In the main paper, we present only a single constraint language. However, conceptually, we work with this language in two distinct stages: first, the constraints directly emitted by lists of rules, then, for use in redundancy and exhaustiveness checking, the constraints which are in the image of the truify and falsify functions and their duals. While irrelevant to the overall theory, to simplify some proofs, it is useful to make this distinction explicit.

In Sec. 1, we present the first stage of constraints, called the *incomplete constraint language*. This consists of any constraint emitted by a pattern, and in particular, includes the ? constraint. In order to define the constraint emitted by a list of rules, we also include $\bot$ and allow taking the $\vee$ of incomplete constraints. At this stage, we often require two versions of each judgment: one describing a determinate result, and one describing a result which is indeterminate due to the presence of the ? constraint.

In turn, in Sec. 2, we discuss those constraints in the image of the truify and falsify functions, as well as their duals. We call this the *complete constraint language*, and it includes almost all of the incomplete language, but excludes the ? constraint. To support the dual operation, we also may take the $\wedge$ of complete constraints, and we add a $\underline{\not n}$ constraint. Due to the absence of ?, judgments related to the complete language do not have to consider indeterminacy, and thus are often simpler than their counterparts in the incomplete language. This is the primary motivation for distinguishing these languages at all.

Finally, we extend Peanut with finite labeled sums in Sec. 6, and discuss necessary changes to static semantics, dynmics semantics, as well as match constraint language with respect to what is presented in the main paper.

1 Incomplete Constraint Language

$\dot\xi \ ::= \ \top \mid \bot \mid ? \mid \underline{n} \mid \texttt{inl}(\dot\xi) \mid \texttt{inr}(\dot\xi) \mid (\dot\xi, \dot\xi) \mid \dot\xi \vee \dot\xi$

$\boxed{\dot\xi : \tau}$ $\quad \dot\xi$ constrains final expressions of type τ

$$\text{CTTruth}\quad \frac{}{\top : \tau} \tag{1a}$$

$$\text{CTFalsity}\quad \frac{}{\bot : \tau} \tag{1b}$$

$$\text{CTUnknown}\quad \frac{}{? : \tau} \tag{1c}$$

$$\text{CTNum}\quad \frac{}{\underline{n} : \texttt{num}} \tag{1d}$$

$$\text{CTInl}\quad \frac{\dot\xi_1 : \tau_1}{\texttt{inl}(\dot\xi_1) : (\tau_1 + \tau_2)} \tag{1e}$$

$$\frac{\dot{\xi}_2 : \tau_2}{\texttt{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)}\ \text{CTInr} \tag{1f}$$

$$\frac{\dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)}\ \text{CTPair} \tag{1g}$$

$$\frac{\dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau}\ \text{CTOr} \tag{1h}$$

$\boxed{\dot{\xi}\ \texttt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\frac{}{\bot\ \texttt{refutable}_?}\ \text{RXFalsity} \tag{2a}$$

$$\frac{}{?\ \texttt{refutable}_?}\ \text{RXUnknown} \tag{2b}$$

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RXNum} \tag{2c}$$

$$\frac{}{\texttt{inl}(\dot{\xi})\ \texttt{refutable}_?}\ \text{RXInl} \tag{2d}$$

$$\frac{}{\texttt{inr}(\dot{\xi})\ \texttt{refutable}_?}\ \text{RXInr} \tag{2e}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\ \text{RXPairL} \tag{2f}$$

$$\frac{\dot{\xi}_2\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\ \text{RXPairR} \tag{2g}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_? \qquad \dot{\xi}_2\ \texttt{refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}\ \text{RXOr} \tag{2h}$$

$\boxed{\dot{\xi}\ \texttt{possible}}$ $\dot{\xi}$ is possible

$$\frac{}{\top\ \texttt{possible}}\ \text{PTruth} \tag{3a}$$

$$\frac{}{?\ \texttt{possible}}\ \text{PUnknown} \tag{3b}$$

$$\frac{}{\underline{n}\ \texttt{possible}}\ \text{PNum} \tag{3c}$$

$$\frac{\dot{\xi}\ \texttt{possible}}{\texttt{inl}(\dot{\xi})\ \texttt{possible}}\ \text{PInl} \tag{3d}$$

$$\frac{\dot{\xi}\ \texttt{possible}}{\texttt{inr}(\dot{\xi})\ \texttt{possible}}\ \text{PInr} \tag{3e}$$

$$\frac{\dot{\xi}_1\ \texttt{possible} \qquad \dot{\xi}_2\ \texttt{possible}}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{possible}}\ \text{PPair} \tag{3f}$$

$$\frac{\dot{\xi}_1\ \texttt{possible}}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{possible}}\ \text{POrL} \tag{3g}$$

$$\frac{\dot{\xi}_2\ \texttt{possible}}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{possible}}\ \text{POrR} \tag{3h}$$

$\boxed{e \models \dot{\xi}}$ e satisfies $\dot{\xi}$

$$\frac{}{e \models \top}\ \text{CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \models \underline{n}}\ \text{CSNum} \tag{4b}$$

$$\frac{e_1 \models \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)}\ \text{CSInl} \tag{4c}$$

$$\frac{e_2 \models \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)}\ \text{CSInr} \tag{4d}$$

$$\frac{e_1 \models \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CSPair} \tag{4e}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \models \dot{\xi}_1 \qquad \texttt{snd}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CSNotIntroPair} \tag{4f}$$

$$\frac{e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrL} \tag{4g}$$

$$\frac{e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrR} \tag{4h}$$

$\boxed{e \models_? \dot{\xi}}$ e may satisfy $\dot{\xi}$

$$\frac{}{e \models_? ?}\ \text{CMSUnknown} \tag{5a}$$

$$\frac{e_1 \models_? \dot{\xi}_1}{\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)}\ \text{CMSInl} \tag{5b}$$

$$\frac{e_2 \models_? \dot{\xi}_2}{\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)}\ \text{CMSInr} \tag{5c}$$

$$\frac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairL} \tag{5d}$$

$$\frac{e_1 \models \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairR} \tag{5e}$$

$$\frac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPair} \tag{5f}$$

$$\frac{e \models_? \dot{\xi}_1 \qquad e \not\models \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrL} \tag{5g}$$

$$\frac{e \not\models \dot{\xi}_1 \qquad e \models_? \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrR} \tag{5h}$$

$$\frac{e\ \mathtt{notintro} \qquad \dot{\xi}\ \mathtt{refutable}_? \qquad \dot{\xi}\ \mathtt{possible}}{e \models_? \dot{\xi}}\ \text{CMSNotIntro} \tag{5i}$$

$\boxed{e \models_?^{\dagger} \dot{\xi}}$ e satisfies or may satisfy $\dot{\xi}$

$$\frac{e \models_? \dot{\xi}}{e \models_?^{\dagger} \dot{\xi}}\ \text{CSMSMay} \tag{6a}$$

$$\frac{e \models \dot{\xi}}{e \models_?^{\dagger} \dot{\xi}}\ \text{CSMSSat} \tag{6b}$$

Lemma 1.0.1. *Assume* e notintro. *If* $e \models_? \dot{\xi}$ *or* $e \not\models \dot{\xi}$ *then* $\dot{\xi}$ $\texttt{refutable}_?$.

Lemma 1.0.2. *If* e notintro *and* $e \models \dot{\xi}$ *then* $\cancel{\dot{\xi}\ \texttt{refutable}_?}$.

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If* $\dot{\xi} : \tau$ *and* $\cdot;\Delta \vdash e : \tau$ *and* e final *then exactly one of the following holds*

1. $e \models \dot{\xi}$
2. $e \models_? \dot{\xi}$
3. $e \not\models^{\dagger}_{?} \dot{\xi}$

Definition 1.1.1 (Entailment of Constraints)**.** *Suppose that* $\dot{\xi}_1 : \tau$ *and* $\dot{\xi}_2 : \tau$. *Then* $\dot{\xi}_1 \models \dot{\xi}_2$ *iff for all* e *such that* $\cdot\,;\Delta \vdash e : \tau$ *and* e val *we have* $e \models^{\dagger}_{?} \dot{\xi}_1$ *implies* $e \models \dot{\xi}_2$

Definition 1.1.2 (Potential Entailment of Constraints)**.** *Suppose that* $\dot{\xi}_1 : \tau$ *and* $\dot{\xi}_2 : \tau$. *Then* $\dot{\xi}_1 \models^{\dagger}_{?} \dot{\xi}_2$ *iff for all* e *such that* $\cdot\,;\Delta \vdash e : \tau$ *and* e final *we have* $e \models^{\dagger}_{?} \dot{\xi}_1$ *implies* $e \models^{\dagger}_{?} \dot{\xi}_2$

Corollary 1.1.1. *Suppose that* $\dot{\xi} : \tau$ *and* $\cdot;\Delta \vdash e : \tau$ *and* e final. *Then* $\top \models^{\dagger}_{?} \dot{\xi}$ *implies* $e \models^{\dagger}_{?} \dot{\xi}$

2 Complete Constraint Language

$\xi \ ::= \ \top \mid \bot \mid \underline{n} \mid \underline{\cancel{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\text{CTTruth}\quad \frac{}{\top : \tau} \tag{7a}$$

$$\text{CTFalsity}\quad \frac{}{\bot : \tau} \tag{7b}$$

$$\text{CTNum}\quad \frac{}{\underline{n} : \texttt{num}} \tag{7c}$$

$$\text{CTNotNum}\quad \frac{}{\underline{\cancel{n}} : \texttt{num}} \tag{7d}$$

$$\text{CTAnd}\quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \tag{7e}$$

$$\text{CTOr}\quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{7f}$$

$$\text{CTInl}\quad \frac{\xi_1 : \tau_1}{\mathtt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{7g}$$

$$\text{CTInr}\quad \frac{\xi_2 : \tau_2}{\mathtt{inr}(\xi_2) : (\tau_1 + \tau_2)} \tag{7h}$$

$$\text{CTPair}\quad \frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \tag{7i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\begin{aligned}
\overline{\top} &= \bot \\
\overline{\bot} &= \top \\
\overline{\underline{n}} &= \cancel{\underline{n}} \\
\overline{\cancel{\underline{n}}} &= \underline{n} \\
\overline{\xi_1 \wedge \xi_2} &= \overline{\xi_1} \vee \overline{\xi_2} \\
\overline{\xi_1 \vee \xi_2} &= \overline{\xi_1} \wedge \overline{\xi_2} \\
\overline{\mathtt{inl}(\xi_1)} &= \mathtt{inl}(\overline{\xi_1}) \vee \mathtt{inr}(\top) \\
\overline{\mathtt{inr}(\xi_2)} &= \mathtt{inr}(\overline{\xi_2}) \vee \mathtt{inl}(\top) \\
\overline{(\xi_1, \xi_2)} &= (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2})
\end{aligned}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\text{CSTruth}\quad \frac{}{e \models \top} \tag{9a}$$

$$\text{CSNum}\quad \frac{}{\underline{n} \models \underline{n}} \tag{9b}$$

$$\text{CSNotNum}\quad \frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}} \tag{9c}$$

$$\text{CSAnd}\quad \frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \tag{9d}$$

$$\text{CSOrL}\quad \frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2} \tag{9e}$$

$$\text{CSOrR}\quad \frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2} \tag{9f}$$

$$\frac{e_1 \models \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)}\ \text{CSInl} \tag{9g}$$

$$\frac{e_2 \models \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)}\ \text{CSInr} \tag{9h}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)}\ \text{CSPair} \tag{9i}$$

Lemma 2.0.1. *Assume e $\texttt{val}$. Then $e \not\models \xi$ iff $e \models \overline{\xi}$.*

Theorem 2.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e $\texttt{val}$ then exactly one of the following holds*

1. $e \models \xi$
2. $e \models \overline{\xi}$

Definition 2.1.1 (Entailment of Constraints)**.** *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot;\Delta \vdash e : \tau$ and e $\texttt{val}$ we have $e \models \xi_1$ implies $e \models \xi_2$*

Lemma 2.1.1 (Material Entailment of Complete Constraints)**.** *$\xi_1 \models \xi_2$ iff $\top \models \overline{\xi_1} \vee \xi_2$.*

2.1 Relationship with Incomplete Constraint Language

Lemma 2.1.2. *Assume that e $\texttt{val}$. Then $e \models_{?}^{\dagger} \dot{\xi}$ iff $e \models \dot{\top}(\dot{\xi})$.*

Lemma 2.1.3. *$e \models \dot{\xi}$ iff $e \models \dot{\bot}(\dot{\xi})$*

Lemma 2.1.4. *Suppose $\dot{\xi} : \tau$. Then $e \models_{?}^{\dagger} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\texttt{final}$ **iff** $e \models_{?}^{\dagger} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\texttt{val}$.*

Theorem 2.2. *$\top \models_{?}^{\dagger} \dot{\xi}$ iff $\top \models \dot{\top}(\dot{\xi})$.*

Theorem 2.3. *$\dot{\xi}_1 \models \dot{\xi}_2$ iff $\dot{\top}(\dot{\xi}_1) \models \dot{\bot}(\dot{\xi}_2)$.*

3 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\frac{}{\underline{n}\ \texttt{val}}\ \text{VNum} \tag{10a}$$

$$\frac{}{\lambda x : \tau.e\ \texttt{val}}\ \text{VLam} \tag{10b}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}}\ \text{VPair} \tag{10c}$$

$$\frac{e\ \texttt{val}}{\texttt{inl}_\tau(e)\ \texttt{val}}\ \text{VInl} \tag{10d}$$

$$\frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}}\ \text{VInr} \tag{10e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{}{(\!|\!)^u\ \texttt{indet}}\ \text{IEHole} \tag{11a}$$

$$\frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}}\ \text{IHole} \tag{11b}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}}\ \text{IAp} \tag{11c}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairL} \tag{11d}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairR} \tag{11e}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPair} \tag{11f}$$

$$\frac{e\ \texttt{final}}{\texttt{fst}(e)\ \texttt{indet}}\ \text{IFst} \tag{11g}$$

$$\frac{e\ \texttt{final}}{\texttt{snd}(e)\ \texttt{indet}}\ \text{ISnd} \tag{11h}$$

$$\frac{e\ \texttt{indet}}{\texttt{inl}_\tau(e)\ \texttt{indet}}\ \text{IInL} \tag{11i}$$

$$\frac{e\ \texttt{indet}}{\texttt{inr}_\tau(e)\ \texttt{indet}}\ \text{IInR} \tag{11j}$$

$$\frac{e\ \texttt{final} \qquad e\ ?\ p_r}{\texttt{match}(e)\ \{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \texttt{indet}}\ \text{IMatch} \tag{11k}$$

$\boxed{e\ \texttt{final}}$ e is final

$$\frac{e\ \texttt{val}}{e\ \texttt{final}}\ \text{FVal} \tag{12a}$$

$$\frac{e\ \texttt{indet}}{e\ \texttt{final}}\ \text{FIndet} \tag{12b}$$

$\boxed{e\ \texttt{notintro}}$ e cannot be a value syntactically

$$\frac{}{(\!|\!)^u\ \texttt{notintro}}\ \text{NVEHole} \tag{13a}$$

$$\frac{}{(\!|e|\!)^u\ \texttt{notintro}}\ \text{NVHole} \tag{13b}$$

$$\frac{}{e_1(e_2)\ \texttt{notintro}}\ \text{NVAp} \tag{13c}$$

$$\frac{}{\texttt{match}(e)\ \{\hat{rs}\}\ \texttt{notintro}}\ \text{NVMatch} \tag{13d}$$

$$\frac{}{\texttt{fst}(e)\ \texttt{notintro}}\ \text{NVFst} \tag{13e}$$

$$\frac{}{\texttt{snd}(e)\ \texttt{notintro}}\ \text{NVSnd} \tag{13f}$$

$\boxed{e' \in \texttt{values}[\Delta](e)}$ e' is one of the possible values of e

$$\frac{e\ \texttt{val} \qquad \cdot\,;\Delta \vdash e : \tau}{e \in \texttt{values}[\Delta](e)}\ \text{IVVal} \tag{14a}$$

$$\frac{e\ \texttt{notintro} \qquad \cdot\,;\Delta \vdash e : \tau \qquad e'\ \texttt{val} \qquad \cdot\,;\Delta \vdash e' : \tau}{e' \in \texttt{values}[\Delta](e)}\ \text{IVIndet} \tag{14b}$$

$$\frac{e'_1 \in \texttt{values}[\Delta](e_1)}{\texttt{inl}_{\tau_2}(e'_1) \in \texttt{values}[\Delta](\texttt{inl}_{\tau_2}(e_1))}\ \text{IVInl} \tag{14c}$$

$$\frac{e'_2 \in \texttt{values}[\Delta](e_2)}{\texttt{inr}_{\tau_1}(e'_2) \in \texttt{values}[\Delta](\texttt{inr}_{\tau_1}(e_2))}\ \text{IVInr} \tag{14d}$$

$$\frac{e'_1 \in \texttt{values}[\Delta](e_1) \qquad e'_2 \in \texttt{values}[\Delta](e_2)}{(e'_1, e'_2) \in \texttt{values}[\Delta]((e_1, e_2))}\ \text{IVPair} \tag{14e}$$

Lemma 3.0.1. *If $e' \in \texttt{values}[\Delta](e)$ and $\cdot\,;\Delta \vdash e : \tau$ then $\cdot\,;\Delta \vdash e' : \tau$.*

Lemma 3.0.2. *If $e' \in \texttt{values}[\Delta](e)$ then e' $\texttt{val}$.*

Lemma 3.0.3. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ then there exists e' such that $e' \in \texttt{values}[\Delta](e)$.*

Lemma 3.0.4. *Assume e $\texttt{final}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$. If $e \not\models^{\dagger}_{?} \dot{\xi}$ then and $e' \in \texttt{values}[\Delta](e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}$.*

Lemma 3.0.5. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$ and there exists e' such that $e' \in \texttt{values}[\Delta](e)$ and $e' \models^{\dagger}_{?} \dot{\xi}$ then $e \models^{\dagger}_{?} \dot{\xi}$.*

$\boxed{\Gamma\,;\Delta \vdash \theta : \Gamma\theta}$ θ is of type $\Gamma\theta$

$$\frac{}{\Gamma\,;\Delta \vdash \emptyset : \cdot}\ \text{STEmpty} \tag{15a}$$

$$\frac{\Gamma\,;\Delta \vdash \theta : \Gamma_\theta \qquad \Gamma\,;\Delta \vdash e : \tau}{\Gamma\,;\Delta \vdash \theta, x/e : \Gamma_\theta, x : \tau}\ \text{STExtend} \tag{15b}$$

$\boxed{p\ \texttt{refutable}_?}$ p is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RNum} \tag{16a}$$

$$\frac{}{(\!|\,|\!)^{w}\ \texttt{refutable}_?}\ \text{REHole} \tag{16b}$$

$$\frac{}{(\!|p|\!)^{w}_{\tau}\ \texttt{refutable}_?}\ \text{RHole} \tag{16c}$$

$$\frac{}{\texttt{inl}(p)\ \texttt{refutable}_?}\ \text{RInl} \tag{16d}$$

$$\frac{}{\texttt{inr}(p)\ \texttt{refutable}_?}\ \text{RInr} \tag{16e}$$

$$\frac{p_1\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairL} \tag{16f}$$

$$\frac{p_2\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairR} \tag{16g}$$

$\boxed{e \rhd p \dashv\!\mid \theta}$ e matches p, emitting θ

$$\frac{}{e \rhd x \dashv\!\mid e/x}\ \text{MVar} \tag{17a}$$

$$\frac{}{e \rhd \_ \dashv\!\mid \cdot}\ \text{MWild} \tag{17b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashv\!\mid \cdot}\ \text{MNum} \tag{17c}$$

$$\frac{e_1 \rhd p_1 \dashv\!\mid \theta_1 \qquad e_2 \rhd p_2 \dashv\!\mid \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2}\ \text{MPair} \tag{17d}$$

$$\frac{e \rhd p \dashv\!\mid \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\mid \theta}\ \text{MInl} \tag{17e}$$

$$\frac{e \rhd p \dashv\!\mid \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\mid \theta}\ \text{MInr} \tag{17f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \rhd p_1 \dashv\!\mid \theta_1 \qquad \texttt{snd}(e) \rhd p_2 \dashv\!\mid \theta_2}{e \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{17g}$$

$\boxed{e\ ?\ p}$ e may match p

$$\frac{}{e\ ?\ (\!|\,|\!)^w}\ \text{MMEHole} \tag{18a}$$

$$\frac{}{e\ ?\ (\!|p|\!)^w_\tau}\ \text{MMHole} \tag{18b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}_?}{e\ ?\ p}\ \text{MMNotIntro} \tag{18c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\!\!| \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{18d}$$

$$\frac{e_1 \rhd p_1 \dashv\!\!| \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{18e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{18f}$$

$$\frac{e \mathrel{?} p}{\mathtt{inl}_\tau(e) \mathrel{?} \mathtt{inl}(p)}\ \text{MMInl} \tag{18g}$$

$$\frac{e \mathrel{?} p}{\mathtt{inr}_\tau(e) \mathrel{?} \mathtt{inr}(p)}\ \text{MMInr} \tag{18h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{19a}$$

$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairL} \tag{19b}$$

$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairR} \tag{19c}$$

$$\frac{}{\mathtt{inr}_\tau(e) \perp \mathtt{inl}(p)}\ \text{NMConfL} \tag{19d}$$

$$\frac{}{\mathtt{inl}_\tau(e) \perp \mathtt{inr}(p)}\ \text{NMConfR} \tag{19e}$$

$$\frac{e \perp p}{\mathtt{inl}_\tau(e) \perp \mathtt{inl}(p)}\ \text{NMInl} \tag{19f}$$

$$\frac{e \perp p}{\mathtt{inr}_\tau(e) \perp \mathtt{inr}(p)}\ \text{NMInr} \tag{19g}$$

$\boxed{(\hat{rs})^\diamond = rs}$ $\quad rs$ can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{20a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{20b}$$

$\boxed{e \mapsto e'}$ $\quad e$ takes a step to e'

$$\text{ITHole}\quad \frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{21a}$$

$$\text{ITApFun}\quad \frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \tag{21b}$$

$$\text{ITApArg}\quad \frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')} \tag{21c}$$

$$\text{ITAp}\quad \frac{e_2\ \texttt{val}}{\lambda x : \tau . e_1(e_2) \mapsto [e_2/x]e_1} \tag{21d}$$

$$\text{ITPairL}\quad \frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \tag{21e}$$

$$\text{ITPairR}\quad \frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \tag{21f}$$

$$\text{ITFstPair}\quad \frac{(e_1, e_2)\ \texttt{final}}{\texttt{fst}((e_1, e_2)) \mapsto e_1} \tag{21g}$$

$$\text{ITSndPair}\quad \frac{(e_1, e_2)\ \texttt{final}}{\texttt{snd}((e_1, e_2)) \mapsto e_2} \tag{21h}$$

$$\text{ITInl}\quad \frac{e \mapsto e'}{\texttt{inl}_\tau(e) \mapsto \texttt{inl}_\tau(e')} \tag{21i}$$

$$\text{ITInr}\quad \frac{e \mapsto e'}{\texttt{inr}_\tau(e) \mapsto \texttt{inr}_\tau(e')} \tag{21j}$$

$$\text{ITExpMatch}\quad \frac{e \mapsto e'}{\texttt{match}(e)\ \{\hat{rs}\} \mapsto \texttt{match}(e')\ \{\hat{rs}\}} \tag{21k}$$

ITSuccMatch

$$\frac{e\ \texttt{final} \qquad e \rhd p_r \dashv\!\vert\, \theta}{\texttt{match}(e)\ \{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{21l}$$

ITFailMatch

$$\frac{e\ \texttt{final} \qquad e \perp p_r}{\texttt{match}(e)\ \{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\ \{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^{\diamond} \mid r' \mid rs'\}} \tag{21m}$$

Lemma 3.0.6. *If* e $\texttt{final}$ *and* e $\texttt{notintro}$ *then* e $\texttt{indet}$*.*

Lemma 3.0.7. *There doesn't exist such an expression* e *such that both* e $\texttt{val}$ *and* e $\texttt{indet}$*.*

Lemma 3.0.8. *There doesn't exist such an expression* e *such that both* e $\texttt{val}$ *and* e $\texttt{notintro}$*.*

Lemma 3.0.9 (Finality)**.** *There doesn't exist such an expression* e *such that both* e $\texttt{final}$ *and* $e \mapsto e'$ *for some* e'

Lemma 3.0.10 (Matching Determinism)**.** *If* e $\texttt{final}$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $\Delta \vdash p : \tau[\xi] \dashv\!\vert\, \Gamma$ *then exactly one of the following holds*

1. $e \rhd p \dashv\!\vert\, \theta$ *for some* θ
2. $e \mathrel{?} p$
3. $e \perp p$

Lemma 3.0.11 (Matching Coherence of Constraint)**.** *Suppose that* $\cdot;\Delta_e \vdash e : \tau$ *and* e $\texttt{final}$ *and* $\Delta \vdash p : \tau[\xi] \dashv\!\vert\, \Gamma$*. Then we have*

1. $e \models \xi$ *iff* $e \rhd p \dashv\!\vert\, \theta$
2. $e \models_? \xi$ *iff* $e \mathrel{?} p$
3. $e \not\models^{\dagger}_{?} \xi$ *iff* $e \perp p$

4 Static Semantics

$$\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & \lambda x : \tau . e \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\ \{\hat{rs}\} \\
 & \mid & (\!|\,|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid _ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid (\!|\,|\!)^w \mid (\!|p|\!)^w_\tau
\end{array}$$

$\boxed{\Gamma\,;\Delta \vdash e : \tau}$ $\quad e$ is of type τ

$$\text{TVar}\quad \frac{}{\Gamma, x : \tau\,; \Delta \vdash x : \tau} \tag{22a}$$

$$\text{TEHole}\quad \frac{}{\Gamma\,; \Delta, u :: \tau \vdash (\!|\!)^{u} : \tau} \tag{22b}$$

$$\text{THole}\quad \frac{\Gamma\,; \Delta, u :: \tau \vdash e : \tau'}{\Gamma\,; \Delta, u :: \tau \vdash (\!|e|\!)^{u} : \tau} \tag{22c}$$

$$\text{TNum}\quad \frac{}{\Gamma\,; \Delta \vdash \underline{n} : \texttt{num}} \tag{22d}$$

$$\text{TLam}\quad \frac{\Gamma, x : \tau_1\,; \Delta \vdash e : \tau_2}{\Gamma\,; \Delta \vdash \lambda x : \tau_1.e : (\tau_1 \to \tau_2)} \tag{22e}$$

$$\text{TAp}\quad \frac{\Gamma\,; \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma\,; \Delta \vdash e_2 : \tau_2}{\Gamma\,; \Delta \vdash e_1(e_2) : \tau} \tag{22f}$$

$$\text{TPair}\quad \frac{\Gamma\,; \Delta \vdash e_1 : \tau_1 \qquad \Gamma\,; \Delta \vdash e_2 : \tau_2}{\Gamma\,; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \tag{22g}$$

$$\text{TFst}\quad \frac{\Gamma\,; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,; \Delta \vdash \texttt{fst}(e) : \tau_1} \tag{22h}$$

$$\text{TSnd}\quad \frac{\Gamma\,; \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,; \Delta \vdash \texttt{snd}(e) : \tau_2} \tag{22i}$$

$$\text{TInl}\quad \frac{\Gamma\,; \Delta \vdash e : \tau_1}{\Gamma\,; \Delta \vdash \texttt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \tag{22j}$$

$$\text{TInr}\quad \frac{\Gamma\,; \Delta \vdash e : \tau_2}{\Gamma\,; \Delta \vdash \texttt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \tag{22k}$$

$$\text{TMatchZPre}\quad \frac{\Gamma\,; \Delta \vdash e : \tau \qquad \Gamma\,; \Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \models^{\dagger}_{?} \xi}{\Gamma\,; \Delta \vdash \texttt{match}(e)\ \{\cdot \mid r \mid rs\} : \tau'} \tag{22l}$$

$$\text{TMatchNZPre}\quad \frac{\begin{array}{c}\Gamma\,; \Delta \vdash e : \tau \\ \Gamma\,; \Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \qquad \Gamma\,; \Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \\ \forall e'.e' \in \texttt{values}[\Delta](e) \Rightarrow e' \not\models^{\dagger}_{?} \xi_{pre} \qquad \top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}\end{array}}{\Gamma\,; \Delta \vdash \texttt{match}(e)\ \{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \tag{22m}$$

$\boxed{\Delta \vdash p : \tau[\xi] \dashv\!\vdash \Gamma}$ p is assigned type τ and emits constraint ξ

$$\frac{}{\cdot \vdash x : \tau[\top] \dashv\!\vdash x : \tau}\ \text{PTVar} \tag{23a}$$

$$\frac{}{\cdot \vdash \_ : \tau[\top] \dashv\!\vdash \cdot}\ \text{PTWild} \tag{23b}$$

$$\frac{}{w :: \tau \vdash (\!|\!)^{w} : \tau[?] \dashv\!\vdash \cdot}\ \text{PTEHole} \tag{23c}$$

$$\frac{\Delta \vdash p : \tau[\xi] \dashv\!\vdash \Gamma}{\Delta, w :: \tau' \vdash (\!|p|\!)^{w}_{\tau} : \tau'[?] \dashv\!\vdash \Gamma}\ \text{PTHole} \tag{23d}$$

$$\frac{}{\cdot \vdash \underline{n} : \texttt{num}[\underline{n}] \dashv\!\vdash \cdot}\ \text{PTNum} \tag{23e}$$

$$\frac{\Delta \vdash p : \tau_1[\xi] \dashv\!\vdash \Gamma}{\Delta \vdash \texttt{inl}(p) : (\tau_1 + \tau_2)[\texttt{inl}(\xi)] \dashv\!\vdash \Gamma}\ \text{PTInl} \tag{23f}$$

$$\frac{\Delta \vdash p : \tau_2[\xi] \dashv\!\vdash \Gamma}{\Delta \vdash \texttt{inr}(p) : (\tau_1 + \tau_2)[\texttt{inr}(\xi)] \dashv\!\vdash \Gamma}\ \text{PTInr} \tag{23g}$$

$$\frac{\Delta_1 \vdash p_1 : \tau_1[\xi_1] \dashv\!\vdash \Gamma_1 \qquad \Delta_2 \vdash p_2 : \tau_2[\xi_2] \dashv\!\vdash \Gamma_2}{\Delta_1 \uplus \Delta_2 \vdash (p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\!\vdash \Gamma_1 \uplus \Gamma_2}\ \text{PTPair} \tag{23h}$$

$\boxed{\Gamma\, ; \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

$$\frac{\Delta_p \vdash p : \tau[\xi] \dashv\!\vdash \Gamma_p \qquad \Gamma \uplus \Gamma_p\, ; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma\, ; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'}\ \text{CTRule} \tag{24a}$$

$\boxed{\Gamma\, ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

$$\frac{\Gamma\, ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\, ; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'}\ \text{CTOneRules} \tag{25a}$$

$$\frac{\Gamma\, ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\, ; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma\, ; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'}\ \text{CTRules} \tag{25b}$$

Lemma 4.0.1. *If* $\Delta \vdash p : \tau[\xi] \dashv\!\vdash \Gamma$ *then* $\xi : \tau$.

Lemma 4.0.2. *If* $\cdot\, ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Lemma 4.0.3. *If* $\cdot\,;\Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Lemma 4.0.4. *If* $\Gamma\,;\Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,;\Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ *then* $\Gamma\,;\Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^{\diamond} : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$

Lemma 4.0.5 (Substitution)**.** *If* $\Gamma, x : \tau\,;\Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta \vdash e : \tau$ *and* e `final` *then* $\Gamma\,;\Delta \vdash [e/x]e_0 : \tau_0$

Lemma 4.0.6 (Simultaneous Substitution)**.** *If* $\Gamma \uplus \Gamma';\Delta \vdash e : \tau$ *and* $\Gamma;\Delta \vdash \theta : \Gamma'$ *then* $\Gamma\,;\Delta \vdash [\theta]e : \tau$

Lemma 4.0.7 (Substitution Typing)**.** *If* $e \rhd p \dashv\!\mid \theta$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $\Delta \vdash p : \tau[\xi] \dashv\!\mid \Gamma$ *and all expressions in* θ *are final then* $\cdot\,;\Delta_e \vdash \theta : \Gamma$

Theorem 4.1 (Preservation)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,;\Delta \vdash e' : \tau$

Theorem 4.2 (Determinism)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

5 Decidability

$\boxed{\dot{\top}(\dot{\xi}) = \xi}$

$$\dot{\top}(\top) = \top \tag{26a}$$
$$\dot{\top}(?) = \top \tag{26b}$$
$$\dot{\top}(\underline{n}) = \underline{n} \tag{26c}$$
$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{26d}$$
$$\dot{\top}(\texttt{inl}(\xi)) = \texttt{inl}(\dot{\top}(\xi)) \tag{26e}$$
$$\dot{\top}(\texttt{inr}(\xi)) = \texttt{inr}(\dot{\top}(\xi)) \tag{26f}$$
$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{26g}$$

$\boxed{\dot{\bot}(\dot{\xi}) = \xi}$

$$\dot{\bot}(\top) = \top \tag{27a}$$

$$\dot{\bot}(?) = \bot \tag{27b}$$

$$\dot{\bot}(\underline{n}) = \underline{n} \tag{27c}$$

$$\dot{\bot}(\xi_1 \vee \xi_2) = \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2) \tag{27d}$$

$$\dot{\bot}(\texttt{inl}(\xi)) = \texttt{inl}(\dot{\bot}(\xi)) \tag{27e}$$

$$\dot{\bot}(\texttt{inr}(\xi)) = \texttt{inr}(\dot{\bot}(\xi)) \tag{27f}$$

$$\dot{\bot}((\xi_1, \xi_2)) = (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \tag{27g}$$

$\boxed{\Xi\ \texttt{incon}}$ A finite set of constraints, Ξ, is inconsistent

CINCTruth

$$\frac{\Xi\ \texttt{incon}}{\Xi, \top\ \texttt{incon}} \tag{28a}$$

CINCFalsity

$$\frac{}{\Xi, \bot\ \texttt{incon}} \tag{28b}$$

CINCNum

$$\frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \texttt{incon}} \tag{28c}$$

CINCNotNum

$$\frac{}{\Xi, \underline{n}, \not{\underline{n}}\ \texttt{incon}} \tag{28d}$$

CINCAnd

$$\frac{\Xi, \xi_1, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \texttt{incon}} \tag{28e}$$

CINCOr

$$\frac{\Xi, \xi_1\ \texttt{incon} \qquad \Xi, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \vee \xi_2\ \texttt{incon}} \tag{28f}$$

CINCInj

$$\frac{}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2)\ \texttt{incon}} \tag{28g}$$

CINCInl

$$\frac{\xi_1, \cdots, \xi_n\ \texttt{incon}}{\texttt{inl}(\xi_1), \cdots, \texttt{inl}(\xi_n)\ \texttt{incon}} \tag{28h}$$

CINCInr

$$\frac{\xi_1, \cdots, \xi_n\ \texttt{incon}}{\texttt{inr}(\xi_1), \cdots, \texttt{inr}(\xi_n)\ \texttt{incon}} \tag{28i}$$

CINCPairL

$$\frac{\xi_{11}, \cdots, \xi_{n1}\ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2})\ \texttt{incon}} \tag{28j}$$

$$\frac{\xi_{12}, \cdots, \xi_{n2} \ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2}) \ \texttt{incon}} \ \text{CINCPairR} \tag{28k}$$

Lemma 5.0.1 (Decidability of Inconsistency). *It is decidable whether* ξ `incon`.

Lemma 5.0.2 (Inconsistency and Entailment of Constraint). $\overline{\xi}$ `incon` *iff* $\top \models \xi$

Theorem 5.1 (Decidability of Exhaustiveness). *It is decidable whether* $\top \models_{?}^{\dagger} \dot{\xi}$.

Theorem 5.2 (Decidability of Redundancy). *It is decidable whether* $\dot{\xi}_1 \models \dot{\xi}_2$.

6 Labeled Sums

In this section, we conservatively extend Peanut with finite labeled sums, which is a practical necessity in general-purpose functional languages.

The first step is to generalize the syntax of binary sums to more than two variants:

$$\begin{array}{lcl} \tau & ::= & \ldots \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ e & ::= & \ldots \mid \mathtt{inj}^{\tau}_{C}(e) \\ p & ::= & \ldots \mid \mathtt{inj}_{C}(p) \\ C & ::= & \underline{\mathbf{C}} \mid (\!|\,|\!)^u \mid (\!|\underline{\mathbf{C}}|\!)^u \end{array}$$

Labeled sums introduce a new sort C for labels, a.k.a. datatype constructors, and a new type-level connective $C_1(\tau_1) + \cdots + C_n(\tau_n)$ for gathering and labeling types. Since we are not usually concerned with the length of any particular sum, we adopt a slightly more general and compact notation $+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}$ for the sum consisting of labels $\mathcal{C} = \{C_i\}_{i \leq n}$ with respective argument types $\{\tau_i\}_{i \leq n}$. The introduction form is the *labeled injection expression* $\mathtt{inj}^{\tau}_{C}(e)$ for injecting expression e into sum τ at label C. The elimination form is the *labeled injection pattern* $\mathtt{inj}^{\tau}_{C}(p)$ for expressions matching pattern p that have been injected into sum τ at label C.

To ensure maximal liveness, we distinguish *concrete labels* $\underline{\mathbf{C}}$ from *label holes* which are either empty $(\!|\,|\!)^u$ or not empty $(\!|\underline{\mathbf{C}}|\!)^u$. Empty holes arise where labels have yet to be constructed, for example during incremental construction of a sum or injection. Non-empty holes operate as membranes around labels that are syntactically malformed or that violate a semantic constraint. Non-empty holes around syntactically valid labels indicate duplication for sum type declarations and non-membership for injections.

Moving on to the dynamic semantics, Figure 1 and 2 extend Peanut's dynamic semantics to support evaluation of labeled injections. The rules in Figure 1 define pattern matching on injections with label holes, subject to the matching determinism conditions imposed by Lemma 3.0.10. Rule MInj is a straightforward rule for matching labeled injections. Rules MMInjTag and MMInjArg allow indeterminate matching against a pattern whose label or argument indeterminately match, respectively. Rule NMInj forbids matching of concrete but unequal labels, and Rules NMInjTag and NMInjArg forbid matching when the arguments do not match. Rules TMMSym, TMMHole, and TMMEHole define indeterminate matching of labels by establishing a partial equivalence among distinct labels when at least one of them is a hole. Figure 2 defines stepping of injection expressions, subject to the determinism conditions imposed by Lemma 3.0.7, 3.0.8 and 3.0.9. These rules are direct adaptations of the corresponding rules (21, 10, and 11) for binary injections.

Moving on to the static semantics, we first extend the typing relations as shown in Figure 3 by generalizing the rules for binary injections from Rules 22, 23, 24, and 25. The rules shown in Figure 3 are essentially the same as their binary counterparts, differing in that instead of assuming all sums contain precisely two fixed labels, we must check whether label C_j with argument type τ_j belongs to the annotated sum type τ.

$\boxed{e \rhd p \dashv\!\!\vdash \theta}$

e matches p, emitting θ

$$\frac{e \rhd p \dashv\!\!\vdash \theta}{\texttt{inj}^{\tau}_{C}(e) \rhd \texttt{inj}_{C}(p) \dashv\!\!\vdash \theta}\ \text{MInj}$$

$\boxed{e ? p}$

e indeterminately matches p

$$\frac{C ? C' \qquad e \not\perp p}{\texttt{inj}^{\tau}_{C}(e) ? \texttt{inj}_{C'}(p)}\ \text{MMInjTag}$$

$$\frac{e ? p}{\texttt{inj}^{\tau}_{C}(e) ? \texttt{inj}_{C}(p)}\ \text{MMInjArg}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{\underline{\mathbf{C}} \neq \underline{\mathbf{C}}'}{\texttt{inj}^{\tau}_{\underline{\mathbf{C}}}(e) \perp \texttt{inj}_{\underline{\mathbf{C}}'}(p)}\ \text{NMInj}$$

$$\frac{C ? C' \qquad e \perp p}{\texttt{inj}^{\tau}_{C}(e) \perp \texttt{inj}_{C'}(p)}\ \text{NMInjTag}$$

$$\frac{e \perp p}{\texttt{inj}^{\tau}_{C}(e) \perp \texttt{inj}_{C}(p)}\ \text{NMInjArg}$$

$\boxed{C ? C'}$ C indeterminately matches C'

$$\frac{C' ? C}{C ? C'}\ \text{TMMSym} \qquad \frac{\llparenthesis\rrparenthesis^{u} \neq C}{\llparenthesis\rrparenthesis^{u} ? C}\ \text{TMMHole} \qquad \frac{\underline{\mathbf{C}} \neq C}{\llparenthesis\underline{\mathbf{C}}\rrparenthesis^{u} ? C}\ \text{TMMEHole}$$

Figure 1: Pattern matching rules for labeled sums

Next, we extend the syntax of constraints to labeled sums:

$$\xi \ ::= \ \ldots \mid \texttt{inj}^{\tau}_{C}(\xi)$$

Figure 4 extends Peanut's constraint checking semantics to support matching on injections into labeled sums. A constraint of the form $\texttt{inj}^{\tau}_{C}(\xi)$ represents injections into sum τ with label C with arguments constrained by ξ. Rule CTInj decides which sum type is constrained by an injection constraint. Rules CSInj, CMSInjTag, and CMSInjArg specify whether and to what extent injections satisfy injection constraints, subject to the coherence conditions imposed by Theorem 1.1 and Lemma 3.0.11. That is, an injection satisfies a constraint only when their labels are equal and the argument expression satisfies the argument constraint.

Finally, we extend the inconsistency relations as shown in Figure 5 and 6. Rule RInjMult establishes that constraints for nontrivial sums are refutable, i.e., that the set of dual constraints is not empty. The dual of a constraint with a concrete label consists of injections of the same label with dual arguments as well as injections of other labels into the sum. A constraint with a label hole is dual to all injections into the sum with concrete labels. The premises of Rule RInjSing merely express the fact that there are no alternatives injections into singleton sums, and none at all into void sums. Rule PTInj establishes that

$\boxed{e \mapsto e'}$ e takes a step to e'

$$\frac{e \mapsto e'}{\texttt{inj}^{\tau}_{C}(e) \mapsto \texttt{inj}^{\tau}_{C}(e')}\ \text{ITInj}$$

$\boxed{e\ \texttt{val}}$ e is a value

$$\frac{e\ \texttt{val}}{\texttt{inj}^{\tau}_{\underline{\mathbf{C}}}(e)\ \texttt{val}}\ \text{VInj}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{e\ \texttt{indet}}{\texttt{inj}^{\tau}_{\underline{\mathbf{C}}}(e)\ \texttt{indet}}\ \text{IEInj}$$

$$\frac{C \neq \underline{\mathbf{C}} \qquad e\ \texttt{final}}{\texttt{inj}^{\tau}_{C}(e)\ \texttt{indet}}\ \text{IEInjHole}$$

Figure 2: Evaluation rules for labeled injections

$\boxed{\Gamma\,;\Delta \vdash e : \tau}$ e is of type τ

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma\,;\Delta \vdash e : \tau_j}{\Gamma\,;\Delta \vdash \texttt{inj}^{\tau}_{C_j}(e) : \tau}\ \text{TInj}$$

$\boxed{\Delta \vdash p : \tau[\xi] \dashv\vert\, \Gamma}$ p is assigned type τ and emits constraint ξ

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta \vdash p : \tau_j[\xi] \dashv\vert\, \Gamma}{\Delta \vdash \texttt{inj}_{C_j}(p) : \tau[\texttt{inj}_{C_j}(\xi)] \dashv\vert\, \Gamma}\ \text{PTInj}$$

Figure 3: Typing rules for labeled sums and injections

it is possible to satisfy a labeled injection constraint whenever it is possible to satisfy its argument constraint. Rule CINCInjTag detects inconsistent labels. Since label holes may match any other label, it suffices to consider only concrete labels. Rule CINCInjArg detects inconsistent argument constraints, regardless of label status. Both rules are necessary to ensure the absence of inconsistent injection constraints.

$\boxed{\xi : \tau}$

ξ constrains final expressions of type τ

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ C_j \in \mathcal{C} \qquad \xi : \tau_j\end{array}}{\texttt{inj}^{\tau}_{C_j}(\xi) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}\ \text{CTInj}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\frac{C = C' \qquad e \models \xi}{\texttt{inj}^{\tau}_{C}(e) \models \texttt{inj}^{\tau}_{C'}(\xi)}\ \text{CSInj}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\overline{\texttt{inj}^{\tau}_{\underline{\mathbf{C}}}(\xi)} = \texttt{inj}^{\tau}_{\underline{\mathbf{C}}}(\overline{\xi}) \vee \left(\bigvee_{C_i \in \mathcal{C} \setminus \{\underline{\mathbf{C}}\}} \texttt{inj}^{\tau}_{C_i}(\top) \right)$$

$$\overline{\texttt{inj}^{\tau}_{\llparenthesis\rrparenthesis^u}(\xi)} = \bigvee_{\underline{\mathbf{C}}_i \in \mathcal{C}} \texttt{inj}^{\tau}_{\underline{\mathbf{C}}_i}(\top)$$

$$\overline{\texttt{inj}^{\tau}_{\llparenthesis\underline{\mathbf{C}}\rrparenthesis^u}(\xi)} = \bigvee_{\underline{\mathbf{C}}_i \in \mathcal{C}} \texttt{inj}^{\tau}_{\underline{\mathbf{C}}_i}(\top)$$

$\left(\text{where } \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\right)$

$\boxed{e \models_? \xi}$ e may satisfy ξ

$$\frac{C \mathbin{?} C' \qquad e \models^{\dagger}_{?} \xi}{\texttt{inj}^{\tau}_{C}(e) \models_? \texttt{inj}^{\tau}_{C'}(\xi)}\ \text{CMSInjTag} \qquad \frac{C = C' \qquad e \models_? \xi}{\texttt{inj}^{\tau}_{C}(e) \models_? \texttt{inj}^{\tau}_{C'}(\xi)}\ \text{CMSInjArg}$$

Figure 4: Constraint satisfaction rules for labeled sums

$\boxed{p\ \texttt{refutable}_?}$ p is refutable

$$\frac{C \in \mathcal{C} \qquad |\mathcal{C}| = 1 \qquad p\ \texttt{refutable}_?}{\texttt{inj}_{C}(p)\ \texttt{refutable}_?}\ \text{RInjSing}$$

$$\frac{C \in \mathcal{C} \qquad |\mathcal{C}| > 1}{\texttt{inj}_{C}(p)\ \texttt{refutable}_?}\ \text{RInjMult}$$

Figure 5: Refutability rules for labeled injections

$\boxed{\dot{\top}(\xi_1) = \xi_2}$

$$\dot{\top}(\mathtt{inj}^{\tau}_{C}(\xi)) = \mathtt{inj}^{\tau}_{C}(\dot{\top}(\xi))$$

$\boxed{\dot{\bot}(\xi_1) = \xi_2}$

$$\dot{\bot}(\mathtt{inj}^{\tau}_{C}(\xi)) = \mathtt{inj}^{\tau}_{C}(\dot{\bot}(\xi))$$

$\boxed{\Xi\ \mathtt{incon}}$

$$\frac{\underline{\mathbf{C}} \neq \underline{\mathbf{C}}'}{\Xi, \mathtt{inj}^{\tau}_{\underline{\mathbf{C}'}}(\xi'), \mathtt{inj}^{\tau}_{\underline{\mathbf{C}}}(\xi)\ \mathtt{incon}}\ \text{CINCInjTag}$$

$$\frac{\{\xi' | \mathtt{inj}^{\tau}_{C'}(\xi') \in \Xi\}, \xi\ \mathtt{incon}}{\Xi, \mathtt{inj}^{\tau}_{C}(\xi)\ \mathtt{incon}}\ \text{CINCInjArg}$$

Figure 6: Inconsistency rules for labeled injections