In the main paper, we present only a single constraint language. However, conceptually, we work with this language in two distinct stages: first, the constraints directly emitted by lists of rules, then, for use in redundancy and exhaustiveness checking, the constraints which are in the image of the truify and falsify functions and their duals. While irrelevant to the overall theory, to simplify some proofs, it is useful to make this distinction explicit.

In Sec. 1, we present the first stage of constraints, called the *incomplete constraint language*. This consists of any constraint emitted by a pattern, and in particular, includes the ? constraint. In order to define the constraint emitted by a list of rules, we also include $\bot$ and allow taking the $\vee$ of incomplete constraints. At this stage, we often require two versions of each judgement: one describing a determinate result, and one describing a result which is indeterminate due to the presence of the ? constraint.

In turn, in Sec. 2, we discuss those constraints in the image of the truify and falsify functions, as well as their duals. We call this the *complete constraint language*, and it includes almost all of the incomplete language, but excludes the ? constraint. To support the dual operation, we also may take the $\wedge$ of complete constraints, and we add a $\not\underline{n}$ constraint. Due to the absence of ?, judgements related to the complete language do not have to consider indeterminacy, and thus are often simpler than their counterparts in the incomplete language. This is the primary motivation for distinguishing these languages at all.

1 Incomplete Constraint Language

$$\dot\xi \;::=\; \top \mid \bot \mid ? \mid \underline{n} \mid \mathtt{inl}(\dot\xi) \mid \mathtt{inr}(\dot\xi) \mid (\dot\xi, \dot\xi) \mid \dot\xi \vee \dot\xi$$

$\boxed{\dot\xi : \tau}$ $\quad\dot\xi$ constrains final expressions of type τ

$$\text{CTTruth}\quad \frac{}{\top : \tau} \tag{1a}$$

$$\text{CTFalsity}\quad \frac{}{\bot : \tau} \tag{1b}$$

$$\text{CTUnknown}\quad \frac{}{? : \tau} \tag{1c}$$

$$\text{CTNum}\quad \frac{}{\underline{n} : \mathtt{num}} \tag{1d}$$

$$\text{CTInl}\quad \frac{\dot\xi_1 : \tau_1}{\mathtt{inl}(\dot\xi_1) : (\tau_1 + \tau_2)} \tag{1e}$$

$$\text{CTInr}\quad \frac{\dot\xi_2 : \tau_2}{\mathtt{inr}(\dot\xi_2) : (\tau_1 + \tau_2)} \tag{1f}$$

$$\frac{\dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)}\ \text{CTPair} \tag{1g}$$

$$\frac{\dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau}\ \text{CTOr} \tag{1h}$$

$\boxed{\dot{\xi}\ \texttt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\frac{}{\bot\ \texttt{refutable}_?}\ \text{RXFalsity} \tag{2a}$$

$$\frac{}{?\ \texttt{refutable}_?}\ \text{RXUnknown} \tag{2b}$$

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RXNum} \tag{2c}$$

$$\frac{}{\texttt{inl}(\dot{\xi})\ \texttt{refutable}_?}\ \text{RXInl} \tag{2d}$$

$$\frac{}{\texttt{inr}(\dot{\xi})\ \texttt{refutable}_?}\ \text{RXInr} \tag{2e}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\ \text{RXPairL} \tag{2f}$$

$$\frac{\dot{\xi}_2\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}\ \text{RXPairR} \tag{2g}$$

$$\frac{\dot{\xi}_1\ \texttt{refutable}_? \qquad \dot{\xi}_2\ \texttt{refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}\ \text{RXOr} \tag{2h}$$

$\boxed{\dot{\xi}\ \texttt{possible}}$ $\dot{\xi}$ is possible

$$\frac{}{\top\ \texttt{possible}}\ \text{PTruth} \tag{3a}$$

$$\frac{}{?\ \texttt{possible}}\ \text{PUnknown} \tag{3b}$$

$$\frac{}{\underline{n}\ \texttt{possible}}\ \text{PNum} \tag{3c}$$

$$\frac{\dot{\xi}\ \texttt{possible}}{\texttt{inl}(\dot{\xi})\ \texttt{possible}}\ \text{PInl} \tag{3d}$$

$$\frac{\dot{\xi}\ \texttt{possible}}{\texttt{inr}(\dot{\xi})\ \texttt{possible}}\ \text{PInr} \tag{3e}$$

$$\frac{\dot{\xi}_1\ \texttt{possible} \qquad \dot{\xi}_2\ \texttt{possible}}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{possible}}\ \text{PPair} \tag{3f}$$

$$\frac{\dot{\xi}_1\ \texttt{possible}}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{possible}}\ \text{POrL} \tag{3g}$$

$$\frac{\dot{\xi}_2\ \texttt{possible}}{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{possible}}\ \text{POrR} \tag{3h}$$

$\boxed{e \dot{\models} \dot{\xi}}$ e satisfies $\dot{\xi}$

$$\frac{}{e \dot{\models} \top}\ \text{CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \dot{\models} \underline{n}}\ \text{CSNum} \tag{4b}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \dot{\models} \texttt{inl}(\dot{\xi}_1)}\ \text{CSInl} \tag{4c}$$

$$\frac{e_2 \dot{\models} \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \dot{\models} \texttt{inr}(\dot{\xi}_2)}\ \text{CSInr} \tag{4d}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1 \qquad e_2 \dot{\models} \dot{\xi}_2}{(e_1, e_2) \dot{\models} (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CSPair} \tag{4e}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \dot{\models} \dot{\xi}_1 \qquad \texttt{snd}(e) \dot{\models} \dot{\xi}_2}{e \dot{\models} (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CSNotIntroPair} \tag{4f}$$

$$\frac{e \dot{\models} \dot{\xi}_1}{e \dot{\models} \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrL} \tag{4g}$$

$$\frac{e \dot{\models} \dot{\xi}_2}{e \dot{\models} \dot{\xi}_1 \vee \dot{\xi}_2} \text{ CSOrR} \tag{4h}$$

$\boxed{e \dot{\models}_? \dot{\xi}}$ e may satisfy $\dot{\xi}$

$$\frac{}{e \dot{\models}_? ?} \text{ CMSUnknown} \tag{5a}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1}{\mathtt{inl}_{\tau_2}(e_1) \dot{\models}_? \mathtt{inl}(\dot{\xi}_1)} \text{ CMSInl} \tag{5b}$$

$$\frac{e_2 \dot{\models}_? \dot{\xi}_2}{\mathtt{inr}_{\tau_1}(e_2) \dot{\models}_? \mathtt{inr}(\dot{\xi}_2)} \text{ CMSInr} \tag{5c}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1 \qquad e_2 \dot{\models} \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)} \text{ CMSPairL} \tag{5d}$$

$$\frac{e_1 \dot{\models} \dot{\xi}_1 \qquad e_2 \dot{\models}_? \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)} \text{ CMSPairR} \tag{5e}$$

$$\frac{e_1 \dot{\models}_? \dot{\xi}_1 \qquad e_2 \dot{\models}_? \dot{\xi}_2}{(e_1, e_2) \dot{\models}_? (\dot{\xi}_1, \dot{\xi}_2)} \text{ CMSPair} \tag{5f}$$

$$\frac{e \dot{\models}_? \dot{\xi}_1 \qquad e \dot{\not\models} \dot{\xi}_2}{e \dot{\models}_? \dot{\xi}_1 \vee \dot{\xi}_2} \text{ CMSOrL} \tag{5g}$$

$$\frac{e \dot{\not\models} \dot{\xi}_1 \qquad e \dot{\models}_? \dot{\xi}_2}{e \dot{\models}_? \dot{\xi}_1 \vee \dot{\xi}_2} \text{ CMSOrR} \tag{5h}$$

$$\frac{e \ \mathtt{notintro} \qquad \dot{\xi} \ \mathtt{refutable}_? \qquad \dot{\xi} \ \mathtt{possible}}{e \dot{\models}_? \dot{\xi}} \text{ CMSNotIntro} \tag{5i}$$

$\boxed{e \dot{\models}_?^{\dagger} \dot{\xi}}$ e satisfies or may satisfy $\dot{\xi}$

$$\frac{e \dot{\models}_? \dot{\xi}}{e \dot{\models}_?^{\dagger} \dot{\xi}} \text{ CSMSMay} \tag{6a}$$

$$\frac{e \dot{\models} \dot{\xi}}{e \dot{\models}_?^{\dagger} \dot{\xi}} \text{ CSMSSat} \tag{6b}$$

Lemma 1.0.1. *Assume e $\texttt{notintro}$. If $e \dot{\models}_? \dot{\xi}$ or $e \dot{\not\models} \dot{\xi}$ then $\dot{\xi}$ $\texttt{refutable}_?$.*

Lemma 1.0.2. *If e $\texttt{notintro}$ and $e \dot{\models} \dot{\xi}$ then $\cancel{\dot{\xi}\ \texttt{refutable}_?}$.*

Theorem 1.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e $\texttt{final}$ then exactly one of the following holds*

1. $e \dot{\models} \dot{\xi}$
2. $e \dot{\models}_? \dot{\xi}$
3. $e \dot{\not\models}^{\dagger}_{?} \dot{\xi}$

Definition 1.1.1 (Entailment of Constraints)**.** *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \dot{\models} \dot{\xi}_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e $\texttt{val}$ we have $e \dot{\models}^{\dagger}_{?} \dot{\xi}_1$ implies $e \dot{\models} \dot{\xi}_2$*

Definition 1.1.2 (Potential Entailment of Constraints)**.** *Suppose that $\dot{\xi}_1 : \tau$ and $\dot{\xi}_2 : \tau$. Then $\dot{\xi}_1 \dot{\models}^{\dagger}_{?} \dot{\xi}_2$ iff for all e such that $\cdot ; \Delta \vdash e : \tau$ and e $\texttt{final}$ we have $e \dot{\models}^{\dagger}_{?} \dot{\xi}_1$ implies $e \dot{\models}^{\dagger}_{?} \dot{\xi}_2$*

Corollary 1.1.1. *Suppose that $\dot{\xi} : \tau$ and $\cdot ; \Delta \vdash e : \tau$ and e $\texttt{final}$. Then $\top \dot{\models}^{\dagger}_{?} \dot{\xi}$ implies $e \dot{\models}^{\dagger}_{?} \dot{\xi}$*

2 Complete Constraint Language

$$\xi \ ::= \ \top \mid \bot \mid \underline{n} \mid \cancel{\underline{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\text{CTTruth} \quad \frac{}{\top : \tau} \tag{7a}$$

$$\text{CTFalsity} \quad \frac{}{\bot : \tau} \tag{7b}$$

$$\text{CTNum} \quad \frac{}{\underline{n} : \texttt{num}} \tag{7c}$$

$$\text{CTNotNum} \quad \frac{}{\cancel{\underline{n}} : \texttt{num}} \tag{7d}$$

$$\text{CTAnd} \quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \tag{7e}$$

$$\text{CTOr} \quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{7f}$$

$$\text{CTInl}\quad \frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{7g}$$

$$\text{CTInr}\quad \frac{\xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)} \tag{7h}$$

$$\text{CTPair}\quad \frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \tag{7i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\begin{aligned}
\overline{\top} &= \bot \\
\overline{\bot} &= \top \\
\overline{\underline{n}} &= \cancel{\underline{n}} \\
\overline{\cancel{\underline{n}}} &= \underline{n} \\
\overline{\xi_1 \wedge \xi_2} &= \overline{\xi_1} \vee \overline{\xi_2} \\
\overline{\xi_1 \vee \xi_2} &= \overline{\xi_1} \wedge \overline{\xi_2} \\
\overline{\texttt{inl}(\xi_1)} &= \texttt{inl}(\overline{\xi_1}) \vee \texttt{inr}(\top) \\
\overline{\texttt{inr}(\xi_2)} &= \texttt{inr}(\overline{\xi_2}) \vee \texttt{inl}(\top) \\
\overline{(\xi_1, \xi_2)} &= (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2})
\end{aligned}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\text{CSTruth}\quad \frac{}{e \models \top} \tag{9a}$$

$$\text{CSNum}\quad \frac{}{\underline{n} \models \underline{n}} \tag{9b}$$

$$\text{CSNotNum}\quad \frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}} \tag{9c}$$

$$\text{CSAnd}\quad \frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \tag{9d}$$

$$\text{CSOrL}\quad \frac{e \models \xi_1}{e \models \xi_1 \vee \xi_2} \tag{9e}$$

$$\text{CSOrR}\quad \frac{e \models \xi_2}{e \models \xi_1 \vee \xi_2} \tag{9f}$$

$$\text{CSInl}\quad \frac{e_1 \models \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)} \tag{9g}$$

$$\text{CSInr}\quad \frac{e_2 \models \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)} \tag{9h}$$

$$\text{CSPair}\quad \frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \tag{9i}$$

Lemma 2.0.1. *Assume e $\texttt{val}$. Then $e \not\models \xi$ iff $e \models \overline{\xi}$.*

Theorem 2.1 (Exclusiveness of Satisfaction Judgment)**.** *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e $\texttt{val}$ then exactly one of the following holds*

1. $e \models \xi$
2. $e \models \overline{\xi}$

Definition 2.1.1 (Entailment of Constraints)**.** *Suppose that $\xi_1 : \tau$ and $\xi_2 : \tau$. Then $\xi_1 \models \xi_2$ iff for all e such that $\cdot;\Delta \vdash e : \tau$ and e $\texttt{val}$ we have $e \models \xi_1$ implies $e \models \xi_2$*

Lemma 2.1.1 (Material Entailment of Complete Constraints)**.** *$\xi_1 \models \xi_2$ iff $\top \models \overline{\xi_1} \vee \xi_2$.*

2.1 Relationship with Incomplete Constraint Language

Lemma 2.1.2. *Assume that e $\texttt{val}$. Then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ iff $e \models \dot{\top}(\dot{\xi})$.*

Lemma 2.1.3. *$e \dot{\models} \dot{\xi}$ iff $e \models \dot{\bot}(\dot{\xi})$*

Lemma 2.1.4. *Suppose $\dot{\xi} : \tau$. Then $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\texttt{final}$ **iff** $e \dot{\models}^{\dagger}_{?} \dot{\xi}$ for all e such that $\cdot\,;\Delta \vdash e : \tau$ and e $\texttt{val}$.*

Theorem 2.2. *$\top \dot{\models}^{\dagger}_{?} \dot{\xi}$ iff $\top \models \dot{\top}(\dot{\xi})$.*

Theorem 2.3. *$\dot{\xi}_1 \dot{\models} \dot{\xi}_2$ iff $\dot{\top}(\dot{\xi}_1) \models \dot{\bot}(\dot{\xi}_2)$.*

3 Dynamic Semantics

$\boxed{e\ \texttt{val}}$ e is a value

$$\frac{}{\underline{n}\ \texttt{val}}\ \text{VNum} \tag{10a}$$

$$\frac{}{\lambda x:\tau.e\ \texttt{val}}\ \text{VLam} \tag{10b}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{val}}\ \text{VPair} \tag{10c}$$

$$\frac{e\ \texttt{val}}{\texttt{inl}_\tau(e)\ \texttt{val}}\ \text{VInl} \tag{10d}$$

$$\frac{e\ \texttt{val}}{\texttt{inr}_\tau(e)\ \texttt{val}}\ \text{VInr} \tag{10e}$$

$\boxed{e\ \texttt{indet}}$ e is indeterminate

$$\frac{}{(\!|\!)^u\ \texttt{indet}}\ \text{IEHole} \tag{11a}$$

$$\frac{e\ \texttt{final}}{(\!|e|\!)^u\ \texttt{indet}}\ \text{IHole} \tag{11b}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{final}}{e_1(e_2)\ \texttt{indet}}\ \text{IAp} \tag{11c}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{val}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairL} \tag{11d}$$

$$\frac{e_1\ \texttt{val} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPairR} \tag{11e}$$

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}}\ \text{IPair} \tag{11f}$$

$$\frac{e\ \texttt{final}}{\texttt{fst}(e)\ \texttt{indet}}\ \text{IFst} \tag{11g}$$

ISnd
$$\frac{e\ \mathtt{final}}{\mathtt{snd}(e)\ \mathtt{indet}} \tag{11h}$$

IInL
$$\frac{e\ \mathtt{indet}}{\mathtt{inl}_\tau(e)\ \mathtt{indet}} \tag{11i}$$

IInR
$$\frac{e\ \mathtt{indet}}{\mathtt{inr}_\tau(e)\ \mathtt{indet}} \tag{11j}$$

IMatch
$$\frac{e\ \mathtt{final} \qquad e\ ?\ p_r}{\mathtt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \mathtt{indet}} \tag{11k}$$

$\boxed{e\ \mathtt{final}}$ e is final

FVal
$$\frac{e\ \mathtt{val}}{e\ \mathtt{final}} \tag{12a}$$

FIndet
$$\frac{e\ \mathtt{indet}}{e\ \mathtt{final}} \tag{12b}$$

$\boxed{e\ \mathtt{notintro}}$ e cannot be a value syntactically

NVEHole
$$\frac{}{⦇⦈^u\ \mathtt{notintro}} \tag{13a}$$

NVHole
$$\frac{}{⦇e⦈^u\ \mathtt{notintro}} \tag{13b}$$

NVAp
$$\frac{}{e_1(e_2)\ \mathtt{notintro}} \tag{13c}$$

NVMatch
$$\frac{}{\mathtt{match}(e)\{\hat{rs}\}\ \mathtt{notintro}} \tag{13d}$$

NVFst
$$\frac{}{\mathtt{fst}(e)\ \mathtt{notintro}} \tag{13e}$$

NVSnd
$$\frac{}{\mathtt{snd}(e)\ \mathtt{notintro}} \tag{13f}$$

$\boxed{e' \in \mathtt{values}[\Delta](e)}$ e' is one of the possible values of e

IVVal
$$\frac{e\ \mathtt{val} \qquad \cdot\, ; \Delta \vdash e : \tau}{e \in \mathtt{values}[\Delta](e)} \tag{14a}$$

$$\frac{e\ \texttt{notintro} \qquad \cdot\,;\Delta \vdash e : \tau \qquad e'\ \texttt{val} \qquad \cdot\,;\Delta \vdash e' : \tau}{e' \in \texttt{values}[\Delta](e)}\ \text{IVIndet} \tag{14b}$$

$$\frac{e_1' \in \texttt{values}[\Delta](e_1)}{\texttt{inl}_{\tau_2}(e_1') \in \texttt{values}[\Delta](\texttt{inl}_{\tau_2}(e_1))}\ \text{IVInl} \tag{14c}$$

$$\frac{e_2' \in \texttt{values}[\Delta](e_2)}{\texttt{inr}_{\tau_1}(e_2') \in \texttt{values}[\Delta](\texttt{inr}_{\tau_1}(e_2))}\ \text{IVInr} \tag{14d}$$

$$\frac{e_1' \in \texttt{values}[\Delta](e_1) \qquad e_2' \in \texttt{values}[\Delta](e_2)}{(e_1', e_2') \in \texttt{values}[\Delta]((e_1, e_2))}\ \text{IVPair} \tag{14e}$$

Lemma 3.0.1. *If $e' \in \texttt{values}[\Delta](e)$ and $\cdot\,;\Delta \vdash e : \tau$ then $\cdot\,;\Delta \vdash e' : \tau$.*

Lemma 3.0.2. *If $e' \in \texttt{values}[\Delta](e)$ then e' $\texttt{val}$.*

Lemma 3.0.3. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ then there exists e' such that $e' \in \texttt{values}[\Delta](e)$.*

Lemma 3.0.4. *Assume e $\texttt{final}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$. If $e \dot{\not\models}{}^{\dagger}_{?} \dot{\xi}$ then and $e' \in \texttt{values}[\Delta](e)$ then $e' \dot{\not\models}{}^{\dagger}_{?} \dot{\xi}$.*

Lemma 3.0.5. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$ and there exists e' such that $e' \in \texttt{values}[\Delta](e)$ and $e' \dot{\models}{}^{\dagger}_{?} \dot{\xi}$ then $e \dot{\models}{}^{\dagger}_{?} \dot{\xi}$.*

$\boxed{\Gamma\,;\Delta \vdash \theta : \Gamma\theta}$ $\quad\theta$ is of type $\Gamma\theta$

$$\frac{}{\Gamma\,;\Delta \vdash \emptyset : \cdot}\ \text{STEmpty} \tag{15a}$$

$$\frac{\Gamma\,;\Delta \vdash \theta : \Gamma_\theta \qquad \Gamma\,;\Delta \vdash e : \tau}{\Gamma\,;\Delta \vdash \theta, x/e : \Gamma_\theta, x : \tau}\ \text{STExtend} \tag{15b}$$

$\boxed{p\ \texttt{refutable}_?}$ $\quad p$ is refutable

$$\frac{}{\underline{n}\ \texttt{refutable}_?}\ \text{RNum} \tag{16a}$$

$$\frac{}{(\!|\,|\!)^w\ \texttt{refutable}_?}\ \text{REHole} \tag{16b}$$

$$\frac{}{(\!|p|\!)^w_\tau\ \texttt{refutable}_?}\ \text{RHole} \tag{16c}$$

$$\frac{}{\texttt{inl}(p)\ \texttt{refutable}_?}\ \text{RInl} \tag{16d}$$

$$\frac{}{\texttt{inr}(p)\ \texttt{refutable}_?}\ \text{RInr} \tag{16e}$$

$$\frac{p_1\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairL} \tag{16f}$$

$$\frac{p_2\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?}\ \text{RPairR} \tag{16g}$$

$\boxed{e \rhd p \dashv\!\!| \theta}$ e matches p, emitting θ

$$\frac{}{e \rhd x \dashv\!\!| e/x}\ \text{MVar} \tag{17a}$$

$$\frac{}{e \rhd \_ \dashv\!\!| \cdot}\ \text{MWild} \tag{17b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashv\!\!| \cdot}\ \text{MNum} \tag{17c}$$

$$\frac{e_1 \rhd p_1 \dashv\!\!| \theta_1 \qquad e_2 \rhd p_2 \dashv\!\!| \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \theta_1 \uplus \theta_2}\ \text{MPair} \tag{17d}$$

$$\frac{e \rhd p \dashv\!\!| \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashv\!\!| \theta}\ \text{MInl} \tag{17e}$$

$$\frac{e \rhd p \dashv\!\!| \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashv\!\!| \theta}\ \text{MInr} \tag{17f}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{fst}(e) \rhd p_1 \dashv\!\!| \theta_1 \qquad \texttt{snd}(e) \rhd p_2 \dashv\!\!| \theta_2}{e \rhd (p_1, p_2) \dashv\!\!| \theta_1 \uplus \theta_2}\ \text{MNotIntroPair} \tag{17g}$$

$\boxed{e \,?\, p}$ e may match p

$$\frac{}{e \,?\, (\!|\!)^w}\ \text{MMEHole} \tag{18a}$$

$$\frac{}{e \,?\, (\!|p|\!)^w_\tau}\ \text{MMHole} \tag{18b}$$

$$\frac{e\ \texttt{notintro} \qquad p\ \texttt{refutable}_?}{e \,?\, p}\ \text{MMNotIntro} \tag{18c}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \rhd p_2 \dashv\!\!\mid \theta_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairL} \tag{18d}$$

$$\frac{e_1 \rhd p_1 \dashv\!\!\mid \theta_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPairR} \tag{18e}$$

$$\frac{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}{(e_1, e_2) \mathrel{?} (p_1, p_2)}\ \text{MMPair} \tag{18f}$$

$$\frac{e \mathrel{?} p}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)}\ \text{MMInl} \tag{18g}$$

$$\frac{e \mathrel{?} p}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)}\ \text{MMInr} \tag{18h}$$

$\boxed{e \perp p}$ e does not match p

$$\frac{n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}}\ \text{NMNum} \tag{19a}$$

$$\frac{e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairL} \tag{19b}$$

$$\frac{e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)}\ \text{NMPairR} \tag{19c}$$

$$\frac{}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)}\ \text{NMConfL} \tag{19d}$$

$$\frac{}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)}\ \text{NMConfR} \tag{19e}$$

$$\frac{e \perp p}{\texttt{inl}_\tau(e) \perp \texttt{inl}(p)}\ \text{NMInl} \tag{19f}$$

$$\frac{e \perp p}{\texttt{inr}_\tau(e) \perp \texttt{inr}(p)}\ \text{NMInr} \tag{19g}$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{20a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{20b}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

$$\text{ITHole}\quad \frac{e \mapsto e'}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{21a}$$

$$\text{ITApFun}\quad \frac{e_1 \mapsto e_1'}{e_1(e_2) \mapsto e_1'(e_2)} \tag{21b}$$

$$\text{ITApArg}\quad \frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{e_1(e_2) \mapsto e_1(e_2')} \tag{21c}$$

$$\text{ITAp}\quad \frac{e_2\ \texttt{val}}{\lambda x : \tau.e_1(e_2) \mapsto [e_2/x]e_1} \tag{21d}$$

$$\text{ITPairL}\quad \frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \tag{21e}$$

$$\text{ITPairR}\quad \frac{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \tag{21f}$$

$$\text{ITFstPair}\quad \frac{(e_1, e_2)\ \texttt{final}}{\texttt{fst}((e_1, e_2)) \mapsto e_1} \tag{21g}$$

$$\text{ITSndPair}\quad \frac{(e_1, e_2)\ \texttt{final}}{\texttt{snd}((e_1, e_2)) \mapsto e_2} \tag{21h}$$

$$\text{ITInl}\quad \frac{e \mapsto e'}{\texttt{inl}_\tau(e) \mapsto \texttt{inl}_\tau(e')} \tag{21i}$$

$$\text{ITInr}\quad \frac{e \mapsto e'}{\texttt{inr}_\tau(e) \mapsto \texttt{inr}_\tau(e')} \tag{21j}$$

$$\text{ITExpMatch}\quad \frac{e \mapsto e'}{\texttt{match}(e)\{\hat{rs}\} \mapsto \texttt{match}(e')\{\hat{rs}\}} \tag{21k}$$

$$\text{ITSuccMatch}\quad \frac{e\ \texttt{final} \qquad e \rhd p_r \dashv\!\vdash \theta}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \tag{21l}$$

$$\text{ITFailMatch}\quad \frac{e\ \texttt{final} \qquad e \perp p_r}{\texttt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^{\diamond} \mid r' \mid rs'\}} \tag{21m}$$

Lemma 3.0.6. *If e `final` and e `notintro` then e `indet`.*

Lemma 3.0.7. *There doesn't exist such an expression e such that both e `val` and e `indet`.*

Lemma 3.0.8. *There doesn't exist such an expression e such that both e `val` and e `notintro`.*

Lemma 3.0.9 (Finality)**.** *There doesn't exist such an expression e such that both e `final` and $e \mapsto e'$ for some e'*

Lemma 3.0.10 (Matching Determinism)**.** *If e `final` and $\cdot\,;\Delta_e \vdash e : \tau$ and $\Delta \vdash p : \tau[\xi] \dashv\!\vdash \Gamma$ then exactly one of the following holds*

1. *$e \rhd p \dashv\!\vdash \theta$ for some θ*
2. *$e \,?\, p$*
3. *$e \perp p$*

Lemma 3.0.11 (Matching Coherence of Constraint)**.** *Suppose that $\cdot\,;\Delta_e \vdash e : \tau$ and e `final` and $\Delta \vdash p : \tau[\xi] \dashv\!\vdash \Gamma$. Then we have*

1. *$e \dot{\models} \xi$ iff $e \rhd p \dashv\!\vdash \theta$*
2. *$e \dot{\models}_? \xi$ iff $e \,?\, p$*
3. *$e \dot{\not\models}{}^{\dagger}_{?} \xi$ iff $e \perp p$*

4 Static Semantics

$$\begin{array}{lcl}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
 & \mid & \lambda x : \tau.e \mid e_1(e_2) \\
 & \mid & (e_1, e_2) \\
 & \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
 & \mid & \llparenthesis\rrparenthesis^u \mid \llparenthesis e \rrparenthesis^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid _ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid \llparenthesis\rrparenthesis^w \mid \llparenthesis p \rrparenthesis^w_\tau
\end{array}$$

$\boxed{\Gamma \,;\, \Delta \vdash e : \tau}$ $\quad e$ is of type τ

TVar

$$\frac{}{\Gamma, x : \tau \,;\, \Delta \vdash x : \tau} \tag{22a}$$

TEHole

$$\frac{}{\Gamma \,;\, \Delta, u :: \tau \vdash \llparenthesis\rrparenthesis^{u} : \tau} \tag{22b}$$

THole

$$\frac{\Gamma \,;\, \Delta, u :: \tau \vdash e : \tau'}{\Gamma \,;\, \Delta, u :: \tau \vdash \llparenthesis e \rrparenthesis^{u} : \tau} \tag{22c}$$

TNum

$$\frac{}{\Gamma \,;\, \Delta \vdash \underline{n} : \mathtt{num}} \tag{22d}$$

TLam

$$\frac{\Gamma, x : \tau_1 \,;\, \Delta \vdash e : \tau_2}{\Gamma \,;\, \Delta \vdash \lambda x : \tau_1 . e : (\tau_1 \to \tau_2)} \tag{22e}$$

TAp

$$\frac{\Gamma \,;\, \Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma \,;\, \Delta \vdash e_2 : \tau_2}{\Gamma \,;\, \Delta \vdash e_1(e_2) : \tau} \tag{22f}$$

TPair

$$\frac{\Gamma \,;\, \Delta \vdash e_1 : \tau_1 \qquad \Gamma \,;\, \Delta \vdash e_2 : \tau_2}{\Gamma \,;\, \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)} \tag{22g}$$

TFst

$$\frac{\Gamma \,;\, \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma \,;\, \Delta \vdash \mathtt{fst}(e) : \tau_1} \tag{22h}$$

TSnd

$$\frac{\Gamma \,;\, \Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma \,;\, \Delta \vdash \mathtt{snd}(e) : \tau_2} \tag{22i}$$

TInl

$$\frac{\Gamma \,;\, \Delta \vdash e : \tau_1}{\Gamma \,;\, \Delta \vdash \mathtt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)} \tag{22j}$$

TInr

$$\frac{\Gamma \,;\, \Delta \vdash e : \tau_2}{\Gamma \,;\, \Delta \vdash \mathtt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)} \tag{22k}$$

TMatchZPre

$$\frac{\Gamma \,;\, \Delta \vdash e : \tau \qquad \Gamma \,;\, \Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \dot{\models}^{\dagger}_{?} \xi}{\Gamma \,;\, \Delta \vdash \mathtt{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \tag{22l}$$

$$\text{TMatchNZPre}\quad \frac{\begin{array}{c}\Gamma\,;\Delta\vdash e:\tau\\ \Gamma\,;\Delta\vdash[\bot]rs_{pre}:\tau[\xi_{pre}]\Rightarrow\tau' \qquad \Gamma\,;\Delta\vdash[\bot\vee\xi_{pre}]r\mid rs_{post}:\tau[\xi_{rest}]\Rightarrow\tau'\\ \forall e'.e'\in\texttt{values}[\Delta](e)\Rightarrow e'\,\dot{\not\models}{}^{\dagger}_{?}\,\xi_{pre} \qquad \top\dot{\models}{}^{\dagger}_{?}\,\xi_{pre}\vee\xi_{rest}\end{array}}{\Gamma\,;\Delta\vdash\texttt{match}(e)\{rs_{pre}\mid r\mid rs_{post}\}:\tau'} \tag{22m}$$

$\boxed{\Delta\vdash p:\tau[\xi]\dashv\!\vdash\Gamma}$ p is assigned type τ and emits constraint ξ

$$\text{PTVar}\quad \frac{}{\cdot\vdash x:\tau[\top]\dashv\!\vdash x:\tau} \tag{23a}$$

$$\text{PTWild}\quad \frac{}{\cdot\vdash \_:\tau[\top]\dashv\!\vdash\cdot} \tag{23b}$$

$$\text{PTEHole}\quad \frac{}{w::\tau\vdash (\!|\,|\!)^{w}:\tau[?]\dashv\!\vdash\cdot} \tag{23c}$$

$$\text{PTHole}\quad \frac{\Delta\vdash p:\tau[\xi]\dashv\!\vdash\Gamma}{\Delta,w::\tau'\vdash (\!|p|\!)^{w}_{\tau}:\tau'[?]\dashv\!\vdash\Gamma} \tag{23d}$$

$$\text{PTNum}\quad \frac{}{\cdot\vdash \underline{n}:\texttt{num}[\underline{n}]\dashv\!\vdash\cdot} \tag{23e}$$

$$\text{PTInl}\quad \frac{\Delta\vdash p:\tau_1[\xi]\dashv\!\vdash\Gamma}{\Delta\vdash\texttt{inl}(p):(\tau_1+\tau_2)[\texttt{inl}(\xi)]\dashv\!\vdash\Gamma} \tag{23f}$$

$$\text{PTInr}\quad \frac{\Delta\vdash p:\tau_2[\xi]\dashv\!\vdash\Gamma}{\Delta\vdash\texttt{inr}(p):(\tau_1+\tau_2)[\texttt{inr}(\xi)]\dashv\!\vdash\Gamma} \tag{23g}$$

$$\text{PTPair}\quad \frac{\Delta_1\vdash p_1:\tau_1[\xi_1]\dashv\!\vdash\Gamma_1 \qquad \Delta_2\vdash p_2:\tau_2[\xi_2]\dashv\!\vdash\Gamma_2}{\Delta_1\uplus\Delta_2\vdash(p_1,p_2):(\tau_1\times\tau_2)[(\xi_1,\xi_2)]\dashv\!\vdash\Gamma_1\uplus\Gamma_2} \tag{23h}$$

$\boxed{\Gamma\,;\Delta\vdash(p\Rightarrow e):\tau[\xi]\Rightarrow\tau'}$ r transforms a final expression of type τ to a final expression of type τ'

$$\text{CTRule}\quad \frac{\Delta_p\vdash p:\tau[\xi]\dashv\!\vdash\Gamma_p \qquad \Gamma\uplus\Gamma_p\,;\Delta\uplus\Delta_p\vdash e:\tau'}{\Gamma\,;\Delta\vdash p\Rightarrow e:\tau[\xi]\Rightarrow\tau'} \tag{24a}$$

$\boxed{\Gamma\,;\Delta\vdash[\xi_{pre}]rs:\tau[\xi_{rs}]\Rightarrow\tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

$$\text{CTOneRules}\quad \frac{\Gamma\,;\Delta\vdash r:\tau[\xi_r]\Rightarrow\tau' \qquad \xi_r\,\dot{\not\models}\,\xi_{pre}}{\Gamma\,;\Delta\vdash[\xi_{pre}](r\mid\cdot):\tau[\xi_r]\Rightarrow\tau'} \tag{25a}$$

CTRules
$$\frac{\Gamma\,;\Delta\vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma\,;\Delta\vdash [\xi_{pre}\vee\xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models^{\!\!\cdot} \xi_{pre}}{\Gamma\,;\Delta\vdash [\xi_{pre}]r \mid rs : \tau[\xi_r\vee\xi_{rs}] \Rightarrow \tau'} \tag{25b}$$

Lemma 4.0.1. *If* $\Delta\vdash p : \tau[\xi] \dashv\!\!\mid \Gamma$ *then* $\xi : \tau$.

Lemma 4.0.2. *If* $\cdot\,;\Delta\vdash r : \tau_1[\xi_r] \Rightarrow \tau$ *then* $\xi_r : \tau_1$.

Lemma 4.0.3. *If* $\cdot\,;\Delta\vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ *then* $\xi_{rs} : \tau_1$.

Lemma 4.0.4. *If* $\Gamma\,;\Delta\vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ *and* $\Gamma\,;\Delta\vdash r : \tau[\xi_r] \Rightarrow \tau'$ *and* $\xi_r \not\models^{\!\!\cdot} \xi_{pre}\vee\xi_{rs}$ *then* $\Gamma\,;\Delta\vdash [\xi_{pre}](rs \mid r \mid \cdot)^{\diamond} : \tau[\xi_{rs}\vee\xi_r] \Rightarrow \tau'$

Lemma 4.0.5 (Substitution)**.** *If* $\Gamma, x : \tau\,;\Delta\vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta\vdash e : \tau$ *and* e `final` *then* $\Gamma\,;\Delta\vdash [e/x]e_0 : \tau_0$

Lemma 4.0.6 (Simultaneous Substitution)**.** *If* $\Gamma\uplus\Gamma';\Delta\vdash e : \tau$ *and* $\Gamma;\Delta\vdash \theta : \Gamma'$ *then* $\Gamma\,;\Delta\vdash [\theta]e : \tau$

Lemma 4.0.7 (Substitution Typing)**.** *If* $e \rhd p \dashv\!\!\mid \theta$ *and* $\cdot\,;\Delta_e\vdash e : \tau$ *and* $\Delta\vdash p : \tau[\xi] \dashv\!\!\mid \Gamma$ *and all expressions in* θ *are final then* $\cdot\,;\Delta_e\vdash\theta : \Gamma$

Theorem 4.1 (Preservation)**.** *If* $\cdot\,;\Delta\vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,;\Delta\vdash e' : \tau$

Theorem 4.2 (Determinism)**.** *If* $\cdot\,;\Delta\vdash e : \tau$ *then exactly one of the following holds*

1. e `val`
2. e `indet`
3. $e \mapsto e'$ *for some unique* e'

5 Decidability

$\boxed{\dot\top(\dot\xi) = \xi}$

$$\dot\top(\top) = \top \tag{26a}$$
$$\dot\top(?) = \top \tag{26b}$$
$$\dot\top(\underline{n}) = \underline{n} \tag{26c}$$
$$\dot\top(\xi_1\vee\xi_2) = \dot\top(\xi_1)\vee\dot\top(\xi_2) \tag{26d}$$
$$\dot\top(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot\top(\xi)) \tag{26e}$$
$$\dot\top(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot\top(\xi)) \tag{26f}$$
$$\dot\top((\xi_1,\xi_2)) = (\dot\top(\xi_1),\dot\top(\xi_2)) \tag{26g}$$

$\boxed{\dot{\perp}(\dot{\xi}) = \xi}$

$$\dot{\perp}(\top) = \top \tag{27a}$$

$$\dot{\perp}(?) = \perp \tag{27b}$$

$$\dot{\perp}(\underline{n}) = \underline{n} \tag{27c}$$

$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \tag{27d}$$

$$\dot{\perp}(\mathtt{inl}(\xi)) = \mathtt{inl}(\dot{\perp}(\xi)) \tag{27e}$$

$$\dot{\perp}(\mathtt{inr}(\xi)) = \mathtt{inr}(\dot{\perp}(\xi)) \tag{27f}$$

$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \tag{27g}$$

$\boxed{\Xi\ \mathtt{incon}}$ A finite set of constraints, Ξ, is inconsistent

$$\begin{array}{c}\text{CINCTruth}\\ \dfrac{\Xi\ \mathtt{incon}}{\Xi, \top\ \mathtt{incon}}\end{array} \tag{28a}$$

$$\begin{array}{c}\text{CINCFalsity}\\ \dfrac{}{\Xi, \perp\ \mathtt{incon}}\end{array} \tag{28b}$$

$$\begin{array}{c}\text{CINCNum}\\ \dfrac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \mathtt{incon}}\end{array} \tag{28c}$$

$$\begin{array}{c}\text{CINCNotNum}\\ \dfrac{}{\Xi, \underline{n}, \not\underline{n}\ \mathtt{incon}}\end{array} \tag{28d}$$

$$\begin{array}{c}\text{CINCAnd}\\ \dfrac{\Xi, \xi_1, \xi_2\ \mathtt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \mathtt{incon}}\end{array} \tag{28e}$$

$$\begin{array}{c}\text{CINCOr}\\ \dfrac{\Xi, \xi_1\ \mathtt{incon} \qquad \Xi, \xi_2\ \mathtt{incon}}{\Xi, \xi_1 \vee \xi_2\ \mathtt{incon}}\end{array} \tag{28f}$$

$$\begin{array}{c}\text{CINCInj}\\ \dfrac{}{\Xi, \mathtt{inl}(\xi_1), \mathtt{inr}(\xi_2)\ \mathtt{incon}}\end{array} \tag{28g}$$

$$\begin{array}{c}\text{CINCInl}\\ \dfrac{\xi_1, \cdots, \xi_n\ \mathtt{incon}}{\mathtt{inl}(\xi_1), \cdots, \mathtt{inl}(\xi_n)\ \mathtt{incon}}\end{array} \tag{28h}$$

$$\begin{array}{c}\text{CINCInr}\\ \dfrac{\xi_1, \cdots, \xi_n\ \mathtt{incon}}{\mathtt{inr}(\xi_1), \cdots, \mathtt{inr}(\xi_n)\ \mathtt{incon}}\end{array} \tag{28i}$$

$$\begin{array}{c}\text{CINCPairL}\\ \dfrac{\xi_{11}, \cdots, \xi_{n1}\ \mathtt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2})\ \mathtt{incon}}\end{array} \tag{28j}$$

CINCPairR

$$\frac{\xi_{12}, \cdots, \xi_{n2} \ \texttt{incon}}{(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22}), \cdots, (\xi_{n1}, \xi_{n2}) \ \texttt{incon}} \tag{28k}$$

Lemma 5.0.1 (Decidability of Inconsistency). *It is decidable whether* ξ `incon`.

Lemma 5.0.2 (Inconsistency and Entailment of Constraint). $\overline{\xi}$ `incon` *iff* $\top \models \xi$

Theorem 5.1 (Decidability of Exhaustiveness). *It is decidable whether* $\top \dot{\models}^{\dagger}_{?} \dot{\xi}$.

Theorem 5.2 (Decidability of Redundancy). *It is decidable whether* $\dot{\xi}_1 \dot{\models} \dot{\xi}_2$.