) on (3) |

□

Lemma 2.0.13. *If $e_1 \models_{\tau}^{\dagger} \xi_1$ and $e_2 \models_{\tau}^{\dagger} \xi_2$ then $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$*

Lemma 2.0.14 (Soundness and Completeness of Refutable Constraints). ξ **refutable** iff $\text{refutable}(\xi) = \text{true}$.

Lemma 2.0.15. *If e **notintro** and ξ **refutable** then either $\dot{\vdash}(\xi)$ **refutable** or $e \models \dot{\vdash}(\xi)$.*

Proof. By structural induction on ξ . □

Lemma 2.0.16. *There does not exist such a constraint $\xi_1 \wedge \xi_2$ such that $\xi_1 \wedge \xi_2$ **refutable**.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \wedge \xi_2$ **refutable** is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 2.0.17. *There does not exist such a constraint $\xi_1 \vee \xi_2$ such that $\xi_1 \vee \xi_2$ **refutable**.*

Proof. By rule induction over Rules (10), we notice that $\xi_1 \vee \xi_2$ **refutable** is in syntactic contradiction with all the cases, hence not derivable. \square

Lemma 2.0.18. *If e **notintro** and $e \models \xi$ then ξ **refutable**.*

Proof.

- | | |
|-------------------------|---------------|
| (1) e notintro | by assumption |
| (2) $e \models \xi$ | by assumption |

By rule induction over Rules (14) on (2).

Case (14a).

- | | |
|------------------|---------------|
| (3) $\xi = \top$ | by assumption |
|------------------|---------------|

Assume \top **refutable**. By rule induction over Rules (10), no case applies due to syntactic contradiction.
Therefore, \top **refutable**.

Case (14e),(14f).

- | | |
|-----------------------------------------|-----------------|
| (3) $\xi = \xi_1 \vee \xi_2$ | by assumption |
| (4) $\xi_1 \vee \xi_2$ refutable | by Lemma 2.0.17 |

Case (14d).

- | | |
|-------------------------------------------|-----------------|
| (3) $\xi = \xi_1 \wedge \xi_2$ | by assumption |
| (4) $\xi_1 \wedge \xi_2$ refutable | by Lemma 2.0.16 |

Case (14j).

- | | |
|-------------------------------------|----------------------|
| (3) $\xi = (\xi_1, \xi_2)$ | by assumption |
| (4) $\text{prl}(e) \models \xi_1$ | by assumption |
| (5) $\text{prr}(e) \models \xi_2$ | by assumption |
| (6) $\text{prl}(e)$ notintro | by Rule (26e) |
| (7) $\text{prr}(e)$ notintro | by Rule (26f) |
| (8) ξ_1 refutable | by IH on (6) and (4) |
| (9) ξ_2 refutable | by IH on (7) and (5) |

Assume (ξ_1, ξ_2) **refutable**. By rule induction over Rules (10) on it, only two cases apply.

Case (10d).

- | | |
|-------------------------------|---------------|
| (10) ξ_1 refutable | by assumption |
|-------------------------------|---------------|

Contradicts (8).

Case (10e).

(10) ξ_2 **refutable**

by assumption

Contradicts (9).

Therefore, ~~(ξ_1, ξ_2) **refutable**~~.

Otherwise.

(3) $e = \underline{n}, \text{inl}_{\tau_2}(e_1), \text{inr}_{\tau_1}(e_2), (e_1, e_2)$

by assumption

By rule induction over Rules (26) on (1), no case applies due to syntactic contradiction.

□

Lemma 2.0.19 (Soundness and Completeness of Satisfaction Judgment). $e \models \xi$ iff $\text{satisfy}(e, \xi) = \text{true}$.

Proof. We prove soundness and completeness separately.

1. Soundness:

(1) $e \models \xi$

by assumption

By rule induction over Rules (14) on (1).

Case (14a).

(2) $\xi = \top$

by assumption

(3) $\text{satisfy}(e, \top) = \text{true}$

by Definition 15a

Case (14b).

(2) $e = \underline{n}$

by assumption

(3) $\xi = \underline{n}$

by assumption

(4) $\text{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$

by Definition 15b

Case (14c).

(2) $e = \underline{n_1}$

by assumption

(3) $\xi = \underline{\neg}$

by assumption

(4) $n_1 \neq n_2$

by assumption

(5) $\text{satisfy}(\underline{n_1}, \underline{\neg}) = (n_1 \neq n_2) = \text{true}$

by Definition 15c on (4)

Case (14d).

(2) $\xi = \xi_1 \wedge \xi_2$

by assumption

(3) $e \models \xi_1$

by assumption

- (4) $e \models \xi_2$ by assumption
- (5) $\text{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)
- (6) $\text{satisfy}(e, \xi_2) = \text{true}$ by IH on (4)
- (7) $\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1)$ and $\text{satisfy}(e, \xi_2) = \text{true}$
by Definition 15d on (5) and (6)

Case (14e).

- (2) $\xi = \xi_1 \vee \xi_2$ by assumption
- (3) $e \models \xi_1$ by assumption
- (4) $\text{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)
- (5) $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$ or $\text{satisfy}(e, \xi_2) = \text{true}$
by Definition 15e on (4)

Case (14f).

- (2) $\xi = \xi_1 \vee \xi_2$ by assumption
- (3) $e \models \xi_2$ by assumption
- (4) $\text{satisfy}(e, \xi_2) = \text{true}$ by IH on (3)
- (5) $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$ or $\text{satisfy}(e, \xi_2) = \text{true}$
by Definition 15e on (4)

Case (14g).

- (2) $e = \text{inl}_{\tau_2}(e_1)$ by assumption
- (3) $\xi = \text{inl}(\xi_1)$ by assumption
- (4) $e_1 \models \xi_1$ by assumption
- (5) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)
- (6) $\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) = \text{true}$
by Definition 15f on (5)

Case (14h).

- (2) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
- (3) $\xi = \text{inl}(\xi_2)$ by assumption
- (4) $e_2 \models \xi_2$ by assumption
- (5) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (4)
- (6) $\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\xi_2)) = \text{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 15g on (5)

Case (14i).

- (2) $e = (e_1, e_2)$ by assumption
- (3) $\xi = (\xi_1, \xi_2)$ by assumption
- (4) $e_1 \models \xi_1$ by assumption
- (5) $e_2 \models \xi_2$ by assumption
- (6) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)
- (7) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (5)
- (8) $\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$
 $\text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) = \text{true}$
by Definition 15h on
(6) and (7)

Case (14j).

- (2) $\xi = (\xi_1, \xi_2)$ by assumption
- (3) $e \text{ notintro}$ by assumption
- (4) $\text{prl}(e) \models \xi_1$ by assumption
- (5) $\text{prr}(e) \models \xi_2$ by assumption
- (6) $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$ by IH on (4)
- (7) $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$ by IH on (5)

By rule induction over Rules (26) on (3).

Otherwise.

- (8) $e = (\emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\})$
by assumption
- (9) $\text{satisfy}(e, (\xi_1, \xi_2)) =$
 $\text{satisfy}(\text{prl}(e), \xi_1) \text{ and } \text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$
by Definition 15 on (6)
and (7)

2. Completeness:

- (1) $\text{satisfy}(e, \xi) = \text{true}$ by assumption

By structural induction on ξ .

Case $\xi = \top$.

- (2) $e \models \top$ by Rule (14a)

Case $\xi = \perp, ?$.

- (2) $\text{satisfy}(e, \xi) = \text{false}$ by Definition 15o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$.

By structural induction on e .

Case $e = \underline{n'}$.
 (2) $n' = n$ by Definition 15b on (1)
 (3) $\underline{n'} \models \underline{n}$ by Rule (14b) on (2)

Otherwise.
 (2) $\text{satisfy}(e, \underline{n}) = \text{false}$ by Definition 15o
 (2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{\mathcal{N}}$.
 By structural induction on e .

Case $e = \underline{n'}$.
 (2) $n' \neq n$ by Definition 15c on (1)
 (3) $\underline{n'} \models \underline{\mathcal{N}}$ by Rule (14c) on (2)

Otherwise.
 (2) $\text{satisfy}(e, \underline{\mathcal{N}}) = \text{false}$ by Definition 15o
 (2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$.
 (2) $\text{satisfy}(e, \xi_1) = \text{true}$ by Definition 15d on (1)
 (3) $\text{satisfy}(e, \xi_2) = \text{true}$ by Definition 15d on (1)
 (4) $e \models \xi_1$ by IH on (2)
 (5) $e \models \xi_2$ by IH on (3)
 (6) $e \models \xi_1 \wedge \xi_2$ by Rule (14d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$.
 (2) $\text{satisfy}(e, \xi_1) \text{ or } \text{satisfy}(e, \xi_2) = \text{true}$ by Definition 15e on (1)

By case analysis on (2).

Case $\text{satisfy}(e, \xi_1) = \text{true}$.
 (3) $\text{satisfy}(e, \xi_1) = \text{true}$ by assumption
 (4) $e \models \xi_1$ by IH on (3)
 (5) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (4)

Case $\text{satisfy}(e, \xi_2) = \text{true}$.
 (3) $\text{satisfy}(e, \xi_2) = \text{true}$ by assumption
 (4) $e \models \xi_2$ by IH on (3)
 (5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

Case $\xi = \text{inl}(\xi_1)$.

By structural induction on e .

Case $e = \text{inl}_{\tau_2}(e_1)$.

- (2) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 15f on (1)
- (3) $e_1 \models \xi_1$ by IH on (2)
- (4) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ by Rule (14g) on (3)

Otherwise.

- (2) $\text{satisfy}(e, \text{inl}(\xi_1)) = \text{false}$ by Definition 15o
- (2) contradicts (1) and thus vacuously true.

Case $\xi = \text{inr}(\xi_2)$.

By structural induction on e .

Case $e = \text{inr}_{\tau_1}(e_2)$.

- (2) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 15g on (1)
- (3) $e_2 \models \xi_2$ by IH on (2)
- (4) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ by Rule (14h) on (3)

Otherwise.

- (2) $\text{satisfy}(e, \text{inr}(\xi_2)) = \text{false}$ by Definition 15o
- (2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$.

By structural induction on e .

Case $e = (e_1, e_2)$.

- (2) $\text{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 15h on (1)
- (3) $\text{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 15h on (1)
- (4) $e_1 \models \xi_1$ by IH on (2)
- (5) $e_2 \models \xi_2$ by IH on (3)
- (6) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (4) and (5)

Case $e = (\llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{rs\})$.

- (2) $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$ by Definition 15h on (1)
- (3) $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$ by Definition 15h on (1)
- (4) $\text{prl}(e) \models \xi_1$ by IH on (2)
- (5) $\text{prr}(e) \models \xi_2$ by IH on (3)
- (6) $e \text{ notintro}$ by each rule in Rules (26)

(7) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14j) on (6) and (4) and (5)

Otherwise.

(2) $\text{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$ by Definition 15o
 (2) contradicts (1) and thus vacuously true.

□

Lemma 2.0.20. $e \not\models \xi$ and $e \not\models_{\text{?}} \xi$ iff $e \not\models_{\text{?}}^{\dagger} \xi$.

Proof. 1. Sufficiency:

(1) $e \not\models \xi$ by assumption
 (2) $e \not\models_{\text{?}} \xi$ by assumption

Assume $e \models_{\text{?}}^{\dagger} \xi$. By rule induction over Rules (17) on it.

Case (17a).

(3) $e \models \xi$ by assumption

Contradicts (1).

Case (17b).

(3) $e \models_{\text{?}} \xi$ by assumption

Contradicts (2).

Therefore, $e \models_{\text{?}}^{\dagger} \xi$ is not derivable.

2. Necessity:

(1) $e \not\models_{\text{?}}^{\dagger} \xi$ by assumption

Assume $e \models \xi$.

(2) $e \models_{\text{?}}^{\dagger} \xi$ by Rule (17b) on assumption

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_{\text{?}} \xi$.

(3) $e \models_{\text{?}}^{\dagger} \xi$ by Rule (17a) on assumption

Contradicts (1). Therefore, $e \not\models_{\text{?}} \xi$.

□

Theorem 2.1 (Exclusiveness of Satisfaction Judgment). *If $\xi : \tau$ and $\cdot; \Delta \vdash e : \tau$ and e final then exactly one of the following holds*

1. $e \models \xi$

2. $e \models_{\tau} \xi$

3. $e \not\models_{\tau}^{\dagger} \xi$

Proof.

(4) $\xi : \tau$ by assumption

(5) $\cdot; \Delta \vdash e : \tau$ by assumption

(6) e **final** by assumption

By rule induction over Rules (8) on (4), we would show one conclusion is derivable while the other two are not.

Case (8a).

(7) $\xi = \top$ by assumption

(8) $e \models \top$ by Rule (14a)

(9) $e \not\models_{\tau} \top$ by Lemma 2.0.3

(10) $e \models_{\tau}^{\dagger} \top$ by Rule (17b) on (8)

Case (8b).

(7) $\xi = \perp$ by assumption

(8) $e \not\models \perp$ by Lemma 2.0.1

(9) $e \not\models_{\tau} \perp$ by Lemma 2.0.2

(10) $e \not\models_{\tau}^{\dagger} \perp$ by Lemma 2.0.20 on
(8) and (9)

Case (1b).

(7) $\xi = ?$ by assumption

(8) $e \not\models ?$ by Lemma 2.0.4

(9) $e \models_{\tau} ?$ by Rule (16a)

(10) $e \models_{\tau}^{\dagger} ?$ by Rule (17a) on (9)

Case (8c).

(7) $\xi = \underline{n_2}$ by assumption

(8) $\tau = \mathbf{num}$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$
by assumption

(10) e **notintro** by Rule
(26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction on ξ .

(11) $e \not\models \underline{n_2}$ by contradiction

(12) $\underline{n_2}$ **refutable** by Rule (10a)

(13) $e \models_{\text{?}} \underline{n_2}$ by Rule (16b) on (10)
and (12)

(14) $e \models_{\text{?}}^{\dagger} \underline{n_2}$ by Rule (17a) on (13)

Case (19d).

(9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \models_{\text{?}} \underline{n_2}$. By rule induction over Rules (16), only one case applies.

Case (16b).

(10) $\underline{n_1}$ **notintro** by assumption

Contradicts Lemma 4.0.4.

(11) $\underline{n_1} \not\models_{\text{?}} \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2 .

Case $n_1 = n_2$.

(12) $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 15

(13) $\underline{n_1} \models \underline{n_2}$ by Lemma 2.0.19 on
(12)

(14) $e \models_{\text{?}}^{\dagger} \underline{n_2}$ by Rule (17b) on (13)

Case $n_1 \neq n_2$.

(12) $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 15

(13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 2.0.19 on
(12)

(14) $e \not\models_{\text{?}}^{\dagger} \underline{n_2}$ by Lemma 2.0.20 on
(11) and (13)

Case (8f).

(7) $\xi = \xi_1 \vee \xi_2$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models? \xi_1$, and $e \not\models?^\dagger \xi_1$ holds. The same goes for ξ_2 . By case analysis on which conclusion holds for ξ_1 and ξ_2 .

Case $e \models \xi_1, e \models \xi_2$.

- | | |
|--------------------------------------------|-----------------------|
| (8) $e \models \xi_1$ | by assumption |
| (9) $e \not\models? \xi_1$ | by assumption |
| (10) $e \models \xi_2$ | by assumption |
| (11) $e \not\models? \xi_2$ | by assumption |
| (12) $e \models \xi_1 \vee \xi_2$ | by Rule (14e) on (8) |
| (13) $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (17b) on (12) |

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- | | |
|------------------------------------------|---------------|
| (14) $\xi_1 \vee \xi_2$ refutable | by assumption |
| Contradicts Lemma 2.0.17. | |

Case (16c).

- | | |
|-------------------------|---------------|
| (14) $e \models? \xi_1$ | by assumption |
| Contradicts (9). | |

Case (16d).

- | | |
|-------------------------|---------------|
| (14) $e \models? \xi_2$ | by assumption |
| Contradicts (11). | |

- | | |
|----------------------------------------|------------------|
| (15) $e \not\models? \xi_1 \vee \xi_2$ | by contradiction |
|----------------------------------------|------------------|

Case $e \models \xi_1, e \models? \xi_2$.

- | | |
|--------------------------------------------|-----------------------|
| (8) $e \models \xi_1$ | by assumption |
| (9) $e \not\models? \xi_1$ | by assumption |
| (10) $e \not\models \xi_2$ | by assumption |
| (11) $e \models? \xi_2$ | by assumption |
| (12) $e \models \xi_1 \vee \xi_2$ | by Rule (14e) on (8) |
| (13) $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (17b) on (12) |

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- | | |
|------------------------------------------|---------------|
| (14) $\xi_1 \vee \xi_2$ refutable | by assumption |
| Contradicts Lemma 2.0.17. | |

Case (16c).

- | | |
|-------------------------|---------------|
| (14) $e \models? \xi_1$ | by assumption |
|-------------------------|---------------|

Contradicts (9).

Case (16d).

(14) $e \not\models \xi_1$

by assumption

Contradicts (8).

(15) $e \not\models? \xi_1 \vee \xi_2$

by contradiction

Case $e \models \xi_1, e \not\models?^\dagger \xi_2$.

(8) $e \models \xi_1$

by assumption

(9) $e \not\models? \xi_1$

by assumption

(10) $e \not\models \xi_2$

by assumption

(11) $e \not\models? \xi_2$

by assumption

(12) $e \models \xi_1 \vee \xi_2$

by Rule (14e) on (8)

(13) $e \models?^\dagger \xi_1 \vee \xi_2$

by Rule (17b) on (12)

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable**

by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models? \xi_1$

by assumption

Contradicts (9).

Case (16d).

(14) $e \not\models \xi_1$

by assumption

Contradicts (8).

(15) $e \not\models? \xi_1 \vee \xi_2$

by contradiction

Case $e \models? \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$

by assumption

(9) $e \models? \xi_1$

by assumption

(10) $e \models \xi_2$

by assumption

(11) $e \not\models? \xi_2$

by assumption

(12) $e \models \xi_1 \vee \xi_2$

by Rule (14f) on (10)

(13) $e \models?^\dagger \xi_1 \vee \xi_2$

by Rule (17b) on (12)

Assume $e \models? \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable**

by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (16d).

(14) $e \models? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models? \xi_1 \vee \xi_2$ by contradiction

Case $e \models? \xi_1, e \models? \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models? \xi_2$ by assumption

(12) $e \models? \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \models? \xi_1, e \not\models?^\dagger \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models? \xi_2$ by assumption

(12) $e \models? \xi_1 \vee \xi_2$ by Rule (16c) on (9) and (10)

(13) $e \models?^\dagger \xi_1 \vee \xi_2$ by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$

by assumption

Contradicts (8).

Case (14f).

(14) $e \models \xi_2$

by assumption

Contradicts (10).

(15) $e \not\models \xi_1 \vee \xi_2$

by contradiction

Case $e \not\models_{\vdash}^{\dagger} \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$

by assumption

(9) $e \not\models_{\vdash} \xi_1$

by assumption

(10) $e \models \xi_2$

by assumption

(11) $e \not\models_{\vdash} \xi_2$

by assumption

(12) $e \models \xi_1 \vee \xi_2$

by Rule (14f) on (10)

(13) $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$

by Rule (17b) on (12)

Assume $e \models_{\vdash} \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable**

by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \not\models \xi_2$

by assumption

Contradicts (10).

Case (16d).

(14) $e \models_{\vdash} \xi_2$

by assumption

Contradicts (11).

(15) $e \not\models_{\vdash} \xi_1 \vee \xi_2$

by contradiction

Case $e \not\models_{\vdash}^{\dagger} \xi_1, e \models_{\vdash} \xi_2$.

(8) $e \not\models \xi_1$

by assumption

(9) $e \not\models_{\vdash} \xi_1$

by assumption

(10) $e \not\models \xi_2$

by assumption

(11) $e \models_{\vdash} \xi_2$

by assumption

(12) $e \models_{\vdash} \xi_1 \vee \xi_2$

by Rule (16d) on (11)
and (8)

(13) $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$

by Rule (17a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14), only two cases apply.

Case (14e).

(14) $e \models \xi_1$ by assumption

Contradicts (8)

Case (14f).

(14) $e \models \xi_2$ by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_{\text{?}} \xi_1, e \not\models_{\text{?}} \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_{\text{?}} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_{\text{?}} \xi_2$ by assumption

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (14) on it, only two cases apply.

Case (14e).

(12) $e \models \xi_1$ by assumption

Contradicts (8).

Case (14f).

(12) $e \models \xi_2$ by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Assume $e \models_{\text{?}} \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(14) $\xi_1 \vee \xi_2$ **refutable** by assumption

Contradicts Lemma 2.0.17.

Case (16c).

(14) $e \models_{\text{?}} \xi_1$ by assumption

Contradicts (9).

Case (16d).

(14) $e \models_{\text{?}} \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_{\text{?}} \xi_1 \vee \xi_2$ by contradiction

(16) $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ by Lemma 2.0.20 on (13) and (15)

Case (8g).

(7) $\xi = \text{inl}(\xi_1)$ by assumption
 (8) $\tau = (\tau_1 + \tau_2)$ by assumption
 (9) $\xi_1 : \tau_1$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(10) $e = \langle \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\} \rangle$ by assumption
 (11) $e \text{ notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \text{inl}(\xi_1)$ by contradiction

By case analysis on the value of $\text{refutable}(\text{inl}(\xi_1))$.

Case $\text{refutable}(\text{inl}(\xi_1)) = \text{true}$.

(13) $\text{refutable}(\text{inl}(\xi_1)) = \text{true}$ by assumption
 (14) $\text{inl}(\xi_1) \text{ refutable}$ by Lemma 2.0.14 on (13)
 (15) $e \models_{\tau} \text{inl}(\xi_1)$ by Rule (16b) on (11) and (14)
 (16) $e \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Rule (17a) on (15)

Case $\text{refutable}(\text{inl}(\xi_1)) = \text{false}$.

(13) $\text{refutable}(\text{inl}(\xi_1)) = \text{false}$ by assumption
 (14) ~~$\text{inl}(\xi_1) \text{ refutable}$~~ by Lemma 2.0.14 on (13)

Assume $e \models_{\tau} \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(15) $\text{inl}(\xi_1) \text{ refutable}$ by assumption
 Contradicts (14).

(16) $e \not\models_{\tau} \text{inl}(\xi_1)$ by contradiction
 (17) $e \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Lemma 2.0.20 on (12) and (16)

Case (19j).

- (10) $e = \text{inl}_{\tau_2}(e_1)$ by assumption
- (11) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
- (12) e_1 **final** by Lemma 4.0.1 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models? \xi_1$, and $e_1 \not\models? \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$.

- (13) $e_1 \models \xi_1$ by assumption
- (14) $e_1 \not\models? \xi_1$ by assumption
- (15) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ by Rule (14g) on (13)
- (16) $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ by Rule (17b) on (15)

Assume $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

- (17) $\text{inl}_{\tau_2}(e_1)$ **notintro** by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

- (17) $e_1 \models? \xi_1$

Contradicts (14).

- (18) $\text{inl}_{\tau_2}(e_1) \not\models? \text{inl}(\xi_1)$ by contradiction

Case $e_1 \models? \xi_1$.

- (13) $e_1 \not\models \xi_1$ by assumption
- (14) $e_1 \models? \xi_1$ by assumption
- (15) $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ by Rule (16e) on (14)
- (16) $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ by Rule (17a) on (15)

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

- (17) $e_1 \models \xi_1$

Contradicts (13).

- (18) $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$ by contradiction

Case $e_1 \not\models? \xi_1$.

- (13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \not\models_{\tau} \xi_1$ by assumption

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, only one case applies.

Case (14g).

(15) $e_1 \models \xi_1$

Contradicts (13).

(16) $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(17) $\text{inl}_{\tau_2}(e_1) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16e).

(17) $e_1 \models_{\tau} \xi_1$

Contradicts (14).

(18) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\xi_1)$ by contradiction

(19) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (19k).

(10) $e = \text{inr}_{\tau_1}(e_2)$ by assumption

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\xi_1)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

(11) $\text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\xi_1)$ by contradiction

Assume $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inl}(\xi_1)$ by contradiction

(14) $\text{inr}_{\tau_1}(e_2) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$ by Lemma 2.0.20 on (11) and (13)

Case (8h).

(7) $\xi = \text{inr}(\xi_2)$ by assumption

- (8) $\tau = (\tau_1 + \tau_2)$ by assumption
 (9) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (10) $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$
 by assumption
 (11) e **notintro** by Rule
 (26a),(26b),(26c),(26d),(26e),(26f)

Assume $e \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (12) $e \not\models \text{inr}(\xi_2)$ by contradiction

By case analysis on the value of $\text{refutable}(\text{inr}(\xi_2))$.

inr is
refutable

Case $\text{refutable}(\text{inr}(\xi_2)) = \text{true}$.

- (13) $\text{refutable}(\text{inr}(\xi_2)) = \text{true}$ by assumption
 (14) **inr**(ξ_2) **refutable** by Lemma 2.0.14 on
 (13)
 (15) $e \models_{\text{?}} \text{inr}(\xi_2)$ by Rule (16b) on (11)
 and (14)
 (16) $e \models_{\text{?}}^{\dagger} \text{inr}(\xi_2)$ by Rule (17a) on (15)

Case $\text{refutable}(\text{inr}(\xi_2)) = \text{false}$.

- (13) $\text{refutable}(\text{inr}(\xi_2)) = \text{false}$ by assumption
 (14) ~~**inr**(ξ_2) **refutable**~~ by Lemma 2.0.14 on
 (13)

Assume $e \models_{\text{?}} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

- (15) **inr**(ξ_2) **refutable** by assumption
 Contradicts (14).

- (16) $e \not\models_{\text{?}} \text{inr}(\xi_2)$ by contradiction
 (17) $e \not\models_{\text{?}}^{\dagger} \text{inr}(\xi_2)$ by Lemma 2.0.20 on
 (12) and (16)

Case (19j).

- (10) $e = \text{inl}_{\tau_2}(e_1)$ by assumption

Assume $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, no case applies due to syntactic contradiction.

- (11) $\text{inl}_{\tau_2}(e_1) \not\models \text{inr}(\xi_2)$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(12) $\text{inl}_{\tau_2}(e_1) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (12), no case applies due to syntactic contradiction.

(13) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inr}(\xi_2)$ by contradiction

(14) $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inr}(\xi_2)$ by Lemma 2.0.20 on (11) and (13)

Case (19k).

(10) $e = \text{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot; \Delta \vdash e_2 : \tau_2$ by assumption

(12) $e_2 \text{ final}$ by Lemma 4.0.2 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_{\tau} \xi_2$, and $e_2 \not\models_{\tau}^{\dagger} \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

(13) $e_2 \models \xi_2$ by assumption

(14) $e_2 \not\models_{\tau} \xi_2$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ by Rule (14g) on (13)

(16) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ by Rule (17b) on (15)

Assume $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16b).

(17) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

(17) $e_2 \models_{\tau} \xi_2$

Contradicts (14).

(18) $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\xi_2)$ by contradiction

Case $e_2 \models_{\tau} \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \models_{\tau} \xi_2$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$ by Rule (16f) on (14)

(16) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ by Rule (17a) on (15)

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

$$(17) \quad e_2 \models \xi_2$$

Contradicts (13).

$$(18) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2) \quad \text{by contradiction}$$

Case $e_2 \not\models_{\tau_1}^{\dagger} \xi_2$.

$$(13) \quad e_2 \not\models \xi_2 \quad \text{by assumption}$$

$$(14) \quad e_2 \not\models_{\tau_1} \xi_2 \quad \text{by assumption}$$

Assume $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14h).

$$(15) \quad e_2 \models \xi_2$$

Contradicts (13).

$$(16) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2) \quad \text{by contradiction}$$

Assume $\text{inr}_{\tau_1}(e_2) \models_{\tau_1} \text{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

$$(17) \quad \text{inr}_{\tau_1}(e_2) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (26) on (17), no case applies due to syntactic contradiction.

Case (16f).

$$(17) \quad e_2 \models_{\tau_1} \xi_2$$

Contradicts (14).

$$(18) \quad \text{inr}_{\tau_1}(e_2) \not\models_{\tau_1} \text{inr}(\xi_2) \quad \text{by contradiction}$$

$$(19) \quad \text{inl}_{\tau_2}(e_1) \not\models_{\tau_2}^{\dagger} \text{inl}(\xi_1) \quad \text{by Lemma 2.0.20 on (16) and (18)}$$

Case (14i).

$$(7) \quad \xi = (\xi_1, \xi_2) \quad \text{by assumption}$$

$$(8) \quad \tau = (\tau_1 \times \tau_2) \quad \text{by assumption}$$

$$(9) \quad \xi_1 : \tau_1 \quad \text{by assumption}$$

$$(10) \quad \xi_2 : \tau_2 \quad \text{by assumption}$$

By rule induction over Rules (19) on (5), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (11) $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$
by assumption
- (12) e notintro
by Rule (26a),(26b),(26c),(26d),(26e),(26f)
- (13) e indet
by Lemma 4.0.8 on (6) and (12)
- (14) $\text{prl}(e)$ indet
by Rule (24g) on (13)
- (15) $\text{prl}(e)$ final
by Rule (25b) on (14)
- (16) $\text{prr}(e)$ indet
by Rule (24h) on (13)
- (17) $\text{prr}(e)$ final
by Rule (25b) on (16)
- (18) $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$
by Rule (19h) on (5)
- (19) $\cdot; \Delta \vdash \text{prr}(e) : \tau_2$
by Rule (19i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\text{prl}(e) \models \xi_1$, $\text{prl}(e) \models? \xi_1$, and $\text{prl}(e) \not\models?^\dagger \xi_1$ holds.

By inductive hypothesis on (10) and (19) and (17), exactly one of $\text{prr}(e) \models \xi_2$, $\text{prr}(e) \models? \xi_2$, and $\text{prr}(e) \not\models?^\dagger \xi_2$ holds.

By case analysis on which conclusion holds for ξ_1 and ξ_2 .

Case $\text{prl}(e) \models \xi_1, \text{prr}(e) \models \xi_2$.

- (20) $\text{prl}(e) \models \xi_1$
by assumption
- (21) $\text{prl}(e) \not\models? \xi_1$
by assumption
- (22) $\text{prr}(e) \models \xi_2$
by assumption
- (23) $\text{prr}(e) \not\models? \xi_2$
by assumption
- (24) $e \models (\xi_1, \xi_2)$
by Rule (14j) on (12) and (20) and (22)
- (25) $e \models?^\dagger (\xi_1, \xi_2)$
by Rule (17b) on (24)
- (26) ~~(ξ_1, ξ_2) refutable~~
by Lemma 2.0.18 on (12) and (24)

Assume $e \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

- (27) (ξ_1, ξ_2) refutable
by assumption
- Contradicts (26).

- (28) $e \not\models? (\xi_1, \xi_2)$
by contradiction

Case $\text{prl}(e) \models \xi_1, \text{prr}(e) \models? \xi_2$.

- (20) $\text{prl}(e) \models \xi_1$
by assumption
- (21) $\text{prl}(e) \not\models? \xi_1$
by assumption
- (22) $\text{prr}(e) \not\models \xi_2$
by assumption
- (23) $\text{prr}(e) \models? \xi_2$
by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{pr}(e) \models \xi_2$ by assumption
Contradicts (22)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

(26) ξ_2 **refutable** by assumption
(27) (ξ_1, ξ_2) **refutable** by Rule (10e) on (26)
(28) $e \models? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)
(29) $e \models?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\text{pr}(e) \models \xi_1, \text{pr}(e) \not\models?^\dagger \xi_2$.

(20) $\text{pr}(e) \models \xi_1$ by assumption
(21) $\text{pr}(e) \not\models? \xi_1$ by assumption
(22) $\text{pr}(e) \not\models \xi_2$ by assumption
(23) $\text{pr}(e) \not\models? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\text{pr}(e) \models \xi_2$ by assumption
Contradicts (22).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) **refutable** by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 **refutable** by assumption
(28) $\text{pr}(e)$ **notintro** by Rule (26e)
(29) $\text{pr}(e) \models? \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

assume no
"or" and
"and" in
pair

(27) ξ_2 refutable	by assumption
(28) $\text{pr}(e)$ notintro	by Rule (26f)
(29) $\text{pr}(e) \models \xi_2$	by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models (\xi_1, \xi_2)$	by contradiction
(31) $e \not\models^\dagger (\xi_1, \xi_2)$	by Lemma 2.0.20 on (25) and (30)

Case $\text{pr}(e) \models \xi_1, \text{pr}(e) \models \xi_2$.

(20) $\text{pr}(e) \not\models \xi_1$	by assumption
(21) $\text{pr}(e) \models \xi_1$	by assumption
(22) $\text{pr}(e) \models \xi_2$	by assumption
(23) $\text{pr}(e) \not\models \xi_2$	by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{pr}(e) \models \xi_1$	by assumption
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Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$	by contradiction
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By rule induction over Rules (16) on (21), only one case applies.

Case (16b).

(26) ξ_1 refutable	by assumption
(27) (ξ_1, ξ_2) refutable	by Rule (10e) on (26)
(28) $e \models (\xi_1, \xi_2)$	by Rule (16b) on (12) and (27)
(29) $e \models^\dagger (\xi_1, \xi_2)$	by Rule (17a) on (28)

Case $\text{pr}(e) \models \xi_1, \text{pr}(e) \models \xi_2$.

(20) $\text{pr}(e) \not\models \xi_1$	by assumption
(21) $\text{pr}(e) \models \xi_1$	by assumption
(22) $\text{pr}(e) \not\models \xi_2$	by assumption
(23) $\text{pr}(e) \models \xi_2$	by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{pr}(e) \models \xi_1$	by assumption
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Contradicts (20).

assume no
"or" and
"and" in
pair

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction
 By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

(26) ξ_2 **refutable** by assumption
 (27) (ξ_1, ξ_2) **refutable** by Rule (10e) on (26)
 (28) $e \models? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)
 (29) $e \models?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\text{prl}(e) \models? \xi_1, \text{prr}(e) \not\models?^\dagger \xi_2$.

(20) $\text{prl}(e) \not\models \xi_1$ by assumption
 (21) $\text{prl}(e) \models? \xi_1$ by assumption
 (22) $\text{prr}(e) \not\models \xi_2$ by assumption
 (23) $\text{prr}(e) \not\models? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{prl}(e) \models \xi_1$ by assumption
 Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction
 By rule induction over Rules (16) on (21), only one case applies.

Case (16b).

(26) ξ_1 **refutable** by assumption
 (27) (ξ_1, ξ_2) **refutable** by Rule (10e) on (26)
 (28) $e \models? (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)
 (29) $e \models?^\dagger (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\text{prl}(e) \not\models?^\dagger \xi_1, \text{prr}(e) \models \xi_2$.

(20) $\text{prl}(e) \not\models \xi_1$ by assumption
 (21) $\text{prl}(e) \not\models? \xi_1$ by assumption
 (22) $\text{prr}(e) \models \xi_2$ by assumption
 (23) $\text{prr}(e) \not\models? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\text{prl}(e) \models \xi_1$ by assumption
 Contradicts (20)

assume no
"or" and
"and" in
pair

assume no
"or" and
"and" in
pair

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_{\text{?}} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) **refutable** by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 **refutable** by assumption

(28) **prl**(e) **notintro** by Rule (26e)

(29) **prl**(e) $\models_{\text{?}} \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 **refutable** by assumption

(28) **prr**(e) **notintro** by Rule (26f)

(29) **prr**(e) $\models_{\text{?}} \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_{\text{?}} (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \xi_1, \text{prr}(e) \models_{\text{?}} \xi_2$.

(20) $\text{prl}(e) \not\models \xi_1$ by assumption

(21) $\text{prl}(e) \not\models_{\text{?}} \xi_1$ by assumption

(22) $\text{prr}(e) \not\models \xi_2$ by assumption

(23) $\text{prr}(e) \models_{\text{?}} \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14), only one case applies.

Case (14j).

(24) $\text{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (16) on (23), only one case applies.

Case (16b).

(26) ξ_2 **refutable** by assumption

(27) (ξ_1, ξ_2) **refutable** by Rule (10e) on (26)

assume no
"or" and
"and" in
pair

(28) $e \models_{\tau} (\xi_1, \xi_2)$ by Rule (16b) on (12) and (27)

(29) $e \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ by Rule (17a) on (28)

Case $\text{prl}(e) \not\models_{\tau}^{\dagger} \xi_1, \text{prl}(e) \not\models_{\tau}^{\dagger} \xi_2$.

(20) $\text{prl}(e) \not\models \xi_1$ by assumption

(21) $\text{prl}(e) \not\models_{\tau} \xi_1$ by assumption

(22) $\text{prl}(e) \not\models \xi_2$ by assumption

(23) $\text{prl}(e) \not\models_{\tau} \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only one case applies.

Case (14j).

(24) $\text{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_{\tau} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16b).

(26) (ξ_1, ξ_2) **refutable** by assumption

By rule induction over Rules (10) on (26), only two cases apply.

Case (10d).

(27) ξ_1 **refutable** by assumption

(28) $\text{prl}(e)$ **notintro** by Rule (26e)

(29) $\text{prl}(e) \models_{\tau} \xi_1$ by Rule (16b) on (28) and (27)

Contradicts (21).

Case (10e).

(27) ξ_2 **refutable** by assumption

(28) $\text{prl}(e)$ **notintro** by Rule (26f)

(29) $\text{prl}(e) \models_{\tau} \xi_2$ by Rule (16b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_{\tau} (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case (19g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption

- (13) $\cdot; \Delta \vdash e_2 : \tau_2$ by assumption
- (14) e_1 **final** by Lemma 4.0.3 on (6)
- (15) e_2 **final** by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models? \xi_1$, and $e_1 \models \overline{\xi_1}$ holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models? \xi_2$, and $e_2 \models \overline{\xi_2}$ holds.

By case analysis on which conclusion holds for ξ_1 and ξ_2 .

Case $e_1 \models \xi_1, e_2 \models \xi_2$.

- (16) $e_1 \models \xi_1$ by assumption
- (17) $e_1 \not\models? \xi_1$ by assumption
- (18) $e_2 \models \xi_2$ by assumption
- (19) $e_2 \not\models? \xi_2$ by assumption
- (20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (16)
and (18)
- (21) $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ by Rule (17b) on (20)

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

- (22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

- (22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

Case (16h).

- (22) $e_2 \models? \xi_2$ by assumption

Contradicts (19).

Case (16i).

- (22) $e_1 \models? \xi_1$ by assumption

Contradicts (17).

- (23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models? \xi_2$.

- (16) $e_1 \models \xi_1$ by assumption
- (17) $e_1 \not\models? \xi_1$ by assumption
- (18) $e_2 \not\models \xi_2$ by assumption
- (19) $e_2 \models? \xi_2$ by assumption
- (20) $(e_1, e_2) \models? (\xi_1, \xi_2)$ by Rule (16h) on (16)
and (19)

(21) $(e_1, e_2) \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$ by Rule (17a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(22) $(e_1, e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models_{\text{?}}^{\dagger} \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_{\text{?}} \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_{\text{?}} \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) $(e_1, e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) $(e_1, e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models_{\text{?}} \xi_1$ by assumption

Contradicts (17).

Case (16h).

(22) $e_2 \models_{\text{?}} \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_1 \models_{\text{?}} \xi_1$ by assumption

Contradicts (17).

- | | | |
|------|--------------------------------------------------|-------------------------------------|
| (23) | $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ | by contradiction |
| (24) | $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ | by Lemma 2.0.20 on
(21) and (23) |

Case $e_1 \models? \xi_1, e_2 \models \xi_2$.

- | | | |
|------|--------------------------------------|-----------------------------------|
| (16) | $e_1 \not\models \xi_1$ | by assumption |
| (17) | $e_1 \models? \xi_1$ | by assumption |
| (18) | $e_2 \models \xi_2$ | by assumption |
| (19) | $e_2 \not\models? \xi_2$ | by assumption |
| (20) | $(e_1, e_2) \models? (\xi_1, \xi_2)$ | by Rule (16g) on (17)
and (18) |

- | | | |
|------|----------------------------------------------|-----------------------|
| (21) | $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ | by Rule (17a) on (20) |
|------|----------------------------------------------|-----------------------|

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

- | | | |
|------|------------------------------|---------------|
| (22) | (e_1, e_2) notintro | by assumption |
|------|------------------------------|---------------|

Contradicts Lemma 4.0.7.

Case (14i).

- | | | |
|------|---------------------|---------------|
| (22) | $e_1 \models \xi_1$ | by assumption |
|------|---------------------|---------------|

Contradicts (16).

- | | | |
|------|-----------------------------------------|------------------|
| (23) | $(e_1, e_2) \not\models (\xi_1, \xi_2)$ | by contradiction |
|------|-----------------------------------------|------------------|

Case $e_1 \models? \xi_1, e_2 \models? \xi_2$.

- | | | |
|------|--------------------------------------|-----------------------------------|
| (16) | $e_1 \not\models \xi_1$ | by assumption |
| (17) | $e_1 \models? \xi_1$ | by assumption |
| (18) | $e_2 \not\models \xi_2$ | by assumption |
| (19) | $e_2 \models? \xi_2$ | by assumption |
| (20) | $(e_1, e_2) \models? (\xi_1, \xi_2)$ | by Rule (16i) on (17)
and (19) |

- | | | |
|------|----------------------------------------------|-----------------------|
| (21) | $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ | by Rule (17a) on (20) |
|------|----------------------------------------------|-----------------------|

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

- | | | |
|------|------------------------------|---------------|
| (22) | (e_1, e_2) notintro | by assumption |
|------|------------------------------|---------------|

Contradicts Lemma 4.0.7.

Case (14i).

- | | | |
|------|---------------------|---------------|
| (22) | $e_1 \models \xi_1$ | by assumption |
|------|---------------------|---------------|

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models? \xi_1, e_2 \not\models?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

Case (16h).

(22) $e_2 \models? \xi_2$ by assumption

Contradicts (19).

Case (16i).

(22) $e_2 \models? \xi_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models?^\dagger \xi_1, e_2 \models \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models? \xi_2$ by assumption
 Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.
Case (14j).
 (20) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.
Case (14i).
 (20) $e_1 \models \xi_1$ by assumption
 Contradicts (16).

 (21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction
 Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.
Case (16b).
 (22) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.
Case (16g).
 (22) $e_1 \models? \xi_1$ by assumption
 Contradicts (17).
Case (16h).
 (22) $e_2 \models? \xi_2$ by assumption
 Contradicts (19).
Case (16i).
 (22) $e_1 \models? \xi_1$ by assumption
 Contradicts (17).

 (23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction
 (24) $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)
Case $e_1 \not\models?^\dagger \xi_1, e_2 \models? \xi_2$.
 (16) $e_1 \not\models \xi_1$ by assumption
 (17) $e_1 \not\models? \xi_1$ by assumption
 (18) $e_2 \not\models \xi_2$ by assumption
 (19) $e_2 \models? \xi_2$ by assumption
 Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.
Case (14j).
 (20) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.
Case (14i).

(20) $e_2 \models \xi_2$ by assumption
 Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.

Case (16g).

(22) $e_1 \models? \xi_1$ by assumption
 Contradicts (17).

Case (16h).

(22) $e_1 \models \xi_1$ by assumption
 Contradicts (16).

Case (16i).

(22) $e_1 \models? \xi_1$ by assumption
 Contradicts (17).

(23) $(e_1, e_2) \not\models? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models?^\dagger \xi_1, e_2 \not\models?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption
 (17) $e_1 \not\models? \xi_1$ by assumption
 (18) $e_2 \not\models \xi_2$ by assumption
 (19) $e_2 \not\models? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (14) on it, only two cases apply.

Case (14j).

(20) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.

Case (14i).

(20) $e_2 \models \xi_2$ by assumption
 Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models? (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, the following cases apply.

Case (16b).

(22) (e_1, e_2) notintro	by assumption
Contradicts Lemma 4.0.7.	
Case (16g).	
(22) $e_1 \models_{\tau} \xi_1$	by assumption
Contradicts (17).	
Case (16h).	
(22) $e_2 \models_{\tau} \xi_2$	by assumption
Contradicts (19).	
Case (16i).	
(22) $e_1 \models_{\tau} \xi_1$	by assumption
Contradicts (17).	
(23) $(e_1, e_2) \not\models_{\tau} (\xi_1, \xi_2)$	by contradiction
(24) $(e_1, e_2) \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$	by Lemma 2.0.20 on (21) and (23)

□

Definition 2.1.1 (Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e **val** we have $e \models_{\tau}^{\dagger} \xi_1$ implies $e \models \xi_2$*

Definition 2.1.2 (Potential Entailment of Constraints). *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models_{\tau}^{\dagger} \xi_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e **final** we have $e \models_{\tau}^{\dagger} \xi_1$ implies $e \models_{\tau}^{\dagger} \xi_2$*

Corollary 2.1.1. *Suppose that $\xi : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e **final**. Then $\top \models_{\tau}^{\dagger} \xi$ implies $e \models_{\tau}^{\dagger} \xi$*

Proof.

(1) $\xi : \tau$	by assumption
(2) $\cdot ; \Gamma \vdash e : \tau$	by assumption
(3) e final	by assumption
(4) $\top \models_{\tau}^{\dagger} \xi$	by assumption
(5) $e_1 \models \top$	by Rule (14a)
(6) $e_1 \models_{\tau}^{\dagger} \top$	by Rule (17b) on (5)
(7) $\top : \tau$	by Rule (8a)
(8) $e_1 \models_{\tau}^{\dagger} \xi_r$	by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

□

3 Static Semantics

$$\begin{aligned}
\tau &::= \text{num} \mid (\tau_1 \rightarrow \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e &::= x \mid \underline{n} \\
&\quad \mid (\lambda x : \tau. e) \mid e_1(e_2) \\
&\quad \mid (e_1, e_2) \\
&\quad \mid \text{inl}_\tau(e) \mid \text{inr}_\tau(e) \mid \text{match}(e)\{\hat{r}s\} \\
&\quad \mid \textcolor{violet}{\mathbb{O}}^u \mid \textcolor{violet}{\langle} e \textcolor{violet}{\rangle}^u \\
\hat{r}s &::= (rs \mid r \mid rs) \\
rs &::= \cdot \mid (r \mid rs') \\
r &::= p \Rightarrow e \\
p &::= x \mid \underline{n} \mid _ \mid (p_1, p_2) \mid \text{inl}(p) \mid \text{inr}(p) \mid \textcolor{violet}{\mathbb{O}}^w \mid \textcolor{violet}{\langle} p \textcolor{violet}{\rangle}^w \\
\boxed{(\hat{r}s)^\diamond = rs} &\quad rs \text{ can be obtained by erasing pointer from } \hat{r}s
\end{aligned}$$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \quad (18a)$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \quad (18b)$$

$$\boxed{\Gamma; \Delta \vdash e : \tau} \quad e \text{ is of type } \tau$$

$$\frac{\text{TVar}}{\Gamma, x : \tau; \Delta \vdash x : \tau} \quad (19a)$$

$$\frac{\text{TEHole}}{\Gamma; \Delta, u :: \tau \vdash \textcolor{violet}{\mathbb{O}}^u : \tau} \quad (19b)$$

$$\frac{\text{THole} \quad \Gamma; \Delta, u :: \tau \vdash e : \tau'}{\Gamma; \Delta, u :: \tau \vdash \textcolor{violet}{\langle} e \textcolor{violet}{\rangle}^u : \tau} \quad (19c)$$

$$\frac{\text{TNum}}{\Gamma; \Delta \vdash \underline{n} : \text{num}} \quad (19d)$$

$$\frac{\text{TLam} \quad \Gamma, x : \tau_1; \Delta \vdash e : \tau_2}{\Gamma; \Delta \vdash (\lambda x : \tau_1. e) : (\tau_1 \rightarrow \tau_2)} \quad (19e)$$

$$\frac{\text{TAp} \quad \Gamma; \Delta \vdash e_1 : (\tau_2 \rightarrow \tau) \quad \Gamma; \Delta \vdash e_2 : \tau_2}{\Gamma; \Delta \vdash e_1(e_2) : \tau} \quad (19f)$$

$$\frac{\text{TPair} \quad \Gamma; \Delta \vdash e_1 : \tau_1 \quad \Gamma; \Delta \vdash e_2 : \tau_2}{\Gamma; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \quad (19g)$$

$$\frac{\text{TPrl} \quad \Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma; \Delta \vdash \text{prl}(e) : \tau_1} \quad (19h)$$

$$\frac{\text{TPrr}}{\Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)} \quad \Gamma; \Delta \vdash \text{pr}(e) : \tau_2 \quad (19i)$$

$$\frac{\text{TInl}}{\Gamma; \Delta \vdash e : \tau_1} \quad \Gamma; \Delta \vdash \text{inl}_{\tau_2}(e) : (\tau_1 + \tau_2) \quad (19j)$$

$$\frac{\text{TInr}}{\Gamma; \Delta \vdash e : \tau_2} \quad \Gamma; \Delta \vdash \text{inr}_{\tau_1}(e) : (\tau_1 + \tau_2) \quad (19k)$$

$$\frac{\text{TMatchZPre} \quad \Gamma; \Delta \vdash e : \tau \quad \Gamma; \Delta \vdash [\perp]r \mid rs : \tau[\xi] \Rightarrow \tau' \quad \top \models_{\tau}^{\dagger} \xi}{\Gamma; \Delta \vdash \text{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \quad (19l)$$

$$\frac{\text{TMatchNZPre} \quad \Gamma; \Delta \vdash e : \tau \quad e \text{ final} \quad \Gamma; \Delta \vdash [\perp]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \quad \Gamma; \Delta \vdash [\perp \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \quad e \not\models_{\tau}^{\dagger} \xi_{pre} \quad \top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}}{\Gamma; \Delta \vdash \text{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \quad (19m)$$

$\boxed{p : \tau[\xi] \dashv \Gamma; \Delta}$ p is assigned type τ and emits constraint ξ

$$\frac{\text{PTVar}}{x : \tau[\top] \dashv \cdot; x : \tau} \quad (20a)$$

$$\frac{\text{PTWild}}{_ : \tau[\top] \dashv \cdot; \cdot} \quad (20b)$$

$$\frac{\text{PTEHole}}{\langle \rangle^w : \tau[?] \dashv \cdot; w :: \tau} \quad (20c)$$

$$\frac{\text{PTHole} \quad p : \tau[\xi] \dashv \Gamma; \Delta}{\langle p \rangle^w : \tau'[?] \dashv \Gamma; \Delta, w :: \tau'} \quad (20d)$$

$$\frac{\text{PTNum}}{\underline{n} : \text{num}[\underline{n}] \dashv \cdot; \cdot} \quad (20e)$$

$$\frac{\text{PTInl} \quad p : \tau_1[\xi] \dashv \Gamma; \Delta}{\text{inl}(p) : (\tau_1 + \tau_2)[\text{inl}(\xi)] \dashv \Gamma; \Delta} \quad (20f)$$

$$\frac{\text{PTInr} \quad p : \tau_2[\xi] \dashv \Gamma; \Delta}{\text{inr}(p) : (\tau_1 + \tau_2)[\text{inr}(\xi)] \dashv \Gamma; \Delta} \quad (20g)$$

$$\frac{\text{PTPair} \quad p_1 : \tau_1[\xi_1] \dashv \Gamma_1; \Delta_1 \quad p_2 : \tau_2[\xi_2] \dashv \Gamma_2; \Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv \Gamma_1 \uplus \Gamma_2; \Delta_1 \uplus \Delta_2} \quad (20h)$$

$$\boxed{\Gamma ; \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'} \quad \begin{array}{l} r \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\frac{\text{CTRrule} \quad p : \tau[\xi] \dashv\!\!\vdash \Gamma_p ; \Delta_p \quad \Gamma \uplus \Gamma_p ; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma ; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \quad (21a)$$

$$\boxed{\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'} \quad \begin{array}{l} rs \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\frac{\text{CTOneRules} \quad \Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \xi_r \not\models \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \quad (22a)$$

$$\frac{\text{CTRrules} \quad \Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \quad \xi_r \not\models \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \quad (22b)$$

Lemma 3.0.1. *If $p : \tau[\xi] \dashv\!\!\vdash \Gamma ; \Delta$ then $\xi : \tau$.*

Proof. By rule induction over Rules (20). □

Lemma 3.0.2. *If $\cdot ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ then $\xi_r : \tau_1$.*

Proof. By rule induction over Rules (21). □

Lemma 3.0.3. *If $\cdot ; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ then $\xi_{rs} : \tau_1$.*

Proof. By rule induction over Rules (22). □

Lemma 3.0.4. *If $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ and $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ and $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ then $\Gamma ; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$*

Proof.

- (1) $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ by assumption
- (2) $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ by assumption
- (3) $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ by assumption

By rule induction over Rules (22) on (1).

Case (22a).

- (4) $rs = r' \mid \cdot$ by assumption
- (5) $\xi_{rs} = \xi'_r$ by assumption
- (6) $\Gamma ; \Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$ by assumption
- (7) $\xi'_r \not\models \xi_{pre}$ by assumption
- (8) $\Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r]r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$ by Rule (22a) on (2) and (3)
- (9) $\Gamma ; \Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$ by Rule (22b) on (6) and (8) and (7)

$$(10) \quad \Gamma ; \Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau' \\ \text{by Definition 18 on (9)}$$

Case (22b).

$$\begin{aligned} (4) \quad rs &= r' \mid rs' && \text{by assumption} \\ (5) \quad \xi_{rs} &= \xi'_r \vee \xi'_{rs} && \text{by assumption} \\ (6) \quad \Gamma ; \Delta \vdash r' : \tau[\xi'_r] &\Rightarrow \tau' && \text{by assumption} \\ (7) \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] &\Rightarrow \tau' && \text{by assumption} \\ (8) \quad \xi'_r &\not\models \xi_{pre} && \text{by assumption} \\ (9) \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^\diamond : \tau[\xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by IH on (7) and (2)} \\ &\text{and (3)} \\ (10) \quad \Gamma ; \Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^\diamond) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by Rule (22b) on (6)} \\ &\text{and (9) and (8)} \\ (11) \quad \Gamma ; \Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by Definition 18 on} \\ &\text{(10)} \end{aligned}$$

□

Lemma 3.0.5 (Substitution). *If $\Gamma, x : \tau ; \Delta \vdash e_0 : \tau_0$ and $\Gamma ; \Delta \vdash e : \tau$ then $\Gamma ; \Delta \vdash [e/x]e_0 : \tau_0$*

Lemma 3.0.6 (Simultaneous Substitution). *If $\Gamma \uplus \Gamma' ; \Delta \vdash e : \tau$ and $\theta : \Gamma'$ then $\Gamma ; \Delta \vdash [\theta]e : \tau$*

Lemma 3.0.7 (Substitution Typing). *If $e \triangleright p \dashv\vdash \theta$ and $\cdot ; \Delta_e \vdash e : \tau$ and $p : \tau[\xi] \dashv\vdash \Gamma ; \Delta$ then $\theta : \Gamma$*

Proof by induction on the derivation of $e \triangleright p \dashv\vdash \theta$.

Theorem 3.1 (Determinism). *If $\cdot ; \Delta \vdash e : \tau$ then exactly one of the following holds*

1. $e \text{ val}$
2. $e \text{ indet}$
3. $e \mapsto e'$ for some unique e'

4 Dynamic Semantics

$\boxed{e \text{ val}}$ e is a value

$$\frac{\text{VNum}}{\underline{n} \text{ val}} \quad (23a)$$

$$\frac{\text{VLam}}{(\lambda x : \tau. e) \text{ val}} \quad (23b)$$

$$\frac{\text{VPair} \quad e_1 \text{ val} \quad e_2 \text{ val}}{(e_1, e_2) \text{ val}} \quad (23c)$$

$$\frac{\text{VInl} \quad e \text{ val}}{\text{inl}_\tau(e) \text{ val}} \quad (23d)$$

$$\frac{\text{VInr} \quad e \text{ val}}{\text{inr}_\tau(e) \text{ val}} \quad (23e)$$

$\boxed{e \text{ indet}}$ e is indeterminate

$$\frac{\text{IEHole}}{\llbracket \cdot \rrbracket^u \text{ indet}} \quad (24a)$$

$$\frac{\text{IHole} \quad e \text{ final}}{\llbracket e \rrbracket^u \text{ indet}} \quad (24b)$$

$$\frac{\text{IAp} \quad e_1 \text{ indet} \quad e_2 \text{ final}}{e_1(e_2) \text{ indet}} \quad (24c)$$

$$\frac{\text{IPairL} \quad e_1 \text{ indet} \quad e_2 \text{ val}}{(e_1, e_2) \text{ indet}} \quad (24d)$$

$$\frac{\text{IPairR} \quad e_1 \text{ val} \quad e_2 \text{ indet}}{(e_1, e_2) \text{ indet}} \quad (24e)$$

$$\frac{\text{IPair} \quad e_1 \text{ indet} \quad e_2 \text{ indet}}{(e_1, e_2) \text{ indet}} \quad (24f)$$

$$\frac{\text{IPrl} \quad e \text{ indet}}{\text{prl}(e) \text{ indet}} \quad (24g)$$

$$\frac{\text{IPrr} \quad e \text{ \texttt{indet}}}{\text{pr}(e) \text{ \texttt{indet}}} \quad (24\text{h})$$

$$\frac{\text{IInL} \quad e \text{ \texttt{indet}}}{\text{inl}_\tau(e) \text{ \texttt{indet}}} \quad (24\text{i})$$

$$\frac{\text{IInR} \quad e \text{ \texttt{indet}}}{\text{inr}_\tau(e) \text{ \texttt{indet}}} \quad (24\text{j})$$

$$\frac{\text{IMatch} \quad e \text{ \texttt{final}} \quad e ? p_r}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \text{ \texttt{indet}}} \quad (24\text{k})$$

$$\boxed{e \text{ \texttt{final}}} \quad e \text{ is final}$$

$$\frac{\text{FVal} \quad e \text{ \texttt{val}}}{e \text{ \texttt{final}}} \quad (25\text{a})$$

$$\frac{\text{FIndet} \quad e \text{ \texttt{indet}}}{e \text{ \texttt{final}}} \quad (25\text{b})$$

$$\boxed{e \text{ \texttt{notintro}}} \quad e \text{ cannot be a value syntactically}$$

$$\frac{\text{NVEHole}}{\text{⋈}^u \text{ \texttt{notintro}}} \quad (26\text{a})$$

$$\frac{\text{NVHole}}{\text{⋈}(e)^u \text{ \texttt{notintro}}} \quad (26\text{b})$$

$$\frac{\text{NVAp}}{e_1(e_2) \text{ \texttt{notintro}}} \quad (26\text{c})$$

$$\frac{\text{NVMatch}}{\text{match}(e)\{\hat{rs}\} \text{ \texttt{notintro}}} \quad (26\text{d})$$

$$\frac{\text{NVPrI}}{\text{prl}(e) \text{ \texttt{notintro}}} \quad (26\text{e})$$

$$\frac{\text{NVPrR}}{\text{pr}(e) \text{ \texttt{notintro}}} \quad (26\text{f})$$

$$\boxed{\theta : \Gamma} \quad \theta \text{ is of type } \Gamma$$

$$\frac{\text{STEmpty}}{\emptyset : \cdot} \quad (27\text{a})$$

$$\frac{\text{STExtend} \quad \theta : \Gamma_\theta \quad \Gamma ; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau} \quad (27b)$$

$\boxed{p \text{ refutable}}$ p is refutable

$$\frac{\text{RNum}}{\underline{n} \text{ refutable}} \quad (28a)$$

$$\frac{\text{REHole}}{\llbracket \rrbracket^w \text{ refutable}} \quad (28b)$$

$$\frac{\text{RHole}}{\llbracket p \rrbracket^w \text{ refutable}} \quad (28c)$$

$$\frac{\text{RInl}}{\text{inl}(p) \text{ refutable}} \quad (28d)$$

$$\frac{\text{RInr}}{\text{inr}(p) \text{ refutable}} \quad (28e)$$

$$\frac{\text{RPairL} \quad p_1 \text{ refutable}}{(p_1, p_2) \text{ refutable}} \quad (28f)$$

$$\frac{\text{RPairR} \quad p_2 \text{ refutable}}{(p_1, p_2) \text{ refutable}} \quad (28g)$$

$\boxed{e \triangleright p \dashv\!\!\parallel \theta}$ e matches p , emitting θ

$$\frac{\text{MVar}}{e \triangleright x \dashv\!\!\parallel e/x} \quad (29a)$$

$$\frac{\text{MWild}}{e \triangleright _ \dashv\!\!\parallel \cdot} \quad (29b)$$

$$\frac{\text{MNum}}{\underline{n} \triangleright \underline{n} \dashv\!\!\parallel \cdot} \quad (29c)$$

$$\frac{\text{MPair} \quad e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1 \quad e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta_1 \uplus \theta_2} \quad (29d)$$

$$\frac{\text{MInl} \quad e \triangleright p \dashv\!\!\parallel \theta}{\text{inl}_\tau(e) \triangleright \text{inl}(p) \dashv\!\!\parallel \theta} \quad (29e)$$

$$\frac{\text{MInr} \quad e \triangleright p \dashv\!\!\parallel \theta}{\text{inr}_\tau(e) \triangleright \text{inr}(p) \dashv\!\!\parallel \theta} \quad (29f)$$

$$\frac{\text{MNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \triangleright p_1 \dashv\!\parallel \theta_1 \quad \text{prl}(e) \triangleright p_2 \dashv\!\parallel \theta_2}{e \triangleright (p_1, p_2) \dashv\!\parallel \theta_1 \uplus \theta_2} \quad (29g)$$

$e ? p$ e may match p

$$\frac{\text{MMEHole}}{e ? \langle \rangle^w} \quad (30a)$$

$$\frac{\text{MMHole}}{e ? \langle p \rangle^w} \quad (30b)$$

$$\frac{\text{MMNotIntro} \quad e \text{ notintro} \quad p \text{ refutable}}{e ? p} \quad (30c)$$

$$\frac{\text{MMPairL} \quad e_1 ? p_1 \quad e_2 \triangleright p_2 \dashv\!\parallel \theta_2}{(e_1, e_2) ? (p_1, p_2)} \quad (30d)$$

$$\frac{\text{MMPairR} \quad e_1 \triangleright p_1 \dashv\!\parallel \theta_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (30e)$$

$$\frac{\text{MMPair} \quad e_1 ? p_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (30f)$$

$$\frac{\text{MMInl} \quad e ? p}{\text{inl}_\tau(e) ? \text{inl}(p)} \quad (30g)$$

$$\frac{\text{MMInr} \quad e ? p}{\text{inr}_\tau(e) ? \text{inr}(p)} \quad (30h)$$

$e \perp p$ e does not match p

$$\frac{\text{NMNum} \quad n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}} \quad (31a)$$

$$\frac{\text{NMPairL} \quad e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)} \quad (31b)$$

$$\frac{\text{NMPairR} \quad e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)} \quad (31c)$$

$$\frac{\text{NMConfl}}{\text{inr}_\tau(e) \perp \text{inl}(p)} \quad (31d)$$

$$\frac{\text{NMConfR}}{\text{inl}_\tau(e) \perp \text{inr}(p)} \quad (31\text{e})$$

$$\frac{\text{NMInl}}{e \perp p} \quad (31\text{f})$$

$$\frac{\text{NMInr}}{e \perp p} \quad (31\text{g})$$

$\boxed{e \mapsto e'}$ e takes a step to e'

$$\frac{\text{ITHole}}{e \mapsto e'} \quad (32\text{a})$$

$$\frac{\text{ITApFun}}{e_1 \mapsto e'_1} \quad (32\text{b})$$

$$\frac{\text{ITApArg}}{e_1 \text{ val } e_2 \mapsto e'_2} \quad (32\text{c})$$

$$\frac{\text{ITAP}}{e_2 \text{ val } (\lambda x : \tau. e_1)(e_2) \mapsto [e_2/x]e_1} \quad (32\text{d})$$

$$\frac{\text{ITPairL}}{e_1 \mapsto e'_1} \quad (32\text{e})$$

$$\frac{\text{ITPairR}}{e_1 \text{ val } e_2 \mapsto e'_2} \quad (32\text{f})$$

$$\frac{\text{ITPrL}}{(e_1, e_2) \text{ final}} \quad (32\text{g})$$

$$\frac{\text{ITPrR}}{(e_1, e_2) \text{ final}} \quad (32\text{h})$$

$$\frac{\text{ITInl}}{e \mapsto e'} \quad (32\text{i})$$

$$\frac{\text{ITInr}}{e \mapsto e'} \quad (32\text{j})$$

$$\frac{\text{ITExpMatch} \quad e \mapsto e'}{\text{match}(e)\{\hat{r}s\} \mapsto \text{match}(e')\{\hat{r}s\}} \quad (32k)$$

$$\frac{\text{ITSuccMatch} \quad e \text{ final} \quad e \triangleright p_r \dashv \parallel \theta}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \quad (32l)$$

$$\frac{\text{ITFailMatch} \quad e \text{ final} \quad e \perp p_r}{\text{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \text{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \quad (32m)$$

Lemma 4.0.1. *If $\text{inl}_{\tau_2}(e_1)$ final then e_1 final.*

Proof. By rule induction over Rules (25) on $\text{inl}_{\tau_2}(e_1)$ final.

Case (25a).

(1) $\text{inl}_{\tau_2}(e_1) \text{ val}$ by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23d).

(2) $e_1 \text{ val}$ by assumption
(3) $e_1 \text{ final}$ by Rule (25a) on (2)

Case (25b).

(1) $\text{inl}_{\tau_2}(e_1) \text{ indet}$ by assumption

By rule induction over Rules (24) on (1), only one case applies.

Case (24i).

(2) $e_1 \text{ indet}$ by assumption
(3) $e_1 \text{ final}$ by Rule (25b) on (2)

□

Lemma 4.0.2. *If $\text{inr}_{\tau_1}(e_2)$ final then e_2 final.*

Proof. By rule induction over Rules (25) on $\text{inr}_{\tau_1}(e_2)$ final.

Case (25a).

(1) $\text{inr}_{\tau_1}(e_2) \text{ val}$ by assumption

By rule induction over Rules (23) on (1), only one case applies.

Case (23d).

- | | |
|------------------------|----------------------|
| (2) e_2 val | by assumption |
| (3) e_2 final | by Rule (25a) on (2) |

Case (25b).

- | | |
|---------------------------------------------|---------------|
| (1) $\text{inr}_{\tau_1}(e_2)$ indet | by assumption |
|---------------------------------------------|---------------|

By rule induction over Rules (24) on (1), only one case applies.

Case (24i).

- | | |
|------------------------|----------------------|
| (2) e_2 indet | by assumption |
| (3) e_2 final | by Rule (25b) on (2) |

□

Lemma 4.0.3. *If (e_1, e_2) **final** then e_1 **final** and e_2 **final**.*

Proof. By rule induction over Rules (25) on (e_1, e_2) **final**.

Case (25a).

- | | |
|-----------------------------|---------------|
| (1) (e_1, e_2) val | by assumption |
|-----------------------------|---------------|

By rule induction over Rules (23) on (1), only one case applies.

Case (23c).

- | | |
|------------------------|----------------------|
| (2) e_1 val | by assumption |
| (3) e_2 val | by assumption |
| (4) e_1 final | by Rule (25a) on (2) |
| (5) e_2 final | by Rule (25a) on (3) |

Case (25b).

- | | |
|-------------------------------|---------------|
| (1) (e_1, e_2) indet | by assumption |
|-------------------------------|---------------|

By rule induction over Rules (24) on (1), only three cases apply.

Case (24d).

- | | |
|------------------------|----------------------|
| (2) e_1 indet | by assumption |
| (3) e_2 val | by assumption |
| (4) e_1 final | by Rule (25b) on (2) |
| (5) e_1 final | by Rule (25a) on (3) |

Case (24e).

- | | |
|------------------------|----------------------|
| (2) e_1 val | by assumption |
| (3) e_2 indet | by assumption |
| (4) e_1 final | by Rule (25a) on (2) |

(5) e_1 **final** by Rule (25b) on (3)

Case (24f).

(2) e_1 **indet** by assumption
 (3) e_2 **indet** by assumption
 (4) e_1 **final** by Rule (25b) on (2)
 (5) e_1 **final** by Rule (25b) on (3)

□

Lemma 4.0.4. *There doesn't exist \underline{n} such that \underline{n} **notintro**.*

Proof. By rule induction over Rules (26) on \underline{n} **notintro**, no case applies due to syntactic contradiction. □

Lemma 4.0.5. *There doesn't exist $\text{inl}_\tau(e)$ such that $\text{inl}_\tau(e)$ **notintro**.*

Proof. By rule induction over Rules (26) on $\text{inl}_\tau(e)$ **notintro**, no case applies due to syntactic contradiction. □

Lemma 4.0.6. *There doesn't exist $\text{inr}_\tau(e)$ such that $\text{inr}_\tau(e)$ **notintro**.*

Proof. By rule induction over Rules (26) on $\text{inr}_\tau(e)$ **notintro**, no case applies due to syntactic contradiction. □

Lemma 4.0.7. *There doesn't exist (e_1, e_2) such that (e_1, e_2) **notintro**.*

Proof. By rule induction over Rules (26) on (e_1, e_2) **notintro**, no case applies due to syntactic contradiction. □

Lemma 4.0.8. *If e **final** and e **notintro** then e **indet**.*

Proof Sketch. By rule induction over Rules (26) on e **notintro**, for each case, by rule induction over Rules (23) on e **val** and we notice that e **val** is not derivable. By rule induction over Rules (25) on e **final**, Rule (25a) result in a contradiction with the fact that e **val** is not derivable while Rule (25b) tells us e **indet**. □

Lemma 4.0.9 (Finality). *There doesn't exist such an expression e such that both e **final** and $e \mapsto e'$ for some e'*

Proof. Assume there exists such an e such that both e **final** and $e \mapsto e'$ for some e' then proof by contradiction.

By rule induction over Rules (25) and Rules (32), *i.e.*, over Rules (23) and Rules (32) and over Rules (24) and Rules (32) respectively. The proof can be done by straightforward observation of syntactic contradictions. □

Lemma 4.0.10 (Matching Determinism). *If e **final** and $\cdot; \Delta_e \vdash e : \tau$ and $p : \tau[\xi] \dashv \vdash \Gamma; \Delta$ then exactly one of the following holds*

1. $e \triangleright p \dashv\!\!\vdash \theta$ for some θ

2. $e ? p$

3. $e \perp p$

Proof.

- | | |
|-----------------------------------------------------|---------------|
| (1) $e \text{ final}$ | by assumption |
| (2) $\cdot; \Delta_e \vdash e : \tau$ | by assumption |
| (3) $p : \tau[\xi] \dashv\!\!\vdash \Gamma; \Delta$ | by assumption |

By rule induction over Rules (20) on (3), we would show one conclusion is derivable while the other two are not.

Case (20a).

- | | |
|-----------------------------------------------|---------------|
| (4) $p = x$ | by assumption |
| (5) $e \triangleright x \dashv\!\!\vdash e/x$ | by Rule (29a) |

Assume $e ? x$. By rule induction over Rules (30) on it, only one case applies.

Case (30c).

- | | |
|---------------------------|---------------|
| (6) $x \text{ refutable}$ | by assumption |
|---------------------------|---------------|

By rule induction over Rules (28) on (6), no case applies due to syntactic contradiction.

- | | |
|------------------------|------------------|
| (7) $e ? \overline{x}$ | by contradiction |
|------------------------|------------------|

Assume $e \perp x$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

- | | |
|----------------------------|------------------|
| (8) $e \perp \overline{x}$ | by contradiction |
|----------------------------|------------------|

Case (20b).

- | | |
|--------------------------------------------------|---------------|
| (4) $p = _$ | by assumption |
| (5) $e \triangleright _ \dashv\!\!\vdash \cdot$ | by Rule (29b) |

Assume $e ? _$. By rule induction over Rules (30) on it, only one case applies.

Case (30c).

- | | |
|----------------------------|---------------|
| (6) $_ \text{ refutable}$ | by assumption |
|----------------------------|---------------|

By rule induction over Rules (28) on (6), no case applies due to syntactic contradiction.

(7) $e \not\vdash _$ by contradiction

Assume $e \perp _$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(8) $e \not\vdash _$ by contradiction

Case (20c).

(4) $p = \langle \rangle^w$ by assumption

(5) $e ? \langle \rangle^w$ by Rule (30a)

Assume $e \triangleright \langle \rangle^w \dashv \vdash \theta$ for some θ . By rule induction over Rules (30) on it, no case applies due to syntactic contradiction.

(6) $e \triangleright \langle \rangle^w \dashv \vdash \theta$ by contradiction

Assume $e \perp \langle \rangle^w$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(7) $e \not\vdash \langle \rangle^w$ by contradiction

Case (20d).

(4) $p = \langle p_0 \rangle^w$ by assumption

(5) $e ? \langle p_0 \rangle^w$ by Rule (30b)

Assume $e \triangleright \langle p_0 \rangle^w \dashv \vdash \theta$ for some θ . By rule induction over Rules (30) on it, no case applies due to syntactic contradiction.

(6) $e \triangleright \langle p_0 \rangle^w \dashv \vdash \theta$ by contradiction

Assume $e \perp \langle p_0 \rangle^w$. By rule induction over Rules (31) on it, no case applies due to syntactic contradiction.

(7) $e \not\vdash \langle p_0 \rangle^w$ by contradiction

Case (20e).

(4) $p = \underline{n_2}$ by assumption

(5) $\tau = \mathbf{num}$ by assumption

(6) $\xi = \underline{n_2}$ by assumption

(7) $\underline{n_2}$ **refutable** by Rule (28a)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (8) $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{rs\}$
by assumption
- (9) e **notintro** by Rule
(26a),(26b),(26c),(26d),(26e),(26f)
- (10) $e ? \underline{n_2}$ by Rule (16b) on (7)
and (9)

Assume $e \triangleright \underline{n_2} \dashv\!\!\parallel \theta$ for some θ . By rule induction over it, no case applies due to syntactic contradiction.

- (11) $e \triangleright \underline{n_2} \dashv\!\!\parallel \theta$ by contradiction

Assume $e \perp \underline{n_2}$. By rule induction over it, no case applies due to syntactic contradiction.

- (12) $e \perp \underline{n_2}$ by contradiction

Case (19d).

- (8) $e = \underline{n_1}$

Assume $\underline{n_1} ? \underline{n_2}$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

- (9) $\underline{n_1}$ **notintro** by assumption

Contradicts Lemma 4.0.4.

- (10) $\underline{n_1} ? \underline{n_2}$ by contradiction

By case analysis on whether $n_1 = n_2$.

Case $n_1 = n_2$.

- (11) $n_1 = n_2$ by assumption
- (12) $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\parallel \cdot$ by Rule (29c)

Assume $\underline{n_1} \perp \underline{n_2}$. By rule induction over Rules (31) on it, only one case applies.

Case (31a).

- (13) $n_1 \neq n_2$ by assumption

Contradicts (11).

- (14) $\underline{n_1} \dashv\!\!\parallel \underline{n_2}$ by contradiction

Case $n_1 \neq n_2$.

- (11) $n_1 \neq n_2$ by assumption

- (12) $\underline{n_1} \perp n_2$ by Rule (31a) on (11)
 Assume $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash \theta$ for some θ . By rule induction over Rules (29) on it, no case applies due to syntactic contradiction.
 (13) $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash \theta$ by contradiction

Case (20f).

- (4) $p = \text{inl}(p_1)$ by assumption
 (5) $\tau = (\tau_1 + \tau_2)$ by assumption
 (6) $\xi = \text{inl}(\xi_1)$ by assumption
 (7) $p_1 : \tau_1[\xi_1] \dashv\!\!\vdash \Gamma ; \Delta$ by assumption
 (8) $\text{inl}(p_1)$ **refutable** by Rule (28d)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (9) $e = \text{inl}^u, \text{inl}^u(e_0), e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$
 by assumption
 (10) e **notintro** by Rule (26a),(26b),(26c),(26d),(26e),(26f)
 (11) $e ? \text{inl}(p_1)$ by Rule (16b) on (8) and (10)

Assume $e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1$ for some θ_1 . By rule induction over it, no case applies due to syntactic contradiction.

- (12) $\underline{e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1}$ by contradiction

Assume $e \perp \text{inl}(p_1)$. By rule induction over it, no case applies due to syntactic contradiction.

- (13) $\underline{e \perp \text{inl}(p_1)}$ by contradiction

Case (19j).

- (9) $e = \text{inl}_{\tau_2}(e_1)$ by assumption
 (10) $\cdot ; \Delta_e \vdash e_1 : \tau_1$ by assumption
 (11) e_1 **final** by Lemma 4.0.1 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ for some θ_1 , $e_1 ? p_1$, and $e_1 \perp p_1$ holds.

By case analysis on which one holds.

Case $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$.

- (12) $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ by assumption
 (13) $\underline{e_1 ? p_1}$ by assumption
 (14) $\underline{e_1 \perp p_1}$ by assumption
 (15) $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1$ by Rule (29e) on (12)

Assume $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(16) $\text{inl}_{\tau_2}(e_1) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (30g).

(16) $e_1 ? p_1$ by assumption

Contradicts (13).

(17) $\frac{\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)}{\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)}$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$. By rule induction over Rules (31) on it, only one case applies.

Case (31f).

(18) $e_1 \perp p_1$ by assumption

Contradicts (14).

(19) $\frac{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}$ by contradiction

Case $e_1 ? p_1$.

(12) $\frac{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}$ by assumption

(13) $e_1 ? p_1$ by assumption

(14) $\frac{e_1 \perp p_1}{e_1 \perp p_1}$ by assumption

(15) $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ by Rule (30g) on (13)

Assume $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta$ for some θ . By rule induction over Rules (29) on it, only one case applies.

Case (29e).

(16) $e_1 \triangleright p_1 \dashv\!\!\dashv \theta$ by assumption

Contradicts (12).

(17) $\frac{\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta}{\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta}$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$. By rule induction over Rules (31) on it, only one case applies.

Case (31f).

(18) $e_1 \perp p_1$ by assumption

Contradicts (14).

(19) $\frac{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}$ by contradiction

Case $e_1 \perp p_1$.

(12) $\frac{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}$ by assumption

(13) $\frac{e_1 ? p_1}{e_1 ? p_1}$ by assumption

(14) $e_1 \perp p_1$ by assumption

(15) $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$ by Rule (31f) on (14)

Assume $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta$ for some θ . By rule induction over Rules (29) on it, only one case applies.

Case (29e).

(16) $e_1 \triangleright p_1 \dashv\!\!\dashv \theta$ by assumption

Contradicts (12).

(17) $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\dashv \theta$ by contradiction

Assume $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(18) $\text{inl}_{\tau_2}(e_1) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (30g).

(18) $e_1 ? p_1$ by assumption

Contradicts (13).

(19) $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ by contradiction

Case (20g).

(4) $p = \text{inr}(p_2)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \text{inr}(\xi_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\!\dashv \Gamma ; \Delta$ by assumption

(8) $\text{inr}(p_2) \text{ refutable}$ by Rule (28e)

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

(9) $e = \text{inl}^u, \text{inl}^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$
by assumption

(10) $e \text{ notintro}$ by Rule
(26a),(26b),(26c),(26d),(26e),(26f)

(11) $e ? \text{inr}(p_2)$ by Rule (16b) on (8)
and (10)

Assume $e \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta_2$ for some θ_2 . By rule induction over it, no case applies due to syntactic contradiction.

(12) $e \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta_2$ by contradiction

Assume $e \perp \text{inr}(p_2)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $e \perp \overline{\text{inr}(p_2)}$ by contradiction

Case (19k).

(9) $e = \text{inr}_{\tau_1}(e_2)$ by assumption

(10) $\cdot; \Delta_e \vdash e_2 : \tau_2$ by assumption

(11) $e_2 \text{ final}$ by Lemma 4.0.2 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$ for some θ_2 , $e_2 ? p_2$, and $e_2 \perp p_2$ holds.

By case analysis on which one holds.

Case $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$.

(12) $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$ by assumption

(13) $e_2 ? \overline{p_2}$ by assumption

(14) $e_2 \perp \overline{p_2}$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta_2$ by Rule (29f) on (12)

Assume $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(16) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.6.

Case (30h).

(16) $e_2 ? p_2$ by assumption

Contradicts (13).

(17) $\overline{\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)}$ by contradiction

Assume $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$. By rule induction over Rules (31) on it, only one case applies.

Case (31g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\overline{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}$ by contradiction

Case $e_2 ? p_2$.

(12) $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$ by assumption

(13) $e_2 ? p_2$ by assumption

(14) $e_2 \perp \overline{p_2}$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ by Rule (30h) on (13)

Assume $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta$ for some θ . By rule induction over Rules (29) on it, only one case applies.

Case (29f).

(16) $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$ by assumption

Contradicts (12).

(17) $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta}{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)}$ by contradiction

Assume $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$. By rule induction over Rules (31) on it, only one case applies.

Case (31g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\frac{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)}$ by contradiction

Case $e_2 \perp p_2$.

(12) $\frac{e_2 \triangleright p_2 \dashv\vdash \theta}{e_2 \triangleright p_2}$ by assumption

(13) $\frac{e_2 \triangleright p_2}{e_2 \perp p_2}$ by assumption

(14) $e_2 \perp p_2$ by assumption

(15) $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$ by Rule (31g) on (14)

Assume $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta$ for some θ . By rule induction over Rules (29) on it, only one case applies.

Case (29f).

(16) $e_2 \triangleright p_2 \dashv\vdash \theta$ by assumption

Contradicts (12).

(17) $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta}{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)}$ by contradiction

Assume $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)$. By rule induction over Rules (30) on it, only two cases apply.

Case (30c).

(18) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.6.

Case (30h).

(18) $e_2 \triangleright p_2$ by assumption

Contradicts (13).

(19) $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2)}{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}$ by contradiction

Case (20h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\tau = (\tau_1 \times \tau_2)$ by assumption

(6) $\xi = (\xi_1, \xi_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

- (9) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 ; \Delta_1$ by assumption
 (10) $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 ; \Delta_2$ by assumption

By rule induction over Rules (19) on (2), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (11) $e = \text{new } u, \text{new } e_0^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$
 by assumption
 (12) e **notintro** by Rule (26a),(26b),(26c),(26d),(26e),(26f)
 (13) e **indet** by Lemma 4.0.8 on (1) and (12)
 (14) $\text{prl}(e)$ **indet** by Rule (24g) on (13)
 (15) $\text{prl}(e)$ **final** by Rule (25b) on (14)
 (16) $\text{prl}(e)$ **indet** by Rule (24h) on (13)
 (17) $\text{prl}(e)$ **final** by Rule (25b) on (16)
 (18) $\cdot ; \Delta \vdash \text{prl}(e) : \tau_1$ by Rule (19h) on (2)
 (19) $\cdot ; \Delta \vdash \text{prl}(e) : \tau_2$ by Rule (19i) on (2)

Assume $e \perp (p_1, p_2)$. By rule induction on it, no case applies due to syntactic contradiction.

- (20) $e \perp \overline{(p_1, p_2)}$ by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$, $\text{prl}(e) ? p_1$, and $\text{prl}(e) \perp p_1$ holds.

By inductive hypothesis on (17) and (19) and (10), exactly one of $\text{prl}(e) \triangleright p_2 \dashv\vdash \theta_2$, $\text{prl}(e) ? p_2$, and $\text{prl}(e) \perp p_2$ holds.

By case analysis on which conclusion holds for p_1 and p_2 . Note that we have already shown $e \perp \overline{(p_1, p_2)}$.

Case $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1, \text{prl}(e) \triangleright p_2 \dashv\vdash \theta_2$.

- (21) $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$ by assumption
 (22) $\overline{\text{prl}(e) ? p_1}$ by assumption
 (23) $\overline{\text{prl}(e) \perp p_1}$ by assumption
 (24) $\text{prl}(e) \triangleright p_2 \dashv\vdash \theta_2$ by assumption
 (25) $\overline{\text{prl}(e) ? p_2}$ by assumption
 (26) $\overline{\text{prl}(e) \perp p_2}$ by assumption
 (27) $e \triangleright (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2$ by Rule (29g) on (12) and (21) and (24)

Assume $e ? (p_1, p_2)$. By rule induction over Rules (30) on it, only one case applies.

Case (30c).

- (28) (p_1, p_2) **refutable** by assumption

By rule induction over Rules (28), only two cases apply.

Case (28f).

- | | |
|-------------------------------|-----------------------------------|
| (29) p_1 refutable | by assumption |
| (30) $\text{prl}(e)$ notintro | by Rule (26e) |
| (31) $\text{prl}(e) ? p_1$ | by Rule (30c) on (29)
and (30) |

Contradicts (22).

Case (28g).

- | | |
|-------------------------------|-----------------------------------|
| (29) p_2 refutable | by assumption |
| (30) $\text{prr}(e)$ notintro | by Rule (26f) |
| (31) $\text{prl}(e) ? p_1$ | by Rule (30c) on (29)
and (30) |

Contradicts (22).

- | | |
|----------------------------------|------------------|
| (32) $e ? (\overline{p_1, p_2})$ | by contradiction |
|----------------------------------|------------------|

Case $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1, \text{prr}(e) ? p_2$.

- | | |
|------------------------------------------------------------------------------|---------------|
| (21) $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$ | by assumption |
| (22) $\overline{\text{prl}(e) ? p_1}$ | by assumption |
| (23) $\overline{\text{prl}(e) \perp p_1}$ | by assumption |
| (24) $\overline{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}$ | by assumption |
| (25) $\text{prr}(e) ? p_2$ | by assumption |
| (26) $\overline{\text{prr}(e) \perp p_2}$ | by assumption |

Assume $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$. By rule induction over Rules (29), only one case applies.

Case (29g).

- | | |
|-------------------------------------------------------------------|---------------|
| (27) $\theta = \theta_1 \uplus \theta_2$ | by assumption |
| (28) $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ | by assumption |

Contradicts (24).

- | | |
|-----------------------------------------------------------------------|------------------|
| (29) $e \triangleright (\overline{p_1, p_2}) \dashv\!\!\vdash \theta$ | by contradiction |
|-----------------------------------------------------------------------|------------------|

By rule induction over Rules (30) on (25), the following cases apply.

Case (30a),(30b).

- | | |
|-----------------------------------------------------------------------|-----------------------------------|
| (30) $p_2 = \llbracket \cdot \rrbracket^w, \llbracket p \rrbracket^w$ | by assumption |
| (31) p_2 refutable | by Rule (28b) and Rule (28c) |
| (32) (p_1, p_2) refutable | by Rule (28g) on (31) |
| (33) $e ? (p_1, p_2)$ | by Rule (30c) on (12)
and (32) |

Case (30c).

- | | |
|----------------------|---------------|
| (30) p_2 refutable | by assumption |
|----------------------|---------------|

- (31) (p_1, p_2) refutable by Rule (28g) on (30)
 (32) $e ? (p_1, p_2)$ by Rule (30c) on (12)
 and (31)

Case $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1, \text{prr}(e) \perp p_2$.

- (21) $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$ by assumption
 (22) $\text{prl}(e) ? p_1$ by assumption
 (23) $\text{prl}(e) \perp p_1$ by assumption
 (24) $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ by assumption
 (25) $\text{prr}(e) ? p_2$ by assumption
 (26) $\text{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (31) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\text{prl}(e) ? p_1, \text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$.

- (21) $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$ by assumption
 (22) $\text{prl}(e) ? p_1$ by assumption
 (23) $\text{prl}(e) \perp p_1$ by assumption
 (24) $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ by assumption
 (25) $\text{prr}(e) ? p_2$ by assumption
 (26) $\text{prr}(e) \perp p_2$ by assumption

Assume $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$. By rule induction over Rules (29), only one case applies.

Case (29g).

- (27) $\theta = \theta_1 \uplus \theta_2$ by assumption
 (28) $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$ by assumption

Contradicts (21).

- (29) $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ by contradiction

By rule induction over Rules (30) on (22), the following cases apply.

Case (30a),(30b).

- (30) $p_1 = \langle \rangle^w, \langle p \rangle^w$ by assumption
 (31) p_1 refutable by Rule (28b) and Rule (28c)
 (32) (p_1, p_2) refutable by Rule (28g) on (31)
 (33) $e ? (p_1, p_2)$ by Rule (30c) on (12)
 and (32)

Case (30c).

- (30) p_1 refutable by assumption
 (31) (p_1, p_2) refutable by Rule (28g) on (30)

(32) $e ? (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case $\text{prl}(e) ? p_1, \text{prr}(e) ? p_2$.

(21) $\frac{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}{\text{prl}(e) ? p_1}$ by assumption
 (22) $\text{prl}(e) ? p_1$ by assumption
 (23) $\frac{\text{prl}(e) \perp p_1}{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}$ by assumption
 (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}{\text{prr}(e) ? p_2}$ by assumption
 (25) $\text{prr}(e) ? p_2$ by assumption
 (26) $\frac{\text{prr}(e) \perp p_2}{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}$ by assumption

Assume $e \triangleright (p_1, p_2) \dashv \vdash \theta$. By rule induction over Rules (29), only one case applies.

Case (29g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1$ by assumption

Contradicts (21).

(29) $\frac{e \triangleright (p_1, p_2) \dashv \vdash \theta}{e \triangleright (p_1, p_2) \dashv \vdash \theta}$ by contradiction

By rule induction over Rules (30) on (22), the following cases apply.

Case (30a),(30b).

(30) $p_1 = \langle \rangle^w, \langle p \rangle^w$ by assumption
 (31) p_1 **refutable** by Rule (28b) and Rule (28c)
 (32) (p_1, p_2) **refutable** by Rule (28g) on (31)
 (33) $e ? (p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c).

(30) p_1 **refutable** by assumption
 (31) (p_1, p_2) **refutable** by Rule (28g) on (30)
 (32) $e ? (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case $\text{prl}(e) ? p_1, \text{prr}(e) \perp p_2$.

(21) $\frac{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}{\text{prl}(e) ? p_1}$ by assumption
 (22) $\text{prl}(e) ? p_1$ by assumption
 (23) $\frac{\text{prl}(e) \perp p_1}{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}$ by assumption
 (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}{\text{prr}(e) ? p_2}$ by assumption
 (25) $\text{prr}(e) ? p_2$ by assumption
 (26) $\text{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (31) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\text{prl}(e) \perp p_1, \text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$.

- | | |
|------------------------------------------------------------------------------------------------------------|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \perp p_1}$ | by assumption |
| (22) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \triangleright p_1}$ | by assumption |
| (23) $\text{prl}(e) \perp p_1$ | by assumption |
| (24) $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ | by assumption |
| (25) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \perp p_2}$ | by assumption |
| (26) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \triangleright p_2}$ | by assumption |

By rule induction over Rules (31) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\text{prl}(e) \perp p_1, \text{prr}(e) ? p_2$.

- | | |
|------------------------------------------------------------------------------------------------------------|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \perp p_1}$ | by assumption |
| (22) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \triangleright p_1}$ | by assumption |
| (23) $\text{prl}(e) \perp p_1$ | by assumption |
| (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \perp p_2}$ | by assumption |
| (25) $\text{prr}(e) ? p_2$ | by assumption |
| (26) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \triangleright p_2}$ | by assumption |

By rule induction over Rules (31) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\text{prl}(e) \perp p_1, \text{prr}(e) \perp p_2$.

- | | |
|------------------------------------------------------------------------------------------------------------|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \perp p_1}$ | by assumption |
| (22) $\frac{\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{prl}(e) \triangleright p_1}$ | by assumption |
| (23) $\text{prl}(e) \perp p_1$ | by assumption |
| (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \perp p_2}$ | by assumption |
| (25) $\text{prr}(e) \perp p_2$ | by assumption |
| (26) $\frac{\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2}{\text{prr}(e) \triangleright p_2}$ | by assumption |

By rule induction over Rules (31) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case (19g).

- | | |
|------------------------------------------|-----------------------|
| (11) $e = (e_1, e_2)$ | by assumption |
| (12) $\cdot; \Delta \vdash e_1 : \tau_1$ | by assumption |
| (13) $\cdot; \Delta \vdash e_2 : \tau_2$ | by assumption |
| (14) e_1 final | by Lemma 4.0.3 on (1) |
| (15) e_2 final | by Lemma 4.0.3 on (1) |

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$, $e_1 ? p_1$, and $e_1 \perp p_1$ holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$, $e_2 ? p_2$, and $e_2 \perp p_2$ holds.

By case analysis on which conclusion holds for p_1 and p_2 .

Case $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1, e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$.

- | | | |
|------|----------------------------------------------------------------------------------|--------------------------------|
| (16) | $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ | by assumption |
| (17) | $\underline{e_1 ? p_1}$ | by assumption |
| (18) | $\underline{e_1 \perp p_1}$ | by assumption |
| (19) | $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ | by assumption |
| (20) | $\underline{e_2 ? p_2}$ | by assumption |
| (21) | $\underline{e_2 \perp p_2}$ | by assumption |
| (22) | $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2$ | by Rule (29d) on (16) and (19) |

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

- | | | |
|------|-------------------------------|---------------|
| (23) | $(e_1, e_2) \text{ notintro}$ | by assumption |
|------|-------------------------------|---------------|

Contradicts Lemma 4.0.7.

Case (30d).

- | | | |
|------|-------------|---------------|
| (23) | $e_1 ? p_1$ | by assumption |
|------|-------------|---------------|

Contradicts (17).

Case (30e).

- | | | |
|------|-------------|---------------|
| (23) | $e_2 ? p_2$ | by assumption |
|------|-------------|---------------|

Contradicts (20).

Case (30f).

- | | | |
|------|-------------|---------------|
| (23) | $e_1 ? p_1$ | by assumption |
|------|-------------|---------------|

Contradicts (17).

- | | | |
|------|---------------------------------------|------------------|
| (24) | $\underline{(e_1, e_2) ? (p_1, p_2)}$ | by contradiction |
|------|---------------------------------------|------------------|

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

- | | | |
|------|-----------------|---------------|
| (25) | $e_1 \perp p_1$ | by assumption |
|------|-----------------|---------------|

Contradicts (18).

Case (31c).

- | | | |
|------|-----------------|---------------|
| (25) | $e_2 \perp p_2$ | by assumption |
|------|-----------------|---------------|

Contradicts (21).

- | | | |
|------|-------------------------------------------|------------------|
| (26) | $\underline{(e_1, e_2) \perp (p_1, p_2)}$ | by contradiction |
|------|-------------------------------------------|------------------|

Case $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1, e_2 ? p_2$.

- | | | |
|------|----------------------------------------------------------------|---------------|
| (16) | $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ | by assumption |
| (17) | $\underline{e_1 ? p_1}$ | by assumption |
| (18) | $\underline{e_1 \perp p_1}$ | by assumption |
| (19) | $\underline{e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2}$ | by assumption |

- (20) $e_2 ? p_2$ by assumption
 (21) $\overline{e_2 \perp p_2}$ by assumption
 (22) $(e_1, e_2) ? (p_1, p_2)$ by Rule (30e) on (16) and (20)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

- (23) $\theta = \theta_1 \uplus \theta_2$
 (24) $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ by assumption
 Contradicts (19).

- (25) $\overline{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

- (26) $e_1 \perp p_1$ by assumption
 Contradicts (18).

Case (31c).

- (26) $e_2 \perp p_2$ by assumption
 Contradicts (21).

- (27) $\overline{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1, e_2 \perp p_2$.

- (16) $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ by assumption
 (17) $\overline{e_1 ? p_1}$ by assumption
 (18) $\overline{e_1 \perp p_1}$ by assumption
 (19) $\overline{e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2}$ by assumption
 (20) $\overline{e_2 ? p_2}$ by assumption
 (21) $e_2 \perp p_2$ by assumption
 (22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31c) on (21)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

- (23) $\theta = \theta_1 \uplus \theta_2$
 (24) $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ by assumption
 Contradicts (19).

- (25) $\overline{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta}$ by contradiction

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) $(e_1, e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

Case (30e).

(26) $e_2 ? p_2$ by assumption

Contradicts (20).

Case (30f).

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

(27) $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

Case $e_1 ? p_1, e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$.

(16) $\frac{e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1}{\text{by assumption}}$

(17) $e_1 ? p_1$ by assumption

(18) $\frac{e_1 \perp p_1}{\text{by assumption}}$

(19) $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ by assumption

(20) $\frac{e_2 ? p_2}{\text{by assumption}}$

(21) $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22) $(e_1, e_2) ? (p_1, p_2)$ by Rule (30d) on (17) and (19)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ by assumption

Contradicts (16).

(25) $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta}{\text{by contradiction}}$

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (31c).

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\frac{(e_1, e_2) \perp (p_1, p_2)}{\text{by contradiction}}$

Case $e_1 ? p_1, e_2 ? p_2$.

(16) $\frac{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}{\text{by assumption}}$

(17) $e_1 ? p_1$ by assumption

(18) $\frac{e_1 \perp p_1}{\text{by assumption}}$

(19) $\frac{e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2}{\text{by assumption}}$

(20) $e_2 ? p_2$ by assumption

(21) $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22) $(e_1, e_2) ? (p_1, p_2)$ by Rule (30f) on (17) and (20)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$ by assumption

Contradicts (19).

(25) $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta}{\text{by contradiction}}$

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (31) on it, only two cases apply.

Case (31b).

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (31c).

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\frac{(e_1, e_2) \perp (p_1, p_2)}{\text{by contradiction}}$

Case $e_1 ? p_1, e_2 \perp p_2$.

(16) $\frac{e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1}{\text{by assumption}}$

(17) $e_1 ? p_1$ by assumption

(18) $\frac{e_1 \perp p_1}{\text{by assumption}}$

(19) $\frac{e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2}{\text{by assumption}}$

(20) $\frac{e_2 ? p_2}{\text{by assumption}}$

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31c) on (21)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by assumption
 Contradicts (19).

(25) $(e_1, e_2) \triangleright \overline{(p_1, p_2) \dashv\!\!\parallel \theta}$ by contradiction

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by assumption
 Contradicts (19).

Case (30e).

(26) $e_2 ? p_2$ by assumption
 Contradicts (20).

Case (30f).

(26) $e_2 ? p_2$ by assumption
 Contradicts (20).

(27) $(e_1, e_2) ? \overline{(p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$.

(16) $\overline{e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1}$ by assumption

(17) $\overline{e_1 ? p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by assumption

(20) $\overline{e_2 ? p_2}$ by assumption

(21) $\overline{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31b) on (18)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ by assumption
 Contradicts (16).

(25) $(e_1, e_2) \triangleright \overline{(p_1, p_2) \dashv\!\!\parallel \theta}$ by contradiction

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

Case (30e).

(26) $e_2 ? p_2$ by assumption

Contradicts (20).

Case (30f).

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

(27) $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

Case $e_1 \perp p_1, e_2 ? p_2$.

(16) $\frac{e_1 \triangleright p_1 \dashv\!\!\!\vdash \theta_1}{\text{by assumption}}$

(17) $\frac{e_1 ? p_1}{\text{by assumption}}$

(18) $e_1 \perp p_1$ by assumption

(19) $\frac{e_2 \triangleright p_2 \dashv\!\!\!\vdash \theta_2}{\text{by assumption}}$

(20) $e_2 ? p_2$ by assumption

(21) $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31b) on (18)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\vdash \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \triangleright p_2 \dashv\!\!\!\vdash \theta_2$ by assumption

Contradicts (19).

(25) $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\vdash \theta}{\text{by contradiction}}$

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) $(e_1, e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_2 \triangleright p_2 \dashv\!\!\!\vdash \theta_2$ by assumption

Contradicts (19).

Case (30e).

(26) $e_1 \triangleright p_1 \dashv\!\!\!\vdash \theta_1$ by assumption

Contradicts (16).

Case (30f).

(26) $e_1 ? p_1$ by assumption
 Contradicts (17).

(27) $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

Case $e_1 \perp p_1, e_2 \perp p_2$.

(16) $\frac{e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1}{\text{by assumption}}$

(17) $\frac{e_1 ? p_1}{\text{by assumption}}$

(18) $e_1 \perp p_1$ by assumption

(19) $\frac{e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}{\text{by assumption}}$

(20) $e_2 ? p_2$ by assumption

(21) $\frac{e_2 \perp p_2}{\text{by assumption}}$

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (31b) on (18)

Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$. By rule induction over Rules (29) on it, only one case applies.

Case (29d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by assumption

Contradicts (19).

(25) $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta}{\text{by contradiction}}$

Assume $(e_1, e_2) ? (p_1, p_2)$. By rule induction over Rules (30) on it, only four cases apply.

Case (30c).

(26) $(e_1, e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (30d).

(26) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by assumption

Contradicts (19).

Case (30e).

(26) $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ by assumption

Contradicts (16).

Case (30f).

(26) $e_1 ? p_1$ by assumption

Contradicts (17).

(27) $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

□

Lemma 4.0.11 (Matching Coherence of Constraint). *Suppose that $\cdot; \Delta_e \vdash e : \tau$ and e final and $p : \tau[\xi] \dashv\!\!\parallel \Gamma; \Delta$. Then we have*

1. $e \models \xi \text{ iff } e \triangleright p \dashv\!\!\vdash \theta$
2. $e \models_{?} \xi \text{ iff } e ? p$
3. $e \not\models_{?}^{\dagger} \xi \text{ iff } e \perp p$

Proof.

- (1) $\cdot; \Delta_e \vdash e : \tau$ by assumption
- (2) $e \text{ final}$ by assumption
- (3) $p : \tau[\xi] \dashv\!\!\vdash \Gamma ; \Delta$ by assumption

Given Lemma 3.0.1, Theorem 2.1, and Lemma 4.0.10, it is sufficient to prove

1. $e \models \xi \text{ iff } e \triangleright p \dashv\!\!\vdash \theta$
2. $e \models_{?} \xi \text{ iff } e ? p$

By rule induction over Rules (20) on (3).

Case (20a).

- (4) $p = x$ by assumption
- (5) $\xi = \top$ by assumption

1. Prove $e \models \top$ implies $e \triangleright x \dashv\!\!\vdash \theta$ for some θ .

- (6) $e \triangleright x \dashv\!\!\vdash e/x$ by Rule (29a)

2. Prove $e \triangleright x \dashv\!\!\vdash \theta$ implies $e \models \top$.

- (6) $e \models \top$ by Rule (14a)

3. Prove $e \models_{?} \top$ implies $e ? x$.

- (6) $e \not\models_{?} \top$ by Lemma 2.0.3

Vacuously true.

4. Prove $e ? x$ implies $e \models_{?} \top$.

By rule induction over Rules (30), we notice that either, $e ? x$ is in syntactic contradiction with all the cases, or the premise x **refutable** is not derivable. Hence, $e ? x$ are not derivable. And thus vacuously true.

Case (20b).

- (4) $p = _$ by assumption
- (5) $\xi = \top$ by assumption

1. Prove $e \models \top$ implies $e \triangleright _ \dashv\!\!\vdash \theta$ for some θ .

- (6) $e \triangleright _ \dashv\!\!\vdash \cdot$ by Rule (29a)

2. Prove $e \triangleright _ \dashv \! \vdash \theta$ implies $e \models \top$.

(6) $e \models \top$ by Rule (14a)

3. Prove $e \models ? \top$ implies $e ? _$.

(6) $e \not\models ? \top$ by Lemma 2.0.3

Vacuously true.

4. Prove $e ? _$ implies $e \models ? \xi$.

By rule induction over Rules (30), we notice that either, $e ? _$ is in syntactic contradiction with all the cases, or the premise $_ \text{refutable}$ is not derivable. Hence, $e ? _$ are not derivable. And thus vacuously true.

Case (20c).

(4) $p = \langle \rangle^w$ by assumption

(5) $\xi = ?$ by assumption

(6) $\bar{\xi} = ?$ by Definition 9

1. Prove $e \models ?$ implies $e \triangleright \langle \rangle^w \dashv \! \vdash \theta$ for some θ .

(7) $e \not\models ?$ by Rule (29a)

Vacuously true.

2. Prove $e \triangleright \langle \rangle^w \dashv \! \vdash \theta$ implies $e \models ?$.

By rule induction over Rules (29), we notice that $e \triangleright \langle \rangle^w \dashv \! \vdash \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models ?$ implies $e ? \langle \rangle^w$.

(7) $e ? \langle \rangle^w$ by Rule (30a)

4. Prove $e ? \langle \rangle^w$ implies $e \models ? ?$.

(7) $e \models ? ?$ by Rule (16a)

Case (20d).

(4) $p = \langle p_0 \rangle^w$ by assumption

(5) $\xi = ?$ by assumption

1. Prove $e \models ?$ implies $e \triangleright \langle p_0 \rangle^w \dashv \! \vdash \theta$ for some θ .

(6) $e \not\models ?$ by Rule (29a)

Vacuously true.

2. Prove $e \triangleright \langle p_0 \rangle^w \dashv\!\! \dashv \theta$ implies $e \models ?$.
By rule induction over Rules (29), we notice that $e \triangleright \langle p_0 \rangle^w \dashv\!\! \dashv \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.
3. Prove $e \models ?$ implies $e ? \langle p_0 \rangle^w$.
(6) $e ? \langle p_0 \rangle^w$ by Rule (30b)
4. Prove $e ? \langle p_0 \rangle^w$ implies $e \models ?$.
(6) $e \models ?$ by Rule (16a)

Case (20e).

- (4) $p = \underline{n}$ by assumption
- (5) $\xi = \underline{n}$ by assumption
1. Prove $e \models \underline{n}$ implies $e \triangleright \underline{n} \dashv\!\! \dashv \theta$ for some θ .
(6) $e \models \underline{n}$ by assumption
By rule induction over Rules (14) on (6), only one case applies.
Case (14b).
(7) $e = \underline{n}$ by assumption
(8) $\underline{n} \triangleright \underline{n} \dashv\!\! \dashv \cdot$ by Rule (29c)
2. Prove $e \triangleright \underline{n} \dashv\!\! \dashv \theta$ implies $e \models \underline{n}$.
(6) $e \triangleright \underline{n} \dashv\!\! \dashv \theta$ by assumption
By rule induction over Rules (29) on (6), only one case applies.
Case (29c).
(7) $e = \underline{n}$ by assumption
(8) $\theta = \cdot$ by assumption
(9) $\underline{n} \models \underline{n}$ by Rule (14b)
3. Prove $e \models ? \underline{n}$ implies $e ? \underline{n}$.
(6) $e \models ? \underline{n}$ by assumption
By rule induction over Rules (16) on (6), only one case applies.
Case (16b).
(7) e **notintro** by assumption
(8) \underline{n} **refutable** by Rule (28a)
(9) $e ? \underline{n}$ by Rule (30c) on (7) and (8)
4. Prove $e ? \underline{n}$ implies $e \models ? \underline{n}$.
(6) $e ? \underline{n}$ by assumption

By rule induction over Rules (30) on (6), only one case applies.

Case (30c).

- | | |
|--------------------------------------|--------------------------------|
| (7) e notintro | by assumption |
| (8) \underline{n} refutable | by Rule (10a) |
| (9) $e \models? \underline{n}$ | by Rule (16) on (7)
and (8) |

Case (20f).

- | | |
|------------------------------------------------------------|---------------|
| (4) $p = \text{inl}(p_1)$ | by assumption |
| (5) $\xi = \text{inl}(\xi_1)$ | by assumption |
| (6) $\tau = (\tau_1 + \tau_2)$ | by assumption |
| (7) $p_1 : \tau_1[\xi_1] \dashv\!\!\vdash \Gamma ; \Delta$ | by assumption |

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- | | |
|-----------------------------------------------------------------------------------------------------------------|------------------------------------------------|
| (8) $e = \text{inl}^u, \text{inl}^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$ | by assumption |
| (9) e notintro | by Rule
(26a),(26b),(26c),(26d),(26e),(26f) |

1. Prove $e \models \text{inl}(\xi_1)$ implies $e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$ for some θ . By rule induction over Rules (14) on $e \models \text{inl}(\xi_1)$, no case applies due to syntactic contradiction.
Therefore, vacuously true.
2. Prove $e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$ implies $e \models \text{inl}(\xi_1)$. By rule induction over Rules (29) on $e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$, no case applies due to syntactic contradiction.
Therefore, vacuously true.
3. Prove $e \models? \text{inl}(\xi_1)$ implies $e ? \text{inl}(p_1)$.

(10) $\text{inl}(p_1)$ refutable	by Rule (28d)
(11) $e ? \text{inl}(p_1)$	by Rule (30c) on (9) and (10)
4. Prove $e ? \text{inl}(p_1)$ implies $e \models? \text{inl}(\xi_1)$.

(10) $\text{inl}(\xi_1)$ refutable	by Rule (10b)
(11) $e \models? \text{inl}(\xi_1)$	by Rule (16b) on (9) and (10)

Case (19j).

- | | |
|--------------------------------------------|-----------------------|
| (8) $e = \text{inl}_{\tau_2}(e_1)$ | by assumption |
| (9) $\cdot ; \Delta_e \vdash e_1 : \tau_1$ | by assumption |
| (10) e_1 final | by Lemma 4.0.1 on (2) |

By inductive hypothesis on (10) and (9) and (7).

$$(11) \quad e_1 \models \xi_1 \text{ iff } e_1 \triangleright p_1 \dashv\!\!\vdash \theta \text{ for some } \theta$$

$$(12) \quad e_1 \models? \xi_1 \text{ iff } e_1 ? p_1$$

1. Prove $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ implies $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$ for some θ .

$$(13) \quad \text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1) \quad \text{by assumption}$$

By rule induction over Rules (14) on (13), only one case applies.

Case (14g).

$$(14) \quad e_1 \models \xi_1 \quad \text{by assumption}$$

$$(15) \quad e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1 \text{ for some } \theta_1 \quad \text{by (11) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1 \quad \text{by Rule (29e) on (15)}$$

2. Prove $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta$ implies $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$.

$$(13) \quad \text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta \quad \text{by assumption}$$

By rule induction over Rules (29) on (13), only one case applies.

Case (29e).

$$(14) \quad e_1 \triangleright p_1 \dashv\!\!\vdash \theta \quad \text{by assumption}$$

$$(15) \quad e_1 \models \xi_1 \quad \text{by (11) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1) \quad \text{by Rule (14g) on (15)}$$

3. Prove $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ implies $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$.

$$(13) \quad \text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1) \quad \text{by assumption}$$

By rule induction over Rules (16) on (13), only two cases apply.

Case (16b).

$$(14) \quad \text{inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

Contradicts Lemma 4.0.5.

Case (16e).

$$(14) \quad e_1 \models? \xi_1 \quad \text{by assumption}$$

$$(15) \quad e_1 ? p_1 \quad \text{by (12) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1) \quad \text{by Rule (30g) on (15)}$$

4. Prove $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ implies $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$.

$$(13) \quad \text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1) \quad \text{by assumption}$$

By rule induction over Rules (30) on (13), only two cases apply.

Case (30c).

$$(14) \quad \text{inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

Contradicts Lemma 4.0.5.

Case (30g).

$$(14) \quad e_1 ? p_1 \quad \text{by assumption}$$

$$(15) \quad e_1 \models? \xi_1 \quad \text{by (12) on (14)}$$

$$(16) \quad \text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1) \quad \text{by Rule (16e) on (15)}$$

Case (20g).

- (4) $p = \text{inr}(p_2)$ by assumption
- (5) $\xi = \text{inr}(\xi_2)$ by assumption
- (6) $\tau = (\tau_1 + \tau_2)$ by assumption
- (7) $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma ; \Delta$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (8) $e = \text{inl}^u, \text{inl}^u(e_0), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$ by assumption
- (9) $e \text{ notintro}$ by Rule (26a),(26b),(26c),(26d),(26e),(26f)

1. Prove $e \models \text{inr}(\xi_2)$ implies $e \triangleright \text{inr}(p_2) \dashv\vdash \theta$ for some θ . By rule induction over Rules (14) on $e \models \text{inr}(\xi_2)$, no case applies due to syntactic contradiction.
Therefore, vacuously true.
2. Prove $e \triangleright \text{inr}(p_2) \dashv\vdash \theta$ implies $e \models \text{inr}(\xi_2)$. By rule induction over Rules (29) on $e \triangleright \text{inr}(p_2) \dashv\vdash \theta$, no case applies due to syntactic contradiction.
Therefore, vacuously true.
3. Prove $e \models_{\text{?}} \text{inr}(\xi_2)$ implies $e \text{ ? inr}(p_2)$.
 - (10) $\text{inr}(p_2) \text{ refutable}$ by Rule (28e)
 - (11) $e \text{ ? inr}(p_2)$ by Rule (30c) on (9) and (10)
4. Prove $e \text{ ? inr}(p_2)$ implies $e \models_{\text{?}} \text{inr}(\xi_2)$.
 - (10) $\text{inr}(\xi_2) \text{ refutable}$ by Rule (10c)
 - (11) $e \models_{\text{?}} \text{inr}(\xi_2)$ by Rule (16b) on (9) and (10)

Case (19k).

- (8) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
- (9) $\cdot ; \Delta_e \vdash e_2 : \tau_2$ by assumption
- (10) $e_2 \text{ final}$ by Lemma 4.0.1 on (2)

By inductive hypothesis on (10) and (9) and (7).

- (11) $e_2 \models \xi_2$ iff $e_2 \triangleright p_2 \dashv\vdash \theta$ for some θ
- (12) $e_2 \models_{\text{?}} \xi_2$ iff $e_2 \text{ ? } p_2$

1. Prove $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ implies $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\vdash \theta$ for some θ .
 - (13) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ by assumption

By rule induction over Rules (14) on (13), only one case applies.

Case (14g).

- (14) $e_2 \models \xi_2$ by assumption
- (15) $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_1$ for some θ_1 by (11) on (14)
- (16) $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta_1$ by Rule (29e) on (15)

2. Prove $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta$ implies $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$.

- (13) $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta$ by assumption

By rule induction over Rules (29) on (13), only one case applies.

Case (29e).

- (14) $e_2 \triangleright p_2 \dashv\!\!\vdash \theta$ by assumption
- (15) $e_2 \models \xi_2$ by (11) on (14)
- (16) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ by Rule (14g) on (15)

3. Prove $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$ implies $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$.

- (13) $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$ by assumption

By rule induction over Rules (16) on (13), only two cases apply.

Case (16b).

- (14) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (16e).

- (14) $e_2 \models? \xi_2$ by assumption
- (15) $e_2 ? p_2$ by (12) on (14)
- (16) $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ by Rule (30g) on (15)

4. Prove $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ implies $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$.

- (13) $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ by assumption

By rule induction over Rules (30) on (13), only two cases apply.

Case (30c).

- (14) $\text{inr}_{\tau_1}(e_2) \text{ notintro}$ by assumption

Contradicts Lemma 4.0.5.

Case (30g).

- (14) $e_2 ? p_2$ by assumption
- (15) $e_2 \models? \xi_2$ by (12) on (14)
- (16) $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\xi_2)$ by Rule (16e) on (15)

Case (20h).

- (4) $p = (p_1, p_2)$ by assumption
- (5) $\xi = (\xi_1, \xi_2)$ by assumption
- (6) $\tau = (\tau_1 \times \tau_2)$ by assumption
- (7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption
- (8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

- (9) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 ; \Delta_1$ by assumption
 (10) $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 ; \Delta_2$ by assumption

By rule induction over Rules (19) on (1), the following cases apply.

Case (19b),(19c),(19f),(19h),(19i),(19l),(19m).

- (11) $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$
 by assumption
 (12) e **notintro** by Rule (26a),(26b),(26c),(26d),(26e),(26f)
 (13) e **indet** by Lemma 4.0.8 on (2) and (12)
 (14) $\text{prl}(e)$ **indet** by Rule (24g) on (13)
 (15) $\text{prl}(e)$ **final** by Rule (25b) on (14)
 (16) $\text{prr}(e)$ **indet** by Rule (24h) on (13)
 (17) $\text{prr}(e)$ **final** by Rule (25b) on (16)
 (18) $\cdot ; \Delta \vdash \text{prl}(e) : \tau_1$ by Rule (19h) on (1)
 (19) $\cdot ; \Delta \vdash \text{prr}(e) : \tau_2$ by Rule (19i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

- (20) $\text{prl}(e) \models \xi_1$ iff $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$ for some θ_1
 (21) $\text{prl}(e) \models? \xi_1$ iff $\text{prl}(e) ? p_1$
 (22) $\text{prr}(e) \models \xi_2$ iff $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$ for some θ_2
 (23) $\text{prr}(e) \models? \xi_2$ iff $\text{prr}(e) ? p_2$

1. Prove $e \models (\xi_1, \xi_2)$ implies $e \triangleright (p_1, p_2) \dashv\vdash \theta$ for some θ .

- (24) $e \models (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (14) on (24), only one case applies.

Case (14j).

- (25) $\text{prl}(e) \models \xi_1$ by assumption
 (26) $\text{prr}(e) \models \xi_2$ by assumption
 (27) $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$ by (20) on (25)
 (28) $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$ by (22) on (26)
 (29) $e \triangleright (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2$ by Rule (29g) on (12) and (27) and (28)

2. Prove $e \triangleright (p_1, p_2) \dashv\vdash \theta$ implies $e \models (\xi_1, \xi_2)$.

- (24) $e \triangleright (p_1, p_2) \dashv\vdash \theta$ by assumption

By rule induction over Rules (29) on (24), only one case applies.

Case (29g).

- (25) $\theta = \theta_1 \uplus \theta_2$ by assumption
 (26) $\text{prl}(e) \triangleright \xi_1 \dashv\vdash \theta_1$ by assumption

- (27) $\text{prr}(e) \triangleright \xi_2 \dashv\!\!\vdash \theta_2$ by assumption
- (28) $\text{prl}(e) \models \xi_1$ by (20) on (26)
- (29) $\text{prr}(e) \models \xi_2$ by (22) on (27)
- (30) $e \models (\xi_1, \xi_2)$ by Rule (14j) on (12) and (28) and (29)

3. Prove $e \models_{\text{?}} (\xi_1, \xi_2)$ implies $e \text{ ? } (p_1, p_2)$.

- (24) $e \models_{\text{?}} (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (16) on (24), only one case applies.

Case (16b).

- (25) (ξ_1, ξ_2) **refutable** by assumption

By rule induction over Rules (10) on (25), only two cases apply.

Case (10d).

- (26) ξ_1 **refutable** by assumption
- (27) $\text{prl}(e)$ **notintro** by Rule (26e)
- (28) $\text{prl}(e) \models_{\text{?}} \xi_1$ by Rule (16b) on (26) and (27)
- (29) $\text{prl}(e) \text{ ? } p_1$ by (21) on (28)

By rule induction over Rules (30) on (29), only three cases apply.

Case (30a),(30b).

- (30) $p_1 = \langle \rangle^w, \langle p_0 \rangle^w$ by assumption
- (31) p_1 **refutable** by Rule (28b) and Rule (28c)
- (32) (p_1, p_2) **refutable** by Rule (28f) on (31)
- (33) $e \text{ ? } (p_1, p_2)$ by Rule (30c) on (12) and (32)

Case (30c).

- (30) p_1 **refutable** by assumption
- (31) (p_1, p_2) **refutable** by Rule (28f) on (30)
- (32) $e \text{ ? } (p_1, p_2)$ by Rule (30c) on (12) and (31)

Case (10e).

- (26) ξ_2 **refutable** by assumption
- (27) $\text{prr}(e)$ **notintro** by Rule (26e)
- (28) $\text{prr}(e) \models_{\text{?}} \xi_2$ by Rule (16b) on (26) and (27)
- (29) $\text{prr}(e) \text{ ? } p_2$ by (23) on (28)

By rule induction over Rules (30) on (29), only three cases apply.

Case (30a),(30b).

- | | |
|-------------------------------------------------------|--------------------------------|
| (30) $p_2 = \langle \rangle^w, \langle p_0 \rangle^w$ | by assumption |
| (31) p_2 refutable | by Rule (28b) and Rule (28c) |
| (32) (p_1, p_2) refutable | by Rule (28g) on (31) |
| (33) $e ? (p_1, p_2)$ | by Rule (30c) on (12) and (32) |

Case (30c).

- | | |
|-----------------------------|--------------------------------|
| (30) p_2 refutable | by assumption |
| (31) (p_1, p_2) refutable | by Rule (28g) on (30) |
| (32) $e ? (p_1, p_2)$ | by Rule (30c) on (12) and (31) |

4. Prove $e ? (p_1, p_2)$ implies $e \models_{\tau} (\xi_1, \xi_2)$.

- | | |
|-----------------------|---------------|
| (24) $e ? (p_1, p_2)$ | by assumption |
|-----------------------|---------------|

By rule induction over Rules (30) on (24), only one case applies.

Case (30c).

- | | |
|-----------------------------|---------------|
| (25) (p_1, p_2) refutable | by assumption |
|-----------------------------|---------------|

By rule induction over Rules (28) on (25), only two cases apply.

Case (28f).

- | | |
|-------------------------------------------|--------------------------------|
| (26) p_1 refutable | by assumption |
| (27) $\text{prl}(e)$ notintro | by Rule (26e) |
| (28) $\text{prl}(e) ? p_1$ | by Rule (30c) on (26) and (27) |
| (29) $\text{prl}(e) \models_{\tau} \xi_1$ | by (21) on (28) |

By rule induction over Rules (16) on (29), only three cases apply.

Case (16a).

- | | |
|----------------------------------------|--------------------------------|
| (30) $\xi_1 = ?$ | by assumption |
| (31) ξ_1 refutable | by Rule (2b) |
| (32) (ξ_1, ξ_2) refutable | by Rule (10d) on (31) |
| (33) $e \models_{\tau} (\xi_1, \xi_2)$ | by Rule (16b) on (12) and (32) |

Case (16b).

- | | |
|----------------------------------------|--------------------------------|
| (30) ξ_1 refutable | by assumption |
| (31) (ξ_1, ξ_2) refutable | by Rule (10d) on (30) |
| (32) $e \models_{\tau} (\xi_1, \xi_2)$ | by Rule (16b) on (12) and (31) |

Case (28g).

- | | |
|-------------------------------|---------------|
| (26) p_2 refutable | by assumption |
| (27) $\text{prr}(e)$ notintro | by Rule (26e) |

(28) $\text{pr}(e) \text{ ? } p_2$ by Rule (30c) on (26)
and (27)

(29) $\text{pr}(e) \models_{\text{?}} \xi_2$ by (23) on (28)

By rule induction over Rules (16) on (29), only three cases apply.

Case (16a).

(30) $\xi_2 = ?$ by assumption

(31) ξ_2 **refutable** by Rule (2b)

(32) (ξ_1, ξ_2) **refutable** by Rule (10e) on (31)

(33) $e \models_{\text{?}} (\xi_1, \xi_2)$ by Rule (16b) on (12)
and (32)

Case (16b).

(30) ξ_2 **refutable** by assumption

(31) (ξ_1, ξ_2) **refutable** by Rule (10e) on (30)

(32) $e \models_{\text{?}} (\xi_1, \xi_2)$ by Rule (16b) on (12)
and (31)

Case (19g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot; \Delta_e \vdash e_1 : \tau_1$ by assumption

(13) $\cdot; \Delta_e \vdash e_2 : \tau_2$ by assumption

(14) e_1 **final** by Lemma 4.0.3 on (2)

(15) e_2 **final** by Lemma 4.0.3 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16) $e_1 \models \xi_1$ iff $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ for some θ_1

(17) $e_1 \models_{\text{?}} \xi_1$ iff $e_1 \text{ ? } p_1$

(18) $e_2 \models \xi_2$ iff $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ for some θ_2

(19) $e_2 \models_{\text{?}} \xi_2$ iff $e_2 \text{ ? } p_2$

1. Prove $(e_1, e_2) \models (\xi_1, \xi_2)$ implies $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ for some θ .

(20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (14) on (20), only two cases apply.

Case (14i).

(21) $e_1 \models \xi_1$ by assumption

(22) $e_2 \models \xi_2$ by assumption

(23) $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$ for some θ_1 by (16) on (21)

(24) $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$ for some θ_2 by (18) on (22)

(25) $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2$ by Rule (29d) on (23)
and (24)

Case (14j).

(21) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

2. Prove $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ implies $(e_1, e_2) \models (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ by assumption

By rule induction over Rules (29) on (20), only two cases apply.

Case (29d).

(21) $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ for some θ_1 by assumption

(22) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ for some θ_2 by assumption

(23) $e_1 \models \xi_1$ by (16) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (14i) on (23) and (24)

Case (29g).

(21) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

3. Prove $(e_1, e_2) \models? (\xi_1, \xi_2)$ implies $(e_1, e_2) ? (p_1, p_2)$.

(20) $(e_1, e_2) \models? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (16) on (20), only four cases apply.

Case (16b).

(21) (e_1, e_2) **notintro** by assumption

Contradicts Lemma 4.0.7.

Case (16g).

(21) $e_1 \models? \xi_1$ by assumption

(22) $e_2 \models \xi_2$ by assumption

(23) $e_1 ? p_1$ by (17) on (21)

(24) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by (18) on (22)

(25) $(e_1, e_2) ? (p_1, p_2)$ by Rule (30d) on (23) and (24)

Case (16h).

(21) $e_1 \models \xi_1$ by assumption

(22) $e_2 \models? \xi_2$ by assumption

(23) $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ by (16) on (21)

(24) $e_2 ? p_2$ by (19) on (22)

(25) $(e_1, e_2) ? (p_1, p_2)$ by Rule (30e) on (23) and (24)

Case (16i).

(21) $e_1 \models? \xi_1$ by assumption

(22) $e_2 \models? \xi_2$ by assumption

(23) $e_1 ? p_1$ by (17) on (21)

- (24) $e_2 ? p_2$ by (19) on (22)
 (25) $(e_1, e_2) ? (p_1, p_2)$ by Rule (30f) on (23) and (24)
4. Prove $(e_1, e_2) ? (p_1, p_2)$ implies $(e_1, e_2) \models? (\xi_1, \xi_2)$.
 (20) $(e_1, e_2) ? (p_1, p_2)$ by assumption
 By rule induction over Rules (30) on (20), only four cases apply.
Case (30c).
 (21) (e_1, e_2) **notintro** by assumption
 Contradicts Lemma 4.0.7.
Case (30d).
 (21) $e_1 ? p_1$ by assumption
 (22) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ by assumption
 (23) $e_1 \models? \xi_1$ by (17) on (21)
 (24) $e_2 \models \xi_2$ by (18) on (22)
 (25) $(e_1, e_2) ? (p_1, p_2)$ by Rule (16g) on (23) and (24)
Case (30e).
 (21) $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ by assumption
 (22) $e_2 ? p_2$ by assumption
 (23) $e_1 \models \xi_1$ by (16) on (21)
 (24) $e_2 \models? \xi_2$ by (19) on (22)
 (25) $(e_1, e_2) ? (p_1, p_2)$ by Rule (16h) on (23) and (24)
Case (30f).
 (21) $e_1 ? p_1$ by assumption
 (22) $e_2 ? p_2$ by assumption
 (23) $e_1 \models? \xi_1$ by (17) on (21)
 (24) $e_2 \models? \xi_2$ by (19) on (22)
 (25) $(e_1, e_2) ? (p_1, p_2)$ by Rule (16i) on (23) and (24)

□

5 Preservation and Progress

Theorem 5.1 (Preservation). *If $\cdot ; \Delta \vdash e : \tau$ and $e \mapsto e'$ then $\cdot ; \Delta \vdash e' : \tau$*

Proof. By rule induction over Rules (19) on typing judgment of e . For simplicity, we only consider two cases for match expressions here.

Case (19l).

- (1) $\cdot ; \Delta \vdash \text{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

- (2) $\text{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$ by assumption
- (3) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
- (4) $\cdot; \Delta \vdash [\perp](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ by assumption
- (5) $\top \models_{\tau}^{\dagger} \xi$ by assumption

By rule induction over Rules (32) on (2).

Case (32k).

- (6) $e' = \text{match}(e'_1)\{\cdot \mid r \mid rs\}$ by assumption
- (7) $e_1 \mapsto e'_1$ by assumption
- (8) $\cdot; \Delta \vdash e'_1 : \tau_1$ by IH on (3) and (7)
- (9) $\cdot; \Delta \vdash \text{match}(e'_1)\{\cdot \mid r \mid rs\} : \tau$ by Rule (19l) on (8) and (4) and (5)

Case (32l).

- (6) $r = p_r \Rightarrow e_r$ by assumption
- (7) $e' = [\theta](e_r)$ by assumption
- (8) $e_1 \triangleright p_r \dashv \theta$ by assumption

By rule induction over Rules (22) on (4).

Case (22a).

- (9) $\xi = \xi_r$ by assumption
- (10) $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (11) $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (10)
- (12) $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (10)
- (13) $\theta : \Gamma_r$ by Lemma 3.0.7 on (3) and (11) and (8)
- (14) $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (12) and (13)

Case (22b).

- (9) $\xi = \xi_r \vee \xi_{rs}$ by assumption
- (10) $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (11) $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (10)
- (12) $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (10)
- (13) $\theta : \Gamma_r$ by Lemma 3.0.7 on (3) and (11) and (8)
- (14) $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (12) and (13)

Case (32m).

- (6) $rs = r' \mid rs'$ by assumption
- (7) $e' = \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by assumption
- (8) $e_1 \text{ final}$ by assumption
- (9) $e_1 \perp p_r$ by assumption

By rule induction over Rules (22) on (4).

Case (22a). Syntactic contradiction of rs .

Case (22b).

- (10) $\xi = \xi_r \vee \xi_{rs}$ by assumption
- (11) $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (12) $\cdot; \Delta \vdash [\perp \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$ by assumption
- (13) $\xi_r \not\equiv \perp$ by assumption
- (14) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (11)
- (15) $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (11)
- (16) $\cdot; \Delta \vdash [\perp](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$ by Rule (22a) on (11) and (13)
- (17) $e_1 \not\equiv_{\tau}^{\dagger} \xi_r$ by Lemma 4.0.11 on (3) and (8) and (14) and (9)
- (18) $\cdot; \Delta \vdash \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$ by Rule (19m) on (3) and (8) and (16) and (12) and (17) and (5)

Case (19m).

- (1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption
- (2) $\cdot; \Delta \vdash \text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption
- (3) $\text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$ by assumption
- (4) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
- (5) $e_1 \text{ final}$ by assumption
- (6) $\cdot; \Delta \vdash [\perp]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$ by assumption
- (7) $\cdot; \Delta \vdash [\perp \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption
- (8) $e_1 \not\equiv_{\tau}^{\dagger} \xi_{pre}$ by assumption

(9) $\top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (32) on (3).

Case (32k).

(10) $e' = \text{match}(e'_1)\{rs_{pre} \mid r \mid rs_{post}\}$ by assumption

(11) $e_1 \mapsto e'_1$ by assumption

By Lemma 4.0.9, (11) contradicts (5).

Case (32l).

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $e' = [\theta](e_r)$ by assumption

(12) $e_1 \triangleright p_r \dashv \theta$ by assumption

By rule induction over Rules (22) on (7).

Case (22a).

(13) $\xi_{rest} = \xi_r$ by assumption

(14) $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (14)

(16) $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (14)

(17) $\theta : \Gamma_r$ by Lemma 3.0.7 on (4) and (15) and (12)

(18) $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (16) and (17)

Case (22b).

(13) $\xi_{rest} = \xi_r \vee \xi_{rs}$ by assumption

(14) $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$ by assumption

(16) $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by assumption

(17) $\theta : \Gamma_r$ by Lemma 3.0.7 on (4) and (15) and (12)

(18) $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (16) and (17)

Case (32m).

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $rs_{post} = r' \mid rs'$ by assumption

(12) $e' = \text{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond \mid r' \mid rs'\}$ by assumption

(13) $e_1 \perp p_r$ by assumption

By rule induction over Rules (22) on (7).

Case (22a). Syntactic contradiction of rs_{post} .

Case (22b).

- (14) $\xi_{rest} = \xi_r \vee \xi_{post}$ by assumption
- (15) $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (16) $\cdot; \Delta \vdash [\perp \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$
by assumption
- (17) $\xi_r \not\vdash \xi_{pre}$ by assumption
- (18) $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (15)
- (19) $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (21a) on (15)
- (20) $\xi_r : \tau_1$ by Lemma 3.0.2 on (15)
- (21) $\xi_{pre} : \tau_1$ by Lemma 3.0.3 on (6)
- (22) $\xi_r \not\vdash \perp \vee \xi_{pre}$ by Lemma 2.0.6 on (20) and (21) and (17)
- (23) $\cdot; \Delta \vdash [\perp](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$
by Lemma 3.0.4 on (6) and (15) and (22)
- (24) $e_1 \not\vdash_{\tau}^\dagger \xi_r$ by Lemma 4.0.11 on (4) and (5) and (18) and (13)
- (25) $e_1 \not\vdash_{\tau}^\dagger \xi_{pre} \vee \xi_r$ by Lemma 2.0.7 on (8) and (24)
- (26) $\cdot; \Delta \vdash \text{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond \mid r' \mid rs'\} : \tau$
by Rule (19m) on (4) and (5) and (23) and (16) and (25) and (9)

□

Theorem 5.2 (Progress). *If $\cdot; \Delta \vdash e : \tau$ then either e final or $e \mapsto e'$ for some e' .*

Proof. By rule induction over Rules (19) on typing judgment of e . For simplicity, we only consider two cases for match expressions here.

Case (19l).

- (1) $\cdot; \Delta \vdash \text{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption
- (2) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
- (3) $\cdot; \Delta \vdash [\perp](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ by assumption
- (4) $\top \vdash_{\tau}^\dagger \xi$ by assumption

By IH on (2).

Case Scrutinee takes a step.

- (5) $e_1 \mapsto e'_1$ by assumption
- (6) $\text{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \text{match}(e'_1)\{\cdot \mid r \mid rs\}$ by Rule (32k) on (5)

Case Scrutinee is final.

- (5) e_1 **final** by assumption

By rule induction over Rules (22) on (3).

Case (22a).

- (6) $rs = \cdot$ by assumption
- (7) $\xi = \xi_r$ by assumption
- (8) $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (8)
- (10) $p_r : \tau_1[\xi_r] \dashv\!\parallel \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (8)
- (11) $e_1 \models_{\tau}^{\dagger} \xi_r$ by Corollary 2.1.1 on (5) and (4)

By rule induction over Rules (17) on (11).

Case (17a).

- (12) $e_1 \models_{\tau} \xi_r$ by assumption
- (13) $e_1 ? p_r$ by Lemma 4.0.11 on (2) and (5) and (10) and (12)
- (14) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ **indet** by Rule (24k) on (5) and (13)
- (15) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ **final** by Rule (25b) on (14)

Case (17b).

- (12) $e_1 \models \xi_r$ by assumption
- (13) $e_1 \triangleright p_r \dashv\!\parallel \theta$ by Lemma 4.0.11 on (2) and (5) and (10) and (12)
- (14) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (32l) on (5) and (13)

Case (22b).

- (6) $rs = r' \mid rs'$ by assumption

- (7) $\xi = \xi_r \vee \xi_{rs}$ by assumption
- (8) $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (8)
- (10) $p_r : \tau_1[\xi_r] \dashv\!\vdash \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (8)

By Lemma 4.0.10 on (2) and (5) and (10).

Case Scrutinee matches pattern.

- (11) $e_1 \triangleright p_r \dashv\!\vdash \theta$ by assumption
- (12) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$ by Rule (32l) on (5) and (11)

Case Scrutinee may matches pattern.

- (11) $e_1 ? p_r$ by assumption
- (12) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}$ **indet** by Rule (24k) on (5) and (11)
- (13) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}$ **final** by Rule (25b) on (12)

Case Scrutinee doesn't matche pattern.

- (11) $e_1 \perp p_r$ by assumption
- (12) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$
 $\mapsto \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by Rule (32m) on (5) and (11)

Case (19m).

- (1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption
- (2) $\cdot; \Delta \vdash \text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption
- (3) $\cdot; \Delta \vdash e_1 : \tau_1$ by assumption
- (4) e_1 **final** by assumption
- (5) $\cdot; \Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption
- (6) $e_1 \not\models_{\tau}^{\dagger} \xi_{pre}$ by assumption
- (7) $\top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (22) on (5).

Case (22a).

- (5) $rs_{post} = \cdot$ by assumption

- (6) $\xi_{rest} = \xi_r$ by assumption
- (7) $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (8) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (7)
- (9) $p_r : \tau_1[\xi_r] \dashv\!\parallel \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (7)
- (10) $e_1 \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_r$ by Corollary 2.1.1 on (4) and (7)
- (11) $e_1 \models_{\tau}^{\dagger} \xi_r$ by Lemma 2.0.8 on (10) and (6)

By rule induction over Rules (17) on (11).

Case (17a).

- (12) $e_1 \models_{\tau} \xi_r$ by assumption
- (13) $e_1 \text{ ? } p_r$ by Lemma 4.0.11 on (3) and (4) and (9) and (12)
- (14) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{indet}$ by Rule (24k) on (4) and (13)
- (15) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{final}$ by Rule (25b) on (14)

Case (17b).

- (12) $e_1 \models \xi_r$ by assumption
- (13) $e_1 \triangleright p_r \dashv\!\parallel \theta$ by Lemma 4.0.11 on (3) and (4) and (9) and (12)
- (14) $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (32l) on (4) and (13)

Case (22b).

- (5) $rs_{post} = r' \mid rs'_{post}$ by assumption
- (6) $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption
- (7) $r = p_r \Rightarrow e_r$ by Inversion of Rule (21a) on (6)
- (8) $p_r : \tau_1[\xi_r] \dashv\!\parallel \Gamma_r ; \Delta_r$ by Inversion of Rule (21a) on (6)

By Lemma 4.0.10 on (3) and (4) and (8).

Case Scrutinee matches pattern.

- (9) $e_1 \triangleright p_r \dashv\!\parallel \theta$ by assumption

$$\begin{aligned}
(10) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r) \\
& \text{by Rule (32l) on (4)} \\
& \text{and (9)}
\end{aligned}$$

Case Scrutinee may matches pattern.

$$\begin{aligned}
(9) \quad & e_1 ? p_r \quad \text{by assumption} \\
(10) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \text{indet} \\
& \text{by Rule (24k) on (4)} \\
& \text{and (9)} \\
(11) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \text{final} \\
& \text{by Rule (25b) on (10)}
\end{aligned}$$

Case Scrutinee doesn't matche pattern.

$$\begin{aligned}
(9) \quad & e_1 \perp p_r \quad \text{by assumption} \\
(10) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\} \\
& \mapsto \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\} \\
& \text{by Rule (32m) on (4)} \\
& \text{and (9)}
\end{aligned}$$

□

6 Decidability

$\Xi \text{ incon}$ A finite set of constraints, Ξ , is inconsistent

$$\begin{array}{c}
\text{CINCTruth} \\
\Xi \text{ incon} \\
\hline
\Xi, \top \text{ incon}
\end{array} \tag{33a}$$

$$\begin{array}{c}
\text{CINCFalse} \\
\Xi, \perp \text{ incon} \\
\hline
\Xi, \perp \text{ incon}
\end{array} \tag{33b}$$

$$\begin{array}{c}
\text{CINCNum} \\
n_1 \neq n_2 \\
\hline
\Xi, \underline{n_1}, \underline{n_2} \text{ incon}
\end{array} \tag{33c}$$

$$\begin{array}{c}
\text{CINCNotNum} \\
\Xi, \underline{n}, \text{not } \underline{n} \text{ incon} \\
\hline
\Xi, \underline{n}, \text{not } \underline{n} \text{ incon}
\end{array} \tag{33d}$$

$$\begin{array}{c}
\text{CINCAnd} \\
\Xi, \xi_1, \xi_2 \text{ incon} \\
\hline
\Xi, \xi_1 \wedge \xi_2 \text{ incon}
\end{array} \tag{33e}$$

$$\begin{array}{c}
\text{CINCOr} \\
\Xi, \xi_1 \text{ incon} \quad \Xi, \xi_2 \text{ incon} \\
\hline
\Xi, \xi_1 \vee \xi_2 \text{ incon}
\end{array} \tag{33f}$$

$$\frac{\text{CINCI}_{\text{nj}}}{\overline{\Xi}, \text{inl}(\xi_1), \text{inr}(\xi_2) \text{ incon}} \quad (33\text{g})$$

$$\frac{\frac{\text{CINCI}_{\text{nl}}}{\Xi \text{ incon}}}{\text{inl}(\Xi) \text{ incon}} \quad (33\text{h})$$

$$\frac{\frac{\text{CINCI}_{\text{nr}}}{\Xi \text{ incon}}}{\text{inr}(\Xi) \text{ incon}} \quad (33\text{i})$$

$$\frac{\frac{\text{CINCPairL}}{\Xi_1 \text{ incon}}}{(\Xi_1, \Xi_2) \text{ incon}} \quad (33\text{j})$$

$$\frac{\frac{\text{CINCPairR}}{\Xi_2 \text{ incon}}}{(\Xi_1, \Xi_2) \text{ incon}} \quad (33\text{k})$$

Lemma 6.0.1 (Decidability of Inconsistency). *Suppose $\dot{\top}(\xi) = \xi$. It is decidable whether $\xi \text{ incon}$.*

Lemma 6.0.2 (Inconsistency and Entailment of Constraint). *Suppose that $\dot{\top}(\xi) = \xi$. Then $\bar{\xi} \text{ incon}$ iff $\top \models \xi$*

Lemma 6.0.3. *If $e \models \xi$ then $e \models \dot{\top}(\xi)$*

Proof. By rule induction over Rules (14), it is obvious to see that $\dot{\top}(\xi) = \xi$. \square

Lemma 6.0.4. *If $e \models_{\text{?}} \xi$ then $e \models_{\text{?}}^{\dot{\top}} \dot{\top}(\xi)$.*

Proof.

$$(11) \quad e \models_{\text{?}} \xi \quad \text{by assumption}$$

By Rule Induction over Rules (16) on (11).

Case (16a).

$$\begin{aligned} (12) \quad \xi &= ? & \text{by assumption} \\ (13) \quad e &\models \top & \text{by Rule (14a)} \\ (14) \quad e &\models_{\text{?}}^{\dot{\top}} \top & \text{by Rule (17b) on (13)} \end{aligned}$$

Case (16b).

$$\begin{aligned} (12) \quad e &\text{ notintro} & \text{by assumption} \\ (13) \quad \xi &\text{ refutable} & \text{by assumption} \end{aligned}$$

By Lemma 2.0.15 on (12) and (13) and case analysis on its conclusion.
By rule induction over Rules (10).

Case $\dot{\vdash}(\xi)$ refutable.

(14) $\dot{\vdash}(\xi)$ **refutable**

by assumption

(15) $e \models_{\dot{?}} \dot{\vdash}(\xi)$

by Rule (16b) on (12)
and (14)

(16) $e \models_{\dot{?}}^{\dagger} \dot{\vdash}(\xi)$

by Rule (17b) on (15)

Case $e \models \dot{\vdash}(\xi)$.

(14) $e \models \dot{\vdash}(\xi)$

by assumption

(15) $e \models_{\dot{?}}^{\dagger} \top$

by Rule (17b) on (14)

Case (16c).

(12) $\xi = \xi_1 \vee \xi_2$

by assumption

(13) $e \models_{\dot{?}} \xi_1$

by assumption

(14) $e \models_{\dot{?}}^{\dagger} \dot{\vdash}(\xi_1)$

by IH on (13)

(15) $e \models_{\dot{?}}^{\dagger} \dot{\vdash}(\xi_1) \vee \dot{\vdash}(\xi_2)$

by Lemma 2.0.10 on
(14)

Case (16d).

(12) $\xi = \xi_1 \vee \xi_2$

by assumption

(13) $e \models_{\dot{?}} \xi_2$

by assumption

(14) $e \models_{\dot{?}}^{\dagger} \dot{\vdash}(\xi_2)$

by IH on (13)

(15) $e \models_{\dot{?}}^{\dagger} \dot{\vdash}(\xi_1) \vee \dot{\vdash}(\xi_2)$

by Lemma 2.0.10 on
(14)

Case (16e).

(12) $e = \text{inl}_{\tau_2}(e_1)$

by assumption

(13) $\xi = \text{inl}(\xi_1)$

by assumption

(14) $e_1 \models_{\dot{?}} \xi_1$

by assumption

(15) $e_1 \models_{\dot{?}}^{\dagger} \dot{\vdash}(\xi_1)$

by IH on (14)

(16) $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\dot{\vdash}(\xi_1))$

by Lemma 2.0.11 on
(15)

Case (16f).

(12) $e = \text{inr}_{\tau_1}(e_2)$

by assumption

(13) $\xi = \text{inr}(\xi_2)$

by assumption

(14)	$e_2 \models_{\tau_1} \xi_2$	by assumption
(15)	$e_2 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_2)$	by IH on (14)
(16)	$\text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\dot{\top}(\xi_2))$	by Lemma 2.0.12 on (15)

Case (16g).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{\tau_1} \xi_1$	by assumption
(15)	$e_2 \models_{\tau_1} \xi_2$	by assumption
(16)	$e_1 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_1)$	by IH on (14)
(17)	$e_2 \models_{\tau_1} \dot{\top}(\xi_2)$	by Lemma 6.0.3 on (15)
(18)	$e_2 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_2)$	by Rule (17b) on (17)
(19)	$(e_1, e_2) \models_{\tau_1}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (16) and (18)

Case (16h).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{\tau_1} \xi_1$	by assumption
(15)	$e_2 \models_{\tau_1} \xi_2$	by assumption
(16)	$e_1 \models_{\tau_1} \dot{\top}(\xi_1)$	by Lemma 6.0.3 on (14)
(17)	$e_1 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_1)$	by Rule (17b) on (16)
(18)	$e_2 \models_{\tau_1}^{\dagger} \dot{\top}(\xi_2)$	by IH on (15)
(19)	$(e_1, e_2) \models_{\tau_1}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 2.0.13 on (17) and (18)

Case (16i).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{\tau_1} \xi_1$	by assumption
(15)	$e_2 \models_{\tau_1} \xi_2$	by assumption

- | | |
|-------------------------------------------------------------------------------------|----------------------------------|
| (16) $e_1 \models_{\vdash}^{\dagger} \dot{\top}(\xi_1)$ | by IH on (14) |
| (17) $e_2 \models_{\vdash}^{\dagger} \dot{\top}(\xi_2)$ | by IH on (15) |
| (18) $(e_1, e_2) \models_{\vdash}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Lemma 2.0.13 on (16) and (17) |

□

Lemma 6.0.5. $e \models_{\vdash}^{\dagger} \xi$ iff $e \models_{\vdash}^{\dagger} \dot{\top}(\xi)$

Proof. 1. Sufficiency:

- | | |
|----------------------------------------|---------------|
| (1) $e \models_{\vdash}^{\dagger} \xi$ | by assumption |
|----------------------------------------|---------------|

By rule induction over Rules (17) on (1)

Case (17b).

- | | |
|----------------------------------------------------|-----------------------|
| (2) $e \models \xi$ | by assumption |
| (3) $e \models \dot{\top}(\xi)$ | by Lemma 6.0.3 on (2) |
| (4) $e \models_{\vdash}^{\dagger} \dot{\top}(\xi)$ | by Rule (17b) on (3) |

Case (17a).

- | | |
|----------------------------------------------------|-----------------------|
| (2) $e \models_{\vdash} \xi$ | by assumption |
| (3) $e \models_{\vdash}^{\dagger} \dot{\top}(\xi)$ | by Lemma 6.0.4 on (2) |

2. Necessity:

- | | |
|----------------------------------------------------|---------------|
| (1) $e \models_{\vdash}^{\dagger} \dot{\top}(\xi)$ | by assumption |
|----------------------------------------------------|---------------|

By structural induction on ξ ,

Case $\xi = \top, \perp, n, \neg$.

- | | |
|----------------------------------------|--------------------------|
| (2) $e \models_{\vdash}^{\dagger} \xi$ | by (1) and Definition 12 |
|----------------------------------------|--------------------------|

Case $\xi = ?$.

- | | |
|--------------------------------------|----------------------|
| (2) $e \models_{\vdash} ?$ | by Rule (16a) |
| (3) $e \models_{\vdash}^{\dagger} ?$ | by Rule (17a) on (2) |

Case $\xi = \xi_1 \vee \xi_2$.

- | | |
|-------------------------------------------------------------------------------|------------------|
| (2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Definition 12 |
|-------------------------------------------------------------------------------|------------------|

By rule induction over Rules (17) on (1),

Case (17b).

(3) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (14) on (3) and two cases apply,

Case (14e).

(4) $e \models \dot{\top}(\xi_1)$ by assumption

(5) $e \models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_1)$ by Rule (17b) on (4)

(6) $e \models_{\dot{?}}^{\dot{\top}} \xi_1$ by IH on (5)

(7) $e \models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case (14f).

(4) $e \models \dot{\top}(\xi_2)$ by assumption

(5) $e \models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_2)$ by Rule (17b) on (4)

(6) $e \models_{\dot{?}}^{\dot{\top}} \xi_2$ by IH on (5)

(7) $e \models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case (17a).

(3) $e \models_{\dot{?}} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (16) on (3) and two cases apply,

Case (16c).

(4) $e \models_{\dot{?}} \dot{\top}(\xi_1)$ by assumption

(5) $e \models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_1)$ by Rule (17a) on (4)

(6) $e \models_{\dot{?}}^{\dot{\top}} \xi_1$ by IH on (5)

(7) $e \models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case (16d).

(4) $e \models_{\dot{?}} \dot{\top}(\xi_2)$ by assumption

(5) $e \models_{\dot{?}}^{\dot{\top}} \dot{\top}(\xi_2)$ by Rule (17a) on (4)

(6) $e \models_{\dot{?}}^{\dot{\top}} \xi_2$ by IH on (5)

(7) $e \models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case $\xi = \text{inl}(\xi_1)$.

(2) $e = \text{inl}_{r_2}(e_1)$ by assumption

(3) $\dot{\top}(\xi) = \text{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (17) on (1),

Case (17b).

(4) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\top}(\xi_1))$ by assumption
 By rule induction over Rules (14) and only one case applies,
Case (14g).

- (5) $e_1 \models \dot{\top}(\xi_1)$ by assumption
- (6) $e_1 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1)$ by Rule (17b) on (5)
- (7) $e_1 \models_{\dot{?}}^{\dagger} \xi_1$ by IH on (6)
- (8) $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\xi_1)$ by Lemma 2.0.11 on (7)

Case (17a).

(4) $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}} \text{inl}(\dot{\top}(\xi_1))$ by assumption
 By rule induction over Rules (16) and only one case applies,
Case (16e).

- (5) $e_1 \models_{\dot{?}} \dot{\top}(\xi_1)$ by assumption
- (6) $e_1 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_1)$ by Rule (17a) on (5)
- (7) $e_1 \models_{\dot{?}}^{\dagger} \xi_1$ by IH on (6)
- (8) $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dagger} \text{inl}(\xi_1)$ by Lemma 2.0.11 on (7)

Case $\xi = \text{inr}(\xi_2)$.

- (2) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
- (3) $\dot{\top}(\xi) = \text{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (17) on (1),

Case (17b).

(4) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\top}(\xi_2))$ by assumption
 By rule induction over Rules (14) and only one case applies,
Case (14h).

- (5) $e_2 \models \dot{\top}(\xi_2)$ by assumption
- (6) $e_2 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_2)$ by Rule (17b) on (5)
- (7) $e_2 \models_{\dot{?}}^{\dagger} \xi_2$ by IH on (6)
- (8) $\text{inr}_{\tau_1}(e_2) \models_{\dot{?}}^{\dagger} \text{inr}(\xi_2)$ by Lemma 2.0.12 on (7)

Case (17a).

(4) $\text{inr}_{\tau_1}(e_2) \models_{\dot{?}} \text{inr}(\dot{\top}(\xi_2))$ by assumption
 By rule induction over Rules (16) and only one case applies,
Case (16f).

- (5) $e_2 \models_{\dot{?}} \dot{\top}(\xi_2)$ by assumption
- (6) $e_2 \models_{\dot{?}}^{\dagger} \dot{\top}(\xi_2)$ by Rule (17a) on (5)
- (7) $e_2 \models_{\dot{?}}^{\dagger} \xi_2$ by IH on (6)

$$(8) \text{ inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{ inr}(\xi_2) \quad \text{by Lemma 2.0.12 on (7)}$$

Case $\xi = (\xi_1, \xi_2)$.

$$(2) e = (e_1, e_2) \quad \text{by assumption}$$

$$(3) \dot{\vdash}(\xi) = \dot{\vdash}(\xi_1) \wedge \dot{\vdash}(\xi_2) \quad \text{by Definition 12}$$

By rule induction over Rules (17) on (1),

Case (17b).

$$(4) (e_1, e_2) \models (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (14) on (4) and only one case applies,

Case (14i).

$$(5) e_1 \models \dot{\vdash}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models \dot{\vdash}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_1) \quad \text{by Rule (17b) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_2) \quad \text{by Rule (17b) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

$$(10) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (8)}$$

$$(11) (e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2) \quad \text{by Lemma 2.0.13 on (9) and (10)}$$

Case (17a).

$$(4) (e_1, e_2) \models_{\tau} (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (16) on (4) and three cases apply,

Case (16g).

$$(5) e_1 \models_{\tau} \dot{\vdash}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models \dot{\vdash}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_1) \quad \text{by Rule (17a) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_2) \quad \text{by Rule (17b) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

$$(10) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (8)}$$

$$(11) (e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2) \quad \text{by Lemma 2.0.13 on (9) and (10)}$$

Case (16h).

$$(5) e_1 \models \dot{\vdash}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models_{\tau} \dot{\vdash}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_1) \quad \text{by Rule (17b) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\vdash}(\xi_2) \quad \text{by Rule (17a) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

- | | |
|-----------------------------------------------------------|---------------------------------|
| (10) $e_2 \models_{\tau}^{\dagger} \xi_2$ | by IH on (8) |
| (11) $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

Case (16i).

- | | |
|-----------------------------------------------------------|---------------------------------|
| (5) $e_1 \models_{\tau} \dot{\top}(\xi_1)$ | by assumption |
| (6) $e_2 \models_{\tau} \dot{\top}(\xi_2)$ | by assumption |
| (7) $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$ | by Rule (17a) on (5) |
| (8) $e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$ | by Rule (17a) on (6) |
| (9) $e_1 \models_{\tau}^{\dagger} \xi_1$ | by IH on (7) |
| (10) $e_2 \models_{\tau}^{\dagger} \xi_2$ | by IH on (8) |
| (11) $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

□

Lemma 6.0.6. Assume $\dot{\top}(\xi) = \xi$. Then $\top \models_{\tau}^{\dagger} \xi$ iff $\top \models \xi$.

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:
2. Necessity:

□

Theorem 6.1. $\top \models_{\tau}^{\dagger} \xi$ iff $\top \models \dot{\top}(\xi)$.

Lemma 6.1.1. Assume that $e \text{ val}$. Then $e \models_{\tau}^{\dagger} \xi$ iff $e \models \dot{\top}(\xi)$

Proof.

- | | |
|---------------------|---------------|
| (1) $e \text{ val}$ | by assumption |
|---------------------|---------------|

We prove sufficiency and necessity separately.

1. Sufficiency:

- | | |
|--------------------------------------|---------------|
| (2) $e \models_{\tau}^{\dagger} \xi$ | by assumption |
|--------------------------------------|---------------|

By rule induction over Rules (17) on (2).

Case (17b).

- | | |
|---------------------------------|-----------------------|
| (3) $e \models \xi$ | by assumption |
| (4) $e \models \dot{\top}(\xi)$ | by Lemma 6.0.3 on (3) |

Case (17a).

- | | |
|----------------------------|---------------|
| (3) $e \models_{\tau} \xi$ | by assumption |
|----------------------------|---------------|

By rule induction over Rules (16) on (3).

Case (16a).

- | | |
|---------------------------------|------------------------------------|
| (4) $\xi = ?$ | by assumption |
| (5) $e \models \dot{\top}(\xi)$ | by Rule (14a) and
Definition 12 |

Case (16b).

- | | |
|--------------------------|---------------|
| (4) $e \text{ notintro}$ | by assumption |
|--------------------------|---------------|

By rule induction over Rules (26) on (4), for each case, by rule induction over Rules (23) on (1), no case applies due to syntactic contradiction.

Case (16c).

- | | |
|------------------------------------------------------------------|----------------------|
| (4) $\xi = \xi_1 \vee \xi_2$ | by assumption |
| (5) $e \models? \xi_1$ | by assumption |
| (6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Equation 12 |
| (7) $e \models?^{\dagger} \xi_1$ | by Rule (17a) on (5) |
| (8) $e \models \dot{\top}(\xi_1)$ | by IH on (7) |
| (9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Rule (14e) on (8) |

Case (16d).

- | | |
|------------------------------------------------------------------|----------------------|
| (4) $\xi = \xi_1 \vee \xi_2$ | by assumption |
| (5) $e \models? \xi_2$ | by assumption |
| (6) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Equation 12 |
| (7) $e \models?^{\dagger} \xi_2$ | by Rule (17a) on (5) |
| (8) $e \models \dot{\top}(\xi_2)$ | by IH on (7) |
| (9) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Rule (14f) on (8) |

Case (16e).

- | | |
|-------------------------------------------------------|----------------------|
| (4) $\xi = \text{inl}(\xi_1)$ | by assumption |
| (5) $e \models? \xi_1$ | by assumption |
| (6) $\dot{\top}(\xi) = \text{inl}(\dot{\top}(\xi_1))$ | by Equation 12 |
| (7) $e \models?^{\dagger} \xi_1$ | by Rule (17a) on (5) |
| (8) $e \models \dot{\top}(\xi_1)$ | by IH on (7) |
| (9) $e \models \text{inl}(\dot{\top}(\xi_1))$ | by Rule (14g) on (8) |

Case (16f).

- | | |
|-------------------------------------------------------|----------------------|
| (4) $\xi = \text{inr}(\xi_2)$ | by assumption |
| (5) $e \models? \xi_2$ | by assumption |
| (6) $\dot{\top}(\xi) = \text{inr}(\dot{\top}(\xi_2))$ | by Equation 12 |
| (7) $e \models?^{\dagger} \xi_2$ | by Rule (17a) on (5) |

- | | |
|-------------------------------------------------|----------------------|
| (8) $e \models \dot{\top}(\xi_2)$ | by IH on (7) |
| (9) $e \models \mathbf{inr}(\dot{\top}(\xi_2))$ | by Rule (14h) on (8) |

Case (16g).

- | | |
|------------------------------------------------------------------|--------------------------------|
| (4) $e = (e_1, e_2)$ | by assumption |
| (5) $\xi = (\xi_1, \xi_2)$ | by assumption |
| (6) $e_1 \models_{\dot{?}} \xi_1$ | by assumption |
| (7) $e_2 \models \xi_2$ | by assumption |
| (8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Equation 12 |
| (9) $e_1 \models_{\dot{?}}^{\dagger} \xi_1$ | by Rule (17a) on (6) |
| (10) $e_2 \models_{\dot{?}}^{\dagger} \xi_2$ | by Rule (17b) on (7) |
| (11) $e_1 \models \dot{\top}(\xi_1)$ | by IH on (9) |
| (12) $e_2 \models \dot{\top}(\xi_2)$ | by IH on (10) |
| (13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (14i) on (11) and (12) |

Case (16h).

- | | |
|------------------------------------------------------------------|--------------------------------|
| (4) $e = (e_1, e_2)$ | by assumption |
| (5) $\xi = (\xi_1, \xi_2)$ | by assumption |
| (6) $e_1 \models \xi_1$ | by assumption |
| (7) $e_2 \models_{\dot{?}} \xi_2$ | by assumption |
| (8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Equation 12 |
| (9) $e_1 \models_{\dot{?}}^{\dagger} \xi_1$ | by Rule (17b) on (6) |
| (10) $e_2 \models_{\dot{?}}^{\dagger} \xi_2$ | by Rule (17a) on (7) |
| (11) $e_1 \models \dot{\top}(\xi_1)$ | by IH on (9) |
| (12) $e_2 \models \dot{\top}(\xi_2)$ | by IH on (10) |
| (13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (14i) on (11) and (12) |

Case (16i).

- | | |
|------------------------------------------------------------------|--------------------------------|
| (4) $e = (e_1, e_2)$ | by assumption |
| (5) $\xi = (\xi_1, \xi_2)$ | by assumption |
| (6) $e_1 \models_{\dot{?}} \xi_1$ | by assumption |
| (7) $e_2 \models_{\dot{?}} \xi_2$ | by assumption |
| (8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Equation 12 |
| (9) $e_1 \models_{\dot{?}}^{\dagger} \xi_1$ | by Rule (17a) on (6) |
| (10) $e_2 \models_{\dot{?}}^{\dagger} \xi_2$ | by Rule (17a) on (7) |
| (11) $e_1 \models \dot{\top}(\xi_1)$ | by IH on (9) |
| (12) $e_2 \models \dot{\top}(\xi_2)$ | by IH on (10) |
| (13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (14i) on (11) and (12) |

2. Necessity:

$$(2) \quad e \models \dot{\top}(\xi) \quad \text{by assumption}$$

By structural induction on ξ .

Case $\xi = \top, \perp, n, \neg$.

$$(3) \quad \xi = \dot{\top}(\xi) \quad \text{by Equation 12}$$

$$(4) \quad e \models_{\dot{?}}^{\dot{?}} \xi \quad \text{by Rule (17b) on (2)}$$

Case $\xi = ?$.

$$(3) \quad e \models_{\dot{?}} ? \quad \text{by Rule (16a)}$$

$$(4) \quad e \models_{\dot{?}}^{\dot{?}} ? \quad \text{by Rule (17a) on (3)}$$

Case $\xi = \xi_1 \wedge \xi_2$.

$$(3) \quad \dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2), only one case applies.

Case (14d).

$$(4) \quad e \models \dot{\top}(\xi_1) \quad \text{by assumption}$$

$$(5) \quad e \models \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(6) \quad e \models_{\dot{?}}^{\dot{?}} \xi_1 \quad \text{by IH on (4)}$$

$$(7) \quad e \models_{\dot{?}}^{\dot{?}} \xi_2 \quad \text{by IH on (5)}$$

$$(8) \quad e \models \xi_1 \wedge \xi_2 \quad \text{by Lemma 2.0.9 on (6) and (7)}$$

Case $\xi = \xi_1 \vee \xi_2$.

$$(3) \quad \dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only two cases apply.

Case (14e).

$$(4) \quad e \models \dot{\top}(\xi_1) \quad \text{by assumption}$$

$$(5) \quad e \models_{\dot{?}}^{\dot{?}} \xi_1 \quad \text{by IH on (4)}$$

$$(6) \quad e \models_{\dot{?}}^{\dot{?}} \xi_1 \vee \xi_2 \quad \text{by Lemma 2.0.10 on (5)}$$

Case (14f).

$$(4) \quad e \models \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(5) \quad e \models_{\dot{?}}^{\dot{?}} \xi_2 \quad \text{by IH on (4)}$$

$$(6) \quad e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2 \quad \text{by Lemma 2.0.10 on (5)}$$

Case $\xi = \text{inl}(\xi_1)$.

$$(3) \quad \dot{\vdash}(\xi) = \text{inl}(\dot{\vdash}(\xi_1)) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only one case applies.

Case (14g).

$$\begin{aligned} (4) \quad e &= \text{inl}_{\tau_2}(e_1) && \text{by assumption} \\ (5) \quad e_1 &\models \dot{\vdash}(\xi_1) && \text{by assumption} \\ (6) \quad e_1 &\models_{\tau}^{\dagger} \xi_1 && \text{by IH on (5)} \\ (7) \quad \text{inl}_{\tau_2}(e_1) &\models_{\tau}^{\dagger} \text{inl}(\xi_1) && \text{by Lemma 2.0.11 on (6)} \end{aligned}$$

Case $\xi = \text{inr}(\xi_2)$.

$$(3) \quad \dot{\vdash}(\xi) = \text{inr}(\dot{\vdash}(\xi_2)) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only one case applies.

Case (14h).

$$\begin{aligned} (4) \quad e &= \text{inr}_{\tau_1}(e_2) && \text{by assumption} \\ (5) \quad e_2 &\models \dot{\vdash}(\xi_2) && \text{by assumption} \\ (6) \quad e_2 &\models_{\tau}^{\dagger} \xi_2 && \text{by IH on (5)} \\ (7) \quad \text{inr}_{\tau_1}(e_2) &\models_{\tau}^{\dagger} \text{inr}(\xi_2) && \text{by Lemma 2.0.12 on (6)} \end{aligned}$$

Case $\xi = (\xi_1, \xi_2)$.

$$(3) \quad \dot{\vdash}(\xi) = (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by Equation 12}$$

By rule induction over Rules (14) on (2) and only one case applies.

Case (14i).

$$\begin{aligned} (4) \quad e &= (e_1, e_2) && \text{by assumption} \\ (5) \quad e_1 &\models \dot{\perp}(\xi_1) && \text{by assumption} \\ (6) \quad e_2 &\models \dot{\perp}(\xi_2) && \text{by assumption} \\ (7) \quad e_1 &\models_{\tau}^{\dagger} \xi_1 && \text{by IH on (5)} \\ (8) \quad e_2 &\models_{\tau}^{\dagger} \xi_2 && \text{by IH on (6)} \\ (9) \quad (e_1, e_2) &\models_{\tau}^{\dagger} (\xi_1, \xi_2) && \text{by Lemma 2.0.13 on (7) and (8)} \end{aligned}$$

□

Lemma 6.1.2. $e \models \xi \text{ iff } e \models \dot{\perp}(\xi)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models \xi$ by assumption

By rule induction over Rules (14) on (1).

Case (14a).

(2) $\xi = \top$ by assumption
 (3) $e \models \dot{\perp}(\top)$ by (1) and Definition 13

Case (14b).

(2) $\xi = \underline{n}$ by assumption
 (3) $e \models \dot{\perp}(\underline{n})$ by (1) and Definition 13

Case (14c).

(2) $\xi = \underline{\neg}$ by assumption
 (3) $e \models \dot{\perp}(\underline{\neg})$ by (1) and Definition 13

Case (14d).

(2) $\xi = \xi_1 \wedge \xi_2$ by assumption
 (3) $e \models \xi_1$ by assumption
 (4) $e \models \xi_2$ by assumption
 (5) $e \models \dot{\perp}(\xi_1)$ by IH on (3)
 (6) $e \models \dot{\perp}(\xi_2)$ by IH on (4)
 (7) $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$ by Rule (14d) on (5) and (6)
 (8) $e \models \dot{\perp}(\xi_1 \wedge \xi_2)$ by (7) and Definition 13

Case (14e).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption
 (3) $e \models \xi_1$ by assumption
 (4) $e \models \dot{\perp}(\xi_1)$ by IH on (3)
 (5) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ by Rule (14e) on (4)
 (6) $e \models \dot{\perp}(\xi_1 \vee \xi_2)$ by (5) and Definition 13

Case (14f).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption
 (3) $e \models \xi_2$ by assumption
 (4) $e \models \dot{\perp}(\xi_2)$ by IH on (3)
 (5) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ by Rule (14f) on (4)

(6) $e \models \dot{\perp}(\xi_1 \vee \xi_2)$ by (5) and Definition 13

Case (14g).

(2) $e = \text{inl}_{\tau_2}(e_1)$ by assumption
 (3) $\xi = \text{inl}(\xi_1)$ by assumption
 (4) $e_1 \models \xi_1$ by assumption
 (5) $e_1 \models \dot{\perp}(\xi_1)$ by IH on (4)
 (6) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\perp}(\xi_1))$ by Rule (14g) on (5)
 (7) $\text{inl}_{\tau_2}(e_1) \models \dot{\perp}(\text{inl}(\xi_1))$ by (6) and Definition 13

Case (14h).

(2) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
 (3) $\xi = \text{inr}(\xi_2)$ by assumption
 (4) $e_2 \models \xi_2$ by assumption
 (5) $e_2 \models \dot{\perp}(\xi_2)$ by IH on (4)
 (6) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\perp}(\xi_2))$ by Rule (14h) on (5)
 (7) $\text{inr}_{\tau_1}(e_2) \models \dot{\perp}(\text{inr}(\xi_2))$ by (6) and Definition 13

Case (14i).

(2) $e = (e_1, e_2)$ by assumption
 (3) $\xi = (\xi_1, \xi_2)$ by assumption
 (4) $e_1 \models \xi_1$ by assumption
 (5) $e_2 \models \xi_2$ by assumption
 (6) $e_1 \models \dot{\perp}(\xi_1)$ by IH on (4)
 (7) $e_2 \models \dot{\perp}(\xi_2)$ by IH on (5)
 (8) $(e_1, e_2) \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$ by Rule (14i) on (6) and (7)
 (9) $(e_1, e_2) \models \dot{\perp}((\xi_1, \xi_2))$ by (8) and Definition 13

2. Necessity:

(1) $e \models \dot{\perp}(\xi)$ by assumption

By structural induction on ξ .

Case $\xi = \top, \perp, \underline{n}, \mathcal{N}$.

(2) $e \models \xi$ by (1) and Definition 13

Case $\xi = ?$.

(2) $e \models \perp$ by (1) and Definition 13

(3) $e \not\models \perp$ by Lemma 2.0.1

(3) contradicts (2).

Case $\xi = \xi_1 \wedge \xi_2$.

(2) $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only case applies.

Case (14d).

(3) $e \models \dot{\perp}(\xi_1)$ by assumption

(4) $e \models \dot{\perp}(\xi_2)$ by assumption

(5) $e \models \xi_1$ by IH on (3)

(6) $e \models \xi_2$ by IH on (4)

(7) $e \models \xi_1 \wedge \xi_2$ by Rule (14d) on (5) and (6)

Case $\xi = \xi_1 \vee \xi_2$.

(2) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only two cases apply.

Case (14e).

(3) $e \models \dot{\perp}(\xi_1)$ by assumption

(4) $e \models \xi_1$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14e) on (4)

Case (14f).

(3) $e \models \dot{\perp}(\xi_2)$ by assumption

(4) $e \models \xi_2$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (14f) on (4)

Case $\xi = \text{inl}(\xi_1)$.

(2) $e \models \text{inl}(\dot{\perp}(\xi_1))$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only one case applies.

Case (14g).

(3) $e = \text{inl}_{\tau_2}(e_1)$ by assumption

(4) $e_1 \models \dot{\perp}(\xi_1)$ by assumption

(5) $e_1 \models \xi_1$ by IH on (4)

(6) $e \models \text{inl}(\xi_1)$ by Rule (14g) on (5)

Case $\xi = \text{inr}(\xi_2)$.

(2) $e \models \text{inr}(\dot{\perp}(\xi_2))$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only one case applies.

Case (14h).

(3) $e = \text{inr}_{\tau_1}(e_2)$ by assumption
 (4) $e_2 \models \dot{\perp}(\xi_2)$ by assumption
 (5) $e_2 \models \xi_2$ by IH on (4)
 (6) $e \models \text{inr}(\xi_2)$ by Rule (14h) on (5)

Case $\xi = (\xi_1, \xi_2)$.

(2) $e \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$ by (1) and Definition 13

By rule induction over Rules (14) on (2) and only case applies.

Case (14i).

(3) $e = (e_1, e_2)$ by assumption
 (4) $e_1 \models \dot{\perp}(\xi_1)$ by assumption
 (5) $e_2 \models \dot{\perp}(\xi_2)$ by assumption
 (6) $e_1 \models \xi_1$ by IH on (4)
 (7) $e_2 \models \xi_2$ by IH on (5)
 (8) $e \models (\xi_1, \xi_2)$ by Rule (14i) on (6) and (7)

□

Lemma 6.1.3. Assume $e \text{ val}$ and $\dot{\top}(\xi) = \xi$. Then $e \not\models \xi$ iff $e \models \bar{\xi}$.

Theorem 6.2. $\xi_r \models \xi_{rs}$ iff $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\perp}(\xi_{rs})$.