1 Match Constraint Language

$\dot{\xi}$ ::= $\top \mid ? \mid \underline{n} \mid \texttt{inl}(\dot{\xi}) \mid \texttt{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$

$\boxed{\dot{\xi} : \tau}$ $\dot{\xi}$ constrains final expressions of type τ

$$\frac{}{\top : \tau}\ \text{CTTruth} \tag{1a}$$

$$\frac{}{? : \tau}\ \text{CTUnknown} \tag{1b}$$

$$\frac{}{\underline{n} : \texttt{num}}\ \text{CTNum} \tag{1c}$$

$$\frac{\dot{\xi}_1 : \tau_1}{\texttt{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)}\ \text{CTInl} \tag{1d}$$

$$\frac{\dot{\xi}_2 : \tau_2}{\texttt{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)}\ \text{CTInr} \tag{1e}$$

$$\frac{\dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)}\ \text{CTPair} \tag{1f}$$

$$\frac{\dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau}\ \text{CTOr} \tag{1g}$$

$\boxed{\dot{\xi}\ \texttt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RXNum} \tag{2a}$$

$$\frac{}{?\ \texttt{refutable}_?}\ \text{RXUnknown} \tag{2b}$$

$$\frac{}{\texttt{inl}(\dot{\xi})\ \texttt{refutable}_?}\ \text{RXInl} \tag{2c}$$

$$\frac{}{\texttt{inr}(\dot{\xi})\ \texttt{refutable}_?}\ \text{RXInr} \tag{2d}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\ \text{RXPairL} \tag{2e}$$

$$\frac{\dot{\xi}_2\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\ \text{RXPairR} \tag{2f}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_? \qquad \dot{\xi}_2\ \texttt{refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}\ \text{RXOr} \tag{2g}$$

$\boxed{\mathit{refutable}_?(\dot{\xi})}$

$$\mathit{refutable}_?(\top) = \text{false} \tag{3a}$$
$$\mathit{refutable}_?(\underline{n}) = \text{true} \tag{3b}$$
$$\mathit{refutable}_?(?) = \text{true} \tag{3c}$$
$$\mathit{refutable}_?(\texttt{inl}(\dot{\xi})) = \text{true} \tag{3d}$$
$$\mathit{refutable}_?(\texttt{inr}(\dot{\xi})) = \text{true} \tag{3e}$$
$$\mathit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \mathit{refutable}_?(\dot{\xi}_1) \text{ or } \mathit{refutable}_?(\dot{\xi}_2) \tag{3f}$$
$$\mathit{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{refutable}_?(\dot{\xi}_1) \text{ and } \mathit{refutable}_?(\dot{\xi}_2) \tag{3g}$$

Lemma 1.0.1 (Soundness and Completeness of Refutable Constraints)**.** *$\dot{\xi}$ $\texttt{refutable}_?$ iff $\mathit{refutable}_?(\dot{\xi}) = \mathit{true}$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $\dot{\xi}\ \texttt{refutable}_?$ — by assumption

   By rule induction over Rules (2) on (1).

   Case (2a)**.**

   (2) $\dot{\xi} = \underline{n}$ — by assumption
   (3) $\mathit{refutable}_?(\underline{n}) = \text{true}$ — by Definition 3

   Case (2b)**.**

   (2) $\dot{\xi} = ?$ — by assumption
   (3) $\mathit{refutable}_?(?) = \text{true}$ — by Definition 3

   Case (2c)**.**

   (2) $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$ — by assumption
   (3) $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1)) = \text{true}$ — by Definition 3

   Case (2d)**.**

   (2) $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$ — by assumption
   (3) $\mathit{refutable}_?(\texttt{inr}(\dot{\xi}_2)) = \text{true}$ — by Definition 3

Case (2e).

(2)	$\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$	by assumption
(3)	$\dot{\xi}_1$ `refutable`$_?$	by assumption
(4)	$\mathit{refutable}_?(\dot{\xi}_1) = \text{true}$	by IH on (3)
(5)	$\mathit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$	by Definition 3 on (4)

Case (2f).

(2)	$\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$	by assumption
(3)	$\dot{\xi}_2$ `refutable`$_?$	by assumption
(4)	$\mathit{refutable}_?(\dot{\xi}_2) = \text{true}$	by IH on (3)
(5)	$\mathit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$	by Definition 3 on (4)

Case (2g).

(2)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3)	$\dot{\xi}_1$ `refutable`$_?$	by assumption
(4)	$\dot{\xi}_2$ `refutable`$_?$	by assumption
(5)	$\mathit{refutable}_?(\dot{\xi}_1) = \text{true}$	by IH on (3)
(6)	$\mathit{refutable}_?(\dot{\xi}_2) = \text{true}$	by IH on (4)
(7)	$\mathit{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \text{true}$	by Definition 3 on (5) and (6)

2. Completeness:

(1)	$\mathit{refutable}_?(\dot{\xi}) = \text{true}$	by assumption

By structural induction on $\dot{\xi}$.

Case $\top$.

(2)	$\mathit{refutable}_?(\top) = \text{false}$	by Definition 3

Contradicts (1).

Case $?$.

(2)	$?$ `refutable`$_?$	by Rule (2b)

Case $\underline{n}$.

(2)	$\underline{n}$ `refutable`$_?$	by Rule (2a)

Case `inl`$(\dot{\xi}_1)$.

(2)	`inl`$(\dot{\xi}_1)$ `refutable`$_?$	by Rule (2c)

Case `inr`$(\dot{\xi}_2)$.

(2)	`inr`$(\dot{\xi}_2)$ `refutable`$_?$	by Rule (2d)

Case $(\dot{\xi}_1, \dot{\xi}_2)$.

(2) $\mathit{refutable}_?(\dot{\xi}_1) = \text{true}$ or $\mathit{refutable}_?(\dot{\xi}_2) = \text{true}$ by Definition 3 on (1)

By case analysis on (2).

Case $\mathit{refutable}_?(\dot{\xi}_1) = \text{true}$**.**

(3) $\mathit{refutable}_?(\dot{\xi}_1) = \text{true}$	by assumption
(4) $\dot{\xi}_1\ \texttt{refutable}_?$	by IH on (3)
(5) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$	by Rule (2e) on (4)

Case $\mathit{refutable}_?(\dot{\xi}_2) = \text{true}$**.**

(3) $\mathit{refutable}_?(\dot{\xi}_2) = \text{true}$	by assumption
(4) $\dot{\xi}_2\ \texttt{refutable}_?$	by IH on (3)
(5) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$	by Rule (2f) on (4)

Case $\dot{\xi}_1 \vee \dot{\xi}_2$**.**

(2) $\mathit{refutable}_?(\dot{\xi}_1) = \text{true}$	by Definition 3 on (1)
(3) $\mathit{refutable}_?(\dot{\xi}_2) = \text{true}$	by Definition 3 on (1)
(4) $\dot{\xi}_1\ \texttt{refutable}_?$	by IH on (2)
(5) $\dot{\xi}_2\ \texttt{refutable}_?$	by IH on (3)
(6) $\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?$	by Rule (2g) on (4) and (5)

□

$\boxed{e \models \dot{\xi}}$ e satisfies $\dot{\xi}$

$$\frac{}{e \models \top}\ \text{CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \models \underline{n}}\ \text{CSNum} \tag{4b}$$

$$\frac{e_1 \models \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)}\ \text{CSInl} \tag{4c}$$

$$\frac{e_2 \models \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)}\ \text{CSInr} \tag{4d}$$

$$\frac{e_1 \models \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CSPair} \tag{4e}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{prl}(e) \models \dot{\xi}_1 \qquad \texttt{prr}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CSNotIntroPair} \tag{4f}$$

$$\frac{e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrL} \tag{4g}$$

$$\frac{e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrR} \tag{4h}$$

$\boxed{\mathit{satisfy}(e, \dot{\xi})}$

$$\mathit{satisfy}(e, \top) = \text{true} \tag{5a}$$

$$\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{5b}$$

$$\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1) \text{ or } \mathit{satisfy}(e, \dot{\xi}_2) \tag{5c}$$

$$\mathit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\dot{\xi}_1)) = \mathit{satisfy}(e_1, \dot{\xi}_1) \tag{5d}$$

$$\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\dot{\xi}_2)) = \mathit{satisfy}(e_2, \dot{\xi}_2) \tag{5e}$$

$$\mathit{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2) \tag{5f}$$

$$\mathit{satisfy}(⦇⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(\mathtt{prl}(⦇⦈^u), \dot{\xi}_1) \text{ and } \mathit{satisfy}(\mathtt{prr}(⦇⦈^u), \dot{\xi}_2) \tag{5g}$$

$$\mathit{satisfy}(⦇e⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(\mathtt{prl}(⦇e⦈^u), \dot{\xi}_1) \text{ and } \mathit{satisfy}(\mathtt{prr}(⦇e⦈^u), \dot{\xi}_2) \tag{5h}$$

$$\begin{aligned}\mathit{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \mathit{satisfy}(\mathtt{prl}(e_1(e_2)), \dot{\xi}_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(e_1(e_2)), \dot{\xi}_2)\end{aligned} \tag{5i}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{match}(e)\{\hat{rs}\}, (\dot{\xi}_1, \dot{\xi}_2)) = {} & \mathit{satisfy}(\mathtt{prl}(\mathtt{match}(e)\{\hat{rs}\}), \dot{\xi}_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{match}(e)\{\hat{rs}\}), \dot{\xi}_2)\end{aligned} \tag{5j}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{prl}(e), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \mathit{satisfy}(\mathtt{prl}(\mathtt{prl}(e)), \dot{\xi}_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{prl}(e)), \dot{\xi}_2)\end{aligned} \tag{5k}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{prr}(e), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \mathit{satisfy}(\mathtt{prl}(\mathtt{prr}(e)), \dot{\xi}_1) \\ & \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{prr}(e)), \dot{\xi}_2)\end{aligned} \tag{5l}$$

$$\text{Otherwise} \quad \mathit{satisfy}(e, \dot{\xi}) = \text{false} \tag{5m}$$

Lemma 1.0.2 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \models \dot{\xi}$ iff $\mathit{satisfy}(e, \dot{\xi}) = \mathit{true}$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models \dot{\xi}$ — by assumption

   By rule induction over Rules (4) on (1).

   Case (4a)**.**

(2) $\dot{\xi} = \top$ by assumption

(3) $\mathit{satisfy}(e, \top) = \text{true}$ by Definition 5a

Case (4b)**.**

(2) $e = \underline{n}$ by assumption

(3) $\dot{\xi} = \underline{n}$ by assumption

(4) $\mathit{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$ by Definition 5b

Case (4g)**.**

(2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(3) $e \models \dot{\xi}_1$ by assumption

(4) $\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$ by IH on (3)

(5) $\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1)$ or $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$
by Definition 5c on (4)

Case (4h)**.**

(2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(3) $e \models \dot{\xi}_2$ by assumption

(4) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$ by IH on (3)

(5) $\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1)$ or $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$
by Definition 5c on (4)

Case (4c)**.**

(2) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\dot{\xi} = \mathtt{inl}(\dot{\xi}_1)$ by assumption

(4) $e_1 \models \dot{\xi}_1$ by assumption

(5) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\dot{\xi}_1)) = \mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$
by Definition 5d on (5)

Case (4d)**.**

(2) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\dot{\xi} = \mathtt{inl}(\dot{\xi}_2)$ by assumption

(4) $e_2 \models \dot{\xi}_2$ by assumption

(5) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\dot{\xi}_2)) = \mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$
by Definition 5e on (5)

Case (4e)**.**

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(4) $e_1 \models \dot{\xi}_1$ by assumption

(5) $e_2 \models \dot{\xi}_2$ by assumption

(6) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by IH on (5)

(8) $\mathit{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) =$
$\mathit{satisfy}(e_1, \dot{\xi}_1)$ and $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$
by Definition 5f on (6) and (7)

Case (4f)**.**

(2) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(3) e $\texttt{notintro}$ by assumption

(4) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(5) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(6) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$ by IH on (5)

By rule induction over Rules (18) on (3).

Otherwise.

(8) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(9) $\mathit{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) =$
$\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1)$ and $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$
by Definition 5 on (6) and (7)

2. Completeness:

(1) $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$ by assumption

By structural induction on $\dot{\xi}$.

Case $\dot{\xi} = \top$**.**

(2) $e \models \top$ by Rule (4a)

Case $\dot{\xi} = \bot, ?$**.**

(2) $\mathit{satisfy}(e, \dot{\xi}) = \text{false}$ by Definition 5m

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \underline{n}$**.**

By structural induction on e.

Case $e = \underline{n'}$**.**

(2) $n' = n$ by Definition 5b on (1)

(3) $\underline{n'} \models \underline{n}$ by Rule (4b) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \underline{n}) = \text{false}$ by Definition 5m

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$**.**

(2) $\mathit{satisfy}(e, \dot{\xi}_1)$ or $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$

by Definition 5c on (1)

By case analysis on (2).

Case $\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$ by assumption

(4) $e \models \dot{\xi}_1$ by IH on (3)

(5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (4g) on (4)

Case $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$ by assumption

(4) $e \models \dot{\xi}_2$ by IH on (3)

(5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (4h) on (4)

Case $\dot{\xi} = \mathtt{inl}(\dot{\xi}_1)$**.**

By structural induction on e.

Case $e = \mathtt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by Definition 5d on (1)

(3) $e_1 \models \dot{\xi}_1$ by IH on (2)

(4) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\xi}_1)$ by Rule (4c) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \mathtt{inl}(\dot{\xi}_1)) = \text{false}$ by Definition 5m

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \mathtt{inr}(\dot{\xi}_2)$**.**

By structural induction on e.

Case $e = \mathtt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by Definition 5e on (1)

(3) $e_2 \models \dot{\xi}_2$ by IH on (2)

(4) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$ by Rule (4d) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \mathtt{inr}(\dot{\xi}_2)) = \text{false}$ by Definition 5m

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$**.**

By structural induction on e.

Case $e = (e_1, e_2)$**.**

(2)	$\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$	by Definition 5f on (1)
(3)	$\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$	by Definition 5f on (1)
(4)	$e_1 \models \dot{\xi}_1$	by IH on (2)
(5)	$e_2 \models \dot{\xi}_2$	by IH on (3)
(6)	$(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (4e) on (4) and (5)

Case $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$**.**

(2)	$\mathit{satisfy}(\mathtt{prl}(e), \dot{\xi}_1) = \text{true}$	by Definition 5f on (1)
(3)	$\mathit{satisfy}(\mathtt{prr}(e), \dot{\xi}_2) = \text{true}$	by Definition 5f on (1)
(4)	$\mathtt{prl}(e) \models \dot{\xi}_1$	by IH on (2)
(5)	$\mathtt{prr}(e) \models \dot{\xi}_2$	by IH on (3)
(6)	e $\mathtt{notintro}$	by each rule in Rules (18)
(7)	$(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (4f) on (6) and (4) and (5)

Otherwise.

(2) $\mathit{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) = \text{false}$ by Definition 5m

(2) contradicts (1) and thus vacuously true.

□

$\boxed{e \models_? \dot{\xi}}$ e may satisfy $\dot{\xi}$

$$\text{CMSUnknown}\quad \frac{}{e \models_? ?} \tag{6a}$$

$$\text{CMSInl}\quad \frac{e_1 \models_? \dot{\xi}_1}{\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)} \tag{6b}$$

$$\text{CMSInr}\quad \frac{e_2 \models_? \dot{\xi}_2}{\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)} \tag{6c}$$

$$\text{CMSPairL}\quad \frac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)} \tag{6d}$$

$$\frac{e_1 \models \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairR} \tag{6e}$$

$$\frac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPair} \tag{6f}$$

$$\frac{e \models_? \dot{\xi}_1 \qquad e \not\models \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrL} \tag{6g}$$

$$\frac{e \not\models \dot{\xi}_1 \qquad e \models_? \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrR} \tag{6h}$$

$$\frac{e\ \texttt{notintro} \qquad \dot{\xi}\ \texttt{refutable}_?}{e \models_? \dot{\xi}}\ \text{CMSNotIntro} \tag{6i}$$

$\boxed{\mathit{maysatisfy}(e, \dot{\xi})}$

$$\mathit{maysatisfy}(e, ?) = \text{true} \tag{7a}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \mathit{maysatisfy}(e_1, \dot{\xi}_1) \tag{7b}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \mathit{maysatisfy}(e_2, \dot{\xi}_2) \tag{7c}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inr}(\dot{\xi}_2)) = \text{false} \tag{7d}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inl}(\dot{\xi}_1)) = \text{false} \tag{7e}$$

$$\begin{aligned} \mathit{maysatisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big) \end{aligned} \tag{7f}$$

$$\begin{aligned} \mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = {} & \Big(\mathit{maysatisfy}(e, \dot{\xi}_1) \text{ and } \Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_2)\Big)\Big) \\ & \text{or } \Big(\Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_1)\Big) \text{ and } \mathit{maysatisfy}(e, \dot{\xi}_2)\Big) \end{aligned} \tag{7g}$$

$$\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{notintro}(e) \text{ and } \mathit{refutable}_?(\dot{\xi}) \tag{7h}$$

Lemma 1.0.3 (Soundness and Completeness of Maybe Satisfaction Judgment)**.** *$e \models_? \dot{\xi}$ iff $\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{true}$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

(1) $e \models_? \dot{\xi}$ by assumption

By rule induction over Rules (6) on (1).

Case (6a).

(2) $\dot{\xi} = ?$ by assumption

(3) $\mathit{maysatisfy}(e, ?) = \text{true}$ by Definition 7a

Case (6b).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$ by assumption

(4) $e_1 \models_? \dot{\xi}_1$ by assumption

(5) $\mathit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)

(6) $\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \text{true}$ by Definition 7b on (5)

Case (6c).

(2) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$ by assumption

(4) $e_2 \models_? \dot{\xi}_2$ by assumption

(5) $\mathit{maysatisfy}(e_2, \dot{\xi}_2) = \text{true}$ by IH on (4)

(6) $\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \text{true}$ by Definition 7c on (5)

Case (6d).

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(4) $e_1 \models_? \dot{\xi}_1$ by assumption

(5) $e_2 \models \dot{\xi}_2$ by assumption

(6) $\mathit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by Lemma 1.0.2 on (5)

(8) $\mathit{maysatisfy}((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$ by Definition 7f on (6) and (7)

Case (6e).

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(4) $e_1 \models \dot{\xi}_1$ by assumption

(5) $e_2 \models_? \dot{\xi}_2$ by assumption

(6) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by Lemma 1.0.2 on (4)

(7) $\mathit{maysatisfy}(e_2, \dot{\xi}_2) = \text{true}$ by IH on (5)

(8) $\mathit{maysatisfy}((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$ by Definition 7f on (6) and (7)

Case (6f)**.**

(2) $e = (e_1, e_2)$	by assumption
(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$	by assumption
(4) $e_1 \models_? \dot{\xi}_1$	by assumption
(5) $e_2 \models_? \dot{\xi}_2$	by assumption
(6) $\mathit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$	by IH on (4)
(7) $\mathit{maysatisfy}(e_2, \dot{\xi}_2) = \text{true}$	by IH on (5)
(8) $\mathit{maysatisfy}((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$	by Definition 7f on (6) and (7)

Case (6g)**.**

(2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3) $e \models_? \dot{\xi}_1$	by assumption
(4) $e \not\models \dot{\xi}_2$	by assumption
(5) $\mathit{maysatisfy}(e, \dot{\xi}_1) = \text{true}$	by IH on (3)
(6) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$	by Lemma 1.0.2 on (4)
(7) $\mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{true}$	by Definition 5c on (5) and (6)

Case (6h)**.**

(2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3) $e \not\models \dot{\xi}_1$	by assumption
(4) $e \models_? \dot{\xi}_2$	by assumption
(5) $\mathit{satisfy}(e, \dot{\xi}_1) = \text{false}$	by Lemma 1.0.2 on (3)
(6) $\mathit{maysatisfy}(e, \dot{\xi}_2) = \text{true}$	by IH on (4)
(7) $\mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{true}$	by Definition 5c on (5) and (6)

Case (6i)**.**

(2) e `notintro`	by assumption
(3) $\dot{\xi}$ `refutable`$_?$	by assumption
(4) $\mathit{notintro}(e) = \text{true}$	by Lemma 3.0.1 on (2)
(5) $\mathit{refutable}_?(\dot{\xi}) = \text{true}$	by Lemma 1.0.1 on (3)
(6) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$	by Definition 7h on (4) and (5)

2. Completeness:

(1) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$	by assumption

By structural induction on $\dot{\xi}$.

Case $\dot{\xi} = \top, \bot$**.**

(2) $\mathit{refutable}_?(\dot{\xi}) = \text{false}$ — by Definition 3

(3) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{false}$ — by Definition 7h and (2)

Contradicts (1) and thus vacuously true.

Case $\dot{\xi} = ?$.

(2) $e \models_? ?$ — by Rule (6a)

Case $\dot{\xi} = \underline{n}$.

(2) $\mathit{notintro}(e) = \text{true}$ — by Definition 7h of (1)

(3) $e\ \mathtt{notintro}$ — by Lemma 3.0.1 on (2)

(4) $\underline{n}\ \mathtt{refutable}_?$ — by Rule (2a)

(5) $e \models_? \underline{n}$ — by Rule (6i) on (3) and (4)

Case $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$.

By case analysis on Definition 7g of (1).

Case $\mathit{maysatisfy}(e, \dot{\xi}_1) = \text{true}$ and $\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$.

(2) $\mathit{maysatisfy}(e, \dot{\xi}_1) = \text{true}$ — by assumption

(3) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$ — by assumption

(4) $e \models_? \dot{\xi}_1$ — by IH on (2)

(5) $e \not\models \dot{\xi}_2$ — by Lemma 1.0.2 on (3)

(6) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (6g) on (4) and (5)

Case $\mathit{satisfy}(e, \dot{\xi}_1) = \text{false}$ and $\mathit{maysatisfy}(e, \dot{\xi}_2) = \text{true}$.

(2) $\mathit{satisfy}(e, \dot{\xi}_1) = \text{false}$ — by assumption

(3) $\mathit{maysatisfy}(e, \dot{\xi}_2) = \text{true}$ — by assumption

(4) $e \not\models \dot{\xi}_1$ — by Lemma 1.0.2 on (2)

(5) $e \models_? \dot{\xi}_2$ — by IH on (3)

(6) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (6h) on (4) and (5)

Case $\dot{\xi} = \mathtt{inl}(\dot{\xi}_1)$.

By structural induction on e.

Case $e = (\!|\,|\!)^u, (\!|e'|\!)^u, e_1(e_2), \mathtt{match}(e')\{\hat{rs}\}, \mathtt{prl}(e'), \mathtt{prr}(e')$.

(2) $\mathit{refutable}_?(\mathtt{inl}(\dot{\xi}_1)) = \text{true}$ — by Definition 7h of (1)

(3) $\mathtt{inl}(\dot{\xi}_1)\ \mathtt{refutable}_?$ — by Lemma 1.0.1 on (2)

(4) $e\ \mathtt{notintro}$ — by Rules (18)

(5) $e \models_? \mathtt{inl}(\dot{\xi}_1)$ — by Rule (6i) on (4) and (3)

Case $e = x, \underline{n}, (\lambda x : \tau . e'), (e_1, e_2)$.

(2) $\mathit{notintro}(e) = \text{false}$ — by Rules (18)

(3) $\mathit{maysatisfy}(e, \mathtt{inl}(\dot{\xi}_1)) = \text{false}$ by Definition 7h on (2)

Contradicts (1).

Case $e = \mathtt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{maysatisfy}(e_1, \dot{\xi}_1)$ by Definition 7b of (1)

(3) $e_1 \models_? \dot{\xi}_1$ by Lemma 1.0.3 on (2)

(4) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6b) on (3)

Case $e = \mathtt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{maysatisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inl}(\dot{\xi}_1)) = \text{false}$
by Definition 7e

Contradicts (1).

Case $\dot{\xi} = \mathtt{inr}(\dot{\xi}_2)$**.**

By structural induction on e.

Case $e = (\!|\!)^u, (\!|e'|\!)^u, e_1(e_2), \mathtt{match}(e')\{\hat{rs}\}, \mathtt{prl}(e'), \mathtt{prr}(e')$**.**

(2) $\mathit{refutable}_?(\mathtt{inr}(\dot{\xi}_2)) = \text{true}$ by Definition 7h of (1)

(3) $\mathtt{inr}(\dot{\xi}_2)\ \mathtt{refutable}_?$ by Lemma 1.0.1 on (2)

(4) $e\ \mathtt{notintro}$ by Rules (18)

(5) $e \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (6i) on (4) and (3)

Case $e = x, \underline{n}, (\lambda x : \tau.e'), (e_1, e_2)$**.**

(2) $\mathit{notintro}(e) = \text{false}$ by Rules (18)

(3) $\mathit{maysatisfy}(e, \mathtt{inr}(\dot{\xi}_2)) = \text{false}$ by Definition 7h on (2)

Contradicts (1).

Case $e = \mathtt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{maysatisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inr}(\dot{\xi}_2)) = \text{false}$
by Definition 7d

Contradicts (1).

Case $e = \mathtt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{maysatisfy}(e_2, \dot{\xi}_2)$ by Definition 7c of (1)

(3) $e_2 \models_? \dot{\xi}_2$ by Lemma 1.0.3 on (2)

(4) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (6c) on (3)

Case $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$**.**

By structural induction on e.

Case $e = (\!|\!)^u, (\!|e'|\!)^u, e_1(e_2), \mathtt{match}(e')\{\hat{rs}\}, \mathtt{prl}(e'), \mathtt{prr}(e')$**.**

(2) $\mathit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$ by Definition 7h of (1)

(3) $(\dot{\xi}_1, \dot{\xi}_2)\ \mathtt{refutable}_?$ by Lemma 1.0.1 on (2)

(4) $e\ \mathtt{notintro}$ by Rules (18)

(5) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6i) on (4) and (3)

Case $e = x, \underline{n}, (\lambda x : \tau . e'), \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2)$**.**

(2) $\textit{notintro}(e) = \text{false}$ by Rules (18)

(3) $\textit{maysatisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) = \text{false}$ by Definition 7h on (2)

Contradicts (1).

Case $e = (e_1, e_2)$**.** By case analysis on Definition 7f on (1).

Case $\textit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ and $\textit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$**.**

(2) $\textit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ by assumption

(3) $\textit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by assumption

(4) $e_1 \models_? \dot{\xi}_1$ by IH on (2)

(5) $e_2 \models \dot{\xi}_2$ by Lemma 1.0.2 on (3)

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6d) on (4) and (5)

Case $\textit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ and $\textit{maysatisfy}(e_2, \dot{\xi}_2) = \text{true}$**.**

(2) $\textit{satisfy}(e_1, \dot{\xi}_1)$ by assumption

(3) $\textit{maysatisfy}(e_2, \dot{\xi}_2)$ by assumption

(4) $e_1 \models \dot{\xi}_1$ by Lemma 1.0.2 on (2)

(5) $e_2 \models_? \dot{\xi}_2$ by IH on (3)

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6e) on (4) and (5)

Case $\textit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ and $\textit{maysatisfy}(e_2, \dot{\xi}_2) = \text{true}$**.**

(2) $\textit{maysatisfy}(e_1, \dot{\xi}_1)$ by assumption

(3) $\textit{maysatisfy}(e_2, \dot{\xi}_2)$ by assumption

(4) $e_1 \models_? \dot{\xi}_1$ by IH on (2)

(5) $e_2 \models_? \dot{\xi}_2$ by IH on (3)

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6f) on (4) and (5)

□

$\boxed{e \models_?^\dagger \dot{\xi}}$ e satisfies or may satisfy $\dot{\xi}$

$$\text{CSMSMay}\quad \frac{e \models_? \dot{\xi}}{e \models_?^\dagger \dot{\xi}} \tag{8a}$$

$$\text{CSMSSat}\quad \frac{e \models \dot{\xi}}{e \models_?^\dagger \dot{\xi}} \tag{8b}$$

$\boxed{\textit{satisfyormay}(e, \dot{\xi})}$

$$\textit{satisfyormay}(e, \dot{\xi}) = \textit{satisfy}(e, \dot{\xi}) \text{ or } \textit{maysatisfy}(e, \dot{\xi}) \tag{9}$$

Lemma 1.0.4 (Soundness and Completeness of Satisfaction or Maybe Satisfaction)**.** $e \models_?^\dagger \dot{\xi}$ *iff* $\mathit{satisfyormay}(e, \dot{\xi})$.

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models_?^\dagger \dot{\xi}$ by assumption

   By rule induction over Rules (8) on (1).

   Case (8b)**.**

   (2) $e \models \dot{\xi}$ by assumption
   (3) $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$ by Lemma 1.0.2 on (2)
   (4) $\mathit{satisfyormay}(e, \dot{\xi}) = \text{true}$ by Definition 9 on (3)

   Case (8a)**.**

   (2) $e \models_? \dot{\xi}$ by assumption
   (3) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$ by Lemma 1.0.3 on (2)
   (4) $\mathit{satisfyormay}(e, \dot{\xi}) = \text{true}$ by Definition 9 on (3)

2. Completeness:

   (1) $\mathit{satisfyormay}(e, \dot{\xi}) = \text{true}$ by assumption

   By case analysis on Definition 9 of (1).

   Case $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$**.**

   (2) $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$ by assumption
   (3) $e \models \dot{\xi}$ by Lemma 1.0.2 on (2)
   (4) $e \models_?^\dagger \dot{\xi}$ by Rule (8b) on (3)

   Case $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$**.**

   (2) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$ by assumption
   (3) $e \models_? \dot{\xi}$ by Lemma 1.0.3 on (2)
   (4) $e \models_?^\dagger \dot{\xi}$ by Rule (8a) on (3)

□

Lemma 1.0.5. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (6) on $e \models_? \top$, only one case applies.

Case (6i)**.**

(1) $\top\ \texttt{refutable}_?$ by assumption

By rule induction over Rules (2) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. □

Lemma 1.0.6. $e \not\models ?$

Proof. By rule induction over Rules (4), we notice that $e \models ?$ is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.7. *$e \not\models_?^\dagger \dot{\xi}_1$ and $e \not\models_?^\dagger \dot{\xi}_2$ iff $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency: to show $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$, we assume $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$.

   (1) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption
   (2) $e \not\models_?^\dagger \dot{\xi}_1$ by assumption
   (3) $e \not\models_?^\dagger \dot{\xi}_2$ by assumption

   By rule induction over Rules (8) on (1).

   Case (8b).

   (4) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

   By rule induction over Rules (4) on (4) and only two of them apply.

   Case (4g).

   (5) $e \models \dot{\xi}_1$ by assumption
   (6) $e \models_?^\dagger \dot{\xi}_1$ by Rule (8b) on (5)

   (6) contradicts (2).

   Case (4h).

   (5) $e \models \dot{\xi}_2$ by assumption
   (6) $e \models_?^\dagger \dot{\xi}_2$ by Rule (8b) on (5)

   (6) contradicts (3).

   Case (8a).

   (4) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

   By rule induction over Rules (6) on (4) and only two of them apply.

   Case (6g).

   (5) $e \models_? \dot{\xi}_1$ by assumption
   (6) $e \models_?^\dagger \dot{\xi}_1$ by Rule (8a) on (5)

   (6) contradicts (2).

   Case (6h).

(5) $e \models_? \dot{\xi}_2$ by assumption

(6) $e \models_?^\dagger \dot{\xi}_2$ by Rule (8a) on (5)

(6) contradicts (3).

The conclusion holds as follows:

(a) $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$

2. Necessity:

(1) $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

We show $e \not\models_?^\dagger \dot{\xi}_1$ and $e \not\models_?^\dagger \dot{\xi}_2$ separately.

(a) To show $e \not\models_?^\dagger \dot{\xi}_1$, we assume $e \models_?^\dagger \dot{\xi}_1$.

(2) $e \models_?^\dagger \dot{\xi}_1$ by assumption

(3) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Lemma 1.0.9 on (2)

Contradicts (1).

(b) To show $e \not\models_?^\dagger \dot{\xi}_2$, we assume $e \models_?^\dagger \dot{\xi}_2$.

(2) $e \models_?^\dagger \dot{\xi}_2$ by assumption

(3) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Lemma 1.0.9 on (2)

Contradicts (1).

In conclusion, $e \not\models_?^\dagger \dot{\xi}_1$ and $e \not\models_?^\dagger \dot{\xi}_2$.

□

Lemma 1.0.8. *If* $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ *and* $e \not\models_?^\dagger \dot{\xi}_1$ *then* $e \models_?^\dagger \dot{\xi}_2$

Proof.

(4) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(5) $e \not\models_?^\dagger \dot{\xi}_1$ by assumption

By rule induction over Rules (8) on (4).

Case (8b)**.**

(6) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By rule induction over Rules (4) on (6) and only two of them apply.

Case (4g)**.**

(7) $e \models \dot{\xi}_1$ by assumption

(8) $e \models_?^\dagger \dot{\xi}_1$ by Rule (8b) on (7)

(8) contradicts (5).

Case (4h).

(7) $e \models \dot{\xi}_2$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8b) on (7)

Case (8a).

(6) $e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By rule induction over Rules (6) on (6) and only two of them apply.

Case (6g).

(7) $e \models_{?} \dot{\xi}_1$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (8a) on (7)

(8) contradicts (5).

Case (6h).

(7) $e \models_{?} \dot{\xi}_2$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8a) on (7)

□

Lemma 1.0.9. *If* $e \models^{\dagger}_{?} \dot{\xi}_1$ *then* $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ *and* $e \models^{\dagger}_{?} \dot{\xi}_2 \vee \dot{\xi}_1$

Proof.

(1) $e \models^{\dagger}_{?} \dot{\xi}_1$ by assumption ,

By rule induction over Rules (8) on (1),

Case (8b).

(2) $e \models \dot{\xi}_1$ by assumption

(3) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (4g) on (2)

(4) $e \models \dot{\xi}_2 \vee \dot{\xi}_1$ by Rule (4h) on (2)

(5) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (8b) on (3)

(6) $e \models^{\dagger}_{?} \dot{\xi}_2 \vee \dot{\xi}_1$ by Rule (8b) on (4)

Case (8a).

(2) $e \models_{?} \dot{\xi}_1$ by assumption

By case analysis on the result of *satisfy*$(e, \dot{\xi}_2)$.

Case true.

(3)	$\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by assumption
(4)	$e \models \dot{\xi}_2$	by Lemma 1.0.2 on (3)
(5)	$e \models \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (4h) on (4)
(6)	$e \models \dot{\xi}_2 \vee \dot{\xi}_1$	by Rule (4g) on (4)
(7)	$e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (8b) on (5)
(8)	$e \models_?^\dagger \dot{\xi}_2 \vee \dot{\xi}_1$	by Rule (8b) on (6)

Case false**.**

(3)	$\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$	by assumption
(4)	$e \not\models \dot{\xi}_2$	by Lemma 1.0.2 on (3)
(5)	$e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (6g) on (2) and (4)
(6)	$e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (8a) on (5)

□

Lemma 1.0.10. $e_1 \models_?^\dagger \dot{\xi}_1$ *iff* $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1)	$e_1 \models_?^\dagger \dot{\xi}_1$	by assumption

By rule induction over Rules (8) on (1).

Case (8b)**.**

(2)	$e_1 \models \dot{\xi}_1$	by assumption
(3)	$\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$	by Rule (4c) on (2)
(4)	$\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$	by Rule (8b) on (3)

Case (8a)**.**

(2)	$e_1 \models_? \dot{\xi}_1$	by assumption
(3)	$\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$	by Rule (6b) on (2)
(4)	$\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$	by Rule (8a) on (3)

2. Necessity:

(1)	$\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$	by assumption

By rule induction over Rules (8) on (1).

Case (8b)**.**

(2) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (4) on (2), only one rule applies.

Case (4c)**.**

(3) $e_1 \models \dot{\xi}_1$ by assumption

(4) $e_1 \models_?^\dagger \dot{\xi}_1$ by Rule (8b) on (3)

Case (8a)**.**

(2) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (6) on (2), only two rules apply.

Case (6b)**.**

(3) $e_1 \models_? \dot{\xi}_1$ by assumption

(4) $e_1 \models_?^\dagger \dot{\xi}_1$ by Rule (8a) on (3)

Case (6i)**.**

(3) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (18) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.11. $e_2 \models_?^\dagger \dot{\xi}_2$ *iff* $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e_2 \models_?^\dagger \dot{\xi}_2$ by assumption

By rule induction over Rules (8) on (1).

Case (8b)**.**

(2) $e_2 \models \dot{\xi}_2$ by assumption

(3) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by Rule (4d) on (2)

(4) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Rule (8b) on (3)

Case (8a)**.**

(2) $e_2 \models_? \dot{\xi}_2$ by assumption

(3) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (6c) on (2)

(4) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (3)

2. Necessity:

(1) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (8) on (1).

Case (8b).

(2) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (4) on (2), only one rule applies.

Case (4d).

(3) $e_2 \models \dot{\xi}_2$ by assumption

(4) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8b) on (3)

Case (8a).

(2) $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (6) on (2), only two rules apply.

Case (6c).

(3) $e_2 \models_{?} \dot{\xi}_2$ by assumption

(4) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8a) on (3)

Case (6i).

(3) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (18) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.12. *$e_1 \models^{\dagger}_{?} \dot{\xi}_1$ and $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ iff $(e_1, e_2) \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by assumption

(2) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by assumption

By rule induction over Rules (8) on (1).

Case (8b).

(3) $e_1 \models \dot{\xi}_1$ by assumption

By rule induction over Rules (8) on (2).

Case (8b).

(4) $e_2 \models \dot{\xi}_2$ by assumption

(5) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (4e) on (3) and (4)

(6) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8b) on (5)

Case (8a).

(4) $e_2 \models_? \dot{\xi}_2$ by assumption

(5) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6e) on (3) and (4)

(6) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (5)

Case (8a)**.**

(4) $e_1 \models_? \dot{\xi}_1$ by assumption

By rule induction over Rules (8) on (2).

Case (8b)**.**

(5) $e_2 \models \dot{\xi}_2$ by assumption

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6d) on (4) and (5)

(7) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (6)

Case (8a)**.**

(5) $e_2 \models_? \dot{\xi}_2$ by assumption

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6e) on (4) and (5)

(7) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (6)

2. Necessity:

(1) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

By rule induction over Rules (8) on (1).

Case (8b)**.**

(2) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

By rule induction over Rules (4) on (2), only one rule applies.

Case (4e)**.**

(3) $e_1 \models \dot{\xi}_1$ by assumption

(4) $e_2 \models \dot{\xi}_2$ by assumption

(5) $e_1 \models_?^\dagger \dot{\xi}_1$ by Rule (8b) on (3)

(6) $e_2 \models_?^\dagger \dot{\xi}_2$ by Rule (8b) on (4)

Case (8a)**.**

(2) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

By rule induction over Rules (6) on (2), only three rules apply.

Case (6d)**.**

(3) $e_1 \models_? \dot{\xi}_1$ by assumption

(4) $e_2 \models \dot{\xi}_2$ by assumption

(5) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (8a) on (3)

(6) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8b) on (4)

Case (6e)**.**

(3) $e_1 \models \dot{\xi}_1$ by assumption

(4) $e_2 \models_{?} \dot{\xi}_2$ by assumption

(5) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (8b) on (3)

(6) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8a) on (4)

Case (6f)**.**

(3) $e_1 \models_{?} \dot{\xi}_1$ by assumption

(4) $e_2 \models_{?} \dot{\xi}_2$ by assumption

(5) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (8a) on (3)

(6) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (8a) on (4)

□

Lemma 1.0.13. *Assume e `notintro`. If $e \models_{?} \dot{\xi}$ or $e \not\models \dot{\xi}$ then $\dot{\xi}$ `refutable`$_{?}$.*

Proof.

(1) e `notintro` by assumption

By case analysis on the premise, which is a disjunction.

Case $e \models_{?} \dot{\xi}$**.**

(2) $e \models_{?} \dot{\xi}$ by assumption

By rule induction over Rules (6) on (2).

Case (6a)**.**

(3) $\dot{\xi} = ?$ by assumption

(4) $?$ `refutable`$_{?}$ by Rule (2b)

Case (6b)**.**

(3) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

By rule induction over Rules (18) on (1), no rule applies due to syntactic contradiction.

Case (6c)**.**

(3) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

By rule induction over Rules (18) on (1), no rule applies due to syntactic contradiction.

Case (6d), (6e), (6f)**.**

(3) $e = (e_1, e_2)$ by assumption

By rule induction over Rules (18) on (1), no rule applies due to syntactic contradiction.

Case (6g).

(3) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(4) $e \models_? \dot{\xi}_1$	by assumption
(5) $e \not\models \dot{\xi}_2$	by assumption
(6) $\dot{\xi}_1$ $\texttt{refutable}_?$	by IH on (1) and (4)
(7) $\dot{\xi}_2$ $\texttt{refutable}_?$	by IH on (1) and (5)
(8) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$	by Rule (2g) on (6) and (7)

Case (6h).

(3) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(4) $e \not\models \dot{\xi}_1$	by assumption
(5) $e \models_? \dot{\xi}_2$	by assumption
(6) $\dot{\xi}_1$ $\texttt{refutable}_?$	by IH on (1) and (4)
(7) $\dot{\xi}_2$ $\texttt{refutable}_?$	by IH on (1) and (5)
(8) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$	by Rule (2g) on (6) and (7)

Case (6i).

(3) $\dot{\xi}$ $\texttt{refutable}_?$	by assumption

Case $e \not\models \dot{\xi}$.

(2) $e \not\models \dot{\xi}$	by assumption

By structural induction on $\dot{\xi}$.

Case $\dot{\xi} = \top$.

(3) $e \models \top$	by Rule (4a)

Contradicts (2).

Case $\dot{\xi} = ?$.

(3) $?$ $\texttt{refutable}_?$	by Rule (2b)

Case $\dot{\xi} = \underline{n}$.

(3) $\underline{n}$ $\texttt{refutable}_?$	by Rule (2a)

Case $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$.

(3) $\texttt{inl}(\dot{\xi}_1)$ $\texttt{refutable}_?$	by Rule (2c)

Case $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$.

(3) $\texttt{inr}(\dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2d)

Case $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$**.**

(3) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (18e)

(4) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (18f)

By case analysis on the value of $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1)$ and $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2)$.

Case true, true**.**

(5) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by assumption

(6) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$ by assumption

(7) $\texttt{prl}(e) \models \dot{\xi}_1$ by Lemma 1.0.2 on (5)

(8) $\texttt{prr}(e) \models \dot{\xi}_2$ by Lemma 1.0.2 on (6)

(9) $e \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (4f) on (1) and (7) and (8)

Contradicts $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$.

Case true, false**.**

(5) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by assumption

(6) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{false}$ by assumption

(7) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by Lemma 1.0.2 on (6)

(8) $\dot{\xi}_2\ \texttt{refutable}_?$ by IH on (4) and (7)

(9) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2f) on (8)

Case false, true**.**

(5) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{false}$ by assumption

(6) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$ by assumption

(7) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by Lemma 1.0.2 on (5)

(8) $\dot{\xi}_1\ \texttt{refutable}_?$ by IH on (3) and (7)

(9) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2e) on (8)

Case false, false**.**

(5) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{false}$ by assumption

(6) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{false}$ by assumption

(7) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by Lemma 1.0.2 on (5)

(8) $\dot{\xi}_1\ \texttt{refutable}_?$ by IH on (3) and (7)

(9) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2e) on (8)

Case $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$**.**

To show that $e \not\models \dot{\xi}_1$, we assume $e \models \dot{\xi}_1$ and obtain a contradiction.

(3) $e \models \dot{\xi}_1$ by assumption

(4) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (4g) on (3)

Contradicts (2). Therefore,

(3)	$e \not\models \dot{\xi}_1$	by contradiction
(4)	$\dot{\xi}_1\ \texttt{refutable}_?$	by IH on (1) and (3)

Similarly, to show that $e \not\models \dot{\xi}_2$, we assume $e \models \dot{\xi}_2$ and obtain a contradiction.

(5)	$e \models \dot{\xi}_2$	by assumption
(6)	$e \models \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (4h) on (5)

Contradicts (2). Therefore,

(5)	$e \not\models \dot{\xi}_2$	by contradiction
(6)	$\dot{\xi}_2\ \texttt{refutable}_?$	by IH on (1) and (5)
(7)	$\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?$	by Rule (2g) on (4) and (6)

□

Lemma 1.0.14. *If* e $\texttt{notintro}$ *and* $e \models \dot{\xi}$ *then* $\cancel{\dot{\xi}\ \texttt{refutable}_?}$.

Proof.
To show $\cancel{\dot{\xi}\ \texttt{refutable}_?}$, we assume $\dot{\xi}\ \texttt{refutable}_?$ and obtain a contradiction.

(8)	e $\texttt{notintro}$	by assumption
(9)	$e \models \dot{\xi}$	by assumption
(10)	$\dot{\xi}\ \texttt{refutable}_?$	by assumption

By rule induction over Rules (4) on (9).

Case (4a)**.**

(11)	$\dot{\xi} = \top$	by assumption

By rule induction over Rules (2), no case applies due to syntactic contradiction.

Case (4g)**.**

(11)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(12)	$e \models \dot{\xi}_1$	by assumption
(13)	$\cancel{\dot{\xi}_1\ \texttt{refutable}_?}$	by IH on (8) and (12)
(14)	$\cancel{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}$	by **??**

By rule induction over Rules (2) on (10), only one rule applies.

Case (2g)**.**

(15)	$\dot{\xi}_1\ \texttt{refutable}_?$	by assumption

Contradicts (13).

Case (4h)**.**

(11) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(12) $e \models \dot{\xi}_2$ by assumption

(13) $\cancel{\dot{\xi}_2\ \texttt{refutable}_?}$ by IH on (8) and (12)

(14) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}$ by **??**

By rule induction over Rules (2) on (10), only one rule applies.

Case (2g)**.**

(15) $\dot{\xi}_2\ \texttt{refutable}_?$ by assumption

Contradicts (13).

Case (4f)**.**

(11) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(12) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(13) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(14) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (18e)

(15) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (18f)

(16) $\cancel{\dot{\xi}_1\ \texttt{refutable}_?}$ by IH on (14) and (12)

(17) $\cancel{\dot{\xi}_2\ \texttt{refutable}_?}$ by IH on (15) and (13)

By rule induction over Rules (2) on it, only two cases apply.

Case (2e)**.**

(18) $\dot{\xi}_1\ \texttt{refutable}_?$ by assumption

Contradicts (16).

Case (2f)**.**

(18) $\dot{\xi}_2\ \texttt{refutable}_?$ by assumption

Contradicts (17).

Otherwise.

(11) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (18) on (8), no case applies due to syntactic contradiction.

□

Lemma 1.0.15. $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ *is not derivable.*

Proof. We prove by assuming $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ and obtaining a contradiction.

(1) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (8) on (1).

Case (8b).

(2) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (4) on (2), no rule applies due to syntactic contradiction.

Case (8a).

(2) $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (6) on (2), only one rule applies.

Case (6i).

(3) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{notintro}$ by assumption

By rule induction over Rules (18) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.16. $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inl}(\dot{\xi}_1)$ *is not derivable.*

Proof. We prove by assuming $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inl}(\dot{\xi}_1)$ and obtaining a contradiction.

(1) $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (8) on (1).

Case (8b).

(2) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (4) on (2), no rule applies due to syntactic contradiction.

Case (8a).

(2) $\mathtt{inr}_{\tau_1}(e_2) \models_{?} \mathtt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (6) on (2), only one rule applies.

Case (6i).

(3) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

By rule induction over Rules (18) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.17. *$e \not\models \dot{\xi}$ and $e \not\models_? \dot{\xi}$ iff $e \not\models_?^\dagger \dot{\xi}$.*

Proof. 1. Sufficiency:

(1) $e \not\models \dot{\xi}$ by assumption

(2) $e \not\models_? \dot{\xi}$ by assumption

Assume $e \models_?^\dagger \dot{\xi}$. By rule induction over Rules (8) on it.

Case (8a).

(3) $e \models \dot{\xi}$ by assumption

Contradicts (1).

Case (8b).

(3) $e \models_? \dot{\xi}$ by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \dot{\xi}$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \dot{\xi}$ by assumption

Assume $e \models \dot{\xi}$.

(2) $e \models_?^\dagger \dot{\xi}$ by Rule (8b) on assumption

Contradicts (1). Therefore, $e \not\models \dot{\xi}$. Assume $e \models_? \dot{\xi}$.

(3) $e \models_?^\dagger \dot{\xi}$ by Rule (8a) on assumption

Contradicts (1). Therefore, $e \not\models_? \dot{\xi}$.

□

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \models \dot{\xi}$
2. $e \models_? \dot{\xi}$
3. $e \not\models_?^\dagger \dot{\xi}$

Proof.

(4) $\dot{\xi} : \tau$ by assumption

(5) $\cdot\, ; \Delta \vdash e : \tau$ by assumption

(6) e `final` by assumption

By rule induction over Rules (1) on (4), we would show one conclusion is derivable while the other two are not.

Case (1a).

(7) $\dot{\xi} = \top$ by assumption

(8) $e \models \top$ by Rule (4a)

(9) $e \not\models_? \top$ by Lemma 1.0.5

(10) $e \models_?^\dagger \top$ by Rule (8b) on (8)

Case (1b).

(7) $\dot{\xi} = ?$ by assumption

(8) $e \not\models ?$ by Lemma 1.0.6

(9) $e \models_? ?$ by Rule (6a)

(10) $e \models_?^\dagger ?$ by Rule (8a) on (9)

Case (1c).

(7) $\dot{\xi} = \underline{n_2}$ by assumption

(8) $\tau = \texttt{num}$ by assumption

By rule induction over Rules (11) on (5), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e `notintro` by Rule (18a),(18b),(18c),(18d),(18e),(18f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (4) on it, no case applies due to syntactic contradiction on $\dot{\xi}$.

(11) $e \not\models \underline{n_2}$ by contradiction

(12) $\underline{n_2}$ $\texttt{refutable}_?$ by Rule (2a)

(13) $e \models_? \underline{n_2}$ by Rule (6i) on (10) and (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (8a) on (13)

Case (11d).

(9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (6), only one case applies.

Case (6i).

(10) $\underline{n_1}$ `notintro` by assumption

Contradicts Lemma 3.0.6.

(11) $\underline{n_1} \not\models_? \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 5

(13) $\underline{n_1} \models \underline{n_2}$ by Lemma 1.0.2 on (12)

(14) $\underline{n_1} \models_?^\dagger \underline{n_2}$ by Rule (8b) on (13)

Case $n_1 \neq n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 5

(13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 1.0.2 on (12)

(14) $\underline{n_1} \not\models_?^\dagger \underline{n_2}$ by Lemma 1.0.17 on (11) and (13)

Case (1g).

(7) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \dot{\xi}_1$, $e \models_? \dot{\xi}_1$, and $e \not\models_?^\dagger \dot{\xi}_1$ holds. The same goes for $\dot{\xi}_2$. By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $e \models \dot{\xi}_1, e \models \dot{\xi}_2$.

(8) $e \models \dot{\xi}_1$ by assumption

(9) $e \not\models_? \dot{\xi}_1$ by assumption

(10) $e \models \dot{\xi}_2$ by assumption

(11) $e \not\models_? \dot{\xi}_2$ by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (4g) on (8)

(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (8b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(14) e `notintro` — by assumption

(15) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ — by assumption

(16) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2 \texttt{ refutable}_?}$ — by Lemma 1.0.14 on (14) and (12)

(15) and (16) are in contradiction with each other.

Case (6g).

(14) $e \models_? \dot{\xi}_1$ — by assumption

Contradicts (9).

Case (6h).

(14) $e \models_? \dot{\xi}_2$ — by assumption

Contradicts (11).

(14) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \models \dot{\xi}_1, e \models_? \dot{\xi}_2$.

(8) $e \models \dot{\xi}_1$ — by assumption

(9) $e \not\models_? \dot{\xi}_1$ — by assumption

(10) $e \not\models \dot{\xi}_2$ — by assumption

(11) $e \models_? \dot{\xi}_2$ — by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (4g) on (8)

(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (8b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(14) e `notintro` — by assumption

(15) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ — by assumption

(16) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2 \texttt{ refutable}_?}$ — by Lemma 1.0.14 on (14) and (12)

(15) and (16) are in contradiction with each other.

Case (6g).

(14) $e \models_? \dot{\xi}_1$ — by assumption

Contradicts (9).

Case (6h).

(14) $e \not\models \dot{\xi}_1$ — by assumption

Contradicts (8).

(14) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \models \dot{\xi}_1, e \not\models_?^{\dagger} \dot{\xi}_2$**.**

(8) $e \models \dot{\xi}_1$ — by assumption

(9) $e \not\models_? \dot{\xi}_1$ — by assumption

(10) $e \not\models \dot{\xi}_2$ — by assumption

(11) $e \not\models_? \dot{\xi}_2$ — by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (4g) on (8)

(13) $e \models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (8b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i)**.**

(14) e `notintro` — by assumption

(15) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ — by assumption

(16) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2 \; \texttt{refutable}_?}$ — by Lemma 1.0.14 on (14) and (12)

(15) and (16) are in contradiction with each other.

Case (6g)**.**

(14) $e \models_? \dot{\xi}_1$ — by assumption

Contradicts (9).

Case (6h)**.**

(14) $e \not\models \dot{\xi}_1$ — by assumption

Contradicts (8).

(14) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \models_? \dot{\xi}_1, e \models \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ — by assumption

(9) $e \models_? \dot{\xi}_1$ — by assumption

(10) $e \models \dot{\xi}_2$ — by assumption

(11) $e \not\models_? \dot{\xi}_2$ — by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (4h) on (10)

(13) $e \models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (8b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i)**.**

(14) e `notintro` — by assumption

(15) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ — by assumption

(16) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}$ — by Lemma 1.0.14 on (14) and (12)

(15) and (16) are in contradiction with each other.

Case (6g)**.**

(14) $e \not\models \dot{\xi}_2$ — by assumption

Contradicts (10).

Case (6h)**.**

(14) $e \models_? \dot{\xi}_2$ — by assumption

Contradicts (11).

(14) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \models_? \dot{\xi}_1, e \models_? \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ — by assumption

(9) $e \models_? \dot{\xi}_1$ — by assumption

(10) $e \not\models \dot{\xi}_2$ — by assumption

(11) $e \models_? \dot{\xi}_2$ — by assumption

(12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (6g) on (9) and (10)

(13) $e \models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (8a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (4), only two cases apply.

Case (4g)**.**

(14) $e \models \dot{\xi}_1$ — by assumption

Contradicts (8)

Case (4h)**.**

(14) $e \models \dot{\xi}_2$ — by assumption

Contradicts (10)

(14) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \models_? \dot{\xi}_1, e \not\models_?^{\dagger} \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ — by assumption

(9) $e \models_? \dot{\xi}_1$ — by assumption

(10) $e \not\models \dot{\xi}_2$ — by assumption

(11) $e \not\models_? \dot{\xi}_2$ — by assumption

(12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (6g) on (9) and (10)

(13) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (8a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (4), only two cases apply.

Case (4g)**.**

(14) $e \models \dot{\xi}_1$ — by assumption

Contradicts (8).

Case (4h)**.**

(14) $e \models \dot{\xi}_2$ — by assumption

Contradicts (10).

(14) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \not\models^{\dagger}_{?} \dot{\xi}_1, e \models \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ — by assumption

(9) $e \not\models_{?} \dot{\xi}_1$ — by assumption

(10) $e \models \dot{\xi}_2$ — by assumption

(11) $e \not\models_{?} \dot{\xi}_2$ — by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (4h) on (10)

(13) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (8b) on (12)

Assume $e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i)**.**

(14) e `notintro` — by assumption

(15) $\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?$ — by assumption

(16) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}$ — by Lemma 1.0.14 on (14) and (12)

(15) and (16) are in contradiction with each other.

Case (6g)**.**

(14) $e \not\models \dot{\xi}_2$ — by assumption

Contradicts (10).

Case (6h)**.**

(14) $e \models_{?} \dot{\xi}_2$ — by assumption

Contradicts (11).

(14) $e \not\models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \not\models^{\dagger}_{?} \dot{\xi}_1, e \models_{?} \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \not\models_? \dot{\xi}_1$ by assumption

(10) $e \not\models \dot{\xi}_2$ by assumption

(11) $e \models_? \dot{\xi}_2$ by assumption

(12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (6h) on (11) and (8)

(13) $e \models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (8a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (4), only two cases apply.

Case (4g).

(14) $e \models \dot{\xi}_1$ by assumption

Contradicts (8)

Case (4h).

(14) $e \models \dot{\xi}_2$ by assumption

Contradicts (10)

(14) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models_?^{\dagger} \dot{\xi}_1, e \not\models_?^{\dagger} \dot{\xi}_2$.

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \not\models_? \dot{\xi}_1$ by assumption

(10) $e \not\models \dot{\xi}_2$ by assumption

(11) $e \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (4) on it, only two cases apply.

Case (4g).

(12) $e \models \dot{\xi}_1$ by assumption

Contradicts (8).

Case (4h).

(12) $e \models \dot{\xi}_2$ by assumption

Contradicts (10).

(13) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(14) e `notintro` by assumption

(15) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

By rule induction over Rules (2) on (15), only one rule applies.

Case (2g).

(16) $\dot{\xi}_1\ \texttt{refutable}_?$ — by assumption

(17) $e \models_? \dot{\xi}_1$ — by Rule (6i) on (14) and (16)

Contradicts (9).

Case (6g).

(14) $e \models_? \dot{\xi}_1$ — by assumption

Contradicts (9).

Case (6h).

(14) $e \models_? \dot{\xi}_2$ — by assumption

Contradicts (11).

(14) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

(15) $e \not\models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Lemma 1.0.17 on (13) and (14)

Case (1d).

(7) $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$ — by assumption

(8) $\tau = (\tau_1 + \tau_2)$ — by assumption

(9) $\dot{\xi}_1 : \tau_1$ — by assumption

By rule induction over Rules (11) on (5), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(10) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ — by assumption

(11) $e\ \texttt{notintro}$ — by Rule (18a),(18b),(18c),(18d),(18e),(18f)

Assume $e \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (4) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inl}(\dot{\xi}_1)$ — by contradiction

(13) $\texttt{inl}(\dot{\xi}_1)\ \texttt{refutable}_?$ — by Rule (2c)

(14) $e \models_? \texttt{inl}(\dot{\xi}_1)$ — by Rule (6i) on (11) and (13)

(15) $e \models_?^{\dagger} \texttt{inl}(\dot{\xi}_1)$ — by Rule (8a) on (14)

Case (11j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ — by assumption

(11) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(12) e_1 $\texttt{final}$ by Lemma 3.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \dot{\xi}_1$, $e_1 \models_? \dot{\xi}_1$, and $e_1 \not\models_?^\dagger \dot{\xi}_1$ holds. By case analysis on which one holds.

Case $e_1 \models \dot{\xi}_1$**.**

(13) $e_1 \models \dot{\xi}_1$ by assumption

(14) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$ by Rule (4c) on (13)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Rule (8b) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (6) on it, only two cases apply.

Case (6i)**.**

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (18) on (17), no case applies due to syntactic contradiction.

Case (6b)**.**

(17) $e_1 \models_? \dot{\xi}_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\dot{\xi}_1)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1$**.**

(13) $e_1 \not\models \dot{\xi}_1$ by assumption

(14) $e_1 \models_? \dot{\xi}_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$ by Rule (6b) on (14)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Rule (8a) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (4) on it, only one case applies.

Case (4c)**.**

(17) $e_1 \models \dot{\xi}_1$

Contradicts (13).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

Case $e_1 \not\models_?^\dagger \dot{\xi}_1$**.**

(13) $e_1 \not\models \dot{\xi}_1$ by assumption

(14) $e_1 \not\models_? \dot{\xi}_1$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (4) on it, only one case applies.

Case (4c).

(15) $e_1 \models \dot{\xi}_1$

Contradicts (13).

(16) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(17) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (18) on (17), no case applies due to syntactic contradiction.

Case (6b).

(17) $e_1 \models_? \dot{\xi}_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\dot{\xi}_1)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Lemma 1.0.17 on (16) and (18)

Case (11k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (4) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(12) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (18) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inl}(\dot{\xi}_1)$ by contradiction

(14) $\texttt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Lemma 1.0.17 on (11) and (13)

Case (1e).

(7) $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (11) on (5), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) e $\texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

Assume $e \models \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (4) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inr}(\dot{\xi}_2)$ by contradiction

(13) $\texttt{inr}(\dot{\xi}_2)$ $\texttt{refutable}_?$ by Rule (2d)

(14) $e \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (11) and (13)

(15) $e \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (14)

Case (11j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (4) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inr}(\dot{\xi}_2)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(12) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (18) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inr}(\dot{\xi}_2)$ by contradiction

(14) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Lemma 1.0.17 on (11) and (13)

Case (11k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(12) e_2 $\texttt{final}$ by Lemma 3.0.4 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \dot{\xi}_2$, $e_2 \models_? \dot{\xi}_2$, and $e_2 \not\models_?^\dagger \dot{\xi}_2$ holds. By case analysis on which one holds.

Case $e_2 \models \dot{\xi}_2$.

(13) $e_2 \models \dot{\xi}_2$ by assumption

(14) $e_2 \not\models_? \dot{\xi}_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by Rule (4c) on (13)

(16) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$ by Rule (8b) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (6) on it, only two cases apply.

Case (6i).

(17) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (18) on (17), no case applies due to syntactic contradiction.

Case (6c).

(17) $e_2 \models_? \dot{\xi}_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inr}(\dot{\xi}_2)$ by contradiction

Case $e_2 \models_? \dot{\xi}_2$.

(13) $e_2 \not\models \dot{\xi}_2$ by assumption

(14) $e_2 \models_? \dot{\xi}_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (6c) on (14)

(16) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (4) on it, only one case applies.

Case (4d).

(17) $e_2 \models \dot{\xi}_2$

Contradicts (13).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\dot{\xi}_2)$ by contradiction

Case $e_2 \not\models^{\dagger}_? \dot{\xi}_2$.

(13) $e_2 \not\models \dot{\xi}_2$ by assumption

(14) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (4) on it, only one case applies.

Case (4d).

(15) $e_2 \models \dot{\xi}_2$

Contradicts (13).

(16) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\dot{\xi}_2)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (18) on (17), no case applies due to syntactic contradiction.

Case (6c).

(17) $e_2 \models_{?} \dot{\xi}_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_{?} \texttt{inr}(\dot{\xi}_2)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_{?}^{\dagger} \texttt{inl}(\dot{\xi}_1)$ by Lemma 1.0.17 on (16) and (18)

Case (4e).

(7) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\dot{\xi}_1 : \tau_1$ by assumption

(10) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (11) on (5), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) $e\ \texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

(13) $e\ \texttt{indet}$ by Lemma 3.0.10 on (6) and (12)

(14) $\texttt{prl}(e)\ \texttt{indet}$ by Rule (16g) on (13)

(15) $\texttt{prl}(e)\ \texttt{final}$ by Rule (17b) on (14)

(16) $\texttt{prr}(e)\ \texttt{indet}$ by Rule (16h) on (13)

(17) $\texttt{prr}(e)\ \texttt{final}$ by Rule (17b) on (16)

(18) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (18e)

(19) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (18f)

(20) $\cdot\,; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (11h) on (5)

(21) $\cdot\,; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (11i) on (5)

By inductive hypothesis on (9) and (20) and (15), exactly one of $\texttt{prl}(e) \models \dot{\xi}_1$, $\texttt{prl}(e) \models_{?} \dot{\xi}_1$, and $\texttt{prl}(e) \not\models_{?}^{\dagger} \dot{\xi}_1$ holds.
By inductive hypothesis on (10) and (21) and (17), exactly one of $\texttt{prr}(e) \models \dot{\xi}_2$, $\texttt{prr}(e) \models_{?} \dot{\xi}_2$, and $\texttt{prr}(e) \not\models_{?}^{\dagger} \dot{\xi}_2$ holds.
By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $\texttt{prl}(e) \models \dot{\xi}_1, \texttt{prr}(e) \models \dot{\xi}_2$.

(22) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(23) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(24) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(25) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

(26) $e \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (4f) on (12) and (22) and (24)

(27) $e \models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8b) on (26)

(28) $\cancel{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}$ by Lemma 1.0.14 on (12) and (26)

Assume $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(29) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by assumption

Contradicts (28).

(30) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $\texttt{prl}(e) \models \dot{\xi}_1, \texttt{prr}(e) \models_? \dot{\xi}_2$.

(22) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(23) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(24) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(25) $\texttt{prr}(e) \models_? \dot{\xi}_2$ by assumption

(26) $\dot{\xi}_2\ \texttt{refutable}_?$ by Lemma 1.0.13 on (19) and (25)

(27) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2f) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6i) on (12) and (27)

(29) $e \models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (28)

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4), only one case applies.

Case (4f).

(30) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

Contradicts (24)

(31) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $\texttt{prl}(e) \models \dot{\xi}_1, \texttt{prr}(e) \not\models^\dagger_? \dot{\xi}_2$.

(22) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(23) $\mathtt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(24) $\mathtt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(25) $\mathtt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only one case applies.

Case (4f).

(26) $\mathtt{prr}(e) \models \dot{\xi}_2$ by assumption

Contradicts (24).

(27) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(28) $(\dot{\xi}_1, \dot{\xi}_2)\ \mathtt{refutable}_?$ by assumption

By rule induction over Rules (2) on (28), only two cases apply.

Case (2e).

(29) $\dot{\xi}_1\ \mathtt{refutable}_?$ by assumption

(30) $\mathtt{prl}(e)\ \mathtt{notintro}$ by Rule (18e)

(31) $\mathtt{prl}(e) \models_? \dot{\xi}_1$ by Rule (6i) on (30) and (29)

Contradicts (23).

Case (2f).

(29) $\dot{\xi}_2\ \mathtt{refutable}_?$ by assumption

(30) $\mathtt{prr}(e)\ \mathtt{notintro}$ by Rule (18f)

(31) $\mathtt{prr}(e) \models_? \dot{\xi}_2$ by Rule (6i) on (30) and (29)

Contradicts (25).

(32) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(33) $e \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (27) and (32)

Case $\mathtt{prl}(e) \models_? \dot{\xi}_1, \mathtt{prr}(e) \models \dot{\xi}_2$.

(22) $\mathtt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(23) $\mathtt{prl}(e) \models_? \dot{\xi}_1$ by assumption

(24) $\mathtt{prr}(e) \models \dot{\xi}_2$ by assumption

(25) $\mathtt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

(26) $\dot{\xi}_1\ \mathtt{refutable}_?$ by Lemma 1.0.13 on (18) and (23)

(27) $(\dot{\xi}_1, \dot{\xi}_2)\ \mathtt{refutable}_?$ by Rule (2f) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (6i) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (8a) on (28)

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4), only one case applies.

Case (4f)**.**

(30) $\texttt{prl}(e) \models \dot{\xi}_1$ — by assumption

Contradicts (22).

(31) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ — by contradiction

Case $\texttt{prl}(e) \models_? \dot{\xi}_1, \texttt{prr}(e) \models_? \dot{\xi}_2$**.**

(22) $\texttt{prl}(e) \not\models \dot{\xi}_1$ — by assumption

(23) $\texttt{prl}(e) \models_? \dot{\xi}_1$ — by assumption

(24) $\texttt{prr}(e) \not\models \dot{\xi}_2$ — by assumption

(25) $\texttt{prr}(e) \models_? \dot{\xi}_2$ — by assumption

(26) $\dot{\xi}_2$ $\texttt{refutable}_?$ — by Lemma 1.0.13 on (18) and (23)

(27) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ — by Rule (2f) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (6i) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (8a) on (28)

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4), only one case applies.

Case (4f)**.**

(30) $\texttt{prl}(e) \models \dot{\xi}_1$ — by assumption

Contradicts (22).

(31) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ — by contradiction

Case $\texttt{prl}(e) \models_? \dot{\xi}_1, \texttt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$**.**

(22) $\texttt{prl}(e) \not\models \dot{\xi}_1$ — by assumption

(23) $\texttt{prl}(e) \models_? \dot{\xi}_1$ — by assumption

(24) $\texttt{prr}(e) \not\models \dot{\xi}_2$ — by assumption

(25) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ — by assumption

(26) $\dot{\xi}_1$ $\texttt{refutable}_?$ — by Lemma 1.0.13 on (18) and (23)

(27) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ — by Rule (2f) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (6i) on (12) and (27)

(29) $e \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (28)

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4), only one case applies.

Case (4f).

(30) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (22)

(31) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $\texttt{prl}(e) \not\models^{\dagger}_{?} \dot{\xi}_1, \texttt{prr}(e) \models \dot{\xi}_2$.

(22) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(23) $\texttt{prl}(e) \not\models_{?} \dot{\xi}_1$ by assumption

(24) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(25) $\texttt{prr}(e) \not\models_{?} \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only one case applies.

Case (4f).

(26) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (22)

(27) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $e \models_{?} (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i).

(28) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_{?}$ by assumption

By rule induction over Rules (2) on (28), only two cases apply.

Case (2e).

(29) $\dot{\xi}_1\ \texttt{refutable}_{?}$ by assumption

(30) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (18e)

(31) $\texttt{prl}(e) \models_{?} \dot{\xi}_1$ by Rule (6i) on (30) and (29)

Contradicts (23).

Case (2f).

(29) $\dot{\xi}_2\ \texttt{refutable}_{?}$ by assumption

(30) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (18f)

(31) $\texttt{prr}(e) \models_{?} \dot{\xi}_2$ by Rule (6i) on (30) and (29)

Contradicts (25).

(32) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(33) $e \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (27) and (32)

Case $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1, \texttt{prr}(e) \models_? \dot{\xi}_2$**.**

(22) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(23) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(24) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(25) $\texttt{prr}(e) \models_? \dot{\xi}_2$ by assumption

(26) $\dot{\xi}_2\ \texttt{refutable}_?$ by Lemma 1.0.13 on (19) and (25)

(27) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2f) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6i) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (28)

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4), only one case applies.

Case (4f)**.**

(30) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (22).

(31) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1, \texttt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$**.**

(22) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(23) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(24) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(25) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only one case applies.

Case (4f)**.**

(26) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (22)

(27) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, only one case applies.

Case (6i)**.**

(28) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by assumption

By rule induction over Rules (2) on (28), only two cases apply.

Case (2e).

(29) $\dot{\xi}_1$ `refutable`$_?$	by assumption
(30) `prl`(e) `notintro`	by Rule (18e)
(31) `prl`$(e) \models_? \dot{\xi}_1$	by Rule (6i) on (30) and (29)

Contradicts (23).

Case (2f).

(29) $\dot{\xi}_2$ `refutable`$_?$	by assumption
(30) `prr`(e) `notintro`	by Rule (18f)
(31) `prr`$(e) \models_? \dot{\xi}_2$	by Rule (6i) on (30) and (29)

Contradicts (25).

(32) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$	by contradiction
(33) $e \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$	by Lemma 1.0.17 on (27) and (32)

Case (11g).

(11) $e = (e_1, e_2)$	by assumption
(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$	by assumption
(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$	by assumption
(14) e_1 `final`	by Lemma 3.0.5 on (6)
(15) e_2 `final`	by Lemma 3.0.5 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \dot{\xi}_1$, $e_1 \models_? \dot{\xi}_1$, and $e_1 \models \overline{\dot{\xi}_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \dot{\xi}_2$, $e_2 \models_? \dot{\xi}_2$, and $e_2 \models \overline{\dot{\xi}_2}$ holds.
By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $e_1 \models \dot{\xi}_1, e_2 \models \dot{\xi}_2$.

(16) $e_1 \models \dot{\xi}_1$	by assumption
(17) $e_1 \not\models_? \dot{\xi}_1$	by assumption
(18) $e_2 \models \dot{\xi}_2$	by assumption
(19) $e_2 \not\models_? \dot{\xi}_2$	by assumption
(20) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (4e) on (16) and (18)
(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (8b) on (20)

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(22) (e_1, e_2) `notintro`	by assumption

Contradicts Lemma 3.0.9.

Case (6d).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (6e).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (6f).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$.

(16) $e_1 \models \dot{\xi}_1$ by assumption
(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption
(18) $e_2 \not\models \dot{\xi}_2$ by assumption
(19) $e_2 \models_? \dot{\xi}_2$ by assumption
(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6e) on (16) and (19)
(21) $(e_1, e_2) \models_?^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e).

(22) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models \dot{\xi}_1, e_2 \not\models_?^{\dagger} \dot{\xi}_2$.

(16) $e_1 \models \dot{\xi}_1$ by assumption
(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption
(18) $e_2 \not\models \dot{\xi}_2$ by assumption
(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e).

(20) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (6d).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (6e).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (6f).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (21) and (23)

Case $e_1 \models_? \dot{\xi}_1, e_2 \models \dot{\xi}_2$.

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \models_? \dot{\xi}_1$ by assumption

(18) $e_2 \models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6d) on (17) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e)**.**

(22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \models_? \dot{\xi}_2$ by assumption

(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6f) on (17) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e)**.**

(22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e)**.**

(20) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (6d)**.**

(22) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

Case (6e)**.**

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (6f)**.**

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (21) and (23)

Case $e_1 \not\models_?^\dagger \dot{\xi}_1, e_2 \models \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e)**.**

(20) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (6d).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (6e).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (6f).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (21) and (23)

Case $e_1 \not\models_?^\dagger \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e).

(20) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (6d).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (6e).

(22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

Case (6f).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (21) and (23)

Case $e_1 \not\models_?^\dagger \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$.

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (4) on it, only two cases apply.

Case (4f).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (4e).

(20) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (6) on it, the following cases apply.

Case (6i).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (6d).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (6e).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (6f).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.17 on (21) and (23)

□

Definition 1.1.1 (Entailment of Constraints). *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \models \dot{\xi}_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e $\texttt{val}$ we have $e \models^{\dagger}_{?} \dot{\xi}_1$ implies $e \models \dot{\xi}_2$*

Definition 1.1.2 (Potential Entailment of Constraints). *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \models^{\dagger}_{?} \dot{\xi}_2$ iff for all e such that $\cdot\,; \Delta \vdash e : \tau$ and e $\texttt{final}$ we have $e \models^{\dagger}_{?} \dot{\xi}_1$ implies $e \models^{\dagger}_{?} \dot{\xi}_2$*

Corollary 1.1.1. *Suppose that $\dot{\xi} : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e $\texttt{final}$. Then $\top \models^{\dagger}_{?} \dot{\xi}$ implies $e \models^{\dagger}_{?} \dot{\xi}$*

Proof.

(1) $\dot{\xi} : \tau$ by assumption

(2) $\cdot\,; \Gamma \vdash e : \tau$ by assumption

(3) e $\texttt{final}$ by assumption

(4) $\top \models^{\dagger}_{?} \dot{\xi}$ by assumption

(5) $e_1 \models \top$ by Rule (4a)

(6) $e_1 \models^{\dagger}_{?} \top$ by Rule (8b) on (5)

(7) $\top : \tau$ by Rule (1a)

(8) $e_1 \models^{\dagger}_{?} \dot{\xi}_r$ by Definition 1.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

□

2 Static Semantics

$$
\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & (\lambda x : \tau . e) \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
 & \mid & (\!|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid (\!|\!)^w \mid (\!|p|\!)^w
\end{array}
$$

$\boxed{(\hat{rs})^{\diamond} = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^{\diamond} = r \mid rs \tag{10a}$$

$$((r' \mid rs') \mid r \mid rs)^{\diamond} = r' \mid (rs' \mid r \mid rs)^{\diamond} \tag{10b}$$

$\boxed{\Gamma \, ; \Delta \vdash e : \tau}$ e is of type τ

TVar
$$\frac{}{\Gamma, x : \tau \, ; \Delta \vdash x : \tau} \tag{11a}$$

TEHole
$$\frac{}{\Gamma \, ; \Delta, u :: \tau \vdash (\!|\!)^u : \tau} \tag{11b}$$

THole
$$\frac{\Gamma \, ; \Delta, u :: \tau \vdash e : \tau'}{\Gamma \, ; \Delta, u :: \tau \vdash (\!|e|\!)^u : \tau} \tag{11c}$$

TNum
$$\frac{}{\Gamma \, ; \Delta \vdash \underline{n} : \texttt{num}} \tag{11d}$$

TLam
$$\frac{\Gamma, x : \tau_1 \, ; \Delta \vdash e : \tau_2}{\Gamma \, ; \Delta \vdash (\lambda x : \tau_1 . e) : (\tau_1 \to \tau_2)} \tag{11e}$$

TAp
$$\frac{\Gamma \, ; \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma \, ; \Delta \vdash e_2 : \tau_2}{\Gamma \, ; \Delta \vdash e_1(e_2) : \tau} \tag{11f}$$

TPair
$$\frac{\Gamma \, ; \Delta \vdash e_1 : \tau_1 \qquad \Gamma \, ; \Delta \vdash e_2 : \tau_2}{\Gamma \, ; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \tag{11g}$$

TPrl
$$\frac{\Gamma \, ; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma \, ; \Delta \vdash \texttt{prl}(e) : \tau_1} \tag{11h}$$

TPrr
$$\frac{\Gamma \, ; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma \, ; \Delta \vdash \texttt{prr}(e) : \tau_2} \tag{11i}$$

TInl
$$\frac{\Gamma \, ; \Delta \vdash e : \tau_1}{\Gamma \, ; \Delta \vdash \texttt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \tag{11j}$$

TInr
$$\frac{\Gamma \, ; \Delta \vdash e : \tau_2}{\Gamma \, ; \Delta \vdash \texttt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \tag{11k}$$

TMatchZPre
$$\frac{\Gamma \, ; \Delta \vdash e : \tau \qquad \Gamma \, ; \Delta \vdash [\bot] r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \models^{\dagger}_{?} \xi}{\Gamma \, ; \Delta \vdash \texttt{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \tag{11l}$$

TMatchNZPre

$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad e\ \texttt{final} \qquad \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \quad \Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \qquad e \not\models^{\dagger}_{?} \xi_{pre} \qquad \top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \tag{11m}$$

$\boxed{p : \tau[\xi] \dashv\!\vdash \Gamma\,;\Delta}$ p is assigned type τ and emits constraint ξ

PTVar

$$\frac{}{x : \tau[\top] \dashv\!\vdash \cdot\,; x : \tau} \tag{12a}$$

PTWild

$$\frac{}{\_ : \tau[\top] \dashv\!\vdash \cdot\,;\cdot} \tag{12b}$$

PTEHole

$$\frac{}{(\!|\,|\!)^{w} : \tau[?] \dashv\!\vdash \cdot\,; w :: \tau} \tag{12c}$$

PTHole

$$\frac{p : \tau[\xi] \dashv\!\vdash \Gamma\,;\Delta}{(\!|p|\!)^{w} : \tau'[?] \dashv\!\vdash \Gamma\,;\Delta, w :: \tau'} \tag{12d}$$

PTNum

$$\frac{}{\underline{n} : \texttt{num}[\underline{n}] \dashv\!\vdash \cdot\,;\cdot} \tag{12e}$$

PTInl

$$\frac{p : \tau_1[\xi] \dashv\!\vdash \Gamma\,;\Delta}{\texttt{inl}(p) : (\tau_1 + \tau_2)[\texttt{inl}(\xi)] \dashv\!\vdash \Gamma\,;\Delta} \tag{12f}$$

PTInr

$$\frac{p : \tau_2[\xi] \dashv\!\vdash \Gamma\,;\Delta}{\texttt{inr}(p) : (\tau_1 + \tau_2)[\texttt{inr}(\xi)] \dashv\!\vdash \Gamma\,;\Delta} \tag{12g}$$

PTPair

$$\frac{p_1 : \tau_1[\xi_1] \dashv\!\vdash \Gamma_1\,;\Delta_1 \qquad p_2 : \tau_2[\xi_2] \dashv\!\vdash \Gamma_2\,;\Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\!\vdash \Gamma_1 \uplus \Gamma_2\,;\Delta_1 \uplus \Delta_2} \tag{12h}$$

$\boxed{\Gamma\,;\Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

CTRule

$$\frac{p : \tau[\xi] \dashv\!\vdash \Gamma_p\,;\Delta_p \qquad \Gamma \uplus \Gamma_p\,;\Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma\,;\Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \tag{13a}$$

$\boxed{\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

CTOneRules

$$\frac{\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,;\Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \tag{14a}$$

CTRules

$$\frac{\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\,;\Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \tag{14b}$$

Lemma 2.0.1. *If* $p : \tau[\xi] \dashv \Gamma\,;\Delta$ *then* $\xi : \tau$.

Proof. By rule induction over Rules (12). ☐

Lemma 2.0.2. *If* $\cdot\,;\Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Proof. By rule induction over Rules (13). ☐

Lemma 2.0.3. *If* $\cdot\,;\Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Proof. By rule induction over Rules (14). ☐

Lemma 2.0.4. *If* $\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma\,;\Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Proof.

(1) $\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ by assumption

(2) $\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ by assumption

(3) $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ by assumption

By rule induction over Rules (14) on (1).

Case (14a).

(4) $rs = r' \mid \cdot$ by assumption

(5) $\xi_{rs} = \xi'_r$ by assumption

(6) $\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$ by assumption

(7) $\xi'_r \not\models \xi_{pre}$ by assumption

(8) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$ by Rule (14a) on (2) and (3)

(9) $\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$ by Rule (14b) on (6) and (8) and (7)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$ by Definition 10 on (9)

Case (14b).

(4) $rs = r' \mid rs'$ by assumption

(5) $\xi_{rs} = \xi'_r \vee \xi'_{rs}$ by assumption

(6) $\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$ by assumption

(7) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] \Rightarrow \tau'$ by assumption

(8) $\xi'_r \not\models \xi_{pre}$ by assumption

(9) $\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^{\diamond} : \tau[\xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by IH on (7) and (2) and (3)

(10) $\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^{\diamond}) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by Rule (14b) on (6) and (9) and (8)

(11) $\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$
by Definition 10 on (10)

□

Lemma 2.0.5 (Substitution). *If* $\Gamma, x : \tau\,;\Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta \vdash e : \tau$ *then* $\Gamma\,;\Delta \vdash [e/x]e_0 : \tau_0$

Lemma 2.0.6 (Simultaneous Substitution). *If* $\Gamma \uplus \Gamma'\,;\Delta \vdash e : \tau$ *and* $\theta : \Gamma'$ *then* $\Gamma\,;\Delta \vdash [\theta]e : \tau$

Lemma 2.0.7 (Substitution Typing). *If* $e \rhd p \dashv\!\!\vdash \theta$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\!\vdash \Gamma\,;\Delta$ *then* $\theta : \Gamma$

Proof by induction on the derivation of $e \rhd p \dashv\!\!\vdash \theta$.

Theorem 2.1 (Determinism). *If* $\cdot\,;\Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

3 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\text{VNum}\quad \frac{}{\underline{n}\ \texttt{val}} \tag{15a}$$

$$\text{VLam}\quad \frac{}{(\lambda x : \tau.e)\ \texttt{val}} \tag{15b}$$

$$\text{VPair}\quad \frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}} \tag{15c}$$

$$\text{VInl}\quad \frac{e\ \texttt{val}}{\texttt{inl}_{\tau}(e)\ \texttt{val}} \tag{15d}$$

$$\frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}}\ \text{VInr} \tag{15e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{}{(\!|\!)^u\ \texttt{indet}}\ \text{IEHole} \tag{16a}$$

$$\frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}}\ \text{IHole} \tag{16b}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}}\ \text{IAp} \tag{16c}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairL} \tag{16d}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairR} \tag{16e}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPair} \tag{16f}$$

$$\frac{e\ \texttt{indet}}{\texttt{prl}(e)\ \texttt{indet}}\ \text{IPrl} \tag{16g}$$

$$\frac{e\ \texttt{indet}}{\texttt{prr}(e)\ \texttt{indet}}\ \text{IPrr} \tag{16h}$$

$$\frac{e\ \texttt{indet}}{\texttt{inl}_\tau(e)\ \texttt{indet}}\ \text{IInL} \tag{16i}$$

$$\frac{e\ \texttt{indet}}{\texttt{inr}_\tau(e)\ \texttt{indet}}\ \text{IInR} \tag{16j}$$

$$\frac{e\ \texttt{final} \qquad e\ ?\ p_r}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \texttt{indet}}\ \text{IMatch} \tag{16k}$$

$\boxed{e\ \texttt{final}}$ e is final

$$\frac{e\ \texttt{val}}{e\ \texttt{final}}\ \text{FVal} \tag{17a}$$

$$\frac{e\ \texttt{indet}}{e\ \texttt{final}}\ \text{FIndet} \tag{17b}$$

$\boxed{e\ \texttt{notintro}}$ e cannot be a value syntactically

$$\frac{}{(\!|\,|\!)^u\ \texttt{notintro}}\ \text{NVEHole} \tag{18a}$$

$$\frac{}{(\!|e|\!)^u\ \texttt{notintro}}\ \text{NVHole} \tag{18b}$$

$$\frac{}{e_1(e_2)\ \texttt{notintro}}\ \text{NVAp} \tag{18c}$$

$$\frac{}{\texttt{match}(e)\{\hat{rs}\}\ \texttt{notintro}}\ \text{NVMatch} \tag{18d}$$

$$\frac{}{\texttt{prl}(e)\ \texttt{notintro}}\ \text{NVPrl} \tag{18e}$$

$$\frac{}{\texttt{prr}(e)\ \texttt{notintro}}\ \text{NVPrr} \tag{18f}$$

$\boxed{\mathit{notintro}(e)}$

$$\mathit{notintro}((\!|\,|\!)^u) = \text{true} \tag{19a}$$

$$\mathit{notintro}((\!|e|\!)^u) = \text{true} \tag{19b}$$

$$\mathit{notintro}(e_1(e_2)) = \text{true} \tag{19c}$$

$$\mathit{notintro}(\texttt{match}(e)\{\hat{rs}\}) = \text{true} \tag{19d}$$

$$\mathit{notintro}(\texttt{prl}(e)) = \text{true} \tag{19e}$$

$$\mathit{notintro}(\texttt{prr}(e)) = \text{true} \tag{19f}$$

$$\text{Otherwise} \quad \mathit{notintro}(e) = \text{false} \tag{19g}$$

Lemma 3.0.1 (Soundness and Completeness of NotIntro Judgment)**.** *e `notintro` iff $\mathit{notintro}(e)$.*

Proof. TODO □

$\boxed{e' \in \texttt{values}(e)}$ e' is one of the possible values of e

$$\frac{e\ \texttt{val} \qquad \cdot\,;\Delta \vdash e : \tau}{e \in \texttt{values}(e)}\ \text{IVVal} \tag{20a}$$

$$\frac{e\ \texttt{notintro} \qquad \cdot\,;\Delta \vdash e : \tau \qquad e'\ \texttt{val} \qquad \cdot\,;\Delta \vdash e' : \tau}{e' \in \texttt{values}(e)}\ \text{IVIndet} \tag{20b}$$

$$\frac{\texttt{inl}_{\tau_2}(e_1)\ \texttt{indet} \qquad \cdot\,;\Delta \vdash \texttt{inl}_{\tau_2}(e_1) : \tau \qquad e_1' \in \texttt{values}(e_1)}{\texttt{inl}_{\tau_2}(e_1') \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))}\ \text{IVInl} \tag{20c}$$

$$\frac{\texttt{inr}_{\tau_1}(e_2)\ \texttt{indet} \qquad \cdot\,;\Delta \vdash \texttt{inr}_{\tau_1}(e_2) : \tau \qquad e_2' \in \texttt{values}(e_2)}{\texttt{inr}_{\tau_1}(e_2') \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))}\ \text{IVInr} \tag{20d}$$

$$\frac{(e_1, e_2)\ \texttt{indet} \qquad \cdot\,;\Delta \vdash (e_1, e_2) : \tau \qquad e_1' \in \texttt{values}(e_1) \qquad e_2' \in \texttt{values}(e_2)}{(e_1', e_2') \in \texttt{values}((e_1, e_2))}\ \text{IVPair} \tag{20e}$$

Lemma 3.0.2. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$ and $e \not\models_?^\dagger \dot{\xi}$ then $e' \not\models_?^\dagger \dot{\xi}$ whenever $e' \in \texttt{values}(e)$.*

Proof.

(1) e $\texttt{indet}$ — by assumption

(2) $\cdot\,;\Delta \vdash e : \tau$ — by assumption

(3) $\dot{\xi} : \tau$ — by assumption

(4) $e \not\models_?^\dagger \dot{\xi}$ — by assumption

By rule induction over Rules (1) on (3).

Case (1a)**.**

(5) $\dot{\xi} = \top$ — by assumption

(6) $e \models \top$ — by Rule (4a)

(7) $e \models_?^\dagger \top$ — by Rule (8b) on (6)

Contradicts (4).

Case (1b)**.**

(5) $\dot{\xi} = ?$ — by assumption

(6) $e \models_? ?$ — by Rule (6a)

(7) $e \models_?^\dagger ?$ — by Rule (8a) on (6)

Contradicts (4).

Case (1c).

(5) $\dot{\xi} = \underline{n}$ by assumption

(6) $\tau = \mathtt{num}$ by assumption

(7) $\underline{n}\ \mathtt{refutable}_?$ by Rule (2a)

By rule induction over Rules (16) on (1).

Case (16a).

(8) $e = (\!|\!)^u$ by assumption

(9) $(\!|\!)^u\ \mathtt{notintro}$ by Rule (18a)

(10) $(\!|\!)^u \models_? \underline{n}$ by Rule (6i) on (9) and (7)

(11) $(\!|\!)^u \models_?^\dagger \underline{n}$ by Rule (8a) on (10)

Contradicts (4).

Case (16b).

(8) $e = (\!|e_1|\!)^u$ by assumption

(9) $(\!|e_1|\!)^u\ \mathtt{notintro}$ by Rule (18b)

(10) $(\!|e_1|\!)^u \models_? \underline{n}$ by Rule (6i) on (9) and (7)

(11) $(\!|e_1|\!)^u \models_?^\dagger \underline{n}$ by Rule (8a) on (10)

Contradicts (4).

Case (16c).

(8) $e = e_1(e_2)$ by assumption

(9) $e_1(e_2)\ \mathtt{notintro}$ by Rule (18c)

(10) $e_1(e_2) \models_? \underline{n}$ by Rule (6i) on (9) and (7)

(11) $e_1(e_2) \models_?^\dagger \underline{n}$ by Rule (8a) on (10)

Contradicts (4).

Case (16g).

(8) $e = \mathtt{prl}(e_1)$ by assumption

(9) $\mathtt{prl}(e_1)\ \mathtt{notintro}$ by Rule (18e)

(10) $\mathtt{prl}(e_1) \models_? \underline{n}$ by Rule (6i) on (9) and (7)

(11) $\mathtt{prl}(e_1) \models_?^\dagger \underline{n}$ by Rule (8a) on (10)

Contradicts (4).

Case (16h).

(8)	$e = \texttt{prr}(e_1)$	by assumption
(9)	$\texttt{prr}(e_1)$ `notintro`	by Rule (18f)
(10)	$\texttt{prr}(e_1) \models_? \underline{n}$	by Rule (6i) on (9) and (7)
(11)	$\texttt{prr}(e_1) \models_?^\dagger \underline{n}$	by Rule (8a) on (10)

Contradicts (4).

Case (16k).

(8)	$e = \texttt{match}(e_1)\{\hat{rs}\}$	by assumption
(9)	$\texttt{match}(e_1)\{\hat{rs}\}$ `notintro`	by Rule (18d)
(10)	$\texttt{match}(e_1)\{\hat{rs}\} \models_? \underline{n}$	by Rule (6i) on (9) and (7)
(11)	$\texttt{match}(e_1)\{\hat{rs}\} \models_?^\dagger \underline{n}$	by Rule (8a) on (10)

Contradicts (4).

Case (16d), (16e), (16f).

(8)	$e = (e_1, e_2)$	by assumption

By rule induction over Rules (11) on (2), no rule applies due to syntactic contradiction.

Case (16i).

(8)	$e = \texttt{inl}_{\tau_2}(e_1)$	by assumption

By rule induction over Rules (11) on (2), no rule applies due to syntactic contradiction.

Case (16j).

(8)	$e = \texttt{inr}_{\tau_1}(e_2)$	by assumption

By rule induction over Rules (11) on (2), no rule applies due to syntactic contradiction.

Case (1d).

(5)	$\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$	by assumption
(6)	$\tau = (\tau_1 + \tau_2)$	by assumption
(7)	$\dot{\xi}_1 : \tau_1$	by assumption
(8)	$\texttt{inl}(\dot{\xi}_1)\ \texttt{refutable}_?$	by Rule (2c)

By rule induction over Rules (16) on (1).

Case (16a).

(9)	$e = (\!\|\!)^u$	by assumption
(10)	$(\!\|\!)^u$ `notintro`	by Rule (18a)

(11) $(\!|\!)^u \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6i) on (10) and (8)

(12) $(\!|\!)^u \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (8a) on (11)

Contradicts (4).

Case (16b).

(9) $e = (\!|e_1|\!)^u$ by assumption

(10) $(\!|e_1|\!)^u\ \mathtt{notintro}$ by Rule (18b)

(11) $(\!|e_1|\!)^u \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6i) on (10) and (8)

(12) $(\!|e_1|\!)^u \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (8a) on (11)

Contradicts (4).

Case (16c).

(9) $e = e_1(e_2)$ by assumption

(10) $e_1(e_2)\ \mathtt{notintro}$ by Rule (18c)

(11) $e_1(e_2) \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6i) on (10) and (8)

(12) $e_1(e_2) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (8a) on (11)

Contradicts (4).

Case (16g).

(9) $e = \mathtt{prl}(e_1)$ by assumption

(10) $\mathtt{prl}(e_1)\ \mathtt{notintro}$ by Rule (18e)

(11) $\mathtt{prl}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6i) on (10) and (8)

(12) $\mathtt{prl}(e_1) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (8a) on (11)

Contradicts (4).

Case (16h).

(9) $e = \mathtt{prr}(e_1)$ by assumption

(10) $\mathtt{prr}(e_1)\ \mathtt{notintro}$ by Rule (18f)

(11) $\mathtt{prr}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6i) on (10) and (8)

(12) $\mathtt{prr}(e_1) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (8a) on (11)

Contradicts (4).

Case (16k).

(9) $e = \mathtt{match}(e_1)\{\hat{rs}\}$ by assumption

(10) $\mathtt{match}(e_1)\{\hat{rs}\}\ \mathtt{notintro}$ by Rule (18d)

(11) $\mathtt{match}(e_1)\{\hat{rs}\} \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (6i) on (10) and (8)

(12) $\texttt{match}(e_1)\{\hat{rs}\} \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Rule (8a) on (11)

Contradicts (4).

Case (16d), (16e), (16f).

(9) $e = (e_1, e_2)$ by assumption

By rule induction over Rules (16) on (1), no rule applies due to syntactic contradiction.

Case (16i).

(9) $e = \texttt{inl}_{\tau_2'}(e_1)$ by assumption

(10) e_1 $\texttt{indet}$ by assumption

By rule induction over Rules (11) on (2), only one rule applies.

Case (11j).

(11) $\tau_2' = \tau_2$ by assumption

(12) $\cdot ; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $e_1 \not\models_?^\dagger \dot{\xi}_1$ by Lemma 1.0.10 on (4)

(14) if $e_1' \in \texttt{values}(e_1)$ then $e_1' \not\models_?^\dagger \dot{\xi}_1$ by IH on (10) and (12) and (7) and (13)

To show if $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ then $e' \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$, we assume $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$.

(15) $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ by assumption

By rule induction over Rules (20) on (15).

Case (20a).

(16) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{val}$ by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(16) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 3.0.7

Case (20c).

(16) $e' = \texttt{inl}_{\tau_2}(e_1')$ by assumption

(17) $e_1' \in \texttt{values}(e_1)$ by assumption

(18) $e_1' \not\models_?^\dagger \dot{\xi}_1$ by (14) on (17)

(19) $\texttt{inl}_{\tau_2}(e_1') \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Lemma 1.0.10 on (18)

Case (16j).

(9) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

To show if $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$ then $e' \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$, we assume $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$.

(10) $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$ by assumption

By rule induction over Rules (20) on (10).

Case (20a).

(11) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{val}$ by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(11) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 3.0.8

Case (20d).

(11) $e' = \texttt{inr}_{\tau_1}(e_2')$ by assumption

(12) $\texttt{inr}_{\tau_1}(e_2') \not\models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Lemma 1.0.16

Case (1e).

(5) $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $\dot{\xi}_2 : \tau_2$ by assumption

(8) $\texttt{inr}(\dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (2d)

By rule induction over Rules (16) on (1).

Case (16a).

(9) $e = (\!|\!)^u$ by assumption

(10) $(\!|\!)^u\ \texttt{notintro}$ by Rule (18a)

(11) $(\!|\!)^u \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (10) and (8)

(12) $(\!|\!)^u \models^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (11)

Contradicts (4).

Case (16b).

(9) $e = (\!|e_1|\!)^u$ by assumption

(10) $(\!|e_1|\!)^u\ \texttt{notintro}$ by Rule (18b)

(11) $(\!|e_1|\!)^u \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (10) and (8)

(12) $(\!|e_1|\!)^u \models^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (11)

Contradicts (4).

Case (16c).

(9) $e = e_1(e_2)$ by assumption

(10) $e_1(e_2)\ \texttt{notintro}$ by Rule (18c)

(11) $e_1(e_2) \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (10) and (8)

(12) $e_1(e_2) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (11)

Contradicts (4).

Case (16g).

(9) $e = \texttt{prl}(e_1)$ by assumption

(10) $\texttt{prl}(e_1)$ $\texttt{notintro}$ by Rule (18e)

(11) $\texttt{prl}(e_1) \models_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (10) and (8)

(12) $\texttt{prl}(e_1) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (11)

Contradicts (4).

Case (16h).

(9) $e = \texttt{prr}(e_1)$ by assumption

(10) $\texttt{prr}(e_1)$ $\texttt{notintro}$ by Rule (18f)

(11) $\texttt{prr}(e_1) \models_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (10) and (8)

(12) $\texttt{prr}(e_1) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (11)

Contradicts (4).

Case (16k).

(9) $e = \texttt{match}(e_1)\{\hat{rs}\}$ by assumption

(10) $\texttt{match}(e_1)\{\hat{rs}\}$ $\texttt{notintro}$ by Rule (18d)

(11) $\texttt{match}(e_1)\{\hat{rs}\} \models_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (6i) on (10) and (8)

(12) $\texttt{match}(e_1)\{\hat{rs}\} \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (8a) on (11)

Contradicts (4).

Case (16d), (16e), (16f).

(9) $e = (e_1, e_2)$ by assumption

By rule induction over Rules (16) on (1), no rule applies due to syntactic contradiction.

Case (16i).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

To show if $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ then $e' \not\models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$, we assume $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$.

(10) $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ by assumption

By rule induction over Rules (20) on (10).

Case (20a).

(11) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{val}$ by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(11) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.7

Case (20c).

(11) $e' = \mathtt{inl}_{\tau_2}(e_1')$ by assumption

(12) $\mathtt{inl}_{\tau_2}(e_1') \not\models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ by Lemma 1.0.15

Case (16j).

(9) $e = \mathtt{inr}_{\tau_1'}(e_2)$ by assumption

(10) $e_2\ \mathtt{indet}$ by assumption

By rule induction over Rules (11) on (2), only one rule applies.

Case (11k).

(11) $\tau_1' = \tau_1$ by assumption

(12) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(13) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.10 on (4)

(14) if $e_2' \in \mathtt{values}(e_2)$ then $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by IH on (10) and (12) and (7) and (13)

To show if $e' \in \mathtt{values}(\mathtt{inr}_{\tau_1}(e_2))$ then $e' \not\models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$, we assume $e' \in \mathtt{values}(\mathtt{inr}_{\tau_1}(e_2))$.

(15) $e' \in \mathtt{values}(\mathtt{inr}_{\tau_1}(e_2))$ by assumption

By rule induction over Rules (20) on (15).

Case (20a).

(16) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{val}$ by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(16) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.8

Case (20d).

(16) $e' = \mathtt{inr}_{\tau_1}(e_2')$ by assumption

(17) $e_2' \in \mathtt{values}(e_2)$ by assumption

(18) $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by (14) on (17)

(19) $\mathtt{inr}_{\tau_1}(e_2') \not\models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ by Lemma 1.0.11 on (18)

Case (1f).

(5) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ by assumption

(7) $\dot{\xi}_1 : \tau_1$ by assumption

(8) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (16) on (1).

Case (16a), (16b), (16c), (16g), (16h), (16k).

(9) $e = (\!|\!)^u, (\!|e_1|\!)^u, e_1(e_2), \texttt{prl}(e_1), \texttt{prr}(e_1), \texttt{match}(e_1)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rules (18)

(11) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (18e)

(12) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (18f)

(13) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (16g) on (1)

(14) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (16h) on (1)

(15) $\cdot\, ; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (11h) on (2)

(16) $\cdot\, ; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (11i) on (2)

By case analysis on the result of *satisfyormay*$(\texttt{prl}(e), \dot{\xi}_1)$.

Case true.

(17) *satisfyormay*$(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by assumption

(18) $\texttt{prl}(e) \models^{\dagger}_{?} \dot{\xi}_1$ by Lemma 1.0.4 on (17)

By case analysis on the result of *satisfyormay*$(\texttt{prr}(e), \dot{\xi}_2)$.

Case true.

(19) *satisfyormay*$(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$ by assumption

(20) $\texttt{prr}(e) \models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.4 on (19)

By rule induction over Rules (8) on (18).

Case (8b).

(21) $\texttt{prl}(e) \models \dot{\xi}_1$ by asssumption

By rule induction over Rules (8) on (20).

Case (8b).

(22) $\texttt{prr}(e) \models \dot{\xi}_2$ by asssumption

(23) $e \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (4f) on (10) and (21) and (22)

(24) $e \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8b) on (23)

Contradicts (4).

Case (8a).

(22) $\texttt{prr}(e) \models_{?} \dot{\xi}_2$ by assumption

(23) $\dot{\xi}_2$ $\texttt{refutable}_?$ by **??** on (12) and (22)

(24) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ by Rule (2f) on (23)

(25) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6i) on (10) and (24)

(26) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (25)

Case (8a)**.**

(21) $\texttt{prl}(e) \models_? \dot{\xi}_1$ by assumption

(22) $\dot{\xi}_1$ $\texttt{refutable}_?$ by **??** on (11) and (21)

(23) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ by Rule (2e) on (22)

(24) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (6i) on (10) and (23)

(25) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (8a) on (24)

Case false**.**

(19) $\mathit{satisfyormay}(\texttt{prr}(e), \dot{\xi}_2) = \text{false}$ by assumption

(20) $\texttt{prr}(e) \models_?^\dagger \dot{\xi}_2$ by Lemma 1.0.4 on (19)

(21) if $e_2' \in \texttt{values}(\texttt{prr}(e))$ then $e_2' \not\models_?^\dagger \dot{\xi}_2$ by IH on (14) and (16) and (8) and (20)

To show if $e' \in \texttt{values}(e)$ then $e' \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}(e)$.

(22) $e' \in \texttt{values}(e)$ by assumption

By rule induction over Rules (20) on (22), only two rules apply.

Case (20a)**.**

(23) e $\texttt{val}$ by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b)**.**

(23) e' $\texttt{val}$ by assumption

(24) $\cdot\,; \Delta \vdash e' : (\tau_1 \times \tau_2)$ by assumption

By rule induction over Rules (15) on (23).

Case (15a)**.**

(25) $e' = \underline{n}$ by assumption

By rule induction over Rules (11) on (24), no rule applies due to syntactic contradiction.

Case (15b)**.**

(25) $e' = (\lambda x : \tau'.e_1')$ by assumption

By rule induction over Rules (11) on (24), no rule applies due to syntactic contradiction.

Case (15c)**.**

(25) $e' = (e_1', e_2')$ by assumption

(26) e_2' `val` by assumption

By rule induction over Rules (11) on (24), only one rule applies.

Case (11g).

(27) $\cdot\,; \Delta \vdash e_2' : \tau_2$ by assumption

(28) $e_2' \in \texttt{values}(\texttt{prr}(e))$ by Rule (20b) on (12) and (16) and (26) and (27)

(29) $e_2' \not\models_?^\dagger \dot{\xi}_2$ by (21) on (28)

(30) $(e_1', e_2') \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (27)

Case (15d).

(25) $e' = \texttt{inl}_{\tau_2}(e_1')$ by assumption

By rule induction over Rules (11) on (24), no rule applies due to syntactic contradiction.

Case (15e).

(25) $e' = \texttt{inr}_{\tau_1}(e_2')$ by assumption

By rule induction over Rules (11) on (24), no rule applies due to syntactic contradiction.

Case false.

(17) $\mathit{satisfyormay}(\texttt{prl}(e), \dot{\xi}_1) = \text{false}$ by assumption

(18) $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1$ by Lemma 1.0.4 on (17)

(19) if $e_1' \in \texttt{values}(\texttt{prl}(e))$ then $e_1' \not\models_?^\dagger \dot{\xi}_1$ by IH on (13) and (15) and (7) and (18)

To show if $e' \in \texttt{values}(e)$ then $e' \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}(e)$.

(20) $e' \in \texttt{values}(e)$ by assumption

By rule induction over Rules (20) on (20), only two rules apply.

Case (20a).

(21) e `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(21) e' `val` by assumption

(22) $\cdot\,; \Delta \vdash e' : (\tau_1 \times \tau_2)$ by assumption

By rule induction over Rules (15) on (21).

Case (15a).

(23) $e' = \underline{n}$ by assumption

By rule induction over Rules (11) on (22), no rule applies due to syntactic contradiction.

Case (15b).

(23) $e' = (\lambda x : \tau'.e'_1)$ by assumption

By rule induction over Rules (11) on (22), no rule applies due to syntactic contradiction.

Case (15c).

(23) $e' = (e'_1, e'_2)$ by assumption

(24) e'_1 `val` by assumption

By rule induction over Rules (11) on (22), only one rule applies.

Case (11g).

(25) $\cdot\,; \Delta \vdash e'_1 : \tau_1$ by assumption

(26) $e'_1 \in \texttt{values}(\texttt{prl}(e))$ by Rule (20b) on (11) and (15) and (24) and (25)

(27) $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by (19) on (26)

(28) $(e'_1, e'_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (27)

Case (15d).

(23) $e' = \texttt{inl}_{\tau_2}(e'_1)$ by assumption

By rule induction over Rules (11) on (22), no rule applies due to syntactic contradiction.

Case (15e).

(23) $e' = \texttt{inr}_{\tau_1}(e'_2)$ by assumption

By rule induction over Rules (11) on (22), no rule applies due to syntactic contradiction.

Case (16d).

(9) $e = (e_1, e_2)$ by assumption

(10) e_1 `indet` by assumption

(11) e_2 `val` by assumption

(12) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ or $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.12 on (4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$.

(13) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by assumption

By rule induction over Rules (11) on (2), only one rule applies.

Case (11g).

(14) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(15) if $e_1' \in \texttt{values}(e_1)$ then $e_1' \not\models^{\dagger}_{?} \dot{\xi}_1$ by IH on (10) and (14) and (7) and (13)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(16) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (20) on (16).

Case (20a).

(17) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(17) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (20e).

(17) $e' = (e_1', e_2')$ by assumption

(18) $e_1' \in \texttt{values}(e_1)$ by assumption

(19) $e_1' \not\models^{\dagger}_{?} \dot{\xi}_1$ by (15) on (18)

(20) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (19)

Case $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$.

(13) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(14) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (20) on (14).

Case (20a).

(15) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(15) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (20e).

(15) $e' = (e_1', e_2')$ by assumption

(16) $e_2' \in \texttt{values}(e_2)$ by assumption

By rule induction over Rules (20) on (16).

Case (20a).

(17) $e_2' = e_2$ by assumption

(18) $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by (17) and (13)

(19) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (18)

Case (20b).

(17) e_2 `notintro` by assumption

Contradicts (11) by Lemma 3.0.12.

Case (20c), (20d), (20e).

(17) e_2 `indet` by assumption

Contradicts (11) by Lemma 3.0.11.

Case (16e).

(9) $e = (e_1, e_2)$ by assumption

(10) e_1 `val` by assumption

(11) e_2 `indet` by assumption

(12) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ or $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.12 on (4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$.

(13) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by assumption

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(14) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (20) on (14).

Case (20a).

(15) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(15) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (20e).

(15) $e' = (e'_1, e'_2)$ by assumption

(16) $e'_1 \in \texttt{values}(e_1)$ by assumption

By rule induction over Rules (20) on (16).

Case (20a).

(17) $e'_1 = e_1$ by assumption

(18) $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by (17) and (13)

(19) $(e'_1, e'_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (18)

Case (20b).

(17) e_1 `notintro` by assumption

Contradicts (10) by Lemma 3.0.12.

Case (20c), (20d), (20e).

(17) e_1 `indet` by assumption

Contradicts (10) by Lemma 3.0.11.

Case $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$.

(13) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

By rule induction over Rules (11) on (2), only one rule applies.

Case (11g).

(14) $\cdot\,;\Delta \vdash e_2 : \tau_2$ by assumption

(15) if $e_2' \in \texttt{values}(e_2)$ then $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by IH on (11) and (14) and (8) and (13)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(16) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (20) on (16).

Case (20a).

(17) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b).

(17) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (20e).

(17) $e' = (e_1', e_2')$ by assumption

(18) $e_2' \in \texttt{values}(e_2)$ by assumption

(19) $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by (15) on (18)

(20) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (19)

Case (16f).

(9) $e = (e_1, e_2)$ by assumption

(10) e_1 `indet` by assumption

(11) e_2 `indet` by assumption

(12) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ or $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.12 on (4)

By rule induction over Rules (11) on (2), only one rule applies.

Case (11g).

(13) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(14) $\cdot\,;\Delta \vdash e_2 : \tau_2$ by assumption

By case analysis on the disjunction in (12).

Case $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$.

(15) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by assumption

(16) if $e_1' \in \texttt{values}(e_1)$ then $e_1' \not\models^{\dagger}_{?} \dot{\xi}_1$ by IH on (10) and (13) and (7) and (15)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(17) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (20) on (17).

Case (20a)**.**

(18) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b)**.**

(18) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (20e)**.**

(18) $e' = (e_1', e_2')$ by assumption

(19) $e_1' \in \texttt{values}(e_1)$ by assumption

(20) $e_1' \not\models^{\dagger}_{?} \dot{\xi}_1$ by (16) on (19)

(21) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (20)

Case $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$**.**

(15) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

(16) if $e_2' \in \texttt{values}(e_2)$ then $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by IH on (11) and (14) and (8) and (15)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(17) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (20) on (17).

Case (20a)**.**

(18) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 3.0.11.

Case (20b)**.**

(18) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (20e)**.**

(18) $e' = (e_1', e_2')$ by assumption

(19) $e_2' \in \texttt{values}(e_2)$ by assumption

(20) $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by (16) on (19)

(21) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 1.0.12 on (20)

Case (16i).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

By rule induction over Rules (11) on (2), no rule applies due to syntactic contradiction.

Case (16j).

(9) $e = \texttt{inr}_{\tau_1'}(e_2)$ by assumption

By rule induction over Rules (11) on (2), no rule applies due to syntactic contradiction.

Case (1g).

(5)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(6)	$\dot{\xi}_1 : \tau_1$	by assumption
(7)	$\dot{\xi}_2 : \tau_2$	by assumption
(8)	$e \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(9)	$e \not\models^{\dagger}_{?} \dot{\xi}_1$	by Lemma 1.0.9 on (8)
(10)	$e \not\models^{\dagger}_{?} \dot{\xi}_2$	by Lemma 1.0.9 on (8)
(11)	if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}_1$	by IH on (1) and (2) and (6) and (9)
(12)	if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}_2$	by IH on (1) and (2) and (7) and (10)

To show that if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$, we assume $e' \in \texttt{values}(e)$.

(13)	$e' \in \texttt{values}(e)$	by assumption
(14)	$e' \not\models^{\dagger}_{?} \dot{\xi}_1$	by (11) on (13)
(15)	$e' \not\models^{\dagger}_{?} \dot{\xi}_2$	by (12) on (13)
(16)	$e' \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$	by Lemma 1.0.9 on (14) and (15)

□

$\boxed{\theta : \Gamma}$ θ is of type Γ

$$\frac{}{\emptyset : \cdot}\ \text{STEmpty} \tag{21a}$$

$$\frac{\theta : \Gamma_\theta \qquad \Gamma ; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau}\ \text{STExtend} \tag{21b}$$

$\boxed{p\ \texttt{refutable}_?}$ p is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RNum} \tag{22a}$$

$$\frac{}{(\!|\,|\!)^w\ \texttt{refutable}_?}\ \text{REHole} \tag{22b}$$

$$\frac{}{(\!|p|\!)^w\ \texttt{refutable}_?}\ \text{RHole} \tag{22c}$$

$$\frac{}{\texttt{inl}(p)\ \texttt{refutable}_?}\ \text{RInl} \tag{22d}$$

$$\frac{}{\texttt{inr}(p)\ \texttt{refutable}_?}\ \text{RInr} \tag{22e}$$

$$\frac{p_1\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairL} \tag{22f}$$

$$\frac{p_2\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairR} \tag{22g}$$

$\boxed{e \rhd p \dashv\!\vert\, \theta}$ e matches p, emitting θ

$$\frac{}{e \rhd x \dashv\!\vert\, e/x}\ \text{MVar} \tag{23a}$$

$$\frac{}{e \rhd \_ \dashv\!\vert\, \cdot}\ \text{MWild} \tag{23b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashv\!\vert\, \cdot}\ \text{MNum} \tag{23c}$$

$$\frac{e_1 \rhd p_1 \dashv\!\vert\, \theta_1 \qquad e_2 \rhd p_2 \dashv\!\vert\, \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2}\ \text{MPair} \tag{23d}$$

$$\frac{e \rhd p \dashv\!\vert\, \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\vert\, \theta}\ \text{MInl} \tag{23e}$$

$$\frac{e \rhd p \dashv\!\vert\, \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\vert\, \theta}\ \text{MInr} \tag{23f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1 \qquad \texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2}{e \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{23g}$$

$\boxed{e \mathrel{?} p}$ e may match p

$$\frac{}{e \mathrel{?} \llparenthesis\rrparenthesis^w}\ \text{MMEHole} \tag{24a}$$

$$\frac{}{e \mathrel{?} \llparenthesis p \rrparenthesis^w}\ \text{MMHole} \tag{24b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}_?}{e \mathrel{?} p}\ \text{MMNotIntro} \tag{24c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\!\mid \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{24d}$$

$$\frac{e_1 \rhd p_1 \dashv\!\mid \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{24e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{24f}$$

$$\frac{e \mathrel{?} p}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)}\ \text{MMInl} \tag{24g}$$

$$\frac{e \mathrel{?} p}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)}\ \text{MMInr} \tag{24h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{25a}$$

$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairL} \tag{25b}$$

$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairR} \tag{25c}$$

$$\frac{}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)}\ \text{NMConfL} \tag{25d}$$

$$\frac{}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)}\ \text{NMConfR} \tag{25e}$$

NMInl
$$\frac{e \perp p}{\mathtt{inr}_\tau(e) \perp \mathtt{inl}(p)} \tag{25f}$$

NMInr
$$\frac{e \perp p}{\mathtt{inl}_\tau(e) \perp \mathtt{inr}(p)} \tag{25g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

ITHole
$$\frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{26a}$$

ITApFun
$$\frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \tag{26b}$$

ITApArg
$$\frac{e_1 \ \mathtt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')} \tag{26c}$$

ITAP
$$\frac{e_2 \ \mathtt{val}}{(\lambda x : \tau . e_1)(e_2) \mapsto [e_2/x]e_1} \tag{26d}$$

ITPairL
$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \tag{26e}$$

ITPairR
$$\frac{e_1 \ \mathtt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \tag{26f}$$

ITPrl
$$\frac{(e_1, e_2) \ \mathtt{final}}{\mathtt{prl}((e_1, e_2)) \mapsto e_1} \tag{26g}$$

ITPrr
$$\frac{(e_1, e_2) \ \mathtt{final}}{\mathtt{prr}((e_1, e_2)) \mapsto e_2} \tag{26h}$$

ITInl
$$\frac{e \mapsto e'}{\mathtt{inl}_\tau(e) \mapsto \mathtt{inl}_\tau(e')} \tag{26i}$$

ITInr
$$\frac{e \mapsto e'}{\mathtt{inr}_\tau(e) \mapsto \mathtt{inr}_\tau(e')} \tag{26j}$$

ITExpMatch
$$\frac{e \mapsto e'}{\mathtt{match}(e)\{\hat{rs}\} \mapsto \mathtt{match}(e')\{\hat{rs}\}} \tag{26k}$$

ITSuccMatch
$$\frac{e\ \texttt{final} \qquad e \rhd p_r \dashv\!\!| \ \theta}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{26l}$$

ITFailMatch
$$\frac{e\ \texttt{final} \qquad e \perp p_r}{\texttt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^{\diamond} \mid r' \mid rs'\}} \tag{26m}$$

Lemma 3.0.3. *If* $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{final}$ *then* e_1 $\texttt{final}$.

Proof. By rule induction over Rules (17) on $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{final}$.

Case (17a).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{val}$ — by assumption

By rule induction over Rules (15) on (17), only one case applies.

Case (15d).

(18) e_1 $\texttt{val}$ — by assumption
(19) e_1 $\texttt{final}$ — by Rule (17a) on (18)

Case (17b).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{indet}$ — by assumption

By rule induction over Rules (16) on (17), only one case applies.

Case (16i).

(18) e_1 $\texttt{indet}$ — by assumption
(19) e_1 $\texttt{final}$ — by Rule (17b) on (18)

□

Lemma 3.0.4. *If* $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{final}$ *then* e_2 $\texttt{final}$.

Proof. By rule induction over Rules (17) on $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{final}$.

Case (17a).

(1) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{val}$ — by assumption

By rule induction over Rules (15) on (1), only one case applies.

Case (15d).

(2) e_2 $\texttt{val}$ — by assumption
(3) e_2 $\texttt{final}$ — by Rule (17a) on (2)

Case (17b).

(1) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{indet}$ by assumption

By rule induction over Rules (16) on (1), only one case applies.

Case (16i)**.**

(2) e_2 $\texttt{indet}$ by assumption
(3) e_2 $\texttt{final}$ by Rule (17b) on (2)

□

Lemma 3.0.5. *If* (e_1, e_2) $\texttt{final}$ *then* e_1 $\texttt{final}$ *and* e_2 $\texttt{final}$*.*

Proof. By rule induction over Rules (17) on (e_1, e_2) $\texttt{final}$.

Case (17a)**.**

(1) (e_1, e_2) $\texttt{val}$ by assumption

By rule induction over Rules (15) on (1), only one case applies.

Case (15c)**.**

(2) e_1 $\texttt{val}$ by assumption
(3) e_2 $\texttt{val}$ by assumption
(4) e_1 $\texttt{final}$ by Rule (17a) on (2)
(5) e_2 $\texttt{final}$ by Rule (17a) on (3)

Case (17b)**.**

(1) (e_1, e_2) $\texttt{indet}$ by assumption

By rule induction over Rules (16) on (1), only three cases apply.

Case (16d)**.**

(2) e_1 $\texttt{indet}$ by assumption
(3) e_2 $\texttt{val}$ by assumption
(4) e_1 $\texttt{final}$ by Rule (17b) on (2)
(5) e_1 $\texttt{final}$ by Rule (17a) on (3)

Case (16e)**.**

(2) e_1 $\texttt{val}$ by assumption
(3) e_2 $\texttt{indet}$ by assumption
(4) e_1 $\texttt{final}$ by Rule (17a) on (2)
(5) e_1 $\texttt{final}$ by Rule (17b) on (3)

Case (16f)**.**

(2) e_1 $\texttt{indet}$ by assumption

(3) e_2 `indet`	by assumption
(4) e_1 `final`	by Rule (17b) on (2)
(5) e_1 `final`	by Rule (17b) on (3)

□

Lemma 3.0.6. *There doesn't exist $\underline{n}$ such that $\underline{n}$ `notintro`.*

Proof. By rule induction over Rules (18) on $\underline{n}$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.7. *There doesn't exist $\texttt{inl}_\tau(e)$ such that $\texttt{inl}_\tau(e)$ `notintro`.*

Proof. By rule induction over Rules (18) on $\texttt{inl}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.8. *There doesn't exist $\texttt{inr}_\tau(e)$ such that $\texttt{inr}_\tau(e)$ `notintro`.*

Proof. By rule induction over Rules (18) on $\texttt{inr}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.9. *There doesn't exist (e_1, e_2) such that (e_1, e_2) `notintro`.*

Proof. By rule induction over Rules (18) on (e_1, e_2) `notintro`, no case applies due to syntactic contradiction. □

Lemma 3.0.10. *If e `final` and e `notintro` then e `indet`.*

Proof Sketch. By rule induction over Rules (18) on e `notintro`, for each case, by rule induction over Rules (15) on e `val` and we notice that e `val` is not derivable. By rule induction over Rules (17) on e `final`, Rule (17a) result in a contradiction with the fact that e `val` is not derivable while Rule (17b) tells us e `indet`. □

Lemma 3.0.11. *There doesn't exist such an expression e such that both e `val` and e `indet`.*

Lemma 3.0.12. *There doesn't exist such an expression e such that both e `val` and e `notintro`.*

Lemma 3.0.13 (Finality)**.** *There doesn't exist such an expression e such that both e `final` and $e \mapsto e'$ for some e'*

Proof. Assume there exists such an e such that both e `final` and $e \mapsto e'$ for some e' then proof by contradiction.

By rule induction over Rules (17) and Rules (26), *i.e.*, over Rules (15) and Rules (26) and over Rules (16) and Rules (26) respectively. The proof can be done by straightforward observation of syntactic contradictions. □

Lemma 3.0.14 (Matching Determinism)**.** *If e `final` and $\cdot\,; \Delta_e \vdash e : \tau$ and $p : \tau[\xi] \dashv\vdash \Gamma\,; \Delta$ then exactly one of the following holds*

1. $e \rhd p \dashv\!\!\vdash \theta$ *for some* θ

2. $e \,?\, p$

3. $e \perp p$

Proof.

(1) e `final` by assumption

(2) $\cdot\,;\Delta_e \vdash e : \tau$ by assumption

(3) $p : \tau[\xi] \dashv\!\!\vdash \Gamma\,;\Delta$ by assumption

By rule induction over Rules (12) on (3), we would show one conclusion is derivable while the other two are not.

Case (12a)**.**

(4) $p = x$ by assumption

(5) $e \rhd x \dashv\!\!\vdash e/x$ by Rule (23a)

Assume $e \,?\, x$. By rule induction over Rules (24) on it, only one case applies.

Case (24c)**.**

(6) x $\texttt{refutable}_?$ by assumption

By rule induction over Rules (22) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \,?\, x}$ by contradiction

Assume $e \perp x$. By rule induction over Rules (25) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp x}$ by contradiction

Case (12b)**.**

(4) $p = \_$ by assumption

(5) $e \rhd \_ \dashv\!\!\vdash \cdot$ by Rule (23b)

Assume $e \,?\, \_$. By rule induction over Rules (24) on it, only one case applies.

Case (24c)**.**

(6) $\_$ $\texttt{refutable}_?$ by assumption

By rule induction over Rules (22) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \mathrel{?} \_}$ by contradiction

Assume $e \perp \_$. By rule induction over Rules (25) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp \_}$ by contradiction

Case (12c)**.**

(4) $p = (\!|\!)^w$ by assumption

(5) $e \mathrel{?} (\!|\!)^w$ by Rule (24a)

Assume $e \rhd (\!|\!)^w \dashv\!| \theta$ for some θ. By rule induction over Rules (24) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd (\!|\!)^w \dashv\!| \theta}$ by contradiction

Assume $e \perp (\!|\!)^w$. By rule induction over Rules (25) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp (\!|\!)^w}$ by contradiction

Case (12d)**.**

(4) $p = (\!|p_0|\!)^w$ by assumption

(5) $e \mathrel{?} (\!|p_0|\!)^w$ by Rule (24b)

Assume $e \rhd (\!|p_0|\!)^w \dashv\!| \theta$ for some θ. By rule induction over Rules (24) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd (\!|p_0|\!)^w \dashv\!| \theta}$ by contradiction

Assume $e \perp (\!|p_0|\!)^w$. By rule induction over Rules (25) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp (\!|p_0|\!)^w}$ by contradiction

Case (12e)**.**

(4) $p = \underline{n_2}$ by assumption

(5) $\tau = \texttt{num}$ by assumption

(6) $\xi = \underline{n_2}$ by assumption

(7) $\underline{n_2}\ \texttt{refutable}_?$ by Rule (22a)

By rule induction over Rules (11) on (2), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(8) $e = \llparenthesis\rrparenthesis^u, \llparenthesis e_0 \rrparenthesis^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) $e\ \texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

(10) $e\ ?\ \underline{n_2}$ by Rule (6i) on (7) and (9)

Assume $e \rhd \underline{n_2} \dashv\vdash \theta$ for some θ. By rule induction over it, no case applies due to syntactic contradiction.

(11) $\cancel{e \rhd \underline{n_2} \dashv\vdash \theta}$ by contradiction

Assume $e \perp \underline{n_2}$. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \perp \underline{n_2}}$ by contradiction

Case (11d).

(8) $e = \underline{n_1}$

Assume $\underline{n_1}\ ?\ \underline{n_2}$. By rule induction over Rules (24) on it, only two cases apply.

Case (24c).

(9) $\underline{n_1}\ \texttt{notintro}$ by assumption

Contradicts Lemma 3.0.6.

(10) $\cancel{\underline{n_1}\ ?\ \underline{n_2}}$ by contradiction

By case analysis on whether $n_1 = n_2$.

Case $n_1 = n_2$.

(11) $n_1 = n_2$ by assumption

(12) $\underline{n_1} \rhd \underline{n_2} \dashv\vdash \cdot$ by Rule (23c)

Assume $\underline{n_1} \perp \underline{n_2}$. By rule induction over Rules (25) on it, only one case applies.

Case (25a).

(13) $n_1 \neq n_2$ by assumption

Contradicts (11).

(14) $\cancel{\underline{n_1} \perp \underline{n_2}}$ by contradiction

Case $n_1 \neq n_2$.

(11) $n_1 \neq n_2$ by assumption

(12) $\underline{n_1} \perp \underline{n_2}$ by Rule (25a) on (11)

Assume $\underline{n_1} \rhd \underline{n_2} \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (23) on it, no case applies due to syntactic contradiction.

(13) $\cancel{\underline{n_1} \rhd \underline{n_2} \dashv\!\vert\, \theta}$ by contradiction

Case (12f).

(4) $p = \texttt{inl}(p_1)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \texttt{inl}(\xi_1)$ by assumption

(7) $p_1 : \tau_1[\xi_1] \dashv\!\vert\, \Gamma ; \Delta$ by assumption

(8) $\texttt{inl}(p_1)\ \texttt{refutable}_?$ by Rule (22d)

By rule induction over Rules (11) on (2), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) $e\ \texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

(11) $e \mathrel{?} \texttt{inl}(p_1)$ by Rule (6i) on (8) and (10)

Assume $e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1$ for some θ_1. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1}$ by contradiction

Assume $e \perp \texttt{inl}(p_1)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inl}(p_1)}$ by contradiction

Case (11j).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(10) $\cdot ; \Delta_e \vdash e_1 : \tau_1$ by assumption

(11) $e_1\ \texttt{final}$ by Lemma 3.0.3 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1, $e_1 \mathrel{?} p_1$, and $e_1 \perp p_1$ holds.
By case analysis on which one holds.

Case $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$.

(12) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

(13) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(14) $\cancel{e_1 \perp p_1}$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta_1$ by Rule (23e) on (12)

Assume $\texttt{inl}_{\tau_2}(e_1) \mathrel{?} \texttt{inl}(p_1)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24c).

(16) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

Contradicts Lemma 3.0.7.

Case (24g).

(16) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (13).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \mathrel{?} \texttt{inl}(p_1)}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (25) on it, only one case applies.

Case (25f).

(18) $e_1 \perp p_1$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ by contradiction

Case $e_1 \mathrel{?} p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(13) $e_1 \mathrel{?} p_1$ by assumption

(14) $\cancel{e_1 \perp p_1}$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \mathrel{?} \texttt{inl}(p_1)$ by Rule (24g) on (13)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (23) on it, only one case applies.

Case (23e).

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (25) on it, only one case applies.

Case (25f).

(18) $e_1 \perp p_1$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ by contradiction

Case $e_1 \perp p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(13) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(14) $e_1 \perp p_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$ by Rule (25f) on (14)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vdash \theta$ for some θ. By rule induction over Rules (23) on it, only one case applies.

Case (23e).

(16) $e_1 \rhd p_1 \dashv\!\vdash \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vdash \theta}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24c).

(18) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 3.0.7.

Case (24g).

(18) $e_1 \,?\, p_1$ by assumption

Contradicts (13).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)}$ by contradiction

Case (12g).

(4) $p = \texttt{inr}(p_2)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \texttt{inr}(\xi_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\vdash \Gamma \,;\, \Delta$ by assumption

(8) $\texttt{inr}(p_2)$ $\texttt{refutable}_?$ by Rule (22e)

By rule induction over Rules (11) on (2), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(9) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

(11) $e \,?\, \texttt{inr}(p_2)$ by Rule (6i) on (8) and (10)

Assume $e \rhd \texttt{inr}(p_2) \dashv\!\vdash \theta_2$ for some θ_2. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inr}(p_2) \dashv\!\vdash \theta_2}$ by contradiction

Assume $e \perp \texttt{inr}(p_2)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \mathtt{inr}(p_2)}$ by contradiction

Case (11k).

(9) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(10) $\cdot\,;\Delta_e \vdash e_2 : \tau_2$ by assumption

(11) $e_2\ \mathtt{final}$ by Lemma 3.0.4 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2, $e_2 \mathrel{?} p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which one holds.

Case $e_2 \rhd p_2 \dashv\!\mid \theta_2$.

(12) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ by assumption

(13) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\mid \theta_2$ by Rule (23f) on (12)

Assume $\mathtt{inr}_{\tau_1}(e_2) \mathrel{?} \mathtt{inr}(p_2)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24c).

(16) $\mathtt{inr}_{\tau_1}(e_2)\ \mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.8.

Case (24h).

(16) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (13).

(17) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \mathrel{?} \mathtt{inr}(p_2)}$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)$. By rule induction over Rules (25) on it, only one case applies.

Case (25g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\mathtt{inr}_{\tau_1}(e_2) \perp \mathtt{inr}(p_2)}$ by contradiction

Case $e_2 \mathrel{?} p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\mid \theta}$ by assumption

(13) $e_2 \mathrel{?} p_2$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \mathrel{?} \mathtt{inr}(p_2)$ by Rule (24h) on (13)

Assume $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\mid \theta$ for some θ. By rule induction over Rules (23) on it, only one case applies.

Case (23f).

(16) $e_2 \rhd p_2 \dashv\!\mid \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\vdash \theta}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$. By rule induction over Rules (25) on it, only one case applies.

Case (25g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)}$ by contradiction

Case $e_2 \perp p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\!\vdash \theta}$ by assumption

(13) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(14) $e_2 \perp p_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$ by Rule (25g) on (14)

Assume $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\vdash \theta$ for some θ. By rule induction over Rules (23) on it, only one case applies.

Case (23f).

(16) $e_2 \rhd p_2 \dashv\!\!\vdash \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\vdash \theta}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)$. By rule induction over Rules (24) on it, only two cases apply.

Case (24c).

(18) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 3.0.8.

Case (24h).

(18) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (13).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)}$ by contradiction

Case (12h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\tau = (\tau_1 \times \tau_2)$ by assumption

(6) $\xi = (\xi_1, \xi_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashV \Gamma_1 ; \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashV \Gamma_2 ; \Delta_2$ by assumption

By rule induction over Rules (11) on (2), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(11) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

(13) e $\texttt{indet}$ by Lemma 3.0.10 on (1) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (16g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (17b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (16h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (17b) on (16)

(18) $\cdot ; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (11h) on (2)

(19) $\cdot ; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (11i) on (2)

Assume $e \perp (p_1, p_2)$. By rule induction on it, no case applies due to syntactic contradiction.

(20) $\cancel{e \perp (p_1, p_2)}$ by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of $\texttt{prl}(e) \rhd p_1 \dashV \theta_1$, $\texttt{prl}(e) \,?\, p_1$, and $\texttt{prl}(e) \perp p_1$ holds.
By inductive hypothesis on (17) and (19) and (10), exactly one of $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$, $\texttt{prr}(e) \,?\, p_2$, and $\texttt{prr}(e) \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2. Note that we have already shown $\cancel{e \perp (p_1, p_2)}$.

Case $\texttt{prl}(e) \rhd p_1 \dashV \theta_1, \texttt{prr}(e) \rhd p_2 \dashV \theta_2$.

(21) $\texttt{prl}(e) \rhd p_1 \dashV \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e) \,?\, p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$ by assumption

(25) $\cancel{\texttt{prr}(e) \,?\, p_2}$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

(27) $e \rhd (p_1, p_2) \dashV \theta_1 \uplus \theta_2$ by Rule (23g) on (12) and (21) and (24)

Assume $e \,?\, (p_1, p_2)$. By rule induction over Rules (24) on it, only one case applies.

Case (24c).

(28) (p_1, p_2) $\texttt{refutable}_?$ by assumption

By rule induction over Rules (22), only two cases apply.

Case (22f).

(29) p_1 $\texttt{refutable}_?$ by assumption

(30) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (18e)

(31) $\texttt{prl}(e)\,?\,p_1$ by Rule (24c) on (29) and (30)

Contradicts (22).

Case (22g).

(29) p_2 $\texttt{refutable}_?$ by assumption

(30) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (18f)

(31) $\texttt{prl}(e)\,?\,p_1$ by Rule (24c) on (29) and (30)

Contradicts (22).

(32) $\cancel{e\,?\,(p_1, p_2)}$ by contradiction

Case $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1, \texttt{prr}(e)\,?\,p_2$.

(21) $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e)\,?\,p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption

(25) $\texttt{prr}(e)\,?\,p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (23), only one case applies.

Case (23g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

Contradicts (24).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

By rule induction over Rules (24) on (25), the following cases apply.

Case (24a),(24b).

(30) $p_2 = (\!|\,|\!)^w, (\!|p|\!)^w$ by assumption

(31) p_2 $\texttt{refutable}_?$ by Rule (22b) and Rule (22c)

(32) (p_1, p_2) $\texttt{refutable}_?$ by Rule (22g) on (31)

(33) $e\,?\,(p_1, p_2)$ by Rule (24c) on (12) and (32)

Case (24c).

(30) p_2 $\texttt{refutable}_?$ by assumption

(31) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (22g) on (30)
(32) $e\ ?\ (p_1, p_2)$ by Rule (24c) on (12) and (31)

Case $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption
(22) $\cancel{\texttt{prl}(e)\ ?\ p_1}$ by assumption
(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption
(25) $\cancel{\texttt{prr}(e)\ ?\ p_2}$ by assumption
(26) $\texttt{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (25) on (26), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e)\ ?\ p_1, \texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1}$ by assumption
(22) $\texttt{prl}(e)\ ?\ p_1$ by assumption
(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption
(24) $\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption
(25) $\cancel{\texttt{prr}(e)\ ?\ p_2}$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (23), only one case applies.

Case (23g)**.**

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption
(28) $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

By rule induction over Rules (24) on (22), the following cases apply.

Case (24a),(24b)**.**

(30) $p_1 = \llparenthesis\rrparenthesis^w, \llparenthesis p \rrparenthesis^w$ by assumption
(31) $p_1\ \texttt{refutable}_?$ by Rule (22b) and Rule (22c)
(32) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (22g) on (31)
(33) $e\ ?\ (p_1, p_2)$ by Rule (24c) on (12) and (32)

Case (24c)**.**

(30) $p_1\ \texttt{refutable}_?$ by assumption
(31) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (22g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (24c) on (12) and (31)

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \mathrel{?} p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (23), only one case applies.

Case (23g)**.**

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\vert\, \theta}$ by contradiction

By rule induction over Rules (24) on (22), the following cases apply.

Case (24a),(24b)**.**

(30) $p_1 = \llparenthesis\rrparenthesis^w, \llparenthesis p \rrparenthesis^w$ by assumption

(31) $p_1\ \texttt{refutable}_?$ by Rule (22b) and Rule (22c)

(32) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (22g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (24c) on (12) and (32)

Case (24c)**.**

(30) $p_1\ \texttt{refutable}_?$ by assumption

(31) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (22g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (24c) on (12) and (31)

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption

(26) $\texttt{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (25) on (26), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \rhd p_2 \dashV \theta_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$ by assumption
(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (25) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \mathrel{?} p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ by assumption
(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (25) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \perp p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ by assumption
(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption
(23) $\texttt{prl}(e) \perp p_1$ by assumption
(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ by assumption
(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption
(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (25) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case (11g).

(11) $e = (e_1, e_2)$ by assumption
(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption
(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption
(14) e_1 $\texttt{final}$ by Lemma 3.0.5 on (1)
(15) e_2 $\texttt{final}$ by Lemma 3.0.5 on (1)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \rhd p_1 \dashV \theta_1$, $e_1 \mathrel{?} p_1$, and $e_1 \perp p_1$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \rhd p_2 \dashV \theta_2$, $e_2 \mathrel{?} p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2.

Case $e_1 \rhd p_1 \dashv\!\!| \, \theta_1, e_2 \rhd p_2 \dashv\!\!| \, \theta_2$.

(16) $e_1 \rhd p_1 \dashv\!\!| \, \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\!| \, \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \, \theta_1 \uplus \theta_2$ by Rule (23d) on (16) and (19)

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (24) on it, only four cases apply.

Case (24c).

(23) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d).

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (24e).

(23) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (24f).

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(24) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (25) on it, only two cases apply.

Case (25b).

(25) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (25c).

(25) $e_2 \perp p_2$ by assumption

Contradicts (21).

(26) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\!| \, \theta_1, e_2 \mathrel{?} p_2$.

(16) $e_1 \rhd p_1 \dashv\!\!| \, \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\!| \, \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (24e) on (16) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (25) on it, only two cases apply.

Case (25b).

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (25c).

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\!\mid \theta_1, e_2 \perp p_2$.

(16) $e_1 \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (25c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (24) on it, only four cases apply.

Case (24c)**.**

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d)**.**

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (24e)**.**

(26) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (24f)**.**

(26) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \rhd p_2 \dashv\!\!| \, \theta_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\!| \, \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\!| \, \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (24d) on (17) and (19)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \, \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\!| \, \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \, \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (25) on it, only two cases apply.

Case (25b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (25c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathbin{?} p_1, e_2 \mathbin{?} p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $e_1 \mathbin{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $e_2 \mathbin{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathbin{?} (p_1, p_2)$ by Rule (24f) on (17) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashV \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (25) on it, only two cases apply.

Case (25b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (25c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathbin{?} p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $e_1 \mathbin{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $\cancel{e_2 \mathbin{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (25c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta}$ by contradiction

Assume $(e_1, e_2)\,?\,(p_1, p_2)$. By rule induction over Rules (24) on it, only four cases apply.

Case (24c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d).

(26) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

Contradicts (19).

Case (24e).

(26) $e_2\,?\,p_2$ by assumption

Contradicts (20).

Case (24f).

(26) $e_2\,?\,p_2$ by assumption

Contradicts (20).

(27) $\cancel{(e_1, e_2)\,?\,(p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \rhd p_2 \dashv\!\vdash \theta_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\vdash \theta_1}$ by assumption

(17) $\cancel{e_1\,?\,p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

(20) $\cancel{e_2\,?\,p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (25b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta}$ by contradiction

Assume $(e_1, e_2)\,?\,(p_1, p_2)$. By rule induction over Rules (24) on it, only four cases apply.

Case (24c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d).

(26) $e_1 \,?\, p_1$ by assumption

Contradicts (17).

Case (24e).

(26) $e_2 \,?\, p_2$ by assumption

Contradicts (20).

Case (24f).

(26) $e_1 \,?\, p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \,?\, (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \,?\, p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\!\!| \theta_1}$ by assumption

(17) $\cancel{e_1 \,?\, p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\!| \theta_2}$ by assumption

(20) $e_2 \,?\, p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (25b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\!| \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2)} \dashv\!\!| \theta$ by contradiction

Assume $(e_1, e_2) \,?\, (p_1, p_2)$. By rule induction over Rules (24) on it, only four cases apply.

Case (24c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d).

(26) $e_2 \rhd p_2 \dashv\!\!| \theta_2$ by assumption

Contradicts (19).

Case (24e).

(26) $e_1 \rhd p_1 \dashv\!\!| \theta_1$ by assumption

Contradicts (16).

Case (24f).

(26) $e_1 \,?\, p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \,?\, (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashV \theta_1}$ by assumption

(17) $\cancel{e_1 \,?\, p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashV \theta_2}$ by assumption

(20) $e_2 \,?\, p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (25b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashV \theta$. By rule induction over Rules (23) on it, only one case applies.

Case (23d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashV \theta}$ by contradiction

Assume $(e_1, e_2) \,?\, (p_1, p_2)$. By rule induction over Rules (24) on it, only four cases apply.

Case (24c)**.**

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d)**.**

(26) $e_2 \rhd p_2 \dashV \theta_2$ by assumption

Contradicts (19).

Case (24e)**.**

(26) $e_1 \rhd p_1 \dashV \theta_1$ by assumption

Contradicts (16).

Case (24f)**.**

(26) $e_1 \,?\, p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \,?\, (p_1, p_2)}$ by contradiction

□

Lemma 3.0.15 (Matching Coherence of Constraint)**.** *Suppose that* $\cdot ; \Delta_e \vdash e : \tau$ *and* e `final` *and* $p : \tau[\xi] \dashV \Gamma ; \Delta$*. Then we have*

1. $e \models \xi$ *iff* $e \rhd p \dashv\vdash \theta$

2. $e \models_? \xi$ *iff* $e \,?\, p$

3. $e \not\models_?^\dagger \xi$ *iff* $e \perp p$

Proof.

(1) $\cdot\,; \Delta_e \vdash e : \tau$ — by assumption

(2) e `final` — by assumption

(3) $p : \tau[\xi] \dashv\vdash \Gamma\,; \Delta$ — by assumption

Given Lemma 2.0.1, Theorem 1.1, and Lemma 3.0.14, it is sufficient to prove

1. $e \models \xi$ iff $e \rhd p \dashv\vdash \theta$
2. $e \models_? \xi$ iff $e \,?\, p$

By rule induction over Rules (12) on (3).

Case (12a)**.**

(4) $p = x$ — by assumption

(5) $\xi = \top$ — by assumption

1. Prove $e \models \top$ implies $e \rhd x \dashv\vdash \theta$ for some θ.

   (6) $e \rhd x \dashv\vdash e/x$ — by Rule (23a)

2. Prove $e \rhd x \dashv\vdash \theta$ implies $e \models \top$.

   (6) $e \models \top$ — by Rule (4a)

3. Prove $e \models_? \top$ implies $e \,?\, x$.

   (6) $e \not\models_? \top$ — by Lemma 1.0.5

   Vacuously true.

4. Prove $e \,?\, x$ implies $e \models_? \top$.

   By rule induction over Rules (24), we notice that either, $e?x$ is in syntactic contradiction with all the cases, or the premise x $\texttt{refutable}_?$ is not derivable. Hence, $e \,?\, x$ are not derivable. And thus vacuously true.

Case (12b)**.**

(4) $p = \_$ — by assumption

(5) $\xi = \top$ — by assumption

1. Prove $e \models \top$ implies $e \rhd \_ \dashv\vdash \theta$ for some θ.

   (6) $e \rhd \_ \dashv\vdash \cdot$ — by Rule (23a)

2. Prove $e \triangleright \_ \dashv\!\vdash \theta$ implies $e \models \top$.

   (6) $e \models \top$ by Rule (4a)

3. Prove $e \models_? \top$ implies $e \mathrel{?} \_$.

   (6) $e \not\models_? \top$ by Lemma 1.0.5

   Vacuously true.

4. Prove $e \mathrel{?} \_$ implies $e \models_? \xi$.

   By rule induction over Rules (24), we notice that either, $e ? \_$ is in syntactic contradiction with all the cases, or the premise $\_\ \texttt{refutable}_?$ is not derivable. Hence, $e \mathrel{?} \_$ are not derivable. And thus vacuously true.

Case (12c)**.**

(4) $p = \llparenthesis\rrparenthesis^w$ by assumption

(5) $\xi = ?$ by assumption

(6) $\bar{\xi} = ?$ by Definition **??**

1. Prove $e \models ?$ implies $e \triangleright \llparenthesis\rrparenthesis^w \dashv\!\vdash \theta$ for some θ.

   (7) $e \not\models ?$ by Rule (23a)

   Vacuously true.

2. Prove $e \triangleright \llparenthesis\rrparenthesis^w \dashv\!\vdash \theta$ implies $e \models ?$.

   By rule induction over Rules (23), we notice that $e \triangleright \llparenthesis\rrparenthesis^w \dashv\!\vdash \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? ?$ implies $e \mathrel{?} \llparenthesis\rrparenthesis^w$.

   (7) $e \mathrel{?} \llparenthesis\rrparenthesis^w$ by Rule (24a)

4. Prove $e \mathrel{?} \llparenthesis\rrparenthesis^w$ implies $e \models_? ?$.

   (7) $e \models_? ?$ by Rule (6a)

Case (12d)**.**

(4) $p = \llparenthesis p_0 \rrparenthesis^w$ by assumption

(5) $\xi = ?$ by assumption

1. Prove $e \models ?$ implies $e \triangleright \llparenthesis p_0 \rrparenthesis^w \dashv\!\vdash \theta$ for some θ.

   (6) $e \not\models ?$ by Rule (23a)

   Vacuously true.

2. Prove $e \rhd (\!|p_0|\!)^w \dashv\vdash \theta$ implies $e \models ?$.
   By rule induction over Rules (23), we notice that $e \rhd (\!|p_0|\!)^w \dashv\vdash \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? ?$ implies $e ? (\!|p_0|\!)^w$.

   (6) $e ? (\!|p_0|\!)^w$ — by Rule (24b)

4. Prove $e ? (\!|p_0|\!)^w$ implies $e \models_? ?$.

   (6) $e \models_? ?$ — by Rule (6a)

Case (12e)**.**

(4) $p = \underline{n}$ — by assumption

(5) $\xi = \underline{n}$ — by assumption

1. Prove $e \models \underline{n}$ implies $e \rhd \underline{n} \dashv\vdash \theta$ for some θ.

   (6) $e \models \underline{n}$ — by assumption

   By rule induction over Rules (4) on (6), only one case applies.

   Case (4b)**.**

   (7) $e = \underline{n}$ — by assumption

   (8) $\underline{n} \rhd \underline{n} \dashv\vdash \cdot$ — by Rule (23c)

2. Prove $e \rhd \underline{n} \dashv\vdash \theta$ implies $e \models \underline{n}$.

   (6) $e \rhd \underline{n} \dashv\vdash \theta$ — by assumption

   By rule induction over Rules (23) on (6), only one case applies.

   Case (23c)**.**

   (7) $e = \underline{n}$ — by assumption

   (8) $\theta = \cdot$ — by assumption

   (9) $\underline{n} \models \underline{n}$ — by Rule (4b)

3. Prove $e \models_? \underline{n}$ implies $e ? \underline{n}$.

   (6) $e \models_? \underline{n}$ — by assumption

   By rule induction over Rules (6) on (6), only one case applies.

   Case (6i)**.**

   (7) e `notintro` — by assumption

   (8) $\underline{n}$ $\texttt{refutable}_?$ — by Rule (22a)

   (9) $e ? \underline{n}$ — by Rule (24c) on (7) and (8)

4. Prove $e ? \underline{n}$ implies $e \models_? \underline{n}$.

   (6) $e ? \underline{n}$ — by assumption

By rule induction over Rules (24) on (6), only one case applies.

Case (24c).

(7)	e `notintro`	by assumption
(8)	$\underline{n}$ $\texttt{refutable}_?$	by Rule (2a)
(9)	$e \models_? \underline{n}$	by Rule (6) on (7) and (8)

Case (12f).

(4)	$p = \texttt{inl}(p_1)$	by assumption
(5)	$\xi = \texttt{inl}(\xi_1)$	by assumption
(6)	$\tau = (\tau_1 + \tau_2)$	by assumption
(7)	$p_1 : \tau_1[\xi_1] \dashv\!\vert\, \Gamma \,;\, \Delta$	by assumption

By rule induction over Rules (11) on (1), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m).

(8)	$e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$	by assumption
(9)	e `notintro`	by Rule (18a),(18b),(18c),(18d),(18e),(18f)

1. Prove $e \models \texttt{inl}(\xi_1)$ implies $e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (4) on $e \models \texttt{inl}(\xi_1)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ implies $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (23) on $e \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \texttt{inl}(\xi_1)$ implies $e \,?\, \texttt{inl}(p_1)$.

(10)	$\texttt{inl}(p_1)$ $\texttt{refutable}_?$	by Rule (22d)
(11)	$e \,?\, \texttt{inl}(p_1)$	by Rule (24c) on (9) and (10)

4. Prove $e \,?\, \texttt{inl}(p_1)$ implies $e \models_? \texttt{inl}(\xi_1)$.

(10)	$\texttt{inl}(\xi_1)$ $\texttt{refutable}_?$	by Rule (2c)
(11)	$e \models_? \texttt{inl}(\xi_1)$	by Rule (6i) on (9) and (10)

Case (11j).

(8)	$e = \texttt{inl}_{\tau_2}(e_1)$	by assumption
(9)	$\cdot \,;\, \Delta_e \vdash e_1 : \tau_1$	by assumption
(10)	e_1 `final`	by Lemma 3.0.3 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\vdash \theta$ for some θ

(12) $e_1 \models_? \xi_1$ iff $e_1 \,?\, p_1$

1. Prove $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\vdash \theta$ for some θ.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by assumption

   By rule induction over Rules (4) on (13), only one case applies.

   Case (4c).

   (14) $e_1 \models \xi_1$ by assumption

   (15) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ for some θ_1 by (11) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\vdash \theta_1$ by Rule (23e) on (15)

2. Prove $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\vdash \theta$ implies $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \rhd \mathtt{inl}(p_1) \dashv\!\vdash \theta$ by assumption

   By rule induction over Rules (23) on (13), only one case applies.

   Case (23e).

   (14) $e_1 \rhd p_1 \dashv\!\vdash \theta$ by assumption

   (15) $e_1 \models \xi_1$ by (11) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by Rule (4c) on (15)

3. Prove $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ by assumption

   By rule induction over Rules (6) on (13), only two cases apply.

   Case (6i).

   (14) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{notintro}$ by assumption

   Contradicts Lemma 3.0.7.

   Case (6b).

   (14) $e_1 \models_? \xi_1$ by assumption

   (15) $e_1 \,?\, p_1$ by (12) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$ by Rule (24g) on (15)

4. Prove $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$.

   (13) $\mathtt{inl}_{\tau_2}(e_1) \,?\, \mathtt{inl}(p_1)$ by assumption

   By rule induction over Rules (24) on (13), only two cases apply.

   Case (24c).

   (14) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{notintro}$ by assumption

   Contradicts Lemma 3.0.7.

   Case (24g).

   (14) $e_1 \,?\, p_1$ by assumption

   (15) $e_1 \models_? \xi_1$ by (12) on (14)

   (16) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\xi_1)$ by Rule (6b) on (15)

Case (12g)**.**

(4) $p = \texttt{inr}(p_2)$ by assumption

(5) $\xi = \texttt{inr}(\xi_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\!\mid \Gamma \,;\, \Delta$ by assumption

By rule induction over Rules (11) on (1), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m)**.**

(8) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e $\texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

1. Prove $e \models \texttt{inr}(\xi_2)$ implies $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ for some θ. By rule induction over Rules (4) on $e \models \texttt{inr}(\xi_2)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ implies $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (23) on $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \texttt{inr}(\xi_2)$ implies $e \,?\, \texttt{inr}(p_2)$.
   (10) $\texttt{inr}(p_2)$ $\texttt{refutable}_?$ by Rule (22e)
   (11) $e \,?\, \texttt{inr}(p_2)$ by Rule (24c) on (9) and (10)
4. Prove $e \,?\, \texttt{inr}(p_2)$ implies $e \models_? \texttt{inr}(\xi_2)$.
   (10) $\texttt{inr}(\xi_2)$ $\texttt{refutable}_?$ by Rule (2d)
   (11) $e \models_? \texttt{inr}(\xi_2)$ by Rule (6i) on (9) and (10)

Case (11k)**.**

(8) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(9) $\cdot \,;\, \Delta_e \vdash e_2 : \tau_2$ by assumption

(10) e_2 $\texttt{final}$ by Lemma 3.0.3 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\!\mid \theta$ for some θ

(12) $e_2 \models_? \xi_2$ iff $e_2 \,?\, p_2$

1. Prove $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ implies $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ for some θ.
   (13) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ by assumption

By rule induction over Rules (4) on (13), only one case applies.

Case (4c).

(14) $e_2 \models \xi_2$ by assumption

(15) $e_2 \rhd p_2 \dashv\!\vert\, \theta_1$ for some θ_1 by (11) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\vert\, \theta_1$ by Rule (23e) on (15)

2. Prove $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\vert\, \theta$ implies $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \rhd \mathtt{inr}(p_2) \dashv\!\vert\, \theta$ by assumption

By rule induction over Rules (23) on (13), only one case applies.

Case (23e).

(14) $e_2 \rhd p_2 \dashv\!\vert\, \theta$ by assumption

(15) $e_2 \models \xi_2$ by (11) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)$ by Rule (4c) on (15)

3. Prove $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ implies $\mathtt{inr}_{\tau_1}(e_2)\, ?\, \mathtt{inr}(p_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by assumption

By rule induction over Rules (6) on (13), only two cases apply.

Case (6i).

(14) $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.7.

Case (6b).

(14) $e_2 \models_? \xi_2$ by assumption

(15) $e_2\, ?\, p_2$ by (12) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2)\, ?\, \mathtt{inr}(p_2)$ by Rule (24g) on (15)

4. Prove $\mathtt{inr}_{\tau_1}(e_2)\, ?\, \mathtt{inr}(p_2)$ implies $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$.

(13) $\mathtt{inr}_{\tau_1}(e_2)\, ?\, \mathtt{inr}(p_2)$ by assumption

By rule induction over Rules (24) on (13), only two cases apply.

Case (24c).

(14) $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{notintro}$ by assumption

Contradicts Lemma 3.0.7.

Case (24g).

(14) $e_2\, ?\, p_2$ by assumption

(15) $e_2 \models_? \xi_2$ by (12) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\xi_2)$ by Rule (6b) on (15)

Case (12h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\xi = (\xi_1, \xi_2)$ by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\!\vert\, \Gamma_1 \,;\, \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\!\vert\, \Gamma_2 \,;\, \Delta_2$ by assumption

By rule induction over Rules (11) on (1), the following cases apply.

Case (11b),(11c),(11f),(11h),(11i),(11l),(11m)**.**

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (18a),(18b),(18c),(18d),(18e),(18f)

(13) e $\texttt{indet}$ by Lemma 3.0.10 on (2) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (16g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (17b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (16h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (17b) on (16)

(18) $\cdot\,;\Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (11h) on (1)

(19) $\cdot\,;\Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (11i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

(20) $\texttt{prl}(e) \models \xi_1$ iff $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1

(21) $\texttt{prl}(e) \models_? \xi_1$ iff $\texttt{prl}(e)\ ?\ p_1$

(22) $\texttt{prr}(e) \models \xi_2$ iff $\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2$ for some θ_2

(23) $\texttt{prr}(e) \models_? \xi_2$ iff $\texttt{prr}(e)\ ?\ p_2$

1. Prove $e \models (\xi_1, \xi_2)$ implies $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ for some θ.

   (24) $e \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (4) on (24), only one case applies.

   Case (4f)**.**

   (25) $\texttt{prl}(e) \models \xi_1$ by assumption

   (26) $\texttt{prr}(e) \models \xi_2$ by assumption

   (27) $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ by (20) on (25)

   (28) $\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2$ by (22) on (26)

   (29) $e \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2$ by Rule (23g) on (12) and (27) and (28)

2. Prove $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ implies $e \models (\xi_1, \xi_2)$.

   (24) $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ by assumption

   By rule induction over Rules (23) on (24), only one case applies.

   Case (23g)**.**

   (25) $\theta = \theta_1 \uplus \theta_2$ by assumption

   (26) $\texttt{prl}(e) \rhd \xi_1 \dashv\!\vert\, \theta_1$ by assumption

(27) $\texttt{prr}(e) \rhd \xi_2 \dashv\!\vert\, \theta_2$ by assumption

(28) $\texttt{prl}(e) \models \xi_1$ by (20) on (26)

(29) $\texttt{prr}(e) \models \xi_2$ by (22) on (27)

(30) $e \models (\xi_1, \xi_2)$ by Rule (4f) on (12) and (28) and (29)

3. Prove $e \models_? (\xi_1, \xi_2)$ implies $e \,?\, (p_1, p_2)$.

(24) $e \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (6) on (24), only one case applies.

Case (6i).

(25) (ξ_1, ξ_2) $\texttt{refutable}_?$ by assumption

By rule induction over Rules (2) on (25), only two cases apply.

Case (2e).

(26) ξ_1 $\texttt{refutable}_?$ by assumption

(27) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (18e)

(28) $\texttt{prl}(e) \models_? \xi_1$ by Rule (6i) on (26) and (27)

(29) $\texttt{prl}(e) \,?\, p_1$ by (21) on (28)

By rule induction over Rules (24) on (29), only three cases apply.

Case (24a),(24b).

(30) $p_1 = (\!|\,|\!)^w, (\!|p_0|\!)^w$ by assumption

(31) p_1 $\texttt{refutable}_?$ by Rule (22b) and Rule (22c)

(32) (p_1, p_2) $\texttt{refutable}_?$ by Rule (22f) on (31)

(33) $e \,?\, (p_1, p_2)$ by Rule (24c) on (12) and (32)

Case (24c).

(30) p_1 $\texttt{refutable}_?$ by assumption

(31) (p_1, p_2) $\texttt{refutable}_?$ by Rule (22f) on (30)

(32) $e \,?\, (p_1, p_2)$ by Rule (24c) on (12) and (31)

Case (2f).

(26) ξ_2 $\texttt{refutable}_?$ by assumption

(27) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (18e)

(28) $\texttt{prr}(e) \models_? \xi_2$ by Rule (6i) on (26) and (27)

(29) $\texttt{prr}(e) \,?\, p_2$ by (23) on (28)

By rule induction over Rules (24) on (29), only three cases apply.

Case (24a),(24b).

(30) $p_2 = (\!|\,|\!)^w, (\!|p_0|\!)^w$ by assumption

(31) p_2 $\texttt{refutable}_?$ by Rule (22b) and Rule (22c)

(32) (p_1, p_2) $\texttt{refutable}_?$ by Rule (22g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (24c) on (12) and (32)

Case (24c)**.**

(30) p_2 $\texttt{refutable}_?$ by assumption

(31) (p_1, p_2) $\texttt{refutable}_?$ by Rule (22g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (24c) on (12) and (31)

4. Prove $e \mathrel{?} (p_1, p_2)$ implies $e \models_? (\xi_1, \xi_2)$.

(24) $e \mathrel{?} (p_1, p_2)$ by assumption

By rule induction over Rules (24) on (24), only one case applies.

Case (24c)**.**

(25) (p_1, p_2) $\texttt{refutable}_?$ by assumption

By rule induction over Rules (22) on (25), only two cases apply.

Case (22f)**.**

(26) p_1 $\texttt{refutable}_?$ by assumption

(27) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (18e)

(28) $\texttt{prl}(e) \mathrel{?} p_1$ by Rule (24c) on (26) and (27)

(29) $\texttt{prl}(e) \models_? \xi_1$ by (21) on (28)

By rule induction over Rules (6) on (29), only three cases apply.

Case (6a)**.**

(30) $\xi_1 = ?$ by assumption

(31) ξ_1 $\texttt{refutable}_?$ by Rule (2b)

(32) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (2e) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (6i) on (12) and (32)

Case (6i)**.**

(30) ξ_1 $\texttt{refutable}_?$ by assumption

(31) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (2e) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (6i) on (12) and (31)

Case (22g)**.**

(26) p_2 $\texttt{refutable}_?$ by assumption

(27) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (18e)

(28) $\texttt{prr}(e) \mathrel{?} p_2$ by Rule (24c) on (26) and (27)

(29) $\texttt{prr}(e) \models_? \xi_2$ by (23) on (28)

By rule induction over Rules (6) on (29), only three cases apply.

Case (6a).

(30) $\xi_2 = ?$ by assumption

(31) ξ_2 $\texttt{refutable}_?$ by Rule (2b)

(32) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (2f) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (6i) on (12) and (32)

Case (6i).

(30) ξ_2 $\texttt{refutable}_?$ by assumption

(31) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (2f) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (6i) on (12) and (31)

Case (11g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ by assumption

(14) e_1 $\texttt{final}$ by Lemma 3.0.5 on (2)

(15) e_2 $\texttt{final}$ by Lemma 3.0.5 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ for some θ_1

(17) $e_1 \models_? \xi_1$ iff $e_1 \mathrel{?} p_1$

(18) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ for some θ_2

(19) $e_2 \models_? \xi_2$ iff $e_2 \mathrel{?} p_2$

1. Prove $(e_1, e_2) \models (\xi_1, \xi_2)$ implies $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta$ for some θ.

   (20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (4) on (20), only two cases apply.

   Case (4e).

   (21) $e_1 \models \xi_1$ by assumption

   (22) $e_2 \models \xi_2$ by assumption

   (23) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ for some θ_1 by (16) on (21)

   (24) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ for some θ_2 by (18) on (22)

   (25) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta_1 \uplus \theta_2$ by Rule (23d) on (23) and (24)

Case (4f).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

2. Prove $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\!\mid \theta$ implies $(e_1, e_2) \models (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!\!\mid \theta$ by assumption

By rule induction over Rules (23) on (20), only two cases apply.

Case (23d).

(21) $e_1 \rhd p_1 \dashv\!\!\!\mid \theta_1$ for some θ_1 by assumption

(22) $e_2 \rhd p_2 \dashv\!\!\!\mid \theta_2$ for some θ_2 by assumption

(23) $e_1 \models \xi_1$ by (16) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (4e) on (23) and (24)

Case (23g).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

3. Prove $(e_1, e_2) \models_? (\xi_1, \xi_2)$ implies $(e_1, e_2) \mathrel{?} (p_1, p_2)$.

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (6) on (20), only four cases apply.

Case (6i).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (6d).

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models \xi_2$ by assumption

(23) $e_1 \mathrel{?} p_1$ by (17) on (21)

(24) $e_2 \rhd p_2 \dashv\!\!\!\mid \theta_2$ by (18) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (24d) on (23) and (24)

Case (6e).

(21) $e_1 \models \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1 \rhd p_1 \dashv\!\!\!\mid \theta_1$ by (16) on (21)

(24) $e_2 \mathrel{?} p_2$ by (19) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (24e) on (23) and (24)

Case (6f).

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1 \mathrel{?} p_1$ by (17) on (21)

(24) $e_2 \mathrel{?} p_2$ by (19) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (24f) on (23) and (24)

4. Prove $(e_1, e_2) \mathrel{?} (p_1, p_2)$ implies $(e_1, e_2) \models_? (\xi_1, \xi_2)$.

(20) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by assumption

By rule induction over Rules (24) on (20), only four cases apply.

Case (24c)**.**

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 3.0.9.

Case (24d)**.**

(21) $e_1 \mathrel{?} p_1$ by assumption

(22) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

(23) $e_1 \models_? \xi_1$ by (17) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (6d) on (23) and (24)

Case (24e)**.**

(21) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ by assumption

(22) $e_2 \mathrel{?} p_2$ by assumption

(23) $e_1 \models \xi_1$ by (16) on (21)

(24) $e_2 \models_? \xi_2$ by (19) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (6e) on (23) and (24)

Case (24f)**.**

(21) $e_1 \mathrel{?} p_1$ by assumption

(22) $e_2 \mathrel{?} p_2$ by assumption

(23) $e_1 \models_? \xi_1$ by (17) on (21)

(24) $e_2 \models_? \xi_2$ by (19) on (22)

(25) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (6f) on (23) and (24)

□

4 Preservation and Progress

Theorem 4.1 (Preservation)**.** *If* $\cdot\,; \Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,; \Delta \vdash e' : \tau$

Proof. By rule induction over Rules (11) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (11l)**.**

(1) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

(2) $\texttt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$	by assumption
(3) $\cdot\,;\Delta \vdash e_1 : \tau_1$	by assumption
(4) $\cdot\,;\Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$	by assumption
(5) $\top \models^{\dagger}_{?} \xi$	by assumption

By rule induction over Rules (26) on (2).

Case (26k)**.**

(6) $e' = \texttt{match}(e_1')\{\cdot \mid r \mid rs\}$	by assumption
(7) $e_1 \mapsto e_1'$	by assumption
(8) $\cdot\,;\Delta \vdash e_1' : \tau_1$	by IH on (3) and (7)
(9) $\cdot\,;\Delta \vdash \texttt{match}(e_1')\{\cdot \mid r \mid rs\} : \tau$	by Rule (11l) on (8) and (4) and (5)

Case (26l)**.**

(6) $r = p_r \Rightarrow e_r$	by assumption
(7) $e' = [\theta](e_r)$	by assumption
(8) $e_1 \rhd p_r \dashv\vert\, \theta$	by assumption

By rule induction over Rules (14) on (4).

Case (14a)**.**

(9) $\xi = \xi_r$	by assumption
(10) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$	by assumption
(11) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r\,;\Delta_r$	by Inversion of Rule (13a) on (10)
(12) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$	by Inversion of Rule (13a) on (10)
(13) $\theta : \Gamma_r$	by Lemma 2.0.7 on (3) and (11) and (8)
(14) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$	by Lemma 2.0.6 on (12) and (13)

Case (14b)**.**

(9) $\xi = \xi_r \vee \xi_{rs}$	by assumption
(10) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$	by assumption
(11) $p_r : \tau_1[\xi_r] \dashv\vert\, \Gamma_r\,;\Delta_r$	by Inversion of Rule (13a) on (10)
(12) $\Gamma_r\,;\Delta \uplus \Delta_r \vdash e_r : \tau$	by Inversion of Rule (13a) on (10)
(13) $\theta : \Gamma_r$	by Lemma 2.0.7 on (3) and (11) and (8)
(14) $\cdot\,;\Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$	by Lemma 2.0.6 on (12) and (13)

Case (26m).

(6) $rs = r' \mid rs'$ by assumption

(7) $e' = \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by assumption

(8) e_1 $\texttt{final}$ by assumption

(9) $e_1 \perp p_r$ by assumption

By rule induction over Rules (14) on (4).

Case (14a). Syntactic contradiction of rs.

Case (14b).

(10) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(11) $\cdot \,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(12) $\cdot \,; \Delta \vdash [\perp \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$ by assumption

(13) $\xi_r \not\models \perp$ by assumption

(14) $p_r : \tau_1[\xi_r] \dashv\!\mid \Gamma_r \,; \Delta_r$ by Inversion of Rule (13a) on (11)

(15) $\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (13a) on (11)

(16) $\cdot \,; \Delta \vdash [\perp](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$ by Rule (14a) on (11) and (13)

(17) $e_1 \not\models^{\dagger}_{?} \xi_r$ by Lemma 3.0.15 on (3) and (8) and (14) and (9)

(18) $\cdot \,; \Delta \vdash \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$ by Rule (11m) on (3) and (8) and (16) and (12) and (17) and (5)

Case (11m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot \,; \Delta \vdash \texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\texttt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$ by assumption

(4) $\cdot \,; \Delta \vdash e_1 : \tau_1$ by assumption

(5) e_1 $\texttt{final}$ by assumption

(6) $\cdot \,; \Delta \vdash [\perp]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$ by assumption

(7) $\cdot \,; \Delta \vdash [\perp \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(8) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(9) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (26) on (3).

Case (26k)**.**

(10) $e' = \texttt{match}(e'_1)\{rs_{pre} \mid r \mid rs_{post}\}$ by assumption

(11) $e_1 \mapsto e'_1$ by assumption

By Lemma 3.0.13, (11) contradicts (5).

Case (26l)**.**

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $e' = [\theta](e_r)$ by assumption

(12) $e_1 \rhd p_r \dashv\!\vert\, \theta$ by assumption

By rule induction over Rules (14) on (7).

Case (14a)**.**

(13) $\xi_{rest} = \xi_r$ by assumption

(14) $\cdot\, ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv\!\vert\, \Gamma_r \, ; \Delta_r$ by Inversion of Rule (13a) on (14)

(16) $\Gamma_r \, ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (13a) on (14)

(17) $\theta : \Gamma_r$ by Lemma 2.0.7 on (4) and (15) and (12)

(18) $\cdot\, ; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 2.0.6 on (16) and (17)

Case (14b)**.**

(13) $\xi_{rest} = \xi_r \vee \xi_{rs}$ by assumption

(14) $\cdot\, ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv\!\vert\, \Gamma_r \, ; \Delta_r$ by assumption

(16) $\Gamma_r \, ; \Delta \uplus \Delta_r \vdash e_r : \tau$ by assumption

(17) $\theta : \Gamma_r$ by Lemma 2.0.7 on (4) and (15) and (12)

(18) $\cdot\, ; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 2.0.6 on (16) and (17)

Case (26m)**.**

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $rs_{post} = r' \mid rs'$ by assumption

(12) $e' = \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\}$ by assumption

(13) $e_1 \perp p_r$ by assumption

By rule induction over Rules (14) on (7).

Case (14a)**.** Syntactic contradiction of rs_{post}.

Case (14b)**.**

(14) $\xi_{rest} = \xi_r \vee \xi_{post}$ by assumption

(15) $\cdot\,;\Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(16) $\cdot\,;\Delta \vdash [\bot \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$ by assumption

(17) $\xi_r \not\models \xi_{pre}$ by assumption

(18) $p_r : \tau_1[\xi_r] \dashv\!\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (13a) on (15)

(19) $\Gamma_r\,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (13a) on (15)

(20) $\xi_r : \tau_1$ by Lemma 2.0.2 on (15)

(21) $\xi_{pre} : \tau_1$ by Lemma 2.0.3 on (6)

(22) $\xi_r \not\models \bot \vee \xi_{pre}$ by Lemma **??** on (20) and (21) and (17)

(23) $\cdot\,;\Delta \vdash [\bot](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$ by Lemma 2.0.4 on (6) and (15) and (22)

(24) $e_1 \not\models^{\dagger}_{?} \xi_r$ by Lemma 3.0.15 on (4) and (5) and (18) and (13)

(25) $e_1 \not\models^{\dagger}_{?} \xi_{pre} \vee \xi_r$ by Lemma 1.0.7 on (8) and (24)

(26) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\} : \tau$ by Rule (11m) on (4) and (5) and (23) and (16) and (25) and (9)

□

Theorem 4.2 (Progress)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then either* e $\texttt{final}$ *or* $e \mapsto e'$ *for some* e'*.*

Proof. By rule induction over Rules (11) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (11l)**.**

(1) $\cdot\,;\Delta \vdash \texttt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

(2) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(3) $\cdot\,;\Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ by assumption

(4) $\top \models^{\dagger}_{?} \xi$ by assumption

By IH on (2).

Case Scrutinee takes a step.

(5) $e_1 \mapsto e_1'$ by assumption

(6) $\texttt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \texttt{match}(e_1')\{\cdot \mid r \mid rs\}$
by Rule (26k) on (5)

Case Scrutinee is final.

(5) e_1 $\texttt{final}$ by assumption

By rule induction over Rules (14) on (3).

Case (14a).

(6) $rs = \cdot$ by assumption

(7) $\xi = \xi_r$ by assumption

(8) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (13a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\!\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (13a) on (8)

(11) $e_1 \models_?^\dagger \xi_r$ by Corollary 1.1.1 on (5) and (4)

By rule induction over Rules (8) on (11).

Case (8a).

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \mathrel{?} p_r$ by Lemma 3.0.15 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{indet}$
by Rule (16k) on (5) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}$ $\texttt{final}$
by Rule (17b) on (14)

Case (8b).

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\!\vdash \theta$ by Lemma 3.0.15 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$
by Rule (26l) on (5) and (13)

Case (14b).

(6) $rs = r' \mid rs'$ by assumption

(7) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(8) $\cdot\, ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (13a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\!| \Gamma_r \,; \Delta_r$ by Inversion of Rule (13a) on (8)

By Lemma 3.0.14 on (2) and (5) and (10).

Case Scrutinee matches pattern.

(11) $e_1 \rhd p_r \dashv\!| \theta$ by assumption

(12) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$ by Rule (26l) on (5) and (11)

Case Scrutinee may matches pattern.

(11) $e_1 \,?\, p_r$ by assumption

(12) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}\ \mathtt{indet}$ by Rule (16k) on (5) and (11)

(13) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}\ \mathtt{final}$ by Rule (17b) on (12)

Case Scrutinee doesn't matche pattern.

(11) $e_1 \perp p_r$ by assumption

(12) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$
$\mapsto \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by Rule (26m) on (5) and (11)

Case (11m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\, ; \Delta \vdash \mathtt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\cdot\, ; \Delta \vdash e_1 : \tau_1$ by assumption

(4) $e_1\ \mathtt{final}$ by assumption

(5) $\cdot\, ; \Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(6) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(7) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (14) on (5).

Case (14a).

(5) $rs_{post} = \cdot$ by assumption

(6) $\xi_{rest} = \xi_r$ by assumption

(7) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(8) $r = p_r \Rightarrow e_r$ by Inversion of Rule (13a) on (7)

(9) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r \,; \Delta_r$ by Inversion of Rule (13a) on (7)

(10) $e_1 \models^{\dagger}_{?} \xi_{pre} \vee \xi_r$ by Corollary 1.1.1 on (4) and (7)

(11) $e_1 \models^{\dagger}_{?} \xi_r$ by Lemma 1.0.8 on (10) and (6)

By rule induction over Rules (8) on (11).

Case (8a)**.**

(12) $e_1 \models_{?} \xi_r$ by assumption

(13) $e_1 \;?\; p_r$ by Lemma 3.0.15 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{indet}$ by Rule (16k) on (4) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{final}$ by Rule (17b) on (14)

Case (8b)**.**

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\!\!\mid \theta$ by Lemma 3.0.15 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (26l) on (4) and (13)

Case (14b)**.**

(5) $rs_{post} = r' \mid rs'_{post}$ by assumption

(6) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(7) $r = p_r \Rightarrow e_r$ by Inversion of Rule (13a) on (6)

(8) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r \,; \Delta_r$ by Inversion of Rule (13a) on (6)

By Lemma 3.0.14 on (3) and (4) and (8).

Case Scrutinee matches pattern**.**

(9) $e_1 \rhd p_r \dashv\!\!\mid \theta$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$
by Rule (26l) on (4) and (9)

Case Scrutinee may matches pattern.

(9) $e_1 \mathbin{?} p_r$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}\ \texttt{indet}$
by Rule (16k) on (4) and (9)

(11) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}\ \texttt{final}$
by Rule (17b) on (10)

Case Scrutinee doesn't matche pattern.

(9) $e_1 \perp p_r$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\}$
$\mapsto \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$
by Rule (26m) on (4) and (9)

□

5 Decidability

$\boxed{\Xi\ \texttt{incon}}$ A finite set of constraints, Ξ, is inconsistent

CINCTruth
$$\frac{\Xi\ \texttt{incon}}{\Xi, \top\ \texttt{incon}} \tag{27a}$$

CINCFalsity
$$\frac{}{\Xi, \bot\ \texttt{incon}} \tag{27b}$$

CINCNum
$$\frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \texttt{incon}} \tag{27c}$$

CINCNotNum
$$\frac{}{\Xi, \underline{n}, \cancel{\underline{n}}\ \texttt{incon}} \tag{27d}$$

CINCAnd
$$\frac{\Xi, \xi_1, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \texttt{incon}} \tag{27e}$$

CINCOr
$$\frac{\Xi, \xi_1\ \texttt{incon} \qquad \Xi, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \vee \xi_2\ \texttt{incon}} \tag{27f}$$

$$\frac{\text{CINCInj}}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2)\ \texttt{incon}} \tag{27g}$$

$$\text{CINCInl}\ \frac{\Xi\ \texttt{incon}}{\texttt{inl}(\Xi)\ \texttt{incon}} \tag{27h}$$

$$\text{CINCInr}\ \frac{\Xi\ \texttt{incon}}{\texttt{inr}(\Xi)\ \texttt{incon}} \tag{27i}$$

$$\text{CINCPairL}\ \frac{\Xi_1\ \texttt{incon}}{(\Xi_1, \Xi_2)\ \texttt{incon}} \tag{27j}$$

$$\text{CINCPairR}\ \frac{\Xi_2\ \texttt{incon}}{(\Xi_1, \Xi_2)\ \texttt{incon}} \tag{27k}$$

Lemma 5.0.1 (Decidability of Inconsistency)**.** *Suppose* $\dot{\top}(\xi) = \xi$*. It is decidable whether* ξ $\texttt{incon}$*.*

Lemma 5.0.2 (Inconsistency and Entailment of Constraint)**.** *Suppose that* $\dot{\top}(\xi) = \xi$*. Then* $\bar{\xi}$ $\texttt{incon}$ *iff* $\top \models \xi$

Lemma 5.0.3. *If* $e \models \xi$ *then* $e \models \dot{\top}(\xi)$

Proof. By rule induction over Rules (4), it is obvious to see that $\dot{\top}(\xi) = \xi$. □

Lemma 5.0.4. *If* $e \models_? \xi$ *then* $e \models_?^\dagger \dot{\top}(\xi)$*.*

Proof.

(11) $e \models_? \xi$	by assumption

By Rule Induction over Rules (6) on (11).

Case (6a)**.**

(12) $\xi = ?$	by assumption
(13) $e \models \top$	by Rule (4a)
(14) $e \models_?^\dagger \top$	by Rule (8b) on (13)

Case (6i)**.**

(12) e $\texttt{notintro}$	by assumption
(13) ξ $\texttt{refutable}_?$	by assumption

By **??** on (12) and (13) and case analysis on its conclusion. By rule induction over Rules (2).

Case $\dot{\top}(\xi)$ $\texttt{refutable}_?$.

(14) $\dot{\top}(\xi)$ $\texttt{refutable}_?$	by assumption
(15) $e \models_? \dot{\top}(\xi)$	by Rule (6i) on (12) and (14)
(16) $e \models_?^\dagger \dot{\top}(\xi)$	by Rule (8b) on (15)

Case $e \models \dot{\top}(\xi)$.

(14) $e \models \dot{\top}(\xi)$	by assumption
(15) $e \models_?^\dagger \top$	by Rule (8b) on (14)

Case (6g).

(12) $\xi = \xi_1 \vee \xi_2$	by assumption
(13) $e \models_? \xi_1$	by assumption
(14) $e \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (13)
(15) $e \models_?^\dagger \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 1.0.9 on (14)

Case (6h).

(12) $\xi = \xi_1 \vee \xi_2$	by assumption
(13) $e \models_? \xi_2$	by assumption
(14) $e \models_?^\dagger \dot{\top}(\xi_2)$	by IH on (13)
(15) $e \models_?^\dagger \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 1.0.9 on (14)

Case (6b).

(12) $e = \texttt{inl}_{\tau_2}(e_1)$	by assumption
(13) $\xi = \texttt{inl}(\xi_1)$	by assumption
(14) $e_1 \models_? \xi_1$	by assumption
(15) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by IH on (14)
(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\top}(\xi_1))$	by Lemma 1.0.10 on (15)

Case (6c).

(12) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption
(13) $\xi = \texttt{inr}(\xi_2)$	by assumption

(14)	$e_2 \models_? \xi_2$	by assumption
(15)	$e_2 \models^{\dagger}_? \dot{\top}(\xi_2)$	by IH on (14)
(16)	$\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_? \mathtt{inr}(\dot{\top}(\xi_2))$	by Lemma 1.0.11 on (15)

Case (6d)**.**

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_2 \models \xi_2$	by assumption
(16)	$e_1 \models^{\dagger}_? \dot{\top}(\xi_1)$	by IH on (14)
(17)	$e_2 \models \dot{\top}(\xi_2)$	by Lemma 5.0.3 on (15)
(18)	$e_2 \models^{\dagger}_? \dot{\top}(\xi_2)$	by Rule (8b) on (17)
(19)	$(e_1, e_2) \models^{\dagger}_? (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 1.0.12 on (16) and (18)

Case (6e)**.**

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models \xi_1$	by assumption
(15)	$e_2 \models_? \xi_2$	by assumption
(16)	$e_1 \models \dot{\top}(\xi_1)$	by Lemma 5.0.3 on (14)
(17)	$e_1 \models^{\dagger}_? \dot{\top}(\xi_1)$	by Rule (8b) on (16)
(18)	$e_2 \models^{\dagger}_? \dot{\top}(\xi_2)$	by IH on (15)
(19)	$(e_1, e_2) \models^{\dagger}_? (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Lemma 1.0.12 on (17) and (18)

Case (6f)**.**

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_? \xi_1$	by assumption
(15)	$e_2 \models_? \xi_2$	by assumption

(16) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by IH on (14)

(17) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by IH on (15)

(18) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Lemma 1.0.12 on (16) and (17)

□

Lemma 5.0.5. $e \models^{\dagger}_{?} \xi$ *iff* $e \models^{\dagger}_{?} \dot{\top}(\xi)$

Proof. 1. Sufficiency:

(1) $e \models^{\dagger}_{?} \xi$ by assumption

By rule induction over Rules (8) on (1)

Case (8b)**.**

(2) $e \models \xi$ by assumption

(3) $e \models \dot{\top}(\xi)$ by Lemma 5.0.3 on (2)

(4) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by Rule (8b) on (3)

Case (8a)**.**

(2) $e \models_{?} \xi$ by assumption

(3) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by Lemma 5.0.4 on (2)

2. Necessity:

(1) $e \models^{\dagger}_{?} \dot{\top}(\xi)$ by assumption

By structural induction on ξ,

Case $\xi = \top, \bot, \underline{n}, \underline{\not n}$**.**

(2) $e \models^{\dagger}_{?} \xi$ by (1) and Definition **??**

Case $\xi = ?$**.**

(2) $e \models_{?} ?$ by Rule (6a)

(3) $e \models^{\dagger}_{?} ?$ by Rule (8a) on (2)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Definition **??**

By rule induction over Rules (8) on (1),

Case (8b).

(3) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (4) on (3) and two cases apply,

Case ($4g$).

(4) $e \models \dot{\top}(\xi_1)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (8b) on (4)

(6) $e \models_?^\dagger \xi_1$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.9 on (6)

Case ($4h$).

(4) $e \models \dot{\top}(\xi_2)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (8b) on (4)

(6) $e \models_?^\dagger \xi_2$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.9 on (6)

Case (8a).

(3) $e \models_? \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (6) on (3) and two cases apply,

Case (6g).

(4) $e \models_? \dot{\top}(\xi_1)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (8a) on (4)

(6) $e \models_?^\dagger \xi_1$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.9 on (6)

Case (6h).

(4) $e \models_? \dot{\top}(\xi_2)$ by assumption

(5) $e \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (8a) on (4)

(6) $e \models_?^\dagger \xi_2$ by IH on (5)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 1.0.9 on (6)

Case $\xi = \texttt{inl}(\xi_1)$.

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (8) on (1),

Case (8b).

(4) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (4) and only one case applies,

Case (4c).

(5) $e_1 \models \dot{\top}(\xi_1)$	by assumption
(6) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by Rule (8b) on (5)
(7) $e_1 \models_?^\dagger \xi_1$	by IH on (6)
(8) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$	by Lemma 1.0.10 on (7)

Case (8a).

(4) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\top}(\xi_1))$	by assumption

By rule induction over Rules (6) and only one case applies,

Case (6b).

(5) $e_1 \models_? \dot{\top}(\xi_1)$	by assumption
(6) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$	by Rule (8a) on (5)
(7) $e_1 \models_?^\dagger \xi_1$	by IH on (6)
(8) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$	by Lemma 1.0.10 on (7)

Case $\xi = \texttt{inr}(\xi_2)$.

(2) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption
(3) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$	by assumption

By rule induction over Rules (8) on (1),

Case (8b).

(4) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\top}(\xi_2))$	by assumption

By rule induction over Rules (4) and only one case applies,

Case (4d).

(5) $e_2 \models \dot{\top}(\xi_2)$	by assumption
(6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by Rule (8b) on (5)
(7) $e_2 \models_?^\dagger \xi_2$	by IH on (6)
(8) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\xi_2)$	by Lemma 1.0.11 on (7)

Case (8a).

(4) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\top}(\xi_2))$	by assumption

By rule induction over Rules (6) and only one case applies,

Case (6c).

(5) $e_2 \models_? \dot{\top}(\xi_2)$	by assumption
(6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$	by Rule (8a) on (5)
(7) $e_2 \models_?^\dagger \xi_2$	by IH on (6)

(8) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 1.0.11 on (7)

Case $\xi = (\xi_1, \xi_2)$**.**

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by Definition **??**

By rule induction over Rules (8) on (1),

Case (8b)**.**

(4) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (4) on (4) and only one case applies,

Case (4e)**.**

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (8b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (8b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.12 on (9) and (10)

Case (8a)**.**

(4) $(e_1, e_2) \models_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (6) on (4) and three cases apply,

Case (6d)**.**

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (8a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (8b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.12 on (9) and (10)

Case (6e)**.**

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (8b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (8a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.12 on (9) and (10)

Case (6f)**.**

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (8a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (8a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.12 on (9) and (10)

□

Lemma 5.0.6. *Assume* $\dot{\top}(\xi) = \xi$*. Then* $\top \models^{\dagger}_{?} \xi$ *iff* $\top \models \xi$*.*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

2. Necessity:

□

Theorem 5.1. $\top \models^{\dagger}_{?} \xi$ *iff* $\top \models \dot{\top}(\xi)$*.*

Lemma 5.1.1. *Assume that* e `val`*. Then* $e \models^{\dagger}_{?} \xi$ *iff* $e \models \dot{\top}(\xi)$

Proof.

(1) e `val` by assumption

We prove sufficiency and necessity separately.

1. Sufficiency:

   (2) $e \models^{\dagger}_{?} \xi$ by assumption

   By rule induction over Rules (8) on (2).

   Case (8b)**.**

   (3) $e \models \xi$ by assumption

   (4) $e \models \dot{\top}(\xi)$ by Lemma 5.0.3 on (3)

   Case (8a)**.**

   (3) $e \models_{?} \xi$ by assumption

By rule induction over Rules (6) on (3).

Case (6a).

(4)	$\xi = ?$	by assumption
(5)	$e \models \dot{\top}(\xi)$	by Rule (4a) and Definition **??**

Case (6i).

(4)	e `notintro`	by assumption

By rule induction over Rules (18) on (4), for each case, by rule induction over Rules (15) on (1), no case applies due to syntactic contradiction.

Case (6g).

(4)	$\xi = \xi_1 \vee \xi_2$	by assumption
(5)	$e \models_? \xi_1$	by assumption
(6)	$\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by **??**
(7)	$e \models_?^\dagger \xi_1$	by Rule (8a) on (5)
(8)	$e \models \dot{\top}(\xi_1)$	by IH on (7)
(9)	$e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (4g) on (8)

Case (6h).

(4)	$\xi = \xi_1 \vee \xi_2$	by assumption
(5)	$e \models_? \xi_2$	by assumption
(6)	$\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by **??**
(7)	$e \models_?^\dagger \xi_2$	by Rule (8a) on (5)
(8)	$e \models \dot{\top}(\xi_2)$	by IH on (7)
(9)	$e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (4h) on (8)

Case (6b).

(4)	$\xi = \texttt{inl}(\xi_1)$	by assumption
(5)	$e \models_? \xi_1$	by assumption
(6)	$\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$	by **??**
(7)	$e \models_?^\dagger \xi_1$	by Rule (8a) on (5)
(8)	$e \models \dot{\top}(\xi_1)$	by IH on (7)
(9)	$e \models \texttt{inl}(\dot{\top}(\xi_1))$	by Rule (4c) on (8)

Case (6c).

(4)	$\xi = \texttt{inr}(\xi_2)$	by assumption
(5)	$e \models_? \xi_2$	by assumption
(6)	$\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$	by **??**
(7)	$e \models_?^\dagger \xi_2$	by Rule (8a) on (5)

(8)	$e \models \dot{\top}(\xi_2)$	by IH on (7)
(9)	$e \models \mathtt{inr}(\dot{\top}(\xi_2))$	by Rule (4d) on (8)

Case (6d)**.**

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models_? \xi_1$	by assumption
(7)	$e_2 \models \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by **??**
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (8a) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (8b) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (4e) on (11) and (12)

Case (6e)**.**

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models \xi_1$	by assumption
(7)	$e_2 \models_? \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by **??**
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (8b) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (8a) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (4e) on (11) and (12)

Case (6f)**.**

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models_? \xi_1$	by assumption
(7)	$e_2 \models_? \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by **??**
(9)	$e_1 \models_?^\dagger \xi_1$	by Rule (8a) on (6)
(10)	$e_2 \models_?^\dagger \xi_2$	by Rule (8a) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13)	$(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (4e) on (11) and (12)

2. Necessity:

   (2) $e \models \dot{\top}(\xi)$ by assumption

   By structural induction on ξ.

   Case $\xi = \top, \bot, \underline{n}, \not\underline{n}$**.**

   (3) $\xi = \dot{\top}(\xi)$ by **??**

   (4) $e \models_{?}^{\dagger} \xi$ by Rule (8b) on (2)

   Case $\xi = ?$**.**

   (3) $e \models_{?} ?$ by Rule (6a)

   (4) $e \models_{?}^{\dagger} ?$ by Rule (8a) on (3)

   Case $\xi = \xi_1 \wedge \xi_2$**.**

   (3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by **??**

   By rule induction over Rules (4) on (2), only one case applies.

   Case (**??**)**.**

   (4) $e \models \dot{\top}(\xi_1)$ by assumption

   (5) $e \models \dot{\top}(\xi_2)$ by assumption

   (6) $e \models_{?}^{\dagger} \xi_1$ by IH on (4)

   (7) $e \models_{?}^{\dagger} \xi_2$ by IH on (5)

   (8) $e \models \xi_1 \wedge \xi_2$ by **??** on (6) and (7)

   Case $\xi = \xi_1 \vee \xi_2$**.**

   (3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by **??**

   By rule induction over Rules (4) on (2) and only two cases apply.

   Case (4g)**.**

   (4) $e \models \dot{\top}(\xi_1)$ by assumption

   (5) $e \models_{?}^{\dagger} \xi_1$ by IH on (4)

   (6) $e \models_{?}^{\dagger} \xi_1 \vee \xi_2$ by Lemma 1.0.9 on (5)

   Case (4h)**.**

   (4) $e \models \dot{\top}(\xi_2)$ by assumption

   (5) $e \models_{?}^{\dagger} \xi_2$ by IH on (4)

   (6) $e \models_{?}^{\dagger} \xi_1 \vee \xi_2$ by Lemma 1.0.9 on (5)

Case $\xi = \texttt{inl}(\xi_1)$**.**

(3) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ by **??**

By rule induction over Rules (4) on (2) and only one case applies.

Case (4c)**.**

(4) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (5)

(7) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$ by Lemma 1.0.10 on (6)

Case $\xi = \texttt{inr}(\xi_2)$**.**

(3) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$ by **??**

By rule induction over Rules (4) on (2) and only one case applies.

Case (4d)**.**

(4) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(5) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (5)

(7) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 1.0.11 on (6)

Case $\xi = (\xi_1, \xi_2)$**.**

(3) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by **??**

By rule induction over Rules (4) on (2) and only one case applies.

Case (4e)**.**

(4) $e = (e_1, e_2)$ by assumption

(5) $e_1 \models \dot{\bot}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\bot}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (5)

(8) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (6)

(9) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 1.0.12 on (7) and (8)

□

Lemma 5.1.2. $e \models \xi$ *iff* $e \models \dot{\bot}(\xi)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models \xi$ by assumption

By rule induction over Rules (4) on (1).

Case (4a).

(2)	$\xi = \top$	by assumption
(3)	$e \models \dot{\bot}(\top)$	by (1) and Definition **??**

Case (4b).

(2)	$\xi = \underline{n}$	by assumption
(3)	$e \models \dot{\bot}(\underline{n})$	by (1) and Definition **??**

Case (**??**).

(2)	$\xi = \cancel{\underline{n}}$	by assumption
(3)	$e \models \dot{\bot}(\cancel{\underline{n}})$	by (1) and Definition **??**

Case (**??**).

(2)	$\xi = \xi_1 \wedge \xi_2$	by assumption
(3)	$e \models \xi_1$	by assumption
(4)	$e \models \xi_2$	by assumption
(5)	$e \models \dot{\bot}(\xi_1)$	by IH on (3)
(6)	$e \models \dot{\bot}(\xi_2)$	by IH on (4)
(7)	$e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$	by Rule (**??**) on (5) and (6)
(8)	$e \models \dot{\bot}(\xi_1 \wedge \xi_2)$	by (7) and Definition **??**

Case (4g).

(2)	$\xi = \xi_1 \vee \xi_2$	by assumption
(3)	$e \models \xi_1$	by assumption
(4)	$e \models \dot{\bot}(\xi_1)$	by IH on (3)
(5)	$e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (4g) on (4)
(6)	$e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition **??**

Case (4h).

(2)	$\xi = \xi_1 \vee \xi_2$	by assumption
(3)	$e \models \xi_2$	by assumption
(4)	$e \models \dot{\bot}(\xi_2)$	by IH on (3)
(5)	$e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (4h) on (4)
(6)	$e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition **??**

Case (4c)**.**

(2)	$e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(3)	$\xi = \mathtt{inl}(\xi_1)$	by assumption
(4)	$e_1 \models \xi_1$	by assumption
(5)	$e_1 \models \dot{\bot}(\xi_1)$	by IH on (4)
(6)	$\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\bot}(\xi_1))$	by Rule (4c) on (5)
(7)	$\mathtt{inl}_{\tau_2}(e_1) \models \dot{\bot}(\mathtt{inl}(\xi_1))$	by (6) and Definition **??**

Case (4d)**.**

(2)	$e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(3)	$\xi = \mathtt{inr}(\xi_2)$	by assumption
(4)	$e_2 \models \xi_2$	by assumption
(5)	$e_2 \models \dot{\bot}(\xi_2)$	by IH on (4)
(6)	$\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\bot}(\xi_2))$	by Rule (4d) on (5)
(7)	$\mathtt{inr}_{\tau_1}(e_2) \models \dot{\bot}(\mathtt{inr}(\xi_2))$	by (6) and Definition **??**

Case (4e)**.**

(2)	$e = (e_1, e_2)$	by assumption
(3)	$\xi = (\xi_1, \xi_2)$	by assumption
(4)	$e_1 \models \xi_1$	by assumption
(5)	$e_2 \models \xi_2$	by assumption
(6)	$e_1 \models \dot{\bot}(\xi_1)$	by IH on (4)
(7)	$e_2 \models \dot{\bot}(\xi_2)$	by IH on (5)
(8)	$(e_1, e_2) \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$	by Rule (4e) on (6) and (7)
(9)	$(e_1, e_2) \models \dot{\bot}((\xi_1, \xi_2))$	by (8) and Definition **??**

2. Necessity:

(1)	$e \models \dot{\bot}(\xi)$	by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\not n}$**.**

(2)	$e \models \xi$	by (1) and Definition **??**

Case $\xi = ?$**.**

(2)	$e \models \bot$	by (1) and Definition **??**

(3) $e \not\models \perp$ by Lemma **??**

(3) contradicts (2).

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$ by (1) and Definition **??**

By rule induction over Rules (4) on (2) and only case applies.

Case (**??**)**.**

(3) $e \models \dot{\perp}(\xi_1)$ by assumption
(4) $e \models \dot{\perp}(\xi_2)$ by assumption
(5) $e \models \xi_1$ by IH on (3)
(6) $e \models \xi_2$ by IH on (4)
(7) $e \models \xi_1 \wedge \xi_2$ by Rule (**??**) on (5) and (6)

Case $\xi = \xi_1 \vee \xi_2$**.**

(2) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ by (1) and Definition **??**

By rule induction over Rules (4) on (2) and only two cases apply.

Case (4g)**.**

(3) $e \models \dot{\perp}(\xi_1)$ by assumption
(4) $e \models \xi_1$ by IH on (3)
(5) $e \models \xi_1 \vee \xi_2$ by Rule (4g) on (4)

Case (4h)**.**

(3) $e \models \dot{\perp}(\xi_2)$ by assumption
(4) $e \models \xi_2$ by IH on (3)
(5) $e \models \xi_1 \vee \xi_2$ by Rule (4h) on (4)

Case $\xi = \mathtt{inl}(\xi_1)$**.**

(2) $e \models \mathtt{inl}(\dot{\perp}(\xi_1))$ by (1) and Definition **??**

By rule induction over Rules (4) on (2) and only one case applies.

Case (4c)**.**

(3) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption
(4) $e_1 \models \dot{\perp}(\xi_1)$ by assumption
(5) $e_1 \models \xi_1$ by IH on (4)
(6) $e \models \mathtt{inl}(\xi_1)$ by Rule (4c) on (5)

Case $\xi = \mathtt{inr}(\xi_2)$**.**

(2) $e \models \mathtt{inr}(\dot{\perp}(\xi_2))$ by (1) and Definition **??**

By rule induction over Rules (4) on (2) and only one case applies.

Case (4d).

(3) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption
(4) $e_2 \models \dot{\perp}(\xi_2)$	by assumption
(5) $e_2 \models \xi_2$	by IH on (4)
(6) $e \models \texttt{inr}(\xi_2)$	by Rule (4d) on (5)

Case $\xi = (\xi_1, \xi_2)$.

(2) $e \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$	by (1) and Definition **??**

By rule induction over Rules (4) on (2) and only case applies.

Case (4e).

(3) $e = (e_1, e_2)$	by assumption
(4) $e_1 \models \dot{\perp}(\xi_1)$	by assumption
(5) $e_2 \models \dot{\perp}(\xi_2)$	by assumption
(6) $e_1 \models \xi_1$	by IH on (4)
(7) $e_2 \models \xi_2$	by IH on (5)
(8) $e \models (\xi_1, \xi_2)$	by Rule (4e) on (6) and (7)

$\square$

Lemma 5.1.3. *Assume e $\texttt{val}$ and $\dot{\top}(\xi) = \xi$. Then $e \not\models \xi$ iff $e \models \bar{\xi}$.*

Theorem 5.2. $\xi_r \models \xi_{rs}$ *iff* $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\perp}(\xi_{rs})$.