1 Match Constraint Language

$$\dot{\xi} \ ::= \ \top \mid ? \mid \underline{n} \mid \texttt{inl}(\dot{\xi}) \mid \texttt{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$$

$\boxed{\dot{\xi} : \tau}$ $\dot{\xi}$ constrains final expressions of type τ

$$\text{CTTruth} \quad \frac{}{\top : \tau} \tag{1a}$$

$$\text{CTUnknown} \quad \frac{}{? : \tau} \tag{1b}$$

$$\text{CTNum} \quad \frac{}{\underline{n} : \texttt{num}} \tag{1c}$$

$$\text{CTInl} \quad \frac{\dot{\xi}_1 : \tau_1}{\texttt{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)} \tag{1d}$$

$$\text{CTInr} \quad \frac{\dot{\xi}_2 : \tau_2}{\texttt{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)} \tag{1e}$$

$$\text{CTPair} \quad \frac{\dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)} \tag{1f}$$

$$\text{CTOr} \quad \frac{\dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau} \tag{1g}$$

$\boxed{\dot{\xi} \ \texttt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\text{RXNum} \quad \frac{}{\underline{n} \ \texttt{refutable}_?} \tag{2a}$$

$$\text{RXUnknown} \quad \frac{}{? \ \texttt{refutable}_?} \tag{2b}$$

$$\text{RXInl} \quad \frac{}{\texttt{inl}(\dot{\xi}) \ \texttt{refutable}_?} \tag{2c}$$

$$\text{RXInr} \quad \frac{}{\texttt{inr}(\dot{\xi}) \ \texttt{refutable}_?} \tag{2d}$$

$$\text{RXPairL} \quad \frac{\dot{\xi}_1 \ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \ \texttt{refutable}_?} \tag{2e}$$

$$\frac{\dot\xi_2\ \texttt{refutable}_?}{(\dot\xi_1, \dot\xi_2)\ \texttt{refutable}_?}\ \text{RXPairR} \tag{2f}$$

$$\frac{\dot\xi_1\ \texttt{refutable}_? \qquad \dot\xi_2\ \texttt{refutable}_?}{\dot\xi_1 \vee \dot\xi_2\ \texttt{refutable}_?}\ \text{RXOr} \tag{2g}$$

$\boxed{\mathit{refutable}_?(\dot\xi)}$

$$\mathit{refutable}_?(\top) = \text{false} \tag{3a}$$
$$\mathit{refutable}_?(\underline{n}) = \text{true} \tag{3b}$$
$$\mathit{refutable}_?(?) = \text{true} \tag{3c}$$
$$\mathit{refutable}_?(\texttt{inl}(\dot\xi)) = \text{true} \tag{3d}$$
$$\mathit{refutable}_?(\texttt{inr}(\dot\xi)) = \text{true} \tag{3e}$$
$$\mathit{refutable}_?((\dot\xi_1, \dot\xi_2)) = \mathit{refutable}_?(\dot\xi_1) \text{ or } \mathit{refutable}_?(\dot\xi_2) \tag{3f}$$
$$\mathit{refutable}_?(\dot\xi_1 \vee \dot\xi_2) = \mathit{refutable}_?(\dot\xi_1) \text{ and } \mathit{refutable}_?(\dot\xi_2) \tag{3g}$$

Lemma 1.0.1 (Soundness and Completeness of Refutable Constraints). $\dot\xi$ $\texttt{refutable}_?$ *iff* $\mathit{refutable}_?(\dot\xi) = \mathit{true}$.

$\boxed{e \dot\models \dot\xi}$ $\quad e$ satisfies $\dot\xi$

$$\frac{}{e \dot\models \top}\ \text{CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \dot\models \underline{n}}\ \text{CSNum} \tag{4b}$$

$$\frac{e_1 \dot\models \dot\xi_1}{\texttt{inl}_{\tau_2}(e_1) \dot\models \texttt{inl}(\dot\xi_1)}\ \text{CSInl} \tag{4c}$$

$$\frac{e_2 \dot\models \dot\xi_2}{\texttt{inr}_{\tau_1}(e_2) \dot\models \texttt{inr}(\dot\xi_2)}\ \text{CSInr} \tag{4d}$$

$$\frac{e_1 \dot\models \dot\xi_1 \qquad e_2 \dot\models \dot\xi_2}{(e_1, e_2) \dot\models (\dot\xi_1, \dot\xi_2)}\ \text{CSPair} \tag{4e}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \dot\models \dot\xi_1 \qquad \texttt{snd}(e) \dot\models \dot\xi_2}{e \dot\models (\dot\xi_1, \dot\xi_2)}\ \text{CSNotIntroPair} \tag{4f}$$

$$\frac{e \dot{\models} \dot{\xi}_1}{e \dot{\models} \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrL} \tag{4g}$$

$$\frac{e \dot{\models} \dot{\xi}_2}{e \dot{\models} \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrR} \tag{4h}$$

$\boxed{\textit{satisfy}(e, \dot{\xi})}$

$$\textit{satisfy}(e, \top) = \text{true} \tag{5a}$$

$$\textit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{5b}$$

$$\textit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \textit{satisfy}(e, \dot{\xi}_1) \text{ or } \textit{satisfy}(e, \dot{\xi}_2) \tag{5c}$$

$$\textit{satisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \textit{satisfy}(e_1, \dot{\xi}_1) \tag{5d}$$

$$\textit{satisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \textit{satisfy}(e_2, \dot{\xi}_2) \tag{5e}$$

$$\textit{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \textit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \textit{satisfy}(e_2, \dot{\xi}_2) \tag{5f}$$

$$\textit{satisfy}(⦇⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \textit{satisfy}(\texttt{fst}(⦇⦈^u), \dot{\xi}_1) \text{ and } \textit{satisfy}(\texttt{snd}(⦇⦈^u), \dot{\xi}_2) \tag{5g}$$

$$\textit{satisfy}(⦇e⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \textit{satisfy}(\texttt{fst}(⦇e⦈^u), \dot{\xi}_1) \text{ and } \textit{satisfy}(\texttt{snd}(⦇e⦈^u), \dot{\xi}_2) \tag{5h}$$

$$\begin{aligned}\textit{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) &= \textit{satisfy}(\texttt{fst}(e_1(e_2)), \dot{\xi}_1) \\ &\quad \text{and } \textit{satisfy}(\texttt{snd}(e_1(e_2)), \dot{\xi}_2)\end{aligned} \tag{5i}$$

$$\begin{aligned}\textit{satisfy}(\texttt{match}(e)\{\hat{rs}\}, (\dot{\xi}_1, \dot{\xi}_2)) &= \textit{satisfy}(\texttt{fst}(\texttt{match}(e)\{\hat{rs}\}), \dot{\xi}_1) \\ &\quad \text{and } \textit{satisfy}(\texttt{snd}(\texttt{match}(e)\{\hat{rs}\}), \dot{\xi}_2)\end{aligned} \tag{5j}$$

$$\begin{aligned}\textit{satisfy}(\texttt{fst}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \textit{satisfy}(\texttt{fst}(\texttt{fst}(e)), \dot{\xi}_1) \\ &\quad \text{and } \textit{satisfy}(\texttt{snd}(\texttt{fst}(e)), \dot{\xi}_2)\end{aligned} \tag{5k}$$

$$\begin{aligned}\textit{satisfy}(\texttt{snd}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \textit{satisfy}(\texttt{fst}(\texttt{snd}(e)), \dot{\xi}_1) \\ &\quad \text{and } \textit{satisfy}(\texttt{snd}(\texttt{snd}(e)), \dot{\xi}_2)\end{aligned} \tag{5l}$$

$$\text{Otherwise} \quad \textit{satisfy}(e, \dot{\xi}) = \text{false} \tag{5m}$$

Lemma 1.0.2 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \dot{\models} \dot{\xi}$ iff $\textit{satisfy}(e, \dot{\xi}) = \textit{true}$.*

$\boxed{e \dot{\models}_? \dot{\xi}}$ $\quad$ e may satisfy $\dot{\xi}$

$$\frac{}{e \dot{\models}_? ?}\ \text{CMSUnknown} \tag{6a}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \dot{\models}_? \texttt{inl}(\dot{\xi}_1)}\ \text{CMSInl} \tag{6b}$$

$$\frac{e_2 \dot{\models}_? \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \dot{\models}_? \texttt{inr}(\dot{\xi}_2)}\ \text{CMSInr} \tag{6c}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1 \qquad e_2 \dot{\models} \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairL} \tag{6d}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1 \qquad e_2 \dot{\models}_? \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairR} \tag{6e}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1 \qquad e_2 \dot{\models}_? \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPair} \tag{6f}$$

$$\frac{e \dot{\models}_? \dot{\xi}_1 \qquad e\ \dot{\not\models}\ \dot{\xi}_2}{e \dot{\models}_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrL} \tag{6g}$$

$$\frac{e\ \dot{\not\models}\ \dot{\xi}_1 \qquad e \dot{\models}_? \dot{\xi}_2}{e \dot{\models}_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrR} \tag{6h}$$

$$\frac{e\ \texttt{notintro} \qquad \dot{\xi}\ \texttt{refutable}_?}{e \dot{\models}_? \dot{\xi}}\ \text{CMSNotIntro} \tag{6i}$$

$\boxed{\mathit{maysatisfy}(e, \dot{\xi})}$

$$\mathit{maysatisfy}(e, ?) = \text{true} \tag{7a}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \mathit{maysatisfy}(e_1, \dot{\xi}_1) \tag{7b}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \mathit{maysatisfy}(e_2, \dot{\xi}_2) \tag{7c}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inr}(\dot{\xi}_2)) = \text{false} \tag{7d}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inl}(\dot{\xi}_1)) = \text{false} \tag{7e}$$

$$\begin{aligned} \mathit{maysatisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big) \end{aligned} \tag{7f}$$

$$\begin{aligned} \mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = {} & \Big(\mathit{maysatisfy}(e, \dot{\xi}_1) \text{ and } \Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_2)\Big)\Big) \\ & \text{or } \Big(\Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_1)\Big) \text{ and } \mathit{maysatisfy}(e, \dot{\xi}_2)\Big) \end{aligned} \tag{7g}$$

$$\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{notintro}(e) \text{ and } \mathit{refutable}_?(\dot{\xi}) \tag{7h}$$

Lemma 1.0.3 (Soundness and Completeness of Maybe Satisfaction Judgment)**.** *$e \dot{\models}_? \dot{\xi}$ iff $maysatisfy(e, \dot{\xi}) = true$.*

$\boxed{e \dot{\models}^{\dagger}_? \dot{\xi}}$ e satisfies or may satisfy $\dot{\xi}$

$$\text{CSMSMay}\quad \frac{e \dot{\models}_? \dot{\xi}}{e \dot{\models}^{\dagger}_? \dot{\xi}} \tag{8a}$$

$$\text{CSMSSat}\quad \frac{e \dot{\models} \dot{\xi}}{e \dot{\models}^{\dagger}_? \dot{\xi}} \tag{8b}$$

$\boxed{satisfyormay(e, \dot{\xi})}$

$$satisfyormay(e, \dot{\xi}) = satisfy(e, \dot{\xi}) \text{ or } maysatisfy(e, \dot{\xi}) \tag{9}$$

Lemma 1.0.4 (Soundness and Completeness of Satisfaction or Maybe Satisfaction)**.** *$e \dot{\models}^{\dagger}_? \dot{\xi}$ iff $satisfyormay(e, \dot{\xi})$.*

Lemma 1.0.5. *If $\dot{\xi} : \tau$ then there exists e such that e $\texttt{val}$ and $\cdot ; \Delta \vdash e : \tau$ and $e \dot{\models}^{\dagger}_? \dot{\xi}$.*

Lemma 1.0.6. *$e \dot{\nvDash}_? \top$*

Lemma 1.0.7. *$e \dot{\nvDash} ?$*

Lemma 1.0.8. *$e \dot{\nvDash}^{\dagger}_? \dot{\xi}_1$ and $e \dot{\nvDash}^{\dagger}_? \dot{\xi}_2$ iff $e \dot{\nvDash}^{\dagger}_? \dot{\xi}_1 \vee \dot{\xi}_2$*

Lemma 1.0.9. *If $e \dot{\models}^{\dagger}_? \dot{\xi}_1 \vee \dot{\xi}_2$ and $e \dot{\nvDash}^{\dagger}_? \dot{\xi}_1$ then $e \dot{\models}^{\dagger}_? \dot{\xi}_2$*

Lemma 1.0.10. *If $e \dot{\models}^{\dagger}_? \dot{\xi}_1$ then $e \dot{\models}^{\dagger}_? \dot{\xi}_1 \vee \dot{\xi}_2$ and $e \dot{\models}^{\dagger}_? \dot{\xi}_2 \vee \dot{\xi}_1$*

Lemma 1.0.11. *$e_1 \dot{\models}^{\dagger}_? \dot{\xi}_1$ iff $\texttt{inl}_{\tau_2}(e_1) \dot{\models}^{\dagger}_? \texttt{inl}(\dot{\xi}_1)$*

Lemma 1.0.12. *$e_2 \dot{\models}^{\dagger}_? \dot{\xi}_2$ iff $\texttt{inr}_{\tau_1}(e_2) \dot{\models}^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$*

Lemma 1.0.13. *$e_1 \dot{\models}^{\dagger}_? \dot{\xi}_1$ and $e_2 \dot{\models}^{\dagger}_? \dot{\xi}_2$ iff $(e_1, e_2) \dot{\models}^{\dagger}_? (\dot{\xi}_1, \dot{\xi}_2)$*

Lemma 1.0.14. *Assume e $\texttt{notintro}$. If $e \dot{\models}_? \dot{\xi}$ or $e \dot{\nvDash} \dot{\xi}$ then $\dot{\xi}$ $\texttt{refutable}_?$.*

Lemma 1.0.15. *If e $\texttt{notintro}$ and $e \dot{\models} \dot{\xi}$ then $\cancel{\dot{\xi}\ \texttt{refutable}_?}$.*

Lemma 1.0.16. *$\texttt{inl}_{\tau_2}(e_1) \dot{\models}^{\dagger}_? \texttt{inr}(\dot{\xi}_2)$ is not derivable.*

Lemma 1.0.17. *$\texttt{inr}_{\tau_1}(e_2) \dot{\models}^{\dagger}_? \texttt{inl}(\dot{\xi}_1)$ is not derivable.*

Lemma 1.0.18. *$e \dot{\nvDash} \dot{\xi}$ and $e \dot{\nvDash}_? \dot{\xi}$ iff $e \dot{\nvDash}^{\dagger}_? \dot{\xi}$.*

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \dot{\models} \dot{\xi}$
2. $e \dot{\models}_{?} \dot{\xi}$
3. $e \; \dot{\not\models}{}^{\dagger}_{?} \dot{\xi}$

Definition 1.1.1 (Entailment of Constraints)**.** *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \dot{\models} \dot{\xi}_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e `val` we have $e \dot{\models}^{\dagger}_{?} \dot{\xi}_1$ implies $e \dot{\models} \dot{\xi}_2$*

Definition 1.1.2 (Potential Entailment of Constraints)**.** *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \dot{\models}^{\dagger}_{?} \dot{\xi}_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \dot{\models}^{\dagger}_{?} \dot{\xi}_1$ implies $e \dot{\models}^{\dagger}_{?} \dot{\xi}_2$*

Corollary 1.1.1. *Suppose that $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e `final`. Then $\top \dot{\models}^{\dagger}_{?} \dot{\xi}$ implies $e \dot{\models}^{\dagger}_{?} \dot{\xi}$*

2 Normal Match Constraint Language

$$\xi \;::=\; \top \mid \bot \mid \underline{n} \mid \underline{\cancel{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\text{CTTruth}\quad \frac{}{\top : \tau} \tag{10a}$$

$$\text{CTFalsity}\quad \frac{}{\bot : \tau} \tag{10b}$$

$$\text{CTNum}\quad \frac{}{\underline{n} : \texttt{num}} \tag{10c}$$

$$\text{CTNotNum}\quad \frac{}{\underline{\cancel{n}} : \texttt{num}} \tag{10d}$$

$$\text{CTAnd}\quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \tag{10e}$$

$$\text{CTOr}\quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{10f}$$

$$\text{CTInl}\quad \frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{10g}$$

$$\text{CTInr}\quad \frac{\xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)} \tag{10h}$$

$$\text{CTPair}\quad \frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \tag{10i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$
\begin{aligned}
\overline{\top} &= \bot \\
\overline{\bot} &= \top \\
\overline{\underline{n}} &= \cancel{\underline{n}} \\
\overline{\cancel{\underline{n}}} &= \underline{n} \\
\overline{\xi_1 \wedge \xi_2} &= \overline{\xi_1} \vee \overline{\xi_2} \\
\overline{\xi_1 \vee \xi_2} &= \overline{\xi_1} \wedge \overline{\xi_2} \\
\overline{\texttt{inl}(\xi_1)} &= \texttt{inl}(\overline{\xi_1}) \vee \texttt{inr}(\top) \\
\overline{\texttt{inr}(\xi_2)} &= \texttt{inr}(\overline{\xi_2}) \vee \texttt{inl}(\top) \\
\overline{(\xi_1, \xi_2)} &= (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \\
\overline{\overline{\xi}} &= \xi
\end{aligned}
$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\text{CSTruth}\quad \frac{}{e \models \top} \tag{12a}$$

$$\text{CSNum}\quad \frac{}{\underline{n} \models \underline{n}} \tag{12b}$$

$$\text{CSNotNum}\quad \frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}} \tag{12c}$$

$$\text{CSAnd}\quad \frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \tag{12d}$$

$$\text{CSOrL}\quad \frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2} \tag{12e}$$

$$\text{CSOrR}\quad \frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2} \tag{12f}$$

$$\text{CSInl}\quad \frac{e_1 \models \xi_1}{\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)} \tag{12g}$$

$$\text{CSInr}\quad \frac{e_2 \models \xi_2}{\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\xi_2)} \tag{12h}$$

$$\text{CSPair}\quad \frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \tag{12i}$$

Lemma 2.0.1. *Assume e $\mathtt{val}$. Then $e \not\models \xi$ iff $e \models \overline{\xi}$.*

Theorem 2.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e $\mathtt{val}$ then exactly one of the following holds*

1. $e \models \xi$
2. $e \models \overline{\xi}$

Lemma 2.1.1. *$e \models \overline{\xi_1} \vee \xi_2$ iff $e \models \xi_2$ whenever $e \models \xi_1$.*

Definition 2.1.1 (Entailment of Constraints)**.** *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot;\Delta \vdash e : \tau$ and e $\mathtt{val}$ we have $e \models \xi_1$ implies $e \models \xi_2$*

Corollary 2.1.1 (Material Entailment of Complete Constraint)**.** *$\xi_1 \models \xi_2$ iff $\top \models \overline{\xi_1} \vee \xi_2$.*

2.1 Relationship with Incomplete Constraint Language

Lemma 2.1.2. *Assume that e $\mathtt{val}$. Then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ iff $e \models \dot{\top}(\dot{\xi})$.*

Lemma 2.1.3. *$e \dot{\models} \dot{\xi}$ iff $e \models \dot{\bot}(\dot{\xi})$*

Lemma 2.1.4. *Suppose $\dot{\xi} : \tau$. Then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\mathtt{final}$ **iff** $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\mathtt{val}$.*

Theorem 2.2. *$\top \dot{\models}^{\dagger}_{?} \dot{\xi}$ iff $\top \models \dot{\top}(\dot{\xi})$.*

Theorem 2.3. *$\dot{\xi}_1 \dot{\models} \dot{\xi}_2$ iff $\dot{\top}(\dot{\xi}_1) \models \dot{\bot}(\dot{\xi}_2)$.*

3 Static Semantics

$$
\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & \lambda x : \tau.e \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
 & \mid & (\!|\,|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid (\!|\,|\!)^w \mid (\!|p|\!)^w_\tau
\end{array}
$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{13a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{13b}$$

$\boxed{\Gamma ; \Delta \vdash e : \tau}$ e is of type τ

$$\frac{}{\Gamma, x : \tau ; \Delta \vdash x : \tau}\ \text{TVar} \tag{14a}$$

$$\frac{}{\Gamma ; \Delta, u :: \tau \vdash (\!|\,|\!)^u : \tau}\ \text{TEHole} \tag{14b}$$

$$\frac{\Gamma ; \Delta, u :: \tau \vdash e : \tau'}{\Gamma ; \Delta, u :: \tau \vdash (\!|e|\!)^u : \tau}\ \text{THole} \tag{14c}$$

$$\frac{}{\Gamma ; \Delta \vdash \underline{n} : \texttt{num}}\ \text{TNum} \tag{14d}$$

$$\frac{\Gamma, x : \tau_1 ; \Delta \vdash e : \tau_2}{\Gamma ; \Delta \vdash \lambda x : \tau_1.e : (\tau_1 \to \tau_2)}\ \text{TLam} \tag{14e}$$

$$\frac{\Gamma ; \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash e_1(e_2) : \tau}\ \text{TAp} \tag{14f}$$

$$\frac{\Gamma ; \Delta \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)}\ \text{TPair} \tag{14g}$$

$$\frac{\Gamma ; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma ; \Delta \vdash \texttt{fst}(e) : \tau_1}\ \text{TFst} \tag{14h}$$

TSnd
$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \texttt{snd}(e) : \tau_2} \tag{14i}$$

TInl
$$\frac{\Gamma\,;\Delta \vdash e : \tau_1}{\Gamma\,;\Delta \vdash \texttt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \tag{14j}$$

TInr
$$\frac{\Gamma\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash \texttt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \tag{14k}$$

TMatchZPre
$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad \Gamma\,;\Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \dot{\models}^{\dagger}_{?} \xi}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \tag{14l}$$

TMatchNZPre
$$\frac{\begin{array}{c}\Gamma\,;\Delta \vdash e : \tau \\ \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \qquad \Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \\ \forall e'.e' \in \texttt{values}[\Delta](e) \Rightarrow e' \dot{\not\models}^{\dagger}_{?} \xi_{pre} \qquad \top \dot{\models}^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}\end{array}}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \tag{14m}$$

$\boxed{\Delta \vdash p : \tau[\xi] \dashv\!\mid \Gamma}$ p is assigned type τ and emits constraint ξ

PTVar
$$\frac{}{\cdot \vdash x : \tau[\top] \dashv\!\mid x : \tau} \tag{15a}$$

PTWild
$$\frac{}{\cdot \vdash \_ : \tau[\top] \dashv\!\mid \cdot} \tag{15b}$$

PTEHole
$$\frac{}{w :: \tau \vdash (\!|\,|\!)^{w} : \tau[?] \dashv\!\mid \cdot} \tag{15c}$$

PTHole
$$\frac{\Delta \vdash p : \tau[\xi] \dashv\!\mid \Gamma}{\Delta, w :: \tau' \vdash (\!|p|\!)^{w}_{\tau} : \tau'[?] \dashv\!\mid \Gamma} \tag{15d}$$

PTNum
$$\frac{}{\cdot \vdash \underline{n} : \texttt{num}[\underline{n}] \dashv\!\mid \cdot} \tag{15e}$$

PTInl
$$\frac{\Delta \vdash p : \tau_1[\xi] \dashv\!\mid \Gamma}{\Delta \vdash \texttt{inl}(p) : (\tau_1 + \tau_2)[\texttt{inl}(\xi)] \dashv\!\mid \Gamma} \tag{15f}$$

PTInr
$$\frac{\Delta \vdash p : \tau_2[\xi] \dashv\!\mid \Gamma}{\Delta \vdash \texttt{inr}(p) : (\tau_1 + \tau_2)[\texttt{inr}(\xi)] \dashv\!\mid \Gamma} \tag{15g}$$

PTPair
$$\frac{\Delta_1 \vdash p_1 : \tau_1[\xi_1] \dashv\!\mid \Gamma_1 \qquad \Delta_2 \vdash p_2 : \tau_2[\xi_2] \dashv\!\mid \Gamma_2}{\Delta_1 \uplus \Delta_2 \vdash (p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\!\mid \Gamma_1 \uplus \Gamma_2} \tag{15h}$$

$\boxed{\Gamma\,;\Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

CTRule
$$\frac{\Delta_p \vdash p : \tau[\xi] \dashv\!\mid \Gamma_p \qquad \Gamma \uplus \Gamma_p\,; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma\,; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \tag{16a}$$

$\boxed{\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

CTOneRules
$$\frac{\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \dot{\not\models} \xi_{pre}}{\Gamma\,; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \tag{17a}$$

CTRules
$$\frac{\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\,; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \dot{\not\models} \xi_{pre}}{\Gamma\,; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \tag{17b}$$

Lemma 3.0.1. *If* $\Delta \vdash p : \tau[\xi] \dashv\!\mid \Gamma$ *then* $\xi : \tau$.

Lemma 3.0.2. *If* $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Lemma 3.0.3. *If* $\cdot\,; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Lemma 3.0.4. *If* $\Gamma\,; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \dot{\not\models} \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma\,; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^{\diamond} : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Lemma 3.0.5 (Substitution)**.** *If* $\Gamma, x : \tau\,; \Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,; \Delta \vdash e : \tau$ *and* e `final` *then* $\Gamma\,; \Delta \vdash [e/x]e_0 : \tau_0$

Lemma 3.0.6 (Simultaneous Substitution)**.** *If* $\Gamma \uplus \Gamma'; \Delta \vdash e : \tau$ *and* $\Gamma; \Delta \vdash \theta : \Gamma'$ *then* $\Gamma\,; \Delta \vdash [\theta]e : \tau$

Lemma 3.0.7 (Substitution Typing)**.** *If* $e \rhd p \dashv\!\mid \theta$ *and* $\cdot\,; \Delta_e \vdash e : \tau$ *and* $\Delta \vdash p : \tau[\xi] \dashv\!\mid \Gamma$ *and all expressions in* θ *are final then* $\cdot\,; \Delta_e \vdash \theta : \Gamma$

Proof by induction on the derivation of $e \rhd p \dashv\!\mid \theta$.

Theorem 3.1 (Determinism)**.** *If* $\cdot\,; \Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

4 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\frac{}{\underline{n}\ \texttt{val}}\ \text{VNum} \tag{18a}$$

$$\frac{}{\lambda x : \tau . e\ \texttt{val}}\ \text{VLam} \tag{18b}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}}\ \text{VPair} \tag{18c}$$

$$\frac{e\ \texttt{val}}{\texttt{inl}_\tau(e)\ \texttt{val}}\ \text{VInl} \tag{18d}$$

$$\frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}}\ \text{VInr} \tag{18e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{}{(\!|\,|\!)^u\ \texttt{indet}}\ \text{IEHole} \tag{19a}$$

$$\frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}}\ \text{IHole} \tag{19b}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}}\ \text{IAp} \tag{19c}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairL} \tag{19d}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairR} \tag{19e}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPair} \tag{19f}$$

$$\frac{e\ \texttt{final}}{\texttt{fst}(e)\ \texttt{indet}}\ \text{IFst} \tag{19g}$$

$$\frac{e\ \texttt{final}}{\texttt{snd}(e)\ \texttt{indet}}\ \text{ISnd} \tag{19h}$$

$$\frac{e\ \texttt{indet}}{\texttt{inl}_\tau(e)\ \texttt{indet}}\ \text{IInL} \tag{19i}$$

$$\frac{e\ \texttt{indet}}{\texttt{inr}_\tau(e)\ \texttt{indet}}\ \text{IInR} \tag{19j}$$

$$\frac{e\ \texttt{final} \qquad e\ ?\ p_r}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \texttt{indet}}\ \text{IMatch} \tag{19k}$$

$\boxed{e\ \texttt{final}}$ e is final

$$\frac{e\ \texttt{val}}{e\ \texttt{final}}\ \text{FVal} \tag{20a}$$

$$\frac{e\ \texttt{indet}}{e\ \texttt{final}}\ \text{FIndet} \tag{20b}$$

Lemma 4.0.1. *If* $\texttt{inl}_\tau(e)\ \texttt{final}$ *then* $e\ \texttt{final}$.

Lemma 4.0.2. *If* $\texttt{inr}_\tau(e)\ \texttt{final}$ *then* $e\ \texttt{final}$.

Lemma 4.0.3. *If* $(e_1, e_2)\ \texttt{final}$ *then* $e_1\ \texttt{final}$ *and* $e_2\ \texttt{final}$.

$\boxed{e\ \texttt{notintro}}$ e cannot be a value syntactically

$$\frac{}{\llparenthesis\rrparenthesis^u\ \texttt{notintro}}\ \text{NVEHole} \tag{21a}$$

$$\frac{}{\llparenthesis e\rrparenthesis^u\ \texttt{notintro}}\ \text{NVHole} \tag{21b}$$

$$\frac{}{e_1(e_2)\ \texttt{notintro}}\ \text{NVAp} \tag{21c}$$

$$\frac{}{\texttt{match}(e)\{\hat{rs}\}\ \texttt{notintro}}\ \text{NVMatch} \tag{21d}$$

$$\frac{}{\texttt{fst}(e)\ \texttt{notintro}}\ \text{NVFst} \tag{21e}$$

$$\frac{}{\texttt{snd}(e)\ \texttt{notintro}}\ \text{NVSnd} \tag{21f}$$

$\boxed{notintro(e)}$

$$notintro(⦇⦈^u) = \text{true} \tag{22a}$$
$$notintro(⦇e⦈^u) = \text{true} \tag{22b}$$
$$notintro(e_1(e_2)) = \text{true} \tag{22c}$$
$$notintro(\texttt{match}(e)\{\hat{rs}\}) = \text{true} \tag{22d}$$
$$notintro(\texttt{fst}(e)) = \text{true} \tag{22e}$$
$$notintro(\texttt{snd}(e)) = \text{true} \tag{22f}$$
$$\text{Otherwise} \quad notintro(e) = \text{false} \tag{22g}$$

Lemma 4.0.4 (Soundness and Completeness of NotIntro Judgment)**.** *e $\texttt{notintro}$ iff $notintro(e)$.*

$\boxed{e' \in \texttt{values}[\Delta](e)}$ e' is one of the possible values of e

$$\text{IVVal}\;\frac{e\ \texttt{val} \qquad \cdot\,;\Delta \vdash e : \tau}{e \in \texttt{values}[\Delta](e)} \tag{23a}$$

$$\text{IVIndet}\;\frac{e\ \texttt{notintro} \qquad \cdot\,;\Delta \vdash e : \tau \qquad e'\ \texttt{val} \qquad \cdot\,;\Delta \vdash e' : \tau}{e' \in \texttt{values}[\Delta](e)} \tag{23b}$$

$$\text{IVInl}\;\frac{e' \in \texttt{values}[\Delta](e)}{\lambda x : \tau.e' \in \texttt{values}[\Delta](\lambda x : \tau.e)} \tag{23c}$$

$$\text{IVInl}\;\frac{e_1' \in \texttt{values}[\Delta](e_1)}{\texttt{inl}_{\tau_2}(e_1') \in \texttt{values}[\Delta](\texttt{inl}_{\tau_2}(e_1))} \tag{23d}$$

$$\text{IVInr}\;\frac{e_2' \in \texttt{values}[\Delta](e_2)}{\texttt{inr}_{\tau_1}(e_2') \in \texttt{values}[\Delta](\texttt{inr}_{\tau_1}(e_2))} \tag{23e}$$

$$\text{IVPair}\;\frac{e_1' \in \texttt{values}[\Delta](e_1) \qquad e_2' \in \texttt{values}[\Delta](e_2)}{(e_1', e_2') \in \texttt{values}[\Delta]((e_1, e_2))} \tag{23f}$$

Lemma 4.0.5. *If $e' \in \texttt{values}[\Delta](e)$ and $\cdot\,;\Delta \vdash e : \tau$ then $\cdot\,;\Delta \vdash e' : \tau$.*

Lemma 4.0.6. *If $e' \in \texttt{values}[\Delta](e)$ then e' $\texttt{val}$.*

Lemma 4.0.7. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ then there exists e' such that $e' \in \texttt{values}[\Delta](e)$.*

Lemma 4.0.8. *Assume e $\texttt{final}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$. Then $e \dot{\not\models}{}^{\dagger}_{?}\dot{\xi}$ iff $\forall e'.e' \in \texttt{values}[\Delta](e) \implies e' \dot{\not\models}{}^{\dagger}_{?}\dot{\xi}$.*

Lemma 4.0.9. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$ and there exists e' such that $e' \in \texttt{values}[\Delta](e)$ and $e' \dot{\models}^{\dagger}_{?} \dot{\xi}$ then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$.*

$\boxed{\Gamma\,;\Delta \vdash \theta : \Gamma\theta}$ θ is of type $\Gamma\theta$

$$\frac{}{\Gamma\,;\Delta \vdash \emptyset : \cdot}\ \text{STEmpty} \tag{24a}$$

$$\frac{\Gamma\,;\Delta \vdash \theta : \Gamma_\theta \qquad \Gamma\,;\Delta \vdash e : \tau}{\Gamma\,;\Delta \vdash \theta, x/e : \Gamma_\theta, x : \tau}\ \text{STExtend} \tag{24b}$$

$\boxed{p\ \texttt{refutable}_?}$ p is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RNum} \tag{25a}$$

$$\frac{}{(\!|\!)^w\ \texttt{refutable}_?}\ \text{REHole} \tag{25b}$$

$$\frac{}{(\!|p|\!)^w_\tau\ \texttt{refutable}_?}\ \text{RHole} \tag{25c}$$

$$\frac{}{\texttt{inl}(p)\ \texttt{refutable}_?}\ \text{RInl} \tag{25d}$$

$$\frac{}{\texttt{inr}(p)\ \texttt{refutable}_?}\ \text{RInr} \tag{25e}$$

$$\frac{p_1\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairL} \tag{25f}$$

$$\frac{p_2\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairR} \tag{25g}$$

$\boxed{e \rhd p \dashv\!\vert\, \theta}$ e matches p, emitting θ

$$\frac{}{e \rhd x \dashv\!\vert\, e/x}\ \text{MVar} \tag{26a}$$

$$\frac{}{e \rhd \_ \dashv\!\vert\, \cdot}\ \text{MWild} \tag{26b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashv\!\vert\, \cdot}\ \text{MNum} \tag{26c}$$

$$\frac{e_1 \rhd p_1 \dashv\!\vert\, \theta_1 \qquad e_2 \rhd p_2 \dashv\!\vert\, \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2}\ \text{MPair} \tag{26d}$$

$$\frac{e \rhd p \dashv\!\vert\, \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\vert\, \theta}\ \text{MInl} \tag{26e}$$

$$\frac{e \rhd p \dashv\!\vert\, \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\vert\, \theta}\ \text{MInr} \tag{26f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \rhd p_1 \dashv\!\vert\, \theta_1 \qquad \texttt{snd}(e) \rhd p_2 \dashv\!\vert\, \theta_2}{e \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{26g}$$

$\boxed{e \mathrel{?} p}$ e may match p

$$\frac{}{e \mathrel{?} \llparenthesis\rrparenthesis^w}\ \text{MMEHole} \tag{27a}$$

$$\frac{}{e \mathrel{?} \llparenthesis p \rrparenthesis^w_\tau}\ \text{MMHole} \tag{27b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}_?}{e \mathrel{?} p}\ \text{MMNotIntro} \tag{27c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\!\vert\, \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{27d}$$

$$\frac{e_1 \rhd p_1 \dashv\!\vert\, \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{27e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{27f}$$

$$\frac{e \mathrel{?} p}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)}\ \text{MMInl} \tag{27g}$$

$$\frac{e \mathrel{?} p}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)}\ \text{MMInr} \tag{27h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{28a}$$

NMPairL
$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)} \tag{28b}$$

NMPairR
$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)} \tag{28c}$$

NMConfL
$$\frac{}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)} \tag{28d}$$

NMConfR
$$\frac{}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{28e}$$

NMInl
$$\frac{e \perp p}{\texttt{inl}_\tau(e) \perp \texttt{inl}(p)} \tag{28f}$$

NMInr
$$\frac{e \perp p}{\texttt{inr}_\tau(e) \perp \texttt{inr}(p)} \tag{28g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

ITHole
$$\frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{29a}$$

ITApFun
$$\frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \tag{29b}$$

ITApArg
$$\frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')} \tag{29c}$$

ITAP
$$\frac{e_2\ \texttt{val}}{\lambda x : \tau.e_1(e_2) \mapsto [e_2/x]e_1} \tag{29d}$$

ITPairL
$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \tag{29e}$$

ITPairR
$$\frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \tag{29f}$$

ITFstPair
$$\frac{(e_1, e_2)\ \texttt{final}}{\texttt{fst}((e_1, e_2)) \mapsto e_1} \tag{29g}$$

$$\text{ITSndPair}\quad \frac{(e_1, e_2)\ \texttt{final}}{\texttt{snd}((e_1, e_2)) \mapsto e_2} \tag{29h}$$

$$\text{ITInl}\quad \frac{e \mapsto e'}{\texttt{inl}_\tau(e) \mapsto \texttt{inl}_\tau(e')} \tag{29i}$$

$$\text{ITInr}\quad \frac{e \mapsto e'}{\texttt{inr}_\tau(e) \mapsto \texttt{inr}_\tau(e')} \tag{29j}$$

$$\text{ITExpMatch}\quad \frac{e \mapsto e'}{\texttt{match}(e)\{\hat{rs}\} \mapsto \texttt{match}(e')\{\hat{rs}\}} \tag{29k}$$

$$\text{ITSuccMatch}\quad \frac{e\ \texttt{final} \qquad e \rhd p_r \dashv\!\vert\ \theta}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{29l}$$

$$\text{ITFailMatch}\quad \frac{e\ \texttt{final} \qquad e \perp p_r}{\texttt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \tag{29m}$$

Lemma 4.0.10. *If* $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{final}$ *then* e_1 $\texttt{final}$*.*

Lemma 4.0.11. *If* $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{final}$ *then* e_2 $\texttt{final}$*.*

Lemma 4.0.12. *If* (e_1, e_2) $\texttt{final}$ *then* e_1 $\texttt{final}$ *and* e_2 $\texttt{final}$*.*

Lemma 4.0.13. *There doesn't exist* $\underline{n}$ *such that* $\underline{n}$ $\texttt{notintro}$*.*

Lemma 4.0.14. *There doesn't exist* $\texttt{inl}_\tau(e)$ *such that* $\texttt{inl}_\tau(e)$ $\texttt{notintro}$*.*

Lemma 4.0.15. *There doesn't exist* $\texttt{inr}_\tau(e)$ *such that* $\texttt{inr}_\tau(e)$ $\texttt{notintro}$*.*

Lemma 4.0.16. *There doesn't exist* (e_1, e_2) *such that* (e_1, e_2) $\texttt{notintro}$*.*

Lemma 4.0.17. *If* e $\texttt{final}$ *and* e $\texttt{notintro}$ *then* e $\texttt{indet}$*.*

Lemma 4.0.18. *There doesn't exist such an expression* e *such that both* e $\texttt{val}$ *and* e $\texttt{indet}$*.*

Lemma 4.0.19. *There doesn't exist such an expression* e *such that both* e $\texttt{val}$ *and* e $\texttt{notintro}$*.*

Lemma 4.0.20 (Finality)**.** *There doesn't exist such an expression* e *such that both* e $\texttt{final}$ *and* $e \mapsto e'$ *for some* e'

Lemma 4.0.21 (Matching Determinism)**.** *If* e $\texttt{final}$ *and* $\cdot\,; \Delta_e \vdash e : \tau$ *and* $\Delta \vdash p : \tau[\xi] \dashv\!\vert\ \Gamma$ *then exactly one of the following holds*

1. $e \rhd p \dashv\!\vert\, \theta$ *for some* θ

2. $e \,?\, p$

3. $e \perp p$

Lemma 4.0.22 (Matching Coherence of Constraint)**.** *Suppose that* $\cdot\,;\Delta_e \vdash e : \tau$ *and* e `final` *and* $\Delta \vdash p : \tau[\xi] \dashv\!\vert\, \Gamma$*. Then we have*

1. $e \dot{\models} \xi$ *iff* $e \rhd p \dashv\!\vert\, \theta$

2. $e \dot{\models}_? \xi$ *iff* $e \,?\, p$

3. $e \dot{\not\models}{}^{\dagger}_{?} \xi$ *iff* $e \perp p$

5 Preservation and Progress

Theorem 5.1 (Preservation)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,;\Delta \vdash e' : \tau$

Theorem 5.2 (Progress)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then either* e `final` *or* $e \mapsto e'$ *for some* e'*.*

6 Decidability

$\boxed{\dot{\top}(\dot{\xi}) = \xi}$

$$\dot{\top}(\top) = \top \tag{30a}$$

$$\dot{\top}(?) = \top \tag{30b}$$

$$\dot{\top}(\underline{n}) = \underline{n} \tag{30c}$$

$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{30d}$$

$$\dot{\top}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\top}(\xi)) \tag{30e}$$

$$\dot{\top}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\top}(\xi)) \tag{30f}$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{30g}$$

$\boxed{\dot{\perp}(\dot{\xi}) = \xi}$

$$\dot{\perp}(\top) = \top \tag{31a}$$

$$\dot{\perp}(?) = \perp \tag{31b}$$

$$\dot{\perp}(\underline{n}) = \underline{n} \tag{31c}$$

$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \tag{31d}$$

$$\dot{\perp}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\perp}(\xi)) \tag{31e}$$

$$\dot{\perp}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\perp}(\xi)) \tag{31f}$$

$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \tag{31g}$$

$\boxed{\Xi\ \texttt{incon}}$ A finite set of constraints, Ξ, is inconsistent

$$\frac{\Xi\ \texttt{incon}}{\Xi, \top\ \texttt{incon}}\ \text{CINCTruth} \tag{32a}$$

$$\frac{}{\Xi, \bot\ \texttt{incon}}\ \text{CINCFalsity} \tag{32b}$$

$$\frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \texttt{incon}}\ \text{CINCNum} \tag{32c}$$

$$\frac{}{\Xi, \underline{n}, \cancel{\underline{n}}\ \texttt{incon}}\ \text{CINCNotNum} \tag{32d}$$

$$\frac{\Xi, \xi_1, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \texttt{incon}}\ \text{CINCAnd} \tag{32e}$$

$$\frac{\Xi, \xi_1\ \texttt{incon} \qquad \Xi, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \vee \xi_2\ \texttt{incon}}\ \text{CINCOr} \tag{32f}$$

$$\frac{}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2)\ \texttt{incon}}\ \text{CINCInj} \tag{32g}$$

$$\frac{\xi_1, \cdots, \xi_n\ \texttt{incon}}{\texttt{inl}(\xi_1), \cdots, \texttt{inl}(\xi_n)\ \texttt{incon}}\ \text{CINCInl} \tag{32h}$$

$$\frac{\xi_1, \cdots, \xi_n\ \texttt{incon}}{\texttt{inr}(\xi_1), \cdots, \texttt{inr}(\xi_n)\ \texttt{incon}}\ \text{CINCInr} \tag{32i}$$

$$\frac{\xi_{11}, \cdots, \xi_{n1}\ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2})\ \texttt{incon}}\ \text{CINCPairL} \tag{32j}$$

$$\frac{\xi_{12}, \cdots, \xi_{n2}\ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2})\ \texttt{incon}}\ \text{CINCPairR} \tag{32k}$$

Lemma 6.0.1 (Decidability of Inconsistency). *It is decidable whether* ξ $\texttt{incon}$.

Lemma 6.0.2 (Inconsistency and Entailment of Constraint). $\overline{\xi}$ $\texttt{incon}$ *iff* $\top \models \xi$

Theorem 6.1 (Decidability of Exhaustiveness). *It is decidable whether* $\top \dot{\models}^{\dagger}_{?} \dot{\xi}$.

Theorem 6.2 (Decidability of Redundancy). *It is decidable whether* $\dot{\xi}_1 \dot{\models} \dot{\xi}_2$.