

# 1 Match Constraint Language

$\dot{\xi} ::= \top \mid \perp \mid ? \mid \underline{n} \mid \text{inl}(\dot{\xi}) \mid \text{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$

$\boxed{\dot{\xi} : \tau}$      $\dot{\xi}$  constrains final expressions of type  $\tau$

CTTruth

$\frac{}{\top : \tau}$

(1a)

CTUnknown

$\frac{}{? : \tau}$

(1b)

CTNum

$\frac{}{\underline{n} : \text{num}}$

(1c)

CTInl

$\frac{\dot{\xi}_1 : \tau_1}{\text{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)}$

(1d)

CTInr

$\frac{\dot{\xi}_2 : \tau_2}{\text{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)}$

(1e)

CTPair

$\frac{\dot{\xi}_1 : \tau_1 \quad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)}$

(1f)

CTOr

$\frac{\dot{\xi}_1 : \tau \quad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau}$

(1g)

$\boxed{\dot{\xi} \text{ refutable?}}$      $\dot{\xi}$  is refutable

RXNum

$\frac{}{\underline{n} \text{ refutable?}}$

(2a)

RXUnknown

$\frac{}{? \text{ refutable?}}$

(2b)

RXInl

$\frac{}{\text{inl}(\dot{\xi}) \text{ refutable?}}$

(2c)

RXInr

$\frac{}{\text{inr}(\dot{\xi}) \text{ refutable?}}$

(2d)

RXPairL

$\frac{\dot{\xi}_1 \text{ refutable?}}{(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}}$

(2e)

$$\frac{\text{RXPairR} \quad \dot{\xi}_2 \text{ refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_?} \quad (2f)$$

$$\frac{\text{RXOr} \quad \dot{\xi}_1 \text{ refutable}_? \quad \dot{\xi}_2 \text{ refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2 \text{ refutable}_?} \quad (2g)$$

$$\boxed{\text{refutable}_?(\dot{\xi})}$$

$$\text{refutable}_?(\underline{n}) = \text{true} \quad (3a)$$

$$\text{refutable}_?(?) = \text{true} \quad (3b)$$

$$\text{refutable}_?(\text{inl}(\dot{\xi})) = \text{refutable}_?(\dot{\xi}) \quad (3c)$$

$$\text{refutable}_?(\text{inr}(\dot{\xi})) = \text{refutable}_?(\dot{\xi}) \quad (3d)$$

$$\text{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{refutable}_?(\dot{\xi}_1) \text{ or } \text{refutable}_?(\dot{\xi}_2) \quad (3e)$$

$$\text{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \text{refutable}_?(\dot{\xi}_1) \text{ and } \text{refutable}_?(\dot{\xi}_2) \quad (3f)$$

$$\text{Otherwise} \quad \text{refutable}_?(\dot{\xi}) = \text{false} \quad (3g)$$

**Lemma 1.0.1** (Soundness and Completeness of Refutable Constraints).  $\dot{\xi} \text{ refutable}_?$  iff  $\text{refutable}_?(\dot{\xi}) = \text{true}$ .

$$\boxed{e \models \dot{\xi}} \quad e \text{ satisfies } \dot{\xi}$$

$$\frac{\text{CSTruth}}{e \models \top} \quad (4a)$$

$$\frac{\text{CSNum}}{\underline{n} \models n} \quad (4b)$$

$$\frac{\text{CSInl} \quad e_1 \models \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)} \quad (4c)$$

$$\frac{\text{CSInr} \quad e_2 \models \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)} \quad (4d)$$

$$\frac{\text{CSPair} \quad e_1 \models \dot{\xi}_1 \quad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)} \quad (4e)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \dot{\xi}_1 \quad \text{prr}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)} \quad (4f)$$

$$\frac{\text{CSOrL} \quad e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \quad (4g)$$

$$\frac{\text{CSOrR} \quad e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2} \quad (4h)$$

$$\boxed{\text{satisfy}(e, \dot{\xi})}$$

$$\text{satisfy}(e, \top) = \text{true} \quad (5a)$$

$$\text{satisfy}(n_1, n_2) = (n_1 = n_2) \quad (5b)$$

$$\text{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{satisfy}(e, \dot{\xi}_1) \text{ or } \text{satisfy}(e, \dot{\xi}_2) \quad (5c)$$

$$\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{satisfy}(e_1, \dot{\xi}_1) \quad (5d)$$

$$\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = \text{satisfy}(e_2, \dot{\xi}_2) \quad (5e)$$

$$\text{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(e_1, \dot{\xi}_1) \text{ and } \text{satisfy}(e_2, \dot{\xi}_2) \quad (5f)$$

$$\text{satisfy}(\mathbb{0}^u, (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(\text{prl}(\mathbb{0}^u), \dot{\xi}_1) \text{ and } \text{satisfy}(\text{prr}(\mathbb{0}^u), \dot{\xi}_2) \quad (5g)$$

$$\text{satisfy}(\llbracket e \rrbracket^u, (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(\text{prl}(\llbracket e \rrbracket^u), \dot{\xi}_1) \text{ and } \text{satisfy}(\text{prr}(\llbracket e \rrbracket^u), \dot{\xi}_2) \quad (5h)$$

$$\begin{aligned} \text{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(e_1(e_2)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{prr}(e_1(e_2)), \dot{\xi}_2) \end{aligned} \quad (5i)$$

$$\begin{aligned} \text{satisfy}(\text{match}(e)\{rs\}, (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{match}(e)\{rs\}), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{prr}(\text{match}(e)\{rs\}), \dot{\xi}_2) \end{aligned} \quad (5j)$$

$$\begin{aligned} \text{satisfy}(\text{prl}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{prl}(e)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{prr}(\text{prl}(e)), \dot{\xi}_2) \end{aligned} \quad (5k)$$

$$\begin{aligned} \text{satisfy}(\text{prr}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \text{satisfy}(\text{prl}(\text{prr}(e)), \dot{\xi}_1) \\ &\text{ and } \text{satisfy}(\text{prr}(\text{prr}(e)), \dot{\xi}_2) \end{aligned} \quad (5l)$$

$$\text{Otherwise } \text{satisfy}(e, \dot{\xi}) = \text{false} \quad (5m)$$

**Lemma 1.0.2** (Soundness and Completeness of Satisfaction Judgment).  $e \models \dot{\xi}$  iff  $\text{satisfy}(e, \dot{\xi}) = \text{true}$ .

*Proof.* We prove soundness and completeness separately.

1. Soundness:

$$(1) \quad e \models \dot{\xi} \quad \text{by assumption}$$

By rule induction over Rules (15) on (1).

**Case (15a).**

- (2)  $\dot{\xi} = \top$  by assumption
- (3)  $satisfy(e, \top) = \text{true}$  by Definition 16a

**Case (15b).**

- (2)  $e = \underline{n}$  by assumption
- (3)  $\dot{\xi} = \underline{n}$  by assumption
- (4)  $satisfy(\underline{n}, \underline{n}) = (n = n) = \text{true}$  by Definition 16b

**Case (15c).**

- (2)  $e = \underline{n_1}$  by assumption
- (3)  $\dot{\xi} = \underline{\underline{p_2}}$  by assumption
- (4)  $n_1 \neq n_2$  by assumption
- (5)  $satisfy(\underline{n_1}, \underline{\underline{p_2}}) = (n_1 \neq n_2) = \text{true}$  by Definition 16c on (4)

**Case (15d).**

- (2)  $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$  by assumption
- (3)  $e \models \dot{\xi}_1$  by assumption
- (4)  $e \models \dot{\xi}_2$  by assumption
- (5)  $satisfy(e, \dot{\xi}_1) = \text{true}$  by IH on (3)
- (6)  $satisfy(e, \dot{\xi}_2) = \text{true}$  by IH on (4)
- (7)  $satisfy(e, \dot{\xi}_1 \wedge \dot{\xi}_2) = satisfy(e, \dot{\xi}_1)$  and  $satisfy(e, \dot{\xi}_2) = \text{true}$  by Definition 16d on (5) and (6)

**Case (15e).**

- (2)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption
- (3)  $e \models \dot{\xi}_1$  by assumption
- (4)  $satisfy(e, \dot{\xi}_1) = \text{true}$  by IH on (3)
- (5)  $satisfy(e, \dot{\xi}_1 \vee \dot{\xi}_2) = satisfy(e, \dot{\xi}_1)$  or  $satisfy(e, \dot{\xi}_2) = \text{true}$  by Definition 16e on (4)

**Case (15f).**

- (2)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption
- (3)  $e \models \dot{\xi}_2$  by assumption
- (4)  $satisfy(e, \dot{\xi}_2) = \text{true}$  by IH on (3)
- (5)  $satisfy(e, \dot{\xi}_1 \vee \dot{\xi}_2) = satisfy(e, \dot{\xi}_1)$  or  $satisfy(e, \dot{\xi}_2) = \text{true}$  by Definition 16e on (4)

**Case (15g).**

- (2)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption
- (3)  $\dot{\xi} = \text{inl}(\dot{\xi}_1)$  by assumption
- (4)  $e_1 \models \dot{\xi}_1$  by assumption
- (5)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by IH on (4)
- (6)  $\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by Definition 16f on (5)

**Case (15h).**

- (2)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption
- (3)  $\dot{\xi} = \text{inl}(\dot{\xi}_2)$  by assumption
- (4)  $e_2 \models \dot{\xi}_2$  by assumption
- (5)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by IH on (4)
- (6)  $\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\dot{\xi}_2)) = \text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by Definition 16g on (5)

**Case (15i).**

- (2)  $e = (e_1, e_2)$  by assumption
- (3)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption
- (4)  $e_1 \models \dot{\xi}_1$  by assumption
- (5)  $e_2 \models \dot{\xi}_2$  by assumption
- (6)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by IH on (5)
- (8)  $\text{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{satisfy}(e_1, \dot{\xi}_1) \text{ and } \text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by Definition 16h on (6) and (7)

**Case (15j).**

- (2)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption
- (3)  $e \text{ notintro}$  by assumption
- (4)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption
- (5)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption
- (6)  $\text{satisfy}(\text{prl}(e), \dot{\xi}_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(\text{prr}(e), \dot{\xi}_2) = \text{true}$  by IH on (5)

By rule induction over Rules (27) on (3).

**Otherwise.**

- (8)  $e = (\emptyset)^u, (\emptyset_{e_0})^u, e_1(e_2), \mathbf{prl}(e_0), \mathbf{prr}(e_0), \mathbf{match}(e_0)\{\hat{r}s\}$   
by assumption
- (9)  $\mathit{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) =$   
 $\mathit{satisfy}(\mathbf{prl}(e), \dot{\xi}_1)$  and  $\mathit{satisfy}(\mathbf{prr}(e), \dot{\xi}_2) = \text{true}$   
by Definition 16 on (6)  
and (7)

2. Completeness:

- (1)  $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$  by assumption

By structural induction on  $\dot{\xi}$ .

**Case**  $\dot{\xi} = \top$ .

- (2)  $e \models \top$  by Rule (15a)

**Case**  $\dot{\xi} = \perp, ?$ .

- (2)  $\mathit{satisfy}(e, \dot{\xi}) = \text{false}$  by Definition 16o

(2) contradicts (1) and thus vacuously true.

**Case**  $\dot{\xi} = \underline{n}$ .

By structural induction on  $e$ .

**Case**  $e = \underline{n'}$ .

- (2)  $\underline{n'} = n$  by Definition 16b on (1)  
(3)  $\underline{n'} \models \underline{n}$  by Rule (15b) on (2)

**Otherwise.**

- (2)  $\mathit{satisfy}(e, \underline{n}) = \text{false}$  by Definition 16o  
(2) contradicts (1) and thus vacuously true.

**Case**  $\dot{\xi} = \underline{\mathcal{N}}$ .

By structural induction on  $e$ .

**Case**  $e = \underline{n'}$ .

- (2)  $\underline{n'} \neq n$  by Definition 16c on (1)  
(3)  $\underline{n'} \models \underline{\mathcal{N}}$  by Rule (15c) on (2)

**Otherwise.**

- (2)  $\mathit{satisfy}(e, \underline{\mathcal{N}}) = \text{false}$  by Definition 16o  
(2) contradicts (1) and thus vacuously true.

**Case**  $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$ .

- (2)  $\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$  by Definition 16d on (1)

- |  |                              |
|--|------------------------------|
| (3) $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$ | by Definition 16d on (1)     |
| (4) $e \models \dot{\xi}_1$                        | by IH on (2)                 |
| (5) $e \models \dot{\xi}_2$                        | by IH on (3)                 |
| (6) $e \models \dot{\xi}_1 \wedge \dot{\xi}_2$     | by Rule (15d) on (4) and (5) |

**Case**  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ .

- |  |                          |
|--|--------------------------|
| (2) $\text{satisfy}(e, \dot{\xi}_1)$ or $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$ | by Definition 16e on (1) |
|--|--------------------------|

By case analysis on (2).

**Case**  $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$ .

- |  |                      |
|--|----------------------|
| (3) $\text{satisfy}(e, \dot{\xi}_1) = \text{true}$ | by assumption        |
| (4) $e \models \dot{\xi}_1$                        | by IH on (3)         |
| (5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$       | by Rule (15e) on (4) |

**Case**  $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$ .

- |  |                      |
|--|----------------------|
| (3) $\text{satisfy}(e, \dot{\xi}_2) = \text{true}$ | by assumption        |
| (4) $e \models \dot{\xi}_2$                        | by IH on (3)         |
| (5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$       | by Rule (15f) on (4) |

**Case**  $\dot{\xi} = \text{inl}(\dot{\xi}_1)$ .

By structural induction on  $e$ .

**Case**  $e = \text{inl}_{\tau_2}(e_1)$ .

- |  |                          |
|--|--------------------------|
| (2) $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$           | by Definition 16f on (1) |
| (3) $e_1 \models \dot{\xi}_1$                                  | by IH on (2)             |
| (4) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$ | by Rule (15g) on (3)     |

**Otherwise.**

- |   |                   |
|---|-------------------|
| (2) $\text{satisfy}(e, \text{inl}(\dot{\xi}_1)) = \text{false}$ | by Definition 16o |
| (2) contradicts (1) and thus vacuously true.                    |                   |

**Case**  $\dot{\xi} = \text{inr}(\dot{\xi}_2)$ .

By structural induction on  $e$ .

**Case**  $e = \text{inr}_{\tau_1}(e_2)$ .

- |  |                          |
|--|--------------------------|
| (2) $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$           | by Definition 16g on (1) |
| (3) $e_2 \models \dot{\xi}_2$                                  | by IH on (2)             |
| (4) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$ | by Rule (15h) on (3)     |

**Otherwise.**

(2)  $\text{satisfy}(e, \text{inr}(\dot{\xi}_2)) = \text{false}$  by Definition 16o

(2) contradicts (1) and thus vacuously true.

**Case**  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ .

By structural induction on  $e$ .

**Case**  $e = (e_1, e_2)$ .

- (2)  $\text{satisfy}(e_1, \dot{\xi}_1) = \text{true}$  by Definition 16h on (1)
- (3)  $\text{satisfy}(e_2, \dot{\xi}_2) = \text{true}$  by Definition 16h on (1)
- (4)  $e_1 \models \dot{\xi}_1$  by IH on (2)
- (5)  $e_2 \models \dot{\xi}_2$  by IH on (3)
- (6)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (15i) on (4) and (5)

**Case**  $e = (\text{!})^u, (\text{!}_{e_0})^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$ .

- (2)  $\text{satisfy}(\text{prl}(e), \dot{\xi}_1) = \text{true}$  by Definition 16h on (1)
- (3)  $\text{satisfy}(\text{prl}(e), \dot{\xi}_2) = \text{true}$  by Definition 16h on (1)
- (4)  $\text{prl}(e) \models \dot{\xi}_1$  by IH on (2)
- (5)  $\text{prl}(e) \models \dot{\xi}_2$  by IH on (3)
- (6)  $e \text{ notintro}$  by each rule in Rules (27)
- (7)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (15j) on (6) and (4) and (5)

**Otherwise.**

(2)  $\text{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) = \text{false}$  by Definition 16o

(2) contradicts (1) and thus vacuously true.

□

$e \models_{\text{?}} \dot{\xi}$

$e$  may satisfy  $\dot{\xi}$

CMSUnknown

$$\frac{}{e \models_{\text{?}} ?} \quad (6a)$$

CMSInl

$$\frac{e_1 \models_{\text{?}} \dot{\xi}_1}{\text{inl}_{\tau_2}(e_1) \models_{\text{?}} \text{inl}(\dot{\xi}_1)} \quad (6b)$$

CMSInr

$$\frac{e_2 \models_{\text{?}} \dot{\xi}_2}{\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\dot{\xi}_2)} \quad (6c)$$



$$\frac{\text{CMSPairL} \quad e_1 \models_{\dot{?}} \dot{\xi}_1 \quad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6d)$$

$$\frac{\text{CMSPairR} \quad e_1 \models \dot{\xi}_1 \quad e_2 \models_{\dot{?}} \dot{\xi}_2}{(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6e)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{\dot{?}} \dot{\xi}_1 \quad e_2 \models_{\dot{?}} \dot{\xi}_2}{(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)} \quad (6f)$$

$$\frac{\text{CMSOrL} \quad e \models_{\dot{?}} \dot{\xi}_1 \quad e \not\models \dot{\xi}_2}{e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2} \quad (6g)$$

$$\frac{\text{CMSOrR} \quad e \not\models \dot{\xi}_1 \quad e \models_{\dot{?}} \dot{\xi}_2}{e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2} \quad (6h)$$

$$\frac{\text{CMSNotIntro} \quad e \text{ notintro} \quad \dot{\xi} \text{ refutable}_{\dot{?}}}{e \models_{\dot{?}} \dot{\xi}} \quad (6i)$$

$$\boxed{\text{maysatisfy}(e, \dot{\xi})}$$

$$\text{maysatisfy}(e, ?) = \text{true} \quad (7a)$$

$$\text{maysatisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) = \text{maysatisfy}(e_1, \dot{\xi}_1) \quad (7b)$$

$$\text{maysatisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) = \text{maysatisfy}(e_2, \dot{\xi}_2) \quad (7c)$$

$$\begin{aligned} \text{maysatisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = & \left( \text{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \text{satisfy}(e_2, \dot{\xi}_2) \right) \\ & \text{or } \left( \text{satisfy}(e_1, \dot{\xi}_1) \text{ and } \text{maysatisfy}(e_2, \dot{\xi}_2) \right) \\ & \text{or } \left( \text{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \text{maysatisfy}(e_2, \dot{\xi}_2) \right) \end{aligned} \quad (7d)$$

$$\begin{aligned} \text{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = & \left( \text{maysatisfy}(e, \dot{\xi}_1) \text{ and } \left( \text{not } \text{satisfy}(e, \dot{\xi}_2) \right) \right) \\ & \text{or } \left( \left( \text{not } \text{satisfy}(e, \dot{\xi}_1) \right) \text{ and } \text{maysatisfy}(e, \dot{\xi}_2) \right) \end{aligned} \quad (7e)$$

$$\text{maysatisfy}(e, \dot{\xi}) = \text{notintro}(e) \text{ and } \text{refutable}_{\dot{?}}(\dot{\xi}) \quad (7f)$$

$$\text{Otherwise } \text{maysatisfy}(e, \dot{\xi}) = \text{false} \quad (7g)$$

**Lemma 1.0.3** (Soundness and Completeness of Maybe Satisfaction Judgment).  
 $e \models_{\dot{?}} \dot{\xi}$  iff  $\text{maysatisfy}(e, \dot{\xi}) = \text{true}$ .

*Proof.* We prove soundness and completeness separately.

1. Soundness:

$$(1) \quad e \models_{\tau} \dot{\xi} \quad \text{by assumption}$$

By rule induction over Rules (17) on (1).

**Case (17a).**

$$(2) \quad \dot{\xi} = ? \quad \text{by assumption}$$

$$(3) \quad \text{maysatisfy}(e, ?) \quad \text{by Equation 7a}$$

**Case (17e).**

$$(2) \quad e = \text{inl}_{\tau_2}(e_1) \quad \text{by assumption}$$

$$(3) \quad \dot{\xi} = \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

$$(4) \quad e_1 \models_{\tau} \dot{\xi}_1 \quad \text{by assumption}$$

$$(5) \quad \text{maysatisfy}(e_1, \dot{\xi}_1) \quad \text{by IH on (4)}$$

$$(6) \quad \text{maysatisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\dot{\xi}_1)) \quad \text{by Equation 7b on (5)}$$

**Case (17f).**

$$(2) \quad e = \text{inr}_{\tau_1}(e_2) \quad \text{by assumption}$$

$$(3) \quad \dot{\xi} = \text{inr}(\dot{\xi}_2) \quad \text{by assumption}$$

$$(4) \quad e_2 \models_{\tau} \dot{\xi}_2 \quad \text{by assumption}$$

$$(5) \quad \text{maysatisfy}(e_2, \dot{\xi}_2) \quad \text{by IH on (4)}$$

$$(6) \quad \text{maysatisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\dot{\xi}_2)) \quad \text{by Equation 7c on (5)}$$

**Case (17g).**

$$(2) \quad e = (e_1, e_2) \quad \text{by assumption}$$

$$(3) \quad \dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

$$(4) \quad e_1 \models_{\tau} \dot{\xi}_1 \quad \text{by assumption}$$

$$(5) \quad e_2 \models_{\tau} \dot{\xi}_2 \quad \text{by assumption}$$

$$(6) \quad \text{maysatisfy}(e_1, \dot{\xi}_1) \quad \text{by IH on (4)}$$

$$(7) \quad \text{satisfy}(e_2, \dot{\xi}_2) \quad \text{by Lemma 2.0.19 on (5)}$$

$$(8) \quad \text{maysatisfy}((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) \quad \text{by Equation 7d on (6) and (7)}$$

**Case (17h).**

$$(2) \quad e = (e_1, e_2) \quad \text{by assumption}$$

$$(3) \quad \dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

$$(4) \quad e_1 \models_{\tau} \dot{\xi}_1 \quad \text{by assumption}$$

$$(5) \quad e_2 \models_{\tau} \dot{\xi}_2 \quad \text{by assumption}$$

$$(6) \quad \text{satisfy}(e_1, \dot{\xi}_1) \quad \text{by Lemma 2.0.19 on (4)}$$

- |   |                               |
|---|-------------------------------|
| (7) $\text{maysatisfy}(e_2, \dot{\xi}_2)$                                       | by IH on (5)                  |
| (8) $\text{maysatisfy}((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2))$ | by Equation 7d on (6) and (7) |

**Case (17i).**

- |   |                               |
|---|-------------------------------|
| (2) $e = (e_1, e_2)$  | by assumption                 |
| (3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$                                    | by assumption                 |
| (4) $e_1 \models_{\text{?}} \dot{\xi}_1$  | by assumption                 |
| (5) $e_2 \models_{\text{?}} \dot{\xi}_2$  | by assumption                 |
| (6) $\text{maysatisfy}(e_1, \dot{\xi}_1)$                                       | by IH on (4)                  |
| (7) $\text{maysatisfy}(e_2, \dot{\xi}_2)$                                       | by IH on (?)                  |
| (8) $\text{maysatisfy}((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2))$ | by Equation 7d on (6) and (7) |

**Case (17c).**

- |  |                                |
|--|--------------------------------|
| (2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$           | by assumption                  |
| (3) $e \models_{\text{?}} \dot{\xi}_1$                   | by assumption                  |
| (4) $e \not\models \dot{\xi}_2$                          | by assumption                  |
| (5) $\text{maysatisfy}(e, \dot{\xi}_1)$                  | by IH on (3)                   |
| (6) $\text{not satisfy}(e, \dot{\xi}_2)$                 | by Lemma 2.0.19 on (4)         |
| (7) $\text{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2)$ | by Equation 16e on (5) and (6) |

**Case (17d).**

- |  |                                |
|--|--------------------------------|
| (2) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$           | by assumption                  |
| (3) $e \not\models \dot{\xi}_1$                          | by assumption                  |
| (4) $e \models_{\text{?}} \dot{\xi}_2$                   | by assumption                  |
| (5) $\text{not satisfy}(e, \dot{\xi}_1)$                 | by Lemma 2.0.19 on (3)         |
| (6) $\text{maysatisfy}(e, \dot{\xi}_2)$                  | by IH on (4)                   |
| (7) $\text{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2)$ | by Equation 16e on (5) and (6) |

**Case (17b).**

- |  |                               |
|--|-------------------------------|
| (2) $e \text{ notintro}$                     | by assumption                 |
| (3) $\dot{\xi} \text{ refutable}_{\text{?}}$ | by assumption                 |
| (4) $\text{notintro}(e)$                     | by Lemma 4.0.1 on (?)         |
| (5) $\text{refutable}_{\text{?}}(\dot{\xi})$ | by Lemma 2.0.14 on (3)        |
| (6) $\text{maysatisfy}(e, \dot{\xi})$        | by Equation 7f on (4) and (5) |

2. Completeness:

□

$$\begin{array}{c}
 \boxed{e \models_{\text{?}}^{\dagger} \dot{\xi}} \quad e \text{ satisfies or may satisfy } \dot{\xi} \\
 \text{CSMSMay} \\
 \frac{e \models_{\text{?}} \dot{\xi}}{e \models_{\text{?}}^{\dagger} \dot{\xi}} \quad (8a) \\
 \text{CSMSSat} \\
 \frac{e \models \dot{\xi}}{e \models_{\text{?}}^{\dagger} \dot{\xi}} \quad (8b)
 \end{array}$$

**Lemma 1.0.4.**

**Lemma 1.0.5.**  $e \not\models \perp$

*Proof.* By rule induction over Rules (15), we notice that  $e \models \perp$  is in syntactic contradiction with all rules, hence not derivable. □

**Lemma 1.0.6.**  $e \not\models_{\text{?}} \perp$

*Proof.* Assume  $e \models_{\text{?}} \perp$ . By rule induction over Rules (17) on  $e \models_{\text{?}} \perp$ , only one case applies.

**Case (17b).**

(1)  $\perp$  **refutable?** by assumption

By rule induction over Rules (11) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \perp$  is not derivable. □

**Lemma 1.0.7.**  $e \not\models_{\text{?}} \top$

*Proof.* Assume  $e \models_{\text{?}} \top$ . By rule induction over Rules (17) on  $e \models_{\text{?}} \top$ , only one case applies.

**Case (17b).**

(1)  $\top$  **refutable?** by assumption

By rule induction over Rules (11) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \top$  is not derivable. □

**Lemma 1.0.8.**  $e \not\models ?$

*Proof.* By rule induction over Rules (15), we notice that  $e \models ?$  is in syntactic contradiction with all the cases, hence not derivable. □

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**Lemma 1.0.9.**  $e \models_{\dot{?}}^{\dagger} \dot{\xi}$  iff  $e \models_{\dot{?}}^{\dagger} \dot{\xi} \vee \perp$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

(1)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}$  by assumption

By rule induction over Rules (18) on (1).

**Case (18a).**

(2)  $e \models_{\dot{?}} \dot{\xi}$  by assumption  
 (3)  $e \models_{\dot{?}} \dot{\xi} \vee \perp$  by Rule (17c) on (2)  
and Lemma 2.0.1  
 (4)  $e \models_{\dot{?}}^{\dagger} \dot{\xi} \vee \perp$  by Rule (18a) on (3)

**Case (18b).**

(2)  $e \models \dot{\xi}$  by assumption  
 (3)  $e \models \dot{\xi} \vee \perp$  by Rule (15e) on (2)  
 (4)  $e \models_{\dot{?}}^{\dagger} \dot{\xi} \vee \perp$  by Rule (18b) on (3)

2. Necessity:

(1)  $e \models_{\dot{?}}^{\dagger} \dot{\xi} \vee \perp$  by assumption

By rule induction over Rules (18) on (1).

**Case (18a).**

(2)  $e \models_{\dot{?}} \dot{\xi} \vee \perp$  by assumption

By rule induction over Rules (17) on (2), only two of them apply.

**Case (17c).**

(3)  $e \models_{\dot{?}} \dot{\xi}$  by assumption  
 (4)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}$  by Rule (18a) on (3)

**Case (17d).**

(3)  $e \models_{\dot{?}} \perp$  by assumption  
 (4)  $e \not\models_{\dot{?}} \perp$  by Lemma 2.0.2  
 (3) contradicts (4).

**Case (18b).**

(2)  $e \models \dot{\xi} \vee \perp$  by assumption

By rule induction over Rules (15) on (2), only two of them apply.

**Case (15e).**

- |  |                      |
|--|----------------------|
| (3) $e \models \dot{\xi}$                  | by assumption        |
| (4) $e \models_{\tau}^{\dagger} \dot{\xi}$ | by Rule (18b) on (3) |

**Case (15f).**

- |                           |                |
|---------------------------|----------------|
| (3) $e \models \perp$     | by assumption  |
| (4) $e \not\models \perp$ | by Lemma 2.0.1 |
| (3) contradicts (4).      |                |

□

**Corollary 1.0.1.**  $\top \models_{\tau}^{\dagger} \dot{\xi} \text{ iff } \top \models_{\tau}^{\dagger} \dot{\xi} \vee \perp$

*Proof.* Follows directly from Definition 2.1.2 and Lemma 2.0.5. □

**Lemma 1.0.10.** *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \not\models \dot{\xi}_2$  iff  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$*

*Proof.*

- |                                     |                             |
|-------------------------------------|-----------------------------|
| (1) $\dot{\xi}_1 : \tau$            | by assumption               |
| (2) $\dot{\xi}_2 : \tau$            | by assumption               |
| (3) $\perp : \tau$                  | by Rule (9b)                |
| (4) $\dot{\xi}_2 \vee \perp : \tau$ | by Rule (9f) on (2) and (3) |

Then we prove sufficiency and necessity separately.

1. Sufficiency:

- |   |               |
|---|---------------|
| (5) $\dot{\xi}_1 \not\models \dot{\xi}_2$ | by assumption |
|---|---------------|

To prove  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$ , assume  $\dot{\xi}_1 \models \dot{\xi}_2 \vee \perp$ .

- |  |               |
|--|---------------|
| (6) $\dot{\xi}_1 \models \dot{\xi}_2 \vee \perp$ | by assumption |
|--|---------------|

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies

- |  |  |
|--|--|
| (7) $e \models \dot{\xi}_2 \vee \perp$ | by Definition 2.1.1 on (1) and (4) and (6) |
|--|--|

By rule induction over Rules (15) on (7).

**Case (15e).**

- |                                       |                            |
|---------------------------------------|----------------------------|
| (8) $e \models \dot{\xi}_2$           | by assumption              |
| (9) $\dot{\xi}_1 \models \dot{\xi}_2$ | by Definition 2.1.1 on (8) |
| (5) contradicts (9).                  |                            |

**Case (15f).**

- (8)  $e \models \perp$  by assumption
- (9)  $e \not\models \perp$  by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

- (a)  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$

2. Necessity:

- (5)  $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \perp$  by assumption

To prove  $\dot{\xi}_1 \not\models \dot{\xi}_2$ , assume  $\dot{\xi}_1 \models \dot{\xi}_2$ .

- (6)  $\dot{\xi}_1 \models \dot{\xi}_2$  by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies

- (7)  $e \models \dot{\xi}_2$  by Definition 2.1.1 on (1) and (2) and (6)
- (8)  $e \models \dot{\xi}_2 \vee \perp$  by Rule (15e) on (7)
- (9)  $\dot{\xi}_1 \models \dot{\xi}_2 \vee \perp$  by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

- (a)  $\dot{\xi}_1 \not\models \dot{\xi}_2$

□

**Lemma 1.0.11.**  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\tau}^{\dagger} \dot{\xi}_2$  iff  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency: to show  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ , we assume  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ .

- (1)  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption
- (2)  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by assumption
- (3)  $e \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by assumption

By rule induction over Rules (18) on (1).

**Case (18b).**

- (4)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (15) on (4) and only two of them apply.

**Case (15e).**

- (5)  $e \models \dot{\xi}_1$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by Rule (18b) on (5)
- (6) contradicts (2).

**Case (15f).**

- (5)  $e \models \dot{\xi}_2$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Rule (18b) on (5)
- (6) contradicts (3).

**Case (18a).**

- (4)  $e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (17) on (4) and only two of them apply.

**Case (17c).**

- (5)  $e \models_{\dot{?}} \dot{\xi}_1$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by Rule (18a) on (5)
- (6) contradicts (2).

**Case (17d).**

- (5)  $e \models_{\dot{?}} \dot{\xi}_2$  by assumption
- (6)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Rule (18a) on (5)
- (6) contradicts (3).

The conclusion holds as follows:

- (a)  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$

2. Necessity:

- (1)  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

We show  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  separately.

- (a) To show  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$ , we assume  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$ .

- (2)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by assumption
- (3)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Lemma 2.0.10 on (2)

Contradicts (1).

- (b) To show  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ , we assume  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .

- (2)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by assumption
- (3)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Lemma 2.0.10 on (2)



Contradicts (1).

In conclusion,  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  and  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .

□

**Lemma 1.0.12.** *If  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  and  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  then  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$*

*Proof.*

(4)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

(5)  $e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by assumption

By rule induction over Rules (18) on (4).

**Case (18b).**

(6)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (15) on (6) and only two of them apply.

**Case (15e).**

(7)  $e \models \dot{\xi}_1$  by assumption

(8)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by Rule (18b) on (7)

(8) contradicts (5).

**Case (15f).**

(7)  $e \models \dot{\xi}_2$  by assumption

(8)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Rule (18b) on (7)

**Case (18a).**

(6)  $e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By rule induction over Rules (17) on (6) and only two of them apply.

**Case (17c).**

(7)  $e \models_{\dot{?}} \dot{\xi}_1$  by assumption

(8)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by Rule (18a) on (7)

(8) contradicts (5).

**Case (17d).**

(7)  $e \models_{\dot{?}} \dot{\xi}_2$  by assumption

(8)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Rule (18a) on (7)

□

**Lemma 1.0.13.** *If  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  then  $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  and  $e \models_{\tau}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$*

*Proof.*

(1)  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  by assumption ,

By rule induction over Rules (18) on (1),

**Case (18b).**

|   |                      |
|---|----------------------|
| (2) $e \models \dot{\xi}_1$                                   | by assumption        |
| (3) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$                  | by Rule (15e) on (2) |
| (4) $e \models \dot{\xi}_2 \vee \dot{\xi}_1$                  | by Rule (15f) on (2) |
| (5) $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ | by Rule (18b) on (3) |
| (6) $e \models_{\tau}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$ | by Rule (18b) on (4) |

**Case (18a).**

(2)  $e \models_{\tau} \dot{\xi}_1$  by assumption

By case analysis on the result of  $satisfy(e, \dot{\xi}_2)$ .

**Case true.**

|   |                        |
|---|------------------------|
| (3) $satisfy(e, \dot{\xi}_2) = \text{true}$                   | by assumption          |
| (4) $e \models \dot{\xi}_2$                                   | by Lemma 2.0.19 on (3) |
| (5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$                  | by Rule (15f) on (4)   |
| (6) $e \models \dot{\xi}_2 \vee \dot{\xi}_1$                  | by Rule (15e) on (4)   |
| (7) $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ | by Rule (18b) on (5)   |
| (8) $e \models_{\tau}^{\dagger} \dot{\xi}_2 \vee \dot{\xi}_1$ | by Rule (18b) on (6)   |

**Case false.**

|   |                              |
|---|------------------------------|
| (3) $satisfy(e, \dot{\xi}_2) = \text{false}$                  | by assumption                |
| (4) $e \not\models \dot{\xi}_2$                               | by Lemma 2.0.19 on (3)       |
| (5) $e \models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2$           | by Rule (17c) on (2) and (4) |
| (6) $e \models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ | by Rule (18a) on (5)         |

□

**Lemma 1.0.14.**  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  iff  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

$$(1) \quad e_1 \models_{\tau}^{\dagger} \dot{\xi}_1 \quad \text{by assumption}$$

By rule induction over Rules (18) on (1).

**Case (18b).**

$$\begin{aligned} (2) \quad e_1 &\models \dot{\xi}_1 && \text{by assumption} \\ (3) \quad \text{inl}_{\tau_2}(e_1) &\models \text{inl}(\dot{\xi}_1) && \text{by Rule (15g) on (2)} \\ (4) \quad \text{inl}_{\tau_2}(e_1) &\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1) && \text{by Rule (18b) on (3)} \end{aligned}$$

**Case (18a).**

$$\begin{aligned} (2) \quad e_1 &\models_{\tau} \dot{\xi}_1 && \text{by assumption} \\ (3) \quad \text{inl}_{\tau_2}(e_1) &\models_{\tau} \text{inl}(\dot{\xi}_1) && \text{by Rule (17e) on (2)} \\ (4) \quad \text{inl}_{\tau_2}(e_1) &\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1) && \text{by Rule (18a) on (3)} \end{aligned}$$

2. Necessity:

$$(1) \quad \text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

By rule induction over Rules (18) on (1).

**Case (18b).**

$$(2) \quad \text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

By rule induction over Rules (15) on (2), only one rule applies.

**Case (15g).**

$$\begin{aligned} (3) \quad e_1 &\models \dot{\xi}_1 && \text{by assumption} \\ (4) \quad e_1 &\models_{\tau}^{\dagger} \dot{\xi}_1 && \text{by Rule (18b) on (3)} \end{aligned}$$

**Case (18a).**

$$(2) \quad \text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1) \quad \text{by assumption}$$

By rule induction over Rules (17) on (2), only two rules apply.

**Case (17e).**

$$\begin{aligned} (3) \quad e_1 &\models_{\tau} \dot{\xi}_1 && \text{by assumption} \\ (4) \quad e_1 &\models_{\tau}^{\dagger} \dot{\xi}_1 && \text{by Rule (18a) on (3)} \end{aligned}$$

**Case (17b).**

$$(3) \quad \text{inl}_{\tau_2}(e_1) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (27) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.15.**  $e_2 \models_{\tau_1}^{\dagger} \dot{\xi}_2 \text{ iff } \text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\dot{\xi}_2)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

$$(1) \quad e_2 \models_{\tau_1}^{\dagger} \dot{\xi}_2 \quad \text{by assumption}$$

By rule induction over Rules (18) on (1).

**Case (18b).**

$$\begin{aligned} (2) \quad e_2 &\models \dot{\xi}_2 && \text{by assumption} \\ (3) \quad \text{inr}_{\tau_1}(e_2) &\models \text{inr}(\dot{\xi}_2) && \text{by Rule (15h) on (2)} \\ (4) \quad \text{inr}_{\tau_1}(e_2) &\models_{\tau_1}^{\dagger} \text{inr}(\dot{\xi}_2) && \text{by Rule (18b) on (3)} \end{aligned}$$

**Case (18a).**

$$\begin{aligned} (2) \quad e_2 &\models_{\tau_1} \dot{\xi}_2 && \text{by assumption} \\ (3) \quad \text{inr}_{\tau_1}(e_2) &\models_{\tau_1} \text{inr}(\dot{\xi}_2) && \text{by Rule (17f) on (2)} \\ (4) \quad \text{inr}_{\tau_1}(e_2) &\models_{\tau_1}^{\dagger} \text{inr}(\dot{\xi}_2) && \text{by Rule (18a) on (3)} \end{aligned}$$

2. Necessity:

$$(1) \quad \text{inr}_{\tau_1}(e_2) \models_{\tau_1}^{\dagger} \text{inr}(\dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (18) on (1).

**Case (18b).**

$$(2) \quad \text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (15) on (2), only one rule applies.

**Case (15h).**

$$\begin{aligned} (3) \quad e_2 &\models \dot{\xi}_2 && \text{by assumption} \\ (4) \quad e_2 &\models_{\tau_1}^{\dagger} \dot{\xi}_2 && \text{by Rule (18b) on (3)} \end{aligned}$$

**Case (18a).**

$$(2) \quad \text{inr}_{\tau_1}(e_2) \models_{\tau_1} \text{inr}(\dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (17) on (2), only two rules apply.

**Case (17f).**

$$\begin{aligned} (3) \quad e_2 &\models_{\tau_1} \dot{\xi}_2 && \text{by assumption} \\ (4) \quad e_2 &\models_{\tau_1}^{\dagger} \dot{\xi}_2 && \text{by Rule (18a) on (3)} \end{aligned}$$

**Case (17b).**

(3)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption  
 By rule induction over Rules (27) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.16.**  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  and  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  iff  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

- (1)  $e_1 \models_{\tau}^{\dagger} \dot{\xi}_1$  by assumption
- (2)  $e_2 \models_{\tau}^{\dagger} \dot{\xi}_2$  by assumption

By rule induction over Rules (18) on (1).

**Case (18b).**

- (3)  $e_1 \models \dot{\xi}_1$  by assumption

By rule induction over Rules (18) on (2).

**Case (18b).**

- (4)  $e_2 \models \dot{\xi}_2$  by assumption
- (5)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (15i) on (3) and (4)
- (6)  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18b) on (5)

**Case (18a).**

- (4)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption
- (5)  $(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17h) on (3) and (4)
- (6)  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (5)

**Case (18a).**

- (4)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption

By rule induction over Rules (18) on (2).

**Case (18b).**

- (5)  $e_2 \models \dot{\xi}_2$  by assumption
- (6)  $(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17g) on (4) and (5)
- (7)  $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (6)

**Case (18a).**

- (5)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption
- (6)  $(e_1, e_2) \models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17h) on (4) and (5)

$$(7) \quad (e_1, e_2) \models_{\text{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by Rule (18a) on (6)}$$

2. Necessity:

$$(1) \quad (e_1, e_2) \models_{\text{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (18) on (1).

**Case (18b).**

$$(2) \quad (e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (15) on (2), only one rule applies.

**Case (15i).**

$$\begin{aligned} (3) \quad e_1 &\models \dot{\xi}_1 && \text{by assumption} \\ (4) \quad e_2 &\models \dot{\xi}_2 && \text{by assumption} \\ (5) \quad e_1 &\models_{\text{?}}^{\dagger} \dot{\xi}_1 && \text{by Rule (18b) on (3)} \\ (6) \quad e_2 &\models_{\text{?}}^{\dagger} \dot{\xi}_2 && \text{by Rule (18b) on (4)} \end{aligned}$$

**Case (18a).**

$$(2) \quad (e_1, e_2) \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2) \quad \text{by assumption}$$

By rule induction over Rules (17) on (2), only three rules apply.

**Case (17g).**

$$\begin{aligned} (3) \quad e_1 &\models_{\text{?}} \dot{\xi}_1 && \text{by assumption} \\ (4) \quad e_2 &\models \dot{\xi}_2 && \text{by assumption} \\ (5) \quad e_1 &\models_{\text{?}}^{\dagger} \dot{\xi}_1 && \text{by Rule (18a) on (3)} \\ (6) \quad e_2 &\models_{\text{?}}^{\dagger} \dot{\xi}_2 && \text{by Rule (18b) on (4)} \end{aligned}$$

**Case (17h).**

$$\begin{aligned} (3) \quad e_1 &\models \dot{\xi}_1 && \text{by assumption} \\ (4) \quad e_2 &\models_{\text{?}} \dot{\xi}_2 && \text{by assumption} \\ (5) \quad e_1 &\models_{\text{?}}^{\dagger} \dot{\xi}_1 && \text{by Rule (18b) on (3)} \\ (6) \quad e_2 &\models_{\text{?}}^{\dagger} \dot{\xi}_2 && \text{by Rule (18a) on (4)} \end{aligned}$$

**Case (17i).**

$$\begin{aligned} (3) \quad e_1 &\models_{\text{?}} \dot{\xi}_1 && \text{by assumption} \\ (4) \quad e_2 &\models_{\text{?}} \dot{\xi}_2 && \text{by assumption} \\ (5) \quad e_1 &\models_{\text{?}}^{\dagger} \dot{\xi}_1 && \text{by Rule (18a) on (3)} \\ (6) \quad e_2 &\models_{\text{?}}^{\dagger} \dot{\xi}_2 && \text{by Rule (18a) on (4)} \end{aligned}$$

□

**Lemma 1.0.17.** *If  $e$  notintro and  $e \models_{\text{?}} \xi$  then  $\xi$  refutable<sub>?</sub>.*

**Lemma 1.0.18.** *There does not exist such a constraint  $\dot{\xi}_1 \wedge \dot{\xi}_2$  such that  $\dot{\xi}_1 \wedge \dot{\xi}_2$  **refutable?**.*

*Proof.* By rule induction over Rules (11), we notice that  $\dot{\xi}_1 \wedge \dot{\xi}_2$  **refutable?** is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 1.0.19.** *There does not exist such a constraint  $\dot{\xi}_1 \vee \dot{\xi}_2$  such that  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable?**.*

*Proof.* By rule induction over Rules (11), we notice that  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable?** is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 1.0.20.** *If  $e$  **notintro** and  $e \models \dot{\xi}$  then  $\dot{\xi}$  ~~**refutable?**~~.*

*Proof.*

- |                           |               |
|---------------------------|---------------|
| (1) $e$ <b>notintro</b>   | by assumption |
| (2) $e \models \dot{\xi}$ | by assumption |

By rule induction over Rules (15) on (2).

**Case (15a).**

- |                        |               |
|------------------------|---------------|
| (3) $\dot{\xi} = \top$ | by assumption |
|------------------------|---------------|

Assume  $\top$  **refutable?**. By rule induction over Rules (11), no case applies due to syntactic contradiction.

Therefore,  $\top$  ~~**refutable?**~~.

**Case (15e),(15f).**

- |  |                 |
|--|-----------------|
| (3) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$                             | by assumption   |
| (4) <del><math>\dot{\xi}_1 \vee \dot{\xi}_2</math> <b>refutable?</b></del> | by Lemma 2.0.17 |

**Case (15d).**

- |  |                 |
|--|-----------------|
| (3) $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$                             | by assumption   |
| (4) <del><math>\dot{\xi}_1 \wedge \dot{\xi}_2</math> <b>refutable?</b></del> | by Lemma 2.0.16 |

**Case (15j).**

- |   |                      |
|---|----------------------|
| (3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$              | by assumption        |
| (4) $\text{prl}(e) \models \dot{\xi}_1$                   | by assumption        |
| (5) $\text{prr}(e) \models \dot{\xi}_2$                   | by assumption        |
| (6) $\text{prl}(e)$ <b>notintro</b>                       | by Rule (27e)        |
| (7) $\text{prr}(e)$ <b>notintro</b>                       | by Rule (27f)        |
| (8) <del><math>\dot{\xi}_1</math> <b>refutable?</b></del> | by IH on (6) and (4) |

(9)  $\dot{\xi}_2 \text{refutable?}$  by IH on (7) and (5)

Assume  $(\dot{\xi}_1, \dot{\xi}_2) \text{refutable?}$ . By rule induction over Rules (11) on it, only two cases apply.

**Case (11d).**

(10)  $\dot{\xi}_1 \text{refutable?}$  by assumption

Contradicts (8).

**Case (11e).**

(10)  $\dot{\xi}_2 \text{refutable?}$  by assumption

Contradicts (9).

Therefore,  $(\dot{\xi}_1, \dot{\xi}_2) \text{refutable?}$ .

**Otherwise.**

(3)  $e = \underline{n}, \text{inl}_{\tau_2}(e_1), \text{inr}_{\tau_1}(e_2), (e_1, e_2)$  by assumption

By rule induction over Rules (27) on (1), no case applies due to syntactic contradiction.

□

**Lemma 1.0.21.**  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  is not derivable.

*Proof.* We prove by assuming  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  and obtaining a contradiction.

(1)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by assumption

By rule induction over Rules (18) on (1).

**Case (18b).**

(2)  $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\dot{\xi}_2)$  by assumption

By rule induction over Rules (15) on (2), no rule applies due to syntactic contradiction.

**Case (18a).**

(2)  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by assumption

By rule induction over Rules (17) on (2), only one rule applies.

**Case (17b).**

(3)  $\text{inl}_{\tau_2}(e_1) \text{notintro}$  by assumption



By rule induction over Rules (27) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.22.**  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  is not derivable.

*Proof.* We prove by assuming  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  and obtaining a contradiction.

(1)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by assumption

By rule induction over Rules (18) on (1).

**Case (18b).**

(2)  $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\dot{\xi}_1)$  by assumption

By rule induction over Rules (15) on (2), no rule applies due to syntactic contradiction.

**Case (18a).**

(2)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\dot{\xi}_1)$  by assumption

By rule induction over Rules (17) on (2), only one rule applies.

**Case (17b).**

(3)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (3), no rule applies due to syntactic contradiction.

□

**Lemma 1.0.23.**  $e \not\models \dot{\xi}$  and  $e \not\models_{\tau} \dot{\xi}$  iff  $e \not\models_{\tau}^{\dagger} \dot{\xi}$ .

*Proof.* 1. Sufficiency:

(1)  $e \not\models \dot{\xi}$  by assumption

(2)  $e \not\models_{\tau} \dot{\xi}$  by assumption

Assume  $e \models_{\tau}^{\dagger} \dot{\xi}$ . By rule induction over Rules (18) on it.

**Case (18a).**

(3)  $e \models \dot{\xi}$  by assumption

Contradicts (1).

**Case (18b).**

(3)  $e \models_{\tau} \dot{\xi}$  by assumption

Contradicts (2).

Therefore,  $e \models_{\tau}^{\dagger} \dot{\xi}$  is not derivable.

2. Necessity:

(1)  $e \not\models_{\tau}^{\dagger} \dot{\xi}$  by assumption

Assume  $e \models \dot{\xi}$ .

(2)  $e \models_{\tau}^{\dagger} \dot{\xi}$  by Rule (18b) on  
assumption

Contradicts (1). Therefore,  $e \not\models \dot{\xi}$ . Assume  $e \models_{\tau} \dot{\xi}$ .

(3)  $e \models_{\tau}^{\dagger} \dot{\xi}$  by Rule (18a) on  
assumption

Contradicts (1). Therefore,  $e \not\models_{\tau} \dot{\xi}$ .

□

**Theorem 1.1** (Exclusiveness of Satisfaction Judgment). *If  $\dot{\xi} : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e$  final then exactly one of the following holds*

1.  $e \models \dot{\xi}$
2.  $e \models_{\tau} \dot{\xi}$
3.  $e \not\models_{\tau}^{\dagger} \dot{\xi}$

*Proof.*

(4)  $\dot{\xi} : \tau$  by assumption

(5)  $\cdot; \Delta \vdash e : \tau$  by assumption

(6)  $e$  final by assumption

By rule induction over Rules (9) on (4), we would show one conclusion is derivable while the other two are not.

**Case (9a).**

(7)  $\dot{\xi} = \top$  by assumption

(8)  $e \models \top$  by Rule (15a)

(9)  $e \not\models_{\tau} \top$  by Lemma 2.0.3

(10)  $e \models_{\tau}^{\dagger} \top$  by Rule (18b) on (8)

**Case (9b).**

|   |                                   |
|---|-----------------------------------|
| (7) $\dot{\xi} = \perp$                         | by assumption                     |
| (8) $e \not\models \perp$                       | by Lemma 2.0.1                    |
| (9) $e \not\models_{\text{?}} \perp$            | by Lemma 2.0.2                    |
| (10) $e \not\models_{\text{?}}^{\dagger} \perp$ | by Lemma 2.0.20 on<br>(8) and (9) |

**Case (1b).**

|   |                      |
|---|----------------------|
| (7) $\dot{\xi} = ?$                     | by assumption        |
| (8) $e \not\models ?$                   | by Lemma 2.0.4       |
| (9) $e \models_{\text{?}} ?$            | by Rule (17a)        |
| (10) $e \models_{\text{?}}^{\dagger} ?$ | by Rule (18a) on (9) |

**Case (9c).**

|                                   |               |
|-----------------------------------|---------------|
| (7) $\dot{\xi} = \underline{n_2}$ | by assumption |
| (8) $\tau = \mathbf{num}$         | by assumption |

By rule induction over Rules (20) on (5), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

|   |  |
|---|--|
| (9) $e = \llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \mathbf{prl}(e_0), \mathbf{prl}(e_0), \mathbf{match}(e_0)\{\hat{r}s\}$ | by assumption                                  |
| (10) $e \mathbf{notintro}$  | by Rule<br>(27a),(27b),(27c),(27d),(27e),(27f) |

Assume  $e \models \underline{n_2}$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction on  $\dot{\xi}$ .

|   |                                   |
|---|-----------------------------------|
| (11) $e \not\models \underline{n_2}$                  | by contradiction                  |
| (12) $\underline{n_2} \mathbf{refutable}_{\text{?}}$  | by Rule (11a)                     |
| (13) $e \models_{\text{?}} \underline{n_2}$           | by Rule (17b) on (10)<br>and (12) |
| (14) $e \models_{\text{?}}^{\dagger} \underline{n_2}$ | by Rule (18a) on (13)             |

**Case (20d).**

|                           |               |
|---------------------------|---------------|
| (9) $e = \underline{n_1}$ | by assumption |
|---------------------------|---------------|

Assume  $\underline{n_1} \models_{\text{?}} \underline{n_2}$ . By rule induction over Rules (17), only one case applies.

**Case (17b).**

(10)  $\underline{n_1}$  **notintro** by assumption  
 Contradicts Lemma 4.0.6.

(11)  $\underline{n_1} \not\models? \underline{n_2}$  by contradiction

By case analysis on whether  $n_1$  is equal to  $n_2$ .

**Case**  $n_1 = n_2$ .

(12)  $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$  by Definition 16  
 (13)  $\underline{n_1} \models \underline{n_2}$  by Lemma 2.0.19 on (12)  
 (14)  $e \models?^\dagger \underline{n_2}$  by Rule (18b) on (13)

**Case**  $n_1 \neq n_2$ .

(12)  $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$  by Definition 16  
 (13)  $\underline{n_1} \not\models \underline{n_2}$  by Lemma 2.0.19 on (12)  
 (14)  $e \not\models?^\dagger \underline{n_2}$  by Lemma 2.0.20 on (11) and (13)

**Case** (9f).

(7)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

By inductive hypothesis on (5) and (6), exactly one of  $e \models \dot{\xi}_1$ ,  $e \models? \dot{\xi}_1$ , and  $e \not\models?^\dagger \dot{\xi}_1$  holds. The same goes for  $\dot{\xi}_2$ . By case analysis on which conclusion holds for  $\dot{\xi}_1$  and  $\dot{\xi}_2$ .

**Case**  $e \models \dot{\xi}_1, e \models \dot{\xi}_2$ .

(8)  $e \models \dot{\xi}_1$  by assumption  
 (9)  $e \not\models? \dot{\xi}_1$  by assumption  
 (10)  $e \models \dot{\xi}_2$  by assumption  
 (11)  $e \not\models? \dot{\xi}_2$  by assumption  
 (12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (15e) on (8)  
 (13)  $e \models?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18b) on (12)

Assume  $e \models? \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

(14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable?** by assumption  
 Contradicts Lemma 2.0.17.

**Case** (17c).

(14)  $e \models? \dot{\xi}_1$  by assumption  
 Contradicts (9).

**Case (17d).**

(14)  $e \models_{\dot{?}} \dot{\xi}_2$  by assumption  
 Contradicts (11).

(15)  $e \not\models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \models \dot{\xi}_1, e \models_{\dot{?}} \dot{\xi}_2$ .**

(8)  $e \models \dot{\xi}_1$  by assumption  
 (9)  $e \not\models_{\dot{?}} \dot{\xi}_1$  by assumption  
 (10)  $e \not\models \dot{\xi}_2$  by assumption  
 (11)  $e \models_{\dot{?}} \dot{\xi}_2$  by assumption  
 (12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (15e) on (8)  
 (13)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18b) on (12)

Assume  $e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable** <sub>$\dot{?}$</sub>  by assumption  
 Contradicts Lemma 2.0.17.

**Case (17c).**

(14)  $e \models_{\dot{?}} \dot{\xi}_1$  by assumption  
 Contradicts (9).

**Case (17d).**

(14)  $e \not\models \dot{\xi}_1$  by assumption  
 Contradicts (8).

(15)  $e \not\models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case  $e \models \dot{\xi}_1, e \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .**

(8)  $e \models \dot{\xi}_1$  by assumption  
 (9)  $e \not\models_{\dot{?}} \dot{\xi}_1$  by assumption  
 (10)  $e \not\models \dot{\xi}_2$  by assumption  
 (11)  $e \not\models_{\dot{?}} \dot{\xi}_2$  by assumption  
 (12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (15e) on (8)  
 (13)  $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18b) on (12)

Assume  $e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

|   |                       |
|---|-----------------------|
| (14) $\dot{\xi}_1 \vee \dot{\xi}_2$ <b>refutable?</b>   | by assumption         |
| Contradicts Lemma 2.0.17.   |                       |
| <b>Case (17c).</b>  |                       |
| (14) $e \models_{\dot{?}} \dot{\xi}_1$  | by assumption         |
| Contradicts (9).  |                       |
| <b>Case (17d).</b>  |                       |
| (14) $e \not\models \dot{\xi}_1$  | by assumption         |
| Contradicts (8).  |                       |
| (15) $e \not\models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$   | by contradiction      |
| <b>Case <math>e \models_{\dot{?}} \dot{\xi}_1, e \models \dot{\xi}_2</math>.</b>  |                       |
| (8) $e \not\models \dot{\xi}_1$   | by assumption         |
| (9) $e \models_{\dot{?}} \dot{\xi}_1$   | by assumption         |
| (10) $e \models \dot{\xi}_2$  | by assumption         |
| (11) $e \not\models_{\dot{?}} \dot{\xi}_2$  | by assumption         |
| (12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$   | by Rule (15f) on (10) |
| (13) $e \models_{\dot{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$   | by Rule (18b) on (12) |
| Assume $e \models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (17) on it, the following cases apply. |                       |
| <b>Case (17b).</b>  |                       |
| (14) $\dot{\xi}_1 \vee \dot{\xi}_2$ <b>refutable?</b>   | by assumption         |
| Contradicts Lemma 2.0.17.   |                       |
| <b>Case (17c).</b>  |                       |
| (14) $e \not\models \dot{\xi}_2$  | by assumption         |
| Contradicts (10).   |                       |
| <b>Case (17d).</b>  |                       |
| (14) $e \models_{\dot{?}} \dot{\xi}_2$  | by assumption         |
| Contradicts (11).   |                       |
| (15) $e \not\models_{\dot{?}} \dot{\xi}_1 \vee \dot{\xi}_2$   | by contradiction      |
| <b>Case <math>e \models_{\dot{?}} \dot{\xi}_1, e \models_{\dot{?}} \dot{\xi}_2</math>.</b>                                      |                       |
| (8) $e \not\models \dot{\xi}_1$   | by assumption         |
| (9) $e \models_{\dot{?}} \dot{\xi}_1$   | by assumption         |
| (10) $e \not\models \dot{\xi}_2$  | by assumption         |
| (11) $e \models_{\dot{?}} \dot{\xi}_2$  | by assumption         |

(12)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17c) on (9) and (10)

(13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18a) on (12)

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (15), only two cases apply.

**Case (15e).**

(14)  $e \models \dot{\xi}_1$  by assumption

Contradicts (8)

**Case (15f).**

(14)  $e \models \dot{\xi}_2$  by assumption

Contradicts (10)

(15)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case**  $e \models_{\text{?}} \dot{\xi}_1, e \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \not\models \dot{\xi}_2$  by assumption

(11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption

(12)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17c) on (9) and (10)

(13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18a) on (12)

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (15), only two cases apply.

**Case (15e).**

(14)  $e \models \dot{\xi}_1$  by assumption

Contradicts (8).

**Case (15f).**

(14)  $e \models \dot{\xi}_2$  by assumption

Contradicts (10).

(15)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case**  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e \models \dot{\xi}_2$ .

(8)  $e \not\models \dot{\xi}_1$  by assumption

(9)  $e \not\models_{\text{?}} \dot{\xi}_1$  by assumption

(10)  $e \models \dot{\xi}_2$  by assumption

(11)  $e \not\models_{\text{?}} \dot{\xi}_2$  by assumption

- (12)  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (15f) on (10)  
 (13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18b) on (12)

Assume  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

- (14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable**<sub>?</sub> by assumption  
 Contradicts Lemma 2.0.17.

**Case (17c).**

- (14)  $e \not\models \dot{\xi}_2$  by assumption  
 Contradicts (10).

**Case (17d).**

- (14)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption  
 Contradicts (11).

- (15)  $e \not\models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case**  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e \models_{\text{?}} \dot{\xi}_2$ .

- (8)  $e \not\models \dot{\xi}_1$  by assumption  
 (9)  $e \not\models_{\text{?}} \dot{\xi}_1$  by assumption  
 (10)  $e \not\models \dot{\xi}_2$  by assumption  
 (11)  $e \models_{\text{?}} \dot{\xi}_2$  by assumption  
 (12)  $e \models_{\text{?}} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (17d) on (11) and (8)  
 (13)  $e \models_{\text{?}}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Rule (18a) on (12)

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (15), only two cases apply.

**Case (15e).**

- (14)  $e \models \dot{\xi}_1$  by assumption  
 Contradicts (8)

**Case (15f).**

- (14)  $e \models \dot{\xi}_2$  by assumption  
 Contradicts (10)

- (15)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

**Case**  $e \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .

- (8)  $e \not\models \dot{\xi}_1$  by assumption



- (9)  $e \not\models_{\tau} \dot{\xi}_1$  by assumption
- (10)  $e \not\models \dot{\xi}_2$  by assumption
- (11)  $e \not\models_{\tau} \dot{\xi}_2$  by assumption

Assume  $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15e).**

- (12)  $e \models \dot{\xi}_1$  by assumption
- Contradicts (8).

**Case (15f).**

- (12)  $e \models \dot{\xi}_2$  by assumption
- Contradicts (10).

- (13)  $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction

Assume  $e \models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

- (14)  $\dot{\xi}_1 \vee \dot{\xi}_2$  **refutable** <sub>$\tau$</sub>  by assumption
- Contradicts Lemma 2.0.17.

**Case (17c).**

- (14)  $e \models_{\tau} \dot{\xi}_1$  by assumption
- Contradicts (9).

**Case (17d).**

- (14)  $e \models_{\tau} \dot{\xi}_2$  by assumption
- Contradicts (11).

- (15)  $e \not\models_{\tau} \dot{\xi}_1 \vee \dot{\xi}_2$  by contradiction
- (16)  $e \not\models_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by Lemma 2.0.20 on (13) and (15)

**Case (9g).**

- (7)  $\dot{\xi} = \text{inl}(\dot{\xi}_1)$  by assumption
- (8)  $\tau = (\tau_1 + \tau_2)$  by assumption
- (9)  $\dot{\xi}_1 : \tau_1$  by assumption

By rule induction over Rules (20) on (5), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

- (10)  $e = \mathbb{0}^u, (\mathbb{0}_0)^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (11)  $e \text{ notintro}$   
by Rule  
(27a),(27b),(27c),(27d),(27e),(27f)

Assume  $e \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

- (12)  $e \not\models \text{inl}(\dot{\xi}_1)$   
by contradiction

By case analysis on the value of  $\text{refutable}_?( \text{inl}(\dot{\xi}_1) )$ .

**Case**  $\text{refutable}_?( \text{inl}(\dot{\xi}_1) ) = \text{true}$ .

- (13)  $\text{refutable}_?( \text{inl}(\dot{\xi}_1) ) = \text{true}$   
by assumption
- (14)  $\text{inl}(\dot{\xi}_1) \text{ refutable}_?$   
by Lemma 2.0.14 on  
(13)
- (15)  $e \models_? \text{inl}(\dot{\xi}_1)$   
by Rule (17b) on (11)  
and (14)
- (16)  $e \models_?^\dagger \text{inl}(\dot{\xi}_1)$   
by Rule (18a) on (15)

**Case**  $\text{refutable}_?( \text{inl}(\dot{\xi}_1) ) = \text{false}$ .

- (13)  $\text{refutable}_?( \text{inl}(\dot{\xi}_1) ) = \text{false}$   
by assumption
- (14)  $\text{inl}(\dot{\xi}_1) \text{ refutable}_?$   
by Lemma 2.0.14 on  
(13)

Assume  $e \models_? \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

- (15)  $\text{inl}(\dot{\xi}_1) \text{ refutable}_?$   
by assumption  
Contradicts (14).
- (16)  $e \not\models_? \text{inl}(\dot{\xi}_1)$   
by contradiction
- (17)  $e \not\models_?^\dagger \text{inl}(\dot{\xi}_1)$   
by Lemma 2.0.20 on  
(12) and (16)

**Case** (20j).

- (10)  $e = \text{inl}_{\tau_2}(e_1)$   
by assumption
- (11)  $\cdot; \Delta \vdash e_1 : \tau_1$   
by assumption
- (12)  $e_1 \text{ final}$   
by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_1 \models \dot{\xi}_1$ ,  $e_1 \models_? \dot{\xi}_1$ , and  $e_1 \not\models_?^\dagger \dot{\xi}_1$  holds. By case analysis on which one holds.

**Case**  $e_1 \models \dot{\xi}_1$ .

- (13)  $e_1 \models \dot{\xi}_1$   
by assumption
- (14)  $e_1 \not\models_? \dot{\xi}_1$   
by assumption

(15)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$  by Rule (15g) on (13)

(16)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (18b) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (17) on it, only two cases apply.

**Case (17b).**

(17)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17e).**

(17)  $e_1 \models_{\tau} \dot{\xi}_1$

Contradicts (14).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\dot{\xi}_1)$  by contradiction

**Case  $e_1 \models_{\tau} \dot{\xi}_1$ .**

(13)  $e_1 \not\models \dot{\xi}_1$  by assumption

(14)  $e_1 \models_{\tau} \dot{\xi}_1$  by assumption

(15)  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$  by Rule (17e) on (14)

(16)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (18a) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15g).**

(17)  $e_1 \models \dot{\xi}_1$

Contradicts (13).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\dot{\xi}_1)$  by contradiction

**Case  $e_1 \not\models_{\tau}^{\dagger} \dot{\xi}_1$ .**

(13)  $e_1 \not\models \dot{\xi}_1$  by assumption

(14)  $e_1 \not\models_{\tau} \dot{\xi}_1$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15g).**

(15)  $e_1 \models \dot{\xi}_1$

Contradicts (13).

(16)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\dot{\xi}_1)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(17)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17e).**

(17)  $e_1 \models_{\tau} \dot{\xi}_1$

Contradicts (14).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\dot{\xi}_1)$  by contradiction

(19)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (16) and (18)

**Case (20k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

(11)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\dot{\xi}_1)$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\dot{\xi}_1)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(12)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inl}(\dot{\xi}_1)$  by contradiction

(14)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (11) and (13)

**Case (9h).**

(7)  $\dot{\xi} = \text{inr}(\dot{\xi}_2)$  by assumption

(8)  $\tau = (\tau_1 + \tau_2)$  by assumption

(9)  $\dot{\xi}_2 : \tau_2$  by assumption

By rule induction over Rules (20) on (5), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

(10)  $e = \text{inl}^u, \text{inr}^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption

(11)  $e \text{ notintro}$  by Rule (27a),(27b),(27c),(27d),(27e),(27f)

Assume  $e \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

(12)  $e \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

By case analysis on the value of  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) )$ .

inr is  
refutable

**Case**  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{true}$ .

(13)  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{true}$  by assumption

(14)  $\text{inr}(\dot{\xi}_2) \text{ refutable}_?$  by Lemma 2.0.14 on (13)

(15)  $e \models_? \text{inr}(\dot{\xi}_2)$  by Rule (17b) on (11) and (14)

(16)  $e \models_?^\dagger \text{inr}(\dot{\xi}_2)$  by Rule (18a) on (15)

**Case**  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{false}$ .

(13)  $\text{refutable}_?( \text{inr}(\dot{\xi}_2) ) = \text{false}$  by assumption

(14)  ~~$\text{inr}(\dot{\xi}_2) \text{ refutable}_?$~~  by Lemma 2.0.14 on (13)

Assume  $e \models_? \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

(15)  $\text{inr}(\dot{\xi}_2) \text{ refutable}_?$  by assumption  
Contradicts (14).

(16)  $e \not\models_? \text{inr}(\dot{\xi}_2)$  by contradiction

(17)  $e \not\models_?^\dagger \text{inr}(\dot{\xi}_2)$  by Lemma 2.0.20 on (12) and (16)

**Case** (20j).

(10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

(11)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_? \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

(12)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inl}_{\tau_2}(e_1) \not\models_? \text{inr}(\dot{\xi}_2)$  by contradiction

(14)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Lemma 2.0.20 on (11) and (13)

**Case (20k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption  
 (11)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption  
 (12)  $e_2$  **final** by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_2 \models \dot{\xi}_2$ ,  $e_2 \models_{\tau} \dot{\xi}_2$ , and  $e_2 \not\models_{\tau}^{\dagger} \dot{\xi}_2$  holds. By case analysis on which one holds.

**Case  $e_2 \models \dot{\xi}_2$ .**

(13)  $e_2 \models \dot{\xi}_2$  by assumption  
 (14)  $e_2 \not\models_{\tau} \dot{\xi}_2$  by assumption  
 (15)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$  by Rule (15g) on (13)  
 (16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (18b) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (17) on it, only two cases apply.

**Case (17b).**

(17)  $\text{inr}_{\tau_1}(e_2)$  **notintro** by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17f).**

(17)  $e_2 \models_{\tau} \dot{\xi}_2$

Contradicts (14).

(18)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\dot{\xi}_2)$  by contradiction

**Case  $e_2 \models_{\tau} \dot{\xi}_2$ .**

(13)  $e_2 \not\models \dot{\xi}_2$  by assumption  
 (14)  $e_2 \models_{\tau} \dot{\xi}_2$  by assumption  
 (15)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (17f) on (14)  
 (16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (18a) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15h).**

(17)  $e_2 \models \dot{\xi}_2$

Contradicts (13).

(18)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

**Case  $e_2 \not\models_{\tau}^{\dagger} \dot{\xi}_2$ .**

(13)  $e_2 \not\models \dot{\xi}_2$  by assumption

(14)  $e_2 \not\models? \dot{\xi}_2$  by assumption

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15h).**

(15)  $e_2 \models \dot{\xi}_2$

Contradicts (13).

(16)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\dot{\xi}_2)$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \models? \text{inr}(\dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(17)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17f).**

(17)  $e_2 \models? \dot{\xi}_2$

Contradicts (14).

(18)  $\text{inr}_{\tau_1}(e_2) \not\models? \text{inr}(\dot{\xi}_2)$  by contradiction

(19)  $\text{inl}_{\tau_2}(e_1) \not\models?^\dagger \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.20 on (16) and (18)

**Case (15i).**

(7)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption

(8)  $\tau = (\tau_1 \times \tau_2)$  by assumption

(9)  $\dot{\xi}_1 : \tau_1$  by assumption

(10)  $\dot{\xi}_2 : \tau_2$  by assumption

By rule induction over Rules (20) on (5), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

(11)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$

by assumption

(12)  $e \text{ notintro}$

by Rule

(27a),(27b),(27c),(27d),(27e),(27f)

(13)  $e \text{ indet}$

by Lemma 4.0.10 on

(6) and (12)

(14)  $\text{prl}(e) \text{ indet}$

by Rule (25g) on (13)

(15)  $\text{prl}(e) \text{ final}$

by Rule (26b) on (14)

(16)  $\text{pr}(e) \text{ indet}$

by Rule (25h) on (13)

- (17)  $\text{pr}(e) \text{ final}$  by Rule (26b) on (16)  
 (18)  $\cdot; \Delta \vdash \text{pr}(e) : \tau_1$  by Rule (20h) on (5)  
 (19)  $\cdot; \Delta \vdash \text{pr}(e) : \tau_2$  by Rule (20i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of  $\text{pr}(e) \models \dot{\xi}_1$ ,  $\text{pr}(e) \models_{\tau} \dot{\xi}_1$ , and  $\text{pr}(e) \not\models_{\tau}^{\dagger} \dot{\xi}_1$  holds.

By inductive hypothesis on (10) and (19) and (17), exactly one of  $\text{pr}(e) \models \dot{\xi}_2$ ,  $\text{pr}(e) \models_{\tau} \dot{\xi}_2$ , and  $\text{pr}(e) \not\models_{\tau}^{\dagger} \dot{\xi}_2$  holds.

By case analysis on which conclusion holds for  $\dot{\xi}_1$  and  $\dot{\xi}_2$ .

**Case**  $\text{pr}(e) \models \dot{\xi}_1, \text{pr}(e) \models \dot{\xi}_2$ .

- (20)  $\text{pr}(e) \models \dot{\xi}_1$  by assumption  
 (21)  $\text{pr}(e) \not\models_{\tau} \dot{\xi}_1$  by assumption  
 (22)  $\text{pr}(e) \models \dot{\xi}_2$  by assumption  
 (23)  $\text{pr}(e) \not\models_{\tau} \dot{\xi}_2$  by assumption  
 (24)  $e \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (15j) on (12) and (20) and (22)  
 (25)  $e \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18b) on (24)  
 (26)  ~~$(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\tau}$~~  by Lemma 2.0.18 on (12) and (24)

Assume  $e \models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

- (27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\tau}$  by assumption  
 Contradicts (26).

- (28)  $e \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case**  $\text{pr}(e) \models \dot{\xi}_1, \text{pr}(e) \models_{\tau} \dot{\xi}_2$ .

- (20)  $\text{pr}(e) \models \dot{\xi}_1$  by assumption  
 (21)  $\text{pr}(e) \not\models_{\tau} \dot{\xi}_1$  by assumption  
 (22)  $\text{pr}(e) \not\models \dot{\xi}_2$  by assumption  
 (23)  $\text{pr}(e) \models_{\tau} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15), only one case applies.

**Case** (15j).

- (24)  $\text{pr}(e) \models \dot{\xi}_2$  by assumption  
 Contradicts (22)

- (25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (17) on (23), only one case applies.



**Case (17b).**

- |   |                                |
|---|--------------------------------|
| (26) $\dot{\xi}_2 \text{ refutable}_?$                | by assumption                  |
| (27) $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_?$ | by Rule (11e) on (26)          |
| (28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$         | by Rule (17b) on (12) and (27) |
| (29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ | by Rule (18a) on (28)          |

**Case  $\text{prl}(e) \models \dot{\xi}_1, \text{prr}(e) \not\models_?^\dagger \dot{\xi}_2$ .**

- |  |               |
|--|---------------|
| (20) $\text{prl}(e) \models \dot{\xi}_1$       | by assumption |
| (21) $\text{prl}(e) \not\models_? \dot{\xi}_1$ | by assumption |
| (22) $\text{prr}(e) \not\models \dot{\xi}_2$   | by assumption |
| (23) $\text{prr}(e) \not\models_? \dot{\xi}_2$ | by assumption |

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15j).**

- |  |               |
|--|---------------|
| (24) $\text{prr}(e) \models \dot{\xi}_2$ | by assumption |
| Contradicts (22).                        |               |

- |   |                  |
|---|------------------|
| (25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ | by contradiction |
|---|------------------|

Assume  $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

- |   |               |
|---|---------------|
| (26) $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_?$ | by assumption |
|---|---------------|

By rule induction over Rules (11) on (26), only two cases apply.

**Case (11d).**

- |  |                                |
|--|--------------------------------|
| (27) $\dot{\xi}_1 \text{ refutable}_?$     | by assumption                  |
| (28) $\text{prl}(e) \text{ notintro}$      | by Rule (27e)                  |
| (29) $\text{prl}(e) \models_? \dot{\xi}_1$ | by Rule (17b) on (28) and (27) |

Contradicts (21).

**Case (11e).**

- |  |                                |
|--|--------------------------------|
| (27) $\dot{\xi}_2 \text{ refutable}_?$     | by assumption                  |
| (28) $\text{prr}(e) \text{ notintro}$      | by Rule (27f)                  |
| (29) $\text{prr}(e) \models_? \dot{\xi}_2$ | by Rule (17b) on (28) and (27) |

Contradicts (23).

- |   |                                  |
|---|----------------------------------|
| (30) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$         | by contradiction                 |
| (31) $e \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ | by Lemma 2.0.20 on (25) and (30) |

**Case**  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1, \text{prr}(e) \models \dot{\xi}_2$ .

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15), only one case applies.

**Case** (15j).

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20).

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (17) on (21), only one case applies.

**Case** (17b).

(26)  $\dot{\xi}_1 \text{ refutable}_{\dot{?}}$  by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by Rule (11e) on (26)

(28)  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17b) on (12) and (27)

(29)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (28)

**Case**  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1, \text{prr}(e) \models_{\dot{?}} \dot{\xi}_2$ .

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15), only one case applies.

**Case** (15j).

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20).

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (17) on (23), only one case applies.

**Case** (17b).

(26)  $\dot{\xi}_2 \text{ refutable}_{\dot{?}}$  by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by Rule (11e) on (26)

(28)  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17b) on (12) and (27)

assume no  
"or" and  
"and" in  
pair

assume no  
"or" and  
"and" in  
pair

(29)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (28)

**Case**  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1, \text{prr}(e) \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \not\models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15), only one case applies.

**Case** (15j).

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

By rule induction over Rules (17) on (21), only one case applies.

**Case** (17b).

(26)  $\dot{\xi}_1 \text{ refutable}_{\dot{?}}$  by assumption

(27)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by Rule (11e) on (26)

(28)  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17b) on (12) and (27)

(29)  $e \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (28)

**Case**  $\text{prl}(e) \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1, \text{prr}(e) \models \dot{\xi}_2$ .

(20)  $\text{prl}(e) \not\models \dot{\xi}_1$  by assumption

(21)  $\text{prl}(e) \not\models_{\dot{?}} \dot{\xi}_1$  by assumption

(22)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption

(23)  $\text{prr}(e) \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case** (15j).

(24)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $e \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

(26)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dot{?}}$  by assumption

By rule induction over Rules (11) on (26), only two cases apply.

assume no  
"or" and  
"and" in  
pair

**Case (11d).**

- |   |                                |
|---|--------------------------------|
| (27) $\dot{\xi}_1$ <b>refutable?</b>            | by assumption                  |
| (28) <b>prl</b> ( $e$ ) <b>notintro</b>         | by Rule (27e)                  |
| (29) <b>prl</b> ( $e$ ) $\models_{\dot{\xi}_1}$ | by Rule (17b) on (28) and (27) |

Contradicts (21).

**Case (11e).**

- |   |                                |
|---|--------------------------------|
| (27) $\dot{\xi}_2$ <b>refutable?</b>            | by assumption                  |
| (28) <b>prr</b> ( $e$ ) <b>notintro</b>         | by Rule (27f)                  |
| (29) <b>prr</b> ( $e$ ) $\models_{\dot{\xi}_2}$ | by Rule (17b) on (28) and (27) |

Contradicts (23).

- |   |                                  |
|---|----------------------------------|
| (30) $e \not\models_{\dot{\xi}_1, \dot{\xi}_2}$           | by contradiction                 |
| (31) $e \not\models_{\dot{\xi}_1, \dot{\xi}_2}^{\dagger}$ | by Lemma 2.0.20 on (25) and (30) |

**Case  $\text{prl}(e) \not\models_{\dot{\xi}_1}^{\dagger}, \text{prr}(e) \models_{\dot{\xi}_2}$ .**

- |   |               |
|---|---------------|
| (20) <b>prl</b> ( $e$ ) $\not\models_{\dot{\xi}_1}$           | by assumption |
| (21) <b>prl</b> ( $e$ ) $\not\models_{\dot{\xi}_1}^{\dagger}$ | by assumption |
| (22) <b>prr</b> ( $e$ ) $\not\models_{\dot{\xi}_2}$           | by assumption |
| (23) <b>prr</b> ( $e$ ) $\models_{\dot{\xi}_2}$               | by assumption |

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15), only one case applies.

**Case (15j).**

- |   |               |
|---|---------------|
| (24) <b>prl</b> ( $e$ ) $\models_{\dot{\xi}_1}$ | by assumption |
|---|---------------|

Contradicts (20).

- |   |                  |
|---|------------------|
| (25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ | by contradiction |
|---|------------------|

By rule induction over Rules (17) on (23), only one case applies.

**Case (17b).**

- |   |                                |
|---|--------------------------------|
| (26) $\dot{\xi}_2$ <b>refutable?</b>                  | by assumption                  |
| (27) $(\dot{\xi}_1, \dot{\xi}_2)$ <b>refutable?</b>   | by Rule (11e) on (26)          |
| (28) $e \models_{\dot{\xi}_1, \dot{\xi}_2}$           | by Rule (17b) on (12) and (27) |
| (29) $e \models_{\dot{\xi}_1, \dot{\xi}_2}^{\dagger}$ | by Rule (18a) on (28)          |

**Case  $\text{prl}(e) \not\models_{\dot{\xi}_1}^{\dagger}, \text{prr}(e) \not\models_{\dot{\xi}_2}^{\dagger}$ .**

- |   |               |
|---|---------------|
| (20) <b>prl</b> ( $e$ ) $\not\models_{\dot{\xi}_1}$           | by assumption |
| (21) <b>prl</b> ( $e$ ) $\not\models_{\dot{\xi}_1}^{\dagger}$ | by assumption |

assume no  
"or" and  
"and" in  
pair

(22)  $\text{pr}(e) \not\models \dot{\xi}_2$  by assumption

(23)  $\text{pr}(e) \not\models? \dot{\xi}_2$  by assumption

Assume  $e \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15j).**

(24)  $\text{pr}(e) \models \dot{\xi}_1$  by assumption

Contradicts (20)

(25)  $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $e \models? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(26)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable?}$  by assumption

By rule induction over Rules (11) on (26), only two cases apply.

**Case (11d).**

(27)  $\dot{\xi}_1 \text{ refutable?}$  by assumption

(28)  $\text{pr}(e) \text{ notintro}$  by Rule (27e)

(29)  $\text{pr}(e) \models? \dot{\xi}_1$  by Rule (17b) on (28) and (27)

Contradicts (21).

**Case (11e).**

(27)  $\dot{\xi}_2 \text{ refutable?}$  by assumption

(28)  $\text{pr}(e) \text{ notintro}$  by Rule (27f)

(29)  $\text{pr}(e) \models? \dot{\xi}_2$  by Rule (17b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models? (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(31)  $e \not\models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (25) and (30)

**Case (20g).**

(11)  $e = (e_1, e_2)$  by assumption

(12)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption

(13)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(14)  $e_1 \text{ final}$  by Lemma 4.0.5 on (6)

(15)  $e_2 \text{ final}$  by Lemma 4.0.5 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of  $e_1 \models \dot{\xi}_1$ ,  $e_1 \models? \dot{\xi}_1$ , and  $e_1 \models \dot{\xi}_1$  holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of  $e_2 \models \dot{\xi}_2$ ,  $e_2 \models? \dot{\xi}_2$ , and  $e_2 \models \dot{\xi}_2$  holds.

By case analysis on which conclusion holds for  $\dot{\xi}_1$  and  $\dot{\xi}_2$ .

**Case**  $e_1 \models \dot{\xi}_1, e_2 \models \dot{\xi}_2$ .

- (16)  $e_1 \models \dot{\xi}_1$  by assumption
- (17)  $e_1 \not\models? \dot{\xi}_1$  by assumption
- (18)  $e_2 \models \dot{\xi}_2$  by assumption
- (19)  $e_2 \not\models? \dot{\xi}_2$  by assumption
- (20)  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (15i) on (16) and (18)
- (21)  $(e_1, e_2) \models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18b) on (20)

Assume  $(e_1, e_2) \models? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

- (22)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.9.

**Case** (17g).

- (22)  $e_1 \models? \dot{\xi}_1$  by assumption

Contradicts (17).

**Case** (17h).

- (22)  $e_2 \models? \dot{\xi}_2$  by assumption

Contradicts (19).

**Case** (17i).

- (22)  $e_1 \models? \dot{\xi}_1$  by assumption

Contradicts (17).

- (23)  $(e_1, e_2) \not\models? (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case**  $e_1 \models \dot{\xi}_1, e_2 \models? \dot{\xi}_2$ .

- (16)  $e_1 \models \dot{\xi}_1$  by assumption
- (17)  $e_1 \not\models? \dot{\xi}_1$  by assumption
- (18)  $e_2 \not\models \dot{\xi}_2$  by assumption
- (19)  $e_2 \models? \dot{\xi}_2$  by assumption
- (20)  $(e_1, e_2) \models? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17h) on (16) and (19)
- (21)  $(e_1, e_2) \models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (20)

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

(22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (15i).**

(22)  $e_2 \models \dot{\xi}_2$  by assumption  
 Contradicts (18).

(23)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case  $e_1 \models \dot{\xi}_1, e_2 \not\models? \dot{\xi}_2$ .**

(16)  $e_1 \models \dot{\xi}_1$  by assumption  
 (17)  $e_1 \not\models? \dot{\xi}_1$  by assumption  
 (18)  $e_2 \not\models \dot{\xi}_2$  by assumption  
 (19)  $e_2 \not\models? \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (15i).**

(20)  $e_2 \models \dot{\xi}_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (17g).**

(22)  $e_1 \models? \dot{\xi}_1$  by assumption  
 Contradicts (17).

**Case (17h).**

(22)  $e_2 \models? \dot{\xi}_2$  by assumption  
 Contradicts (19).

**Case (17i).**

(22)  $e_1 \models? \dot{\xi}_1$  by assumption  
 Contradicts (17).

(23)  $(e_1, e_2) \not\models? (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \models_{\dot{?}} \dot{\xi}_1, e_2 \models \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \models \dot{\xi}_2$  by assumption

(19)  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

(20)  $(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17g) on (17) and (18)

(21)  $(e_1, e_2) \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (20)

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (15i).

(22)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(23)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

**Case**  $e_1 \models_{\dot{?}} \dot{\xi}_1, e_2 \models_{\dot{?}} \dot{\xi}_2$ .

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption

(19)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

(20)  $(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17i) on (17) and (19)

(21)  $(e_1, e_2) \models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (20)

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (15i).

(22)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(23)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction



**Case**  $e_1 \models_{\dot{?}} \dot{\xi}_1, e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .

- (16)  $e_1 \not\models \dot{\xi}_1$  by assumption
- (17)  $e_1 \models_{\dot{?}} \dot{\xi}_1$  by assumption
- (18)  $e_2 \not\models \dot{\xi}_2$  by assumption
- (19)  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

- (20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (15i).

- (20)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

- (21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

- (22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (17g).

- (22)  $e_2 \models \dot{\xi}_2$  by assumption

Contradicts (18).

**Case** (17h).

- (22)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

Contradicts (19).

**Case** (17i).

- (22)  $e_2 \models_{\dot{?}} \dot{\xi}_2$  by assumption

Contradicts (19).

- (23)  $(e_1, e_2) \not\models_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

- (24)  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1, e_2 \models \dot{\xi}_2$ .

- (16)  $e_1 \not\models \dot{\xi}_1$  by assumption
- (17)  $e_1 \not\models_{\dot{?}} \dot{\xi}_1$  by assumption
- (18)  $e_2 \models \dot{\xi}_2$  by assumption
- (19)  $e_2 \not\models_{\dot{?}} \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (15i).**

(20)  $e_1 \models \dot{\xi}_1$  by assumption

Contradicts (16).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (17g).**

(22)  $e_1 \models? \dot{\xi}_1$  by assumption

Contradicts (17).

**Case (17h).**

(22)  $e_2 \models? \dot{\xi}_2$  by assumption

Contradicts (19).

**Case (17i).**

(22)  $e_1 \models? \dot{\xi}_1$  by assumption

Contradicts (17).

(23)  $(e_1, e_2) \not\models? (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on (21) and (23)

**Case  $e_1 \not\models?^\dagger \dot{\xi}_1, e_2 \models? \dot{\xi}_2$ .**

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \not\models? \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption

(19)  $e_2 \models? \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (15i).**

(20)  $e_2 \models \dot{\xi}_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction  
 Assume  $(e_1, e_2) \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (17g).**

(22)  $e_1 \models_{\text{?}} \dot{\xi}_1$  by assumption  
 Contradicts (17).

**Case (17h).**

(22)  $e_1 \models \dot{\xi}_1$  by assumption  
 Contradicts (16).

**Case (17i).**

(22)  $e_1 \models_{\text{?}} \dot{\xi}_1$  by assumption  
 Contradicts (17).

(23)  $(e_1, e_2) \not\models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

(24)  $(e_1, e_2) \not\models_{\text{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.20 on  
 (21) and (23)

**Case  $e_1 \not\models_{\text{?}}^{\dagger} \dot{\xi}_1, e_2 \not\models_{\text{?}}^{\dagger} \dot{\xi}_2$ .**

(16)  $e_1 \not\models \dot{\xi}_1$  by assumption

(17)  $e_1 \not\models_{\text{?}} \dot{\xi}_1$  by assumption

(18)  $e_2 \not\models \dot{\xi}_2$  by assumption

(19)  $e_2 \not\models_{\text{?}} \dot{\xi}_2$  by assumption

Assume  $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (15i).**

(20)  $e_2 \models \dot{\xi}_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\text{?}} (\dot{\xi}_1, \dot{\xi}_2)$ . By rule induction over Rules (17) on it, the following cases apply.

|                          |  |                                     |
|--------------------------|--|-------------------------------------|
| <b>Case (17b).</b>       |  |                                     |
| (22)                     | $(e_1, e_2) \text{ notintro}$  | by assumption                       |
| Contradicts Lemma 4.0.9. |  |                                     |
| <b>Case (17g).</b>       |  |                                     |
| (22)                     | $e_1 \models_{\tau} \dot{\xi}_1$                                     | by assumption                       |
| Contradicts (17).        |  |                                     |
| <b>Case (17h).</b>       |  |                                     |
| (22)                     | $e_2 \models_{\tau} \dot{\xi}_2$                                     | by assumption                       |
| Contradicts (19).        |  |                                     |
| <b>Case (17i).</b>       |  |                                     |
| (22)                     | $e_1 \models_{\tau} \dot{\xi}_1$                                     | by assumption                       |
| Contradicts (17).        |  |                                     |
| (23)                     | $(e_1, e_2) \not\models_{\tau} (\dot{\xi}_1, \dot{\xi}_2)$           | by contradiction                    |
| (24)                     | $(e_1, e_2) \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ | by Lemma 2.0.20 on<br>(21) and (23) |

□

**Definition 1.1.1** (Entailment of Constraints). *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \models \dot{\xi}_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e \text{ val}$  we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies  $e \models \dot{\xi}_2$*

**Definition 1.1.2** (Potential Entailment of Constraints). *Suppose that  $\dot{\xi}_1 : \tau$  and  $\dot{\xi}_2 : \tau$ . Then  $\dot{\xi}_1 \models_{\tau}^{\dagger} \dot{\xi}_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e \text{ final}$  we have  $e \models_{\tau}^{\dagger} \dot{\xi}_1$  implies  $e \models_{\tau}^{\dagger} \dot{\xi}_2$*

**Corollary 1.1.1.** *Suppose that  $\dot{\xi} : \tau$  and  $\cdot ; \Delta \vdash e : \tau$  and  $e \text{ final}$ . Then  $\top \models_{\tau}^{\dagger} \dot{\xi}$  implies  $e \models_{\tau}^{\dagger} \dot{\xi}$*

*Proof.*

|     |  |   |
|-----|--|---|
| (1) | $\dot{\xi} : \tau$                         | by assumption   |
| (2) | $\cdot ; \Gamma \vdash e : \tau$           | by assumption   |
| (3) | $e \text{ final}$                          | by assumption   |
| (4) | $\top \models_{\tau}^{\dagger} \dot{\xi}$  | by assumption   |
| (5) | $e_1 \models \top$                         | by Rule (15a)   |
| (6) | $e_1 \models_{\tau}^{\dagger} \top$        | by Rule (18b) on (5)  |
| (7) | $\top : \tau$                              | by Rule (9a)  |
| (8) | $e_1 \models_{\tau}^{\dagger} \dot{\xi}_r$ | by Definition 2.1.2 of<br>(4) on (7) and (1) and<br>(2) and (3) and (6) |

□

## 2 Match Constraint Language

$\xi ::= \top \mid \perp \mid \underline{n} \mid \underline{\mathcal{N}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \text{inl}(\xi) \mid \text{inr}(\xi) \mid (\xi_1, \xi_2)$   
 $\boxed{\xi : \tau}$      $\xi$  constrains final expressions of type  $\tau$

$$\frac{\text{CTTruth}}{\overline{\top : \tau}} \quad (9a)$$

$$\frac{\text{CTFalsity}}{\overline{\perp : \tau}} \quad (9b)$$

$$\frac{\text{CTNum}}{\overline{\underline{n} : \text{num}}} \quad (9c)$$

$$\frac{\text{CTNotNum}}{\overline{\underline{\mathcal{N}} : \text{num}}} \quad (9d)$$

$$\frac{\text{CTAnd} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\overline{\xi_1 \wedge \xi_2 : \tau}} \quad (9e)$$

$$\frac{\text{CTOr} \quad \xi_1 : \tau \quad \xi_2 : \tau}{\overline{\xi_1 \vee \xi_2 : \tau}} \quad (9f)$$

$$\frac{\text{CTInl} \quad \xi_1 : \tau_1}{\overline{\text{inl}(\xi_1) : (\tau_1 + \tau_2)}} \quad (9g)$$

$$\frac{\text{CTInr} \quad \xi_2 : \tau_2}{\overline{\text{inr}(\xi_2) : (\tau_1 + \tau_2)}} \quad (9h)$$

$$\frac{\text{CTPair} \quad \xi_1 : \tau_1 \quad \xi_2 : \tau_2}{\overline{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)}} \quad (9i)$$

$\boxed{\overline{\xi_1} = \xi_2}$     dual of  $\xi_1$  is  $\xi_2$

$$\overline{\top} = \perp \quad (10a)$$

$$\overline{\perp} = \top \quad (10b)$$

$$\overline{n} = \neg \quad (10c)$$

$$\overline{\neg} = n \quad (10d)$$

$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \quad (10e)$$

$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \quad (10f)$$

$$\overline{\text{inl}(\xi_1)} = \text{inl}(\overline{\xi_1}) \vee \text{inr}(\top) \quad (10g)$$

$$\overline{\text{inr}(\xi_2)} = \text{inr}(\overline{\xi_2}) \vee \text{inl}(\top) \quad (10h)$$

$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \quad (10i)$$

$\xi \text{ refutable?}$

$\xi$  is refutable

$$\frac{\text{RXNum}}{n \text{ refutable?}} \quad (11a)$$

$$\frac{\text{RXInl}}{\text{inl}(\xi) \text{ refutable?}} \quad (11b)$$

$$\frac{\text{RXInr}}{\text{inr}(\xi) \text{ refutable?}} \quad (11c)$$

$$\frac{\text{RXPairL} \quad \xi_1 \text{ refutable?}}{(\xi_1, \xi_2) \text{ refutable?}} \quad (11d)$$

$$\frac{\text{RXPairR} \quad \xi_2 \text{ refutable?}}{(\xi_1, \xi_2) \text{ refutable?}} \quad (11e)$$

$$\frac{\text{RXOr} \quad \xi_1 \text{ refutable?} \quad \xi_2 \text{ refutable?}}{\xi_1 \vee \xi_2 \text{ refutable?}} \quad (11f)$$

$\text{refutable?}(\xi)$

$$\text{refutable?}(\overline{n}) = \text{true} \quad (12a)$$

$$\text{refutable?}(\overline{\neg}) = \text{true} \quad (12b)$$

$$\text{refutable?}(\top) = \text{true} \quad (12c)$$

$$\text{refutable?}(\text{inl}(\xi)) = \text{refutable?}(\xi) \quad (12d)$$

$$\text{refutable?}(\text{inr}(\xi)) = \text{refutable?}(\xi) \quad (12e)$$

$$\text{refutable?}((\xi_1, \xi_2)) = \text{refutable?}(\xi_1) \text{ or } \text{refutable?}(\xi_2) \quad (12f)$$

$$\text{refutable?}(\xi_1 \vee \xi_2) = \text{refutable?}(\xi_1) \text{ and } \text{refutable?}(\xi_2) \quad (12g)$$

$$\text{Otherwise } \text{refutable?}(\xi) = \text{false} \quad (12h)$$

$$\boxed{\dot{\top}(\xi_1) = \xi_2}$$

$$\dot{\top}(\top) = \top \quad (13a)$$

$$\dot{\top}(\perp) = \perp \quad (13b)$$

$$\dot{\top}(?) = \top \quad (13c)$$

$$\dot{\top}(\underline{n}) = \underline{n} \quad (13d)$$

$$\dot{\top}(\underline{\mathscr{N}}) = \underline{\mathscr{N}} \quad (13e)$$

$$\dot{\top}(\xi_1 \wedge \xi_2) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \quad (13f)$$

$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \quad (13g)$$

$$\dot{\top}(\mathbf{inl}(\xi)) = \mathbf{inl}(\dot{\top}(\xi)) \quad (13h)$$

$$\dot{\top}(\mathbf{inr}(\xi)) = \mathbf{inr}(\dot{\top}(\xi)) \quad (13i)$$

$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \quad (13j)$$

$$\boxed{\dot{\perp}(\xi_1) = \xi_2}$$

$$\dot{\perp}(\top) = \top \quad (14a)$$

$$\dot{\perp}(\perp) = \perp \quad (14b)$$

$$\dot{\perp}(?) = \perp \quad (14c)$$

$$\dot{\perp}(\underline{n}) = \underline{n} \quad (14d)$$

$$\dot{\perp}(\underline{\mathscr{N}}) = \underline{\mathscr{N}} \quad (14e)$$

$$\dot{\perp}(\xi_1 \wedge \xi_2) = \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2) \quad (14f)$$

$$\dot{\perp}(\xi_1 \vee \xi_2) = \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2) \quad (14g)$$

$$\dot{\perp}(\mathbf{inl}(\xi)) = \mathbf{inl}(\dot{\perp}(\xi)) \quad (14h)$$

$$\dot{\perp}(\mathbf{inr}(\xi)) = \mathbf{inr}(\dot{\perp}(\xi)) \quad (14i)$$

$$\dot{\perp}((\xi_1, \xi_2)) = (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2)) \quad (14j)$$

$$\boxed{e \models \xi} \quad e \text{ satisfies } \xi$$

$$\frac{\text{CSTruth}}{e \models \top} \quad (15a)$$

$$\frac{\text{CSNum}}{\underline{n} \models \underline{n}} \quad (15b)$$

$$\frac{\text{CSNotNum}}{\underline{n_1} \models \underline{\mathscr{N}_2}} \quad (15c)$$

$$\frac{\text{CSAnd} \quad e \models \xi_1 \quad e \models \xi_2}{e \models \xi_1 \wedge \xi_2} \quad (15d)$$

$$\frac{\text{CSOrL} \quad e \models \xi_1}{e \models \xi_1 \vee \xi_2} \quad (15e)$$

$$\frac{\text{CSOrR} \quad e \models \xi_2}{e \models \xi_1 \vee \xi_2} \quad (15f)$$

$$\frac{\text{CSInl} \quad e_1 \models \xi_1}{\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)} \quad (15g)$$

$$\frac{\text{CSInr} \quad e_2 \models \xi_2}{\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)} \quad (15h)$$

$$\frac{\text{CSPair} \quad e_1 \models \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)} \quad (15i)$$

$$\frac{\text{CSNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \models \xi_1 \quad \text{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)} \quad (15j)$$

$satisfy(e, \xi)$



$$\text{satisfy}(e, \top) = \text{true} \quad (16a)$$

$$\text{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \quad (16b)$$

$$\text{satisfy}(\underline{n_1}, \underline{\neg n_2}) = (n_1 \neq n_2) \quad (16c)$$

$$\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1) \text{ and } \text{satisfy}(e, \xi_2) \quad (16d)$$

$$\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1) \text{ or } \text{satisfy}(e, \xi_2) \quad (16e)$$

$$\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) \quad (16f)$$

$$\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inr}(\xi_2)) = \text{satisfy}(e_2, \xi_2) \quad (16g)$$

$$\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) \quad (16h)$$

$$\text{satisfy}(\llbracket \cdot \rrbracket^u, (\xi_1, \xi_2)) = \text{satisfy}(\text{prl}(\llbracket \cdot \rrbracket^u), \xi_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket \cdot \rrbracket^u), \xi_2) \quad (16i)$$

$$\text{satisfy}(\llbracket e \rrbracket^u, (\xi_1, \xi_2)) = \text{satisfy}(\text{prl}(\llbracket e \rrbracket^u), \xi_1) \text{ and } \text{satisfy}(\text{pr}(\llbracket e \rrbracket^u), \xi_2) \quad (16j)$$

$$\begin{aligned} \text{satisfy}(e_1(e_2), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(e_1(e_2)), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(e_1(e_2)), \xi_2) \end{aligned} \quad (16k)$$

$$\begin{aligned} \text{satisfy}(\text{match}(e)\{\hat{r}s\}, (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{match}(e)\{\hat{r}s\}), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{match}(e)\{\hat{r}s\}), \xi_2) \end{aligned} \quad (16l)$$

$$\begin{aligned} \text{satisfy}(\text{prl}(e), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{prl}(e)), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{prl}(e)), \xi_2) \end{aligned} \quad (16m)$$

$$\begin{aligned} \text{satisfy}(\text{pr}(e), (\xi_1, \xi_2)) &= \text{satisfy}(\text{prl}(\text{pr}(e)), \xi_1) \\ &\text{ and } \text{satisfy}(\text{pr}(\text{pr}(e)), \xi_2) \end{aligned} \quad (16n)$$

$$\text{Otherwise } \text{satisfy}(e, \xi) = \text{false} \quad (16o)$$

$$\boxed{e \models? \xi} \quad e \text{ may satisfy } \xi$$

$$\frac{\text{CMSUnknown}}{e \models? ?} \quad (17a)$$

$$\frac{\text{CMSNotIntro} \quad e \text{ notintro} \quad \xi \text{ refutable?}}{e \models? \xi} \quad (17b)$$

$$\frac{\text{CMSOrL} \quad \frac{e \models? \xi_1 \quad e \not\models \xi_2}{e \models? \xi_1 \vee \xi_2}}{e \models? \xi_1 \vee \xi_2} \quad (17c)$$

$$\frac{\text{CMSOrR} \quad \frac{e \not\models \xi_1 \quad e \models? \xi_2}{e \models? \xi_1 \vee \xi_2}}{e \models? \xi_1 \vee \xi_2} \quad (17d)$$

$$\frac{\text{CMSInl} \quad \frac{e_1 \models? \xi_1}{\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)}}{\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)} \quad (17e)$$

$$\frac{\text{CMSInr} \quad e_2 \models_{\text{?}} \xi_2}{\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\xi_2)} \quad (17f)$$

$$\frac{\text{CMSPairL} \quad e_1 \models_{\text{?}} \xi_1 \quad e_2 \models \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (17g)$$

$$\frac{\text{CMSPairR} \quad e_1 \models \xi_1 \quad e_2 \models_{\text{?}} \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (17h)$$

$$\frac{\text{CMSPair} \quad e_1 \models_{\text{?}} \xi_1 \quad e_2 \models_{\text{?}} \xi_2}{(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)} \quad (17i)$$

$$\boxed{e \models_{\text{?}}^{\dagger} \xi} \quad e \text{ satisfies or may satisfy } \xi$$

$$\frac{\text{CSMSMay} \quad e \models_{\text{?}} \xi}{e \models_{\text{?}}^{\dagger} \xi} \quad (18a)$$

$$\frac{\text{CSMSSat} \quad e \models \xi}{e \models_{\text{?}}^{\dagger} \xi} \quad (18b)$$

**Lemma 2.0.1.**  $e \not\models \perp$

*Proof.* By rule induction over Rules (15), we notice that  $e \models \perp$  is in syntactic contradiction with all rules, hence not derivable.  $\square$

**Lemma 2.0.2.**  $e \not\models_{\text{?}} \perp$

*Proof.* Assume  $e \models_{\text{?}} \perp$ . By rule induction over Rules (17) on  $e \models_{\text{?}} \perp$ , only one case applies.

**Case (17b).**

$$(1) \quad \perp \text{ refutable}_{\text{?}} \quad \text{by assumption}$$

By rule induction over Rules (11) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \perp$  is not derivable.  $\square$

**Lemma 2.0.3.**  $e \not\models_{\text{?}} \top$

*Proof.* Assume  $e \models_{\text{?}} \top$ . By rule induction over Rules (17) on  $e \models_{\text{?}} \top$ , only one case applies.

**Case (17b).**

$$(1) \quad \top \text{ refutable}_{\text{?}} \quad \text{by assumption}$$

By rule induction over Rules (11) on (1), no case applies due to syntactic contradiction.

Therefore,  $e \models_{\text{?}} \top$  is not derivable.  $\square$

**Lemma 2.0.4.**  $e \not\models_{\text{?}}$

*Proof.* By rule induction over Rules (15), we notice that  $e \models_{\text{?}}$  is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 2.0.5.**  $e \models_{\text{?}}^{\dagger} \xi \text{ iff } e \models_{\text{?}}^{\dagger} \xi \vee \perp$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

(1)  $e \models_{\text{?}}^{\dagger} \xi$  by assumption

By rule induction over Rules (18) on (1).

**Case (18a).**

(2)  $e \models_{\text{?}} \xi$  by assumption  
(3)  $e \models_{\text{?}} \xi \vee \perp$  by Rule (17c) on (2)  
and Lemma 2.0.1  
(4)  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$  by Rule (18a) on (3)

**Case (18b).**

(2)  $e \models \xi$  by assumption  
(3)  $e \models \xi \vee \perp$  by Rule (15e) on (2)  
(4)  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$  by Rule (18b) on (3)

2. Necessity:

(1)  $e \models_{\text{?}}^{\dagger} \xi \vee \perp$  by assumption

By rule induction over Rules (18) on (1).

**Case (18a).**

(2)  $e \models_{\text{?}} \xi \vee \perp$  by assumption

By rule induction over Rules (17) on (2), only two of them apply.

**Case (17c).**

(3)  $e \models_{\text{?}} \xi$  by assumption  
(4)  $e \models_{\text{?}}^{\dagger} \xi$  by Rule (18a) on (3)

**Case (17d).**

(3)  $e \models_{\tau} \perp$

by assumption

(4)  $e \not\models_{\tau} \perp$

by Lemma 2.0.2

(3) contradicts (4).

**Case (18b).**

(2)  $e \models \xi \vee \perp$

by assumption

By rule induction over Rules (15) on (2), only two of them apply.

**Case (15e).**

(3)  $e \models \xi$

by assumption

(4)  $e \models_{\tau}^{\dagger} \xi$

by Rule (18b) on (3)

**Case (15f).**

(3)  $e \models \perp$

by assumption

(4)  $e \not\models \perp$

by Lemma 2.0.1

(3) contradicts (4).

□

**Corollary 2.0.1.**  $\top \models_{\tau}^{\dagger} \xi \text{ iff } \top \models_{\tau}^{\dagger} \xi \vee \perp$

*Proof.* By Definition 2.1.2 and Lemma 2.0.5.

□

**Lemma 2.0.6.** *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \not\models \xi_2$  iff  $\xi_1 \not\models \xi_2 \vee \perp$*

*Proof.*

(1)  $\xi_1 : \tau$

by assumption

(2)  $\xi_2 : \tau$

by assumption

(3)  $\perp : \tau$

by Rule (9b)

(4)  $\xi_2 \vee \perp : \tau$

by Rule (9f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5)  $\xi_1 \not\models \xi_2$

by assumption

To prove  $\xi_1 \not\models \xi_2 \vee \perp$ , assume  $\xi_1 \models \xi_2 \vee \perp$ .

(6)  $\xi_1 \models \xi_2 \vee \perp$

by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies

(7)  $e \models \xi_2 \vee \perp$

by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (15) on (7).

**Case (15e).**

- (8)  $e \models \xi_2$  by assumption
- (9)  $\xi_1 \models \xi_2$  by Definition 2.1.1 on (8)

(5) contradicts (9).

**Case (15f).**

- (8)  $e \models \perp$  by assumption
- (9)  $e \not\models \perp$  by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

- (a)  $\xi_1 \not\models \xi_2 \vee \perp$

2. Necessity:

- (5)  $\xi_1 \not\models \xi_2 \vee \perp$  by assumption

To prove  $\xi_1 \not\models \xi_2$ , assume  $\xi_1 \models \xi_2$ .

- (6)  $\xi_1 \models \xi_2$  by assumption

For all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies

- (7)  $e \models \xi_2$  by Definition 2.1.1 on (1) and (2) and (6)
- (8)  $e \models \xi_2 \vee \perp$  by Rule (15e) on (7)
- (9)  $\xi_1 \models \xi_2 \vee \perp$  by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

- (a)  $\xi_1 \not\models \xi_2$

□

**Lemma 2.0.7.** *If  $e \not\models_{\tau}^{\dagger} \xi_1$  and  $e \not\models_{\tau}^{\dagger} \xi_2$  then  $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$*

*Proof.* Assume, for the sake of contradiction, that  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$ .

- (1)  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by assumption
- (2)  $e \not\models_{\tau}^{\dagger} \xi_1$  by assumption
- (3)  $e \not\models_{\tau}^{\dagger} \xi_2$  by assumption

By rule induction over Rules (18) on (1).

**Case (18b).**

$$(4) \quad e \models \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (15) on (4) and only two of them apply.

**Case (15e).**

$$(5) \quad e \models \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (18b) on (5)}$$

(6) contradicts (2).

**Case (15f).**

$$(5) \quad e \models \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (18b) on (5)}$$

(6) contradicts (3).

**Case (18a).**

$$(4) \quad e \models_{\vdash} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

By rule induction over Rules (17) on (4) and only two of them apply.

**Case (17c).**

$$(5) \quad e \models_{\vdash} \xi_1 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_1 \quad \text{by Rule (18a) on (5)}$$

(6) contradicts (2).

**Case (17d).**

$$(5) \quad e \models_{\vdash} \xi_2 \quad \text{by assumption}$$

$$(6) \quad e \models_{\vdash}^{\dagger} \xi_2 \quad \text{by Rule (18a) on (5)}$$

(6) contradicts (3).

The conclusion holds as follows:

$$1. \quad e \not\models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$$

□

**Lemma 2.0.8.** *If  $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$  and  $e \not\models_{\vdash}^{\dagger} \xi_1$  then  $e \models_{\vdash}^{\dagger} \xi_2$*

*Proof.*

$$(1) \quad e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2 \quad \text{by assumption}$$

$$(2) \quad e \not\models_{\vdash}^{\dagger} \xi_1 \quad \text{by assumption}$$

By rule induction over Rules (18) on (1).

**Case (18b).**

(3)  $e \models \xi_1 \vee \xi_2$  by assumption

By rule induction over Rules (15) on (3) and only two of them apply.

**Case (15e).**

(4)  $e \models \xi_1$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_1$  by Rule (18b) on (4)

(5) contradicts (2).

**Case (15f).**

(4)  $e \models \xi_2$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_2$  by Rule (18b) on (4)

**Case (18a).**

(3)  $e \models_{\neg} \xi_1 \vee \xi_2$  by assumption

By rule induction over Rules (17) on (3) and only two of them apply.

**Case (17c).**

(4)  $e \models_{\neg} \xi_1$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_1$  by Rule (18a) on (4)

(5) contradicts (2).

**Case (17d).**

(4)  $e \models_{\neg} \xi_2$  by assumption

(5)  $e \models_{\neg}^{\dagger} \xi_2$  by Rule (18a) on (4)

□

**Lemma 2.0.9.** *If  $e \models_{\neg}^{\dagger} \xi_1$  and  $e \models_{\neg}^{\dagger} \xi_2$  then  $e \models_{\neg}^{\dagger} \xi_1 \wedge \xi_2$*

**Lemma 2.0.10.** *If  $e \models_{\neg}^{\dagger} \xi_1$  then  $e \models_{\neg}^{\dagger} \xi_1 \vee \xi_2$  and  $e \models_{\neg}^{\dagger} \xi_2 \vee \xi_1$*

*Proof.*

(1)  $e \models_{\neg}^{\dagger} \xi_1$  by assumption ,

By rule induction over Rules (18) on (1),

**Case (18b).**

(2)  $e \models \xi_1$  by assumption

(3)  $e \models \xi_1 \vee \xi_2$  by Rule (15e) on (2)

(4)  $e \models \xi_2 \vee \xi_1$  by Rule (15f) on (2)

- (5)  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by Rule (18b) on (3)
- (6)  $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$  by Rule (18b) on (4)

**Case (18a).**

- (2)  $e \models_{\tau} \xi_1$  by assumption

By case analysis on the result of  $\text{satisfy}(e, \xi_2)$ .

**Case true.**

- (3)  $\text{satisfy}(e, \xi_2) = \text{true}$  by assumption
- (4)  $e \models \xi_2$  by Lemma 2.0.19 on (3)
- (5)  $e \models \xi_1 \vee \xi_2$  by Rule (15f) on (4)
- (6)  $e \models \xi_2 \vee \xi_1$  by Rule (15e) on (4)
- (7)  $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by Rule (18b) on (5)
- (8)  $e \models_{\tau}^{\dagger} \xi_2 \vee \xi_1$  by Rule (18b) on (6)

**Case false.**

- (3)  $\text{satisfy}(e, \xi_2) = \text{false}$  by assumption
- (4)  $e \not\models \xi_2$  by Lemma 2.0.19 on (3)
- (5)  $e \models_{\tau} \xi_1 \vee \xi_2$  by Rule (17c) on (2) and (4)
- (6)  $e \models_{\tau}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by Rule (18a) on (5)

□

**Lemma 2.0.11.** *If  $e_1 \models_{\tau}^{\dagger} \xi_1$  then  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$*

*Proof.*

- (1)  $e_1 \models_{\tau}^{\dagger} \xi_1$  by assumption

By rule induction over Rules (18) on (1).

**Case (18b).**

- (2)  $e_1 \models \xi_1$  by assumption
- (3)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (15g) on (2)
- (4)  $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Rule (18b) on (3)



**Case (18a).**

- |   |                      |
|---|----------------------|
| (2) $e_1 \models_{\tau} \xi_1$  | by assumption        |
| (3) $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\xi_1)$           | by Rule (17e) on (2) |
| (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ | by Rule (18a) on (3) |

□

**Lemma 2.0.12.** *If  $e_2 \models_{\tau}^{\dagger} \xi_2$  then  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$*

*Proof.*

- |  |               |
|--|---------------|
| (1) $e_2 \models_{\tau}^{\dagger} \xi_2$ | by assumption |
|--|---------------|

By rule induction over Rules (18) on (1).

**Case (18b).**

- |   |                      |
|---|----------------------|
| (2) $e_2 \models \xi_2$   | by assumption        |
| (3) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$                  | by Rule (15h) on (2) |
| (4) $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ | by Rule (18b) on (3) |

**Case (18a).**

- |   |                      |
|---|----------------------|
| (2) $e_2 \models_{\tau} \xi_2$  | by assumption        |
| (3) $\text{inl}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$           | by Rule (17f) on (2) |
| (4) $\text{inl}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$ | by Rule (18a) on (3) |

□

**Lemma 2.0.13.** *If  $e_1 \models_{\tau}^{\dagger} \xi_1$  and  $e_2 \models_{\tau}^{\dagger} \xi_2$  then  $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$*

**Lemma 2.0.14** (Soundness and Completeness of Refutable Constraints).  $\xi \text{ refutable}_{\tau}$  iff  $\text{refutable}_{\tau}(\xi) = \text{true}$ .

**Lemma 2.0.15.** *If  $e \text{ notintro}$  and  $\xi \text{ refutable}_{\tau}$  then either  $\dagger(\xi) \text{ refutable}_{\tau}$  or  $e \models \dagger(\xi)$ .*

*Proof.* By structural induction on  $\xi$ . □

**Lemma 2.0.16.** *There does not exist such a constraint  $\xi_1 \wedge \xi_2$  such that  $\xi_1 \wedge \xi_2 \text{ refutable}_{\tau}$ .*

*Proof.* By rule induction over Rules (11), we notice that  $\xi_1 \wedge \xi_2 \text{ refutable}_{\tau}$  is in syntactic contradiction with all the cases, hence not derivable. □

**Lemma 2.0.17.** *There does not exist such a constraint  $\xi_1 \vee \xi_2$  such that  $\xi_1 \vee \xi_2 \text{ refutable}_{\tau}$ .*

*Proof.* By rule induction over Rules (11), we notice that  $\xi_1 \vee \xi_2$  **refutable?** is in syntactic contradiction with all the cases, hence not derivable.  $\square$

**Lemma 2.0.18.** *If  $e$  **notintro** and  $e \models \xi$  then  $\xi$  ~~**refutable?**~~.*

*Proof.*

- |                         |               |
|-------------------------|---------------|
| (1) $e$ <b>notintro</b> | by assumption |
| (2) $e \models \xi$     | by assumption |

By rule induction over Rules (15) on (2).

**Case (15a).**

- |                  |               |
|------------------|---------------|
| (3) $\xi = \top$ | by assumption |
|------------------|---------------|

Assume  $\top$  **refutable?**. By rule induction over Rules (11), no case applies due to syntactic contradiction.  
Therefore,  $\top$  ~~**refutable?**~~.

**Case (15e),(15f).**

- |   |                 |
|---|-----------------|
| (3) $\xi = \xi_1 \vee \xi_2$                        | by assumption   |
| (4) $\xi_1 \vee \xi_2$ <del><b>refutable?</b></del> | by Lemma 2.0.17 |

**Case (15d).**

- |   |                 |
|---|-----------------|
| (3) $\xi = \xi_1 \wedge \xi_2$                        | by assumption   |
| (4) $\xi_1 \wedge \xi_2$ <del><b>refutable?</b></del> | by Lemma 2.0.16 |

**Case (15j).**

- |  |                      |
|--|----------------------|
| (3) $\xi = (\xi_1, \xi_2)$               | by assumption        |
| (4) $\text{prl}(e) \models \xi_1$        | by assumption        |
| (5) $\text{prr}(e) \models \xi_2$        | by assumption        |
| (6) $\text{prl}(e)$ <b>notintro</b>      | by Rule (27e)        |
| (7) $\text{prr}(e)$ <b>notintro</b>      | by Rule (27f)        |
| (8) $\xi_1$ <del><b>refutable?</b></del> | by IH on (6) and (4) |
| (9) $\xi_2$ <del><b>refutable?</b></del> | by IH on (7) and (5) |

Assume  $(\xi_1, \xi_2)$  **refutable?**. By rule induction over Rules (11) on it, only two cases apply.

**Case (11d).**

- |                                |               |
|--------------------------------|---------------|
| (10) $\xi_1$ <b>refutable?</b> | by assumption |
|--------------------------------|---------------|

Contradicts (8).

**Case (11e).**

(10)  $\xi_2$  **refutable?** by assumption

Contradicts (9).

Therefore,  $(\xi_1, \xi_2)$  ~~**refutable?**~~.

**Otherwise.**

(3)  $e = \underline{n}, \text{inl}_{\tau_2}(e_1), \text{inr}_{\tau_1}(e_2), (e_1, e_2)$  by assumption

By rule induction over Rules (27) on (1), no case applies due to syntactic contradiction.

□

**Lemma 2.0.19** (Soundness and Completeness of Satisfaction Judgment).  $e \models \xi$  iff  $\text{satisfy}(e, \xi) = \text{true}$ .

*Proof.* We prove soundness and completeness separately.

1. Soundness:

(1)  $e \models \xi$  by assumption

By rule induction over Rules (15) on (1).

**Case (15a).**

(2)  $\xi = \top$  by assumption

(3)  $\text{satisfy}(e, \top) = \text{true}$  by Definition 16a

**Case (15b).**

(2)  $e = \underline{n}$  by assumption

(3)  $\xi = \underline{n}$  by assumption

(4)  $\text{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$  by Definition 16b

**Case (15c).**

(2)  $e = \underline{n_1}$  by assumption

(3)  $\xi = \underline{\neg \bot}$  by assumption

(4)  $n_1 \neq n_2$  by assumption

(5)  $\text{satisfy}(\underline{n_1}, \underline{\neg \bot}) = (n_1 \neq n_2) = \text{true}$  by Definition 16c on (4)

**Case (15d).**

(2)  $\xi = \xi_1 \wedge \xi_2$  by assumption

- (3)  $e \models \xi_1$  by assumption
- (4)  $e \models \xi_2$  by assumption
- (5)  $\text{satisfy}(e, \xi_1) = \text{true}$  by IH on (3)
- (6)  $\text{satisfy}(e, \xi_2) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(e, \xi_1 \wedge \xi_2) = \text{satisfy}(e, \xi_1)$  and  $\text{satisfy}(e, \xi_2) = \text{true}$   
by Definition 16d on (5) and (6)

**Case (15e).**

- (2)  $\xi = \xi_1 \vee \xi_2$  by assumption
- (3)  $e \models \xi_1$  by assumption
- (4)  $\text{satisfy}(e, \xi_1) = \text{true}$  by IH on (3)
- (5)  $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$  or  $\text{satisfy}(e, \xi_2) = \text{true}$   
by Definition 16e on (4)

**Case (15f).**

- (2)  $\xi = \xi_1 \vee \xi_2$  by assumption
- (3)  $e \models \xi_2$  by assumption
- (4)  $\text{satisfy}(e, \xi_2) = \text{true}$  by IH on (3)
- (5)  $\text{satisfy}(e, \xi_1 \vee \xi_2) = \text{satisfy}(e, \xi_1)$  or  $\text{satisfy}(e, \xi_2) = \text{true}$   
by Definition 16e on (4)

**Case (15g).**

- (2)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption
- (3)  $\xi = \text{inl}(\xi_1)$  by assumption
- (4)  $e_1 \models \xi_1$  by assumption
- (5)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by IH on (4)
- (6)  $\text{satisfy}(\text{inl}_{\tau_2}(e_1), \text{inl}(\xi_1)) = \text{satisfy}(e_1, \xi_1) = \text{true}$   
by Definition 16f on (5)

**Case (15h).**

- (2)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption
- (3)  $\xi = \text{inl}(\xi_2)$  by assumption
- (4)  $e_2 \models \xi_2$  by assumption
- (5)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by IH on (4)
- (6)  $\text{satisfy}(\text{inr}_{\tau_1}(e_2), \text{inl}(\xi_2)) = \text{satisfy}(e_2, \xi_2) = \text{true}$   
by Definition 16g on (5)

**Case (15i).**

- (2)  $e = (e_1, e_2)$  by assumption
- (3)  $\xi = (\xi_1, \xi_2)$  by assumption
- (4)  $e_1 \models \xi_1$  by assumption
- (5)  $e_2 \models \xi_2$  by assumption
- (6)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by IH on (5)
- (8)  $\text{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$   
 $\text{satisfy}(e_1, \xi_1) \text{ and } \text{satisfy}(e_2, \xi_2) = \text{true}$   
by Definition 16h on (6) and (7)

**Case (15j).**

- (2)  $\xi = (\xi_1, \xi_2)$  by assumption
- (3)  $e \text{ notintro}$  by assumption
- (4)  $\text{prl}(e) \models \xi_1$  by assumption
- (5)  $\text{prr}(e) \models \xi_2$  by assumption
- (6)  $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$  by IH on (4)
- (7)  $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$  by IH on (5)

By rule induction over Rules (27) on (3).

**Otherwise.**

- (8)  $e = (\emptyset^u, (\emptyset_0)^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\})$   
by assumption
- (9)  $\text{satisfy}(e, (\xi_1, \xi_2)) =$   
 $\text{satisfy}(\text{prl}(e), \xi_1) \text{ and } \text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$   
by Definition 16 on (6) and (7)

2. Completeness:

- (1)  $\text{satisfy}(e, \xi) = \text{true}$  by assumption

By structural induction on  $\xi$ .

**Case  $\xi = \top$ .**

- (2)  $e \models \top$  by Rule (15a)

**Case  $\xi = \perp, ?$ .**

- (2)  $\text{satisfy}(e, \xi) = \text{false}$  by Definition 16a

(2) contradicts (1) and thus vacuously true.

**Case  $\xi = \underline{n}$ .**

By structural induction on  $e$ .

**Case**  $e = \underline{n'}$ .  
 (2)  $n' = n$  by Definition 16b on (1)  
 (3)  $\underline{n'} \models \underline{n}$  by Rule (15b) on (2)

**Otherwise.**  
 (2)  $\text{satisfy}(e, \underline{n}) = \text{false}$  by Definition 16o  
 (2) contradicts (1) and thus vacuously true.

**Case**  $\xi = \underline{\mathcal{N}}$ .  
 By structural induction on  $e$ .

**Case**  $e = \underline{n'}$ .  
 (2)  $n' \neq n$  by Definition 16c on (1)  
 (3)  $\underline{n'} \models \underline{\mathcal{N}}$  by Rule (15c) on (2)

**Otherwise.**  
 (2)  $\text{satisfy}(e, \underline{\mathcal{N}}) = \text{false}$  by Definition 16o  
 (2) contradicts (1) and thus vacuously true.

**Case**  $\xi = \xi_1 \wedge \xi_2$ .  
 (2)  $\text{satisfy}(e, \xi_1) = \text{true}$  by Definition 16d on (1)  
 (3)  $\text{satisfy}(e, \xi_2) = \text{true}$  by Definition 16d on (1)  
 (4)  $e \models \xi_1$  by IH on (2)  
 (5)  $e \models \xi_2$  by IH on (3)  
 (6)  $e \models \xi_1 \wedge \xi_2$  by Rule (15d) on (4) and (5)

**Case**  $\xi = \xi_1 \vee \xi_2$ .  
 (2)  $\text{satisfy}(e, \xi_1) \text{ or } \text{satisfy}(e, \xi_2) = \text{true}$  by Definition 16e on (1)

By case analysis on (2).

**Case**  $\text{satisfy}(e, \xi_1) = \text{true}$ .  
 (3)  $\text{satisfy}(e, \xi_1) = \text{true}$  by assumption  
 (4)  $e \models \xi_1$  by IH on (3)  
 (5)  $e \models \xi_1 \vee \xi_2$  by Rule (15e) on (4)

**Case**  $\text{satisfy}(e, \xi_2) = \text{true}$ .  
 (3)  $\text{satisfy}(e, \xi_2) = \text{true}$  by assumption  
 (4)  $e \models \xi_2$  by IH on (3)  
 (5)  $e \models \xi_1 \vee \xi_2$  by Rule (15f) on (4)

**Case  $\xi = \text{inl}(\xi_1)$ .**

By structural induction on  $e$ .

**Case  $e = \text{inl}_{\tau_2}(e_1)$ .**

- (2)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by Definition 16f on (1)
- (3)  $e_1 \models \xi_1$  by IH on (2)
- (4)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (15g) on (3)

**Otherwise.**

- (2)  $\text{satisfy}(e, \text{inl}(\xi_1)) = \text{false}$  by Definition 16o
- (2) contradicts (1) and thus vacuously true.

**Case  $\xi = \text{inr}(\xi_2)$ .**

By structural induction on  $e$ .

**Case  $e = \text{inr}_{\tau_1}(e_2)$ .**

- (2)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by Definition 16g on (1)
- (3)  $e_2 \models \xi_2$  by IH on (2)
- (4)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by Rule (15h) on (3)

**Otherwise.**

- (2)  $\text{satisfy}(e, \text{inr}(\xi_2)) = \text{false}$  by Definition 16o
- (2) contradicts (1) and thus vacuously true.

**Case  $\xi = (\xi_1, \xi_2)$ .**

By structural induction on  $e$ .

**Case  $e = (e_1, e_2)$ .**

- (2)  $\text{satisfy}(e_1, \xi_1) = \text{true}$  by Definition 16h on (1)
- (3)  $\text{satisfy}(e_2, \xi_2) = \text{true}$  by Definition 16h on (1)
- (4)  $e_1 \models \xi_1$  by IH on (2)
- (5)  $e_2 \models \xi_2$  by IH on (3)
- (6)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (15i) on (4) and (5)

**Case  $e = (\llbracket \cdot \rrbracket^u, \llbracket e_0 \rrbracket^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\})$ .**

- (2)  $\text{satisfy}(\text{prl}(e), \xi_1) = \text{true}$  by Definition 16h on (1)
- (3)  $\text{satisfy}(\text{prr}(e), \xi_2) = \text{true}$  by Definition 16h on (1)
- (4)  $\text{prl}(e) \models \xi_1$  by IH on (2)
- (5)  $\text{prr}(e) \models \xi_2$  by IH on (3)
- (6)  $e \text{ notintro}$  by each rule in Rules (27)

(7)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (15j) on (6) and (4) and (5)

**Otherwise.**

(2)  $\text{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$  by Definition 16o  
 (2) contradicts (1) and thus vacuously true.

□

**Lemma 2.0.20.**  $e \not\models \xi$  and  $e \not\models_{\text{?}} \xi$  iff  $e \not\models_{\text{?}}^{\dagger} \xi$ .

*Proof.* 1. Sufficiency:

(1)  $e \not\models \xi$  by assumption  
 (2)  $e \not\models_{\text{?}} \xi$  by assumption

Assume  $e \models_{\text{?}}^{\dagger} \xi$ . By rule induction over Rules (18) on it.

**Case (18a).**

(3)  $e \models \xi$  by assumption

Contradicts (1).

**Case (18b).**

(3)  $e \models_{\text{?}} \xi$  by assumption

Contradicts (2).

Therefore,  $e \models_{\text{?}}^{\dagger} \xi$  is not derivable.

2. Necessity:

(1)  $e \not\models_{\text{?}}^{\dagger} \xi$  by assumption

Assume  $e \models \xi$ .

(2)  $e \models_{\text{?}}^{\dagger} \xi$  by Rule (18b) on assumption

Contradicts (1). Therefore,  $e \not\models \xi$ . Assume  $e \models_{\text{?}} \xi$ .

(3)  $e \models_{\text{?}}^{\dagger} \xi$  by Rule (18a) on assumption

Contradicts (1). Therefore,  $e \not\models_{\text{?}} \xi$ .

□

**Theorem 2.1** (Exclusiveness of Satisfaction Judgment). *If  $\xi : \tau$  and  $\cdot; \Delta \vdash e : \tau$  and  $e$  final then exactly one of the following holds*

1.  $e \models \xi$



2.  $e \models_{\tau} \xi$

3.  $e \not\models_{\tau}^{\dagger} \xi$

*Proof.*

(4)  $\xi : \tau$  by assumption

(5)  $\cdot; \Delta \vdash e : \tau$  by assumption

(6)  $e$  **final** by assumption

By rule induction over Rules (9) on (4), we would show one conclusion is derivable while the other two are not.

**Case (9a).**

(7)  $\xi = \top$  by assumption

(8)  $e \models \top$  by Rule (15a)

(9)  $e \not\models_{\tau} \top$  by Lemma 2.0.3

(10)  $e \models_{\tau}^{\dagger} \top$  by Rule (18b) on (8)

**Case (9b).**

(7)  $\xi = \perp$  by assumption

(8)  $e \not\models \perp$  by Lemma 2.0.1

(9)  $e \not\models_{\tau} \perp$  by Lemma 2.0.2

(10)  $e \not\models_{\tau}^{\dagger} \perp$  by Lemma 2.0.20 on (8) and (9)

**Case (1b).**

(7)  $\xi = ?$  by assumption

(8)  $e \not\models ?$  by Lemma 2.0.4

(9)  $e \models_{\tau} ?$  by Rule (17a)

(10)  $e \models_{\tau}^{\dagger} ?$  by Rule (18a) on (9)

**Case (9c).**

(7)  $\xi = \underline{n_2}$  by assumption

(8)  $\tau = \mathbf{num}$  by assumption

By rule induction over Rules (20) on (5), the following cases apply.

**Case** (20b),(20c),(20f),(20h),(20i),(20l),(20m).

(9)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption

(10)  $e$  **notintro** by Rule  
(27a),(27b),(27c),(27d),(27e),(27f)

Assume  $e \models \underline{n_2}$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction on  $\xi$ .

(11)  $e \not\models \underline{n_2}$  by contradiction

(12)  $\underline{n_2}$  **refutable?** by Rule (11a)

(13)  $e \models_{\text{?}} \underline{n_2}$  by Rule (17b) on (10)  
and (12)

(14)  $e \models_{\text{?}}^{\dagger} \underline{n_2}$  by Rule (18a) on (13)

**Case** (20d).

(9)  $e = \underline{n_1}$  by assumption

Assume  $\underline{n_1} \models_{\text{?}} \underline{n_2}$ . By rule induction over Rules (17), only one case applies.

**Case** (17b).

(10)  $\underline{n_1}$  **notintro** by assumption

Contradicts Lemma 4.0.6.

(11)  $\underline{n_1} \not\models_{\text{?}} \underline{n_2}$  by contradiction

By case analysis on whether  $n_1$  is equal to  $n_2$ .

**Case**  $n_1 = n_2$ .

(12)  $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$  by Definition 16

(13)  $\underline{n_1} \models \underline{n_2}$  by Lemma 2.0.19 on  
(12)

(14)  $e \models_{\text{?}}^{\dagger} \underline{n_2}$  by Rule (18b) on (13)

**Case**  $n_1 \neq n_2$ .

(12)  $\text{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$  by Definition 16

(13)  $\underline{n_1} \not\models \underline{n_2}$  by Lemma 2.0.19 on  
(12)

(14)  $e \not\models_{\text{?}}^{\dagger} \underline{n_2}$  by Lemma 2.0.20 on  
(11) and (13)

**Case** (9f).

(7)  $\xi = \xi_1 \vee \xi_2$  by assumption

By inductive hypothesis on (5) and (6), exactly one of  $e \models \xi_1$ ,  $e \models? \xi_1$ , and  $e \not\models?^\dagger \xi_1$  holds. The same goes for  $\xi_2$ . By case analysis on which conclusion holds for  $\xi_1$  and  $\xi_2$ .

**Case**  $e \models \xi_1, e \models \xi_2$ .

- |      |                                       |                       |
|------|---------------------------------------|-----------------------|
| (8)  | $e \models \xi_1$                     | by assumption         |
| (9)  | $e \not\models? \xi_1$                | by assumption         |
| (10) | $e \models \xi_2$                     | by assumption         |
| (11) | $e \not\models? \xi_2$                | by assumption         |
| (12) | $e \models \xi_1 \vee \xi_2$          | by Rule (15e) on (8)  |
| (13) | $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (18b) on (12) |

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

- |                           |                                      |               |
|---------------------------|--------------------------------------|---------------|
| (14)                      | $\xi_1 \vee \xi_2$ <b>refutable?</b> | by assumption |
| Contradicts Lemma 2.0.17. |                                      |               |

**Case** (17c).

- |                  |                    |               |
|------------------|--------------------|---------------|
| (14)             | $e \models? \xi_1$ | by assumption |
| Contradicts (9). |                    |               |

**Case** (17d).

- |                   |                    |               |
|-------------------|--------------------|---------------|
| (14)              | $e \models? \xi_2$ | by assumption |
| Contradicts (11). |                    |               |

- |      |                                   |                  |
|------|-----------------------------------|------------------|
| (15) | $e \not\models? \xi_1 \vee \xi_2$ | by contradiction |
|------|-----------------------------------|------------------|

**Case**  $e \models \xi_1, e \models? \xi_2$ .

- |      |                                       |                       |
|------|---------------------------------------|-----------------------|
| (8)  | $e \models \xi_1$                     | by assumption         |
| (9)  | $e \not\models? \xi_1$                | by assumption         |
| (10) | $e \not\models \xi_2$                 | by assumption         |
| (11) | $e \models? \xi_2$                    | by assumption         |
| (12) | $e \models \xi_1 \vee \xi_2$          | by Rule (15e) on (8)  |
| (13) | $e \models?^\dagger \xi_1 \vee \xi_2$ | by Rule (18b) on (12) |

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

- |                           |                                      |               |
|---------------------------|--------------------------------------|---------------|
| (14)                      | $\xi_1 \vee \xi_2$ <b>refutable?</b> | by assumption |
| Contradicts Lemma 2.0.17. |                                      |               |

**Case** (17c).

- |      |                    |               |
|------|--------------------|---------------|
| (14) | $e \models? \xi_1$ | by assumption |
|------|--------------------|---------------|

Contradicts (9).

**Case (17d).**

(14)  $e \not\models \xi_1$  by assumption

Contradicts (8).

(15)  $e \not\models? \xi_1 \vee \xi_2$  by contradiction

**Case**  $e \models \xi_1, e \not\models?^\dagger \xi_2$ .

(8)  $e \models \xi_1$  by assumption

(9)  $e \not\models? \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \not\models? \xi_2$  by assumption

(12)  $e \models \xi_1 \vee \xi_2$  by Rule (15e) on (8)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$  by Rule (18b) on (12)

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(14)  $\xi_1 \vee \xi_2$  **refutable?** by assumption

Contradicts Lemma 2.0.17.

**Case (17c).**

(14)  $e \models? \xi_1$  by assumption

Contradicts (9).

**Case (17d).**

(14)  $e \not\models \xi_1$  by assumption

Contradicts (8).

(15)  $e \not\models? \xi_1 \vee \xi_2$  by contradiction

**Case**  $e \models? \xi_1, e \models \xi_2$ .

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \models? \xi_1$  by assumption

(10)  $e \models \xi_2$  by assumption

(11)  $e \not\models? \xi_2$  by assumption

(12)  $e \models \xi_1 \vee \xi_2$  by Rule (15f) on (10)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$  by Rule (18b) on (12)

Assume  $e \models? \xi_1 \vee \xi_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(14)  $\xi_1 \vee \xi_2$  **refutable?** by assumption

Contradicts Lemma 2.0.17.

**Case (17c).**

(14)  $e \not\models \xi_2$  by assumption

Contradicts (10).

**Case (17d).**

(14)  $e \models? \xi_2$  by assumption

Contradicts (11).

(15)  $e \not\models? \xi_1 \vee \xi_2$  by contradiction

**Case  $e \models? \xi_1, e \models? \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \models? \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \models? \xi_2$  by assumption

(12)  $e \models? \xi_1 \vee \xi_2$  by Rule (17c) on (9) and (10)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$  by Rule (18a) on (12)

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (15), only two cases apply.

**Case (15e).**

(14)  $e \models \xi_1$  by assumption

Contradicts (8)

**Case (15f).**

(14)  $e \models \xi_2$  by assumption

Contradicts (10)

(15)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

**Case  $e \models? \xi_1, e \not\models?^\dagger \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \models? \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \not\models? \xi_2$  by assumption

(12)  $e \models? \xi_1 \vee \xi_2$  by Rule (17c) on (9) and (10)

(13)  $e \models?^\dagger \xi_1 \vee \xi_2$  by Rule (18a) on (12)

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (15), only two cases apply.

**Case (15e).**

(14)  $e \models \xi_1$

by assumption

Contradicts (8).

**Case (15f).**

(14)  $e \models \xi_2$

by assumption

Contradicts (10).

(15)  $e \not\models \xi_1 \vee \xi_2$

by contradiction

**Case  $e \not\models_{\vdash}^{\dagger} \xi_1, e \models \xi_2$ .**

(8)  $e \not\models \xi_1$

by assumption

(9)  $e \not\models_{\vdash} \xi_1$

by assumption

(10)  $e \models \xi_2$

by assumption

(11)  $e \not\models_{\vdash} \xi_2$

by assumption

(12)  $e \models \xi_1 \vee \xi_2$

by Rule (15f) on (10)

(13)  $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$

by Rule (18b) on (12)

Assume  $e \models_{\vdash} \xi_1 \vee \xi_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(14)  $\xi_1 \vee \xi_2$  **refutable**<sub>?</sub>

by assumption

Contradicts Lemma 2.0.17.

**Case (17c).**

(14)  $e \not\models \xi_2$

by assumption

Contradicts (10).

**Case (17d).**

(14)  $e \models_{\vdash} \xi_2$

by assumption

Contradicts (11).

(15)  $e \not\models_{\vdash} \xi_1 \vee \xi_2$

by contradiction

**Case  $e \not\models_{\vdash}^{\dagger} \xi_1, e \models_{\vdash} \xi_2$ .**

(8)  $e \not\models \xi_1$

by assumption

(9)  $e \not\models_{\vdash} \xi_1$

by assumption

(10)  $e \not\models \xi_2$

by assumption

(11)  $e \models_{\vdash} \xi_2$

by assumption

(12)  $e \models_{\vdash} \xi_1 \vee \xi_2$

by Rule (17d) on (11)  
and (8)

(13)  $e \models_{\vdash}^{\dagger} \xi_1 \vee \xi_2$

by Rule (18a) on (12)

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (15), only two cases apply.

**Case (15e).**

(14)  $e \models \xi_1$  by assumption

Contradicts (8)

**Case (15f).**

(14)  $e \models \xi_2$  by assumption

Contradicts (10)

(15)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

**Case  $e \not\models_{\text{?}} \xi_1, e \not\models_{\text{?}} \xi_2$ .**

(8)  $e \not\models \xi_1$  by assumption

(9)  $e \not\models_{\text{?}} \xi_1$  by assumption

(10)  $e \not\models \xi_2$  by assumption

(11)  $e \not\models_{\text{?}} \xi_2$  by assumption

Assume  $e \models \xi_1 \vee \xi_2$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15e).**

(12)  $e \models \xi_1$  by assumption

Contradicts (8).

**Case (15f).**

(12)  $e \models \xi_2$  by assumption

Contradicts (10).

(13)  $e \not\models \xi_1 \vee \xi_2$  by contradiction

Assume  $e \models_{\text{?}} \xi_1 \vee \xi_2$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(14)  $\xi_1 \vee \xi_2$  **refutable**<sub>?</sub> by assumption

Contradicts Lemma 2.0.17.

**Case (17c).**

(14)  $e \models_{\text{?}} \xi_1$  by assumption

Contradicts (9).

**Case (17d).**

(14)  $e \models_{\text{?}} \xi_2$  by assumption

Contradicts (11).

(15)  $e \not\models_{\text{?}} \xi_1 \vee \xi_2$  by contradiction

(16)  $e \not\models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  by Lemma 2.0.20 on (13) and (15)

**Case (9g).**

(7)  $\xi = \text{inl}(\xi_1)$  by assumption  
 (8)  $\tau = (\tau_1 + \tau_2)$  by assumption  
 (9)  $\xi_1 : \tau_1$  by assumption

By rule induction over Rules (20) on (5), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

(10)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$  by assumption  
 (11)  $e \text{ notintro}$  by Rule (27a),(27b),(27c),(27d),(27e),(27f)

Assume  $e \models \text{inl}(\xi_1)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

(12)  $e \not\models \text{inl}(\xi_1)$  by contradiction

By case analysis on the value of  $\text{refutable}_{\tau}(\text{inl}(\xi_1))$ .

**Case  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{true}$ .**

(13)  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{true}$  by assumption  
 (14)  $\text{inl}(\xi_1) \text{ refutable}_{\tau}$  by Lemma 2.0.14 on (13)  
 (15)  $e \models_{\tau} \text{inl}(\xi_1)$  by Rule (17b) on (11) and (14)  
 (16)  $e \models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Rule (18a) on (15)

**Case  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{false}$ .**

(13)  $\text{refutable}_{\tau}(\text{inl}(\xi_1)) = \text{false}$  by assumption  
 (14)  ~~$\text{inl}(\xi_1) \text{ refutable}_{\tau}$~~  by Lemma 2.0.14 on (13)

Assume  $e \models_{\tau} \text{inl}(\xi_1)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(15)  $\text{inl}(\xi_1) \text{ refutable}_{\tau}$  by assumption  
 Contradicts (14).

(16)  $e \not\models_{\tau} \text{inl}(\xi_1)$  by contradiction  
 (17)  $e \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.20 on (12) and (16)



**Case (20j).**

- (10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption
- (11)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (12)  $e_1$  **final** by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_1 \models \xi_1$ ,  $e_1 \models? \xi_1$ , and  $e_1 \not\models? \xi_1$  holds. By case analysis on which one holds.

**Case  $e_1 \models \xi_1$ .**

- (13)  $e_1 \models \xi_1$  by assumption
- (14)  $e_1 \not\models? \xi_1$  by assumption
- (15)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (15g) on (13)
- (16)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (18b) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ . By rule induction over Rules (17) on it, only two cases apply.

**Case (17b).**

- (17)  $\text{inl}_{\tau_2}(e_1)$  **notintro** by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17e).**

- (17)  $e_1 \models? \xi_1$

Contradicts (14).

- (18)  $\text{inl}_{\tau_2}(e_1) \not\models? \text{inl}(\xi_1)$  by contradiction

**Case  $e_1 \models? \xi_1$ .**

- (13)  $e_1 \not\models \xi_1$  by assumption
- (14)  $e_1 \models? \xi_1$  by assumption
- (15)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (17e) on (14)
- (16)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (18a) on (15)

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15g).**

- (17)  $e_1 \models \xi_1$

Contradicts (13).

- (18)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$  by contradiction

**Case  $e_1 \not\models? \xi_1$ .**

- (13)  $e_1 \not\models \xi_1$  by assumption

(14)  $e_1 \not\models_{\tau} \xi_1$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15g).**

(15)  $e_1 \models \xi_1$

Contradicts (13).

(16)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inl}(\xi_1)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\xi_1)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(17)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17e).**

(17)  $e_1 \models_{\tau} \xi_1$

Contradicts (14).

(18)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inl}(\xi_1)$  by contradiction

(19)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.20 on (16) and (18)

**Case (20k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inl}(\xi_1)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

(11)  $\text{inr}_{\tau_1}(e_2) \not\models \text{inl}(\xi_1)$  by contradiction

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inl}(\xi_1)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(12)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inl}(\xi_1)$  by contradiction

(14)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau}^{\dagger} \text{inl}(\xi_1)$  by Lemma 2.0.20 on (11) and (13)

**Case (9h).**

(7)  $\xi = \text{inr}(\xi_2)$  by assumption

- (8)  $\tau = (\tau_1 + \tau_2)$  by assumption  
 (9)  $\xi_2 : \tau_2$  by assumption

By rule induction over Rules (20) on (5), the following cases apply.

**Case** (20b),(20c),(20f),(20h),(20i),(20l),(20m).

- (10)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
 by assumption  
 (11)  $e$  **notintro** by Rule  
 (27a),(27b),(27c),(27d),(27e),(27f)

Assume  $e \models \text{inr}(\xi_2)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

- (12)  $e \not\models \text{inr}(\xi_2)$  by contradiction

By case analysis on the value of  $\text{refutable}_?( \text{inr}(\xi_2) )$ .

inr is  
refutable

**Case**  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{true}$ .

- (13)  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{true}$  by assumption  
 (14)  $\text{inr}(\xi_2)$  **refutable?** by Lemma 2.0.14 on  
 (13)  
 (15)  $e \models? \text{inr}(\xi_2)$  by Rule (17b) on (11)  
 and (14)  
 (16)  $e \models?^\dagger \text{inr}(\xi_2)$  by Rule (18a) on (15)

**Case**  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{false}$ .

- (13)  $\text{refutable}_?( \text{inr}(\xi_2) ) = \text{false}$  by assumption  
 (14)  ~~$\text{inr}(\xi_2)$  **refutable?**~~ by Lemma 2.0.14 on  
 (13)

Assume  $e \models? \text{inr}(\xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

- (15)  $\text{inr}(\xi_2)$  **refutable?** by assumption  
 Contradicts (14).

- (16)  $e \not\models? \text{inr}(\xi_2)$  by contradiction  
 (17)  $e \not\models?^\dagger \text{inr}(\xi_2)$  by Lemma 2.0.20 on  
 (12) and (16)

**Case** (20j).

- (10)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

Assume  $\text{inl}_{\tau_2}(e_1) \models \text{inr}(\xi_2)$ . By rule induction over Rules (15) on it, no case applies due to syntactic contradiction.

- (11)  $\text{inl}_{\tau_2}(e_1) \not\models \text{inr}(\xi_2)$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inr}(\xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(12)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (12), no case applies due to syntactic contradiction.

(13)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau} \text{inr}(\xi_2)$  by contradiction

(14)  $\text{inl}_{\tau_2}(e_1) \not\models_{\tau}^{\dagger} \text{inr}(\xi_2)$  by Lemma 2.0.20 on (11) and (13)

**Case (20k).**

(10)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

(11)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(12)  $e_2 \text{ final}$  by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of  $e_2 \models \xi_2$ ,  $e_2 \models_{\tau} \xi_2$ , and  $e_2 \not\models_{\tau}^{\dagger} \xi_2$  holds. By case analysis on which one holds.

**Case  $e_2 \models \xi_2$ .**

(13)  $e_2 \models \xi_2$  by assumption

(14)  $e_2 \not\models_{\tau} \xi_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by Rule (15g) on (13)

(16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$  by Rule (18b) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$ . By rule induction over Rules (17) on it, only two cases apply.

**Case (17b).**

(17)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17f).**

(17)  $e_2 \models_{\tau} \xi_2$

Contradicts (14).

(18)  $\text{inr}_{\tau_1}(e_2) \not\models_{\tau} \text{inr}(\xi_2)$  by contradiction

**Case  $e_2 \models_{\tau} \xi_2$ .**

(13)  $e_2 \not\models \xi_2$  by assumption

(14)  $e_2 \models_{\tau} \xi_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$  by Rule (17f) on (14)

(16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2)$  by Rule (18a) on (15)

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15h).**

$$(17) \quad e_2 \models \xi_2$$

Contradicts (13).

$$(18) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2) \quad \text{by contradiction}$$

**Case  $e_2 \not\models_{\tau_1}^{\dagger} \xi_2$ .**

$$(13) \quad e_2 \not\models \xi_2 \quad \text{by assumption}$$

$$(14) \quad e_2 \not\models_{\tau_1} \xi_2 \quad \text{by assumption}$$

Assume  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15h).**

$$(15) \quad e_2 \models \xi_2$$

Contradicts (13).

$$(16) \quad \text{inr}_{\tau_1}(e_2) \not\models \text{inr}(\xi_2) \quad \text{by contradiction}$$

Assume  $\text{inr}_{\tau_1}(e_2) \models_{\tau_1} \text{inr}(\xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

$$(17) \quad \text{inr}_{\tau_1}(e_2) \text{ notintro} \quad \text{by assumption}$$

By rule induction over Rules (27) on (17), no case applies due to syntactic contradiction.

**Case (17f).**

$$(17) \quad e_2 \models_{\tau_1} \xi_2$$

Contradicts (14).

$$(18) \quad \text{inr}_{\tau_1}(e_2) \not\models_{\tau_1} \text{inr}(\xi_2) \quad \text{by contradiction}$$

$$(19) \quad \text{inl}_{\tau_2}(e_1) \not\models_{\tau_2}^{\dagger} \text{inl}(\xi_1) \quad \text{by Lemma 2.0.20 on (16) and (18)}$$

**Case (15i).**

$$(7) \quad \xi = (\xi_1, \xi_2) \quad \text{by assumption}$$

$$(8) \quad \tau = (\tau_1 \times \tau_2) \quad \text{by assumption}$$

$$(9) \quad \xi_1 : \tau_1 \quad \text{by assumption}$$

$$(10) \quad \xi_2 : \tau_2 \quad \text{by assumption}$$

By rule induction over Rules (20) on (5), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

- (11)  $e = \emptyset^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (12)  $e$  notintro  
by Rule (27a),(27b),(27c),(27d),(27e),(27f)
- (13)  $e$  indet  
by Lemma 4.0.10 on (6) and (12)
- (14)  $\text{prl}(e)$  indet  
by Rule (25g) on (13)
- (15)  $\text{prl}(e)$  final  
by Rule (26b) on (14)
- (16)  $\text{prr}(e)$  indet  
by Rule (25h) on (13)
- (17)  $\text{prr}(e)$  final  
by Rule (26b) on (16)
- (18)  $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$   
by Rule (20h) on (5)
- (19)  $\cdot; \Delta \vdash \text{prr}(e) : \tau_2$   
by Rule (20i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of  $\text{prl}(e) \models \xi_1$ ,  $\text{prl}(e) \models? \xi_1$ , and  $\text{prl}(e) \not\models?^\dagger \xi_1$  holds.

By inductive hypothesis on (10) and (19) and (17), exactly one of  $\text{prr}(e) \models \xi_2$ ,  $\text{prr}(e) \models? \xi_2$ , and  $\text{prr}(e) \not\models?^\dagger \xi_2$  holds.

By case analysis on which conclusion holds for  $\xi_1$  and  $\xi_2$ .

**Case**  $\text{prl}(e) \models \xi_1, \text{prr}(e) \models \xi_2$ .

- (20)  $\text{prl}(e) \models \xi_1$   
by assumption
- (21)  $\text{prl}(e) \not\models? \xi_1$   
by assumption
- (22)  $\text{prr}(e) \models \xi_2$   
by assumption
- (23)  $\text{prr}(e) \not\models? \xi_2$   
by assumption
- (24)  $e \models (\xi_1, \xi_2)$   
by Rule (15j) on (12) and (20) and (22)
- (25)  $e \models?^\dagger (\xi_1, \xi_2)$   
by Rule (18b) on (24)
- (26)  ~~$(\xi_1, \xi_2)$  refutable?~~  
by Lemma 2.0.18 on (12) and (24)

Assume  $e \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

- (27)  $(\xi_1, \xi_2)$  refutable?  
by assumption
- Contradicts (26).

- (28)  $e \not\models? (\xi_1, \xi_2)$   
by contradiction

**Case**  $\text{prl}(e) \models \xi_1, \text{prr}(e) \models? \xi_2$ .

- (20)  $\text{prl}(e) \models \xi_1$   
by assumption
- (21)  $\text{prl}(e) \not\models? \xi_1$   
by assumption
- (22)  $\text{prr}(e) \not\models \xi_2$   
by assumption
- (23)  $\text{prr}(e) \models? \xi_2$   
by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15), only one case applies.

**Case (15j).**

(24)  $\text{prr}(e) \models \xi_2$  by assumption  
Contradicts (22)

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

By rule induction over Rules (17) on (23), only one case applies.

**Case (17b).**

(26)  $\xi_2 \text{ refutable?}$  by assumption  
(27)  $(\xi_1, \xi_2) \text{ refutable?}$  by Rule (11e) on (26)  
(28)  $e \models? (\xi_1, \xi_2)$  by Rule (17b) on (12) and (27)  
(29)  $e \models?^\dagger (\xi_1, \xi_2)$  by Rule (18a) on (28)

**Case  $\text{prl}(e) \models \xi_1, \text{prr}(e) \not\models?^\dagger \xi_2$ .**

(20)  $\text{prl}(e) \models \xi_1$  by assumption  
(21)  $\text{prl}(e) \not\models? \xi_1$  by assumption  
(22)  $\text{prr}(e) \not\models \xi_2$  by assumption  
(23)  $\text{prr}(e) \not\models? \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15j).**

(24)  $\text{prr}(e) \models \xi_2$  by assumption  
Contradicts (22).

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $e \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(26)  $(\xi_1, \xi_2) \text{ refutable?}$  by assumption

By rule induction over Rules (11) on (26), only two cases apply.

**Case (11d).**

(27)  $\xi_1 \text{ refutable?}$  by assumption  
(28)  $\text{prl}(e) \text{ notintro}$  by Rule (27e)  
(29)  $\text{prl}(e) \models? \xi_1$  by Rule (17b) on (28) and (27)

Contradicts (21).

**Case (11e).**

assume no  
"or" and  
"and" in  
pair

|  |                                   |
|--|-----------------------------------|
| (27) $\xi_2$ <b>refutable</b> <sub>?</sub> | by assumption                     |
| (28) <b>pr</b> $r(e)$ <b>notintro</b>      | by Rule (27f)                     |
| (29) <b>pr</b> $r(e) \models_? \xi_2$      | by Rule (17b) on (28)<br>and (27) |

Contradicts (23).

|   |                                     |
|---|-------------------------------------|
| (30) $e \not\models_? (\xi_1, \xi_2)$         | by contradiction                    |
| (31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ | by Lemma 2.0.20 on<br>(25) and (30) |

**Case**  $\text{prl}(e) \models_? \xi_1, \text{pr}r(e) \models \xi_2$ .

|  |               |
|--|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$   | by assumption |
| (21) $\text{prl}(e) \models_? \xi_1$     | by assumption |
| (22) $\text{pr}r(e) \models \xi_2$       | by assumption |
| (23) $\text{pr}r(e) \not\models_? \xi_2$ | by assumption |

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15), only one case applies.

**Case** (15j).

|                                    |               |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
|------------------------------------|---------------|

Contradicts (20).

|                                     |                  |
|-------------------------------------|------------------|
| (25) $e \not\models (\xi_1, \xi_2)$ | by contradiction |
|-------------------------------------|------------------|

By rule induction over Rules (17) on (21), only one case applies.

**Case** (17b).

|   |                                   |
|---|-----------------------------------|
| (26) $\xi_1$ <b>refutable</b> <sub>?</sub>          | by assumption                     |
| (27) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (11e) on (26)             |
| (28) $e \models_? (\xi_1, \xi_2)$                   | by Rule (17b) on (12)<br>and (27) |
| (29) $e \models_?^\dagger (\xi_1, \xi_2)$           | by Rule (18a) on (28)             |

assume no  
"or" and  
"and" in  
pair

**Case**  $\text{prl}(e) \models_? \xi_1, \text{pr}r(e) \models_? \xi_2$ .

|  |               |
|--|---------------|
| (20) $\text{prl}(e) \not\models \xi_1$ | by assumption |
| (21) $\text{prl}(e) \models_? \xi_1$   | by assumption |
| (22) $\text{pr}r(e) \not\models \xi_2$ | by assumption |
| (23) $\text{pr}r(e) \models_? \xi_2$   | by assumption |

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15), only one case applies.

**Case** (15j).

|                                    |               |
|------------------------------------|---------------|
| (24) $\text{prl}(e) \models \xi_1$ | by assumption |
|------------------------------------|---------------|

Contradicts (20).



(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction  
 By rule induction over Rules (17) on (23), only one case applies.

**Case (17b).**

(26)  $\xi_2$  **refutable**? by assumption  
 (27)  $(\xi_1, \xi_2)$  **refutable**? by Rule (11e) on (26)  
 (28)  $e \models? (\xi_1, \xi_2)$  by Rule (17b) on (12) and (27)  
 (29)  $e \models?^\dagger (\xi_1, \xi_2)$  by Rule (18a) on (28)

**Case**  $\text{prl}(e) \models? \xi_1, \text{prr}(e) \not\models?^\dagger \xi_2$ .

(20)  $\text{prl}(e) \not\models \xi_1$  by assumption  
 (21)  $\text{prl}(e) \models? \xi_1$  by assumption  
 (22)  $\text{prr}(e) \not\models \xi_2$  by assumption  
 (23)  $\text{prr}(e) \not\models? \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15), only one case applies.

**Case (15j).**

(24)  $\text{prl}(e) \models \xi_1$  by assumption  
 Contradicts (20)

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction  
 By rule induction over Rules (17) on (21), only one case applies.

**Case (17b).**

(26)  $\xi_1$  **refutable**? by assumption  
 (27)  $(\xi_1, \xi_2)$  **refutable**? by Rule (11e) on (26)  
 (28)  $e \models? (\xi_1, \xi_2)$  by Rule (17b) on (12) and (27)  
 (29)  $e \models?^\dagger (\xi_1, \xi_2)$  by Rule (18a) on (28)

**Case**  $\text{prl}(e) \not\models?^\dagger \xi_1, \text{prr}(e) \models \xi_2$ .

(20)  $\text{prl}(e) \not\models \xi_1$  by assumption  
 (21)  $\text{prl}(e) \not\models? \xi_1$  by assumption  
 (22)  $\text{prr}(e) \models \xi_2$  by assumption  
 (23)  $\text{prr}(e) \not\models? \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case (15j).**

(24)  $\text{prl}(e) \models \xi_1$  by assumption  
 Contradicts (20)

assume no  
"or" and  
"and" in  
pair

assume no  
"or" and  
"and" in  
pair

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $e \models_{\text{?}} (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case (17b).**

(26)  $(\xi_1, \xi_2) \text{ refutable}_{\text{?}}$  by assumption

By rule induction over Rules (11) on (26), only two cases apply.

**Case (11d).**

(27)  $\xi_1 \text{ refutable}_{\text{?}}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (27e)

(29)  $\text{prl}(e) \models_{\text{?}} \xi_1$  by Rule (17b) on (28) and (27)

Contradicts (21).

**Case (11e).**

(27)  $\xi_2 \text{ refutable}_{\text{?}}$  by assumption

(28)  $\text{prr}(e) \text{ notintro}$  by Rule (27f)

(29)  $\text{prr}(e) \models_{\text{?}} \xi_2$  by Rule (17b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models_{\text{?}} (\xi_1, \xi_2)$  by contradiction

(31)  $e \not\models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$  by Lemma 2.0.20 on (25) and (30)

**Case  $\text{prl}(e) \not\models_{\text{?}}^{\dagger} \xi_1, \text{prr}(e) \models_{\text{?}} \xi_2$ .**

(20)  $\text{prl}(e) \not\models \xi_1$  by assumption

(21)  $\text{prl}(e) \not\models_{\text{?}} \xi_1$  by assumption

(22)  $\text{prr}(e) \not\models \xi_2$  by assumption

(23)  $\text{prr}(e) \models_{\text{?}} \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15), only one case applies.

**Case (15j).**

(24)  $\text{prl}(e) \models \xi_1$  by assumption

Contradicts (20).

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

By rule induction over Rules (17) on (23), only one case applies.

**Case (17b).**

(26)  $\xi_2 \text{ refutable}_{\text{?}}$  by assumption

(27)  $(\xi_1, \xi_2) \text{ refutable}_{\text{?}}$  by Rule (11e) on (26)

assume no  
"or" and  
"and" in  
pair

(28)  $e \models_{\tau} (\xi_1, \xi_2)$  by Rule (17b) on (12) and (27)

(29)  $e \models_{\tau}^{\dagger} (\xi_1, \xi_2)$  by Rule (18a) on (28)

**Case**  $\text{prl}(e) \not\models_{\tau}^{\dagger} \xi_1, \text{prl}(e) \not\models_{\tau}^{\dagger} \xi_2$ .

(20)  $\text{prl}(e) \not\models \xi_1$  by assumption

(21)  $\text{prl}(e) \not\models_{\tau} \xi_1$  by assumption

(22)  $\text{prl}(e) \not\models \xi_2$  by assumption

(23)  $\text{prl}(e) \not\models_{\tau} \xi_2$  by assumption

Assume  $e \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only one case applies.

**Case** (15j).

(24)  $\text{prl}(e) \models \xi_1$  by assumption

Contradicts (20)

(25)  $e \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $e \models_{\tau} (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, only one case applies.

**Case** (17b).

(26)  $(\xi_1, \xi_2) \text{ refutable}_{\tau}$  by assumption

By rule induction over Rules (11) on (26), only two cases apply.

**Case** (11d).

(27)  $\xi_1 \text{ refutable}_{\tau}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (27e)

(29)  $\text{prl}(e) \models_{\tau} \xi_1$  by Rule (17b) on (28) and (27)

Contradicts (21).

**Case** (11e).

(27)  $\xi_2 \text{ refutable}_{\tau}$  by assumption

(28)  $\text{prl}(e) \text{ notintro}$  by Rule (27f)

(29)  $\text{prl}(e) \models_{\tau} \xi_2$  by Rule (17b) on (28) and (27)

Contradicts (23).

(30)  $e \not\models_{\tau} (\xi_1, \xi_2)$  by contradiction

(31)  $e \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$  by Lemma 2.0.20 on (25) and (30)

**Case** (20g).

(11)  $e = (e_1, e_2)$  by assumption

(12)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption

- (13)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption
- (14)  $e_1$  **final** by Lemma 4.0.5 on (6)
- (15)  $e_2$  **final** by Lemma 4.0.5 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of  $e_1 \models \xi_1$ ,  $e_1 \models? \xi_1$ , and  $e_1 \models \overline{\xi_1}$  holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of  $e_2 \models \xi_2$ ,  $e_2 \models? \xi_2$ , and  $e_2 \models \overline{\xi_2}$  holds.

By case analysis on which conclusion holds for  $\xi_1$  and  $\xi_2$ .

**Case**  $e_1 \models \xi_1, e_2 \models \xi_2$ .

- (16)  $e_1 \models \xi_1$  by assumption
- (17)  $e_1 \not\models? \xi_1$  by assumption
- (18)  $e_2 \models \xi_2$  by assumption
- (19)  $e_2 \not\models? \xi_2$  by assumption
- (20)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (15i) on (16) and (18)
- (21)  $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$  by Rule (18b) on (20)

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

- (22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (17g).

- (22)  $e_1 \models? \xi_1$  by assumption

Contradicts (17).

**Case** (17h).

- (22)  $e_2 \models? \xi_2$  by assumption

Contradicts (19).

**Case** (17i).

- (22)  $e_1 \models? \xi_1$  by assumption

Contradicts (17).

- (23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction

**Case**  $e_1 \models \xi_1, e_2 \models? \xi_2$ .

- (16)  $e_1 \models \xi_1$  by assumption
- (17)  $e_1 \not\models? \xi_1$  by assumption
- (18)  $e_2 \not\models \xi_2$  by assumption
- (19)  $e_2 \models? \xi_2$  by assumption
- (20)  $(e_1, e_2) \models? (\xi_1, \xi_2)$  by Rule (17h) on (16) and (19)

(21)  $(e_1, e_2) \models_{\text{?}}^{\dagger} (\xi_1, \xi_2)$  by Rule (18a) on (20)

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (15i).**

(22)  $e_2 \models \xi_2$  by assumption

Contradicts (18).

(23)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

**Case**  $e_1 \models \xi_1, e_2 \not\models_{\text{?}}^{\dagger} \xi_2$ .

(16)  $e_1 \models \xi_1$  by assumption

(17)  $e_1 \not\models_{\text{?}} \xi_1$  by assumption

(18)  $e_2 \not\models \xi_2$  by assumption

(19)  $e_2 \not\models_{\text{?}} \xi_2$  by assumption

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (15i).**

(20)  $e_2 \models \xi_2$  by assumption

Contradicts (18).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models_{\text{?}} (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (17g).**

(22)  $e_1 \models_{\text{?}} \xi_1$  by assumption

Contradicts (17).

**Case (17h).**

(22)  $e_2 \models_{\text{?}} \xi_2$  by assumption

Contradicts (19).

**Case (17i).**

(22)  $e_1 \models_{\text{?}} \xi_1$  by assumption

Contradicts (17).

- |      |  |                                     |
|------|--|-------------------------------------|
| (23) | $(e_1, e_2) \not\models? (\xi_1, \xi_2)$         | by contradiction                    |
| (24) | $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$ | by Lemma 2.0.20 on<br>(21) and (23) |

**Case**  $e_1 \models? \xi_1, e_2 \models \xi_2$ .

- |      |                                      |                                   |
|------|--------------------------------------|-----------------------------------|
| (16) | $e_1 \not\models \xi_1$              | by assumption                     |
| (17) | $e_1 \models? \xi_1$                 | by assumption                     |
| (18) | $e_2 \models \xi_2$                  | by assumption                     |
| (19) | $e_2 \not\models? \xi_2$             | by assumption                     |
| (20) | $(e_1, e_2) \models? (\xi_1, \xi_2)$ | by Rule (17g) on (17)<br>and (18) |

- |      |  |                       |
|------|--|-----------------------|
| (21) | $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ | by Rule (18a) on (20) |
|------|--|-----------------------|

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

- |      |                              |               |
|------|------------------------------|---------------|
| (22) | $(e_1, e_2)$ <b>notintro</b> | by assumption |
|------|------------------------------|---------------|

Contradicts Lemma 4.0.9.

**Case** (15i).

- |      |                     |               |
|------|---------------------|---------------|
| (22) | $e_1 \models \xi_1$ | by assumption |
|------|---------------------|---------------|

Contradicts (16).

- |      |   |                  |
|------|---|------------------|
| (23) | $(e_1, e_2) \not\models (\xi_1, \xi_2)$ | by contradiction |
|------|---|------------------|

**Case**  $e_1 \models? \xi_1, e_2 \models? \xi_2$ .

- |      |                                      |                                   |
|------|--------------------------------------|-----------------------------------|
| (16) | $e_1 \not\models \xi_1$              | by assumption                     |
| (17) | $e_1 \models? \xi_1$                 | by assumption                     |
| (18) | $e_2 \not\models \xi_2$              | by assumption                     |
| (19) | $e_2 \models? \xi_2$                 | by assumption                     |
| (20) | $(e_1, e_2) \models? (\xi_1, \xi_2)$ | by Rule (17i) on (17)<br>and (19) |

- |      |  |                       |
|------|--|-----------------------|
| (21) | $(e_1, e_2) \models?^\dagger (\xi_1, \xi_2)$ | by Rule (18a) on (20) |
|------|--|-----------------------|

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

- |      |                              |               |
|------|------------------------------|---------------|
| (22) | $(e_1, e_2)$ <b>notintro</b> | by assumption |
|------|------------------------------|---------------|

Contradicts Lemma 4.0.9.

**Case** (15i).

- |      |                     |               |
|------|---------------------|---------------|
| (22) | $e_1 \models \xi_1$ | by assumption |
|------|---------------------|---------------|

Contradicts (16).

(23)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

**Case**  $e_1 \models? \xi_1, e_2 \not\models?^\dagger \xi_2$ .

(16)  $e_1 \not\models \xi_1$  by assumption

(17)  $e_1 \models? \xi_1$  by assumption

(18)  $e_2 \not\models \xi_2$  by assumption

(19)  $e_2 \not\models? \xi_2$  by assumption

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case** (15j).

(20)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (15i).

(20)  $e_1 \models \xi_1$  by assumption

Contradicts (16).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case** (17b).

(22)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case** (17g).

(22)  $e_2 \models \xi_2$  by assumption

Contradicts (18).

**Case** (17h).

(22)  $e_2 \models? \xi_2$  by assumption

Contradicts (19).

**Case** (17i).

(22)  $e_2 \models? \xi_2$  by assumption

Contradicts (19).

(23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction

(24)  $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$  by Lemma 2.0.20 on (21) and (23)

**Case**  $e_1 \not\models?^\dagger \xi_1, e_2 \models \xi_2$ .

(16)  $e_1 \not\models \xi_1$  by assumption

(17)  $e_1 \not\models? \xi_1$  by assumption

(18)  $e_2 \models \xi_2$  by assumption

(19)  $e_2 \not\models? \xi_2$  by assumption  
 Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.  
**Case (15j).**  
 (20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.  
**Case (15i).**  
 (20)  $e_1 \models \xi_1$  by assumption  
 Contradicts (16).  
  
 (21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction  
 Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, the following cases apply.  
**Case (17b).**  
 (22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.  
**Case (17g).**  
 (22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).  
**Case (17h).**  
 (22)  $e_2 \models? \xi_2$  by assumption  
 Contradicts (19).  
**Case (17i).**  
 (22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).  
  
 (23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction  
 (24)  $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$  by Lemma 2.0.20 on (21) and (23)  
**Case  $e_1 \not\models?^\dagger \xi_1, e_2 \models? \xi_2$ .**  
 (16)  $e_1 \not\models \xi_1$  by assumption  
 (17)  $e_1 \not\models? \xi_1$  by assumption  
 (18)  $e_2 \not\models \xi_2$  by assumption  
 (19)  $e_2 \models? \xi_2$  by assumption  
 Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.  
**Case (15j).**  
 (20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.  
**Case (15i).**



(20)  $e_2 \models \xi_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction  
 Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

(22)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (17g).**

(22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).

**Case (17h).**

(22)  $e_1 \models \xi_1$  by assumption  
 Contradicts (16).

**Case (17i).**

(22)  $e_1 \models? \xi_1$  by assumption  
 Contradicts (17).

(23)  $(e_1, e_2) \not\models? (\xi_1, \xi_2)$  by contradiction

(24)  $(e_1, e_2) \not\models?^\dagger (\xi_1, \xi_2)$  by Lemma 2.0.20 on (21) and (23)

**Case  $e_1 \not\models?^\dagger \xi_1, e_2 \not\models?^\dagger \xi_2$ .**

(16)  $e_1 \not\models \xi_1$  by assumption

(17)  $e_1 \not\models? \xi_1$  by assumption

(18)  $e_2 \not\models \xi_2$  by assumption

(19)  $e_2 \not\models? \xi_2$  by assumption

Assume  $(e_1, e_2) \models (\xi_1, \xi_2)$ . By rule induction over Rules (15) on it, only two cases apply.

**Case (15j).**

(20)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.

**Case (15i).**

(20)  $e_2 \models \xi_2$  by assumption  
 Contradicts (18).

(21)  $(e_1, e_2) \not\models (\xi_1, \xi_2)$  by contradiction

Assume  $(e_1, e_2) \models? (\xi_1, \xi_2)$ . By rule induction over Rules (17) on it, the following cases apply.

**Case (17b).**

|   |                                     |
|---|-------------------------------------|
| (22) $(e_1, e_2)$ <b>notintro</b>                             | by assumption                       |
| Contradicts Lemma 4.0.9.                                      |                                     |
| <b>Case (17g).</b>  |                                     |
| (22) $e_1 \models_{\tau} \xi_1$                               | by assumption                       |
| Contradicts (17).   |                                     |
| <b>Case (17h).</b>  |                                     |
| (22) $e_2 \models_{\tau} \xi_2$                               | by assumption                       |
| Contradicts (19).   |                                     |
| <b>Case (17i).</b>  |                                     |
| (22) $e_1 \models_{\tau} \xi_1$                               | by assumption                       |
| Contradicts (17).   |                                     |
| (23) $(e_1, e_2) \not\models_{\tau} (\xi_1, \xi_2)$           | by contradiction                    |
| (24) $(e_1, e_2) \not\models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.20 on<br>(21) and (23) |

□

**Definition 2.1.1** (Entailment of Constraints). *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \models \xi_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **val** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies  $e \models \xi_2$*

**Definition 2.1.2** (Potential Entailment of Constraints). *Suppose that  $\xi_1 : \tau$  and  $\xi_2 : \tau$ . Then  $\xi_1 \models_{\tau}^{\dagger} \xi_2$  iff for all  $e$  such that  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final** we have  $e \models_{\tau}^{\dagger} \xi_1$  implies  $e \models_{\tau}^{\dagger} \xi_2$*

**Corollary 2.1.1.** *Suppose that  $\xi : \tau$  and  $\cdot ; \Delta \vdash e : \tau$  and  $e$  **final**. Then  $\top \models_{\tau}^{\dagger} \xi$  implies  $e \models_{\tau}^{\dagger} \xi$*

*Proof.*

|  |   |
|--|---|
| (1) $\xi : \tau$                         | by assumption   |
| (2) $\cdot ; \Gamma \vdash e : \tau$     | by assumption   |
| (3) $e$ <b>final</b>                     | by assumption   |
| (4) $\top \models_{\tau}^{\dagger} \xi$  | by assumption   |
| (5) $e_1 \models \top$                   | by Rule (15a)   |
| (6) $e_1 \models_{\tau}^{\dagger} \top$  | by Rule (18b) on (5)  |
| (7) $\top : \tau$                        | by Rule (9a)  |
| (8) $e_1 \models_{\tau}^{\dagger} \xi_r$ | by Definition 2.1.2 of<br>(4) on (7) and (1) and<br>(2) and (3) and (6) |

□

### 3 Static Semantics

$$\begin{aligned}
\tau &::= \mathbf{num} \mid (\tau_1 \rightarrow \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e &::= x \mid \underline{n} \\
&\quad \mid (\lambda x : \tau. e) \mid e_1(e_2) \\
&\quad \mid (e_1, e_2) \\
&\quad \mid \mathbf{inl}_\tau(e) \mid \mathbf{inr}_\tau(e) \mid \mathbf{match}(e)\{\hat{r}s\} \\
&\quad \mid \mathbb{0}^u \mid \mathbb{e}^u \\
\hat{r}s &::= (rs \mid r \mid rs) \\
rs &::= \cdot \mid (r \mid rs') \\
r &::= p \Rightarrow e \\
p &::= x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \mathbf{inl}(p) \mid \mathbf{inr}(p) \mid \mathbb{0}^w \mid \mathbb{p}^w \\
\boxed{(\hat{r}s)^\diamond = rs} &\quad rs \text{ can be obtained by erasing pointer from } \hat{r}s
\end{aligned}$$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \quad (19a)$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \quad (19b)$$

$$\boxed{\Gamma; \Delta \vdash e : \tau} \quad e \text{ is of type } \tau$$

$$\begin{array}{c}
\text{TVar} \\
\hline
\Gamma, x : \tau; \Delta \vdash x : \tau
\end{array} \quad (20a)$$

$$\begin{array}{c}
\text{TEHole} \\
\hline
\Gamma; \Delta, u :: \tau \vdash \mathbb{0}^u : \tau
\end{array} \quad (20b)$$

$$\begin{array}{c}
\text{THole} \\
\hline
\Gamma; \Delta, u :: \tau \vdash e : \tau' \\
\hline
\Gamma; \Delta, u :: \tau \vdash \mathbb{e}^u : \tau
\end{array} \quad (20c)$$

$$\begin{array}{c}
\text{TNum} \\
\hline
\Gamma; \Delta \vdash \underline{n} : \mathbf{num}
\end{array} \quad (20d)$$

$$\begin{array}{c}
\text{TLam} \\
\hline
\Gamma, x : \tau_1; \Delta \vdash e : \tau_2 \\
\hline
\Gamma; \Delta \vdash (\lambda x : \tau_1. e) : (\tau_1 \rightarrow \tau_2)
\end{array} \quad (20e)$$

$$\begin{array}{c}
\text{TAp} \\
\hline
\Gamma; \Delta \vdash e_1 : (\tau_2 \rightarrow \tau) \quad \Gamma; \Delta \vdash e_2 : \tau_2 \\
\hline
\Gamma; \Delta \vdash e_1(e_2) : \tau
\end{array} \quad (20f)$$

$$\begin{array}{c}
\text{TPair} \\
\hline
\Gamma; \Delta \vdash e_1 : \tau_1 \quad \Gamma; \Delta \vdash e_2 : \tau_2 \\
\hline
\Gamma; \Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)
\end{array} \quad (20g)$$

$$\begin{array}{c}
\text{TPrl} \\
\hline
\Gamma; \Delta \vdash e : (\tau_1 \times \tau_2) \\
\hline
\Gamma; \Delta \vdash \mathbf{prl}(e) : \tau_1
\end{array} \quad (20h)$$

$$\frac{\text{TPrr}}{\Gamma; \Delta \vdash e : (\tau_1 \times \tau_2)} \quad \Gamma; \Delta \vdash \text{pr}(e) : \tau_2 \quad (20i)$$

$$\frac{\text{TInl}}{\Gamma; \Delta \vdash e : \tau_1} \quad \Gamma; \Delta \vdash \text{inl}_{\tau_2}(e) : (\tau_1 + \tau_2) \quad (20j)$$

$$\frac{\text{TInr}}{\Gamma; \Delta \vdash e : \tau_2} \quad \Gamma; \Delta \vdash \text{inr}_{\tau_1}(e) : (\tau_1 + \tau_2) \quad (20k)$$

$$\frac{\text{TMatchZPre} \quad \Gamma; \Delta \vdash e : \tau \quad \Gamma; \Delta \vdash [\perp]r \mid rs : \tau[\xi] \Rightarrow \tau' \quad \top \models_{\tau}^{\dagger} \xi}{\Gamma; \Delta \vdash \text{match}(e)\{\cdot \mid r \mid rs\} : \tau'} \quad (20l)$$

$$\frac{\text{TMatchNZPre} \quad \Gamma; \Delta \vdash e : \tau \quad e \text{ final} \quad \Gamma; \Delta \vdash [\perp]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \quad \Gamma; \Delta \vdash [\perp \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \quad e \not\models_{\tau}^{\dagger} \xi_{pre} \quad \top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}}{\Gamma; \Delta \vdash \text{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'} \quad (20m)$$

$\boxed{p : \tau[\xi] \dashv \Gamma; \Delta}$   $p$  is assigned type  $\tau$  and emits constraint  $\xi$

$$\frac{\text{PTVar}}{x : \tau[\top] \dashv \cdot; x : \tau} \quad (21a)$$

$$\frac{\text{PTWild}}{\_ : \tau[\top] \dashv \cdot; \cdot} \quad (21b)$$

$$\frac{\text{PTEHole}}{\langle \rangle^w : \tau[?] \dashv \cdot; w :: \tau} \quad (21c)$$

$$\frac{\text{PTHole} \quad p : \tau[\xi] \dashv \Gamma; \Delta}{\langle p \rangle^w : \tau'[?] \dashv \Gamma; \Delta, w :: \tau'} \quad (21d)$$

$$\frac{\text{PTNum}}{\underline{n} : \text{num}[\underline{n}] \dashv \cdot; \cdot} \quad (21e)$$

$$\frac{\text{PTInl} \quad p : \tau_1[\xi] \dashv \Gamma; \Delta}{\text{inl}(p) : (\tau_1 + \tau_2)[\text{inl}(\xi)] \dashv \Gamma; \Delta} \quad (21f)$$

$$\frac{\text{PTInr} \quad p : \tau_2[\xi] \dashv \Gamma; \Delta}{\text{inr}(p) : (\tau_1 + \tau_2)[\text{inr}(\xi)] \dashv \Gamma; \Delta} \quad (21g)$$

$$\frac{\text{PTPair} \quad p_1 : \tau_1[\xi_1] \dashv \Gamma_1; \Delta_1 \quad p_2 : \tau_2[\xi_2] \dashv \Gamma_2; \Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv \Gamma_1 \uplus \Gamma_2; \Delta_1 \uplus \Delta_2} \quad (21h)$$

$$\boxed{\Gamma ; \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'} \quad \begin{array}{l} r \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\frac{\text{CTRrule} \quad p : \tau[\xi] \dashv\!\!\vdash \Gamma_p ; \Delta_p \quad \Gamma \uplus \Gamma_p ; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma ; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \quad (22a)$$

$$\boxed{\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'} \quad \begin{array}{l} rs \text{ transforms a final expression of type } \tau \\ \text{to a final expression of type } \tau' \end{array}$$

$$\frac{\text{CTOneRules} \quad \Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \xi_r \not\models \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \quad (23a)$$

$$\frac{\text{CTRrules} \quad \Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \quad \Gamma ; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \quad \xi_r \not\models \xi_{pre}}{\Gamma ; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \quad (23b)$$

**Lemma 3.0.1.** *If  $p : \tau[\xi] \dashv\!\!\vdash \Gamma ; \Delta$  then  $\xi : \tau$ .*

*Proof.* By rule induction over Rules (21). □

**Lemma 3.0.2.** *If  $\cdot ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  then  $\xi_r : \tau_1$ .*

*Proof.* By rule induction over Rules (22). □

**Lemma 3.0.3.** *If  $\cdot ; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$  then  $\xi_{rs} : \tau_1$ .*

*Proof.* By rule induction over Rules (23). □

**Lemma 3.0.4.** *If  $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$  and  $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$  and  $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$  then  $\Gamma ; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$*

*Proof.*

- (1)  $\Gamma ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$  by assumption
- (2)  $\Gamma ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$  by assumption
- (3)  $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$  by assumption

By rule induction over Rules (23) on (1).

**Case (23a).**

- (4)  $rs = r' \mid \cdot$  by assumption
- (5)  $\xi_{rs} = \xi'_r$  by assumption
- (6)  $\Gamma ; \Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$  by assumption
- (7)  $\xi'_r \not\models \xi_{pre}$  by assumption
- (8)  $\Gamma ; \Delta \vdash [\xi_{pre} \vee \xi'_r]r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$  by Rule (23a) on (2) and (3)
- (9)  $\Gamma ; \Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$  by Rule (23b) on (6) and (8) and (7)

$$(10) \quad \Gamma; \Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau' \\ \text{by Definition 19 on (9)}$$

**Case (23b).**

$$\begin{aligned} (4) \quad rs &= r' \mid rs' && \text{by assumption} \\ (5) \quad \xi_{rs} &= \xi'_r \vee \xi'_{rs} && \text{by assumption} \\ (6) \quad \Gamma; \Delta \vdash r' : \tau[\xi'_r] &\Rightarrow \tau' && \text{by assumption} \\ (7) \quad \Gamma; \Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] &\Rightarrow \tau' && \text{by assumption} \\ (8) \quad \xi'_r &\not\models \xi_{pre} && \text{by assumption} \\ (9) \quad \Gamma; \Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^\diamond : \tau[\xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by IH on (7) and (2)} \\ &\text{and (3)} \\ (10) \quad \Gamma; \Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^\diamond) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by Rule (23b) on (6)} \\ &\text{and (9) and (8)} \\ (11) \quad \Gamma; \Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^\diamond : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] &\Rightarrow \tau' \\ &\text{by Definition 19 on} \\ &\text{(10)} \end{aligned}$$

□

**Lemma 3.0.5** (Substitution). *If  $\Gamma, x : \tau; \Delta \vdash e_0 : \tau_0$  and  $\Gamma; \Delta \vdash e : \tau$  then  $\Gamma; \Delta \vdash [e/x]e_0 : \tau_0$*

**Lemma 3.0.6** (Simultaneous Substitution). *If  $\Gamma \uplus \Gamma'; \Delta \vdash e : \tau$  and  $\theta : \Gamma'$  then  $\Gamma; \Delta \vdash [\theta]e : \tau$*

**Lemma 3.0.7** (Substitution Typing). *If  $e \triangleright p \dashv\vdash \theta$  and  $\cdot; \Delta_e \vdash e : \tau$  and  $p : \tau[\xi] \dashv\vdash \Gamma; \Delta$  then  $\theta : \Gamma$*

Proof by induction on the derivation of  $e \triangleright p \dashv\vdash \theta$ .

**Theorem 3.1** (Determinism). *If  $\cdot; \Delta \vdash e : \tau$  then exactly one of the following holds*

1.  $e \text{ val}$
2.  $e \text{ indet}$
3.  $e \mapsto e'$  for some unique  $e'$

## 4 Dynamic Semantics

$\boxed{e \text{ val}}$      $e$  is a value

$$\frac{\text{VNum}}{\underline{n \text{ val}}} \quad (24a)$$

$$\frac{\text{VLam}}{(\lambda x : \tau. e) \text{ val}} \quad (24b)$$

$$\frac{\text{VPair} \quad e_1 \text{ val} \quad e_2 \text{ val}}{(e_1, e_2) \text{ val}} \quad (24c)$$

$$\frac{\text{VInl} \quad e \text{ val}}{\text{inl}_\tau(e) \text{ val}} \quad (24d)$$

$$\frac{\text{VInr} \quad e \text{ val}}{\text{inr}_\tau(e) \text{ val}} \quad (24e)$$

$\boxed{e \text{ indet}}$      $e$  is indeterminate

$$\frac{\text{IEHole}}{\llbracket \cdot \rrbracket^u \text{ indet}} \quad (25a)$$

$$\frac{\text{IHole} \quad e \text{ final}}{\llbracket e \rrbracket^u \text{ indet}} \quad (25b)$$

$$\frac{\text{IAp} \quad e_1 \text{ indet} \quad e_2 \text{ final}}{e_1(e_2) \text{ indet}} \quad (25c)$$

$$\frac{\text{IPairL} \quad e_1 \text{ indet} \quad e_2 \text{ val}}{(e_1, e_2) \text{ indet}} \quad (25d)$$

$$\frac{\text{IPairR} \quad e_1 \text{ val} \quad e_2 \text{ indet}}{(e_1, e_2) \text{ indet}} \quad (25e)$$

$$\frac{\text{IPair} \quad e_1 \text{ indet} \quad e_2 \text{ indet}}{(e_1, e_2) \text{ indet}} \quad (25f)$$

$$\frac{\text{IPrl} \quad e \text{ indet}}{\text{prl}(e) \text{ indet}} \quad (25g)$$

$$\frac{\text{IPrr} \quad e \text{ \texttt{indet}}}{\text{pr}(e) \text{ \texttt{indet}}} \quad (25\text{h})$$

$$\frac{\text{IInL} \quad e \text{ \texttt{indet}}}{\text{inl}_\tau(e) \text{ \texttt{indet}}} \quad (25\text{i})$$

$$\frac{\text{IInR} \quad e \text{ \texttt{indet}}}{\text{inr}_\tau(e) \text{ \texttt{indet}}} \quad (25\text{j})$$

$$\frac{\text{IMatch} \quad e \text{ \texttt{final}} \quad e ? p_r}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \text{ \texttt{indet}}} \quad (25\text{k})$$

$$\boxed{e \text{ \texttt{final}}} \quad e \text{ is final}$$

$$\frac{\text{FVal} \quad e \text{ \texttt{val}}}{e \text{ \texttt{final}}} \quad (26\text{a})$$

$$\frac{\text{FIndet} \quad e \text{ \texttt{indet}}}{e \text{ \texttt{final}}} \quad (26\text{b})$$

$$\boxed{e \text{ \texttt{notintro}}} \quad e \text{ cannot be a value syntactically}$$

$$\frac{\text{NVEHole}}{\llbracket \cdot \rrbracket^u \text{ \texttt{notintro}}} \quad (27\text{a})$$

$$\frac{\text{NVHole}}{\llbracket e \rrbracket^u \text{ \texttt{notintro}}} \quad (27\text{b})$$

$$\frac{\text{NVAp}}{e_1(e_2) \text{ \texttt{notintro}}} \quad (27\text{c})$$

$$\frac{\text{NVMatch}}{\text{match}(e)\{\hat{rs}\} \text{ \texttt{notintro}}} \quad (27\text{d})$$

$$\frac{\text{NVPrI}}{\text{prl}(e) \text{ \texttt{notintro}}} \quad (27\text{e})$$

$$\frac{\text{NVPrR}}{\text{pr}(e) \text{ \texttt{notintro}}} \quad (27\text{f})$$

$$\boxed{\text{notintro}(e)}$$



$$\text{notintro}(\emptyset^u) = \text{true} \quad (28a)$$

$$\text{notintro}(\langle e \rangle^u) = \text{true} \quad (28b)$$

$$\text{notintro}(e_1(e_2)) = \text{true} \quad (28c)$$

$$\text{notintro}(\text{match}(e)\{\hat{r}s\}) = \text{true} \quad (28d)$$

$$\text{notintro}(\text{prl}(e)) = \text{true} \quad (28e)$$

$$\text{notintro}(\text{prrr}(e)) = \text{true} \quad (28f)$$

$$\text{Otherwise } \text{notintro}(e) = \text{false} \quad (28g)$$

**Lemma 4.0.1** (Soundness and Completeness of NotIntro Judgment).  $e \text{ notintro}$  iff  $\text{notintro}(e)$ .

*Proof.* TODO □

$$\boxed{e' \in \text{values}(e)} \quad e' \text{ is one of the possible values of } e$$

$$\frac{\text{IVVal} \quad e \text{ val} \quad \cdot; \Delta \vdash e : \tau}{e \in \text{values}(e)} \quad (29a)$$

$$\frac{\text{IVIndet} \quad e \text{ notintro} \quad \cdot; \Delta \vdash e : \tau \quad e' \text{ val} \quad \cdot; \Delta \vdash e' : \tau}{e' \in \text{values}(e)} \quad (29b)$$

$$\frac{\text{IVInl} \quad \text{inl}_{\tau_2}(e_1) \text{ indet} \quad \cdot; \Delta \vdash \text{inl}_{\tau_2}(e_1) : \tau \quad e'_1 \in \text{values}(e_1)}{\text{inl}_{\tau_2}(e'_1) \in \text{values}(\text{inl}_{\tau_2}(e_1))} \quad (29c)$$

$$\frac{\text{IVInr} \quad \text{inr}_{\tau_1}(e_2) \text{ indet} \quad \cdot; \Delta \vdash \text{inr}_{\tau_1}(e_2) : \tau \quad e'_2 \in \text{values}(e_2)}{\text{inr}_{\tau_1}(e'_2) \in \text{values}(\text{inr}_{\tau_1}(e_2))} \quad (29d)$$

$$\frac{\text{IVPair} \quad (e_1, e_2) \text{ indet} \quad \cdot; \Delta \vdash (e_1, e_2) : \tau \quad e'_1 \in \text{values}(e_1) \quad e'_2 \in \text{values}(e_2)}{(e'_1, e'_2) \in \text{values}((e_1, e_2))} \quad (29e)$$

**Lemma 4.0.2.** If  $e \text{ indet}$  and  $\cdot; \Delta \vdash e : \tau$  and  $\dot{\xi} : \tau$  and  $e \not\vdash_{\tau}^{\dagger} \dot{\xi}$  then  $e' \not\vdash_{\tau}^{\dagger} \dot{\xi}$  whenever  $e' \in \text{values}(e)$ .

*Proof.*

- |   |               |
|---|---------------|
| (1) $e \text{ indet}$                         | by assumption |
| (2) $\cdot; \Delta \vdash e : \tau$           | by assumption |
| (3) $\dot{\xi} : \tau$                        | by assumption |
| (4) $e \not\vdash_{\tau}^{\dagger} \dot{\xi}$ | by assumption |

By rule induction over Rules (9) on (3).

**Case (9a).**

- |                                       |                      |
|---------------------------------------|----------------------|
| (5) $\dot{\xi} = \top$                | by assumption        |
| (6) $e \models \top$                  | by Rule (15a)        |
| (7) $e \models_{\top}^{\dagger} \top$ | by Rule (18b) on (6) |

Contradicts (4).

**Case (1b).**

- |                                    |                      |
|------------------------------------|----------------------|
| (5) $\dot{\xi} = ?$                | by assumption        |
| (6) $e \models_{\top} ?$           | by Rule (17a)        |
| (7) $e \models_{\top}^{\dagger} ?$ | by Rule (18a) on (6) |

Contradicts (4).

**Case (9c).**

- |  |               |
|--|---------------|
| (5) $\dot{\xi} = \underline{n}$        | by assumption |
| (6) $\tau = \text{num}$                | by assumption |
| (7) $\underline{n} \text{ refutable?}$ | by Rule (11a) |

By rule induction over Rules (25) on (1).

**Case (25a).**

- |  |                                 |
|--|---------------------------------|
| (8) $e = \mathbb{O}^u$                                     | by assumption                   |
| (9) $\mathbb{O}^u \text{ notintro}$                        | by Rule (27a)                   |
| (10) $\mathbb{O}^u \models_{\top} \underline{n}$           | by Rule (17b) on (9)<br>and (7) |
| (11) $\mathbb{O}^u \models_{\top}^{\dagger} \underline{n}$ | by Rule (18a) on (10)           |

Contradicts (4).

**Case (25b).**

- |   |                                 |
|---|---------------------------------|
| (8) $e = \langle e_1 \rangle^u$                                     | by assumption                   |
| (9) $\langle e_1 \rangle^u \text{ notintro}$                        | by Rule (27b)                   |
| (10) $\langle e_1 \rangle^u \models_{\top} \underline{n}$           | by Rule (17b) on (9)<br>and (7) |
| (11) $\langle e_1 \rangle^u \models_{\top}^{\dagger} \underline{n}$ | by Rule (18a) on (10)           |

Contradicts (4).

**Case (25c).**

- |                                 |               |
|---------------------------------|---------------|
| (8) $e = e_1(e_2)$              | by assumption |
| (9) $e_1(e_2) \text{ notintro}$ | by Rule (27c) |

(10)  $e_1(e_2) \models_{\tau} \underline{n}$  by Rule (17b) on (9) and (7)

(11)  $e_1(e_2) \models_{\tau}^{\dagger} \underline{n}$  by Rule (18a) on (10)

Contradicts (4).

**Case (25g).**

(8)  $e = \text{prl}(e_1)$  by assumption

(9)  $\text{prl}(e_1) \text{ notintro}$  by Rule (27e)

(10)  $\text{prl}(e_1) \models_{\tau} \underline{n}$  by Rule (17b) on (9) and (7)

(11)  $\text{prl}(e_1) \models_{\tau}^{\dagger} \underline{n}$  by Rule (18a) on (10)

Contradicts (4).

**Case (25h).**

(8)  $e = \text{prr}(e_1)$  by assumption

(9)  $\text{prr}(e_1) \text{ notintro}$  by Rule (27f)

(10)  $\text{prr}(e_1) \models_{\tau} \underline{n}$  by Rule (17b) on (9) and (7)

(11)  $\text{prr}(e_1) \models_{\tau}^{\dagger} \underline{n}$  by Rule (18a) on (10)

Contradicts (4).

**Case (25k).**

(8)  $e = \text{match}(e_1)\{\hat{r}s\}$  by assumption

(9)  $\text{match}(e_1)\{\hat{r}s\} \text{ notintro}$  by Rule (27d)

(10)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau} \underline{n}$  by Rule (17b) on (9) and (7)

(11)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau}^{\dagger} \underline{n}$  by Rule (18a) on (10)

Contradicts (4).

**Case (25d), (25e), (25f).**

(8)  $e = (e_1, e_2)$  by assumption

By rule induction over Rules (20) on (2), no rule applies due to syntactic contradiction.

**Case (25i).**

(8)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

By rule induction over Rules (20) on (2), no rule applies due to syntactic contradiction.

**Case (25j).**

(8)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

By rule induction over Rules (20) on (2), no rule applies due to syntactic contradiction.

**Case (9g).**

- |  |               |
|--|---------------|
| (5) $\dot{\xi} = \text{inl}(\dot{\xi}_1)$        | by assumption |
| (6) $\tau = (\tau_1 + \tau_2)$                   | by assumption |
| (7) $\dot{\xi}_1 : \tau_1$                       | by assumption |
| (8) $\text{inl}(\dot{\xi}_1) \text{ refutable?}$ | by Rule (11b) |

By rule induction over Rules (25) on (1).

**Case (25a).**

- |  |                               |
|--|-------------------------------|
| (9) $e = \mathbb{0}^u$   | by assumption                 |
| (10) $\mathbb{0}^u \text{ notintro}$                                     | by Rule (27a)                 |
| (11) $\mathbb{0}^u \models_{\text{?}} \text{inl}(\dot{\xi}_1)$           | by Rule (17b) on (10) and (8) |
| (12) $\mathbb{0}^u \models_{\text{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25b).**

- |   |                               |
|---|-------------------------------|
| (9) $e = \langle e_1 \rangle^u$   | by assumption                 |
| (10) $\langle e_1 \rangle^u \text{ notintro}$                                     | by Rule (27b)                 |
| (11) $\langle e_1 \rangle^u \models_{\text{?}} \text{inl}(\dot{\xi}_1)$           | by Rule (17b) on (10) and (8) |
| (12) $\langle e_1 \rangle^u \models_{\text{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25c).**

- |  |                               |
|--|-------------------------------|
| (9) $e = e_1(e_2)$   | by assumption                 |
| (10) $e_1(e_2) \text{ notintro}$                                     | by Rule (27c)                 |
| (11) $e_1(e_2) \models_{\text{?}} \text{inl}(\dot{\xi}_1)$           | by Rule (17b) on (10) and (8) |
| (12) $e_1(e_2) \models_{\text{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25g).**

- |   |                               |
|---|-------------------------------|
| (9) $e = \text{prl}(e_1)$   | by assumption                 |
| (10) $\text{prl}(e_1) \text{ notintro}$                                     | by Rule (27e)                 |
| (11) $\text{prl}(e_1) \models_{\text{?}} \text{inl}(\dot{\xi}_1)$           | by Rule (17b) on (10) and (8) |
| (12) $\text{prl}(e_1) \models_{\text{?}}^{\dagger} \text{inl}(\dot{\xi}_1)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25h).**

- (9)  $e = \text{pr}(e_1)$  by assumption
- (10)  $\text{pr}(e_1) \text{ notintro}$  by Rule (27f)
- (11)  $\text{pr}(e_1) \models_{\tau} \text{inl}(\dot{\xi}_1)$  by Rule (17b) on (10) and (8)
- (12)  $\text{pr}(e_1) \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (18a) on (11)

Contradicts (4).

**Case (25k).**

- (9)  $e = \text{match}(e_1)\{\hat{r}s\}$  by assumption
- (10)  $\text{match}(e_1)\{\hat{r}s\} \text{ notintro}$  by Rule (27d)
- (11)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau} \text{inl}(\dot{\xi}_1)$  by Rule (17b) on (10) and (8)
- (12)  $\text{match}(e_1)\{\hat{r}s\} \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Rule (18a) on (11)

Contradicts (4).

**Case (25d), (25e), (25f).**

- (9)  $e = (e_1, e_2)$  by assumption

By rule induction over Rules (25) on (1), no rule applies due to syntactic contradiction.

**Case (25i).**

- (9)  $e = \text{inl}_{\tau_2'}(e_1)$  by assumption
- (10)  $e_1 \text{ indet}$  by assumption

By rule induction over Rules (20) on (2), only one rule applies.

**Case (20j).**

- (11)  $\tau_2' = \tau_2$  by assumption
- (12)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (13)  $e_1 \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by Lemma 2.0.11 on (4)
- (14) if  $e_1' \in \text{values}(e_1)$  then  $e_1' \not\models_{\tau}^{\dagger} \dot{\xi}_1$  by IH on (10) and (12) and (7) and (13)

To show if  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  then  $e' \models_{\tau}^{\dagger} \text{inl}(\dot{\xi}_1)$ , we assume  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$ .

- (15)  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  by assumption

By rule induction over Rules (29) on (15).

**Case (29a).**

- (16)  $\text{inl}_{\tau_2}(e_1) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

- (16)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7

**Case (29c).**

- (16)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption
- (17)  $e'_1 \in \text{values}(e_1)$  by assumption
- (18)  $e'_1 \not\models_{\tau_2}^{\dagger} \dot{\xi}_1$  by (14) on (17)
- (19)  $\text{inl}_{\tau_2}(e'_1) \not\models_{\tau_2}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 2.0.11 on (18)

**Case (25j).**

- (9)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption

To show if  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  then  $e' \not\models_{\tau_1}^{\dagger} \text{inl}(\dot{\xi}_1)$ , we assume  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$ .

- (10)  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  by assumption

By rule induction over Rules (29) on (10).

**Case (29a).**

- (11)  $\text{inr}_{\tau_1}(e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

- (11)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8

**Case (29d).**

- (11)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption
- (12)  $\text{inr}_{\tau_1}(e'_2) \not\models_{\tau_1}^{\dagger} \text{inl}(\dot{\xi}_1)$  by Lemma 1.0.22

**Case (9h).**

- (5)  $\dot{\xi} = \text{inr}(\dot{\xi}_2)$  by assumption
- (6)  $\tau = (\tau_1 + \tau_2)$  by assumption
- (7)  $\dot{\xi}_2 : \tau_2$  by assumption
- (8)  $\text{inr}(\dot{\xi}_2) \text{ refutable?}$  by Rule (11c)

By rule induction over Rules (25) on (1).

**Case (25a).**

- (9)  $e = \mathbb{O}^u$  by assumption
- (10)  $\mathbb{O}^u \text{ notintro}$  by Rule (27a)
- (11)  $\mathbb{O}^u \models_{\tau} \text{inr}(\dot{\xi}_2)$  by Rule (17b) on (10) and (8)
- (12)  $\mathbb{O}^u \models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Rule (18a) on (11)

Contradicts (4).

**Case (25b).**

- |  |                               |
|--|-------------------------------|
| (9) $e = \langle e_1 \rangle^u$  | by assumption                 |
| (10) $\langle e_1 \rangle^u \text{ notintro}$                                    | by Rule (27b)                 |
| (11) $\langle e_1 \rangle^u \models_{\dot{?}} \text{inr}(\dot{\xi}_2)$           | by Rule (17b) on (10) and (8) |
| (12) $\langle e_1 \rangle^u \models_{\dot{?}}^{\dagger} \text{inr}(\dot{\xi}_2)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25c).**

- |   |                               |
|---|-------------------------------|
| (9) $e = e_1(e_2)$  | by assumption                 |
| (10) $e_1(e_2) \text{ notintro}$                                    | by Rule (27c)                 |
| (11) $e_1(e_2) \models_{\dot{?}} \text{inr}(\dot{\xi}_2)$           | by Rule (17b) on (10) and (8) |
| (12) $e_1(e_2) \models_{\dot{?}}^{\dagger} \text{inr}(\dot{\xi}_2)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25g).**

- |  |                               |
|--|-------------------------------|
| (9) $e = \text{prl}(e_1)$  | by assumption                 |
| (10) $\text{prl}(e_1) \text{ notintro}$                                    | by Rule (27e)                 |
| (11) $\text{prl}(e_1) \models_{\dot{?}} \text{inr}(\dot{\xi}_2)$           | by Rule (17b) on (10) and (8) |
| (12) $\text{prl}(e_1) \models_{\dot{?}}^{\dagger} \text{inr}(\dot{\xi}_2)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25h).**

- |  |                               |
|--|-------------------------------|
| (9) $e = \text{prr}(e_1)$  | by assumption                 |
| (10) $\text{prr}(e_1) \text{ notintro}$                                    | by Rule (27f)                 |
| (11) $\text{prr}(e_1) \models_{\dot{?}} \text{inr}(\dot{\xi}_2)$           | by Rule (17b) on (10) and (8) |
| (12) $\text{prr}(e_1) \models_{\dot{?}}^{\dagger} \text{inr}(\dot{\xi}_2)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25k).**

- |  |                               |
|--|-------------------------------|
| (9) $e = \text{match}(e_1)\{\hat{r}s\}$  | by assumption                 |
| (10) $\text{match}(e_1)\{\hat{r}s\} \text{ notintro}$                                    | by Rule (27d)                 |
| (11) $\text{match}(e_1)\{\hat{r}s\} \models_{\dot{?}} \text{inr}(\dot{\xi}_2)$           | by Rule (17b) on (10) and (8) |
| (12) $\text{match}(e_1)\{\hat{r}s\} \models_{\dot{?}}^{\dagger} \text{inr}(\dot{\xi}_2)$ | by Rule (18a) on (11)         |

Contradicts (4).

**Case (25d), (25e), (25f).**

- |                      |               |
|----------------------|---------------|
| (9) $e = (e_1, e_2)$ | by assumption |
|----------------------|---------------|

By rule induction over Rules (25) on (1), no rule applies due to syntactic contradiction.

**Case (25i).**

(9)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

To show if  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  then  $e' \not\vdash_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$ , we assume  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$ .

(10)  $e' \in \text{values}(\text{inl}_{\tau_2}(e_1))$  by assumption

By rule induction over Rules (29) on (10).

**Case (29a).**

(11)  $\text{inl}_{\tau_2}(e_1) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(11)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7

**Case (29c).**

(11)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption

(12)  $\text{inl}_{\tau_2}(e'_1) \not\vdash_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Lemma 1.0.21

**Case (25j).**

(9)  $e = \text{inr}_{\tau'_1}(e_2)$  by assumption

(10)  $e_2 \text{ indet}$  by assumption

By rule induction over Rules (20) on (2), only one rule applies.

**Case (20k).**

(11)  $\tau'_1 = \tau_1$  by assumption

(12)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(13)  $e_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.11 on (4)

(14) if  $e'_2 \in \text{values}(e_2)$  then  $e'_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$   
by IH on (10) and (12)  
and (7) and (13)

To show if  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  then  $e' \not\vdash_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$ , we assume  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$ .

(15)  $e' \in \text{values}(\text{inr}_{\tau_1}(e_2))$  by assumption

By rule induction over Rules (29) on (15).

**Case (29a).**

(16)  $\text{inr}_{\tau_1}(e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(16)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption



Contradicts Lemma 4.0.8

**Case (29d).**

- (16)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption
- (17)  $e'_2 \in \text{values}(e_2)$  by assumption
- (18)  $e'_2 \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by (14) on (17)
- (19)  $\text{inr}_{\tau_1}(e'_2) \not\models_{\tau}^{\dagger} \text{inr}(\dot{\xi}_2)$  by Lemma 2.0.12 on (18)

**Case (9i).**

- (5)  $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$  by assumption
- (6)  $\tau = (\tau_1 \times \tau_2)$  by assumption
- (7)  $\dot{\xi}_1 : \tau_1$  by assumption
- (8)  $\dot{\xi}_2 : \tau_2$  by assumption

By rule induction over Rules (25) on (1).

**Case (25a), (25b), (25c), (25g), (25h), (25k).**

- (9)  $e = \text{ll}^u, \text{ll}_{e_1}^u, e_1(e_2), \text{prl}(e_1), \text{prr}(e_1), \text{match}(e_1)\{\text{rs}\}$  by assumption
- (10)  $e \text{ notintro}$  by Rules (27)
- (11)  $\text{prl}(e) \text{ notintro}$  by Rule (27e)
- (12)  $\text{prr}(e) \text{ notintro}$  by Rule (27f)
- (13)  $\text{prl}(e) \text{ indet}$  by Rule (25g) on (1)
- (14)  $\text{prr}(e) \text{ indet}$  by Rule (25h) on (1)
- (15)  $\cdot; \Delta \vdash \text{prl}(e) : \tau_1$  by Rule (20h) on (2)
- (16)  $\cdot; \Delta \vdash \text{prr}(e) : \tau_2$  by Rule (20i) on (2)

By case analysis on the result of  $\text{satisfyormay}(\text{prl}(e), \dot{\xi}_1)$ .

**Case true.**

- (17)  $\text{satisfyormay}(\text{prl}(e), \dot{\xi}_1) = \text{true}$  by assumption
- (18)  $\text{prl}(e) \models_{\tau}^{\dagger} \dot{\xi}_1$  by Lemma 1.0.4 on (17)

By case analysis on the result of  $\text{satisfyormay}(\text{prr}(e), \dot{\xi}_2)$ .

**Case true.**

- (19)  $\text{satisfyormay}(\text{prr}(e), \dot{\xi}_2) = \text{true}$  by assumption
- (20)  $\text{prr}(e) \models_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 1.0.4 on (19)

By rule induction over Rules (18) on (18).

**Case (18b).**

(21)  $\text{prl}(e) \models \dot{\xi}_1$  by assumption

By rule induction over Rules (18) on (20).

**Case (18b).**

(22)  $\text{prr}(e) \models \dot{\xi}_2$  by assumption

(23)  $e \models (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (15j) on (10) and (21) and (22)

(24)  $e \models_{\dagger}^? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18b) on (23)

Contradicts (4).

**Case (18a).**

(22)  $\text{prr}(e) \models_{\dagger}^? \dot{\xi}_2$  by assumption

(23)  $\dot{\xi}_2 \text{ refutable}_{\dagger}^?$  by Lemma 1.0.17 on (12) and (22)

(24)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dagger}^?$  by Rule (11e) on (23)

(25)  $e \models_{\dagger}^? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17b) on (10) and (24)

(26)  $e \models_{\dagger}^? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (25)

**Case (18a).**

(21)  $\text{prl}(e) \models_{\dagger}^? \dot{\xi}_1$  by assumption

(22)  $\dot{\xi}_1 \text{ refutable}_{\dagger}^?$  by Lemma 1.0.17 on (11) and (21)

(23)  $(\dot{\xi}_1, \dot{\xi}_2) \text{ refutable}_{\dagger}^?$  by Rule (11d) on (22)

(24)  $e \models_{\dagger}^? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (17b) on (10) and (23)

(25)  $e \models_{\dagger}^? (\dot{\xi}_1, \dot{\xi}_2)$  by Rule (18a) on (24)

**Case false.**

(19)  $\text{satisfyormay}(\text{prr}(e), \dot{\xi}_2) = \text{false}$

by assumption

(20)  $\text{prr}(e) \models_{\dagger}^? \dot{\xi}_2$  by Lemma 1.0.4 on (19)

(21) if  $e'_2 \in \text{values}(\text{prr}(e))$  then  $e'_2 \not\models_{\dagger}^? \dot{\xi}_2$   
by IH on (14) and (16) and (8) and (20)

To show if  $e' \in \text{values}(e)$  then  $e' \not\models_{\dagger}^? (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}(e)$ .

(22)  $e' \in \text{values}(e)$  by assumption

By rule induction over Rules (29) on (22), only two rules apply.

**Case (29a).**

(23)  $e \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(23)  $e' \text{ val}$  by assumption

(24)  $\cdot; \Delta \vdash e' : (\tau_1 \times \tau_2)$  by assumption

By rule induction over Rules (24) on (23).

**Case (24a).**

(25)  $e' = \underline{n}$  by assumption

By rule induction over Rules (20) on (24), no rule applies due to syntactic contradiction.

**Case (24b).**

(25)  $e' = (\lambda x : \tau'. e'_1)$  by assumption

By rule induction over Rules (20) on (24), no rule applies due to syntactic contradiction.

**Case (24c).**

(25)  $e' = (e'_1, e'_2)$  by assumption

(26)  $e'_2 \text{ val}$  by assumption

By rule induction over Rules (20) on (24), only one rule applies.

**Case (20g).**

(27)  $\cdot; \Delta \vdash e'_2 : \tau_2$  by assumption

(28)  $e'_2 \in \text{values}(\text{pr}(e))$  by Rule (29b) on (12) and (16) and (26) and (27)

(29)  $e'_2 \not\models_{\tau}^{\dagger} \dot{\xi}_2$  by (21) on (28)

(30)  $(e'_1, e'_2) \not\models_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (27)

**Case (24d).**

(25)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption

By rule induction over Rules (20) on (24), no rule applies due to syntactic contradiction.

**Case (24e).**

(25)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption

By rule induction over Rules (20) on (24), no rule applies due to syntactic contradiction.

**Case false.**

(17)  $\text{satisfyormay}(\text{prl}(e), \dot{\xi}_1) = \text{false}$

by assumption

(18)  $\text{prl}(e) \not\models_{\tau}^{\dagger} \dot{\xi}_1$

by Lemma 1.0.4 on (17)

(19) if  $e'_1 \in \text{values}(\text{prl}(e))$  then  $e'_1 \not\models_{\tau}^{\dagger} \dot{\xi}_1$

by IH on (13) and (15) and (7) and (18)

To show if  $e' \in \text{values}(e)$  then  $e' \not\vdash_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}(e)$ .

(20)  $e' \in \text{values}(e)$  by assumption

By rule induction over Rules (29) on (20), only two rules apply.

**Case (29a).**

(21)  $e \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(21)  $e' \text{ val}$  by assumption

(22)  $\cdot; \Delta \vdash e' : (\tau_1 \times \tau_2)$  by assumption

By rule induction over Rules (24) on (21).

**Case (24a).**

(23)  $e' = \underline{n}$  by assumption

By rule induction over Rules (20) on (22), no rule applies due to syntactic contradiction.

**Case (24b).**

(23)  $e' = (\lambda x : \tau'. e'_1)$  by assumption

By rule induction over Rules (20) on (22), no rule applies due to syntactic contradiction.

**Case (24c).**

(23)  $e' = (e'_1, e'_2)$  by assumption

(24)  $e'_1 \text{ val}$  by assumption

By rule induction over Rules (20) on (22), only one rule applies.

**Case (20g).**

(25)  $\cdot; \Delta \vdash e'_1 : \tau_1$  by assumption

(26)  $e'_1 \in \text{values}(\text{prl}(e))$  by Rule (29b) on (11) and (15) and (24) and (25)

(27)  $e'_1 \not\vdash_{\dot{?}}^{\dagger} \dot{\xi}_1$  by (19) on (26)

(28)  $(e'_1, e'_2) \not\vdash_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (27)

**Case (24d).**

(23)  $e' = \text{inl}_{\tau_2}(e'_1)$  by assumption

By rule induction over Rules (20) on (22), no rule applies due to syntactic contradiction.

**Case (24e).**

(23)  $e' = \text{inr}_{\tau_1}(e'_2)$  by assumption

By rule induction over Rules (20) on (22), no rule applies due to syntactic contradiction.

**Case (25d).**

- (9)  $e = (e_1, e_2)$  by assumption
- (10)  $e_1$  **indet** by assumption
- (11)  $e_2$  **val** by assumption
- (12)  $e_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  or  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.13 on (4)

By case analysis on the disjunction in (12).

**Case  $e_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$ .**

- (13)  $e_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by assumption

By rule induction over Rules (20) on (2), only one rule applies.

**Case (20g).**

- (14)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (15) if  $e'_1 \in \text{values}(e_1)$  then  $e'_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$   
by IH on (10) and (14) and (7) and (13)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}((e_1, e_2))$ .

- (16)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (29) on (16).

**Case (29a).**

- (17)  $(e_1, e_2)$  **val** by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

- (17)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (29e).**

- (17)  $e' = (e'_1, e'_2)$  by assumption
- (18)  $e'_1 \in \text{values}(e_1)$  by assumption
- (19)  $e'_1 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_1$  by (15) on (18)
- (20)  $(e'_1, e'_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (19)

**Case  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$ .**

- (13)  $e_2 \not\models_{\dot{?}}^{\dagger} \dot{\xi}_2$  by assumption

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{?}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \text{values}((e_1, e_2))$ .

- (14)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (29) on (14).

**Case (29a).**

- (15)  $(e_1, e_2)$  **val** by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(15)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (29e).**

(15)  $e' = (e'_1, e'_2)$  by assumption

(16)  $e'_2 \in \mathbf{values}(e_2)$  by assumption

By rule induction over Rules (29) on (16).

**Case (29a).**

(17)  $e'_2 = e_2$  by assumption

(18)  $e'_2 \not\vdash_{\dot{\xi}_2}^{\dot{\xi}_1}$  by (17) and (13)

(19)  $(e'_1, e'_2) \not\vdash_{\dot{\xi}_2}^{\dot{\xi}_1} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (18)

**Case (29b).**

(17)  $e_2$  **notintro** by assumption

Contradicts (11) by Lemma 4.0.12.

**Case (29c), (29d), (29e).**

(17)  $e_2$  **indet** by assumption

Contradicts (11) by Lemma 4.0.11.

**Case (25e).**

(9)  $e = (e_1, e_2)$  by assumption

(10)  $e_1$  **val** by assumption

(11)  $e_2$  **indet** by assumption

(12)  $e_1 \not\vdash_{\dot{\xi}_1}^{\dot{\xi}_2}$  or  $e_2 \not\vdash_{\dot{\xi}_2}^{\dot{\xi}_1}$  by Lemma 2.0.13 on (4)

By case analysis on the disjunction in (12).

**Case  $e_1 \not\vdash_{\dot{\xi}_1}^{\dot{\xi}_2}$ .**

(13)  $e_1 \not\vdash_{\dot{\xi}_1}^{\dot{\xi}_2}$  by assumption

To show that if  $e' \in \mathbf{values}((e_1, e_2))$  then  $(e_1, e_2) \not\vdash_{\dot{\xi}_2}^{\dot{\xi}_1} (\dot{\xi}_1, \dot{\xi}_2)$ , we assume  $e' \in \mathbf{values}((e_1, e_2))$ .

(14)  $e' \in \mathbf{values}((e_1, e_2))$  by assumption

By rule induction over Rules (29) on (14).

**Case (29a).**

(15)  $(e_1, e_2)$  **val** by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(15)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (29e).**

(15)  $e' = (e'_1, e'_2)$  by assumption

(16)  $e'_1 \in \text{values}(e_1)$  by assumption

By rule induction over Rules (29) on (16).

**Case (29a).**

(17)  $e'_1 = e_1$  by assumption

(18)  $e'_1 \not\vdash_{\dot{?}} \dot{\xi}_1$  by (17) and (13)

(19)  $(e'_1, e'_2) \not\vdash_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (18)

**Case (29b).**

(17)  $e_1 \text{ notintro}$  by assumption

Contradicts (10) by Lemma 4.0.12.

**Case (29c), (29d), (29e).**

(17)  $e_1 \text{ indet}$  by assumption

Contradicts (10) by Lemma 4.0.11.

**Case  $e_2 \not\vdash_{\dot{?}} \dot{\xi}_2$ .**

(13)  $e_2 \not\vdash_{\dot{?}} \dot{\xi}_2$  by assumption

By rule induction over Rules (20) on (2), only one rule applies.

**Case (20g).**

(14)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

(15) if  $e'_2 \in \text{values}(e_2)$  then  $e'_2 \not\vdash_{\dot{?}} \dot{\xi}_2$   
by IH on (11) and (14)  
and (8) and (13)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\vdash_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$ ,  
we assume  $e' \in \text{values}((e_1, e_2))$ .

(16)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (29) on (16).

**Case (29a).**

(17)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(17)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.9.

**Case (29e).**

(17)  $e' = (e'_1, e'_2)$  by assumption

(18)  $e'_2 \in \text{values}(e_2)$  by assumption

(19)  $e'_2 \not\vdash_{\dot{?}} \dot{\xi}_2$  by (15) on (18)

(20)  $(e'_1, e'_2) \not\vdash_{\dot{?}} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (19)

**Case (25f).**

- (9)  $e = (e_1, e_2)$  by assumption
- (10)  $e_1 \text{ indet}$  by assumption
- (11)  $e_2 \text{ indet}$  by assumption
- (12)  $e_1 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_1$  or  $e_2 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.13 on  
(4)

By rule induction over Rules (20) on (2), only one rule applies.

**Case (20g).**

- (13)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (14)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption

By case analysis on the disjunction in (12).

**Case  $e_1 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_1$ .**

- (15)  $e_1 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_1$  by assumption
- (16) if  $e'_1 \in \text{values}(e_1)$  then  $e'_1 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_1$  by IH on (10) and (13)  
and (7) and (15)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{\tau}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ ,  
we assume  $e' \in \text{values}((e_1, e_2))$ .

- (17)  $e' \in \text{values}((e_1, e_2))$  by assumption

By rule induction over Rules (29) on (17).

**Case (29a).**

- (18)  $(e_1, e_2) \text{ val}$  by assumption
- Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

- (18)  $(e_1, e_2) \text{ notintro}$  by assumption
- Contradicts Lemma 4.0.9.

**Case (29e).**

- (18)  $e' = (e'_1, e'_2)$  by assumption
- (19)  $e'_1 \in \text{values}(e_1)$  by assumption
- (20)  $e'_1 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_1$  by (16) on (19)
- (21)  $(e'_1, e'_2) \not\models_{\dot{\tau}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on  
(20)

**Case  $e_2 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_2$ .**

- (15)  $e_2 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_2$  by assumption
- (16) if  $e'_2 \in \text{values}(e_2)$  then  $e'_2 \not\models_{\dot{\tau}}^{\dagger} \dot{\xi}_2$  by IH on (11) and (14)  
and (8) and (15)

To show that if  $e' \in \text{values}((e_1, e_2))$  then  $(e_1, e_2) \not\models_{\dot{\tau}}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ ,  
we assume  $e' \in \text{values}((e_1, e_2))$ .

- (17)  $e' \in \text{values}((e_1, e_2))$  by assumption



By rule induction over Rules (29) on (17).

**Case (29a).**

(18)  $(e_1, e_2) \text{ val}$  by assumption

Contradicts (1) by Lemma 4.0.11.

**Case (29b).**

(18)  $(e_1, e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.9.

**Case (29e).**

(18)  $e' = (e'_1, e'_2)$  by assumption

(19)  $e'_2 \in \text{values}(e_2)$  by assumption

(20)  $e'_2 \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by (16) on (19)

(21)  $(e'_1, e'_2) \not\vdash_{\tau}^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$  by Lemma 2.0.13 on (20)

**Case (25i).**

(9)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

By rule induction over Rules (20) on (2), no rule applies due to syntactic contradiction.

**Case (25j).**

(9)  $e = \text{inr}_{\tau'_1}(e_2)$  by assumption

By rule induction over Rules (20) on (2), no rule applies due to syntactic contradiction.

**Case (9f).**

(5)  $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

(6)  $\dot{\xi}_1 : \tau_1$  by assumption

(7)  $\dot{\xi}_2 : \tau_2$  by assumption

(8)  $e \not\vdash_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$  by assumption

(9)  $e \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  by Lemma 2.0.10 on (8)

(10)  $e \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by Lemma 2.0.10 on (8)

(11) if  $e' \in \text{values}(e)$  then  $e' \not\vdash_{\tau}^{\dagger} \dot{\xi}_1$  by IH on (1) and (2) and (6) and (9)

(12) if  $e' \in \text{values}(e)$  then  $e' \not\vdash_{\tau}^{\dagger} \dot{\xi}_2$  by IH on (1) and (2) and (7) and (10)

To show that if  $e' \in \text{values}(e)$  then  $e' \not\vdash_{\tau}^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ , we assume  $e' \in \text{values}(e)$ .

- |   |                                  |
|---|----------------------------------|
| (13) $e' \in \mathbf{values}(e)$                              | by assumption                    |
| (14) $e' \not\vdash_{\dot{\xi}_1}^{\dagger}$                  | by (11) on (13)                  |
| (15) $e' \not\vdash_{\dot{\xi}_2}^{\dagger}$                  | by (12) on (13)                  |
| (16) $e' \not\vdash_{\dot{\xi}_1 \vee \dot{\xi}_2}^{\dagger}$ | by Lemma 2.0.10 on (14) and (15) |

□

$\theta : \Gamma$      $\theta$  is of type  $\Gamma$

$$\frac{\text{STEmpty}}{\overline{\emptyset : \cdot}} \quad (30a)$$

$$\frac{\text{STExtend} \quad \theta : \Gamma_{\theta} \quad \Gamma ; \Delta \vdash e : \tau}{\theta, x/e : \Gamma_{\theta}, x : \tau} \quad (30b)$$

$p \text{ refutable?}$      $p$  is refutable

$$\frac{\text{RNum}}{\overline{n \text{ refutable?}}} \quad (31a)$$

$$\frac{\text{REHole}}{\overline{\textcolor{violet}{\langle \rangle}^w \text{ refutable?}}} \quad (31b)$$

$$\frac{\text{RHole}}{\overline{\textcolor{violet}{\langle p \rangle}^w \text{ refutable?}}} \quad (31c)$$

$$\frac{\text{RInl}}{\overline{\text{inl}(p) \text{ refutable?}}} \quad (31d)$$

$$\frac{\text{RInr}}{\overline{\text{inr}(p) \text{ refutable?}}} \quad (31e)$$

$$\frac{\text{RPairL} \quad p_1 \text{ refutable?}}{\overline{(p_1, p_2) \text{ refutable?}}} \quad (31f)$$

$$\frac{\text{RPairR} \quad p_2 \text{ refutable?}}{\overline{(p_1, p_2) \text{ refutable?}}} \quad (31g)$$

$e \triangleright p \dashv\!\!\parallel \theta$      $e$  matches  $p$ , emitting  $\theta$

$$\frac{\text{MVar}}{\overline{e \triangleright x \dashv\!\!\parallel e/x}} \quad (32a)$$

$$\frac{\text{MWild}}{e \triangleright \_ \dashv\!\! \dashv \cdot} \quad (32b)$$

$$\frac{\text{MNum}}{\underline{n} \triangleright \underline{n} \dashv\!\! \dashv \cdot} \quad (32c)$$

$$\frac{\text{MPair} \quad e_1 \triangleright p_1 \dashv\!\! \dashv \theta_1 \quad e_2 \triangleright p_2 \dashv\!\! \dashv \theta_2}{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\! \dashv \theta_1 \uplus \theta_2} \quad (32d)$$

$$\frac{\text{MInl} \quad e \triangleright p \dashv\!\! \dashv \theta}{\text{inl}_\tau(e) \triangleright \text{inl}(p) \dashv\!\! \dashv \theta} \quad (32e)$$

$$\frac{\text{MInr} \quad e \triangleright p \dashv\!\! \dashv \theta}{\text{inr}_\tau(e) \triangleright \text{inr}(p) \dashv\!\! \dashv \theta} \quad (32f)$$

$$\frac{\text{MNotIntroPair} \quad e \text{ notintro} \quad \text{prl}(e) \triangleright p_1 \dashv\!\! \dashv \theta_1 \quad \text{prr}(e) \triangleright p_2 \dashv\!\! \dashv \theta_2}{e \triangleright (p_1, p_2) \dashv\!\! \dashv \theta_1 \uplus \theta_2} \quad (32g)$$

$\boxed{e ? p}$      $e$  may match  $p$

$$\frac{\text{MMEHole}}{e ? \langle \rangle^w} \quad (33a)$$

$$\frac{\text{MMHole}}{e ? \langle p \rangle^w} \quad (33b)$$

$$\frac{\text{MMNotIntro} \quad e \text{ notintro} \quad p \text{ refutable?}}{e ? p} \quad (33c)$$

$$\frac{\text{MMPairL} \quad e_1 ? p_1 \quad e_2 \triangleright p_2 \dashv\!\! \dashv \theta_2}{(e_1, e_2) ? (p_1, p_2)} \quad (33d)$$

$$\frac{\text{MMPairR} \quad e_1 \triangleright p_1 \dashv\!\! \dashv \theta_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (33e)$$

$$\frac{\text{MMPair} \quad e_1 ? p_1 \quad e_2 ? p_2}{(e_1, e_2) ? (p_1, p_2)} \quad (33f)$$

$$\frac{\text{MMInl} \quad e ? p}{\text{inl}_\tau(e) ? \text{inl}(p)} \quad (33g)$$

$$\frac{\text{MMInr} \quad e ? p}{\text{inr}_\tau(e) ? \text{inr}(p)} \quad (33h)$$

$\boxed{e \perp p}$       $e$  does not match  $p$

$$\frac{\text{NMNum} \quad n_1 \neq n_2}{\underline{n_1} \perp \underline{n_2}} \quad (34a)$$

$$\frac{\text{NMPairL} \quad e_1 \perp p_1}{(e_1, e_2) \perp (p_1, p_2)} \quad (34b)$$

$$\frac{\text{NMPairR} \quad e_2 \perp p_2}{(e_1, e_2) \perp (p_1, p_2)} \quad (34c)$$

$$\frac{\text{NMConfl}}{\text{inr}_\tau(e) \perp \text{inl}(p)} \quad (34d)$$

$$\frac{\text{NMConfr}}{\text{inl}_\tau(e) \perp \text{inr}(p)} \quad (34e)$$

$$\frac{\text{NMInl} \quad e \perp p}{\text{inr}_\tau(e) \perp \text{inl}(p)} \quad (34f)$$

$$\frac{\text{NMInr} \quad e \perp p}{\text{inl}_\tau(e) \perp \text{inr}(p)} \quad (34g)$$

$\boxed{e \mapsto e'}$       $e$  takes a step to  $e'$

$$\frac{\text{ITHole} \quad e \mapsto e'}{\llbracket e \rrbracket^u \mapsto \llbracket e' \rrbracket^u} \quad (35a)$$

$$\frac{\text{ITApFun} \quad e_1 \mapsto e'_1}{e_1(e_2) \mapsto e'_1(e_2)} \quad (35b)$$

$$\frac{\text{ITApArg} \quad e_1 \text{ val} \quad e_2 \mapsto e'_2}{e_1(e_2) \mapsto e_1(e'_2)} \quad (35c)$$

$$\frac{\text{ITAP} \quad e_2 \text{ val}}{(\lambda x : \tau. e_1)(e_2) \mapsto [e_2/x]e_1} \quad (35d)$$

$$\frac{\text{ITPairL} \quad e_1 \mapsto e'_1}{(e_1, e_2) \mapsto (e'_1, e_2)} \quad (35e)$$

$$\frac{\text{ITPairR} \quad e_1 \text{ val} \quad e_2 \mapsto e'_2}{(e_1, e_2) \mapsto (e_1, e'_2)} \quad (35f)$$

$$\frac{\text{ITPrl} \quad (e_1, e_2) \text{ final}}{\text{prl}((e_1, e_2)) \mapsto e_1} \quad (35g)$$

$$\frac{\text{ITPrr} \quad (e_1, e_2) \text{ final}}{\text{prr}((e_1, e_2)) \mapsto e_2} \quad (35h)$$

$$\frac{\text{ITInl} \quad e \mapsto e'}{\text{inl}_\tau(e) \mapsto \text{inl}_\tau(e')} \quad (35i)$$

$$\frac{\text{ITInr} \quad e \mapsto e'}{\text{inr}_\tau(e) \mapsto \text{inr}_\tau(e')} \quad (35j)$$

$$\frac{\text{ITExpMatch} \quad e \mapsto e'}{\text{match}(e)\{\hat{r}s\} \mapsto \text{match}(e')\{\hat{r}s\}} \quad (35k)$$

$$\frac{\text{ITSuccMatch} \quad e \text{ final} \quad e \triangleright p_r \dashv \parallel \theta}{\text{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \quad (35l)$$

$$\frac{\text{ITFailMatch} \quad e \text{ final} \quad e \perp p_r}{\text{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \text{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid \cdot\}^\diamond \mid r' \mid rs'\}} \quad (35m)$$

**Lemma 4.0.3.** *If  $\text{inl}_{\tau_2}(e_1) \text{ final}$  then  $e_1 \text{ final}$ .*

*Proof.* By rule induction over Rules (26) on  $\text{inl}_{\tau_2}(e_1) \text{ final}$ .

**Case (26a).**

$$(17) \quad \text{inl}_{\tau_2}(e_1) \text{ val} \quad \text{by assumption}$$

By rule induction over Rules (24) on (17), only one case applies.

**Case (24d).**

$$(18) \quad e_1 \text{ val} \quad \text{by assumption}$$

$$(19) \quad e_1 \text{ final} \quad \text{by Rule (26a) on (18)}$$

**Case (26b).**

$$(17) \quad \text{inl}_{\tau_2}(e_1) \text{ indet} \quad \text{by assumption}$$

By rule induction over Rules (25) on (17), only one case applies.

**Case (25i).**

(18)  $e_1$  **indet**

by assumption

(19)  $e_1$  **final**

by Rule (26b) on (18)

□

**Lemma 4.0.4.** *If  $\text{inr}_{\tau_1}(e_2)$  **final** then  $e_2$  **final**.*

*Proof.* By rule induction over Rules (26) on  $\text{inr}_{\tau_1}(e_2)$  **final**.

**Case (26a).**

(1)  $\text{inr}_{\tau_1}(e_2)$  **val**

by assumption

By rule induction over Rules (24) on (1), only one case applies.

**Case (24d).**

(2)  $e_2$  **val**

by assumption

(3)  $e_2$  **final**

by Rule (26a) on (2)

**Case (26b).**

(1)  $\text{inr}_{\tau_1}(e_2)$  **indet**

by assumption

By rule induction over Rules (25) on (1), only one case applies.

**Case (25i).**

(2)  $e_2$  **indet**

by assumption

(3)  $e_2$  **final**

by Rule (26b) on (2)

□

**Lemma 4.0.5.** *If  $(e_1, e_2)$  **final** then  $e_1$  **final** and  $e_2$  **final**.*

*Proof.* By rule induction over Rules (26) on  $(e_1, e_2)$  **final**.

**Case (26a).**

(1)  $(e_1, e_2)$  **val**

by assumption

By rule induction over Rules (24) on (1), only one case applies.

**Case (24c).**

(2)  $e_1$  **val**

by assumption

(3)  $e_2$  **val**

by assumption

(4)  $e_1$  **final**

by Rule (26a) on (2)

(5)  $e_2$  **final**

by Rule (26a) on (3)

**Case (26b).**

(1)  $(e_1, e_2)$  **indet** by assumption

By rule induction over Rules (25) on (1), only three cases apply.

**Case (25d).**

(2)  $e_1$  **indet** by assumption  
 (3)  $e_2$  **val** by assumption  
 (4)  $e_1$  **final** by Rule (26b) on (2)  
 (5)  $e_1$  **final** by Rule (26a) on (3)

**Case (25e).**

(2)  $e_1$  **val** by assumption  
 (3)  $e_2$  **indet** by assumption  
 (4)  $e_1$  **final** by Rule (26a) on (2)  
 (5)  $e_1$  **final** by Rule (26b) on (3)

**Case (25f).**

(2)  $e_1$  **indet** by assumption  
 (3)  $e_2$  **indet** by assumption  
 (4)  $e_1$  **final** by Rule (26b) on (2)  
 (5)  $e_1$  **final** by Rule (26b) on (3)

□

**Lemma 4.0.6.** *There doesn't exist  $\underline{n}$  such that  $\underline{n}$  **notintro**.*

*Proof.* By rule induction over Rules (27) on  $\underline{n}$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.7.** *There doesn't exist  $\text{inl}_\tau(e)$  such that  $\text{inl}_\tau(e)$  **notintro**.*

*Proof.* By rule induction over Rules (27) on  $\text{inl}_\tau(e)$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.8.** *There doesn't exist  $\text{inr}_\tau(e)$  such that  $\text{inr}_\tau(e)$  **notintro**.*

*Proof.* By rule induction over Rules (27) on  $\text{inr}_\tau(e)$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.9.** *There doesn't exist  $(e_1, e_2)$  such that  $(e_1, e_2)$  **notintro**.*

*Proof.* By rule induction over Rules (27) on  $(e_1, e_2)$  **notintro**, no case applies due to syntactic contradiction. □

**Lemma 4.0.10.** *If  $e$  **final** and  $e$  **notintro** then  $e$  **indet**.*

*Proof Sketch.* By rule induction over Rules (27) on  $e$  **notintro**, for each case, by rule induction over Rules (24) on  $e$  **val** and we notice that  $e$  **val** is not derivable. By rule induction over Rules (26) on  $e$  **final**, Rule (26a) result in a contradiction with the fact that  $e$  **val** is not derivable while Rule (26b) tells us  $e$  **indet**.  $\square$

**Lemma 4.0.11.** *There doesn't exist such an expression  $e$  such that both  $e$  **val** and  $e$  **indet**.*

**Lemma 4.0.12.** *There doesn't exist such an expression  $e$  such that both  $e$  **val** and  $e$  **notintro**.*

**Lemma 4.0.13** (Finality). *There doesn't exist such an expression  $e$  such that both  $e$  **final** and  $e \mapsto e'$  for some  $e'$*

*Proof.* Assume there exists such an  $e$  such that both  $e$  **final** and  $e \mapsto e'$  for some  $e'$  then proof by contradiction.

By rule induction over Rules (26) and Rules (35), *i.e.*, over Rules (24) and Rules (35) and over Rules (25) and Rules (35) respectively. The proof can be done by straightforward observation of syntactic contradictions.  $\square$

**Lemma 4.0.14** (Matching Determinism). *If  $e$  **final** and  $\cdot; \Delta_e \vdash e : \tau$  and  $p : \tau[\xi] \dashv \vdash \Gamma; \Delta$  then exactly one of the following holds*

1.  $e \triangleright p \dashv \vdash \theta$  for some  $\theta$
2.  $e ? p$
3.  $e \perp p$

*Proof.*

- |  |               |
|--|---------------|
| (1) $e$ <b>final</b>                             | by assumption |
| (2) $\cdot; \Delta_e \vdash e : \tau$            | by assumption |
| (3) $p : \tau[\xi] \dashv \vdash \Gamma; \Delta$ | by assumption |

By rule induction over Rules (21) on (3), we would show one conclusion is derivable while the other two are not.

**Case (21a).**

- |  |               |
|--|---------------|
| (4) $p = x$                                | by assumption |
| (5) $e \triangleright x \dashv \vdash e/x$ | by Rule (32a) |

Assume  $e ? x$ . By rule induction over Rules (33) on it, only one case applies.

**Case (33c).**

- |                           |               |
|---------------------------|---------------|
| (6) $x$ <b>refutable?</b> | by assumption |
|---------------------------|---------------|



By rule induction over Rules (31) on (6), no case applies due to syntactic contradiction.

(7)  $e \not\vdash x$  by contradiction

Assume  $e \perp x$ . By rule induction over Rules (34) on it, no case applies due to syntactic contradiction.

(8)  $e \not\perp x$  by contradiction

**Case (21b).**

(4)  $p = \_$  by assumption

(5)  $e \triangleright \_ \dashv\!\!\vdash$  by Rule (32b)

Assume  $e ? \_$ . By rule induction over Rules (33) on it, only one case applies.

**Case (33c).**

(6)  $\_ \text{refutable?}$  by assumption

By rule induction over Rules (31) on (6), no case applies due to syntactic contradiction.

(7)  $e \not\vdash \_$  by contradiction

Assume  $e \perp \_$ . By rule induction over Rules (34) on it, no case applies due to syntactic contradiction.

(8)  $e \not\perp \_$  by contradiction

**Case (21c).**

(4)  $p = \mathbb{O}^w$  by assumption

(5)  $e ? \mathbb{O}^w$  by Rule (33a)

Assume  $e \triangleright \mathbb{O}^w \dashv\!\!\vdash \theta$  for some  $\theta$ . By rule induction over Rules (33) on it, no case applies due to syntactic contradiction.

(6)  $e \triangleright \mathbb{O}^w \dashv\!\!\vdash \theta$  by contradiction

Assume  $e \perp \mathbb{O}^w$ . By rule induction over Rules (34) on it, no case applies due to syntactic contradiction.

(7)  $e \not\perp \mathbb{O}^w$  by contradiction

**Case (21d).**

- (4)  $p = \langle p_0 \rangle^w$  by assumption
- (5)  $e ? \langle p_0 \rangle^w$  by Rule (33b)

Assume  $e \triangleright \langle p_0 \rangle^w \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over Rules (33) on it, no case applies due to syntactic contradiction.

- (6)  $e \triangleright \cancel{\langle p_0 \rangle^w} \dashv\!\!\dashv \theta$  by contradiction

Assume  $e \perp \langle p_0 \rangle^w$ . By rule induction over Rules (34) on it, no case applies due to syntactic contradiction.

- (7)  $e \perp \cancel{\langle p_0 \rangle^w}$  by contradiction

**Case (21e).**

- (4)  $p = \underline{n_2}$  by assumption
- (5)  $\tau = \text{num}$  by assumption
- (6)  $\xi = \underline{n_2}$  by assumption
- (7)  $\underline{n_2} \text{ refutable?}$  by Rule (31a)

By rule induction over Rules (20) on (2), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

- (8)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{pr}(e_0), \text{match}(e_0)\{\hat{r}s\}$  by assumption
- (9)  $e \text{ notintro}$  by Rule  
(27a),(27b),(27c),(27d),(27e),(27f)
- (10)  $e ? \underline{n_2}$  by Rule (17b) on (7)  
and (9)

Assume  $e \triangleright \underline{n_2} \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over it, no case applies due to syntactic contradiction.

- (11)  $e \triangleright \cancel{\underline{n_2}} \dashv\!\!\dashv \theta$  by contradiction

Assume  $e \perp \underline{n_2}$ . By rule induction over it, no case applies due to syntactic contradiction.

- (12)  $e \perp \cancel{\underline{n_2}}$  by contradiction

**Case (20d).**

- (8)  $e = \underline{n_1}$

Assume  $\underline{n_1} ? \underline{n_2}$ . By rule induction over Rules (33) on it, only two cases apply.

**Case (33c).**

(9)  $\underline{n_1}$  **notintro** by assumption  
 Contradicts Lemma 4.0.6.

(10)  $\underline{n_1} \not\vdash \underline{n_2}$  by contradiction

By case analysis on whether  $n_1 = n_2$ .

**Case**  $n_1 = n_2$ .

(11)  $n_1 = n_2$  by assumption

(12)  $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash$  by Rule (32c)

Assume  $\underline{n_1} \perp \underline{n_2}$ . By rule induction over Rules (34) on it, only one case applies.

**Case** (34a).

(13)  $n_1 \neq n_2$  by assumption

Contradicts (11).

(14)  $\underline{n_1} \not\vdash \underline{n_2}$  by contradiction

**Case**  $n_1 \neq n_2$ .

(11)  $n_1 \neq n_2$  by assumption

(12)  $\underline{n_1} \perp \underline{n_2}$  by Rule (34a) on (11)

Assume  $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash \theta$  for some  $\theta$ . By rule induction over Rules (32) on it, no case applies due to syntactic contradiction.

(13)  $\underline{n_1} \triangleright \underline{n_2} \dashv\!\!\vdash \theta$  by contradiction

**Case** (21f).

(4)  $p = \text{inl}(p_1)$  by assumption

(5)  $\tau = (\tau_1 + \tau_2)$  by assumption

(6)  $\xi = \text{inl}(\xi_1)$  by assumption

(7)  $p_1 : \tau_1[\xi_1] \dashv\!\!\vdash \Gamma ; \Delta$  by assumption

(8)  $\text{inl}(p_1)$  **refutable?** by Rule (31d)

By rule induction over Rules (20) on (2), the following cases apply.

**Case** (20b),(20c),(20f),(20h),(20i),(20l),(20m).

(9)  $e = \text{inl}^u, \text{inl}^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
 by assumption

(10)  $e$  **notintro** by Rule  
 (27a),(27b),(27c),(27d),(27e),(27f)

(11)  $e ? \text{inl}(p_1)$  by Rule (17b) on (8)  
 and (10)

Assume  $e \triangleright \text{inl}(p_1) \dashv\!\!\vdash \theta_1$  for some  $\theta_1$ . By rule induction over it, no case applies due to syntactic contradiction.

(12)  $e \triangleright \overline{\text{inl}(p_1) \dashv \theta_1}$  by contradiction

Assume  $e \perp \text{inl}(p_1)$ . By rule induction over it, no case applies due to syntactic contradiction.

(13)  $e \perp \overline{\text{inl}(p_1)}$  by contradiction

**Case (20j).**

(9)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption

(10)  $\cdot; \Delta_e \vdash e_1 : \tau_1$  by assumption

(11)  $e_1 \text{ final}$  by Lemma 4.0.3 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of  $e_1 \triangleright p_1 \dashv \theta_1$  for some  $\theta_1$ ,  $e_1 ? p_1$ , and  $e_1 \perp p_1$  holds.

By case analysis on which one holds.

**Case  $e_1 \triangleright p_1 \dashv \theta_1$ .**

(12)  $e_1 \triangleright p_1 \dashv \theta_1$  by assumption

(13)  $e_1 ? p_1$  by assumption

(14)  $e_1 \perp p_1$  by assumption

(15)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv \theta_1$  by Rule (32e) on (12)

Assume  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ . By rule induction over Rules (33) on it, only two cases apply.

**Case (33c).**

(16)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7.

**Case (33g).**

(16)  $e_1 ? p_1$  by assumption

Contradicts (13).

(17)  $\overline{\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)}$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$ . By rule induction over Rules (34) on it, only one case applies.

**Case (34f).**

(18)  $e_1 \perp p_1$  by assumption

Contradicts (14).

(19)  $\overline{\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)}$  by contradiction

**Case  $e_1 ? p_1$ .**

(12)  $e_1 \triangleright p_1 \dashv \theta_1$  by assumption

(13)  $e_1 ? p_1$  by assumption

(14)  $e_1 \perp p_1$  by assumption

(15)  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  by Rule (33g) on (13)

Assume  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32e).**

(16)  $e_1 \triangleright p_1 \dashv\vdash \theta$  by assumption

Contradicts (12).

(17)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$ . By rule induction over Rules (34) on it, only one case applies.

**Case (34f).**

(18)  $e_1 \perp p_1$  by assumption

Contradicts (14).

(19)  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$  by contradiction

**Case  $e_1 \perp p_1$ .**

(12)  $e_1 \triangleright p_1 \dashv\vdash \theta_1$  by assumption

(13)  $e_1 ? p_1$  by assumption

(14)  $e_1 \perp p_1$  by assumption

(15)  $\text{inl}_{\tau_2}(e_1) \perp \text{inl}(p_1)$  by Rule (34f) on (14)

Assume  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32e).**

(16)  $e_1 \triangleright p_1 \dashv\vdash \theta$  by assumption

Contradicts (12).

(17)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  by contradiction

Assume  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$ . By rule induction over Rules (33) on it, only two cases apply.

**Case (33c).**

(18)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7.

**Case (33g).**

(18)  $e_1 ? p_1$  by assumption

Contradicts (13).

(19)  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  by contradiction

**Case (21g).**

(4)  $p = \text{inr}(p_2)$  by assumption

(5)  $\tau = (\tau_1 + \tau_2)$  by assumption

- (6)  $\xi = \text{inr}(\xi_2)$  by assumption
- (7)  $p_2 : \tau_2[\xi_2] \dashv \vdash \Gamma ; \Delta$  by assumption
- (8)  $\text{inr}(p_2) \text{ refutable?}$  by Rule (31e)

By rule induction over Rules (20) on (2), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

- (9)  $e = \text{inl}^u, \text{inl}^u(e_0), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$  by assumption
- (10)  $e \text{ notintro}$  by Rule (27a),(27b),(27c),(27d),(27e),(27f)
- (11)  $e ? \text{inr}(p_2)$  by Rule (17b) on (8) and (10)

Assume  $e \triangleright \text{inr}(p_2) \dashv \vdash \theta_2$  for some  $\theta_2$ . By rule induction over it, no case applies due to syntactic contradiction.

- (12)  $e \triangleright \text{inr}(p_2) \dashv \vdash \theta_2$  by contradiction

Assume  $e \perp \text{inr}(p_2)$ . By rule induction over it, no case applies due to syntactic contradiction.

- (13)  $e \perp \text{inr}(p_2)$  by contradiction

**Case (20k).**

- (9)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption
- (10)  $\cdot ; \Delta_e \vdash e_2 : \tau_2$  by assumption
- (11)  $e_2 \text{ final}$  by Lemma 4.0.4 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of  $e_2 \triangleright p_2 \dashv \vdash \theta_2$  for some  $\theta_2$ ,  $e_2 ? p_2$ , and  $e_2 \perp p_2$  holds.

By case analysis on which one holds.

**Case  $e_2 \triangleright p_2 \dashv \vdash \theta_2$ .**

- (12)  $e_2 \triangleright p_2 \dashv \vdash \theta_2$  by assumption
- (13)  $e_2 ? p_2$  by assumption
- (14)  $e_2 \perp p_2$  by assumption
- (15)  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv \vdash \theta_2$  by Rule (32f) on (12)

Assume  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ . By rule induction over Rules (33) on it, only two cases apply.

**Case (33c).**

- (16)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (33h).**

- (16)  $e_2 ? p_2$  by assumption

Contradicts (13).

(17)  $\frac{\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)}{\text{by contradiction}}$

Assume  $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$ . By rule induction over Rules (34) on it, only one case applies.

**Case (34g).**

(18)  $e_2 \perp p_2$  by assumption

Contradicts (14).

(19)  $\frac{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}{\text{by contradiction}}$

**Case  $e_2 ? p_2$ .**

(12)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$  by assumption

(13)  $e_2 ? p_2$  by assumption

(14)  $e_2 \perp p_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$  by Rule (33h) on (13)

Assume  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32f).**

(16)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$  by assumption

Contradicts (12).

(17)  $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta}{\text{by contradiction}}$

Assume  $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$ . By rule induction over Rules (34) on it, only one case applies.

**Case (34g).**

(18)  $e_2 \perp p_2$  by assumption

Contradicts (14).

(19)  $\frac{\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)}{\text{by contradiction}}$

**Case  $e_2 \perp p_2$ .**

(12)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$  by assumption

(13)  $e_2 ? p_2$  by assumption

(14)  $e_2 \perp p_2$  by assumption

(15)  $\text{inr}_{\tau_1}(e_2) \perp \text{inr}(p_2)$  by Rule (34g) on (14)

Assume  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta$  for some  $\theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32f).**

(16)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta$  by assumption

Contradicts (12).

(17)  $\frac{\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\dashv \theta}{\text{by contradiction}}$

Assume  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$ . By rule induction over Rules (33) on it, only two cases apply.

**Case (33c).**

(18)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.8.

**Case (33h).**

(18)  $e_2 ? p_2$  by assumption

Contradicts (13).

(19)  $\text{inr}_{\tau_1}(e_2) ? \text{inr}(p_2)$  by contradiction

**Case (21h).**

(4)  $p = (p_1, p_2)$  by assumption

(5)  $\tau = (\tau_1 \times \tau_2)$  by assumption

(6)  $\xi = (\xi_1, \xi_2)$  by assumption

(7)  $\Gamma = \Gamma_1 \uplus \Gamma_2$  by assumption

(8)  $\Delta = \Delta_1 \uplus \Delta_2$  by assumption

(9)  $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 ; \Delta_1$  by assumption

(10)  $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 ; \Delta_2$  by assumption

By rule induction over Rules (20) on (2), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

(11)  $e = \text{inr}^u, \text{inl}^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$

by assumption

(12)  $e \text{ notintro}$

by Rule

(27a),(27b),(27c),(27d),(27e),(27f)

(13)  $e \text{ indet}$

by Lemma 4.0.10 on

(1) and (12)

(14)  $\text{prl}(e) \text{ indet}$

by Rule (25g) on (13)

(15)  $\text{prl}(e) \text{ final}$

by Rule (26b) on (14)

(16)  $\text{prr}(e) \text{ indet}$

by Rule (25h) on (13)

(17)  $\text{prr}(e) \text{ final}$

by Rule (26b) on (16)

(18)  $\cdot ; \Delta \vdash \text{prl}(e) : \tau_1$

by Rule (20h) on (2)

(19)  $\cdot ; \Delta \vdash \text{prr}(e) : \tau_2$

by Rule (20i) on (2)

Assume  $e \perp (p_1, p_2)$ . By rule induction on it, no case applies due to syntactic contradiction.

(20)  $e \perp (p_1, p_2)$

by contradiction



By inductive hypothesis on (15) and (18) and (9), exactly one of  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$ ,  $\text{prl}(e) ? p_1$ , and  $\text{prl}(e) \perp p_1$  holds.  
 By inductive hypothesis on (17) and (19) and (10), exactly one of  $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ ,  $\text{prr}(e) ? p_2$ , and  $\text{prr}(e) \perp p_2$  holds.  
 By case analysis on which conclusion holds for  $p_1$  and  $p_2$ . Note that we have already shown  $e \perp (p_1, p_2)$ .

**Case**  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1, \text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ .

- |      |   |  |
|------|---|--|
| (21) | $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$            | by assumption                              |
| (22) | $\text{prl}(e) ? p_1$   | by assumption                              |
| (23) | $\text{prl}(e) \perp p_1$   | by assumption                              |
| (24) | $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$            | by assumption                              |
| (25) | $\text{prr}(e) ? p_2$   | by assumption                              |
| (26) | $\text{prr}(e) \perp p_2$   | by assumption                              |
| (27) | $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2$ | by Rule (32g) on (12)<br>and (21) and (24) |

Assume  $e ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only one case applies.

**Case** (33c).

- |      |                                 |               |
|------|---------------------------------|---------------|
| (28) | $(p_1, p_2) \text{ refutable?}$ | by assumption |
|------|---------------------------------|---------------|

By rule induction over Rules (31), only two cases apply.

**Case** (31f).

- |      |                                  |                                   |
|------|----------------------------------|-----------------------------------|
| (29) | $p_1 \text{ refutable?}$         | by assumption                     |
| (30) | $\text{prl}(e) \text{ notintro}$ | by Rule (27e)                     |
| (31) | $\text{prl}(e) ? p_1$            | by Rule (33c) on (29)<br>and (30) |

Contradicts (22).

**Case** (31g).

- |      |                                  |                                   |
|------|----------------------------------|-----------------------------------|
| (29) | $p_2 \text{ refutable?}$         | by assumption                     |
| (30) | $\text{prr}(e) \text{ notintro}$ | by Rule (27f)                     |
| (31) | $\text{prl}(e) ? p_1$            | by Rule (33c) on (29)<br>and (30) |

Contradicts (22).

- |      |                  |                  |
|------|------------------|------------------|
| (32) | $e ? (p_1, p_2)$ | by contradiction |
|------|------------------|------------------|

**Case**  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1, \text{prr}(e) ? p_2$ .

- |      |  |               |
|------|--|---------------|
| (21) | $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$ | by assumption |
| (22) | $\text{prl}(e) ? p_1$  | by assumption |
| (23) | $\text{prl}(e) \perp p_1$                                    | by assumption |
| (24) | $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$ | by assumption |
| (25) | $\text{prr}(e) ? p_2$  | by assumption |
| (26) | $\text{prr}(e) \perp p_2$                                    | by assumption |

Assume  $e \triangleright (p_1, p_2) \dashv\vdash \theta$ . By rule induction over Rules (32), only one case applies.

**Case (32g).**

(27)  $\theta = \theta_1 \uplus \theta_2$  by assumption

(28)  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  by assumption

Contradicts (24).

(29)  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  by contradiction

By rule induction over Rules (33) on (25), the following cases apply.

**Case (33a),(33b).**

(30)  $p_2 = \langle \rangle^w, \langle p \rangle^w$  by assumption

(31)  $p_2 \text{ refutable?}$  by Rule (31b) and Rule (31c)

(32)  $(p_1, p_2) \text{ refutable?}$  by Rule (31g) on (31)

(33)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (32)

**Case (33c).**

(30)  $p_2 \text{ refutable?}$  by assumption

(31)  $(p_1, p_2) \text{ refutable?}$  by Rule (31g) on (30)

(32)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (31)

**Case  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1, \text{prr}(e) \perp p_2$ .**

(21)  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  by assumption

(22)  $\text{prl}(e) ? p_1$  by assumption

(23)  $\text{prl}(e) \perp p_1$  by assumption

(24)  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  by assumption

(25)  $\text{prr}(e) ? p_2$  by assumption

(26)  $\text{prr}(e) \perp p_2$  by assumption

By rule induction over Rules (34) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case  $\text{prl}(e) ? p_1, \text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$ .**

(21)  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  by assumption

(22)  $\text{prl}(e) ? p_1$  by assumption

(23)  $\text{prl}(e) \perp p_1$  by assumption

(24)  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  by assumption

(25)  $\text{prr}(e) ? p_2$  by assumption

(26)  $\text{prr}(e) \perp p_2$  by assumption

Assume  $e \triangleright (p_1, p_2) \dashv\vdash \theta$ . By rule induction over Rules (32), only one case applies.

**Case (32g).**

(27)  $\theta = \theta_1 \uplus \theta_2$  by assumption

(28)  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption

Contradicts (21).

(29)  $e \triangleright \overline{(p_1, p_2) \dashv\!\!\vdash \theta}$  by contradiction

By rule induction over Rules (33) on (22), the following cases apply.

**Case (33a),(33b).**

(30)  $p_1 = \langle \rangle^w, \langle p \rangle^w$  by assumption

(31)  $p_1$  **refutable?** by Rule (31b) and Rule (31c)

(32)  $(p_1, p_2)$  **refutable?** by Rule (31g) on (31)

(33)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (32)

**Case (33c).**

(30)  $p_1$  **refutable?** by assumption

(31)  $(p_1, p_2)$  **refutable?** by Rule (31g) on (30)

(32)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (31)

**Case  $\text{prl}(e) ? p_1, \text{prr}(e) ? p_2$ .**

(21)  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption

(22)  $\text{prl}(e) ? p_1$  by assumption

(23)  $\text{prl}(e) \perp p_1$  by assumption

(24)  $\text{prr}(e) \triangleright p_2 \dashv\!\!\vdash \theta_2$  by assumption

(25)  $\text{prr}(e) ? p_2$  by assumption

(26)  $\text{prr}(e) \perp p_2$  by assumption

Assume  $e \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$ . By rule induction over Rules (32), only one case applies.

**Case (32g).**

(27)  $\theta = \theta_1 \uplus \theta_2$  by assumption

(28)  $\text{prl}(e) \triangleright p_1 \dashv\!\!\vdash \theta_1$  by assumption

Contradicts (21).

(29)  $e \triangleright \overline{(p_1, p_2) \dashv\!\!\vdash \theta}$  by contradiction

By rule induction over Rules (33) on (22), the following cases apply.

**Case (33a),(33b).**

(30)  $p_1 = \langle \rangle^w, \langle p \rangle^w$  by assumption

(31)  $p_1$  **refutable?** by Rule (31b) and Rule (31c)

- |                                      |                       |
|--------------------------------------|-----------------------|
| (32) $(p_1, p_2) \text{ refutable?}$ | by Rule (31g) on (31) |
| (33) $e ? (p_1, p_2)$                | by Rule (33c) on (12) |
|                                      | and (32)              |

**Case (33c).**

- |                                      |                       |
|--------------------------------------|-----------------------|
| (30) $p_1 \text{ refutable?}$        | by assumption         |
| (31) $(p_1, p_2) \text{ refutable?}$ | by Rule (31g) on (30) |
| (32) $e ? (p_1, p_2)$                | by Rule (33c) on (12) |
|                                      | and (31)              |

**Case  $\text{prl}(e) ? p_1, \text{prr}(e) \perp p_2$ .**

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}{\text{prl}(e) ? p_1}$     | by assumption |
| (22) $\text{prl}(e) ? p_1$   | by assumption |
| (23) $\frac{\text{prl}(e) \perp p_1}{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}$ | by assumption |
| (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}{\text{prr}(e) ? p_2}$     | by assumption |
| (25) $\text{prr}(e) ? p_2$   | by assumption |
| (26) $\text{prr}(e) \perp p_2$   | by assumption |

By rule induction over Rules (34) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case  $\text{prl}(e) \perp p_1, \text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2$ .**

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}{\text{prl}(e) ? p_1}$     | by assumption |
| (22) $\text{prl}(e) ? p_1$   | by assumption |
| (23) $\text{prl}(e) \perp p_1$   | by assumption |
| (24) $\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2$                                 | by assumption |
| (25) $\frac{\text{prr}(e) ? p_2}{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}$     | by assumption |
| (26) $\frac{\text{prr}(e) \perp p_2}{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}$ | by assumption |

By rule induction over Rules (34) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case  $\text{prl}(e) \perp p_1, \text{prr}(e) ? p_2$ .**

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}{\text{prl}(e) ? p_1}$ | by assumption |
| (22) $\text{prl}(e) ? p_1$   | by assumption |
| (23) $\text{prl}(e) \perp p_1$   | by assumption |
| (24) $\frac{\text{prr}(e) \triangleright p_2 \dashv \vdash \theta_2}{\text{prr}(e) ? p_2}$ | by assumption |
| (25) $\text{prr}(e) ? p_2$   | by assumption |
| (26) $\frac{\text{prr}(e) \perp p_2}{\text{prr}(e) ? p_2}$                                 | by assumption |

By rule induction over Rules (34) on (23), no case applies due to syntactic contradiction.

Therefore, vacuously true.

**Case  $\text{prl}(e) \perp p_1, \text{prr}(e) \perp p_2$ .**

- |  |               |
|--|---------------|
| (21) $\frac{\text{prl}(e) \triangleright p_1 \dashv \vdash \theta_1}{\text{prl}(e) ? p_1}$ | by assumption |
|--|---------------|

- (22)  $\frac{\text{prl}(e) ? p_1}{\text{prl}(e) \perp p_1}$  by assumption
- (23)  $\text{prl}(e) \perp p_1$  by assumption
- (24)  $\frac{\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2}{\text{prr}(e) ? p_2}$  by assumption
- (25)  $\text{prr}(e) ? p_2$  by assumption
- (26)  $\frac{\text{prr}(e) \perp p_2}{\text{prr}(e) \dashv\vdash \theta_2}$  by assumption

By rule induction over Rules (34) on (23), no case applies due to syntactic contradiction.  
Therefore, vacuously true.

**Case (20g).**

- (11)  $e = (e_1, e_2)$  by assumption
- (12)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption
- (13)  $\cdot; \Delta \vdash e_2 : \tau_2$  by assumption
- (14)  $e_1$  **final** by Lemma 4.0.5 on (1)
- (15)  $e_2$  **final** by Lemma 4.0.5 on (1)

By inductive hypothesis on (9) and (12) and (14), exactly one of  $e_1 \triangleright p_1 \dashv\vdash \theta_1$ ,  $e_1 ? p_1$ , and  $e_1 \perp p_1$  holds.

By inductive hypothesis on (10) and (13) and (15), exactly one of  $e_2 \triangleright p_2 \dashv\vdash \theta_2$ ,  $e_2 ? p_2$ , and  $e_2 \perp p_2$  holds.

By case analysis on which conclusion holds for  $p_1$  and  $p_2$ .

**Case**  $e_1 \triangleright p_1 \dashv\vdash \theta_1, e_2 \triangleright p_2 \dashv\vdash \theta_2$ .

- (16)  $e_1 \triangleright p_1 \dashv\vdash \theta_1$  by assumption
- (17)  $\frac{e_1 ? p_1}{e_1 \perp p_1}$  by assumption
- (18)  $\frac{e_1 \perp p_1}{e_1 \dashv\vdash \theta_1}$  by assumption
- (19)  $e_2 \triangleright p_2 \dashv\vdash \theta_2$  by assumption
- (20)  $\frac{e_2 ? p_2}{e_2 \perp p_2}$  by assumption
- (21)  $\frac{e_2 \perp p_2}{e_2 \dashv\vdash \theta_2}$  by assumption
- (22)  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2$  by Rule (32d) on (16) and (19)

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only four cases apply.

**Case (33c).**

- (23)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (33d).**

- (23)  $e_1 ? p_1$  by assumption

Contradicts (17).

**Case (33e).**

- (23)  $e_2 ? p_2$  by assumption

Contradicts (20).

**Case (33f).**

(23)  $e_1 ? p_1$  by assumption  
 Contradicts (17).

(24)  $\overline{(e_1, e_2) ? (p_1, p_2)}$  by contradiction

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (34) on it, only two cases apply.

**Case (34b).**

(25)  $e_1 \perp p_1$  by assumption  
 Contradicts (18).

**Case (34c).**

(25)  $e_2 \perp p_2$  by assumption  
 Contradicts (21).

(26)  $\overline{(e_1, e_2) \perp (p_1, p_2)}$  by contradiction

**Case**  $e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1, e_2 ? p_2$ .

(16)  $e_1 \triangleright p_1 \dashv\!\!\dashv \theta_1$  by assumption

(17)  $\overline{e_1 ? p_1}$  by assumption

(18)  $\overline{e_1 \perp p_1}$  by assumption

(19)  $\overline{e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2}$  by assumption

(20)  $e_2 ? p_2$  by assumption

(21)  $\overline{e_2 \perp p_2}$  by assumption

(22)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (33e) on (16) and (20)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_2 \triangleright p_2 \dashv\!\!\dashv \theta_2$  by assumption

Contradicts (19).

(25)  $\overline{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\dashv \theta}$  by contradiction

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (34) on it, only two cases apply.

**Case (34b).**

(26)  $e_1 \perp p_1$  by assumption  
 Contradicts (18).

**Case (34c).**

(26)  $e_2 \perp p_2$  by assumption  
 Contradicts (21).

|  |                       |
|--|-----------------------|
| (27) $\frac{(e_1, e_2) \perp (p_1, p_2)}{\text{by contradiction}}$   | by contradiction      |
| <b>Case</b> $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1, e_2 \perp p_2$ .   |                       |
| (16) $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$   | by assumption         |
| (17) $\frac{e_1 ? p_1}{\text{by assumption}}$  | by assumption         |
| (18) $\frac{e_1 \perp p_1}{\text{by assumption}}$  | by assumption         |
| (19) $\frac{e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}{\text{by assumption}}$  | by assumption         |
| (20) $\frac{e_2 ? p_2}{\text{by assumption}}$  | by assumption         |
| (21) $e_2 \perp p_2$   | by assumption         |
| (22) $(e_1, e_2) \perp (p_1, p_2)$   | by Rule (34c) on (21) |
| Assume $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ . By rule induction over Rules (32) on it, only one case applies. |                       |
| <b>Case</b> (32d).   |                       |
| (23) $\theta = \theta_1 \uplus \theta_2$   |                       |
| (24) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$   | by assumption         |
| Contradicts (19).  |                       |
| (25) $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta}{\text{by contradiction}}$                                     | by contradiction      |
| Assume $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only four cases apply.   |                       |
| <b>Case</b> (33c).   |                       |
| (26) $(e_1, e_2) \text{ notintro}$   | by assumption         |
| Contradicts Lemma 4.0.9.   |                       |
| <b>Case</b> (33d).   |                       |
| (26) $e_1 ? p_1$   | by assumption         |
| Contradicts (17).  |                       |
| <b>Case</b> (33e).   |                       |
| (26) $e_2 ? p_2$   | by assumption         |
| Contradicts (20).  |                       |
| <b>Case</b> (33f).   |                       |
| (26) $e_1 ? p_1$   | by assumption         |
| Contradicts (17).  |                       |
| (27) $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$   | by contradiction      |
| <b>Case</b> $e_1 ? p_1, e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ .   |                       |
| (16) $\frac{e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1}{\text{by assumption}}$  | by assumption         |
| (17) $e_1 ? p_1$   | by assumption         |
| (18) $\frac{e_1 \perp p_1}{\text{by assumption}}$  | by assumption         |
| (19) $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$   | by assumption         |
| (20) $\frac{e_2 ? p_2}{\text{by assumption}}$  | by assumption         |
| (21) $\frac{e_2 \perp p_2}{\text{by assumption}}$  | by assumption         |

(22)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (33d) on (17)  
and (19)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$  by assumption

Contradicts (16).

(25)  $(e_1, e_2) \triangleright \overline{(p_1, p_2) \dashv\!\!\parallel \theta}$  by contradiction

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (34) on it, only two cases apply.

**Case (34b).**

(26)  $e_1 \perp p_1$  by assumption

Contradicts (18).

**Case (34c).**

(26)  $e_2 \perp p_2$  by assumption

Contradicts (21).

(27)  $\overline{(e_1, e_2) \perp (p_1, p_2)}$  by contradiction

**Case  $e_1 ? p_1, e_2 ? p_2$ .**

(16)  $\overline{e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1}$  by assumption

(17)  $e_1 ? p_1$  by assumption

(18)  $\overline{e_1 \perp p_1}$  by assumption

(19)  $\overline{e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2}$  by assumption

(20)  $e_2 ? p_2$  by assumption

(21)  $\overline{e_2 \perp p_2}$  by assumption

(22)  $(e_1, e_2) ? (p_1, p_2)$  by Rule (33f) on (17)  
and (20)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption

Contradicts (19).

(25)  $(e_1, e_2) \triangleright \overline{(p_1, p_2) \dashv\!\!\parallel \theta}$  by contradiction

Assume  $(e_1, e_2) \perp (p_1, p_2)$ . By rule induction over Rules (34) on it, only two cases apply.

**Case (34b).**

(26)  $e_1 \perp p_1$  by assumption



Contradicts (18).

**Case (34c).**

(26)  $e_2 \perp p_2$

by assumption

Contradicts (21).

(27)  $\frac{(e_1, e_2) \perp (p_1, p_2)}{\text{by contradiction}}$

by contradiction

**Case  $e_1 ? p_1, e_2 \perp p_2$ .**

(16)  $\frac{e_1 \triangleright p_1 \dashv\vdash \theta_1}{\text{by assumption}}$

by assumption

(17)  $e_1 ? p_1$

by assumption

(18)  $\frac{e_1 \perp p_1}{\text{by assumption}}$

by assumption

(19)  $\frac{e_2 \triangleright p_2 \dashv\vdash \theta_2}{\text{by assumption}}$

by assumption

(20)  $\frac{e_2 ? p_2}{\text{by assumption}}$

by assumption

(21)  $e_2 \perp p_2$

by assumption

(22)  $(e_1, e_2) \perp (p_1, p_2)$

by Rule (34c) on (21)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\vdash \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

(23)  $\theta = \theta_1 \uplus \theta_2$

(24)  $e_2 \triangleright p_2 \dashv\vdash \theta_2$

by assumption

Contradicts (19).

(25)  $\frac{(e_1, e_2) \triangleright (p_1, p_2) \dashv\vdash \theta}{\text{by contradiction}}$

by contradiction

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only four cases apply.

**Case (33c).**

(26)  $(e_1, e_2) \text{ notintro}$

by assumption

Contradicts Lemma 4.0.9.

**Case (33d).**

(26)  $e_2 \triangleright p_2 \dashv\vdash \theta_2$

by assumption

Contradicts (19).

**Case (33e).**

(26)  $e_2 ? p_2$

by assumption

Contradicts (20).

**Case (33f).**

(26)  $e_2 ? p_2$

by assumption

Contradicts (20).

(27)  $\frac{(e_1, e_2) ? (p_1, p_2)}{\text{by contradiction}}$

by contradiction

**Case  $e_1 \perp p_1, e_2 \triangleright p_2 \dashv\vdash \theta_2$ .**

(16)  $\frac{e_1 \triangleright p_1 \dashv\vdash \theta_1}{\text{by assumption}}$

by assumption

- (17)  $\cancel{e_1 ? p_1}$  by assumption
- (18)  $e_1 \perp p_1$  by assumption
- (19)  $e_2 \triangleright p_2 \dashv\!\!\!\dashv \theta_2$  by assumption
- (20)  $\cancel{e_2 ? p_2}$  by assumption
- (21)  $\cancel{e_2 \perp p_2}$  by assumption
- (22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (34b) on (18)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\dashv \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

- (23)  $\theta = \theta_1 \uplus \theta_2$
  - (24)  $e_1 \triangleright p_1 \dashv\!\!\!\dashv \theta_1$  by assumption
- Contradicts (16).

- (25)  $\cancel{(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\dashv \theta}$  by contradiction

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only four cases apply.

**Case (33c).**

- (26)  $(e_1, e_2)$  **notintro** by assumption
- Contradicts Lemma 4.0.9.

**Case (33d).**

- (26)  $e_1 ? p_1$  by assumption
- Contradicts (17).

**Case (33e).**

- (26)  $e_2 ? p_2$  by assumption
- Contradicts (20).

**Case (33f).**

- (26)  $e_1 ? p_1$  by assumption
- Contradicts (17).

- (27)  $\cancel{(e_1, e_2) ? (p_1, p_2)}$  by contradiction

**Case  $e_1 \perp p_1, e_2 ? p_2$ .**

- (16)  $\cancel{e_1 \triangleright p_1 \dashv\!\!\!\dashv \theta_1}$  by assumption
- (17)  $\cancel{e_1 ? p_1}$  by assumption
- (18)  $e_1 \perp p_1$  by assumption
- (19)  $\cancel{e_2 \triangleright p_2 \dashv\!\!\!\dashv \theta_2}$  by assumption
- (20)  $e_2 ? p_2$  by assumption
- (21)  $\cancel{e_2 \perp p_2}$  by assumption
- (22)  $(e_1, e_2) \perp (p_1, p_2)$  by Rule (34b) on (18)

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\!\dashv \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

$$(23) \quad \theta = \theta_1 \uplus \theta_2$$

$$(24) \quad e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2 \quad \text{by assumption}$$

Contradicts (19).

$$(25) \quad \frac{(e_1, e_2) \triangleright (\overline{p_1, p_2}) \dashv\!\!\parallel \theta}{\text{by contradiction}}$$

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only four cases apply.

**Case (33c).**

$$(26) \quad (e_1, e_2) \text{ notintro} \quad \text{by assumption}$$

Contradicts Lemma 4.0.9.

**Case (33d).**

$$(26) \quad e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2 \quad \text{by assumption}$$

Contradicts (19).

**Case (33e).**

$$(26) \quad e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1 \quad \text{by assumption}$$

Contradicts (16).

**Case (33f).**

$$(26) \quad e_1 ? p_1 \quad \text{by assumption}$$

Contradicts (17).

$$(27) \quad \frac{(e_1, e_2) ? (\overline{p_1, p_2})}{\text{by contradiction}}$$

**Case**  $e_1 \perp p_1, e_2 \perp p_2$ .

$$(16) \quad \frac{e_1 \triangleright \overline{p_1} \dashv\!\!\parallel \theta_1}{\text{by assumption}}$$

$$(17) \quad \frac{e_1 ? \overline{p_1}}{\text{by assumption}}$$

$$(18) \quad e_1 \perp p_1 \quad \text{by assumption}$$

$$(19) \quad \frac{e_2 \triangleright \overline{p_2} \dashv\!\!\parallel \theta_2}{\text{by assumption}}$$

$$(20) \quad e_2 ? p_2 \quad \text{by assumption}$$

$$(21) \quad \frac{e_2 \perp \overline{p_2}}{\text{by assumption}}$$

$$(22) \quad (e_1, e_2) \perp (p_1, p_2) \quad \text{by Rule (34b) on (18)}$$

Assume  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\parallel \theta$ . By rule induction over Rules (32) on it, only one case applies.

**Case (32d).**

$$(23) \quad \theta = \theta_1 \uplus \theta_2$$

$$(24) \quad e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2 \quad \text{by assumption}$$

Contradicts (19).

$$(25) \quad \frac{(e_1, e_2) \triangleright (\overline{p_1, p_2}) \dashv\!\!\parallel \theta}{\text{by contradiction}}$$

Assume  $(e_1, e_2) ? (p_1, p_2)$ . By rule induction over Rules (33) on it, only four cases apply.

**Case (33c).**  
 (26)  $(e_1, e_2)$  **notintro** by assumption  
 Contradicts Lemma 4.0.9.  
**Case (33d).**  
 (26)  $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$  by assumption  
 Contradicts (19).  
**Case (33e).**  
 (26)  $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$  by assumption  
 Contradicts (16).  
**Case (33f).**  
 (26)  $e_1 ? p_1$  by assumption  
 Contradicts (17).  
 (27)  $\underline{(e_1, e_2) ? (p_1, p_2)}$  by contradiction

□

**Lemma 4.0.15** (Matching Coherence of Constraint). *Suppose that  $\cdot; \Delta_e \vdash e : \tau$  and  $e$  **final** and  $p : \tau[\xi] \dashv\!\!\parallel \Gamma; \Delta$ . Then we have*

1.  $e \models \xi$  iff  $e \triangleright p \dashv\!\!\parallel \theta$
2.  $e \models_{?} \xi$  iff  $e ? p$
3.  $e \not\models_{?}^{\dagger} \xi$  iff  $e \perp p$

*Proof.*

- (1)  $\cdot; \Delta_e \vdash e : \tau$  by assumption
- (2)  $e$  **final** by assumption
- (3)  $p : \tau[\xi] \dashv\!\!\parallel \Gamma; \Delta$  by assumption

Given Lemma 3.0.1, Theorem 2.1, and Lemma 4.0.14, it is sufficient to prove

1.  $e \models \xi$  iff  $e \triangleright p \dashv\!\!\parallel \theta$
2.  $e \models_{?} \xi$  iff  $e ? p$

By rule induction over Rules (21) on (3).

**Case (21a).**

- (4)  $p = x$  by assumption
- (5)  $\xi = \top$  by assumption

1. Prove  $e \models \top$  implies  $e \triangleright x \dashv\!\!\parallel \theta$  for some  $\theta$ .

- (6)  $e \triangleright x \dashv\!\!\parallel e/x$  by Rule (32a)

2. Prove  $e \triangleright x \dashv\!\!\vdash \theta$  implies  $e \models \top$ .

(6)  $e \models \top$  by Rule (15a)

3. Prove  $e \models_{\text{?}} \top$  implies  $e \text{? } x$ .

(6)  $e \not\models_{\text{?}} \top$  by Lemma 2.0.3

Vacuously true.

4. Prove  $e \text{? } x$  implies  $e \models_{\text{?}} \top$ .

By rule induction over Rules (33), we notice that either,  $e \text{? } x$  is in syntactic contradiction with all the cases, or the premise  $x$  **refutable**<sub>?</sub> is not derivable. Hence,  $e \text{? } x$  are not derivable. And thus vacuously true.

**Case (21b).**

(4)  $p = \_$  by assumption

(5)  $\xi = \top$  by assumption

1. Prove  $e \models \top$  implies  $e \triangleright \_ \dashv\!\!\vdash \theta$  for some  $\theta$ .

(6)  $e \triangleright \_ \dashv\!\!\vdash \cdot$  by Rule (32a)

2. Prove  $e \triangleright \_ \dashv\!\!\vdash \theta$  implies  $e \models \top$ .

(6)  $e \models \top$  by Rule (15a)

3. Prove  $e \models_{\text{?}} \top$  implies  $e \text{? } \_$ .

(6)  $e \not\models_{\text{?}} \top$  by Lemma 2.0.3

Vacuously true.

4. Prove  $e \text{? } \_$  implies  $e \models_{\text{?}} \xi$ .

By rule induction over Rules (33), we notice that either,  $e \text{? } \_$  is in syntactic contradiction with all the cases, or the premise  $\_$  **refutable**<sub>?</sub> is not derivable. Hence,  $e \text{? } \_$  are not derivable. And thus vacuously true.

**Case (21c).**

(4)  $p = \mathbb{I}^w$  by assumption

(5)  $\xi = ?$  by assumption

(6)  $\bar{\xi} = ?$  by Definition 10

1. Prove  $e \models ?$  implies  $e \triangleright \mathbb{I}^w \dashv\!\!\vdash \theta$  for some  $\theta$ .

(7)  $e \not\models ?$  by Rule (32a)

Vacuously true.

2. Prove  $e \triangleright \langle \rangle^w \dashv \parallel \theta$  implies  $e \models ?$ .  
By rule induction over Rules (32), we notice that  $e \triangleright \langle \rangle^w \dashv \parallel \theta$  is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.
3. Prove  $e \models ?$  implies  $e ? \langle \rangle^w$ .  
(7)  $e ? \langle \rangle^w$  by Rule (33a)
4. Prove  $e ? \langle \rangle^w$  implies  $e \models ?$ .  
(7)  $e \models ?$  by Rule (17a)

**Case (21d).**

- (4)  $p = \langle p_0 \rangle^w$  by assumption
- (5)  $\xi = ?$  by assumption
1. Prove  $e \models ?$  implies  $e \triangleright \langle p_0 \rangle^w \dashv \parallel \theta$  for some  $\theta$ .  
(6)  $e \not\models ?$  by Rule (32a)  
Vacuously true.
2. Prove  $e \triangleright \langle p_0 \rangle^w \dashv \parallel \theta$  implies  $e \models ?$ .  
By rule induction over Rules (32), we notice that  $e \triangleright \langle p_0 \rangle^w \dashv \parallel \theta$  is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.
3. Prove  $e \models ?$  implies  $e ? \langle p_0 \rangle^w$ .  
(6)  $e ? \langle p_0 \rangle^w$  by Rule (33b)
4. Prove  $e ? \langle p_0 \rangle^w$  implies  $e \models ?$ .  
(6)  $e \models ?$  by Rule (17a)

**Case (21e).**

- (4)  $p = \underline{n}$  by assumption
- (5)  $\xi = \underline{n}$  by assumption
1. Prove  $e \models \underline{n}$  implies  $e \triangleright \underline{n} \dashv \parallel \theta$  for some  $\theta$ .  
(6)  $e \models \underline{n}$  by assumption  
By rule induction over Rules (15) on (6), only one case applies.  
**Case (15b).**  
(7)  $e = \underline{n}$  by assumption  
(8)  $\underline{n} \triangleright \underline{n} \dashv \parallel$  by Rule (32c)
2. Prove  $e \triangleright \underline{n} \dashv \parallel \theta$  implies  $e \models \underline{n}$ .  
(6)  $e \triangleright \underline{n} \dashv \parallel \theta$  by assumption

By rule induction over Rules (32) on (6), only one case applies.

**Case (32c).**

- |   |               |
|---|---------------|
| (7) $e = \underline{n}$                   | by assumption |
| (8) $\theta = \cdot$                      | by assumption |
| (9) $\underline{n} \models \underline{n}$ | by Rule (15b) |

3. Prove  $e \models_{\tau} \underline{n}$  implies  $e \text{ ? } \underline{n}$ .

- |                                      |               |
|--------------------------------------|---------------|
| (6) $e \models_{\tau} \underline{n}$ | by assumption |
|--------------------------------------|---------------|

By rule induction over Rules (17) on (6), only one case applies.

**Case (17b).**

- |  |                              |
|--|------------------------------|
| (7) $e \text{ notintro}$               | by assumption                |
| (8) $\underline{n} \text{ refutable?}$ | by Rule (31a)                |
| (9) $e \text{ ? } \underline{n}$       | by Rule (33c) on (7) and (8) |

4. Prove  $e \text{ ? } \underline{n}$  implies  $e \models_{\tau} \underline{n}$ .

- |                                  |               |
|----------------------------------|---------------|
| (6) $e \text{ ? } \underline{n}$ | by assumption |
|----------------------------------|---------------|

By rule induction over Rules (33) on (6), only one case applies.

**Case (33c).**

- |  |                             |
|--|-----------------------------|
| (7) $e \text{ notintro}$               | by assumption               |
| (8) $\underline{n} \text{ refutable?}$ | by Rule (11a)               |
| (9) $e \models_{\tau} \underline{n}$   | by Rule (17) on (7) and (8) |

**Case (21f).**

- |  |               |
|--|---------------|
| (4) $p = \text{inl}(p_1)$                              | by assumption |
| (5) $\xi = \text{inl}(\xi_1)$                          | by assumption |
| (6) $\tau = (\tau_1 + \tau_2)$                         | by assumption |
| (7) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma ; \Delta$ | by assumption |

By rule induction over Rules (20) on (1), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

- |   |   |
|---|---|
| (8) $e = \text{Ⓜ}^u, \text{Ⓜ}_{e_0}^u, e_1(e_2), \text{prl}(e_0), \text{prl}(e_0), \text{match}(e_0)\{\hat{r}s\}$ | by assumption                               |
| (9) $e \text{ notintro}$  | by Rule (27a),(27b),(27c),(27d),(27e),(27f) |

1. Prove  $e \models \text{inl}(\xi_1)$  implies  $e \triangleright \text{inl}(p_1) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (15) on  $e \models \text{inl}(\xi_1)$ , no case applies due to syntactic contradiction.  
Therefore, vacuously true.

2. Prove  $e \triangleright \text{inl}(p_1) \dashv\vdash \theta$  implies  $e \models \text{inl}(\xi_1)$ . By rule induction over Rules (32) on  $e \triangleright \text{inl}(p_1) \dashv\vdash \theta$ , no case applies due to syntactic contradiction.

Therefore, vacuously true.

3. Prove  $e \models_{\text{?}} \text{inl}(\xi_1)$  implies  $e \text{?} \text{inl}(p_1)$ .
  - (10)  $\text{inl}(p_1) \text{ refutable?}$  by Rule (31d)
  - (11)  $e \text{?} \text{inl}(p_1)$  by Rule (33c) on (9) and (10)
4. Prove  $e \text{?} \text{inl}(p_1)$  implies  $e \models_{\text{?}} \text{inl}(\xi_1)$ .
  - (10)  $\text{inl}(\xi_1) \text{ refutable?}$  by Rule (11b)
  - (11)  $e \models_{\text{?}} \text{inl}(\xi_1)$  by Rule (17b) on (9) and (10)

**Case (20j).**

- (8)  $e = \text{inl}_{\tau_2}(e_1)$  by assumption
- (9)  $\cdot; \Delta_e \vdash e_1 : \tau_1$  by assumption
- (10)  $e_1 \text{ final}$  by Lemma 4.0.3 on (2)

By inductive hypothesis on (10) and (9) and (7).

- (11)  $e_1 \models \xi_1$  iff  $e_1 \triangleright p_1 \dashv\vdash \theta$  for some  $\theta$
- (12)  $e_1 \models_{\text{?}} \xi_1$  iff  $e_1 \text{?} p_1$

1. Prove  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  implies  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  for some  $\theta$ .

- (13)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by assumption

By rule induction over Rules (15) on (13), only one case applies.

**Case (15g).**

- (14)  $e_1 \models \xi_1$  by assumption
- (15)  $e_1 \triangleright p_1 \dashv\vdash \theta_1$  for some  $\theta_1$  by (11) on (14)
- (16)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta_1$  by Rule (32e) on (15)

2. Prove  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  implies  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$ .

- (13)  $\text{inl}_{\tau_2}(e_1) \triangleright \text{inl}(p_1) \dashv\vdash \theta$  by assumption

By rule induction over Rules (32) on (13), only one case applies.

**Case (32e).**

- (14)  $e_1 \triangleright p_1 \dashv\vdash \theta$  by assumption
- (15)  $e_1 \models \xi_1$  by (11) on (14)
- (16)  $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\xi_1)$  by Rule (15g) on (15)

3. Prove  $\text{inl}_{\tau_2}(e_1) \models_{\text{?}} \text{inl}(\xi_1)$  implies  $\text{inl}_{\tau_2}(e_1) \text{?} \text{inl}(p_1)$ .

- (13)  $\text{inl}_{\tau_2}(e_1) \models_{\text{?}} \text{inl}(\xi_1)$  by assumption

By rule induction over Rules (17) on (13), only two cases apply.

**Case (17b).**

- (14)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption



Contradicts Lemma 4.0.7.

**Case (17e).**

- (14)  $e_1 \models? \xi_1$  by assumption
- (15)  $e_1 ? p_1$  by (12) on (14)
- (16)  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  by Rule (33g) on (15)

4. Prove  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  implies  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$ .

- (13)  $\text{inl}_{\tau_2}(e_1) ? \text{inl}(p_1)$  by assumption

By rule induction over Rules (33) on (13), only two cases apply.

**Case (33c).**

- (14)  $\text{inl}_{\tau_2}(e_1) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7.

**Case (33g).**

- (14)  $e_1 ? p_1$  by assumption
- (15)  $e_1 \models? \xi_1$  by (12) on (14)
- (16)  $\text{inl}_{\tau_2}(e_1) \models? \text{inl}(\xi_1)$  by Rule (17e) on (15)

**Case (21g).**

- (4)  $p = \text{inr}(p_2)$  by assumption
- (5)  $\xi = \text{inr}(\xi_2)$  by assumption
- (6)  $\tau = (\tau_1 + \tau_2)$  by assumption
- (7)  $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma ; \Delta$  by assumption

By rule induction over Rules (20) on (1), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

- (8)  $e = \langle \rangle^u, \langle e_0 \rangle^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption
- (9)  $e \text{ notintro}$  by Rule (27a),(27b),(27c),(27d),(27e),(27f)

1. Prove  $e \models \text{inr}(\xi_2)$  implies  $e \triangleright \text{inr}(p_2) \dashv\vdash \theta$  for some  $\theta$ . By rule induction over Rules (15) on  $e \models \text{inr}(\xi_2)$ , no case applies due to syntactic contradiction.

Therefore, vacuously true.

2. Prove  $e \triangleright \text{inr}(p_2) \dashv\vdash \theta$  implies  $e \models \text{inr}(\xi_2)$ . By rule induction over Rules (32) on  $e \triangleright \text{inr}(p_2) \dashv\vdash \theta$ , no case applies due to syntactic contradiction.

Therefore, vacuously true.

3. Prove  $e \models? \text{inr}(\xi_2)$  implies  $e ? \text{inr}(p_2)$ .

- (10)  $\text{inr}(p_2) \text{ refutable?}$  by Rule (31e)
- (11)  $e ? \text{inr}(p_2)$  by Rule (33c) on (9) and (10)

4. Prove  $e \text{ ? inr}(p_2)$  implies  $e \models_{\text{?}} \text{inr}(\xi_2)$ .
  - (10)  $\text{inr}(\xi_2)$  **refutable?** by Rule (11c)
  - (11)  $e \models_{\text{?}} \text{inr}(\xi_2)$  by Rule (17b) on (9) and (10)

**Case (20k).**

- (8)  $e = \text{inr}_{\tau_1}(e_2)$  by assumption
- (9)  $\cdot; \Delta_e \vdash e_2 : \tau_2$  by assumption
- (10)  $e_2$  **final** by Lemma 4.0.3 on (2)

By inductive hypothesis on (10) and (9) and (7).

- (11)  $e_2 \models \xi_2$  iff  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta$  for some  $\theta$
- (12)  $e_2 \models_{\text{?}} \xi_2$  iff  $e_2 \text{ ? } p_2$

1. Prove  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  implies  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta$  for some  $\theta$ .

- (13)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by assumption

By rule induction over Rules (15) on (13), only one case applies.

**Case (15g).**

- (14)  $e_2 \models \xi_2$  by assumption
- (15)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$  by (11) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta_1$  by Rule (32e) on (15)

2. Prove  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta$  implies  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$ .

- (13)  $\text{inr}_{\tau_1}(e_2) \triangleright \text{inr}(p_2) \dashv\!\!\vdash \theta$  by assumption

By rule induction over Rules (32) on (13), only one case applies.

**Case (32e).**

- (14)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta$  by assumption
- (15)  $e_2 \models \xi_2$  by (11) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\xi_2)$  by Rule (15g) on (15)

3. Prove  $\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\xi_2)$  implies  $\text{inr}_{\tau_1}(e_2) \text{ ? } \text{inr}(p_2)$ .

- (13)  $\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\xi_2)$  by assumption

By rule induction over Rules (17) on (13), only two cases apply.

**Case (17b).**

- (14)  $\text{inr}_{\tau_1}(e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.7.

**Case (17e).**

- (14)  $e_2 \models_{\text{?}} \xi_2$  by assumption
- (15)  $e_2 \text{ ? } p_2$  by (12) on (14)
- (16)  $\text{inr}_{\tau_1}(e_2) \text{ ? } \text{inr}(p_2)$  by Rule (33g) on (15)

4. Prove  $\text{inr}_{\tau_1}(e_2) \text{ ? } \text{inr}(p_2)$  implies  $\text{inr}_{\tau_1}(e_2) \models_{\text{?}} \text{inr}(\xi_2)$ .

- (13)  $\text{inr}_{\tau_1}(e_2) \text{ ? } \text{inr}(p_2)$  by assumption

By rule induction over Rules (33) on (13), only two cases apply.

**Case (33c).**

(14)  $\text{inr}_{\tau_1}(e_2) \text{ notintro}$  by assumption

Contradicts Lemma 4.0.7.

**Case (33g).**

(14)  $e_2 ? p_2$  by assumption

(15)  $e_2 \models_{\tau} \xi_2$  by (12) on (14)

(16)  $\text{inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\xi_2)$  by Rule (17e) on (15)

**Case (21h).**

(4)  $p = (p_1, p_2)$  by assumption

(5)  $\xi = (\xi_1, \xi_2)$  by assumption

(6)  $\tau = (\tau_1 \times \tau_2)$  by assumption

(7)  $\Gamma = \Gamma_1 \uplus \Gamma_2$  by assumption

(8)  $\Delta = \Delta_1 \uplus \Delta_2$  by assumption

(9)  $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 ; \Delta_1$  by assumption

(10)  $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 ; \Delta_2$  by assumption

By rule induction over Rules (20) on (1), the following cases apply.

**Case (20b),(20c),(20f),(20h),(20i),(20l),(20m).**

(11)  $e = \text{Ⓜ}^u, \text{Ⓜ}_{e_0}^u, e_1(e_2), \text{prl}(e_0), \text{prr}(e_0), \text{match}(e_0)\{\hat{r}s\}$   
by assumption

(12)  $e \text{ notintro}$  by Rule (27a),(27b),(27c),(27d),(27e),(27f)

(13)  $e \text{ indet}$  by Lemma 4.0.10 on (2) and (12)

(14)  $\text{prl}(e) \text{ indet}$  by Rule (25g) on (13)

(15)  $\text{prl}(e) \text{ final}$  by Rule (26b) on (14)

(16)  $\text{prr}(e) \text{ indet}$  by Rule (25h) on (13)

(17)  $\text{prr}(e) \text{ final}$  by Rule (26b) on (16)

(18)  $\cdot ; \Delta \vdash \text{prl}(e) : \tau_1$  by Rule (20h) on (1)

(19)  $\cdot ; \Delta \vdash \text{prr}(e) : \tau_2$  by Rule (20i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

(20)  $\text{prl}(e) \models \xi_1$  iff  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  for some  $\theta_1$

(21)  $\text{prl}(e) \models_{\tau} \xi_1$  iff  $\text{prl}(e) ? p_1$

(22)  $\text{prr}(e) \models \xi_2$  iff  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  for some  $\theta_2$

(23)  $\text{prr}(e) \models_{\tau} \xi_2$  iff  $\text{prr}(e) ? p_2$

1. Prove  $e \models (\xi_1, \xi_2)$  implies  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  for some  $\theta$ .
 

(24)  $e \models (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (15) on (24), only one case applies.

**Case (15j).**

(25)  $\text{prl}(e) \models \xi_1$  by assumption

(26)  $\text{prr}(e) \models \xi_2$  by assumption

(27)  $\text{prl}(e) \triangleright p_1 \dashv\vdash \theta_1$  by (20) on (25)

(28)  $\text{prr}(e) \triangleright p_2 \dashv\vdash \theta_2$  by (22) on (26)

(29)  $e \triangleright (p_1, p_2) \dashv\vdash \theta_1 \uplus \theta_2$  by Rule (32g) on (12) and (27) and (28)
2. Prove  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  implies  $e \models (\xi_1, \xi_2)$ .
 

(24)  $e \triangleright (p_1, p_2) \dashv\vdash \theta$  by assumption

By rule induction over Rules (32) on (24), only one case applies.

**Case (32g).**

(25)  $\theta = \theta_1 \uplus \theta_2$  by assumption

(26)  $\text{prl}(e) \triangleright \xi_1 \dashv\vdash \theta_1$  by assumption

(27)  $\text{prr}(e) \triangleright \xi_2 \dashv\vdash \theta_2$  by assumption

(28)  $\text{prl}(e) \models \xi_1$  by (20) on (26)

(29)  $\text{prr}(e) \models \xi_2$  by (22) on (27)

(30)  $e \models (\xi_1, \xi_2)$  by Rule (15j) on (12) and (28) and (29)
3. Prove  $e \models? (\xi_1, \xi_2)$  implies  $e \triangleright? (p_1, p_2)$ .
 

(24)  $e \models? (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (17) on (24), only one case applies.

**Case (17b).**

(25)  $(\xi_1, \xi_2) \text{ refutable?}$  by assumption

By rule induction over Rules (11) on (25), only two cases apply.

**Case (11d).**

(26)  $\xi_1 \text{ refutable?}$  by assumption

(27)  $\text{prl}(e) \text{ notintro}$  by Rule (27e)

(28)  $\text{prl}(e) \models? \xi_1$  by Rule (17b) on (26) and (27)

(29)  $\text{prl}(e) \triangleright? p_1$  by (21) on (28)

By rule induction over Rules (33) on (29), only three cases apply.

**Case (33a),(33b).**

(30)  $p_1 = \langle \rangle^w, \langle p_0 \rangle^w$  by assumption

(31)  $p_1 \text{ refutable?}$  by Rule (31b) and Rule (31c)

(32)  $(p_1, p_2) \text{ refutable?}$  by Rule (31f) on (31)

(33)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (32)

**Case (33c).**

(30)  $p_1$  **refutable?** by assumption  
 (31)  $(p_1, p_2)$  **refutable?** by Rule (31f) on (30)  
 (32)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (31)

**Case (11e).**

(26)  $\xi_2$  **refutable?** by assumption  
 (27)  $\text{pr}(e)$  **notintro** by Rule (27e)  
 (28)  $\text{pr}(e) \models? \xi_2$  by Rule (17b) on (26) and (27)  
 (29)  $\text{pr}(e) ? p_2$  by (23) on (28)

By rule induction over Rules (33) on (29), only three cases apply.

**Case (33a),(33b).**

(30)  $p_2 = \langle \rangle^w, \langle p_0 \rangle^w$  by assumption  
 (31)  $p_2$  **refutable?** by Rule (31b) and Rule (31c)  
 (32)  $(p_1, p_2)$  **refutable?** by Rule (31g) on (31)  
 (33)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (32)

**Case (33c).**

(30)  $p_2$  **refutable?** by assumption  
 (31)  $(p_1, p_2)$  **refutable?** by Rule (31g) on (30)  
 (32)  $e ? (p_1, p_2)$  by Rule (33c) on (12) and (31)

4. Prove  $e ? (p_1, p_2)$  implies  $e \models? (\xi_1, \xi_2)$ .

(24)  $e ? (p_1, p_2)$  by assumption

By rule induction over Rules (33) on (24), only one case applies.

**Case (33c).**

(25)  $(p_1, p_2)$  **refutable?** by assumption

By rule induction over Rules (31) on (25), only two cases apply.

**Case (31f).**

(26)  $p_1$  **refutable?** by assumption  
 (27)  $\text{pr}(e)$  **notintro** by Rule (27e)  
 (28)  $\text{pr}(e) ? p_1$  by Rule (33c) on (26) and (27)  
 (29)  $\text{pr}(e) \models? \xi_1$  by (21) on (28)

By rule induction over Rules (17) on (29), only three cases apply.

**Case (17a).**

- |   |                                |
|---|--------------------------------|
| (30) $\xi_1 = ?$                                    | by assumption                  |
| (31) $\xi_1$ <b>refutable</b> <sub>?</sub>          | by Rule (2b)                   |
| (32) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (11d) on (31)          |
| (33) $e \models_? (\xi_1, \xi_2)$                   | by Rule (17b) on (12) and (32) |

**Case (17b).**

- |   |                                |
|---|--------------------------------|
| (30) $\xi_1$ <b>refutable</b> <sub>?</sub>          | by assumption                  |
| (31) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (11d) on (30)          |
| (32) $e \models_? (\xi_1, \xi_2)$                   | by Rule (17b) on (12) and (31) |

**Case (31g).**

- |  |                                |
|--|--------------------------------|
| (26) $p_2$ <b>refutable</b> <sub>?</sub> | by assumption                  |
| (27) <b>pr</b> $r(e)$ <b>notintro</b>    | by Rule (27e)                  |
| (28) <b>pr</b> $r(e)$ $? p_2$            | by Rule (33c) on (26) and (27) |
| (29) <b>pr</b> $r(e) \models_? \xi_2$    | by (23) on (28)                |

By rule induction over Rules (17) on (29), only three cases apply.

**Case (17a).**

- |   |                                |
|---|--------------------------------|
| (30) $\xi_2 = ?$                                    | by assumption                  |
| (31) $\xi_2$ <b>refutable</b> <sub>?</sub>          | by Rule (2b)                   |
| (32) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (11e) on (31)          |
| (33) $e \models_? (\xi_1, \xi_2)$                   | by Rule (17b) on (12) and (32) |

**Case (17b).**

- |   |                                |
|---|--------------------------------|
| (30) $\xi_2$ <b>refutable</b> <sub>?</sub>          | by assumption                  |
| (31) $(\xi_1, \xi_2)$ <b>refutable</b> <sub>?</sub> | by Rule (11e) on (30)          |
| (32) $e \models_? (\xi_1, \xi_2)$                   | by Rule (17b) on (12) and (31) |

**Case (20g).**

- |  |                       |
|--|-----------------------|
| (11) $e = (e_1, e_2)$                      | by assumption         |
| (12) $\cdot; \Delta_e \vdash e_1 : \tau_1$ | by assumption         |
| (13) $\cdot; \Delta_e \vdash e_2 : \tau_2$ | by assumption         |
| (14) $e_1$ <b>final</b>                    | by Lemma 4.0.5 on (2) |
| (15) $e_2$ <b>final</b>                    | by Lemma 4.0.5 on (2) |

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

- (16)  $e_1 \models \xi_1$  iff  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$
- (17)  $e_1 \models? \xi_1$  iff  $e_1 ? p_1$
- (18)  $e_2 \models \xi_2$  iff  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  for some  $\theta_2$
- (19)  $e_2 \models? \xi_2$  iff  $e_2 ? p_2$

1. Prove  $(e_1, e_2) \models (\xi_1, \xi_2)$  implies  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  for some  $\theta$ .

(20)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (15) on (20), only two cases apply.

**Case (15i).**

- (21)  $e_1 \models \xi_1$  by assumption
- (22)  $e_2 \models \xi_2$  by assumption
- (23)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$  by (16) on (21)
- (24)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  for some  $\theta_2$  by (18) on (22)
- (25)  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta_1 \uplus \theta_2$  by Rule (32d) on (23) and (24)

**Case (15j).**

- (21)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

2. Prove  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  implies  $(e_1, e_2) \models (\xi_1, \xi_2)$ .

(20)  $(e_1, e_2) \triangleright (p_1, p_2) \dashv\!\!\vdash \theta$  by assumption

By rule induction over Rules (32) on (20), only two cases apply.

**Case (32d).**

- (21)  $e_1 \triangleright p_1 \dashv\!\!\vdash \theta_1$  for some  $\theta_1$  by assumption
- (22)  $e_2 \triangleright p_2 \dashv\!\!\vdash \theta_2$  for some  $\theta_2$  by assumption
- (23)  $e_1 \models \xi_1$  by (16) on (21)
- (24)  $e_2 \models \xi_2$  by (18) on (22)
- (25)  $(e_1, e_2) \models (\xi_1, \xi_2)$  by Rule (15i) on (23) and (24)

**Case (32g).**

- (21)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

3. Prove  $(e_1, e_2) \models? (\xi_1, \xi_2)$  implies  $(e_1, e_2) ? (p_1, p_2)$ .

(20)  $(e_1, e_2) \models? (\xi_1, \xi_2)$  by assumption

By rule induction over Rules (17) on (20), only four cases apply.

**Case (17b).**

- (21)  $(e_1, e_2)$  **notintro** by assumption

Contradicts Lemma 4.0.9.

**Case (17g).**

- (21)  $e_1 \models? \xi_1$  by assumption

|      |   |                                   |
|------|---|-----------------------------------|
| (22) | $e_2 \models \xi_2$                                   | by assumption                     |
| (23) | $e_1 ? p_1$   | by (17) on (21)                   |
| (24) | $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ | by (18) on (22)                   |
| (25) | $(e_1, e_2) ? (p_1, p_2)$                             | by Rule (33d) on (23)<br>and (24) |

**Case (17h).**

|      |   |                                   |
|------|---|-----------------------------------|
| (21) | $e_1 \models \xi_1$                                   | by assumption                     |
| (22) | $e_2 \models? \xi_2$                                  | by assumption                     |
| (23) | $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ | by (16) on (21)                   |
| (24) | $e_2 ? p_2$   | by (19) on (22)                   |
| (25) | $(e_1, e_2) ? (p_1, p_2)$                             | by Rule (33e) on (23)<br>and (24) |

**Case (17i).**

|      |                           |                                   |
|------|---------------------------|-----------------------------------|
| (21) | $e_1 \models? \xi_1$      | by assumption                     |
| (22) | $e_2 \models? \xi_2$      | by assumption                     |
| (23) | $e_1 ? p_1$               | by (17) on (21)                   |
| (24) | $e_2 ? p_2$               | by (19) on (22)                   |
| (25) | $(e_1, e_2) ? (p_1, p_2)$ | by Rule (33f) on (23)<br>and (24) |

4. Prove  $(e_1, e_2) ? (p_1, p_2)$  implies  $(e_1, e_2) \models? (\xi_1, \xi_2)$ .

|      |                           |               |
|------|---------------------------|---------------|
| (20) | $(e_1, e_2) ? (p_1, p_2)$ | by assumption |
|------|---------------------------|---------------|

By rule induction over Rules (33) on (20), only four cases apply.

**Case (33c).**

|      |                               |               |
|------|-------------------------------|---------------|
| (21) | $(e_1, e_2) \text{ notintro}$ | by assumption |
|------|-------------------------------|---------------|

Contradicts Lemma 4.0.9.

**Case (33d).**

|      |   |                                   |
|------|---|-----------------------------------|
| (21) | $e_1 ? p_1$   | by assumption                     |
| (22) | $e_2 \triangleright p_2 \dashv\!\!\parallel \theta_2$ | by assumption                     |
| (23) | $e_1 \models? \xi_1$                                  | by (17) on (21)                   |
| (24) | $e_2 \models \xi_2$                                   | by (18) on (22)                   |
| (25) | $(e_1, e_2) ? (p_1, p_2)$                             | by Rule (17g) on (23)<br>and (24) |

**Case (33e).**

|      |   |                                   |
|------|---|-----------------------------------|
| (21) | $e_1 \triangleright p_1 \dashv\!\!\parallel \theta_1$ | by assumption                     |
| (22) | $e_2 ? p_2$   | by assumption                     |
| (23) | $e_1 \models \xi_1$                                   | by (16) on (21)                   |
| (24) | $e_2 \models? \xi_2$                                  | by (19) on (22)                   |
| (25) | $(e_1, e_2) ? (p_1, p_2)$                             | by Rule (17h) on (23)<br>and (24) |

**Case (33f).**



|      |                            |                                   |
|------|----------------------------|-----------------------------------|
| (21) | $e_1 ? p_1$                | by assumption                     |
| (22) | $e_2 ? p_2$                | by assumption                     |
| (23) | $e_1 \models_{\tau} \xi_1$ | by (17) on (21)                   |
| (24) | $e_2 \models_{\tau} \xi_2$ | by (19) on (22)                   |
| (25) | $(e_1, e_2) ? (p_1, p_2)$  | by Rule (17i) on (23)<br>and (24) |

□

## 5 Preservation and Progress

**Theorem 5.1** (Preservation). *If  $\cdot ; \Delta \vdash e : \tau$  and  $e \mapsto e'$  then  $\cdot ; \Delta \vdash e' : \tau$*

*Proof.* By rule induction over Rules (20) on typing judgment of  $e$ . For simplicity, we only consider two cases for match expressions here.

**Case (20l).**

|     |   |               |
|-----|---|---------------|
| (1) | $\cdot ; \Delta \vdash \text{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$  | by assumption |
| (2) | $\text{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$                    | by assumption |
| (3) | $\cdot ; \Delta \vdash e_1 : \tau_1$                                      | by assumption |
| (4) | $\cdot ; \Delta \vdash [\perp](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ | by assumption |
| (5) | $\top \models_{\tau}^{\dagger} \xi$                                       | by assumption |

By rule induction over Rules (35) on (2).

**Case (35k).**

|     |   |   |
|-----|---|---|
| (6) | $e' = \text{match}(e'_1)\{\cdot \mid r \mid rs\}$                         | by assumption                           |
| (7) | $e_1 \mapsto e'_1$  | by assumption                           |
| (8) | $\cdot ; \Delta \vdash e'_1 : \tau_1$                                     | by IH on (3) and (7)                    |
| (9) | $\cdot ; \Delta \vdash \text{match}(e'_1)\{\cdot \mid r \mid rs\} : \tau$ | by Rule (20l) on (8)<br>and (4) and (5) |

**Case (35l).**

|     |  |               |
|-----|--|---------------|
| (6) | $r = p_r \Rightarrow e_r$              | by assumption |
| (7) | $e' = [\theta](e_r)$                   | by assumption |
| (8) | $e_1 \triangleright p_r \dashv \theta$ | by assumption |

By rule induction over Rules (23) on (4).

**Case (23a).**

|      |  |                                       |
|------|--|---------------------------------------|
| (9)  | $\xi = \xi_r$  | by assumption                         |
| (10) | $\cdot ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ | by assumption                         |
| (11) | $p_r : \tau_1[\xi_r] \dashv \Gamma_r ; \Delta_r$                               | by Inversion of Rule<br>(22a) on (10) |

- (12)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (22a) on (10)
- (13)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (3) and (11) and (8)
- (14)  $\cdot ; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (12) and (13)

**Case (23b).**

- (9)  $\xi = \xi_r \vee \xi_{rs}$  by assumption
- (10)  $\cdot ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (11)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by Inversion of Rule (22a) on (10)
- (12)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (22a) on (10)
- (13)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (3) and (11) and (8)
- (14)  $\cdot ; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (12) and (13)

**Case (35m).**

- (6)  $rs = r' \mid rs'$  by assumption
- (7)  $e' = \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$  by assumption
- (8)  $e_1 \text{ final}$  by assumption
- (9)  $e_1 \perp p_r$  by assumption

By rule induction over Rules (23) on (4).

**Case (23a).** Syntactic contradiction of  $rs$ .

**Case (23b).**

- (10)  $\xi = \xi_r \vee \xi_{rs}$  by assumption
- (11)  $\cdot ; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (12)  $\cdot ; \Delta \vdash [\perp \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$  by assumption
- (13)  $\xi_r \not\models \perp$  by assumption
- (14)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by Inversion of Rule (22a) on (11)
- (15)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (22a) on (11)
- (16)  $\cdot ; \Delta \vdash [\perp](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$  by Rule (23a) on (11) and (13)
- (17)  $e_1 \not\models_{\tau}^{\dagger} \xi_r$  by Lemma 4.0.15 on (3) and (8) and (14) and (9)

- (18)  $\cdot; \Delta \vdash \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$   
by Rule (20m) on (3)  
and (8) and (16) and  
(12) and (17) and (5)

**Case (20m).**

- (1)  $rs_{pre} = r_{pre} \mid rs'_{pre}$  by assumption  
(2)  $\cdot; \Delta \vdash \text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$  by assumption  
(3)  $\text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$  by assumption  
(4)  $\cdot; \Delta \vdash e_1 : \tau_1$  by assumption  
(5)  $e_1$  final by assumption  
(6)  $\cdot; \Delta \vdash [\perp]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$  by assumption  
(7)  $\cdot; \Delta \vdash [\perp \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$   
by assumption  
(8)  $e_1 \not\models_{\tau}^{\dagger} \xi_{pre}$  by assumption  
(9)  $\top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest}$  by assumption

By rule induction over Rules (35) on (3).

**Case (35k).**

- (10)  $e' = \text{match}(e'_1)\{rs_{pre} \mid r \mid rs_{post}\}$  by assumption  
(11)  $e_1 \mapsto e'_1$  by assumption

By Lemma 4.0.13, (11) contradicts (5).

**Case (35l).**

- (10)  $r = p_r \Rightarrow e_r$  by assumption  
(11)  $e' = [\theta](e_r)$  by assumption  
(12)  $e_1 \triangleright p_r \dashv \parallel \theta$  by assumption

By rule induction over Rules (23) on (7).

**Case (23a).**

- (13)  $\xi_{rest} = \xi_r$  by assumption  
(14)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption  
(15)  $p_r : \tau_1[\xi_r] \dashv \parallel \Gamma_r ; \Delta_r$  by Inversion of Rule (22a) on (14)  
(16)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (22a) on (14)  
(17)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (4) and (15) and (12)  
(18)  $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (16) and (17)

**Case (23b).**

- (13)  $\xi_{rest} = \xi_r \vee \xi_{rs}$  by assumption
- (14)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (15)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by assumption
- (16)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by assumption
- (17)  $\theta : \Gamma_r$  by Lemma 3.0.7 on (4) and (15) and (12)
- (18)  $\cdot; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$  by Lemma 3.0.6 on (16) and (17)

**Case (35m).**

- (10)  $r = p_r \Rightarrow e_r$  by assumption
- (11)  $rs_{post} = r' \mid rs'$  by assumption
- (12)  $e' = \text{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond \mid r' \mid rs'\}$  by assumption
- (13)  $e_1 \perp p_r$  by assumption

By rule induction over Rules (23) on (7).

**Case (23a).** Syntactic contradiction of  $rs_{post}$ .

**Case (23b).**

- (14)  $\xi_{rest} = \xi_r \vee \xi_{post}$  by assumption
- (15)  $\cdot; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (16)  $\cdot; \Delta \vdash [\perp \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$  by assumption
- (17)  $\xi_r \not\equiv \xi_{pre}$  by assumption
- (18)  $p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r$  by Inversion of Rule (22a) on (15)
- (19)  $\Gamma_r ; \Delta \uplus \Delta_r \vdash e_r : \tau$  by Inversion of Rule (22a) on (15)
- (20)  $\xi_r : \tau_1$  by Lemma 3.0.2 on (15)
- (21)  $\xi_{pre} : \tau_1$  by Lemma 3.0.3 on (6)
- (22)  $\xi_r \not\equiv \perp \vee \xi_{pre}$  by Lemma 2.0.6 on (20) and (21) and (17)
- (23)  $\cdot; \Delta \vdash [\perp](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$  by Lemma 3.0.4 on (6) and (15) and (22)
- (24)  $e_1 \not\equiv_{\tau}^{\dagger} \xi_r$  by Lemma 4.0.15 on (4) and (5) and (18) and (13)
- (25)  $e_1 \not\equiv_{\tau}^{\dagger} \xi_{pre} \vee \xi_r$  by Lemma 2.0.7 on (8) and (24)

$$\begin{aligned}
(26) \quad & \cdot; \Delta \vdash \text{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^\diamond \mid r' \mid rs'\} : \tau \\
& \text{by Rule (20m) on (4)} \\
& \text{and (5) and (23) and} \\
& \text{(16) and (25) and (9)}
\end{aligned}$$

□

**Theorem 5.2** (Progress). *If  $\cdot; \Delta \vdash e : \tau$  then either  $e$  final or  $e \mapsto e'$  for some  $e'$ .*

*Proof.* By rule induction over Rules (20) on typing judgment of  $e$ . For simplicity, we only consider two cases for match expressions here.

**Case (20l).**

$$\begin{aligned}
(1) \quad & \cdot; \Delta \vdash \text{match}(e_1)\{\cdot \mid r \mid rs\} : \tau && \text{by assumption} \\
(2) \quad & \cdot; \Delta \vdash e_1 : \tau_1 && \text{by assumption} \\
(3) \quad & \cdot; \Delta \vdash [\perp](r \mid rs) : \tau_1[\xi] \Rightarrow \tau && \text{by assumption} \\
(4) \quad & \top \models_{\tau}^{\dagger} \xi && \text{by assumption}
\end{aligned}$$

By IH on (2).

**Case** Scrutinee takes a step.

$$\begin{aligned}
(5) \quad & e_1 \mapsto e'_1 && \text{by assumption} \\
(6) \quad & \text{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \text{match}(e'_1)\{\cdot \mid r \mid rs\} && \text{by Rule (35k) on (5)}
\end{aligned}$$

**Case** Scrutinee is final.

$$(5) \quad e_1 \text{ final} \quad \text{by assumption}$$

By rule induction over Rules (23) on (3).

**Case (23a).**

$$\begin{aligned}
(6) \quad & rs = \cdot && \text{by assumption} \\
(7) \quad & \xi = \xi_r && \text{by assumption} \\
(8) \quad & \cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau && \text{by assumption} \\
(9) \quad & r = p_r \Rightarrow e_r && \text{by Inversion of Rule (22a) on (8)} \\
(10) \quad & p_r : \tau_1[\xi_r] \dashv\vdash \Gamma_r ; \Delta_r && \text{by Inversion of Rule (22a) on (8)} \\
(11) \quad & e_1 \models_{\tau}^{\dagger} \xi_r && \text{by Corollary 2.1.1 on (5) and (4)}
\end{aligned}$$

By rule induction over Rules (18) on (11).

**Case (18a).**

$$(12) \quad e_1 \models_{\tau} \xi_r \quad \text{by assumption}$$

- (13)  $e_1 ? p_r$  by Lemma 4.0.15 on  
(2) and (5) and (10)  
and (12)
- (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{indet}$   
by Rule (25k) on (5)  
and (13)
- (15)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{final}$   
by Rule (26b) on (14)

**Case (18b).**

- (12)  $e_1 \models \xi_r$  by assumption
- (13)  $e_1 \triangleright p_r \dashv\!\!\parallel \theta$  by Lemma 4.0.15 on  
(2) and (5) and (10)  
and (12)
- (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$   
by Rule (35l) on (5)  
and (13)

**Case (23b).**

- (6)  $rs = r' \mid rs'$  by assumption
- (7)  $\xi = \xi_r \vee \xi_{rs}$  by assumption
- (8)  $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  by assumption
- (9)  $r = p_r \Rightarrow e_r$  by Inversion of Rule  
(22a) on (8)
- (10)  $p_r : \tau_1[\xi_r] \dashv\!\!\parallel \Gamma_r ; \Delta_r$  by Inversion of Rule  
(22a) on (8)

By Lemma 4.0.14 on (2) and (5) and (10).

**Case Scrutinee matches pattern.**

- (11)  $e_1 \triangleright p_r \dashv\!\!\parallel \theta$  by assumption
- (12)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$   
by Rule (35l) on (5)  
and (11)

**Case Scrutinee may matches pattern.**

- (11)  $e_1 ? p_r$  by assumption
- (12)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \text{indet}$   
by Rule (25k) on (5)  
and (11)
- (13)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \text{final}$   
by Rule (26b) on (12)

**Case Scrutinee doesn't matche pattern.**

- (11)  $e_1 \perp p_r$  by assumption

$$\begin{aligned}
(12) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\} \\
& \mapsto \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} \\
& \text{by Rule (35m) on (5)} \\
& \text{and (11)}
\end{aligned}$$

**Case (20m).**

$$\begin{aligned}
(1) \quad & rs_{pre} = r_{pre} \mid rs'_{pre} && \text{by assumption} \\
(2) \quad & \cdot; \Delta \vdash \text{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau && \text{by assumption} \\
(3) \quad & \cdot; \Delta \vdash e_1 : \tau_1 && \text{by assumption} \\
(4) \quad & e_1 \text{ final} && \text{by assumption} \\
(5) \quad & \cdot; \Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau && \text{by assumption} \\
(6) \quad & e_1 \not\models_{\tau}^{\dagger} \xi_{pre} && \text{by assumption} \\
(7) \quad & \top \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_{rest} && \text{by assumption}
\end{aligned}$$

By rule induction over Rules (23) on (5).

**Case (23a).**

$$\begin{aligned}
(5) \quad & rs_{post} = \cdot && \text{by assumption} \\
(6) \quad & \xi_{rest} = \xi_r && \text{by assumption} \\
(7) \quad & \cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau && \text{by assumption} \\
(8) \quad & r = p_r \Rightarrow e_r && \text{by Inversion of Rule (22a) on (7)} \\
(9) \quad & p_r : \tau_1[\xi_r] \dashv\!\parallel \Gamma_r ; \Delta_r && \text{by Inversion of Rule (22a) on (7)} \\
(10) \quad & e_1 \models_{\tau}^{\dagger} \xi_{pre} \vee \xi_r && \text{by Corollary 2.1.1 on (4) and (7)} \\
(11) \quad & e_1 \models_{\tau}^{\dagger} \xi_r && \text{by Lemma 2.0.8 on (10) and (6)}
\end{aligned}$$

By rule induction over Rules (18) on (11).

**Case (18a).**

$$\begin{aligned}
(12) \quad & e_1 \models_{\tau}^{\dagger} \xi_r && \text{by assumption} \\
(13) \quad & e_1 \text{ ? } p_r && \text{by Lemma 4.0.15 on (3) and (4) and (9) and (12)} \\
(14) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{ indet} && \text{by Rule (25k) on (4) and (13)} \\
(15) \quad & \text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \text{ final} && \text{by Rule (26b) on (14)}
\end{aligned}$$

**Case (18b).**

- (12)  $e_1 \models \xi_r$  by assumption  
 (13)  $e_1 \triangleright p_r \dashv\!\parallel \theta$  by Lemma 4.0.15 on (3) and (4) and (9) and (12)  
 (14)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$  by Rule (35l) on (4) and (13)

**Case (23b).**

- (5)  $rs_{post} = r' \mid rs'_{post}$  by assumption  
 (6)  $\cdot; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$  by assumption  
 (7)  $r = p_r \Rightarrow e_r$  by Inversion of Rule (22a) on (6)  
 (8)  $p_r : \tau_1[\xi_r] \dashv\!\parallel \Gamma_r; \Delta_r$  by Inversion of Rule (22a) on (6)

By Lemma 4.0.14 on (3) and (4) and (8).

**Case Scrutinee matches pattern.**

- (9)  $e_1 \triangleright p_r \dashv\!\parallel \theta$  by assumption  
 (10)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$  by Rule (35l) on (4) and (9)

**Case Scrutinee may matches pattern.**

- (9)  $e_1 ? p_r$  by assumption  
 (10)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \text{indet}$  by Rule (25k) on (4) and (9)  
 (11)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \text{final}$  by Rule (26b) on (10)

**Case Scrutinee doesn't matche pattern.**

- (9)  $e_1 \perp p_r$  by assumption  
 (10)  $\text{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\} \mapsto \text{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$  by Rule (35m) on (4) and (9)

□

## 6 Decidability

$\Xi \text{ incon}$  A finite set of constraints,  $\Xi$ , is inconsistent



$$\frac{\text{CINCTruth} \quad \Xi \text{ incon}}{\Xi, \top \text{ incon}} \quad (36a)$$

$$\frac{\text{CINCFalse}}{\Xi, \perp \text{ incon}} \quad (36b)$$

$$\frac{\text{CINCNum} \quad n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2} \text{ incon}} \quad (36c)$$

$$\frac{\text{CINCNotNum}}{\Xi, \underline{n}, \neg \text{incon}} \quad (36d)$$

$$\frac{\text{CINCAnd} \quad \Xi, \xi_1, \xi_2 \text{ incon}}{\Xi, \xi_1 \wedge \xi_2 \text{ incon}} \quad (36e)$$

$$\frac{\text{CINCOr} \quad \Xi, \xi_1 \text{ incon} \quad \Xi, \xi_2 \text{ incon}}{\Xi, \xi_1 \vee \xi_2 \text{ incon}} \quad (36f)$$

$$\frac{\text{CINCInj}}{\Xi, \text{inl}(\xi_1), \text{inr}(\xi_2) \text{ incon}} \quad (36g)$$

$$\frac{\text{CINCInl} \quad \Xi \text{ incon}}{\text{inl}(\Xi) \text{ incon}} \quad (36h)$$

$$\frac{\text{CINCInr} \quad \Xi \text{ incon}}{\text{inr}(\Xi) \text{ incon}} \quad (36i)$$

$$\frac{\text{CINCPairL} \quad \Xi_1 \text{ incon}}{(\Xi_1, \Xi_2) \text{ incon}} \quad (36j)$$

$$\frac{\text{CINCPairR} \quad \Xi_2 \text{ incon}}{(\Xi_1, \Xi_2) \text{ incon}} \quad (36k)$$

**Lemma 6.0.1** (Decidability of Inconsistency). *Suppose  $\dot{\top}(\xi) = \xi$ . It is decidable whether  $\xi \text{ incon}$ .*

**Lemma 6.0.2** (Inconsistency and Entailment of Constraint). *Suppose that  $\dot{\top}(\xi) = \xi$ . Then  $\bar{\xi} \text{ incon}$  iff  $\top \models \xi$*

**Lemma 6.0.3.** *If  $e \models \xi$  then  $e \models \dot{\top}(\xi)$*

*Proof.* By rule induction over Rules (15), it is obvious to see that  $\dot{\top}(\xi) = \xi$ .  $\square$

**Lemma 6.0.4.** *If  $e \models_{\text{?}} \xi$  then  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$ .*

*Proof.*

(11)  $e \models_{\text{?}} \xi$  by assumption

By Rule Induction over Rules (17) on (11).

**Case (17a).**

(12)  $\xi = ?$  by assumption

(13)  $e \models \top$  by Rule (15a)

(14)  $e \models_{\text{?}}^{\dagger} \top$  by Rule (18b) on (13)

**Case (17b).**

(12)  $e \text{ notintro}$  by assumption

(13)  $\xi \text{ refutable}_{\text{?}}$  by assumption

By Lemma 2.0.15 on (12) and (13) and case analysis on its conclusion.  
By rule induction over Rules (11).

**Case  $\dot{\top}(\xi) \text{ refutable}_{\text{?}}$ .**

(14)  $\dot{\top}(\xi) \text{ refutable}_{\text{?}}$  by assumption

(15)  $e \models_{\text{?}} \dot{\top}(\xi)$  by Rule (17b) on (12)  
and (14)

(16)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$  by Rule (18b) on (15)

**Case  $e \models \dot{\top}(\xi)$ .**

(14)  $e \models \dot{\top}(\xi)$  by assumption

(15)  $e \models_{\text{?}}^{\dagger} \top$  by Rule (18b) on (14)

**Case (17c).**

(12)  $\xi = \xi_1 \vee \xi_2$  by assumption

(13)  $e \models_{\text{?}} \xi_1$  by assumption

(14)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi_1)$  by IH on (13)

(15)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by Lemma 2.0.10 on  
(14)

**Case (17d).**

(12)  $\xi = \xi_1 \vee \xi_2$  by assumption

|      |   |                         |
|------|---|-------------------------|
| (13) | $e \models_{\tau} \xi_2$  | by assumption           |
| (14) | $e \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$                        | by IH on (13)           |
| (15) | $e \models_{\tau}^{\dagger} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Lemma 2.0.10 on (14) |

**Case (17e).**

|      |   |                         |
|------|---|-------------------------|
| (12) | $e = \mathbf{inl}_{\tau_2}(e_1)$  | by assumption           |
| (13) | $\xi = \mathbf{inl}(\xi_1)$   | by assumption           |
| (14) | $e_1 \models_{\tau} \xi_1$  | by assumption           |
| (15) | $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$                                      | by IH on (14)           |
| (16) | $\mathbf{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \mathbf{inl}(\dot{\top}(\xi_1))$ | by Lemma 2.0.11 on (15) |

**Case (17f).**

|      |   |                         |
|------|---|-------------------------|
| (12) | $e = \mathbf{inr}_{\tau_1}(e_2)$  | by assumption           |
| (13) | $\xi = \mathbf{inr}(\xi_2)$   | by assumption           |
| (14) | $e_2 \models_{\tau} \xi_2$  | by assumption           |
| (15) | $e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$                                      | by IH on (14)           |
| (16) | $\mathbf{inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \mathbf{inr}(\dot{\top}(\xi_2))$ | by Lemma 2.0.12 on (15) |

**Case (17g).**

|      |  |                                  |
|------|--|----------------------------------|
| (12) | $e = (e_1, e_2)$   | by assumption                    |
| (13) | $\xi = (\xi_1, \xi_2)$   | by assumption                    |
| (14) | $e_1 \models_{\tau} \xi_1$   | by assumption                    |
| (15) | $e_2 \models_{\tau} \xi_2$   | by assumption                    |
| (16) | $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$                             | by IH on (14)                    |
| (17) | $e_2 \models_{\tau} \dot{\top}(\xi_2)$                                       | by Lemma 6.0.3 on (15)           |
| (18) | $e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$                             | by Rule (18b) on (17)            |
| (19) | $(e_1, e_2) \models_{\tau}^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Lemma 2.0.13 on (16) and (18) |

**Case (17h).**

|   |                                  |
|---|----------------------------------|
| (12) $e = (e_1, e_2)$   | by assumption                    |
| (13) $\xi = (\xi_1, \xi_2)$   | by assumption                    |
| (14) $e_1 \models \xi_1$  | by assumption                    |
| (15) $e_2 \models? \xi_2$   | by assumption                    |
| (16) $e_1 \models \dot{\top}(\xi_1)$  | by Lemma 6.0.3 on (14)           |
| (17) $e_1 \models?^{\dagger} \dot{\top}(\xi_1)$                             | by Rule (18b) on (16)            |
| (18) $e_2 \models?^{\dagger} \dot{\top}(\xi_2)$                             | by IH on (15)                    |
| (19) $(e_1, e_2) \models?^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Lemma 2.0.13 on (17) and (18) |

**Case (17i).**

|   |                                  |
|---|----------------------------------|
| (12) $e = (e_1, e_2)$   | by assumption                    |
| (13) $\xi = (\xi_1, \xi_2)$   | by assumption                    |
| (14) $e_1 \models? \xi_1$   | by assumption                    |
| (15) $e_2 \models? \xi_2$   | by assumption                    |
| (16) $e_1 \models?^{\dagger} \dot{\top}(\xi_1)$                             | by IH on (14)                    |
| (17) $e_2 \models?^{\dagger} \dot{\top}(\xi_2)$                             | by IH on (15)                    |
| (18) $(e_1, e_2) \models?^{\dagger} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Lemma 2.0.13 on (16) and (17) |

□

**Lemma 6.0.5.**  $e \models?^{\dagger} \xi$  iff  $e \models?^{\dagger} \dot{\top}(\xi)$

*Proof.* 1. Sufficiency:

|                                |               |
|--------------------------------|---------------|
| (1) $e \models?^{\dagger} \xi$ | by assumption |
|--------------------------------|---------------|

By rule induction over Rules (18) on (1)

**Case (18b).**

|  |                       |
|--|-----------------------|
| (2) $e \models \xi$                        | by assumption         |
| (3) $e \models \dot{\top}(\xi)$            | by Lemma 6.0.3 on (2) |
| (4) $e \models?^{\dagger} \dot{\top}(\xi)$ | by Rule (18b) on (3)  |

**Case (18a).**

|                      |               |
|----------------------|---------------|
| (2) $e \models? \xi$ | by assumption |
|----------------------|---------------|

(3)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$  by Lemma 6.0.4 on (2)

2. Necessity:

(1)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi)$  by assumption

By structural induction on  $\xi$ ,

**Case**  $\xi = \top, \perp, \underline{n}, \underline{\neg}$ .

(2)  $e \models_{\text{?}}^{\dagger} \xi$  by (1) and Definition 13

**Case**  $\xi = ?$ .

(2)  $e \models_{\text{?}} ?$  by Rule (17a)  
 (3)  $e \models_{\text{?}}^{\dagger} ?$  by Rule (18a) on (2)

**Case**  $\xi = \xi_1 \vee \xi_2$ .

(2)  $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by Definition 13

By rule induction over Rules (18) on (1),

**Case** (18b).

(3)  $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by assumption

By rule induction over Rules (15) on (3) and two cases apply,

**Case** (15e).

(4)  $e \models \dot{\top}(\xi_1)$  by assumption  
 (5)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi_1)$  by Rule (18b) on (4)  
 (6)  $e \models_{\text{?}}^{\dagger} \xi_1$  by IH on (5)  
 (7)  $e \models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$  by Lemma 2.0.10 on (6)

**Case** (15f).

(4)  $e \models \dot{\top}(\xi_2)$  by assumption  
 (5)  $e \models_{\text{?}}^{\dagger} \dot{\top}(\xi_2)$  by Rule (18b) on (4)  
 (6)  $e \models_{\text{?}}^{\dagger} \xi_2$  by IH on (5)  
 (7)  $e \models_{\text{?}}^{\dagger} \xi_1 \vee \xi_2$  by Lemma 2.0.10 on (6)

**Case** (18a).

(3)  $e \models_{\text{?}} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by assumption

By rule induction over Rules (17) on (3) and two cases apply,

**Case** (17c).

|  |                        |
|--|------------------------|
| (4) $e \models_{\tau} \dot{\top}(\xi_1)$           | by assumption          |
| (5) $e \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$ | by Rule (18a) on (4)   |
| (6) $e \models_{\tau}^{\dagger} \xi_1$             | by IH on (5)           |
| (7) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  | by Lemma 2.0.10 on (6) |

**Case (17d).**

|  |                        |
|--|------------------------|
| (4) $e \models_{\tau} \dot{\top}(\xi_2)$           | by assumption          |
| (5) $e \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$ | by Rule (18a) on (4)   |
| (6) $e \models_{\tau}^{\dagger} \xi_2$             | by IH on (5)           |
| (7) $e \models_{\tau}^{\dagger} \xi_1 \vee \xi_2$  | by Lemma 2.0.10 on (6) |

**Case  $\xi = \text{inl}(\xi_1)$ .**

|   |               |
|---|---------------|
| (2) $e = \text{inl}_{\tau_2}(e_1)$                    | by assumption |
| (3) $\dot{\top}(\xi) = \text{inl}(\dot{\top}(\xi_1))$ | by assumption |

By rule induction over Rules (18) on (1),

**Case (18b).**

|  |               |
|--|---------------|
| (4) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\top}(\xi_1))$ | by assumption |
|--|---------------|

By rule induction over Rules (15) and only one case applies,

**Case (15g).**

|   |                        |
|---|------------------------|
| (5) $e_1 \models \dot{\top}(\xi_1)$                                       | by assumption          |
| (6) $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$                      | by Rule (18b) on (5)   |
| (7) $e_1 \models_{\tau}^{\dagger} \xi_1$                                  | by IH on (6)           |
| (8) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ | by Lemma 2.0.11 on (7) |

**Case (18a).**

|   |               |
|---|---------------|
| (4) $\text{inl}_{\tau_2}(e_1) \models_{\tau} \text{inl}(\dot{\top}(\xi_1))$ | by assumption |
|---|---------------|

By rule induction over Rules (17) and only one case applies,

**Case (17e).**

|   |                        |
|---|------------------------|
| (5) $e_1 \models_{\tau} \dot{\top}(\xi_1)$                                | by assumption          |
| (6) $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$                      | by Rule (18a) on (5)   |
| (7) $e_1 \models_{\tau}^{\dagger} \xi_1$                                  | by IH on (6)           |
| (8) $\text{inl}_{\tau_2}(e_1) \models_{\tau}^{\dagger} \text{inl}(\xi_1)$ | by Lemma 2.0.11 on (7) |

**Case  $\xi = \text{inr}(\xi_2)$ .**

|   |               |
|---|---------------|
| (2) $e = \text{inr}_{\tau_1}(e_2)$                    | by assumption |
| (3) $\dot{\top}(\xi) = \text{inr}(\dot{\top}(\xi_2))$ | by assumption |

By rule induction over Rules (18) on (1),

**Case (18b).**

$$(4) \text{ inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\top}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (15) and only one case applies,

**Case (15h).**

$$(5) e_2 \models \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(6) e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2) \quad \text{by Rule (18b) on (5)}$$

$$(7) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (6)}$$

$$(8) \text{ inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2) \quad \text{by Lemma 2.0.12 on (7)}$$

**Case (18a).**

$$(4) \text{ inr}_{\tau_1}(e_2) \models_{\tau} \text{inr}(\dot{\top}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (17) and only one case applies,

**Case (17f).**

$$(5) e_2 \models_{\tau} \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(6) e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2) \quad \text{by Rule (18a) on (5)}$$

$$(7) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (6)}$$

$$(8) \text{ inr}_{\tau_1}(e_2) \models_{\tau}^{\dagger} \text{inr}(\xi_2) \quad \text{by Lemma 2.0.12 on (7)}$$

**Case  $\xi = (\xi_1, \xi_2)$ .**

$$(2) e = (e_1, e_2) \quad \text{by assumption}$$

$$(3) \dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \quad \text{by Definition 13}$$

By rule induction over Rules (18) on (1),

**Case (18b).**

$$(4) (e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \quad \text{by assumption}$$

By rule induction over Rules (15) on (4) and only one case applies,

**Case (15i).**

$$(5) e_1 \models \dot{\top}(\xi_1) \quad \text{by assumption}$$

$$(6) e_2 \models \dot{\top}(\xi_2) \quad \text{by assumption}$$

$$(7) e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1) \quad \text{by Rule (18b) on (5)}$$

$$(8) e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2) \quad \text{by Rule (18b) on (6)}$$

$$(9) e_1 \models_{\tau}^{\dagger} \xi_1 \quad \text{by IH on (7)}$$

$$(10) e_2 \models_{\tau}^{\dagger} \xi_2 \quad \text{by IH on (8)}$$

$$(11) (e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2) \quad \text{by Lemma 2.0.13 on (9) and (10)}$$

**Case (18a).**

(4)  $(e_1, e_2) \models_{\tau} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$  by assumption  
 By rule induction over Rules (17) on (4) and three cases apply,

**Case (17g).**

- |      |  |                                 |
|------|--|---------------------------------|
| (5)  | $e_1 \models_{\tau} \dot{\top}(\xi_1)$               | by assumption                   |
| (6)  | $e_2 \models_{\tau} \dot{\top}(\xi_2)$               | by assumption                   |
| (7)  | $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$     | by Rule (18a) on (5)            |
| (8)  | $e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$     | by Rule (18b) on (6)            |
| (9)  | $e_1 \models_{\tau}^{\dagger} \xi_1$                 | by IH on (7)                    |
| (10) | $e_2 \models_{\tau}^{\dagger} \xi_2$                 | by IH on (8)                    |
| (11) | $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

**Case (17h).**

- |      |  |                                 |
|------|--|---------------------------------|
| (5)  | $e_1 \models_{\tau} \dot{\top}(\xi_1)$               | by assumption                   |
| (6)  | $e_2 \models_{\tau} \dot{\top}(\xi_2)$               | by assumption                   |
| (7)  | $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$     | by Rule (18b) on (5)            |
| (8)  | $e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$     | by Rule (18a) on (6)            |
| (9)  | $e_1 \models_{\tau}^{\dagger} \xi_1$                 | by IH on (7)                    |
| (10) | $e_2 \models_{\tau}^{\dagger} \xi_2$                 | by IH on (8)                    |
| (11) | $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

**Case (17i).**

- |      |  |                                 |
|------|--|---------------------------------|
| (5)  | $e_1 \models_{\tau} \dot{\top}(\xi_1)$               | by assumption                   |
| (6)  | $e_2 \models_{\tau} \dot{\top}(\xi_2)$               | by assumption                   |
| (7)  | $e_1 \models_{\tau}^{\dagger} \dot{\top}(\xi_1)$     | by Rule (18a) on (5)            |
| (8)  | $e_2 \models_{\tau}^{\dagger} \dot{\top}(\xi_2)$     | by Rule (18a) on (6)            |
| (9)  | $e_1 \models_{\tau}^{\dagger} \xi_1$                 | by IH on (7)                    |
| (10) | $e_2 \models_{\tau}^{\dagger} \xi_2$                 | by IH on (8)                    |
| (11) | $(e_1, e_2) \models_{\tau}^{\dagger} (\xi_1, \xi_2)$ | by Lemma 2.0.13 on (9) and (10) |

□

**Lemma 6.0.6.** Assume  $\dot{\top}(\xi) = \xi$ . Then  $\top \models_{\tau}^{\dagger} \xi$  iff  $\top \models \xi$ .

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:
2. Necessity:

□



**Theorem 6.1.**  $\top \models_{\top}^{\dagger} \xi$  iff  $\top \models \dot{\top}(\xi)$ .

**Lemma 6.1.1.** Assume that  $e \text{ val}$ . Then  $e \models_{\top}^{\dagger} \xi$  iff  $e \models \dot{\top}(\xi)$

*Proof.*

(1)  $e \text{ val}$  by assumption

We prove sufficiency and necessity separately.

1. Sufficiency:

(2)  $e \models_{\top}^{\dagger} \xi$  by assumption

By rule induction over Rules (18) on (2).

**Case (18b).**

(3)  $e \models \xi$  by assumption  
(4)  $e \models \dot{\top}(\xi)$  by Lemma 6.0.3 on (3)

**Case (18a).**

(3)  $e \models_{\top} \xi$  by assumption

By rule induction over Rules (17) on (3).

**Case (17a).**

(4)  $\xi = ?$  by assumption  
(5)  $e \models \dot{\top}(\xi)$  by Rule (15a) and  
Definition 13

**Case (17b).**

(4)  $e \text{ notintro}$  by assumption

By rule induction over Rules (27) on (4), for each case, by rule induction over Rules (24) on (1), no case applies due to syntactic contradiction.

**Case (17c).**

(4)  $\xi = \xi_1 \vee \xi_2$  by assumption  
(5)  $e \models_{\top} \xi_1$  by assumption  
(6)  $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by Equation 13  
(7)  $e \models_{\top}^{\dagger} \xi_1$  by Rule (18a) on (5)  
(8)  $e \models \dot{\top}(\xi_1)$  by IH on (7)  
(9)  $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$  by Rule (15e) on (8)

**Case (17d).**

(4)  $\xi = \xi_1 \vee \xi_2$  by assumption  
(5)  $e \models_{\top} \xi_2$  by assumption

|     |  |                      |
|-----|--|----------------------|
| (6) | $\dot{\vdash}(\xi) = \dot{\vdash}(\xi_1) \vee \dot{\vdash}(\xi_2)$ | by Equation 13       |
| (7) | $e \models_{\dot{?}}^{\dot{\vdash}} \xi_2$                         | by Rule (18a) on (5) |
| (8) | $e \models \dot{\vdash}(\xi_2)$                                    | by IH on (7)         |
| (9) | $e \models \dot{\vdash}(\xi_1) \vee \dot{\vdash}(\xi_2)$           | by Rule (15f) on (8) |

**Case (17e).**

|     |   |                      |
|-----|---|----------------------|
| (4) | $\xi = \mathbf{inl}(\xi_1)$                             | by assumption        |
| (5) | $e \models_{\dot{?}} \xi_1$                             | by assumption        |
| (6) | $\dot{\vdash}(\xi) = \mathbf{inl}(\dot{\vdash}(\xi_1))$ | by Equation 13       |
| (7) | $e \models_{\dot{?}}^{\dot{\vdash}} \xi_1$              | by Rule (18a) on (5) |
| (8) | $e \models \dot{\vdash}(\xi_1)$                         | by IH on (7)         |
| (9) | $e \models \mathbf{inl}(\dot{\vdash}(\xi_1))$           | by Rule (15g) on (8) |

**Case (17f).**

|     |   |                      |
|-----|---|----------------------|
| (4) | $\xi = \mathbf{inr}(\xi_2)$                             | by assumption        |
| (5) | $e \models_{\dot{?}} \xi_2$                             | by assumption        |
| (6) | $\dot{\vdash}(\xi) = \mathbf{inr}(\dot{\vdash}(\xi_2))$ | by Equation 13       |
| (7) | $e \models_{\dot{?}}^{\dot{\vdash}} \xi_2$              | by Rule (18a) on (5) |
| (8) | $e \models \dot{\vdash}(\xi_2)$                         | by IH on (7)         |
| (9) | $e \models \mathbf{inr}(\dot{\vdash}(\xi_2))$           | by Rule (15h) on (8) |

**Case (17g).**

|      |  |                                |
|------|--|--------------------------------|
| (4)  | $e = (e_1, e_2)$   | by assumption                  |
| (5)  | $\xi = (\xi_1, \xi_2)$   | by assumption                  |
| (6)  | $e_1 \models_{\dot{?}} \xi_1$                                    | by assumption                  |
| (7)  | $e_2 \models \xi_2$  | by assumption                  |
| (8)  | $\dot{\vdash}(\xi) = (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2))$ | by Equation 13                 |
| (9)  | $e_1 \models_{\dot{?}}^{\dot{\vdash}} \xi_1$                     | by Rule (18a) on (6)           |
| (10) | $e_2 \models_{\dot{?}}^{\dot{\vdash}} \xi_2$                     | by Rule (18b) on (7)           |
| (11) | $e_1 \models \dot{\vdash}(\xi_1)$                                | by IH on (9)                   |
| (12) | $e_2 \models \dot{\vdash}(\xi_2)$                                | by IH on (10)                  |
| (13) | $(e_1, e_2) \models (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2))$  | by Rule (15i) on (11) and (12) |

**Case (17h).**

|     |  |                |
|-----|--|----------------|
| (4) | $e = (e_1, e_2)$   | by assumption  |
| (5) | $\xi = (\xi_1, \xi_2)$   | by assumption  |
| (6) | $e_1 \models \xi_1$  | by assumption  |
| (7) | $e_2 \models_{\dot{?}} \xi_2$                                    | by assumption  |
| (8) | $\dot{\vdash}(\xi) = (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2))$ | by Equation 13 |

|      |   |                                |
|------|---|--------------------------------|
| (9)  | $e_1 \models_{\dot{?}}^{\dagger} \xi_1$                     | by Rule (18b) on (6)           |
| (10) | $e_2 \models_{\dot{?}}^{\dagger} \xi_2$                     | by Rule (18a) on (7)           |
| (11) | $e_1 \models \dot{\top}(\xi_1)$                             | by IH on (9)                   |
| (12) | $e_2 \models \dot{\top}(\xi_2)$                             | by IH on (10)                  |
| (13) | $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (15i) on (11) and (12) |

**Case (17i).**

|      |   |                                |
|------|---|--------------------------------|
| (4)  | $e = (e_1, e_2)$  | by assumption                  |
| (5)  | $\xi = (\xi_1, \xi_2)$                                      | by assumption                  |
| (6)  | $e_1 \models_{\dot{?}} \xi_1$                               | by assumption                  |
| (7)  | $e_2 \models_{\dot{?}} \xi_2$                               | by assumption                  |
| (8)  | $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$  | by Equation 13                 |
| (9)  | $e_1 \models_{\dot{?}}^{\dagger} \xi_1$                     | by Rule (18a) on (6)           |
| (10) | $e_2 \models_{\dot{?}}^{\dagger} \xi_2$                     | by Rule (18a) on (7)           |
| (11) | $e_1 \models \dot{\top}(\xi_1)$                             | by IH on (9)                   |
| (12) | $e_2 \models \dot{\top}(\xi_2)$                             | by IH on (10)                  |
| (13) | $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ | by Rule (15i) on (11) and (12) |

2. Necessity:

|     |                             |               |
|-----|-----------------------------|---------------|
| (2) | $e \models \dot{\top}(\xi)$ | by assumption |
|-----|-----------------------------|---------------|

By structural induction on  $\xi$ .

**Case  $\xi = \top, \perp, n, \neg$ .**

|     |                                     |                      |
|-----|-------------------------------------|----------------------|
| (3) | $\xi = \dot{\top}(\xi)$             | by Equation 13       |
| (4) | $e \models_{\dot{?}}^{\dagger} \xi$ | by Rule (18b) on (2) |

**Case  $\xi = ?$ .**

|     |                                   |                      |
|-----|-----------------------------------|----------------------|
| (3) | $e \models_{\dot{?}} ?$           | by Rule (17a)        |
| (4) | $e \models_{\dot{?}}^{\dagger} ?$ | by Rule (18a) on (3) |

**Case  $\xi = \xi_1 \wedge \xi_2$ .**

|     |  |                |
|-----|--|----------------|
| (3) | $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ | by Equation 13 |
|-----|--|----------------|

By rule induction over Rules (15) on (2), only one case applies.

**Case (15d).**

|     |                               |               |
|-----|-------------------------------|---------------|
| (4) | $e \models \dot{\top}(\xi_1)$ | by assumption |
|-----|-------------------------------|---------------|

- |  |                               |
|--|-------------------------------|
| (5) $e \models \dot{\top}(\xi_2)$            | by assumption                 |
| (6) $e \models_{\dot{?}}^{\dot{\top}} \xi_1$ | by IH on (4)                  |
| (7) $e \models_{\dot{?}}^{\dot{\top}} \xi_2$ | by IH on (5)                  |
| (8) $e \models \xi_1 \wedge \xi_2$           | by Lemma 2.0.9 on (6) and (7) |

**Case  $\xi = \xi_1 \vee \xi_2$ .**

- |  |                |
|--|----------------|
| (3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ | by Equation 13 |
|--|----------------|

By rule induction over Rules (15) on (2) and only two cases apply.

**Case (15e).**

- |   |                        |
|---|------------------------|
| (4) $e \models \dot{\top}(\xi_1)$                       | by assumption          |
| (5) $e \models_{\dot{?}}^{\dot{\top}} \xi_1$            | by IH on (4)           |
| (6) $e \models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2$ | by Lemma 2.0.10 on (5) |

**Case (15f).**

- |   |                        |
|---|------------------------|
| (4) $e \models \dot{\top}(\xi_2)$                       | by assumption          |
| (5) $e \models_{\dot{?}}^{\dot{\top}} \xi_2$            | by IH on (4)           |
| (6) $e \models_{\dot{?}}^{\dot{\top}} \xi_1 \vee \xi_2$ | by Lemma 2.0.10 on (5) |

**Case  $\xi = \text{inl}(\xi_1)$ .**

- |   |                |
|---|----------------|
| (3) $\dot{\top}(\xi) = \text{inl}(\dot{\top}(\xi_1))$ | by Equation 13 |
|---|----------------|

By rule induction over Rules (15) on (2) and only one case applies.

**Case (15g).**

- |   |                        |
|---|------------------------|
| (4) $e = \text{inl}_{\tau_2}(e_1)$  | by assumption          |
| (5) $e_1 \models \dot{\top}(\xi_1)$   | by assumption          |
| (6) $e_1 \models_{\dot{?}}^{\dot{\top}} \xi_1$                                  | by IH on (5)           |
| (7) $\text{inl}_{\tau_2}(e_1) \models_{\dot{?}}^{\dot{\top}} \text{inl}(\xi_1)$ | by Lemma 2.0.11 on (6) |

**Case  $\xi = \text{inr}(\xi_2)$ .**

- |   |                |
|---|----------------|
| (3) $\dot{\top}(\xi) = \text{inr}(\dot{\top}(\xi_2))$ | by Equation 13 |
|---|----------------|

By rule induction over Rules (15) on (2) and only one case applies.

**Case (15h).**

- |   |                        |
|---|------------------------|
| (4) $e = \text{inr}_{\tau_1}(e_2)$  | by assumption          |
| (5) $e_2 \models \dot{\top}(\xi_2)$   | by assumption          |
| (6) $e_2 \models_{\dot{?}}^{\dot{\top}} \xi_2$                                  | by IH on (5)           |
| (7) $\text{inr}_{\tau_1}(e_2) \models_{\dot{?}}^{\dot{\top}} \text{inr}(\xi_2)$ | by Lemma 2.0.12 on (6) |

**Case**  $\xi = (\xi_1, \xi_2)$ .

$$(3) \quad \dot{\vdash}(\xi) = (\dot{\vdash}(\xi_1), \dot{\vdash}(\xi_2)) \quad \text{by Equation 13}$$

By rule induction over Rules (15) on (2) and only one case applies.

**Case** (15i).

$$\begin{array}{ll} (4) \quad e = (e_1, e_2) & \text{by assumption} \\ (5) \quad e_1 \models \dot{\perp}(\xi_1) & \text{by assumption} \\ (6) \quad e_2 \models \dot{\perp}(\xi_2) & \text{by assumption} \\ (7) \quad e_1 \models_{\dot{?}}^{\dot{\vdash}} \xi_1 & \text{by IH on (5)} \\ (8) \quad e_2 \models_{\dot{?}}^{\dot{\vdash}} \xi_2 & \text{by IH on (6)} \\ (9) \quad (e_1, e_2) \models_{\dot{?}}^{\dot{\vdash}} (\xi_1, \xi_2) & \text{by Lemma 2.0.13 on} \\ & \text{(7) and (8)} \end{array}$$

□

**Lemma 6.1.2.**  $e \models \xi$  iff  $e \models \dot{\perp}(\xi)$

*Proof.* We prove sufficiency and necessity separately.

1. Sufficiency:

$$(1) \quad e \models \xi \quad \text{by assumption}$$

By rule induction over Rules (15) on (1).

**Case** (15a).

$$\begin{array}{ll} (2) \quad \xi = \top & \text{by assumption} \\ (3) \quad e \models \dot{\perp}(\top) & \text{by (1) and Definition} \\ & 14 \end{array}$$

**Case** (15b).

$$\begin{array}{ll} (2) \quad \xi = \underline{n} & \text{by assumption} \\ (3) \quad e \models \dot{\perp}(\underline{n}) & \text{by (1) and Definition} \\ & 14 \end{array}$$

**Case** (15c).

$$\begin{array}{ll} (2) \quad \xi = \underline{\neg} & \text{by assumption} \\ (3) \quad e \models \dot{\perp}(\underline{\neg}) & \text{by (1) and Definition} \\ & 14 \end{array}$$

**Case** (15d).

$$\begin{array}{ll} (2) \quad \xi = \xi_1 \wedge \xi_2 & \text{by assumption} \\ (3) \quad e \models \xi_1 & \text{by assumption} \\ (4) \quad e \models \xi_2 & \text{by assumption} \\ (5) \quad e \models \dot{\perp}(\xi_1) & \text{by IH on (3)} \end{array}$$

|  |                              |
|--|------------------------------|
| (6) $e \models \dot{\perp}(\xi_2)$                           | by IH on (4)                 |
| (7) $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$ | by Rule (15d) on (5) and (6) |
| (8) $e \models \dot{\perp}(\xi_1 \wedge \xi_2)$              | by (7) and Definition 14     |

**Case (15e).**

|  |                          |
|--|--------------------------|
| (2) $\xi = \xi_1 \vee \xi_2$                               | by assumption            |
| (3) $e \models \xi_1$                                      | by assumption            |
| (4) $e \models \dot{\perp}(\xi_1)$                         | by IH on (3)             |
| (5) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ | by Rule (15e) on (4)     |
| (6) $e \models \dot{\perp}(\xi_1 \vee \xi_2)$              | by (5) and Definition 14 |

**Case (15f).**

|  |                          |
|--|--------------------------|
| (2) $\xi = \xi_1 \vee \xi_2$                               | by assumption            |
| (3) $e \models \xi_2$                                      | by assumption            |
| (4) $e \models \dot{\perp}(\xi_2)$                         | by IH on (3)             |
| (5) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ | by Rule (15f) on (4)     |
| (6) $e \models \dot{\perp}(\xi_1 \vee \xi_2)$              | by (5) and Definition 14 |

**Case (15g).**

|   |                          |
|---|--------------------------|
| (2) $e = \text{inl}_{\tau_2}(e_1)$                                    | by assumption            |
| (3) $\xi = \text{inl}(\xi_1)$   | by assumption            |
| (4) $e_1 \models \xi_1$   | by assumption            |
| (5) $e_1 \models \dot{\perp}(\xi_1)$                                  | by IH on (4)             |
| (6) $\text{inl}_{\tau_2}(e_1) \models \text{inl}(\dot{\perp}(\xi_1))$ | by Rule (15g) on (5)     |
| (7) $\text{inl}_{\tau_2}(e_1) \models \dot{\perp}(\text{inl}(\xi_1))$ | by (6) and Definition 14 |

**Case (15h).**

|   |                          |
|---|--------------------------|
| (2) $e = \text{inr}_{\tau_1}(e_2)$                                    | by assumption            |
| (3) $\xi = \text{inr}(\xi_2)$   | by assumption            |
| (4) $e_2 \models \xi_2$   | by assumption            |
| (5) $e_2 \models \dot{\perp}(\xi_2)$                                  | by IH on (4)             |
| (6) $\text{inr}_{\tau_1}(e_2) \models \text{inr}(\dot{\perp}(\xi_2))$ | by Rule (15h) on (5)     |
| (7) $\text{inr}_{\tau_1}(e_2) \models \dot{\perp}(\text{inr}(\xi_2))$ | by (6) and Definition 14 |

**Case (15i).**

|                      |               |
|----------------------|---------------|
| (2) $e = (e_1, e_2)$ | by assumption |
|----------------------|---------------|

|   |                              |
|---|------------------------------|
| (3) $\xi = (\xi_1, \xi_2)$  | by assumption                |
| (4) $e_1 \models \xi_1$   | by assumption                |
| (5) $e_2 \models \xi_2$   | by assumption                |
| (6) $e_1 \models \dot{\perp}(\xi_1)$                              | by IH on (4)                 |
| (7) $e_2 \models \dot{\perp}(\xi_2)$                              | by IH on (5)                 |
| (8) $(e_1, e_2) \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$ | by Rule (15i) on (6) and (7) |
| (9) $(e_1, e_2) \models \dot{\perp}((\xi_1, \xi_2))$              | by (8) and Definition 14     |

2. Necessity:

|                                  |               |
|----------------------------------|---------------|
| (1) $e \models \dot{\perp}(\xi)$ | by assumption |
|----------------------------------|---------------|

By structural induction on  $\xi$ .

**Case**  $\xi = \top, \perp, n, \neg$ .

|                     |                          |
|---------------------|--------------------------|
| (2) $e \models \xi$ | by (1) and Definition 14 |
|---------------------|--------------------------|

**Case**  $\xi = ?$ .

|                       |                          |
|-----------------------|--------------------------|
| (2) $e \models \perp$ | by (1) and Definition 14 |
|-----------------------|--------------------------|

|                           |                |
|---------------------------|----------------|
| (3) $e \not\models \perp$ | by Lemma 2.0.1 |
|---------------------------|----------------|

(3) contradicts (2).

**Case**  $\xi = \xi_1 \wedge \xi_2$ .

|  |                          |
|--|--------------------------|
| (2) $e \models \dot{\perp}(\xi_1) \wedge \dot{\perp}(\xi_2)$ | by (1) and Definition 14 |
|--|--------------------------|

By rule induction over Rules (15) on (2) and only case applies.

**Case** (15d).

|                                    |                              |
|------------------------------------|------------------------------|
| (3) $e \models \dot{\perp}(\xi_1)$ | by assumption                |
| (4) $e \models \dot{\perp}(\xi_2)$ | by assumption                |
| (5) $e \models \xi_1$              | by IH on (3)                 |
| (6) $e \models \xi_2$              | by IH on (4)                 |
| (7) $e \models \xi_1 \wedge \xi_2$ | by Rule (15d) on (5) and (6) |

**Case**  $\xi = \xi_1 \vee \xi_2$ .

|  |                          |
|--|--------------------------|
| (2) $e \models \dot{\perp}(\xi_1) \vee \dot{\perp}(\xi_2)$ | by (1) and Definition 14 |
|--|--------------------------|

By rule induction over Rules (15) on (2) and only two cases apply.

**Case** (15e).

|                                    |                      |
|------------------------------------|----------------------|
| (3) $e \models \dot{\perp}(\xi_1)$ | by assumption        |
| (4) $e \models \xi_1$              | by IH on (3)         |
| (5) $e \models \xi_1 \vee \xi_2$   | by Rule (15e) on (4) |

**Case (15f).**

|                                    |                      |
|------------------------------------|----------------------|
| (3) $e \models \dot{\perp}(\xi_2)$ | by assumption        |
| (4) $e \models \xi_2$              | by IH on (3)         |
| (5) $e \models \xi_1 \vee \xi_2$   | by Rule (15f) on (4) |

**Case  $\xi = \text{inl}(\xi_1)$ .**

|  |                          |
|--|--------------------------|
| (2) $e \models \text{inl}(\dot{\perp}(\xi_1))$ | by (1) and Definition 14 |
|--|--------------------------|

By rule induction over Rules (15) on (2) and only one case applies.

**Case (15g).**

|                                      |                      |
|--------------------------------------|----------------------|
| (3) $e = \text{inl}_{\tau_2}(e_1)$   | by assumption        |
| (4) $e_1 \models \dot{\perp}(\xi_1)$ | by assumption        |
| (5) $e_1 \models \xi_1$              | by IH on (4)         |
| (6) $e \models \text{inl}(\xi_1)$    | by Rule (15g) on (5) |

**Case  $\xi = \text{inr}(\xi_2)$ .**

|  |                          |
|--|--------------------------|
| (2) $e \models \text{inr}(\dot{\perp}(\xi_2))$ | by (1) and Definition 14 |
|--|--------------------------|

By rule induction over Rules (15) on (2) and only one case applies.

**Case (15h).**

|                                      |                      |
|--------------------------------------|----------------------|
| (3) $e = \text{inr}_{\tau_1}(e_2)$   | by assumption        |
| (4) $e_2 \models \dot{\perp}(\xi_2)$ | by assumption        |
| (5) $e_2 \models \xi_2$              | by IH on (4)         |
| (6) $e \models \text{inr}(\xi_2)$    | by Rule (15h) on (5) |

**Case  $\xi = (\xi_1, \xi_2)$ .**

|  |                          |
|--|--------------------------|
| (2) $e \models (\dot{\perp}(\xi_1), \dot{\perp}(\xi_2))$ | by (1) and Definition 14 |
|--|--------------------------|

By rule induction over Rules (15) on (2) and only case applies.

**Case (15i).**

|                                      |                              |
|--------------------------------------|------------------------------|
| (3) $e = (e_1, e_2)$                 | by assumption                |
| (4) $e_1 \models \dot{\perp}(\xi_1)$ | by assumption                |
| (5) $e_2 \models \dot{\perp}(\xi_2)$ | by assumption                |
| (6) $e_1 \models \xi_1$              | by IH on (4)                 |
| (7) $e_2 \models \xi_2$              | by IH on (5)                 |
| (8) $e \models (\xi_1, \xi_2)$       | by Rule (15i) on (6) and (7) |



□

**Lemma 6.1.3.** *Assume  $e \text{ val}$  and  $\dot{\top}(\xi) = \xi$ . Then  $e \not\models \xi$  iff  $e \models \bar{\xi}$ .*

**Theorem 6.2.**  $\xi_r \models \xi_{rs}$  iff  $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\perp}(\xi_{rs})$ .