1 Match Constraint Language

$\dot{\xi} \ ::= \ \top \mid \bot \mid ? \mid \underline{n} \mid \texttt{inl}(\dot{\xi}) \mid \texttt{inr}(\dot{\xi}) \mid (\dot{\xi}, \dot{\xi}) \mid \dot{\xi} \vee \dot{\xi}$

$\boxed{\dot{\xi} : \tau}$ $\dot{\xi}$ constrains final expressions of type τ

$$\text{CTTruth}\quad \frac{}{\top : \tau} \tag{1a}$$

$$\text{CTUnknown}\quad \frac{}{? : \tau} \tag{1b}$$

$$\text{CTNum}\quad \frac{}{\underline{n} : \texttt{num}} \tag{1c}$$

$$\text{CTInl}\quad \frac{\dot{\xi}_1 : \tau_1}{\texttt{inl}(\dot{\xi}_1) : (\tau_1 + \tau_2)} \tag{1d}$$

$$\text{CTInr}\quad \frac{\dot{\xi}_2 : \tau_2}{\texttt{inr}(\dot{\xi}_2) : (\tau_1 + \tau_2)} \tag{1e}$$

$$\text{CTPair}\quad \frac{\dot{\xi}_1 : \tau_1 \qquad \dot{\xi}_2 : \tau_2}{(\dot{\xi}_1, \dot{\xi}_2) : (\tau_1 \times \tau_2)} \tag{1f}$$

$$\text{CTOr}\quad \frac{\dot{\xi}_1 : \tau \qquad \dot{\xi}_2 : \tau}{\dot{\xi}_1 \vee \dot{\xi}_2 : \tau} \tag{1g}$$

$\boxed{\dot{\xi}\ \texttt{refutable}_?}$ $\dot{\xi}$ is refutable

$$\text{RXNum}\quad \frac{}{\underline{n}\ \texttt{refutable}_?} \tag{2a}$$

$$\text{RXUnknown}\quad \frac{}{?\ \texttt{refutable}_?} \tag{2b}$$

$$\text{RXInl}\quad \frac{}{\texttt{inl}(\dot{\xi})\ \texttt{refutable}_?} \tag{2c}$$

$$\text{RXInr}\quad \frac{}{\texttt{inr}(\dot{\xi})\ \texttt{refutable}_?} \tag{2d}$$

$$\text{RXPairL}\quad \frac{\dot{\xi}_1\ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?} \tag{2e}$$

$$\frac{\dot{\xi}_2 \ \texttt{refutable}_?}{(\dot{\xi}_1, \dot{\xi}_2) \ \texttt{refutable}_?} \ \text{RXPairR} \tag{2f}$$

$$\frac{\dot{\xi}_1 \ \texttt{refutable}_? \qquad \dot{\xi}_2 \ \texttt{refutable}_?}{\dot{\xi}_1 \vee \dot{\xi}_2 \ \texttt{refutable}_?} \ \text{RXOr} \tag{2g}$$

$\boxed{\mathit{refutable}_?(\dot{\xi})}$

$$\mathit{refutable}_?(\underline{n}) = \text{true} \tag{3a}$$

$$\mathit{refutable}_?(?) = \text{true} \tag{3b}$$

$$\mathit{refutable}_?(\texttt{inl}(\dot{\xi})) = \mathit{refutable}_?(\dot{\xi}) \tag{3c}$$

$$\mathit{refutable}_?(\texttt{inr}(\dot{\xi})) = \mathit{refutable}_?(\dot{\xi}) \tag{3d}$$

$$\mathit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \mathit{refutable}_?(\dot{\xi}_1) \text{ or } \mathit{refutable}_?(\dot{\xi}_2) \tag{3e}$$

$$\mathit{refutable}_?(\dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{refutable}_?(\dot{\xi}_1) \text{ and } \mathit{refutable}_?(\dot{\xi}_2) \tag{3f}$$

$$\text{Otherwise} \quad \mathit{refutable}_?(\dot{\xi}) = \text{false} \tag{3g}$$

Lemma 1.0.1 (Soundness and Completeness of Refutable Constraints). $\dot{\xi}$ $\texttt{refutable}_?$ *iff* $\mathit{refutable}_?(\dot{\xi}) = \mathit{true}$.

$\boxed{e \models \dot{\xi}}$ e satisfies $\dot{\xi}$

$$\frac{}{e \models \top} \ \text{CSTruth} \tag{4a}$$

$$\frac{}{\underline{n} \models \underline{n}} \ \text{CSNum} \tag{4b}$$

$$\frac{e_1 \models \dot{\xi}_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)} \ \text{CSInl} \tag{4c}$$

$$\frac{e_2 \models \dot{\xi}_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)} \ \text{CSInr} \tag{4d}$$

$$\frac{e_1 \models \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)} \ \text{CSPair} \tag{4e}$$

$$\frac{e \ \texttt{notintro} \qquad \texttt{prl}(e) \models \dot{\xi}_1 \qquad \texttt{prr}(e) \models \dot{\xi}_2}{e \models (\dot{\xi}_1, \dot{\xi}_2)} \ \text{CSNotIntroPair} \tag{4f}$$

$$\frac{e \models \dot{\xi}_1}{e \models \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrL} \tag{4g}$$

$$\frac{e \models \dot{\xi}_2}{e \models \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CSOrR} \tag{4h}$$

$\boxed{\mathit{satisfy}(e, \dot{\xi})}$

$$\mathit{satisfy}(e, \top) = \text{true} \tag{5a}$$

$$\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{5b}$$

$$\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1) \text{ or } \mathit{satisfy}(e, \dot{\xi}_2) \tag{5c}$$

$$\mathit{satisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inl}(\dot{\xi}_1)) = \mathit{satisfy}(e_1, \dot{\xi}_1) \tag{5d}$$

$$\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\dot{\xi}_2)) = \mathit{satisfy}(e_2, \dot{\xi}_2) \tag{5e}$$

$$\mathit{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2) \tag{5f}$$

$$\mathit{satisfy}(⦇⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(\mathtt{prl}(⦇⦈^u), \dot{\xi}_1) \text{ and } \mathit{satisfy}(\mathtt{prr}(⦇⦈^u), \dot{\xi}_2) \tag{5g}$$

$$\mathit{satisfy}(⦇e⦈^u, (\dot{\xi}_1, \dot{\xi}_2)) = \mathit{satisfy}(\mathtt{prl}(⦇e⦈^u), \dot{\xi}_1) \text{ and } \mathit{satisfy}(\mathtt{prr}(⦇e⦈^u), \dot{\xi}_2) \tag{5h}$$

$$\begin{aligned}\mathit{satisfy}(e_1(e_2), (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{prl}(e_1(e_2)), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{prr}(e_1(e_2)), \dot{\xi}_2)\end{aligned} \tag{5i}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{match}(e)\{\hat{rs}\}, (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{prl}(\mathtt{match}(e)\{\hat{rs}\}), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{match}(e)\{\hat{rs}\}), \dot{\xi}_2)\end{aligned} \tag{5j}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{prl}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{prl}(\mathtt{prl}(e)), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{prl}(e)), \dot{\xi}_2)\end{aligned} \tag{5k}$$

$$\begin{aligned}\mathit{satisfy}(\mathtt{prr}(e), (\dot{\xi}_1, \dot{\xi}_2)) &= \mathit{satisfy}(\mathtt{prl}(\mathtt{prr}(e)), \dot{\xi}_1) \\ &\quad \text{and } \mathit{satisfy}(\mathtt{prr}(\mathtt{prr}(e)), \dot{\xi}_2)\end{aligned} \tag{5l}$$

$$\text{Otherwise} \quad \mathit{satisfy}(e, \dot{\xi}) = \text{false} \tag{5m}$$

Lemma 1.0.2 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \models \dot{\xi}$ iff $\mathit{satisfy}(e, \dot{\xi}) = \mathit{true}$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models \dot{\xi}$ — by assumption

   By rule induction over Rules (16) on (1).

   Case (16a)**.**

(2)	$\dot{\xi} = \top$	by assumption
(3)	$\mathit{satisfy}(e, \top) = \text{true}$	by Definition 17a

Case (16b)**.**

(2)	$e = \underline{n}$	by assumption
(3)	$\dot{\xi} = \underline{n}$	by assumption
(4)	$\mathit{satisfy}(\underline{n}, \underline{n}) = (n = n) = \text{true}$	by Definition 17b

Case (16c)**.**

(2)	$e = \underline{n_1}$	by assumption
(3)	$\dot{\xi} = \underline{\cancel{n_2}}$	by assumption
(4)	$n_1 \neq n_2$	by assumption
(5)	$\mathit{satisfy}(\underline{n_1}, \underline{\cancel{n_2}}) = (n_1 \neq n_2) = \text{true}$	by Definition 17c on (4)

Case (16d)**.**

(2)	$\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$	by assumption
(3)	$e \models \dot{\xi}_1$	by assumption
(4)	$e \models \dot{\xi}_2$	by assumption
(5)	$\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$	by IH on (3)
(6)	$\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by IH on (4)
(7)	$\mathit{satisfy}(e, \dot{\xi}_1 \wedge \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1)$ and $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by Definition 17d on (5) and (6)

Case (16e)**.**

(2)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3)	$e \models \dot{\xi}_1$	by assumption
(4)	$\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$	by IH on (3)
(5)	$\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1)$ or $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by Definition 17e on (4)

Case (16f)**.**

(2)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3)	$e \models \dot{\xi}_2$	by assumption
(4)	$\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by IH on (3)
(5)	$\mathit{satisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \mathit{satisfy}(e, \dot{\xi}_1)$ or $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by Definition 17e on (4)

Case (16g).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$ by assumption

(4) $e_1 \models \dot{\xi}_1$ by assumption

(5) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$
by Definition 17f on (5)

Case (16h).

(2) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\dot{\xi} = \texttt{inl}(\dot{\xi}_2)$ by assumption

(4) $e_2 \models \dot{\xi}_2$ by assumption

(5) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$
by Definition 17g on (5)

Case (16i).

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(4) $e_1 \models \dot{\xi}_1$ by assumption

(5) $e_2 \models \dot{\xi}_2$ by assumption

(6) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by IH on (5)

(8) $\mathit{satisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) =$
$\mathit{satisfy}(e_1, \dot{\xi}_1)$ and $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$
by Definition 17h on (6) and (7)

Case (16j).

(2) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(3) e $\texttt{notintro}$ by assumption

(4) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(5) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(6) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$ by IH on (5)

By rule induction over Rules (28) on (3).

Otherwise.

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$
by assumption

(9) $\mathit{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) =$
$\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1)$ and $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$
by Definition 17 on (6) and (7)

2. Completeness:

(1) $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$ by assumption

By structural induction on $\dot{\xi}$.

Case $\dot{\xi} = \top$.

(2) $e \models \top$ by Rule (16a)

Case $\dot{\xi} = \bot, ?$.

(2) $\mathit{satisfy}(e, \dot{\xi}) = \text{false}$ by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \underline{n}$.

By structural induction on e.

Case $e = \underline{n'}$.

(2) $n' = n$ by Definition 17b on (1)

(3) $\underline{n'} \models \underline{n}$ by Rule (16b) on (2)

Otherwise.

(2) $\mathit{satisfy}(e, \underline{n}) = \text{false}$ by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$.

(2) $\mathit{satisfy}(e, \dot{\xi}_1)$ or $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$
by Definition 17e on (1)

By case analysis on (2).

Case $\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$.

(3) $\mathit{satisfy}(e, \dot{\xi}_1) = \text{true}$ by assumption

(4) $e \models \dot{\xi}_1$ by IH on (3)

(5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16e) on (4)

Case $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$.

(3) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$ by assumption

(4) $e \models \dot{\xi}_2$ by IH on (3)
(5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16f) on (4)

Case $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$**.**
By structural induction on e.

Case $e = \texttt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by Definition 17f on (1)
(3) $e_1 \models \dot{\xi}_1$ by IH on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$ by Rule (16g) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \texttt{inl}(\dot{\xi}_1)) = \text{false}$ by Definition 17o
(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$**.**
By structural induction on e.

Case $e = \texttt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by Definition 17g on (1)
(3) $e_2 \models \dot{\xi}_2$ by IH on (2)
(4) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by Rule (16h) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \texttt{inr}(\dot{\xi}_2)) = \text{false}$ by Definition 17o
(2) contradicts (1) and thus vacuously true.

Case $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$**.**
By structural induction on e.

Case $e = (e_1, e_2)$**.**

(2) $\mathit{satisfy}(e_1, \dot{\xi}_1) = \text{true}$ by Definition 17h on (1)
(3) $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$ by Definition 17h on (1)
(4) $e_1 \models \dot{\xi}_1$ by IH on (2)
(5) $e_2 \models \dot{\xi}_2$ by IH on (3)
(6) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16i) on (4) and (5)

Case $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$**.**

(2) $\mathit{satisfy}(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by Definition 17h on (1)
(3) $\mathit{satisfy}(\texttt{prr}(e), \dot{\xi}_2) = \text{true}$ by Definition 17h on (1)

(4) $\mathtt{prl}(e) \models \dot{\xi}_1$ — by IH on (2)

(5) $\mathtt{prr}(e) \models \dot{\xi}_2$ — by IH on (3)

(6) e $\mathtt{notintro}$ — by each rule in Rules (28)

(7) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (16j) on (6) and (4) and (5)

Otherwise.

(2) $\mathit{satisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) = \text{false}$ — by Definition 17o

(2) contradicts (1) and thus vacuously true.

□

$\boxed{e \models_? \dot{\xi}}$ e may satisfy $\dot{\xi}$

$$\dfrac{}{e \models_? ?}\ \text{CMSUnknown} \tag{6a}$$

$$\dfrac{e_1 \models_? \dot{\xi}_1}{\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)}\ \text{CMSInl} \tag{6b}$$

$$\dfrac{e_2 \models_? \dot{\xi}_2}{\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)}\ \text{CMSInr} \tag{6c}$$

$$\dfrac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairL} \tag{6d}$$

$$\dfrac{e_1 \models \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPairR} \tag{6e}$$

$$\dfrac{e_1 \models_? \dot{\xi}_1 \qquad e_2 \models_? \dot{\xi}_2}{(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)}\ \text{CMSPair} \tag{6f}$$

$$\dfrac{e \models_? \dot{\xi}_1 \qquad e \not\models \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrL} \tag{6g}$$

$$\dfrac{e \not\models \dot{\xi}_1 \qquad e \models_? \dot{\xi}_2}{e \models_? \dot{\xi}_1 \vee \dot{\xi}_2}\ \text{CMSOrR} \tag{6h}$$

$$\dfrac{e\ \mathtt{notintro} \qquad \dot{\xi}\ \mathtt{refutable}_?}{e \models_? \dot{\xi}}\ \text{CMSNotIntro} \tag{6i}$$

$\boxed{\mathit{maysatisfy}(e, \dot{\xi})}$

$$\mathit{maysatisfy}(e, ?) = \text{true} \tag{7a}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \mathit{maysatisfy}(e_1, \dot{\xi}_1) \tag{7b}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\dot{\xi}_2)) = \mathit{maysatisfy}(e_2, \dot{\xi}_2) \tag{7c}$$

$$\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inr}(\dot{\xi}_2)) = \text{false} \tag{7d}$$

$$\mathit{maysatisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inl}(\dot{\xi}_1)) = \text{false} \tag{7e}$$

$$\begin{aligned}\mathit{maysatisfy}((e_1, e_2), (\dot{\xi}_1, \dot{\xi}_2)) = {} & \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{satisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{satisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big) \\ & \text{or } \Big(\mathit{maysatisfy}(e_1, \dot{\xi}_1) \text{ and } \mathit{maysatisfy}(e_2, \dot{\xi}_2)\Big)\end{aligned} \tag{7f}$$

$$\begin{aligned}\mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = {} & \Big(\mathit{maysatisfy}(e, \dot{\xi}_1) \text{ and } \Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_2)\Big)\Big) \\ & \text{or } \Big(\Big(\text{not } \mathit{satisfy}(e, \dot{\xi}_1)\Big) \text{ and } \mathit{maysatisfy}(e, \dot{\xi}_2)\Big)\end{aligned} \tag{7g}$$

$$\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{notintro}(e) \text{ and } \mathit{refutable}_?(\dot{\xi}) \tag{7h}$$

Lemma 1.0.3 (Soundness and Completeness of Maybe Satisfaction Judgment)**.** *$e \models_? \dot{\xi}$ iff $\mathit{maysatisfy}(e, \dot{\xi}) = \mathit{true}$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

   (1) $e \models_? \dot{\xi}$ — by assumption

   By rule induction over Rules (18) on (1).

   Case (18a)**.**

   (2) $\dot{\xi} = ?$ — by assumption
   (3) $\mathit{maysatisfy}(e, ?) = \text{true}$ — by Definition 7a

   Case (18e)**.**

   (2) $e = \texttt{inl}_{\tau_2}(e_1)$ — by assumption
   (3) $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$ — by assumption
   (4) $e_1 \models_? \dot{\xi}_1$ — by assumption
   (5) $\mathit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ — by IH on (4)
   (6) $\mathit{maysatisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\dot{\xi}_1)) = \text{true}$ — by Definition 7b on (5)

   Case (18f)**.**

(2)	$e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(3)	$\dot{\xi} = \mathtt{inr}(\dot{\xi}_2)$	by assumption
(4)	$e_2 \models_? \dot{\xi}_2$	by assumption
(5)	$maysatisfy(e_2, \dot{\xi}_2) = \text{true}$	by IH on (4)
(6)	$maysatisfy(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\dot{\xi}_2)) = \text{true}$	by Definition 7c on (5)

Case (18g)**.**

(2)	$e = (e_1, e_2)$	by assumption
(3)	$\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$	by assumption
(4)	$e_1 \models_? \dot{\xi}_1$	by assumption
(5)	$e_2 \models \dot{\xi}_2$	by assumption
(6)	$maysatisfy(e_1, \dot{\xi}_1) = \text{true}$	by IH on (4)
(7)	$satisfy(e_2, \dot{\xi}_2) = \text{true}$	by Lemma 2.0.19 on (5)
(8)	$maysatisfy((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$	by Definition 7f on (6) and (7)

Case (18h)**.**

(2)	$e = (e_1, e_2)$	by assumption
(3)	$\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$	by assumption
(4)	$e_1 \models \dot{\xi}_1$	by assumption
(5)	$e_2 \models_? \dot{\xi}_2$	by assumption
(6)	$satisfy(e_1, \dot{\xi}_1) = \text{true}$	by Lemma 2.0.19 on (4)
(7)	$maysatisfy(e_2, \dot{\xi}_2) = \text{true}$	by IH on (5)
(8)	$maysatisfy((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$	by Definition 7f on (6) and (7)

Case (18i)**.**

(2)	$e = (e_1, e_2)$	by assumption
(3)	$\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$	by assumption
(4)	$e_1 \models_? \dot{\xi}_1$	by assumption
(5)	$e_2 \models_? \dot{\xi}_2$	by assumption
(6)	$maysatisfy(e_1, \dot{\xi}_1) = \text{true}$	by IH on (4)
(7)	$maysatisfy(e_2, \dot{\xi}_2) = \text{true}$	by IH on (5)
(8)	$maysatisfy((\dot{\xi}_1, \dot{\xi}_2), (\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$	by Definition 7f on (6) and (7)

Case (18c)**.**

(2)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3)	$e \models_? \dot{\xi}_1$	by assumption

(4)	$e \not\models \dot{\xi}_2$	by assumption
(5)	$\mathit{maysatisfy}(e, \dot{\xi}_1) = \text{true}$	by IH on (3)
(6)	$\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$	by Lemma 2.0.19 on (4)
(7)	$\mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{true}$	by Definition 17e on (5) and (6)

Case (18d)**.**

(2)	$\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$	by assumption
(3)	$e \not\models \dot{\xi}_1$	by assumption
(4)	$e \models_? \dot{\xi}_2$	by assumption
(5)	$\mathit{satisfy}(e, \dot{\xi}_1) = \text{false}$	by Lemma 2.0.19 on (3)
(6)	$\mathit{maysatisfy}(e, \dot{\xi}_2) = \text{true}$	by IH on (4)
(7)	$\mathit{maysatisfy}(e, \dot{\xi}_1 \vee \dot{\xi}_2) = \text{true}$	by Definition 17e on (5) and (6)

Case (18b)**.**

(2)	e `notintro`	by assumption
(3)	$\dot{\xi}$ $\texttt{refutable}_?$	by assumption
(4)	$\mathit{notintro}(e) = \text{true}$	by Lemma 4.0.1 on (2)
(5)	$\mathit{refutable}_?(\dot{\xi}) = \text{true}$	by Lemma 2.0.14 on (3)
(6)	$\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$	by Definition 7h on (4) and (5)

2. Completeness:

(1)	$\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$	by assumption

By structural induction on $\dot{\xi}$.

Case $\dot{\xi} = \top, \bot$**.**

(2)	$\mathit{refutable}_?(\dot{\xi}) = \text{false}$	by Definition 13
(3)	$\mathit{maysatisfy}(e, \dot{\xi}) = \text{false}$	by Definition 7h and (2)

Contradicts (1) and thus vacuously true.

Case $\dot{\xi} = ?$**.**

(2)	$e \models_? ?$	by Rule (18a)

Case $\dot{\xi} = \underline{n}$**.**

(2)	$\mathit{notintro}(e) = \text{true}$	by Definition 7h of (1)
(3)	e `notintro`	by Lemma 4.0.1 on (2)

(4) $\underline{n}$ $\texttt{refutable}_?$ by Rule (12a)

(5) $e \models_? \underline{n}$ by Rule (18b) on (3) and (4)

Case $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$**.**

By case analysis on Definition 7g of (1).

Case $\mathit{maysatisfy}(e, \dot{\xi}_1) = \text{true}$ and $\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$**.**

(2) $\mathit{maysatisfy}(e, \dot{\xi}_1) = \text{true}$	by assumption
(3) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$	by assumption
(4) $e \models_? \dot{\xi}_1$	by IH on (2)
(5) $e \not\models \dot{\xi}_2$	by Lemma 2.0.19 on (3)
(6) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (18c) on (4) and (5)

Case $\mathit{satisfy}(e, \dot{\xi}_1) = \text{false}$ and $\mathit{maysatisfy}(e, \dot{\xi}_2) = \text{true}$**.**

(2) $\mathit{satisfy}(e, \dot{\xi}_1) = \text{false}$	by assumption
(3) $\mathit{maysatisfy}(e, \dot{\xi}_2) = \text{true}$	by assumption
(4) $e \not\models \dot{\xi}_1$	by Lemma 2.0.19 on (2)
(5) $e \models_? \dot{\xi}_2$	by IH on (3)
(6) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (18d) on (4) and (5)

Case $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$**.**

By structural induction on e.

Case $e = (\!|\,|\!)^u, (\!|e'|\!)^u, e_1(e_2), \texttt{match}(e')\{\hat{rs}\}, \texttt{prl}(e'), \texttt{prr}(e')$**.**

(2) $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1)) = \text{true}$	by Definition 7h of (1)
(3) $\texttt{inl}(\dot{\xi}_1)$ $\texttt{refutable}_?$	by Lemma 2.0.14 on (2)
(4) e $\texttt{notintro}$	by Rules (28)
(5) $e \models_? \texttt{inl}(\dot{\xi}_1)$	by Rule (18b) on (4) and (3)

Case $e = x, \underline{n}, (\lambda x : \tau . e'), (e_1, e_2)$**.**

(2) $\mathit{notintro}(e) = \text{false}$	by Rules (28)
(3) $\mathit{maysatisfy}(e, \texttt{inl}(\dot{\xi}_1)) = \text{false}$	by Definition 7h on (2)

Contradicts (1).

Case $e = \texttt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{maysatisfy}(e_1, \dot{\xi}_1)$	by Definition 7b of (1)
(3) $e_1 \models_? \dot{\xi}_1$	by Lemma 1.0.3 on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$	by Rule (18e) on (3)

Case $e = \texttt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{maysatisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inl}(\dot{\xi}_1)) = \text{false}$ by Definition 7e

Contradicts (1).

Case $\dot{\xi} = \mathtt{inr}(\dot{\xi}_2)$**.**

By structural induction on e.

Case $e = (\!|\,|\!)^u, (\!|e'|\!)^u, e_1(e_2), \mathtt{match}(e')\{\hat{rs}\}, \mathtt{prl}(e'), \mathtt{prr}(e')$**.**

(2) $\mathit{refutable}_?(\mathtt{inr}(\dot{\xi}_2)) = \text{true}$ by Definition 7h of (1)

(3) $\mathtt{inr}(\dot{\xi}_2)\ \mathtt{refutable}_?$ by Lemma 2.0.14 on (2)

(4) $e\ \mathtt{notintro}$ by Rules (28)

(5) $e \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (18b) on (4) and (3)

Case $e = x, \underline{n}, (\lambda x : \tau.e'), (e_1, e_2)$**.**

(2) $\mathit{notintro}(e) = \text{false}$ by Rules (28)

(3) $\mathit{maysatisfy}(e, \mathtt{inr}(\dot{\xi}_2)) = \text{false}$ by Definition 7h on (2)

Contradicts (1).

Case $e = \mathtt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{maysatisfy}(\mathtt{inl}_{\tau_2}(e_1), \mathtt{inr}(\dot{\xi}_2)) = \text{false}$ by Definition 7d

Contradicts (1).

Case $e = \mathtt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{maysatisfy}(e_2, \dot{\xi}_2)$ by Definition 7c of (1)

(3) $e_2 \models_? \dot{\xi}_2$ by Lemma 1.0.3 on (2)

(4) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (18f) on (3)

Case $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$**.**

By structural induction on e.

Case $e = (\!|\,|\!)^u, (\!|e'|\!)^u, e_1(e_2), \mathtt{match}(e')\{\hat{rs}\}, \mathtt{prl}(e'), \mathtt{prr}(e')$**.**

(2) $\mathit{refutable}_?((\dot{\xi}_1, \dot{\xi}_2)) = \text{true}$ by Definition 7h of (1)

(3) $(\dot{\xi}_1, \dot{\xi}_2)\ \mathtt{refutable}_?$ by Lemma 2.0.14 on (2)

(4) $e\ \mathtt{notintro}$ by Rules (28)

(5) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (4) and (3)

Case $e = x, \underline{n}, (\lambda x : \tau.e'), \mathtt{inl}_{\tau_2}(e_1), \mathtt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{notintro}(e) = \text{false}$ by Rules (28)

(3) $\mathit{maysatisfy}(e, (\dot{\xi}_1, \dot{\xi}_2)) = \text{false}$ by Definition 7h on (2)

Contradicts (1).

Case $e = (e_1, e_2)$**.** By case analysis on Definition 7f on (1).

Case $\mathit{maysatisfy}(e_1, \dot{\xi}_1) = \text{true}$ and $\mathit{satisfy}(e_2, \dot{\xi}_2) = \text{true}$**.**

(2)	$maysatisfy(e_1, \dot{\xi}_1) = \text{true}$	by assumption
(3)	$satisfy(e_2, \dot{\xi}_2) = \text{true}$	by assumption
(4)	$e_1 \models_? \dot{\xi}_1$	by IH on (2)
(5)	$e_2 \models \dot{\xi}_2$	by Lemma 2.0.19 on (3)
(6)	$(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (18g) on (4) and (5)

Case $satisfy(e_1, \dot{\xi}_1) = \text{true}$ and $maysatisfy(e_2, \dot{\xi}_2) = \text{true}$**.**

(2)	$satisfy(e_1, \dot{\xi}_1)$	by assumption
(3)	$maysatisfy(e_2, \dot{\xi}_2)$	by assumption
(4)	$e_1 \models \dot{\xi}_1$	by Lemma 2.0.19 on (2)
(5)	$e_2 \models_? \dot{\xi}_2$	by IH on (3)
(6)	$(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (18h) on (4) and (5)

Case $maysatisfy(e_1, \dot{\xi}_1) = \text{true}$ and $maysatisfy(e_2, \dot{\xi}_2) = \text{true}$**.**

(2)	$maysatisfy(e_1, \dot{\xi}_1)$	by assumption
(3)	$maysatisfy(e_2, \dot{\xi}_2)$	by assumption
(4)	$e_1 \models_? \dot{\xi}_1$	by IH on (2)
(5)	$e_2 \models_? \dot{\xi}_2$	by IH on (3)
(6)	$(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$	by Rule (18i) on (4) and (5)

□

$\boxed{e \models_?^\dagger \dot{\xi}}$ e satisfies or may satisfy $\dot{\xi}$

$$\text{CSMSMay}\quad \frac{e \models_? \dot{\xi}}{e \models_?^\dagger \dot{\xi}} \tag{8a}$$

$$\text{CSMSSat}\quad \frac{e \models \dot{\xi}}{e \models_?^\dagger \dot{\xi}} \tag{8b}$$

$\boxed{satisfyormay(e, \dot{\xi})}$

$$satisfyormay(e, \dot{\xi}) = satisfy(e, \dot{\xi}) \text{ or } maysatisfy(e, \dot{\xi}) \tag{9}$$

Lemma 1.0.4 (Soundness and Completeness of Satisfaction or Maybe Satisfaction)**.** *$e \models_?^\dagger \dot{\xi}$ iff $satisfyormay(e, \dot{\xi})$.*

Proof. We prove soundness and completeness separately.

1. Soundness:

(1) $e \models^{\dagger}_{?} \dot{\xi}$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $e \models \dot{\xi}$ by assumption
(3) $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$ by Lemma 2.0.19 on (2)
(4) $\mathit{satisfyormay}(e, \dot{\xi}) = \text{true}$ by Definition 9 on (3)

Case (19a)**.**

(2) $e \models_{?} \dot{\xi}$ by assumption
(3) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$ by Lemma 1.0.3 on (2)
(4) $\mathit{satisfyormay}(e, \dot{\xi}) = \text{true}$ by Definition 9 on (3)

2. Completeness:

(1) $\mathit{satisfyormay}(e, \dot{\xi}) = \text{true}$ by assumption

By case analysis on Definition 9 of (1).

Case $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$**.**

(2) $\mathit{satisfy}(e, \dot{\xi}) = \text{true}$ by assumption
(3) $e \models \dot{\xi}$ by Lemma 2.0.19 on (2)
(4) $e \models^{\dagger}_{?} \dot{\xi}$ by Rule (19b) on (3)

Case $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$**.**

(2) $\mathit{maysatisfy}(e, \dot{\xi}) = \text{true}$ by assumption
(3) $e \models_{?} \dot{\xi}$ by Lemma 1.0.3 on (2)
(4) $e \models^{\dagger}_{?} \dot{\xi}$ by Rule (19a) on (3)

□

Lemma 1.0.5. $e \not\models \bot$

Proof. By rule induction over Rules (16), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. □

Lemma 1.0.6. $e \not\models_{?} \bot$

Proof. Assume $e \models_{?} \bot$. By rule induction over Rules (18) on $e \models_{?} \bot$, only one case applies.

Case (18b)**.**

(1) $\bot$ $\texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. $\square$

Lemma 1.0.7. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (18) on $e \models_? \top$, only one case applies.

Case (18b)**.**

(1) $\top$ $\texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. $\square$

Lemma 1.0.8. $e \not\models ?$

Proof. By rule induction over Rules (16), we notice that $e \models ?$ is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 1.0.9. $e \models_?^\dagger \dot{\xi}$ *iff* $e \models_?^\dagger \dot{\xi} \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

   (1) $e \models_?^\dagger \dot{\xi}$ by assumption

   By rule induction over Rules (19) on (1).

   Case (19a)**.**

(2)	$e \models_? \dot{\xi}$	by assumption
(3)	$e \models_? \dot{\xi} \vee \bot$	by Rule (18c) on (2) and Lemma 2.0.1
(4)	$e \models_?^\dagger \dot{\xi} \vee \bot$	by Rule (19a) on (3)

   Case (19b)**.**

(2)	$e \models \dot{\xi}$	by assumption
(3)	$e \models \dot{\xi} \vee \bot$	by Rule (16e) on (2)
(4)	$e \models_?^\dagger \dot{\xi} \vee \bot$	by Rule (19b) on (3)

2. Necessity:

(1) $e \models^{\dagger}_{?} \dot{\xi} \vee \bot$ by assumption

By rule induction over Rules (19) on (1).

Case (19a).

(2) $e \models_{?} \dot{\xi} \vee \bot$ by assumption

By rule induction over Rules (18) on (2), only two of them apply.

Case (18c).

(3) $e \models_{?} \dot{\xi}$ by assumption

(4) $e \models^{\dagger}_{?} \dot{\xi}$ by Rule (19a) on (3)

Case (18d).

(3) $e \models_{?} \bot$ by assumption

(4) $e \not\models_{?} \bot$ by Lemma 2.0.2

(3) contradicts (4).

Case (19b).

(2) $e \models \dot{\xi} \vee \bot$ by assumption

By rule induction over Rules (16) on (2), only two of them apply.

Case (16e).

(3) $e \models \dot{\xi}$ by assumption

(4) $e \models^{\dagger}_{?} \dot{\xi}$ by Rule (19b) on (3)

Case (16f).

(3) $e \models \bot$ by assumption

(4) $e \not\models \bot$ by Lemma 2.0.1

(3) contradicts (4).

□

Corollary 1.0.1. $\top \models^{\dagger}_{?} \dot{\xi}$ *iff* $\top \models^{\dagger}_{?} \dot{\xi} \vee \bot$

Proof. Follows directly from Definition 2.1.2 and Lemma 2.0.5. □

Lemma 1.0.10. *Suppose that* $\dot{\xi}_1 : \tau$ *and* $\dot{\xi}_2 : \tau$. *Then* $\dot{\xi}_1 \not\models \dot{\xi}_2$ *iff* $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \bot$

Proof.

(1) $\dot{\xi}_1 : \tau$ by assumption

(2) $\dot{\xi}_2 : \tau$ by assumption

(3) $\bot : \tau$ by Rule (10b)

(4) $\dot{\xi}_2 \vee \bot : \tau$ by Rule (10f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

   (5) $\dot{\xi}_1 \not\models \dot{\xi}_2$ by assumption

   To prove $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \bot$, assume $\dot{\xi}_1 \models \dot{\xi}_2 \vee \bot$.

   (6) $\dot{\xi}_1 \models \dot{\xi}_2 \vee \bot$ by assumption

   For all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \dot{\xi}_1$ implies

   (7) $e \models \dot{\xi}_2 \vee \bot$ by Definition 2.1.1 on (1) and (4) and (6)

   By rule induction over Rules (16) on (7).

   Case (16e).

   (8) $e \models \dot{\xi}_2$ by assumption
   (9) $\dot{\xi}_1 \models \dot{\xi}_2$ by Definition 2.1.1 on (8)

   (5) contradicts (9).

   Case (16f).

   (8) $e \models \bot$ by assumption
   (9) $e \not\models \bot$ by Lemma 2.0.1

   (8) contradicts (9).

   The conclusion holds as follows:

   (a) $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \bot$

2. Necessity:

   (5) $\dot{\xi}_1 \not\models \dot{\xi}_2 \vee \bot$ by assumption

   To prove $\dot{\xi}_1 \not\models \dot{\xi}_2$, assume $\dot{\xi}_1 \models \dot{\xi}_2$.

   (6) $\dot{\xi}_1 \models \dot{\xi}_2$ by assumption

   For all e such that $\cdot ; \Delta \vdash e : \tau$ and e `final` we have $e \models^{\dagger}_{?} \dot{\xi}_1$ implies

   (7) $e \models \dot{\xi}_2$ by Definition 2.1.1 on (1) and (2) and (6)

   (8) $e \models \dot{\xi}_2 \vee \bot$ by Rule (16e) on (7)

(9) $\dot{\xi}_1 \models \dot{\xi}_2 \vee \bot$ by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

(a) $\dot{\xi}_1 \not\models \dot{\xi}_2$

$\square$

Lemma 1.0.11. *$e \not\models^{\dagger}_{?} \dot{\xi}_1$ and $e \not\models^{\dagger}_{?} \dot{\xi}_2$ iff $e \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency: to show $e \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$, we assume $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$.

(1) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \dot{\xi}_1$ by assumption

(3) $e \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(4) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By rule induction over Rules (16) on (4) and only two of them apply.

Case (16e)**.**

(5) $e \models \dot{\xi}_1$ by assumption

(6) $e \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (19b) on (5)

(6) contradicts (2).

Case (16f)**.**

(5) $e \models \dot{\xi}_2$ by assumption

(6) $e \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (19b) on (5)

(6) contradicts (3).

Case (19a)**.**

(4) $e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By rule induction over Rules (18) on (4) and only two of them apply.

Case (18c)**.**

(5) $e \models_{?} \dot{\xi}_1$ by assumption

(6) $e \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (19a) on (5)

(6) contradicts (2).

Case (18d)**.**

(5) $e \models_? \dot{\xi}_2$ by assumption

(6) $e \models_?^\dagger \dot{\xi}_2$ by Rule (19a) on (5)

(6) contradicts (3).

The conclusion holds as follows:

(a) $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$

2. Necessity:

(1) $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

We show $e \not\models_?^\dagger \dot{\xi}_1$ and $e \not\models_?^\dagger \dot{\xi}_2$ separately.

(a) To show $e \not\models_?^\dagger \dot{\xi}_1$, we assume $e \models_?^\dagger \dot{\xi}_1$.

(2) $e \models_?^\dagger \dot{\xi}_1$ by assumption

(3) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Lemma 2.0.10 on (2)

Contradicts (1).

(b) To show $e \not\models_?^\dagger \dot{\xi}_2$, we assume $e \models_?^\dagger \dot{\xi}_2$.

(2) $e \models_?^\dagger \dot{\xi}_2$ by assumption

(3) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Lemma 2.0.10 on (2)

Contradicts (1).

In conclusion, $e \not\models_?^\dagger \dot{\xi}_1$ and $e \not\models_?^\dagger \dot{\xi}_2$.

□

Lemma 1.0.12. *If $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ and $e \not\models_?^\dagger \dot{\xi}_1$ then $e \models_?^\dagger \dot{\xi}_2$*

Proof.

(4) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(5) $e \not\models_?^\dagger \dot{\xi}_1$ by assumption

By rule induction over Rules (19) on (4).

Case (19b)**.**

(6) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By rule induction over Rules (16) on (6) and only two of them apply.

Case (16e)**.**

(7) $e \models \dot{\xi}_1$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (19b) on (7)

(8) contradicts (5).

Case (16f)**.**

(7) $e \models \dot{\xi}_2$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (19b) on (7)

Case (19a)**.**

(6) $e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

By rule induction over Rules (18) on (6) and only two of them apply.

Case (18c)**.**

(7) $e \models_{?} \dot{\xi}_1$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (19a) on (7)

(8) contradicts (5).

Case (18d)**.**

(7) $e \models_{?} \dot{\xi}_2$ by assumption

(8) $e \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (19a) on (7)

$\square$

Lemma 1.0.13. *If* $e \models^{\dagger}_{?} \dot{\xi}_1$ *then* $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ *and* $e \models^{\dagger}_{?} \dot{\xi}_2 \vee \dot{\xi}_1$

Proof.

(1) $e \models^{\dagger}_{?} \dot{\xi}_1$ by assumption ,

By rule induction over Rules (19) on (1),

Case (19b)**.**

(2) $e \models \dot{\xi}_1$ by assumption

(3) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16e) on (2)

(4) $e \models \dot{\xi}_2 \vee \dot{\xi}_1$ by Rule (16f) on (2)

(5) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19b) on (3)

(6) $e \models^{\dagger}_{?} \dot{\xi}_2 \vee \dot{\xi}_1$ by Rule (19b) on (4)

Case (19a)**.**

(2) $e \models_{?} \dot{\xi}_1$ by assumption

By case analysis on the result of $\mathit{satisfy}(e, \dot{\xi}_2)$.

Case true**.**

(3) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{true}$	by assumption
(4) $e \models \dot{\xi}_2$	by Lemma 2.0.19 on (3)
(5) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (16f) on (4)
(6) $e \models \dot{\xi}_2 \vee \dot{\xi}_1$	by Rule (16e) on (4)
(7) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (19b) on (5)
(8) $e \models_?^\dagger \dot{\xi}_2 \vee \dot{\xi}_1$	by Rule (19b) on (6)

Case false**.**

(3) $\mathit{satisfy}(e, \dot{\xi}_2) = \text{false}$	by assumption
(4) $e \not\models \dot{\xi}_2$	by Lemma 2.0.19 on (3)
(5) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (18c) on (2) and (4)
(6) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$	by Rule (19a) on (5)

□

Lemma 1.0.14. $e_1 \models_?^\dagger \dot{\xi}_1$ *iff* $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e_1 \models_?^\dagger \dot{\xi}_1$	by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $e_1 \models \dot{\xi}_1$	by assumption
(3) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\xi}_1)$	by Rule (16g) on (2)
(4) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$	by Rule (19b) on (3)

Case (19a)**.**

(2) $e_1 \models_? \dot{\xi}_1$	by assumption
(3) $\mathtt{inl}_{\tau_2}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$	by Rule (18e) on (2)
(4) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$	by Rule (19a) on (3)

2. Necessity:

(1) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (16) on (2), only one rule applies.

Case (16g)**.**

(3) $e_1 \models \dot{\xi}_1$ by assumption

(4) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (19b) on (3)

Case (19a)**.**

(2) $\texttt{inl}_{\tau_2}(e_1) \models_{?} \texttt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (18) on (2), only two rules apply.

Case (18e)**.**

(3) $e_1 \models_{?} \dot{\xi}_1$ by assumption

(4) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by Rule (19a) on (3)

Case (18b)**.**

(3) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (28) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.15. $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ *iff* $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $e_2 \models \dot{\xi}_2$ by assumption

(3) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by Rule (16h) on (2)

(4) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (19b) on (3)

Case (19a)**.**

(2) $e_2 \models_{?} \dot{\xi}_2$ by assumption

(3) $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (18f) on (2)

(4) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (19a) on (3)

2. Necessity:

(1) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (16) on (2), only one rule applies.

Case (16h)**.**

(3) $e_2 \models \dot{\xi}_2$ by assumption

(4) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (19b) on (3)

Case (19a)**.**

(2) $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (18) on (2), only two rules apply.

Case (18f)**.**

(3) $e_2 \models_{?} \dot{\xi}_2$ by assumption

(4) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by Rule (19a) on (3)

Case (18b)**.**

(3) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (28) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.16. *$e_1 \models^{\dagger}_{?} \dot{\xi}_1$ and $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ iff $(e_1, e_2) \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e_1 \models^{\dagger}_{?} \dot{\xi}_1$ by assumption

(2) $e_2 \models^{\dagger}_{?} \dot{\xi}_2$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(3) $e_1 \models \dot{\xi}_1$ by assumption

By rule induction over Rules (19) on (2).

Case (19b)**.**

(4) $e_2 \models \dot{\xi}_2$ — by assumption

(5) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (16i) on (3) and (4)

(6) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (19b) on (5)

Case (19a)**.**

(4) $e_2 \models_? \dot{\xi}_2$ — by assumption

(5) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (18h) on (3) and (4)

(6) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (19a) on (5)

Case (19a)**.**

(4) $e_1 \models_? \dot{\xi}_1$ — by assumption

By rule induction over Rules (19) on (2).

Case (19b)**.**

(5) $e_2 \models \dot{\xi}_2$ — by assumption

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (18g) on (4) and (5)

(7) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (19a) on (6)

Case (19a)**.**

(5) $e_2 \models_? \dot{\xi}_2$ — by assumption

(6) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (18h) on (4) and (5)

(7) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (19a) on (6)

2. Necessity:

(1) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ — by assumption

By rule induction over Rules (16) on (2), only one rule applies.

Case (16i)**.**

(3) $e_1 \models \dot{\xi}_1$ — by assumption

(4) $e_2 \models \dot{\xi}_2$ — by assumption

(5) $e_1 \models_?^\dagger \dot{\xi}_1$ — by Rule (19b) on (3)

(6) $e_2 \models_?^\dagger \dot{\xi}_2$ — by Rule (19b) on (4)

Case (19a)**.**

(2) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

By rule induction over Rules (18) on (2), only three rules apply.

Case (18g)**.**

(3) $e_1 \models_? \dot{\xi}_1$ by assumption
(4) $e_2 \models \dot{\xi}_2$ by assumption
(5) $e_1 \models_?^\dagger \dot{\xi}_1$ by Rule (19a) on (3)
(6) $e_2 \models_?^\dagger \dot{\xi}_2$ by Rule (19b) on (4)

Case (18h)**.**

(3) $e_1 \models \dot{\xi}_1$ by assumption
(4) $e_2 \models_? \dot{\xi}_2$ by assumption
(5) $e_1 \models_?^\dagger \dot{\xi}_1$ by Rule (19b) on (3)
(6) $e_2 \models_?^\dagger \dot{\xi}_2$ by Rule (19a) on (4)

Case (18i)**.**

(3) $e_1 \models_? \dot{\xi}_1$ by assumption
(4) $e_2 \models_? \dot{\xi}_2$ by assumption
(5) $e_1 \models_?^\dagger \dot{\xi}_1$ by Rule (19a) on (3)
(6) $e_2 \models_?^\dagger \dot{\xi}_2$ by Rule (19a) on (4)

□

Lemma 1.0.17. *If e `notintro` and $e \models_? \xi$ then ξ `refutable`$_?$.*

Lemma 1.0.18. *There does not exist such a constraint $\dot{\xi}_1 \wedge \dot{\xi}_2$ such that $\dot{\xi}_1 \wedge \dot{\xi}_2$ `refutable`$_?$.*

Proof. By rule induction over Rules (12), we notice that $\dot{\xi}_1 \wedge \dot{\xi}_2$ `refutable`$_?$ is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.19. *There does not exist such a constraint $\dot{\xi}_1 \vee \dot{\xi}_2$ such that $\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_?$.*

Proof. By rule induction over Rules (12), we notice that $\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_?$ is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 1.0.20. *If e `notintro` and $e \models \dot{\xi}$ then $\cancel{\dot{\xi}\ \texttt{refutable}_?}$.*

Proof.

(1) e `notintro` by assumption
(2) $e \models \dot{\xi}$ by assumption

By rule induction over Rules (16) on (2).

Case (16a)**.**

(3) $\dot{\xi} = \top$ — by assumption

Assume $\top$ $\texttt{refutable}_?$. By rule induction over Rules (12), no case applies due to syntactic contradiction.
Therefore, $\cancel{\top\ \texttt{refutable}_?}$.

Case (16e),(16f).

(3) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ — by assumption

(4) $\cancel{\dot{\xi}_1 \vee \dot{\xi}_2\ \texttt{refutable}_?}$ — by Lemma 2.0.17

Case (16d).

(3) $\dot{\xi} = \dot{\xi}_1 \wedge \dot{\xi}_2$ — by assumption

(4) $\cancel{\dot{\xi}_1 \wedge \dot{\xi}_2\ \texttt{refutable}_?}$ — by Lemma 2.0.16

Case (16j).

(3) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ — by assumption

(4) $\texttt{prl}(e) \models \dot{\xi}_1$ — by assumption

(5) $\texttt{prr}(e) \models \dot{\xi}_2$ — by assumption

(6) $\texttt{prl}(e)$ $\texttt{notintro}$ — by Rule (28e)

(7) $\texttt{prr}(e)$ $\texttt{notintro}$ — by Rule (28f)

(8) $\cancel{\dot{\xi}_1\ \texttt{refutable}_?}$ — by IH on (6) and (4)

(9) $\cancel{\dot{\xi}_2\ \texttt{refutable}_?}$ — by IH on (7) and (5)

Assume $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$. By rule induction over Rules (12) on it, only two cases apply.

Case (12d).

(10) $\dot{\xi}_1$ $\texttt{refutable}_?$ — by assumption

Contradicts (8).

Case (12e).

(10) $\dot{\xi}_2$ $\texttt{refutable}_?$ — by assumption

Contradicts (9).

Therefore, $\cancel{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}$.

Otherwise.

(3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ — by assumption

By rule induction over Rules (28) on (1), no case applies due to syntactic contradiction.

□

Lemma 1.0.21. $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ *is not derivable.*

Proof. We prove by assuming $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ and obtaining a contradiction.

(1) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (16) on (2), no rule applies due to syntactic contradiction.

Case (19a)**.**

(2) $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inr}(\dot{\xi}_2)$ by assumption

By rule induction over Rules (18) on (2), only one rule applies.

Case (18b)**.**

(3) $\mathtt{inl}_{\tau_2}(e_1)\ \mathtt{notintro}$ by assumption

By rule induction over Rules (28) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.22. $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inl}(\dot{\xi}_1)$ *is not derivable.*

Proof. We prove by assuming $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inl}(\dot{\xi}_1)$ and obtaining a contradiction.

(1) $\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(2) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (16) on (2), no rule applies due to syntactic contradiction.

Case (19a)**.**

(2) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inl}(\dot{\xi}_1)$ by assumption

By rule induction over Rules (18) on (2), only one rule applies.

Case (18b)**.**

(3) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (28) on (3), no rule applies due to syntactic contradiction.

□

Lemma 1.0.23. *$e \not\models \dot{\xi}$ and $e \not\models_? \dot{\xi}$ iff $e \not\models_?^\dagger \dot{\xi}$.*

Proof. 1. Sufficiency:

(1) $e \not\models \dot{\xi}$ by assumption

(2) $e \not\models_? \dot{\xi}$ by assumption

Assume $e \models_?^\dagger \dot{\xi}$. By rule induction over Rules (19) on it.

Case (19a)**.**

(3) $e \models \dot{\xi}$ by assumption

Contradicts (1).

Case (19b)**.**

(3) $e \models_? \dot{\xi}$ by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \dot{\xi}$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \dot{\xi}$ by assumption

Assume $e \models \dot{\xi}$.

(2) $e \models_?^\dagger \dot{\xi}$ by Rule (19b) on assumption

Contradicts (1). Therefore, $e \not\models \dot{\xi}$. Assume $e \models_? \dot{\xi}$.

(3) $e \models_?^\dagger \dot{\xi}$ by Rule (19a) on assumption

Contradicts (1). Therefore, $e \not\models_? \dot{\xi}$.

□

Theorem 1.1 (Exclusiveness of Satisfaction Judgment). *If $\dot{\xi} : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \models \dot{\xi}$
2. $e \models_{?} \dot{\xi}$
3. $e \not\models_{?}^{\dagger} \dot{\xi}$

Proof.

(4) $\dot{\xi} : \tau$	by assumption
(5) $\cdot\,;\Delta \vdash e : \tau$	by assumption
(6) e `final`	by assumption

By rule induction over Rules (10) on (4), we would show one conclusion is derivable while the other two are not.

Case (10a).

(7) $\dot{\xi} = \top$	by assumption
(8) $e \models \top$	by Rule (16a)
(9) $e \not\models_{?} \top$	by Lemma 2.0.3
(10) $e \models_{?}^{\dagger} \top$	by Rule (19b) on (8)

Case (10b).

(7) $\dot{\xi} = \bot$	by assumption
(8) $e \not\models \bot$	by Lemma 2.0.1
(9) $e \not\models_{?} \bot$	by Lemma 2.0.2
(10) $e \not\models_{?}^{\dagger} \bot$	by Lemma 2.0.20 on (8) and (9)

Case (1b).

(7) $\dot{\xi} = ?$	by assumption
(8) $e \not\models ?$	by Lemma 2.0.4
(9) $e \models_{?} ?$	by Rule (18a)
(10) $e \models_{?}^{\dagger} ?$	by Rule (19a) on (9)

Case (10c).

(7) $\dot{\xi} = \underline{n_2}$ by assumption

(8) $\tau = \texttt{num}$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

Assume $e \models \underline{n_2}$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction on $\dot{\xi}$.

(11) $e \not\models \underline{n_2}$ by contradiction

(12) $\underline{n_2}$ $\texttt{refutable}_?$ by Rule (12a)

(13) $e \models_? \underline{n_2}$ by Rule (18b) on (10) and (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (19a) on (13)

Case (21d).

(9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \models_? \underline{n_2}$. By rule induction over Rules (18), only one case applies.

Case (18b).

(10) $\underline{n_1}$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.6.

(11) $\underline{n_1} \not\models_? \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 17

(13) $\underline{n_1} \models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \models_?^\dagger \underline{n_2}$ by Rule (19b) on (13)

Case $n_1 \neq n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 17

(13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \not\models_?^\dagger \underline{n_2}$ by Lemma 2.0.20 on (11) and (13)

Case (10f).

(7) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ — by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \dot{\xi}_1$, $e \models_? \dot{\xi}_1$, and $e \not\models_?^{\dagger} \dot{\xi}_1$ holds. The same goes for $\dot{\xi}_2$. By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $e \models \dot{\xi}_1, e \models \dot{\xi}_2$.

(8) $e \models \dot{\xi}_1$ — by assumption
(9) $e \not\models_? \dot{\xi}_1$ — by assumption
(10) $e \models \dot{\xi}_2$ — by assumption
(11) $e \not\models_? \dot{\xi}_2$ — by assumption
(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (16e) on (8)
(13) $e \models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (19b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_?$ — by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_? \dot{\xi}_1$ — by assumption

Contradicts (9).

Case (18d).

(14) $e \models_? \dot{\xi}_2$ — by assumption

Contradicts (11).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ — by contradiction

Case $e \models \dot{\xi}_1, e \models_? \dot{\xi}_2$.

(8) $e \models \dot{\xi}_1$ — by assumption
(9) $e \not\models_? \dot{\xi}_1$ — by assumption
(10) $e \not\models \dot{\xi}_2$ — by assumption
(11) $e \models_? \dot{\xi}_2$ — by assumption
(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (16e) on (8)
(13) $e \models_?^{\dagger} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Rule (19b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_? \dot{\xi}_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \not\models \dot{\xi}_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models \dot{\xi}_1, e \not\models_?^\dagger \dot{\xi}_2$**.**

(8) $e \models \dot{\xi}_1$ by assumption
(9) $e \not\models_? \dot{\xi}_1$ by assumption
(10) $e \not\models \dot{\xi}_2$ by assumption
(11) $e \not\models_? \dot{\xi}_2$ by assumption
(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16e) on (8)
(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_? \dot{\xi}_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \not\models \dot{\xi}_1$ by assumption

Contradicts (8).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models_? \dot{\xi}_1, e \models \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ by assumption
(9) $e \models_? \dot{\xi}_1$ by assumption
(10) $e \models \dot{\xi}_2$ by assumption
(11) $e \not\models_? \dot{\xi}_2$ by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16f) on (10)

(13) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19b) on (12)

Assume $e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(14) $\dot{\xi}_1 \vee \dot{\xi}_2$ `refutable`$_{?}$ by assumption

Contradicts Lemma 2.0.17.

Case (18c)**.**

(14) $e \not\models \dot{\xi}_2$ by assumption

Contradicts (10).

Case (18d)**.**

(14) $e \models_{?} \dot{\xi}_2$ by assumption

Contradicts (11).

(15) $e \not\models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models_{?} \dot{\xi}_1, e \models_{?} \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \models_{?} \dot{\xi}_1$ by assumption

(10) $e \not\models \dot{\xi}_2$ by assumption

(11) $e \models_{?} \dot{\xi}_2$ by assumption

(12) $e \models_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (18c) on (9) and (10)

(13) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16), only two cases apply.

Case (16e)**.**

(14) $e \models \dot{\xi}_1$ by assumption

Contradicts (8)

Case (16f)**.**

(14) $e \models \dot{\xi}_2$ by assumption

Contradicts (10)

(15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \models_{?} \dot{\xi}_1, e \not\models^{\dagger}_{?} \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \models_? \dot{\xi}_1$ by assumption

(10) $e \not\models \dot{\xi}_2$ by assumption

(11) $e \not\models_? \dot{\xi}_2$ by assumption

(12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (18c) on (9) and (10)

(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16), only two cases apply.

Case (16e)**.**

(14) $e \models \dot{\xi}_1$ by assumption

Contradicts (8).

Case (16f)**.**

(14) $e \models \dot{\xi}_2$ by assumption

Contradicts (10).

(15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models_?^\dagger \dot{\xi}_1, e \models \dot{\xi}_2$**.**

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \not\models_? \dot{\xi}_1$ by assumption

(10) $e \models \dot{\xi}_2$ by assumption

(11) $e \not\models_? \dot{\xi}_2$ by assumption

(12) $e \models \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (16f) on (10)

(13) $e \models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19b) on (12)

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(14) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

Case (18c)**.**

(14) $e \not\models \dot{\xi}_2$ by assumption

Contradicts (10).

Case (18d)**.**

(14) $e \models_? \dot{\xi}_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \dot{\xi}_1, e \models_? \dot{\xi}_2$.

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \not\models_? \dot{\xi}_1$ by assumption

(10) $e \not\models \dot{\xi}_2$ by assumption

(11) $e \models_? \dot{\xi}_2$ by assumption

(12) $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (18d) on (11) and (8)

(13) $e \models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by Rule (19a) on (12)

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16), only two cases apply.

Case (16e).

(14) $e \models \dot{\xi}_1$ by assumption

Contradicts (8)

Case (16f).

(14) $e \models \dot{\xi}_2$ by assumption

Contradicts (10)

(15) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Case $e \not\models^{\dagger}_{?} \dot{\xi}_1, e \not\models^{\dagger}_{?} \dot{\xi}_2$.

(8) $e \not\models \dot{\xi}_1$ by assumption

(9) $e \not\models_? \dot{\xi}_1$ by assumption

(10) $e \not\models \dot{\xi}_2$ by assumption

(11) $e \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (16) on it, only two cases apply.

Case (16e).

(12) $e \models \dot{\xi}_1$ by assumption

Contradicts (8).

Case (16f).

(12) $e \models \dot{\xi}_2$ by assumption

Contradicts (10).

(13) $e \not\models \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

Assume $e \models_? \dot{\xi}_1 \vee \dot{\xi}_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\dot{\xi}_1 \vee \dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_? \dot{\xi}_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \models_? \dot{\xi}_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \dot{\xi}_1 \vee \dot{\xi}_2$ by contradiction

(16) $e \not\models_?^\dagger \dot{\xi}_1 \vee \dot{\xi}_2$ by Lemma 2.0.20 on (13) and (15)

Case (10g).

(7) $\dot{\xi} = \texttt{inl}(\dot{\xi}_1)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\dot{\xi}_1 : \tau_1$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(10) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

Assume $e \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

By case analysis on the value of $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1))$.

Case $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1)) = \text{true}$.

(13) $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1)) = \text{true}$ by assumption

(14) $\texttt{inl}(\dot{\xi}_1)$ $\texttt{refutable}_?$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inl}(\dot{\xi}_1)$ by Rule (18b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Rule (19a) on (15)

Case $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1)) = \text{false}$.

(13) $\mathit{refutable}_?(\texttt{inl}(\dot{\xi}_1)) = \text{false}$ by assumption

(14) $\cancel{\texttt{inl}(\dot{\xi}_1)\ \texttt{refutable}_?}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b)**.**

(15) $\texttt{inl}(\dot{\xi}_1)\ \texttt{refutable}_?$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inl}(\dot{\xi}_1)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on (12) and (16)

Case (21j)**.**

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(11) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(12) $e_1\ \texttt{final}$ by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \dot{\xi}_1$, $e_1 \models_? \dot{\xi}_1$, and $e_1 \not\models_?^\dagger \dot{\xi}_1$ holds. By case analysis on which one holds.

Case $e_1 \models \dot{\xi}_1$**.**

(13) $e_1 \models \dot{\xi}_1$ by assumption

(14) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$ by Rule (16g) on (13)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Rule (19b) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (18) on it, only two cases apply.

Case (18b)**.**

(17) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18e)**.**

(17) $e_1 \models_? \dot{\xi}_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\dot{\xi}_1)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1$**.**

(13) $e_1 \not\models \dot{\xi}_1$ by assumption

(14) $e_1 \models_? \dot{\xi}_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\dot{\xi}_1)$ by Rule (18e) on (14)

(16) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Rule (19a) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16g)**.**

(17) $e_1 \models \dot{\xi}_1$

Contradicts (13).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

Case $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$**.**

(13) $e_1 \not\models \dot{\xi}_1$ by assumption

(14) $e_1 \not\models_{?} \dot{\xi}_1$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16g)**.**

(15) $e_1 \models \dot{\xi}_1$

Contradicts (13).

(16) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_{?} \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b)**.**

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18e)**.**

(17) $e_1 \models_{?} \dot{\xi}_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_{?} \texttt{inl}(\dot{\xi}_1)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on (16) and (18)

Case (21k)**.**

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inl}(\dot{\xi}_1)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inl}(\dot{\xi}_1)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(12) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inl}(\dot{\xi}_1)$ by contradiction

(14) $\texttt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on (11) and (13)

Case (10h).

(7) $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(10) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

Assume $e \models \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inr}(\dot{\xi}_2)$ by contradiction

By case analysis on the value of $\mathit{refutable}_?(\texttt{inr}(\dot{\xi}_2))$. inr is refutable

Case $\mathit{refutable}_?(\texttt{inr}(\dot{\xi}_2)) = \text{true}$.

(13) $\mathit{refutable}_?(\texttt{inr}(\dot{\xi}_2)) = \text{true}$ by assumption

(14) $\texttt{inr}(\dot{\xi}_2)\ \texttt{refutable}_?$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inr}(\dot{\xi}_2)$ by Rule (18b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Rule (19a) on (15)

Case $\mathit{refutable}_?(\texttt{inr}(\dot{\xi}_2)) = \text{false}$.

(13) $\mathit{refutable}_?(\texttt{inr}(\dot{\xi}_2)) = \text{false}$ by assumption

(14) $\cancel{\texttt{inr}(\dot{\xi}_2)\ \texttt{refutable}_?}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(15) $\texttt{inr}(\dot{\xi}_2)\ \texttt{refutable}_?$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inr}(\dot{\xi}_2)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Lemma 2.0.20 on (12) and (16)

Case (21j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inr}(\dot{\xi}_2)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(12) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inr}(\dot{\xi}_2)$ by contradiction

(14) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Lemma 2.0.20 on (11) and (13)

Case (21k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(12) $e_2\ \texttt{final}$ by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \dot{\xi}_2$, $e_2 \models_? \dot{\xi}_2$, and $e_2 \not\models_?^\dagger \dot{\xi}_2$ holds. By case analysis on which one holds.

Case $e_2 \models \dot{\xi}_2$.

(13) $e_2 \models \dot{\xi}_2$ by assumption

(14) $e_2 \not\models_? \dot{\xi}_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\dot{\xi}_2)$ by Rule (16g) on (13)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Rule (19b) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\dot{\xi}_2)$. By rule induction over Rules (18) on it, only two cases apply.

Case (18b).

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18f).

(17) $e_2 \models_? \dot{\xi}_2$

Contradicts (14).

(18) $\mathtt{inr}_{\tau_1}(e_2) \not\models_? \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Case $e_2 \models_? \dot{\xi}_2$.

(13) $e_2 \not\models \dot{\xi}_2$ by assumption

(14) $e_2 \models_? \dot{\xi}_2$ by assumption

(15) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$ by Rule (18f) on (14)

(16) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$ by Rule (19a) on (15)

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16h).

(17) $e_2 \models \dot{\xi}_2$

Contradicts (13).

(18) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Case $e_2 \not\models_?^\dagger \dot{\xi}_2$.

(13) $e_2 \not\models \dot{\xi}_2$ by assumption

(14) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16h).

(15) $e_2 \models \dot{\xi}_2$

Contradicts (13).

(16) $\mathtt{inr}_{\tau_1}(e_2) \not\models \mathtt{inr}(\dot{\xi}_2)$ by contradiction

Assume $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(17) $\mathtt{inr}_{\tau_1}(e_2)$ $\mathtt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18f).

(17) $e_2 \models_? \dot{\xi}_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inr}(\dot{\xi}_2)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_?^\dagger \texttt{inl}(\dot{\xi}_1)$ by Lemma 2.0.20 on (16) and (18)

Case (16i).

(7) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\dot{\xi}_1 : \tau_1$ by assumption

(10) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(11) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(13) e $\texttt{indet}$ by Lemma 4.0.10 on (6) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (26g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (27b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (26h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (27b) on (16)

(18) $\cdot\,;\Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (21h) on (5)

(19) $\cdot\,;\Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (21i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\texttt{prl}(e) \models \dot{\xi}_1$, $\texttt{prl}(e) \models_? \dot{\xi}_1$, and $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\texttt{prr}(e) \models \dot{\xi}_2$, $\texttt{prr}(e) \models_? \dot{\xi}_2$, and $\texttt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$ holds.
By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $\texttt{prl}(e) \models \dot{\xi}_1, \texttt{prr}(e) \models \dot{\xi}_2$.

(20) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

(24) $e \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16j) on (12) and (20) and (22)

(25) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19b) on (24)

(26) $\cancel{(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?}$ by Lemma 2.0.18 on (12) and (24)

Assume $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(27) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by assumption

Contradicts (26).

(28) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $\texttt{prl}(e) \models \dot{\xi}_1, \texttt{prr}(e) \models_? \dot{\xi}_2$.

(20) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

Contradicts (22)

(25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

By rule induction over Rules (18) on (23), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) $\dot{\xi}_2\ \texttt{refutable}_?$ by assumption

(27) $(\dot{\xi}_1, \dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \models \dot{\xi}_1, \texttt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$.

(20) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16j).

(24) $\texttt{prr}(e) \models \dot{\xi}_2$ by assumption

Contradicts (22).

(25) $e \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

Assume $e \models_? (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(26) $(\dot\xi_1, \dot\xi_2)$ `refutable`$_?$ by assumption

By rule induction over Rules (12) on (26), only two cases apply.

Case (12d).

(27) $\dot\xi_1$ `refutable`$_?$ by assumption

(28) `prl`(e) `notintro` by Rule (28e)

(29) `prl`$(e) \models_? \dot\xi_1$ by Rule (18b) on (28) and (27)

Contradicts (21).

Case (12e).

(27) $\dot\xi_2$ `refutable`$_?$ by assumption

(28) `prr`(e) `notintro` by Rule (28f)

(29) `prr`$(e) \models_? \dot\xi_2$ by Rule (18b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\dot\xi_1, \dot\xi_2)$ by contradiction

(31) $e \not\models_?^{\dagger} (\dot\xi_1, \dot\xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case `prl`$(e) \models_? \dot\xi_1$, `prr`$(e) \models \dot\xi_2$.

(20) `prl`$(e) \not\models \dot\xi_1$ by assumption

(21) `prl`$(e) \models_? \dot\xi_1$ by assumption

(22) `prr`$(e) \models \dot\xi_2$ by assumption

(23) `prr`$(e) \not\models_? \dot\xi_2$ by assumption

Assume $e \models (\dot\xi_1, \dot\xi_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) `prl`$(e) \models \dot\xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\dot\xi_1, \dot\xi_2)$ by contradiction

By rule induction over Rules (18) on (21), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) $\dot\xi_1$ `refutable`$_?$ by assumption

(27) $(\dot\xi_1, \dot\xi_2)$ `refutable`$_?$ by Rule (12e) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \models_? \dot{\xi}_1, \texttt{prr}(e) \models_? \dot{\xi}_2$.

(20) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (20).

(25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

By rule induction over Rules (18) on (23), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) $\dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

(27) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \models_? \dot{\xi}_1, \texttt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$.

(20) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (20)

(25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

By rule induction over Rules (18) on (21), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) $\dot{\xi}_1$ $\texttt{refutable}_?$ by assumption

(27) $(\dot{\xi}_1, \dot{\xi}_2)$ `refutable`$_?$ — by Rule (12e) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (19a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1, \texttt{prr}(e) \models \dot{\xi}_2$**.**

(20) $\texttt{prl}(e) \not\models \dot{\xi}_1$ — by assumption

(21) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ — by assumption

(22) $\texttt{prr}(e) \models \dot{\xi}_2$ — by assumption

(23) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ — by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16j)**.**

(24) $\texttt{prl}(e) \models \dot{\xi}_1$ — by assumption

Contradicts (20)

(25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ — by contradiction

Assume $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b)**.**

(26) $(\dot{\xi}_1, \dot{\xi}_2)$ `refutable`$_?$ — by assumption

By rule induction over Rules (12) on (26), only two cases apply.

Case (12d)**.**

(27) $\dot{\xi}_1$ `refutable`$_?$ — by assumption

(28) $\texttt{prl}(e)$ `notintro` — by Rule (28e)

(29) $\texttt{prl}(e) \models_? \dot{\xi}_1$ — by Rule (18b) on (28) and (27)

Contradicts (21).

Case (12e)**.**

(27) $\dot{\xi}_2$ `refutable`$_?$ — by assumption

(28) $\texttt{prr}(e)$ `notintro` — by Rule (28f)

(29) $\texttt{prr}(e) \models_? \dot{\xi}_2$ — by Rule (18b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by contradiction

(31) $e \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Lemma 2.0.20 on (25) and (30)

Case $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1, \texttt{prr}(e) \models_? \dot{\xi}_2$**.**

(20) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16), only one case applies.

Case (16j)**.**

(24) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (20).

(25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

By rule induction over Rules (18) on (23), only one case applies.

assume no "or" and "and" in pair

Case (18b)**.**

(26) $\dot{\xi}_2$ $\texttt{refutable}_?$ by assumption

(27) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \dot{\xi}_1, \texttt{prr}(e) \not\models_?^\dagger \dot{\xi}_2$**.**

(20) $\texttt{prl}(e) \not\models \dot{\xi}_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \dot{\xi}_1$ by assumption

(22) $\texttt{prr}(e) \not\models \dot{\xi}_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \dot{\xi}_2$ by assumption

Assume $e \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16j)**.**

(24) $\texttt{prl}(e) \models \dot{\xi}_1$ by assumption

Contradicts (20)

(25) $e \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $e \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b)**.**

(26) $(\dot{\xi}_1, \dot{\xi}_2)$ $\texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (26), only two cases apply.

Case (12d)**.**

(27) $\dot{\xi}_1$ $\texttt{refutable}_?$ by assumption

(28) $\texttt{prl}(e)$ $\texttt{notintro}$ — by Rule (28e)

(29) $\texttt{prl}(e) \models_? \dot{\xi}_1$ — by Rule (18b) on (28) and (27)

Contradicts (21).

Case (12e)**.**

(27) $\dot{\xi}_2$ $\texttt{refutable}_?$ — by assumption

(28) $\texttt{prr}(e)$ $\texttt{notintro}$ — by Rule (28f)

(29) $\texttt{prr}(e) \models_? \dot{\xi}_2$ — by Rule (18b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ — by contradiction

(31) $e \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Lemma 2.0.20 on (25) and (30)

Case (21g)**.**

(11) $e = (e_1, e_2)$ — by assumption

(12) $\cdot ; \Delta \vdash e_1 : \tau_1$ — by assumption

(13) $\cdot ; \Delta \vdash e_2 : \tau_2$ — by assumption

(14) e_1 $\texttt{final}$ — by Lemma 4.0.5 on (6)

(15) e_2 $\texttt{final}$ — by Lemma 4.0.5 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \dot{\xi}_1$, $e_1 \models_? \dot{\xi}_1$, and $e_1 \models \overline{\dot{\xi}_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \dot{\xi}_2$, $e_2 \models_? \dot{\xi}_2$, and $e_2 \models \overline{\dot{\xi}_2}$ holds.
By case analysis on which conclusion holds for $\dot{\xi}_1$ and $\dot{\xi}_2$.

Case $e_1 \models \dot{\xi}_1, e_2 \models \dot{\xi}_2$**.**

(16) $e_1 \models \dot{\xi}_1$ — by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ — by assumption

(18) $e_2 \models \dot{\xi}_2$ — by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ — by assumption

(20) $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (16i) on (16) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ — by Rule (19b) on (20)

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(22) (e_1, e_2) $\texttt{notintro}$ — by assumption

Contradicts Lemma 4.0.9.

Case (18g)**.**

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$.

(16) $e_1 \models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \models_? \dot{\xi}_2$ by assumption

(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18h) on (16) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(22) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$.

(16) $e_1 \models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \models_? \dot{\xi}_1, e_2 \models \dot{\xi}_2$.

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \models_? \dot{\xi}_1$ by assumption

(18) $e_2 \models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18g) on (17) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \models_? \dot{\xi}_2$ by assumption

(20) $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18i) on (17) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (20)

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i)**.**

(22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Case $e_1 \models_? \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i)**.**

(20) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

Case (18h).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^{\dagger} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^{\dagger} \dot{\xi}_1, e_2 \models \dot{\xi}_2$.

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \dot{\xi}_1, e_2 \models_? \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_1 \models \dot{\xi}_1$ by assumption

Contradicts (16).

Case (18i).

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \dot{\xi}_1, e_2 \not\models_?^\dagger \dot{\xi}_2$**.**

(16) $e_1 \not\models \dot{\xi}_1$ by assumption

(17) $e_1 \not\models_? \dot{\xi}_1$ by assumption

(18) $e_2 \not\models \dot{\xi}_2$ by assumption

(19) $e_2 \not\models_? \dot{\xi}_2$ by assumption

Assume $(e_1, e_2) \models (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i)**.**

(20) $e_2 \models \dot{\xi}_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\dot{\xi}_1, \dot{\xi}_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g)**.**

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

Case (18h)**.**

(22) $e_2 \models_? \dot{\xi}_2$ by assumption

Contradicts (19).

Case (18i)**.**

(22) $e_1 \models_? \dot{\xi}_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\dot{\xi}_1, \dot{\xi}_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.20 on (21) and (23)

□

Definition 1.1.1 (Entailment of Constraints). *Suppose that* $\dot{\xi}_1 : \tau$ *and* $\dot{\xi}_2 : \tau$*. Then* $\dot{\xi}_1 \models \dot{\xi}_2$ *iff for all* e *such that* $\cdot\,; \Delta \vdash e : \tau$ *and* e $\texttt{val}$ *we have* $e \models^{\dagger}_{?} \dot{\xi}_1$ *implies* $e \models \dot{\xi}_2$

Definition 1.1.2 (Potential Entailment of Constraints). *Suppose that* $\dot{\xi}_1 : \tau$ *and* $\dot{\xi}_2 : \tau$*. Then* $\dot{\xi}_1 \models^{\dagger}_{?} \dot{\xi}_2$ *iff for all* e *such that* $\cdot\,; \Delta \vdash e : \tau$ *and* e $\texttt{final}$ *we have* $e \models^{\dagger}_{?} \dot{\xi}_1$ *implies* $e \models^{\dagger}_{?} \dot{\xi}_2$

Corollary 1.1.1. *Suppose that* $\dot{\xi} : \tau$ *and* $\cdot;\Delta \vdash e : \tau$ *and* e $\texttt{final}$*. Then* $\top \models^{\dagger}_{?} \dot{\xi}$ *implies* $e \models^{\dagger}_{?} \dot{\xi}$

Proof.

(1)	$\dot{\xi} : \tau$	by assumption
(2)	$\cdot\,; \Gamma \vdash e : \tau$	by assumption
(3)	e $\texttt{final}$	by assumption
(4)	$\top \models^{\dagger}_{?} \dot{\xi}$	by assumption
(5)	$e_1 \models \top$	by Rule (16a)
(6)	$e_1 \models^{\dagger}_{?} \top$	by Rule (19b) on (5)
(7)	$\top : \tau$	by Rule (10a)
(8)	$e_1 \models^{\dagger}_{?} \dot{\xi}_r$	by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

□

2 Match Constraint Language

$\xi \; ::= \; \top \mid \bot \mid \underline{n} \mid \cancel{\underline{n}} \mid \xi_1 \wedge \xi_2 \mid \xi_1 \vee \xi_2 \mid \texttt{inl}(\xi) \mid \texttt{inr}(\xi) \mid (\xi_1, \xi_2)$

$\boxed{\xi : \tau}$ ξ constrains final expressions of type τ

$$\text{CTTruth}\quad \frac{}{\top : \tau} \tag{10a}$$

$$\text{CTFalsity}\quad \frac{}{\bot : \tau} \tag{10b}$$

$$\text{CTNum}\quad \frac{}{\underline{n} : \texttt{num}} \tag{10c}$$

$$\text{CTNotNum}\quad \frac{}{\cancel{\underline{n}} : \texttt{num}} \tag{10d}$$

$$\text{CTAnd}\quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \wedge \xi_2 : \tau} \tag{10e}$$

$$\text{CTOr}\quad \frac{\xi_1 : \tau \qquad \xi_2 : \tau}{\xi_1 \vee \xi_2 : \tau} \tag{10f}$$

$$\text{CTInl}\quad \frac{\xi_1 : \tau_1}{\texttt{inl}(\xi_1) : (\tau_1 + \tau_2)} \tag{10g}$$

$$\text{CTInr}\quad \frac{\xi_2 : \tau_2}{\texttt{inr}(\xi_2) : (\tau_1 + \tau_2)} \tag{10h}$$

$$\text{CTPair}\quad \frac{\xi_1 : \tau_1 \qquad \xi_2 : \tau_2}{(\xi_1, \xi_2) : (\tau_1 \times \tau_2)} \tag{10i}$$

$\boxed{\overline{\xi_1} = \xi_2}$ dual of ξ_1 is ξ_2

$$\overline{\top} = \bot \tag{11a}$$

$$\overline{\bot} = \top \tag{11b}$$

$$\overline{\underline{n}} = \cancel{\underline{n}} \tag{11c}$$

$$\overline{\cancel{\underline{n}}} = \underline{n} \tag{11d}$$

$$\overline{\xi_1 \wedge \xi_2} = \overline{\xi_1} \vee \overline{\xi_2} \tag{11e}$$

$$\overline{\xi_1 \vee \xi_2} = \overline{\xi_1} \wedge \overline{\xi_2} \tag{11f}$$

$$\overline{\texttt{inl}(\xi_1)} = \texttt{inl}(\overline{\xi_1}) \vee \texttt{inr}(\top) \tag{11g}$$

$$\overline{\texttt{inr}(\xi_2)} = \texttt{inr}(\overline{\xi_2}) \vee \texttt{inl}(\top) \tag{11h}$$

$$\overline{(\xi_1, \xi_2)} = (\xi_1, \overline{\xi_2}) \vee (\overline{\xi_1}, \xi_2) \vee (\overline{\xi_1}, \overline{\xi_2}) \tag{11i}$$

$\boxed{\xi\ \texttt{refutable}_?}$ ξ is refutable

$$\text{RXNum}\quad \frac{}{\underline{n}\ \texttt{refutable}_?} \tag{12a}$$

$$\text{RXInl}\quad \frac{}{\texttt{inl}(\xi)\ \texttt{refutable}_?} \tag{12b}$$

$$\text{RXInr}\quad \frac{}{\texttt{inr}(\xi)\ \texttt{refutable}_?} \tag{12c}$$

$$\text{RXPairL}\quad \frac{\xi_1\ \texttt{refutable}_?}{(\xi_1, \xi_2)\ \texttt{refutable}_?} \tag{12d}$$

$$\text{RXPairR}\quad \frac{\xi_2\ \texttt{refutable}_?}{(\xi_1, \xi_2)\ \texttt{refutable}_?} \tag{12e}$$

$$\text{RXOr}\quad \frac{\xi_1\ \texttt{refutable}_? \qquad \xi_2\ \texttt{refutable}_?}{\xi_1 \vee \xi_2\ \texttt{refutable}_?} \tag{12f}$$

$\boxed{\mathit{refutable}_?(\xi)}$

$$\mathit{refutable}_?(\underline{n}) = \text{true} \tag{13a}$$
$$\mathit{refutable}_?(\cancel{\underline{n}}) = \text{true} \tag{13b}$$
$$\mathit{refutable}_?(?) = \text{true} \tag{13c}$$
$$\mathit{refutable}_?(\texttt{inl}(\xi)) = \mathit{refutable}_?(\xi) \tag{13d}$$
$$\mathit{refutable}_?(\texttt{inr}(\xi)) = \mathit{refutable}_?(\xi) \tag{13e}$$
$$\mathit{refutable}_?((\xi_1, \xi_2)) = \mathit{refutable}_?(\xi_1) \text{ or } \mathit{refutable}_?(\xi_2) \tag{13f}$$
$$\mathit{refutable}_?(\xi_1 \vee \xi_2) = \mathit{refutable}_?(\xi_1) \text{ and } \mathit{refutable}_?(\xi_2) \tag{13g}$$
$$\text{Otherwise} \quad \mathit{refutable}_?(\xi) = \text{false} \tag{13h}$$

$\boxed{\dot{\top}(\xi_1) = \xi_2}$

$$\dot{\top}(\top) = \top \tag{14a}$$
$$\dot{\top}(\bot) = \bot \tag{14b}$$
$$\dot{\top}(?) = \top \tag{14c}$$
$$\dot{\top}(\underline{n}) = \underline{n} \tag{14d}$$
$$\dot{\top}(\cancel{\underline{n}}) = \cancel{\underline{n}} \tag{14e}$$
$$\dot{\top}(\xi_1 \wedge \xi_2) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2) \tag{14f}$$
$$\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2) \tag{14g}$$
$$\dot{\top}(\texttt{inl}(\xi)) = \texttt{inl}(\dot{\top}(\xi)) \tag{14h}$$
$$\dot{\top}(\texttt{inr}(\xi)) = \texttt{inr}(\dot{\top}(\xi)) \tag{14i}$$
$$\dot{\top}((\xi_1, \xi_2)) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2)) \tag{14j}$$

$\boxed{\dot{\bot}(\xi_1) = \xi_2}$

$$\dot{\bot}(\top) = \top \tag{15a}$$
$$\dot{\bot}(\bot) = \bot \tag{15b}$$
$$\dot{\bot}(?) = \bot \tag{15c}$$
$$\dot{\bot}(\underline{n}) = \underline{n} \tag{15d}$$
$$\dot{\bot}(\cancel{\underline{n}}) = \cancel{\underline{n}} \tag{15e}$$
$$\dot{\bot}(\xi_1 \land \xi_2) = \dot{\bot}(\xi_1) \land \dot{\bot}(\xi_2) \tag{15f}$$
$$\dot{\bot}(\xi_1 \lor \xi_2) = \dot{\bot}(\xi_1) \lor \dot{\bot}(\xi_2) \tag{15g}$$
$$\dot{\bot}(\texttt{inl}(\xi)) = \texttt{inl}(\dot{\bot}(\xi)) \tag{15h}$$
$$\dot{\bot}(\texttt{inr}(\xi)) = \texttt{inr}(\dot{\bot}(\xi)) \tag{15i}$$
$$\dot{\bot}((\xi_1, \xi_2)) = (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2)) \tag{15j}$$

$\boxed{e \models \xi}$ e satisfies ξ

$$\text{CSTruth}\quad \frac{}{e \models \top} \tag{16a}$$

$$\text{CSNum}\quad \frac{}{\underline{n} \models \underline{n}} \tag{16b}$$

$$\text{CSNotNum}\quad \frac{n_1 \neq n_2}{\underline{n_1} \models \cancel{\underline{n_2}}} \tag{16c}$$

$$\text{CSAnd}\quad \frac{e \models \xi_1 \qquad e \models \xi_2}{e \models \xi_1 \land \xi_2} \tag{16d}$$

$$\text{CSOrL}\quad \frac{e \models \xi_1}{e \models \xi_1 \lor \xi_2} \tag{16e}$$

$$\text{CSOrR}\quad \frac{e \models \xi_2}{e \models \xi_1 \lor \xi_2} \tag{16f}$$

$$\text{CSInl}\quad \frac{e_1 \models \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)} \tag{16g}$$

$$\text{CSInr}\quad \frac{e_2 \models \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)} \tag{16h}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models (\xi_1, \xi_2)}\ \text{CSPair} \tag{16i}$$

$$\frac{e\ \texttt{notintro} \qquad \texttt{prl}(e) \models \xi_1 \qquad \texttt{prr}(e) \models \xi_2}{e \models (\xi_1, \xi_2)}\ \text{CSNotIntroPair} \tag{16j}$$

$\boxed{\mathit{satisfy}(e, \xi)}$

$$\mathit{satisfy}(e, \top) = \text{true} \tag{17a}$$
$$\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = (n_1 = n_2) \tag{17b}$$
$$\mathit{satisfy}(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) \tag{17c}$$
$$\mathit{satisfy}(e, \xi_1 \wedge \xi_2) = \mathit{satisfy}(e, \xi_1) \text{ and } \mathit{satisfy}(e, \xi_2) \tag{17d}$$
$$\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1) \text{ or } \mathit{satisfy}(e, \xi_2) \tag{17e}$$
$$\mathit{satisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\xi_1)) = \mathit{satisfy}(e_1, \xi_1) \tag{17f}$$
$$\mathit{satisfy}(\texttt{inr}_{\tau_1}(e_2), \texttt{inr}(\xi_2)) = \mathit{satisfy}(e_2, \xi_2) \tag{17g}$$
$$\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) = \mathit{satisfy}(e_1, \xi_1) \text{ and } \mathit{satisfy}(e_2, \xi_2) \tag{17h}$$
$$\mathit{satisfy}(⦇⦈^u, (\xi_1, \xi_2)) = \mathit{satisfy}(\texttt{prl}(⦇⦈^u), \xi_1) \text{ and } \mathit{satisfy}(\texttt{prr}(⦇⦈^u), \xi_2) \tag{17i}$$
$$\mathit{satisfy}(⦇e⦈^u, (\xi_1, \xi_2)) = \mathit{satisfy}(\texttt{prl}(⦇e⦈^u), \xi_1) \text{ and } \mathit{satisfy}(\texttt{prr}(⦇e⦈^u), \xi_2) \tag{17j}$$
$$\begin{aligned}\mathit{satisfy}(e_1(e_2), (\xi_1, \xi_2)) &= \mathit{satisfy}(\texttt{prl}(e_1(e_2)), \xi_1) \\ &\quad \text{and } \mathit{satisfy}(\texttt{prr}(e_1(e_2)), \xi_2)\end{aligned} \tag{17k}$$
$$\begin{aligned}\mathit{satisfy}(\texttt{match}(e)\{\hat{rs}\}, (\xi_1, \xi_2)) &= \mathit{satisfy}(\texttt{prl}(\texttt{match}(e)\{\hat{rs}\}), \xi_1) \\ &\quad \text{and } \mathit{satisfy}(\texttt{prr}(\texttt{match}(e)\{\hat{rs}\}), \xi_2)\end{aligned} \tag{17l}$$
$$\begin{aligned}\mathit{satisfy}(\texttt{prl}(e), (\xi_1, \xi_2)) &= \mathit{satisfy}(\texttt{prl}(\texttt{prl}(e)), \xi_1) \\ &\quad \text{and } \mathit{satisfy}(\texttt{prr}(\texttt{prl}(e)), \xi_2)\end{aligned} \tag{17m}$$
$$\begin{aligned}\mathit{satisfy}(\texttt{prr}(e), (\xi_1, \xi_2)) &= \mathit{satisfy}(\texttt{prl}(\texttt{prr}(e)), \xi_1) \\ &\quad \text{and } \mathit{satisfy}(\texttt{prr}(\texttt{prr}(e)), \xi_2)\end{aligned} \tag{17n}$$
$$\text{Otherwise} \quad \mathit{satisfy}(e, \xi) = \text{false} \tag{17o}$$

$\boxed{e \models_? \xi}$ e may satisfy ξ

$$\frac{}{e \models_? ?}\ \text{CMSUnknown} \tag{18a}$$

$$\frac{e\ \texttt{notintro} \qquad \xi\ \texttt{refutable}_?}{e \models_? \xi}\ \text{CMSNotIntro} \tag{18b}$$

$$\frac{e \models_? \xi_1 \qquad e \not\models \xi_2}{e \models_? \xi_1 \vee \xi_2}\ \text{CMSOrL} \tag{18c}$$

$$\frac{e \not\models \xi_1 \qquad e \models_? \xi_2}{e \models_? \xi_1 \vee \xi_2} \text{ CMSOrR} \tag{18d}$$

$$\frac{e_1 \models_? \xi_1}{\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)} \text{ CMSInl} \tag{18e}$$

$$\frac{e_2 \models_? \xi_2}{\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)} \text{ CMSInr} \tag{18f}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \text{ CMSPairL} \tag{18g}$$

$$\frac{e_1 \models \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \text{ CMSPairR} \tag{18h}$$

$$\frac{e_1 \models_? \xi_1 \qquad e_2 \models_? \xi_2}{(e_1, e_2) \models_? (\xi_1, \xi_2)} \text{ CMSPair} \tag{18i}$$

$\boxed{e \models_?^\dagger \xi}$ e satisfies or may satisfy ξ

$$\frac{e \models_? \xi}{e \models_?^\dagger \xi} \text{ CSMSMay} \tag{19a}$$

$$\frac{e \models \xi}{e \models_?^\dagger \xi} \text{ CSMSSat} \tag{19b}$$

Lemma 2.0.1. $e \not\models \bot$

Proof. By rule induction over Rules (16), we notice that $e \models \bot$ is in syntactic contradiction with all rules, hence not derivable. □

Lemma 2.0.2. $e \not\models_? \bot$

Proof. Assume $e \models_? \bot$. By rule induction over Rules (18) on $e \models_? \bot$, only one case applies.

Case (18b)**.**

(1) $\bot\ \texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \bot$ is not derivable. □

Lemma 2.0.3. $e \not\models_? \top$

Proof. Assume $e \models_? \top$. By rule induction over Rules (18) on $e \models_? \top$, only one case applies.

Case (18b)**.**

(1) $\top$ $\texttt{refutable}_?$	by assumption

By rule induction over Rules (12) on (1), no case applies due to syntactic contradiction.

Therefore, $e \models_? \top$ is not derivable. □

Lemma 2.0.4. $e \not\models ?$

Proof. By rule induction over Rules (16), we notice that $e \models ?$ is in syntactic contradiction with all the cases, hence not derivable. □

Lemma 2.0.5. $e \models_?^\dagger \xi$ *iff* $e \models_?^\dagger \xi \vee \bot$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models_?^\dagger \xi$	by assumption

By rule induction over Rules (19) on (1).

Case (19a)**.**

(2) $e \models_? \xi$	by assumption
(3) $e \models_? \xi \vee \bot$	by Rule (18c) on (2) and Lemma 2.0.1
(4) $e \models_?^\dagger \xi \vee \bot$	by Rule (19a) on (3)

Case (19b)**.**

(2) $e \models \xi$	by assumption
(3) $e \models \xi \vee \bot$	by Rule (16e) on (2)
(4) $e \models_?^\dagger \xi \vee \bot$	by Rule (19b) on (3)

2. Necessity:

(1) $e \models_?^\dagger \xi \vee \bot$	by assumption

By rule induction over Rules (19) on (1).

Case (19a)**.**

(2) $e \models_? \xi \vee \bot$ by assumption

By rule induction over Rules (18) on (2), only two of them apply.

Case (18c)**.**

(3) $e \models_? \xi$ by assumption

(4) $e \models_?^\dagger \xi$ by Rule (19a) on (3)

Case (18d)**.**

(3) $e \models_? \bot$ by assumption

(4) $e \not\models_? \bot$ by Lemma 2.0.2

(3) contradicts (4).

Case (19b)**.**

(2) $e \models \xi \vee \bot$ by assumption

By rule induction over Rules (16) on (2), only two of them apply.

Case (16e)**.**

(3) $e \models \xi$ by assumption

(4) $e \models_?^\dagger \xi$ by Rule (19b) on (3)

Case (16f)**.**

(3) $e \models \bot$ by assumption

(4) $e \not\models \bot$ by Lemma 2.0.1

(3) contradicts (4).

□

Corollary 2.0.1. $\top \models_?^\dagger \xi$ *iff* $\top \models_?^\dagger \xi \vee \bot$

Proof. By Definition 2.1.2 and Lemma 2.0.5. □

Lemma 2.0.6. *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$*. Then* $\xi_1 \not\models \xi_2$ *iff* $\xi_1 \not\models \xi_2 \vee \bot$

Proof.

(1) $\xi_1 : \tau$ by assumption

(2) $\xi_2 : \tau$ by assumption

(3) $\bot : \tau$ by Rule (10b)

(4) $\xi_2 \vee \bot : \tau$ by Rule (10f) on (2) and (3)

Then we prove sufficiency and necessity separately.

1. Sufficiency:

(5) $\xi_1 \not\models \xi_2$ by assumption

To prove $\xi_1 \not\models \xi_2 \vee \bot$, assume $\xi_1 \models \xi_2 \vee \bot$.

(6) $\xi_1 \models \xi_2 \vee \bot$ — by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies

(7) $e \models \xi_2 \vee \bot$ — by Definition 2.1.1 on (1) and (4) and (6)

By rule induction over Rules (16) on (7).

Case (16e)**.**

(8) $e \models \xi_2$ — by assumption

(9) $\xi_1 \models \xi_2$ — by Definition 2.1.1 on (8)

(5) contradicts (9).

Case (16f)**.**

(8) $e \models \bot$ — by assumption

(9) $e \not\models \bot$ — by Lemma 2.0.1

(8) contradicts (9).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2 \vee \bot$

2. Necessity:

(5) $\xi_1 \not\models \xi_2 \vee \bot$ — by assumption

To prove $\xi_1 \not\models \xi_2$, assume $\xi_1 \models \xi_2$.

(6) $\xi_1 \models \xi_2$ — by assumption

For all e such that $\cdot\,; \Delta \vdash e : \tau$ and e `final` we have $e \models_?^\dagger \xi_1$ implies

(7) $e \models \xi_2$ — by Definition 2.1.1 on (1) and (2) and (6)

(8) $e \models \xi_2 \vee \bot$ — by Rule (16e) on (7)

(9) $\xi_1 \models \xi_2 \vee \bot$ — by Definition 2.1.1 on (8)

(9) contradicts (5).

The conclusion holds as follows:

(a) $\xi_1 \not\models \xi_2$

□

Lemma 2.0.7. *If* $e \not\models^{\dagger}_{?} \xi_1$ *and* $e \not\models^{\dagger}_{?} \xi_2$ *then* $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

Proof. Assume, for the sake of contradiction, that $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$.

(1) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by assumption

(2) $e \not\models^{\dagger}_{?} \xi_1$ by assumption

(3) $e \not\models^{\dagger}_{?} \xi_2$ by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(4) $e \models \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (16) on (4) and only two of them apply.

Case (16e)**.**

(5) $e \models \xi_1$ by assumption

(6) $e \models^{\dagger}_{?} \xi_1$ by Rule (19b) on (5)

(6) contradicts (2).

Case (16f)**.**

(5) $e \models \xi_2$ by assumption

(6) $e \models^{\dagger}_{?} \xi_2$ by Rule (19b) on (5)

(6) contradicts (3).

Case (19a)**.**

(4) $e \models_{?} \xi_1 \vee \xi_2$ by assumption

By rule induction over Rules (18) on (4) and only two of them apply.

Case (18c)**.**

(5) $e \models_{?} \xi_1$ by assumption

(6) $e \models^{\dagger}_{?} \xi_1$ by Rule (19a) on (5)

(6) contradicts (2).

Case (18d)**.**

(5) $e \models_{?} \xi_2$ by assumption

(6) $e \models^{\dagger}_{?} \xi_2$ by Rule (19a) on (5)

(6) contradicts (3).

The conclusion holds as follows:

1. $e \not\models^{\dagger}_{?} \xi_1 \vee \xi_2$

$\square$

Lemma 2.0.8. *If* $e \models_?^\dagger \xi_1 \vee \xi_2$ *and* $e \not\models_?^\dagger \xi_1$ *then* $e \models_?^\dagger \xi_2$

Proof.

(1) $e \models_?^\dagger \xi_1 \vee \xi_2$ — by assumption

(2) $e \not\models_?^\dagger \xi_1$ — by assumption

By rule induction over Rules (19) on (1).

Case (19b)**.**

(3) $e \models \xi_1 \vee \xi_2$ — by assumption

By rule induction over Rules (16) on (3) and only two of them apply.

Case (16e)**.**

(4) $e \models \xi_1$ — by assumption

(5) $e \models_?^\dagger \xi_1$ — by Rule (19b) on (4)

(5) contradicts (2).

Case (16f)**.**

(4) $e \models \xi_2$ — by assumption

(5) $e \models_?^\dagger \xi_2$ — by Rule (19b) on (4)

Case (19a)**.**

(3) $e \models_? \xi_1 \vee \xi_2$ — by assumption

By rule induction over Rules (18) on (3) and only two of them apply.

Case (18c)**.**

(4) $e \models_? \xi_1$ — by assumption

(5) $e \models_?^\dagger \xi_1$ — by Rule (19a) on (4)

(5) contradicts (2).

Case (18d)**.**

(4) $e \models_? \xi_2$ — by assumption

(5) $e \models_?^\dagger \xi_2$ — by Rule (19a) on (4)

$\square$

Lemma 2.0.9. *If* $e \models_?^\dagger \xi_1$ *and* $e \models_?^\dagger \xi_2$ *then* $e \models_?^\dagger \xi_1 \wedge \xi_2$

Lemma 2.0.10. *If* $e \models_?^\dagger \xi_1$ *then* $e \models_?^\dagger \xi_1 \vee \xi_2$ *and* $e \models_?^\dagger \xi_2 \vee \xi_1$

Proof.

(1) $e \models_?^\dagger \xi_1$ by assumption ,

By rule induction over Rules (19) on (1),

Case (19b)**.**

(2) $e \models \xi_1$ by assumption

(3) $e \models \xi_1 \vee \xi_2$ by Rule (16e) on (2)

(4) $e \models \xi_2 \vee \xi_1$ by Rule (16f) on (2)

(5) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (19b) on (3)

(6) $e \models_?^\dagger \xi_2 \vee \xi_1$ by Rule (19b) on (4)

Case (19a)**.**

(2) $e \models_? \xi_1$ by assumption

By case analysis on the result of $\mathit{satisfy}(e, \xi_2)$.

Case true**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by assumption

(4) $e \models \xi_2$ by Lemma 2.0.19 on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (16f) on (4)

(6) $e \models \xi_2 \vee \xi_1$ by Rule (16e) on (4)

(7) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (19b) on (5)

(8) $e \models_?^\dagger \xi_2 \vee \xi_1$ by Rule (19b) on (6)

Case false**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{false}$ by assumption

(4) $e \not\models \xi_2$ by Lemma 2.0.19 on (3)

(5) $e \models_? \xi_1 \vee \xi_2$ by Rule (18c) on (2) and (4)

(6) $e \models_?^\dagger \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Rule (19a) on (5)

□

Lemma 2.0.11. *If* $e_1 \models_?^\dagger \xi_1$ *then* $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$

Proof.

(1) $e_1 \models_?^\dagger \xi_1$ by assumption

By rule induction over Rules (19) on (1).

Case (19b).

(2) $e_1 \models \xi_1$	by assumption
(3) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$	by Rule (16g) on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$	by Rule (19b) on (3)

Case (19a).

(2) $e_1 \models_{?} \xi_1$	by assumption
(3) $\texttt{inl}_{\tau_2}(e_1) \models_{?} \texttt{inl}(\xi_1)$	by Rule (18e) on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$	by Rule (19a) on (3)

□

Lemma 2.0.12. *If* $e_2 \models^{\dagger}_{?} \xi_2$ *then* $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$

Proof.

(1) $e_2 \models^{\dagger}_{?} \xi_2$	by assumption

By rule induction over Rules (19) on (1).

Case (19b).

(2) $e_2 \models \xi_2$	by assumption
(3) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$	by Rule (16h) on (2)
(4) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$	by Rule (19b) on (3)

Case (19a).

(2) $e_2 \models_{?} \xi_2$	by assumption
(3) $\texttt{inl}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\xi_2)$	by Rule (18f) on (2)
(4) $\texttt{inl}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$	by Rule (19a) on (3)

□

Lemma 2.0.13. *If* $e_1 \models^{\dagger}_{?} \xi_1$ *and* $e_2 \models^{\dagger}_{?} \xi_2$ *then* $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$

Lemma 2.0.14 (Soundness and Completeness of Refutable Constraints)**.** ξ $\texttt{refutable}_{?}$ *iff* $\mathit{refutable}_{?}(\xi) = \mathit{true}$.

Lemma 2.0.15. *If* e $\texttt{notintro}$ *and* ξ $\texttt{refutable}_{?}$ *then either* $\dot{\top}(\xi)$ $\texttt{refutable}_{?}$ *or* $e \models \dot{\top}(\xi)$.

Proof. By structural induction on ξ. $\square$

Lemma 2.0.16. *There does not exist such a constraint $\xi_1 \wedge \xi_2$ such that $\xi_1 \wedge \xi_2$ $\texttt{refutable}_?$.*

Proof. By rule induction over Rules (12), we notice that $\xi_1 \wedge \xi_2$ $\texttt{refutable}_?$ is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 2.0.17. *There does not exist such a constraint $\xi_1 \vee \xi_2$ such that $\xi_1 \vee \xi_2$ $\texttt{refutable}_?$.*

Proof. By rule induction over Rules (12), we notice that $\xi_1 \vee \xi_2$ $\texttt{refutable}_?$ is in syntactic contradiction with all the cases, hence not derivable. $\square$

Lemma 2.0.18. *If e $\texttt{notintro}$ and $e \models \xi$ then $\cancel{\xi\ \texttt{refutable}_?}$.*

Proof.

(1) e $\texttt{notintro}$ — by assumption

(2) $e \models \xi$ — by assumption

By rule induction over Rules (16) on (2).

Case (16a).

(3) $\xi = \top$ — by assumption

Assume $\top$ $\texttt{refutable}_?$. By rule induction over Rules (12), no case applies due to syntactic contradiction.
Therefore, $\cancel{\top\ \texttt{refutable}_?}$.

Case (16e),(16f).

(3) $\xi = \xi_1 \vee \xi_2$ — by assumption

(4) $\cancel{\xi_1 \vee \xi_2\ \texttt{refutable}_?}$ — by Lemma 2.0.17

Case (16d).

(3) $\xi = \xi_1 \wedge \xi_2$ — by assumption

(4) $\cancel{\xi_1 \wedge \xi_2\ \texttt{refutable}_?}$ — by Lemma 2.0.16

Case (16j).

(3) $\xi = (\xi_1, \xi_2)$ — by assumption

(4) $\texttt{prl}(e) \models \xi_1$ — by assumption

(5) $\texttt{prr}(e) \models \xi_2$ — by assumption

(6) $\texttt{prl}(e)$ $\texttt{notintro}$ — by Rule (28e)

(7) $\texttt{prr}(e)$ $\texttt{notintro}$ — by Rule (28f)

(8) $\cancel{\xi_1\ \texttt{refutable}_?}$ by IH on (6) and (4)

(9) $\cancel{\xi_2\ \texttt{refutable}_?}$ by IH on (7) and (5)

Assume $(\xi_1, \xi_2)\ \texttt{refutable}_?$. By rule induction over Rules (12) on it, only two cases apply.

Case (12d).

(10) $\xi_1\ \texttt{refutable}_?$ by assumption

Contradicts (8).

Case (12e).

(10) $\xi_2\ \texttt{refutable}_?$ by assumption

Contradicts (9).

Therefore, $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}_?}$.

Otherwise.

(3) $e = \underline{n}, \texttt{inl}_{\tau_2}(e_1), \texttt{inr}_{\tau_1}(e_2), (e_1, e_2)$ by assumption

By rule induction over Rules (28) on (1), no case applies due to syntactic contradiction.

□

Lemma 2.0.19 (Soundness and Completeness of Satisfaction Judgment)**.** *$e \models \xi$ iff satisfy$(e, \xi) =$ true.*

Proof. We prove soundness and completeness separately.

1. Soundness:

(1) $e \models \xi$ by assumption

By rule induction over Rules (16) on (1).

Case (16a).

(2) $\xi = \top$ by assumption
(3) *satisfy*$(e, \top) =$ true by Definition 17a

Case (16b).

(2) $e = \underline{n}$ by assumption
(3) $\xi = \underline{n}$ by assumption
(4) *satisfy*$(\underline{n}, \underline{n}) = (n = n) =$ true by Definition 17b

Case (16c).

(2) $e = \underline{n_1}$ by assumption
(3) $\xi = \cancel{\underline{n_2}}$ by assumption
(4) $n_1 \neq n_2$ by assumption
(5) $\mathit{satisfy}(\underline{n_1}, \cancel{\underline{n_2}}) = (n_1 \neq n_2) = \text{true}$ by Definition 17c on (4)

Case (16d).

(2) $\xi = \xi_1 \wedge \xi_2$ by assumption
(3) $e \models \xi_1$ by assumption
(4) $e \models \xi_2$ by assumption
(5) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)
(6) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by IH on (4)
(7) $\mathit{satisfy}(e, \xi_1 \wedge \xi_2) = \mathit{satisfy}(e, \xi_1)$ and $\mathit{satisfy}(e, \xi_2) = \text{true}$
by Definition 17d on (5) and (6)

Case (16e).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption
(3) $e \models \xi_1$ by assumption
(4) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by IH on (3)
(5) $\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$
by Definition 17e on (4)

Case (16f).

(2) $\xi = \xi_1 \vee \xi_2$ by assumption
(3) $e \models \xi_2$ by assumption
(4) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by IH on (3)
(5) $\mathit{satisfy}(e, \xi_1 \vee \xi_2) = \mathit{satisfy}(e, \xi_1)$ or $\mathit{satisfy}(e, \xi_2) = \text{true}$
by Definition 17e on (4)

Case (16g).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption
(3) $\xi = \texttt{inl}(\xi_1)$ by assumption
(4) $e_1 \models \xi_1$ by assumption
(5) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)
(6) $\mathit{satisfy}(\texttt{inl}_{\tau_2}(e_1), \texttt{inl}(\xi_1)) = \mathit{satisfy}(e_1, \xi_1) = \text{true}$
by Definition 17f on (5)

Case (16h).

(2) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\xi = \mathtt{inl}(\xi_2)$ by assumption

(4) $e_2 \models \xi_2$ by assumption

(5) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (4)

(6) $\mathit{satisfy}(\mathtt{inr}_{\tau_1}(e_2), \mathtt{inr}(\xi_2)) = \mathit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 17g on (5)

Case (16i)**.**

(2) $e = (e_1, e_2)$ by assumption

(3) $\xi = (\xi_1, \xi_2)$ by assumption

(4) $e_1 \models \xi_1$ by assumption

(5) $e_2 \models \xi_2$ by assumption

(6) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by IH on (5)

(8) $\mathit{satisfy}((e_1, e_2), (\xi_1, \xi_2)) =$
$\mathit{satisfy}(e_1, \xi_1)$ and $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 17h on (6) and (7)

Case (16j)**.**

(2) $\xi = (\xi_1, \xi_2)$ by assumption

(3) e $\mathtt{notintro}$ by assumption

(4) $\mathtt{prl}(e) \models \xi_1$ by assumption

(5) $\mathtt{prr}(e) \models \xi_2$ by assumption

(6) $\mathit{satisfy}(\mathtt{prl}(e), \xi_1) = \text{true}$ by IH on (4)

(7) $\mathit{satisfy}(\mathtt{prr}(e), \xi_2) = \text{true}$ by IH on (5)

By rule induction over Rules (28) on (3).

Otherwise.

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) $\mathit{satisfy}(e, (\xi_1, \xi_2)) =$
$\mathit{satisfy}(\mathtt{prl}(e), \xi_1)$ and $\mathit{satisfy}(\mathtt{prr}(e), \xi_2) = \text{true}$ by Definition 17 on (6) and (7)

2. Completeness:

(1) $\mathit{satisfy}(e, \xi) = \text{true}$ by assumption

By structural induction on ξ.

Case $\xi = \top$**.**

(2) $e \models \top$ — by Rule (16a)

Case $\xi = \bot, ?$.

(2) $\textit{satisfy}(e, \xi) = \text{false}$ — by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{n}$.

By structural induction on e.

Case $e = \underline{n'}$.

(2) $n' = n$ — by Definition 17b on (1)

(3) $\underline{n'} \models \underline{n}$ — by Rule (16b) on (2)

Otherwise.

(2) $\textit{satisfy}(e, \underline{n}) = \text{false}$ — by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \underline{\cancel{n}}$.

By structural induction on e.

Case $e = \underline{n'}$.

(2) $n' \neq n$ — by Definition 17c on (1)

(3) $\underline{n'} \models \underline{\cancel{n}}$ — by Rule (16c) on (2)

Otherwise.

(2) $\textit{satisfy}(e, \underline{\cancel{n}}) = \text{false}$ — by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \xi_1 \wedge \xi_2$.

(2) $\textit{satisfy}(e, \xi_1) = \text{true}$ — by Definition 17d on (1)

(3) $\textit{satisfy}(e, \xi_2) = \text{true}$ — by Definition 17d on (1)

(4) $e \models \xi_1$ — by IH on (2)

(5) $e \models \xi_2$ — by IH on (3)

(6) $e \models \xi_1 \wedge \xi_2$ — by Rule (16d) on (4) and (5)

Case $\xi = \xi_1 \vee \xi_2$.

(2) $\textit{satisfy}(e, \xi_1)$ or $\textit{satisfy}(e, \xi_2) = \text{true}$ — by Definition 17e on (1)

By case analysis on (2).

Case $\textit{satisfy}(e, \xi_1) = \text{true}$.

(3) $\mathit{satisfy}(e, \xi_1) = \text{true}$ by assumption
(4) $e \models \xi_1$ by IH on (3)
(5) $e \models \xi_1 \vee \xi_2$ by Rule (16e) on (4)

Case $\mathit{satisfy}(e, \xi_2) = \text{true}$**.**

(3) $\mathit{satisfy}(e, \xi_2) = \text{true}$ by assumption
(4) $e \models \xi_2$ by IH on (3)
(5) $e \models \xi_1 \vee \xi_2$ by Rule (16f) on (4)

Case $\xi = \texttt{inl}(\xi_1)$**.**

By structural induction on e.

Case $e = \texttt{inl}_{\tau_2}(e_1)$**.**

(2) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 17f on (1)
(3) $e_1 \models \xi_1$ by IH on (2)
(4) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ by Rule (16g) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \texttt{inl}(\xi_1)) = \text{false}$ by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\xi = \texttt{inr}(\xi_2)$**.**

By structural induction on e.

Case $e = \texttt{inr}_{\tau_1}(e_2)$**.**

(2) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 17g on (1)
(3) $e_2 \models \xi_2$ by IH on (2)
(4) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ by Rule (16h) on (3)

Otherwise.

(2) $\mathit{satisfy}(e, \texttt{inr}(\xi_2)) = \text{false}$ by Definition 17o

(2) contradicts (1) and thus vacuously true.

Case $\xi = (\xi_1, \xi_2)$**.**

By structural induction on e.

Case $e = (e_1, e_2)$**.**

(2) $\mathit{satisfy}(e_1, \xi_1) = \text{true}$ by Definition 17h on (1)
(3) $\mathit{satisfy}(e_2, \xi_2) = \text{true}$ by Definition 17h on (1)
(4) $e_1 \models \xi_1$ by IH on (2)
(5) $e_2 \models \xi_2$ by IH on (3)
(6) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (16i) on (4) and (5)

Case $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$.

(2) $\mathit{satisfy}(\texttt{prl}(e), \xi_1) = \text{true}$ — by Definition 17h on (1)

(3) $\mathit{satisfy}(\texttt{prr}(e), \xi_2) = \text{true}$ — by Definition 17h on (1)

(4) $\texttt{prl}(e) \models \xi_1$ — by IH on (2)

(5) $\texttt{prr}(e) \models \xi_2$ — by IH on (3)

(6) e $\texttt{notintro}$ — by each rule in Rules (28)

(7) $(e_1, e_2) \models (\xi_1, \xi_2)$ — by Rule (16j) on (6) and (4) and (5)

Otherwise.

(2) $\mathit{satisfy}(e, (\xi_1, \xi_2)) = \text{false}$ — by Definition 17o

(2) contradicts (1) and thus vacuously true.

□

Lemma 2.0.20. *$e \not\models \xi$ and $e \not\models_? \xi$ iff $e \not\models_?^\dagger \xi$.*

Proof. 1. Sufficiency:

(1) $e \not\models \xi$ — by assumption

(2) $e \not\models_? \xi$ — by assumption

Assume $e \models_?^\dagger \xi$. By rule induction over Rules (19) on it.

Case (19a).

(3) $e \models \xi$ — by assumption

Contradicts (1).

Case (19b).

(3) $e \models_? \xi$ — by assumption

Contradicts (2).

Therefore, $e \models_?^\dagger \xi$ is not derivable.

2. Necessity:

(1) $e \not\models_?^\dagger \xi$ — by assumption

Assume $e \models \xi$.

(2) $e \models_?^\dagger \xi$ — by Rule (19b) on assumption

Contradicts (1). Therefore, $e \not\models \xi$. Assume $e \models_? \xi$.

(3) $e \models^{\dagger}_{?} \xi$ by Rule (19a) on assumption

Contradicts (1). Therefore, $e \not\models_{?} \xi$.

□

Theorem 2.1 (Exclusiveness of Satisfaction Judgment). *If $\xi : \tau$ and $\cdot;\Delta \vdash e : \tau$ and e `final` then exactly one of the following holds*

1. $e \models \xi$
2. $e \models_{?} \xi$
3. $e \not\models^{\dagger}_{?} \xi$

Proof.

(4) $\xi : \tau$ by assumption

(5) $\cdot\,;\Delta \vdash e : \tau$ by assumption

(6) e `final` by assumption

By rule induction over Rules (10) on (4), we would show one conclusion is derivable while the other two are not.

Case (10a).

(7) $\xi = \top$ by assumption

(8) $e \models \top$ by Rule (16a)

(9) $e \not\models_{?} \top$ by Lemma 2.0.3

(10) $e \models^{\dagger}_{?} \top$ by Rule (19b) on (8)

Case (10b).

(7) $\xi = \bot$ by assumption

(8) $e \not\models \bot$ by Lemma 2.0.1

(9) $e \not\models_{?} \bot$ by Lemma 2.0.2

(10) $e \not\models^{\dagger}_{?} \bot$ by Lemma 2.0.20 on (8) and (9)

Case (1b).

(7) $\xi = ?$ by assumption

(8) $e \not\models ?$ by Lemma 2.0.4

(9) $e \models_{?} ?$ by Rule (18a)

(10) $e \vDash^{\dagger}_{?} ?$ by Rule (19a) on (9)

Case (10c).

(7) $\xi = \underline{n_2}$ by assumption

(8) $\tau = \texttt{num}$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

Assume $e \vDash \underline{n_2}$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction on ξ.

(11) $e \nvDash \underline{n_2}$ by contradiction

(12) $\underline{n_2}\ \texttt{refutable}_?$ by Rule (12a)

(13) $e \vDash_? \underline{n_2}$ by Rule (18b) on (10) and (12)

(14) $e \vDash^{\dagger}_{?} \underline{n_2}$ by Rule (19a) on (13)

Case (21d).

(9) $e = \underline{n_1}$ by assumption

Assume $\underline{n_1} \vDash_? \underline{n_2}$. By rule induction over Rules (18), only one case applies.

Case (18b).

(10) $\underline{n_1}\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.6.

(11) $\underline{n_1} \nvDash_? \underline{n_2}$ by contradiction

By case analysis on whether n_1 is equal to n_2.

Case $n_1 = n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{true}$ by Definition 17

(13) $\underline{n_1} \vDash \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \vDash^{\dagger}_{?} \underline{n_2}$ by Rule (19b) on (13)

Case $n_1 \neq n_2$.

(12) $\mathit{satisfy}(\underline{n_1}, \underline{n_2}) = \text{false}$ by Definition 17

(13) $\underline{n_1} \not\models \underline{n_2}$ by Lemma 2.0.19 on (12)

(14) $e \not\models^{\dagger}_{?} \underline{n_2}$ by Lemma 2.0.20 on (11) and (13)

Case (10f).

(7) $\xi = \xi_1 \vee \xi_2$ by assumption

By inductive hypothesis on (5) and (6), exactly one of $e \models \xi_1$, $e \models_? \xi_1$, and $e \not\models^{\dagger}_{?} \xi_1$ holds. The same goes for ξ_2. By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e \models \xi_1, e \models \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (16e) on (8)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (19b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\xi_1 \vee \xi_2\ \texttt{refutable}_?$ by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \models_? \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (16e) on (8)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (19b) on (12)

Assume $e \models_{?} \xi_1 \vee \xi_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\xi_1 \vee \xi_2\ \texttt{refutable}_{?}$ by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_{?} \xi_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_{?} \xi_1 \vee \xi_2$ by contradiction

Case $e \models \xi_1, e \not\models^{\dagger}_{?} \xi_2$.

(8) $e \models \xi_1$ by assumption

(9) $e \not\models_{?} \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_{?} \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (16e) on (8)

(13) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Rule (19b) on (12)

Assume $e \models_{?} \xi_1 \vee \xi_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\xi_1 \vee \xi_2\ \texttt{refutable}_{?}$ by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_{?} \xi_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \not\models \xi_1$ by assumption

Contradicts (8).

(15) $e \not\models_{?} \xi_1 \vee \xi_2$ by contradiction

Case $e \models_{?} \xi_1, e \models \xi_2$.

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_{?} \xi_1$ by assumption

(10)	$e \models \xi_2$	by assumption
(11)	$e \not\models_? \xi_2$	by assumption
(12)	$e \models \xi_1 \vee \xi_2$	by Rule (16f) on (10)
(13)	$e \models_?^{\dagger} \xi_1 \vee \xi_2$	by Rule (19b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(14)	$\xi_1 \vee \xi_2$ `refutable`$_?$	by assumption

Contradicts Lemma 2.0.17.

Case (18c)**.**

(14)	$e \not\models \xi_2$	by assumption

Contradicts (10).

Case (18d)**.**

(14)	$e \models_? \xi_2$	by assumption

Contradicts (11).

(15)	$e \not\models_? \xi_1 \vee \xi_2$	by contradiction

Case $e \models_? \xi_1, e \models_? \xi_2$**.**

(8)	$e \not\models \xi_1$	by assumption
(9)	$e \models_? \xi_1$	by assumption
(10)	$e \not\models \xi_2$	by assumption
(11)	$e \models_? \xi_2$	by assumption
(12)	$e \models_? \xi_1 \vee \xi_2$	by Rule (18c) on (9) and (10)
(13)	$e \models_?^{\dagger} \xi_1 \vee \xi_2$	by Rule (19a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (16), only two cases apply.

Case (16e)**.**

(14)	$e \models \xi_1$	by assumption

Contradicts (8)

Case (16f)**.**

(14)	$e \models \xi_2$	by assumption

Contradicts (10)

(15)	$e \not\models \xi_1 \vee \xi_2$	by contradiction

Case $e \models_? \xi_1, e \not\models_?^{\dagger} \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \models_? \xi_1$ by assumption

(10) $e \not\models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ by Rule (18c) on (9) and (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (19a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (16), only two cases apply.

Case (16e)**.**

(14) $e \models \xi_1$ by assumption

Contradicts (8).

Case (16f)**.**

(14) $e \models \xi_2$ by assumption

Contradicts (10).

(15) $e \not\models \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \models \xi_2$**.**

(8) $e \not\models \xi_1$ by assumption

(9) $e \not\models_? \xi_1$ by assumption

(10) $e \models \xi_2$ by assumption

(11) $e \not\models_? \xi_2$ by assumption

(12) $e \models \xi_1 \vee \xi_2$ by Rule (16f) on (10)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ by Rule (19b) on (12)

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(14) $\xi_1 \vee \xi_2$ `refutable`$_?$ by assumption

Contradicts Lemma 2.0.17.

Case (18c)**.**

(14) $e \not\models \xi_2$ by assumption

Contradicts (10).

Case (18d)**.**

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

Case $e \not\models_?^\dagger \xi_1, e \models_? \xi_2$.

(8) $e \not\models \xi_1$ — by assumption

(9) $e \not\models_? \xi_1$ — by assumption

(10) $e \not\models \xi_2$ — by assumption

(11) $e \models_? \xi_2$ — by assumption

(12) $e \models_? \xi_1 \vee \xi_2$ — by Rule (18d) on (11) and (8)

(13) $e \models_?^\dagger \xi_1 \vee \xi_2$ — by Rule (19a) on (12)

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (16), only two cases apply.

Case (16e).

(14) $e \models \xi_1$ — by assumption

Contradicts (8)

Case (16f).

(14) $e \models \xi_2$ — by assumption

Contradicts (10)

(15) $e \not\models \xi_1 \vee \xi_2$ — by contradiction

Case $e \not\models_?^\dagger \xi_1, e \not\models_?^\dagger \xi_2$.

(8) $e \not\models \xi_1$ — by assumption

(9) $e \not\models_? \xi_1$ — by assumption

(10) $e \not\models \xi_2$ — by assumption

(11) $e \not\models_? \xi_2$ — by assumption

Assume $e \models \xi_1 \vee \xi_2$. By rule induction over Rules (16) on it, only two cases apply.

Case (16e).

(12) $e \models \xi_1$ — by assumption

Contradicts (8).

Case (16f).

(12) $e \models \xi_2$ — by assumption

Contradicts (10).

(13) $e \not\models \xi_1 \vee \xi_2$ — by contradiction

Assume $e \models_? \xi_1 \vee \xi_2$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(14) $\xi_1 \vee \xi_2$ $\texttt{refutable}_?$ — by assumption

Contradicts Lemma 2.0.17.

Case (18c).

(14) $e \models_? \xi_1$ by assumption

Contradicts (9).

Case (18d).

(14) $e \models_? \xi_2$ by assumption

Contradicts (11).

(15) $e \not\models_? \xi_1 \vee \xi_2$ by contradiction

(16) $e \not\models_?^\dagger \xi_1 \vee \xi_2$ by Lemma 2.0.20 on (13) and (15)

Case (10g).

(7) $\xi = \texttt{inl}(\xi_1)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(10) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

Assume $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inl}(\xi_1)$ by contradiction

By case analysis on the value of $\mathit{refutable}_?(\texttt{inl}(\xi_1))$.

Case $\mathit{refutable}_?(\texttt{inl}(\xi_1)) = \text{true}$.

(13) $\mathit{refutable}_?(\texttt{inl}(\xi_1)) = \text{true}$ by assumption

(14) $\texttt{inl}(\xi_1)$ $\texttt{refutable}_?$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inl}(\xi_1)$ by Rule (18b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (19a) on (15)

Case $\mathit{refutable}_?(\texttt{inl}(\xi_1)) = \text{false}$.

(13) $\mathit{refutable}_?(\texttt{inl}(\xi_1)) = \text{false}$ by assumption

(14) $\cancel{\texttt{inl}(\xi_1)\ \texttt{refutable}_?}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(15) $\texttt{inl}(\xi_1)\ \texttt{refutable}_?$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(17) $e \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (12) and (16)

Case (21j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(11) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(12) $e_1\ \texttt{final}$ by Lemma 4.0.3 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \not\models_?^\dagger \xi_1$ holds. By case analysis on which one holds.

Case $e_1 \models \xi_1$.

(13) $e_1 \models \xi_1$ by assumption

(14) $e_1 \not\models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ by Rule (16g) on (13)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (19b) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$. By rule induction over Rules (18) on it, only two cases apply.

Case (18b).

(17) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18e).

(17) $e_1 \models_? \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \models_? \xi_1$.

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \models_? \xi_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \models_? \texttt{inl}(\xi_1)$ by Rule (18e) on (14)

(16) $\texttt{inl}_{\tau_2}(e_1) \models_?^\dagger \texttt{inl}(\xi_1)$ by Rule (19a) on (15)

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16g).

(17) $e_1 \models \xi_1$

Contradicts (13).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Case $e_1 \not\models^{\dagger}_{?} \xi_1$.

(13) $e_1 \not\models \xi_1$ by assumption

(14) $e_1 \not\models_{?} \xi_1$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, only one case applies.

Case (16g).

(15) $e_1 \models \xi_1$

Contradicts (13).

(16) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inl}(\xi_1)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_{?} \texttt{inl}(\xi_1)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18e).

(17) $e_1 \models_{?} \xi_1$

Contradicts (14).

(18) $\texttt{inl}_{\tau_2}(e_1) \not\models_{?} \texttt{inl}(\xi_1)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models^{\dagger}_{?} \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (21k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inl}(\xi_1)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inl}(\xi_1)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inl}(\xi_1)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(12) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

By rule induction over Rules (28) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inr}_{\tau_1}(e_2) \not\models_? \texttt{inl}(\xi_1)$ by contradiction

(14) $\texttt{inr}_{\tau_1}(e_2) \not\models_?^\dagger \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (11) and (13)

Case (10h)**.**

(7) $\xi = \texttt{inr}(\xi_2)$ by assumption

(8) $\tau = (\tau_1 + \tau_2)$ by assumption

(9) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m)**.**

(10) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(11) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

Assume $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(12) $e \not\models \texttt{inr}(\xi_2)$ by contradiction

By case analysis on the value of $\textit{refutable}_?(\texttt{inr}(\xi_2))$. inr is refutable

Case $\textit{refutable}_?(\texttt{inr}(\xi_2)) = \text{true}$**.**

(13) $\textit{refutable}_?(\texttt{inr}(\xi_2)) = \text{true}$ by assumption

(14) $\texttt{inr}(\xi_2)\ \texttt{refutable}_?$ by Lemma 2.0.14 on (13)

(15) $e \models_? \texttt{inr}(\xi_2)$ by Rule (18b) on (11) and (14)

(16) $e \models_?^\dagger \texttt{inr}(\xi_2)$ by Rule (19a) on (15)

Case $\textit{refutable}_?(\texttt{inr}(\xi_2)) = \text{false}$**.**

(13) $\textit{refutable}_?(\texttt{inr}(\xi_2)) = \text{false}$ by assumption

(14) $\cancel{\texttt{inr}(\xi_2)\ \texttt{refutable}_?}$ by Lemma 2.0.14 on (13)

Assume $e \models_? \texttt{inr}(\xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b)**.**

(15) $\texttt{inr}(\xi_2)\ \texttt{refutable}_?$ by assumption

Contradicts (14).

(16) $e \not\models_? \texttt{inr}(\xi_2)$ by contradiction

(17) $e \not\models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 2.0.20 on (12) and (16)

Case (21j).

(10) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

Assume $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, no case applies due to syntactic contradiction.

(11) $\texttt{inl}_{\tau_2}(e_1) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \models_{?} \texttt{inr}(\xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(12) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (12), no case applies due to syntactic contradiction.

(13) $\texttt{inl}_{\tau_2}(e_1) \not\models_{?} \texttt{inr}(\xi_2)$ by contradiction

(14) $\texttt{inl}_{\tau_2}(e_1) \not\models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 2.0.20 on (11) and (13)

Case (21k).

(10) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(11) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(12) $e_2\ \texttt{final}$ by Lemma 4.0.4 on (6)

By inductive hypothesis on (9) and (11) and (12), exactly one of $e_2 \models \xi_2$, $e_2 \models_{?} \xi_2$, and $e_2 \not\models^{\dagger}_{?} \xi_2$ holds. By case analysis on which one holds.

Case $e_2 \models \xi_2$.

(13) $e_2 \models \xi_2$ by assumption

(14) $e_2 \not\models_{?} \xi_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ by Rule (16g) on (13)

(16) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Rule (19b) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\xi_2)$. By rule induction over Rules (18) on it, only two cases apply.

Case (18b).

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18f).

(17) $e_2 \models_{?} \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_{?} \texttt{inr}(\xi_2)$ by contradiction

Case $e_2 \models_{?} \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \models_{?} \xi_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\xi_2)$ by Rule (18f) on (14)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_{?}^{\dagger} \texttt{inr}(\xi_2)$ by Rule (19a) on (15)

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16h).

(17) $e_2 \models \xi_2$

Contradicts (13).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Case $e_2 \not\models_{?}^{\dagger} \xi_2$.

(13) $e_2 \not\models \xi_2$ by assumption

(14) $e_2 \not\models_{?} \xi_2$ by assumption

Assume $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16h).

(15) $e_2 \models \xi_2$

Contradicts (13).

(16) $\texttt{inr}_{\tau_1}(e_2) \not\models \texttt{inr}(\xi_2)$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \models_{?} \texttt{inr}(\xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(17) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

By rule induction over Rules (28) on (17), no case applies due to syntactic contradiction.

Case (18f).

(17) $e_2 \models_{?} \xi_2$

Contradicts (14).

(18) $\texttt{inr}_{\tau_1}(e_2) \not\models_{?} \texttt{inr}(\xi_2)$ by contradiction

(19) $\texttt{inl}_{\tau_2}(e_1) \not\models_{?}^{\dagger} \texttt{inl}(\xi_1)$ by Lemma 2.0.20 on (16) and (18)

Case (16i).

(7) $\xi = (\xi_1, \xi_2)$ by assumption

(8) $\tau = (\tau_1 \times \tau_2)$ by assumption

(9) $\xi_1 : \tau_1$ by assumption

(10) $\xi_2 : \tau_2$ by assumption

By rule induction over Rules (21) on (5), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(11) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(13) e $\texttt{indet}$ by Lemma 4.0.10 on (6) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (26g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (27b) on (14)

(16) $\texttt{prr}(e)$ $\texttt{indet}$ by Rule (26h) on (13)

(17) $\texttt{prr}(e)$ $\texttt{final}$ by Rule (27b) on (16)

(18) $\cdot\,; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (21h) on (5)

(19) $\cdot\,; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (21i) on (5)

By inductive hypothesis on (9) and (18) and (15), exactly one of $\texttt{prl}(e) \models \xi_1$, $\texttt{prl}(e) \models_? \xi_1$, and $\texttt{prl}(e) \not\models_?^\dagger \xi_1$ holds.
By inductive hypothesis on (10) and (19) and (17), exactly one of $\texttt{prr}(e) \models \xi_2$, $\texttt{prr}(e) \models_? \xi_2$, and $\texttt{prr}(e) \not\models_?^\dagger \xi_2$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

(24) $e \models (\xi_1, \xi_2)$ by Rule (16j) on (12) and (20) and (22)

(25) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19b) on (24)

(26) $\cancel{(\xi_1, \xi_2)\ \texttt{refutable}_?}$ by Lemma 2.0.18 on (12) and (24)

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(27) (ξ_1, ξ_2) $\texttt{refutable}_?$ by assumption

Contradicts (26).

(28) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \models_? \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (18) on (23), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) ξ_2 $\texttt{refutable}_?$ by assumption

(27) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \models \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\texttt{prl}(e) \models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16j).

(24) $\texttt{prr}(e) \models \xi_2$ by assumption

Contradicts (22).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(26) (ξ_1, ξ_2) $\texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (26), only two cases apply.

Case (12d).

(27) ξ_1 `refutable`$_?$ by assumption

(28) `prl`(e) `notintro` by Rule (28e)

(29) `prl`$(e) \models_? \xi_1$ by Rule (18b) on (28) and (27)

Contradicts (21).

Case (12e).

(27) ξ_2 `refutable`$_?$ by assumption

(28) `prr`(e) `notintro` by Rule (28f)

(29) `prr`$(e) \models_? \xi_2$ by Rule (18b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case `prl`$(e) \models_? \xi_1$, `prr`$(e) \models \xi_2$.

(20) `prl`$(e) \not\models \xi_1$ by assumption

(21) `prl`$(e) \models_? \xi_1$ by assumption

(22) `prr`$(e) \models \xi_2$ by assumption

(23) `prr`$(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) `prl`$(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (18) on (21), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) ξ_1 `refutable`$_?$ by assumption

(27) (ξ_1, ξ_2) `refutable`$_?$ by Rule (12e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19a) on (28)

Case `prl`$(e) \models_? \xi_1$, `prr`$(e) \models_? \xi_2$.

(20) `prl`$(e) \not\models \xi_1$ by assumption

(21) `prl`$(e) \models_? \xi_1$ by assumption

(22) `prr`$(e) \not\models \xi_2$ by assumption

(23) `prr`$(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (18) on (23), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) ξ_2 $\texttt{refutable}_?$ by assumption

(27) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \models_? \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (18) on (21), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) ξ_1 $\texttt{refutable}_?$ by assumption

(27) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \xi_1, \texttt{prr}(e) \models \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(26) $(\xi_1, \xi_2)\ \texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (26), only two cases apply.

Case (12d).

(27) $\xi_1\ \texttt{refutable}_?$ by assumption

(28) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (28e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (18b) on (28) and (27)

Contradicts (21).

Case (12e).

(27) $\xi_2\ \texttt{refutable}_?$ by assumption

(28) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (28f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (18b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case $\texttt{prl}(e) \not\models_?^\dagger \xi_1, \texttt{prr}(e) \models_? \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16), only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20).

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

By rule induction over Rules (18) on (23), only one case applies.

assume no "or" and "and" in pair

Case (18b).

(26) ξ_2 $\texttt{refutable}_?$ by assumption

(27) (ξ_1, ξ_2) $\texttt{refutable}_?$ by Rule (12e) on (26)

(28) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (27)

(29) $e \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19a) on (28)

Case $\texttt{prl}(e) \not\models_?^\dagger \xi_1, \texttt{prr}(e) \not\models_?^\dagger \xi_2$.

(20) $\texttt{prl}(e) \not\models \xi_1$ by assumption

(21) $\texttt{prl}(e) \not\models_? \xi_1$ by assumption

(22) $\texttt{prr}(e) \not\models \xi_2$ by assumption

(23) $\texttt{prr}(e) \not\models_? \xi_2$ by assumption

Assume $e \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only one case applies.

Case (16j).

(24) $\texttt{prl}(e) \models \xi_1$ by assumption

Contradicts (20)

(25) $e \not\models (\xi_1, \xi_2)$ by contradiction

Assume $e \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, only one case applies.

Case (18b).

(26) (ξ_1, ξ_2) $\texttt{refutable}_?$ by assumption

By rule induction over Rules (12) on (26), only two cases apply.

Case (12d).

(27) ξ_1 $\texttt{refutable}_?$ by assumption

(28) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (28e)

(29) $\texttt{prl}(e) \models_? \xi_1$ by Rule (18b) on (28) and (27)

Contradicts (21).

Case (12e).

(27) ξ_2 $\texttt{refutable}_?$ by assumption

(28) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (28f)

(29) $\texttt{prr}(e) \models_? \xi_2$ by Rule (18b) on (28) and (27)

Contradicts (23).

(30) $e \not\models_? (\xi_1, \xi_2)$ by contradiction

(31) $e \not\models^\dagger_? (\xi_1, \xi_2)$ by Lemma 2.0.20 on (25) and (30)

Case (21g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(14) e_1 `final` by Lemma 4.0.5 on (6)

(15) e_2 `final` by Lemma 4.0.5 on (6)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \models \xi_1$, $e_1 \models_? \xi_1$, and $e_1 \models \overline{\xi_1}$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \models \xi_2$, $e_2 \models_? \xi_2$, and $e_2 \models \overline{\xi_2}$ holds.
By case analysis on which conclusion holds for ξ_1 and ξ_2.

Case $e_1 \models \xi_1, e_2 \models \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (16i) on (16) and (18)

(21) $(e_1, e_2) \models^\dagger_? (\xi_1, \xi_2)$ by Rule (19b) on (20)

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by Rule (18h) on (16) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ by Rule (19a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \xi_1$ — by assumption

Contradicts (17).

Case (18h).

(22) $e_2 \models_? \xi_2$ — by assumption

Contradicts (19).

Case (18i).

(22) $e_1 \models_? \xi_1$ — by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ — by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ — by Lemma 2.0.20 on (21) and (23)

Case $e_1 \models_? \xi_1, e_2 \models \xi_2$.

(16) $e_1 \not\models \xi_1$ — by assumption

(17) $e_1 \models_? \xi_1$ — by assumption

(18) $e_2 \models \xi_2$ — by assumption

(19) $e_2 \not\models_? \xi_2$ — by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ — by Rule (18g) on (17) and (18)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ — by Rule (19a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(22) (e_1, e_2) `notintro` — by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(22) $e_1 \models \xi_1$ — by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ — by contradiction

Case $e_1 \models_? \xi_1, e_2 \models_? \xi_2$.

(16) $e_1 \not\models \xi_1$ — by assumption

(17) $e_1 \models_? \xi_1$ — by assumption

(18) $e_2 \not\models \xi_2$ — by assumption

(19) $e_2 \models_? \xi_2$ — by assumption

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ — by Rule (18i) on (17) and (19)

(21) $(e_1, e_2) \models_?^\dagger (\xi_1, \xi_2)$ — by Rule (19a) on (20)

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(23) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Case $e_1 \models_? \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_2 \models \xi_2$ by assumption

Contradicts (18).

Case (18h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models^\dagger_? (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models^\dagger_? \xi_1, e_2 \models \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j)**.**

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i)**.**

(20) $e_1 \models \xi_1$ by assumption

Contradicts (16).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b)**.**

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (18h)**.**

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (18i)**.**

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models^\dagger_? (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models^\dagger_? \xi_1, e_2 \models_? \xi_2$**.**

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_1 \models \xi_1$ by assumption

Contradicts (16).

Case (18i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models_?^\dagger (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

Case $e_1 \not\models_?^\dagger \xi_1, e_2 \not\models_?^\dagger \xi_2$.

(16) $e_1 \not\models \xi_1$ by assumption

(17) $e_1 \not\models_? \xi_1$ by assumption

(18) $e_2 \not\models \xi_2$ by assumption

(19) $e_2 \not\models_? \xi_2$ by assumption

Assume $(e_1, e_2) \models (\xi_1, \xi_2)$. By rule induction over Rules (16) on it, only two cases apply.

Case (16j).

(20) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (16i).

(20) $e_2 \models \xi_2$ by assumption

Contradicts (18).

(21) $(e_1, e_2) \not\models (\xi_1, \xi_2)$ by contradiction

Assume $(e_1, e_2) \models_? (\xi_1, \xi_2)$. By rule induction over Rules (18) on it, the following cases apply.

Case (18b).

(22) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

Case (18h).

(22) $e_2 \models_? \xi_2$ by assumption

Contradicts (19).

Case (18i).

(22) $e_1 \models_? \xi_1$ by assumption

Contradicts (17).

(23) $(e_1, e_2) \not\models_? (\xi_1, \xi_2)$ by contradiction

(24) $(e_1, e_2) \not\models^\dagger_? (\xi_1, \xi_2)$ by Lemma 2.0.20 on (21) and (23)

□

Definition 2.1.1 (Entailment of Constraints)**.** *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$. *Then* $\xi_1 \models \xi_2$ *iff for all* e *such that* $\cdot\,; \Delta \vdash e : \tau$ *and* e `val` *we have* $e \models^\dagger_? \xi_1$ *implies* $e \models \xi_2$

Definition 2.1.2 (Potential Entailment of Constraints)**.** *Suppose that* $\xi_1 : \tau$ *and* $\xi_2 : \tau$. *Then* $\xi_1 \models^\dagger_? \xi_2$ *iff for all* e *such that* $\cdot\,; \Delta \vdash e : \tau$ *and* e `final` *we have* $e \models^\dagger_? \xi_1$ *implies* $e \models^\dagger_? \xi_2$

Corollary 2.1.1. *Suppose that* $\xi : \tau$ *and* $\cdot; \Delta \vdash e : \tau$ *and* e `final`. *Then* $\top \models^\dagger_? \xi$ *implies* $e \models^\dagger_? \xi$

Proof.

(1) $\xi : \tau$ by assumption

(2) $\cdot\,; \Gamma \vdash e : \tau$ by assumption

(3) e `final` by assumption

(4) $\top \models^{\dagger}_{?} \xi$ by assumption

(5) $e_1 \models \top$ by Rule (16a)

(6) $e_1 \models^{\dagger}_{?} \top$ by Rule (19b) on (5)

(7) $\top : \tau$ by Rule (10a)

(8) $e_1 \models^{\dagger}_{?} \xi_r$ by Definition 2.1.2 of (4) on (7) and (1) and (2) and (3) and (6)

$\square$

3 Static Semantics

$$
\begin{array}{lll}
\tau & ::= & \texttt{num} \mid (\tau_1 \to \tau_2) \mid (\tau_1 \times \tau_2) \mid (\tau_1 + \tau_2) \\
e & ::= & x \mid \underline{n} \\
& \mid & (\lambda x : \tau . e) \mid e_1(e_2) \\
& \mid & (e_1, e_2) \\
& \mid & \texttt{inl}_\tau(e) \mid \texttt{inr}_\tau(e) \mid \texttt{match}(e)\{\hat{rs}\} \\
& \mid & (\!|\!)^u \mid (\!|e|\!)^u \\
\hat{rs} & ::= & (rs \mid r \mid rs) \\
rs & ::= & \cdot \mid (r \mid rs') \\
r & ::= & p \Rightarrow e \\
p & ::= & x \mid \underline{n} \mid \_ \mid (p_1, p_2) \mid \texttt{inl}(p) \mid \texttt{inr}(p) \mid (\!|\!)^w \mid (\!|p|\!)^w
\end{array}
$$

$\boxed{(\hat{rs})^\diamond = rs}$ rs can be obtained by erasing pointer from $\hat{rs}$

$$(\cdot \mid r \mid rs)^\diamond = r \mid rs \tag{20a}$$

$$((r' \mid rs') \mid r \mid rs)^\diamond = r' \mid (rs' \mid r \mid rs)^\diamond \tag{20b}$$

$\boxed{\Gamma\,;\Delta \vdash e : \tau}$ e is of type τ

$$\text{TVar}\quad \frac{}{\Gamma, x : \tau\,; \Delta \vdash x : \tau} \tag{21a}$$

$$\text{TEHole}\quad \frac{}{\Gamma\,; \Delta, u :: \tau \vdash (\!|\!)^u : \tau} \tag{21b}$$

$$\text{THole}\quad \frac{\Gamma\,; \Delta, u :: \tau \vdash e : \tau'}{\Gamma\,; \Delta, u :: \tau \vdash (\!|e|\!)^u : \tau} \tag{21c}$$

$$\text{TNum}\quad \frac{}{\Gamma\,; \Delta \vdash \underline{n} : \texttt{num}} \tag{21d}$$

$$\text{TLam}\quad \frac{\Gamma, x : \tau_1\,; \Delta \vdash e : \tau_2}{\Gamma\,; \Delta \vdash (\lambda x : \tau_1 . e) : (\tau_1 \to \tau_2)} \tag{21e}$$

$$\frac{\Gamma\,;\Delta \vdash e_1 : (\tau_2 \to \tau) \qquad \Gamma\,;\Delta \vdash e_2 : \tau_2}{\Gamma\,;\Delta \vdash e_1(e_2) : \tau}\;\text{TAp} \tag{21f}$$

$$\frac{\Gamma\,;\Delta \vdash e_1 : \tau_1 \qquad \Gamma\,;\Delta \vdash e_2 : \tau_2}{\Gamma\,;\Delta \vdash (e_1, e_2) : (\tau_1 \times \tau_2)}\;\text{TPair} \tag{21g}$$

$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \texttt{prl}(e) : \tau_1}\;\text{TPrl} \tag{21h}$$

$$\frac{\Gamma\,;\Delta \vdash e : (\tau_1 \times \tau_2)}{\Gamma\,;\Delta \vdash \texttt{prr}(e) : \tau_2}\;\text{TPrr} \tag{21i}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau_1}{\Gamma\,;\Delta \vdash \texttt{inl}_{\tau_2}(e) : (\tau_1 + \tau_2)}\;\text{TInl} \tag{21j}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau_2}{\Gamma\,;\Delta \vdash \texttt{inr}_{\tau_1}(e) : (\tau_1 + \tau_2)}\;\text{TInr} \tag{21k}$$

$$\frac{\Gamma\,;\Delta \vdash e : \tau \qquad \Gamma\,;\Delta \vdash [\bot]r \mid rs : \tau[\xi] \Rightarrow \tau' \qquad \top \models^{\dagger}_{?} \xi}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{\cdot \mid r \mid rs\} : \tau'}\;\text{TMatchZPre} \tag{21l}$$

$$\frac{\begin{array}{c}\Gamma\,;\Delta \vdash e : \tau \qquad e\ \texttt{final} \qquad \Gamma\,;\Delta \vdash [\bot]rs_{pre} : \tau[\xi_{pre}] \Rightarrow \tau' \\ \Gamma\,;\Delta \vdash [\bot \vee \xi_{pre}]r \mid rs_{post} : \tau[\xi_{rest}] \Rightarrow \tau' \qquad e \not\models^{\dagger}_{?} \xi_{pre} \qquad \top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}\end{array}}{\Gamma\,;\Delta \vdash \texttt{match}(e)\{rs_{pre} \mid r \mid rs_{post}\} : \tau'}\;\text{TMatchNZPre} \tag{21m}$$

$\boxed{p : \tau[\xi] \dashV \Gamma\,;\Delta}$ p is assigned type τ and emits constraint ξ

$$\frac{}{x : \tau[\top] \dashV \cdot\,; x : \tau}\;\text{PTVar} \tag{22a}$$

$$\frac{}{\_ : \tau[\top] \dashV \cdot\,;\cdot}\;\text{PTWild} \tag{22b}$$

$$\frac{}{\llparenthesis\rrparenthesis^{w} : \tau[?] \dashV \cdot\,; w :: \tau}\;\text{PTEHole} \tag{22c}$$

$$\frac{p : \tau[\xi] \dashV \Gamma\,;\Delta}{\llparenthesis p \rrparenthesis^{w} : \tau'[?] \dashV \Gamma\,;\Delta, w :: \tau'}\;\text{PTHole} \tag{22d}$$

$$\frac{}{\underline{n} : \texttt{num}[\underline{n}] \dashV \cdot\,;\cdot}\;\text{PTNum} \tag{22e}$$

$$\frac{p : \tau_1[\xi] \dashv\mid \Gamma \, ; \Delta}{\texttt{inl}(p) : (\tau_1 + \tau_2)[\texttt{inl}(\xi)] \dashv\mid \Gamma \, ; \Delta} \;\text{PTInl} \tag{22f}$$

$$\frac{p : \tau_2[\xi] \dashv\mid \Gamma \, ; \Delta}{\texttt{inr}(p) : (\tau_1 + \tau_2)[\texttt{inr}(\xi)] \dashv\mid \Gamma \, ; \Delta} \;\text{PTInr} \tag{22g}$$

$$\frac{p_1 : \tau_1[\xi_1] \dashv\mid \Gamma_1 \, ; \Delta_1 \qquad p_2 : \tau_2[\xi_2] \dashv\mid \Gamma_2 \, ; \Delta_2}{(p_1, p_2) : (\tau_1 \times \tau_2)[(\xi_1, \xi_2)] \dashv\mid \Gamma_1 \uplus \Gamma_2 \, ; \Delta_1 \uplus \Delta_2} \;\text{PTPair} \tag{22h}$$

$\boxed{\Gamma \, ; \Delta \vdash (p \Rightarrow e) : \tau[\xi] \Rightarrow \tau'}$ r transforms a final expression of type τ to a final expression of type τ'

$$\frac{p : \tau[\xi] \dashv\mid \Gamma_p \, ; \Delta_p \qquad \Gamma \uplus \Gamma_p \, ; \Delta \uplus \Delta_p \vdash e : \tau'}{\Gamma \, ; \Delta \vdash p \Rightarrow e : \tau[\xi] \Rightarrow \tau'} \;\text{CTRule} \tag{23a}$$

$\boxed{\Gamma \, ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'}$ rs transforms a final expression of type τ to a final expression of type τ'

$$\frac{\Gamma \, ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma \, ; \Delta \vdash [\xi_{pre}](r \mid \cdot) : \tau[\xi_r] \Rightarrow \tau'} \;\text{CTOneRules} \tag{24a}$$

$$\frac{\Gamma \, ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau' \qquad \Gamma \, ; \Delta \vdash [\xi_{pre} \vee \xi_r]rs : \tau[\xi_{rs}] \Rightarrow \tau' \qquad \xi_r \not\models \xi_{pre}}{\Gamma \, ; \Delta \vdash [\xi_{pre}]r \mid rs : \tau[\xi_r \vee \xi_{rs}] \Rightarrow \tau'} \;\text{CTRules} \tag{24b}$$

Lemma 3.0.1. *If $p : \tau[\xi] \dashv\mid \Gamma \, ; \Delta$ then $\xi : \tau$.*

Proof. By rule induction over Rules (22). ☐

Lemma 3.0.2. *If $\cdot \, ; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ then $\xi_r : \tau_1$.*

Proof. By rule induction over Rules (23). ☐

Lemma 3.0.3. *If $\cdot \, ; \Delta \vdash [\xi_{pre}]rs : \tau_1[\xi_{rs}] \Rightarrow \tau$ then $\xi_{rs} : \tau_1$.*

Proof. By rule induction over Rules (24). ☐

Lemma 3.0.4. *If $\Gamma \, ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ and $\Gamma \, ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ and $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ then $\Gamma \, ; \Delta \vdash [\xi_{pre}](rs \mid r \mid \cdot)^\diamond : \tau[\xi_{rs} \vee \xi_r] \Rightarrow \tau'$*

Proof.

(1) $\Gamma \, ; \Delta \vdash [\xi_{pre}]rs : \tau[\xi_{rs}] \Rightarrow \tau'$ by assumption

(2) $\Gamma \, ; \Delta \vdash r : \tau[\xi_r] \Rightarrow \tau'$ by assumption

(3) $\xi_r \not\models \xi_{pre} \vee \xi_{rs}$ by assumption

By rule induction over Rules (24) on (1).

Case (24a)**.**

(4)	$rs = r' \mid \cdot$	by assumption
(5)	$\xi_{rs} = \xi'_r$	by assumption
(6)	$\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$	by assumption
(7)	$\xi'_r \not\models \xi_{pre}$	by assumption
(8)	$\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]r \mid \cdot : \tau[\xi_r] \Rightarrow \tau'$	by Rule (24a) on (2) and (3)
(9)	$\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid r \mid \cdot) : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$	by Rule (24b) on (6) and (8) and (7)
(10)	$\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid \cdot) \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi_r] \Rightarrow \tau'$	by Definition 20 on (9)

Case (24b)**.**

(4)	$rs = r' \mid rs'$	by assumption
(5)	$\xi_{rs} = \xi'_r \vee \xi'_{rs}$	by assumption
(6)	$\Gamma\,;\Delta \vdash r' : \tau[\xi'_r] \Rightarrow \tau'$	by assumption
(7)	$\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r]rs' : \tau[\xi'_{rs}] \Rightarrow \tau'$	by assumption
(8)	$\xi'_r \not\models \xi_{pre}$	by assumption
(9)	$\Gamma\,;\Delta \vdash [\xi_{pre} \vee \xi'_r](rs' \mid r \mid \cdot)^{\diamond} : \tau[\xi'_{rs} \vee \xi_r] \Rightarrow \tau'$	by IH on (7) and (2) and (3)
(10)	$\Gamma\,;\Delta \vdash [\xi_{pre}](r' \mid (rs' \mid r \mid \cdot)^{\diamond}) : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$	by Rule (24b) on (6) and (9) and (8)
(11)	$\Gamma\,;\Delta \vdash [\xi_{pre}]((r' \mid rs') \mid r \mid \cdot)^{\diamond} : \tau[\xi'_r \vee \xi'_{rs} \vee \xi_r] \Rightarrow \tau'$	by Definition 20 on (10)

$\square$

Lemma 3.0.5 (Substitution)**.** *If* $\Gamma, x : \tau\,;\Delta \vdash e_0 : \tau_0$ *and* $\Gamma\,;\Delta \vdash e : \tau$ *then* $\Gamma\,;\Delta \vdash [e/x]e_0 : \tau_0$

Lemma 3.0.6 (Simultaneous Substitution)**.** *If* $\Gamma \uplus \Gamma'\,;\Delta \vdash e : \tau$ *and* $\theta : \Gamma'$ *then* $\Gamma\,;\Delta \vdash [\theta]e : \tau$

Lemma 3.0.7 (Substitution Typing)**.** *If* $e \rhd p \dashv\!\!\mid \theta$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\!\mid \Gamma\,;\Delta$ *then* $\theta : \Gamma$

Proof by induction on the derivation of $e \rhd p \dashv\!\!\mid \theta$.

Theorem 3.1 (Determinism)**.** *If* $\cdot\,;\Delta \vdash e : \tau$ *then exactly one of the following holds*

1. e `val`

2. e `indet`

3. $e \mapsto e'$ *for some unique* e'

4 Dynamic Semantics

$\boxed{e\ \mathtt{val}}$ e is a value

$$\frac{}{\underline{n}\ \mathtt{val}}\ \text{VNum} \tag{25a}$$

$$\frac{}{(\lambda x : \tau.e)\ \mathtt{val}}\ \text{VLam} \tag{25b}$$

$$\frac{e_1\ \mathtt{val} \qquad e_2\ \mathtt{val}}{(e_1, e_2)\ \mathtt{val}}\ \text{VPair} \tag{25c}$$

$$\frac{e\ \mathtt{val}}{\mathtt{inl}_\tau(e)\ \mathtt{val}}\ \text{VInl} \tag{25d}$$

$$\frac{e\ \mathtt{val}}{\mathtt{inr}_\tau(e)\ \mathtt{val}}\ \text{VInr} \tag{25e}$$

$\boxed{e\ \mathtt{indet}}$ e is indeterminate

$$\frac{}{(\!|\!)^u\ \mathtt{indet}}\ \text{IEHole} \tag{26a}$$

$$\frac{e\ \mathtt{final}}{(\!|e|\!)^u\ \mathtt{indet}}\ \text{IHole} \tag{26b}$$

$$\frac{e_1\ \mathtt{indet} \qquad e_2\ \mathtt{final}}{e_1(e_2)\ \mathtt{indet}}\ \text{IAp} \tag{26c}$$

$$\frac{e_1\ \mathtt{indet} \qquad e_2\ \mathtt{val}}{(e_1, e_2)\ \mathtt{indet}}\ \text{IPairL} \tag{26d}$$

$$\frac{e_1\ \mathtt{val} \qquad e_2\ \mathtt{indet}}{(e_1, e_2)\ \mathtt{indet}}\ \text{IPairR} \tag{26e}$$

IPair

$$\frac{e_1\ \texttt{indet} \qquad e_2\ \texttt{indet}}{(e_1, e_2)\ \texttt{indet}} \tag{26f}$$

IPrl

$$\frac{e\ \texttt{indet}}{\texttt{prl}(e)\ \texttt{indet}} \tag{26g}$$

IPrr

$$\frac{e\ \texttt{indet}}{\texttt{prr}(e)\ \texttt{indet}} \tag{26h}$$

IInL

$$\frac{e\ \texttt{indet}}{\texttt{inl}_\tau(e)\ \texttt{indet}} \tag{26i}$$

IInR

$$\frac{e\ \texttt{indet}}{\texttt{inr}_\tau(e)\ \texttt{indet}} \tag{26j}$$

IMatch

$$\frac{e\ \texttt{final} \qquad e\ ?\ p_r}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\}\ \texttt{indet}} \tag{26k}$$

$\boxed{e\ \texttt{final}}$ e is final

FVal

$$\frac{e\ \texttt{val}}{e\ \texttt{final}} \tag{27a}$$

FIndet

$$\frac{e\ \texttt{indet}}{e\ \texttt{final}} \tag{27b}$$

$\boxed{e\ \texttt{notintro}}$ e cannot be a value syntactically

NVEHole

$$\frac{}{(\!|\,|\!)^u\ \texttt{notintro}} \tag{28a}$$

NVHole

$$\frac{}{(\!|e|\!)^u\ \texttt{notintro}} \tag{28b}$$

NVAp

$$\frac{}{e_1(e_2)\ \texttt{notintro}} \tag{28c}$$

NVMatch

$$\frac{}{\texttt{match}(e)\{\hat{rs}\}\ \texttt{notintro}} \tag{28d}$$

NVPrl

$$\frac{}{\texttt{prl}(e)\ \texttt{notintro}} \tag{28e}$$

$$\frac{\text{NVPrr}}{\texttt{prr}(e)\ \texttt{notintro}} \tag{28f}$$

$\boxed{\textit{notintro}(e)}$

$$\textit{notintro}(\llparenthesis\rrparenthesis^u) = \text{true} \tag{29a}$$
$$\textit{notintro}(\llparenthesis e\rrparenthesis^u) = \text{true} \tag{29b}$$
$$\textit{notintro}(e_1(e_2)) = \text{true} \tag{29c}$$
$$\textit{notintro}(\texttt{match}(e)\{\hat{rs}\}) = \text{true} \tag{29d}$$
$$\textit{notintro}(\texttt{prl}(e)) = \text{true} \tag{29e}$$
$$\textit{notintro}(\texttt{prr}(e)) = \text{true} \tag{29f}$$
$$\text{Otherwise} \quad \textit{notintro}(e) = \text{false} \tag{29g}$$

Lemma 4.0.1 (Soundness and Completeness of NotIntro Judgment)**.** *e $\texttt{notintro}$ iff $\textit{notintro}(e)$.*

Proof. TODO □

$\boxed{e' \in \texttt{values}(e)}$ e' is one of the possible values of e

$$\frac{\text{IVVal} \quad e\ \texttt{val} \qquad \cdot\,;\Delta \vdash e : \tau}{e \in \texttt{values}(e)} \tag{30a}$$

$$\frac{\text{IVIndet} \quad e\ \texttt{notintro} \qquad \cdot\,;\Delta \vdash e : \tau \qquad e'\ \texttt{val} \qquad \cdot\,;\Delta \vdash e' : \tau}{e' \in \texttt{values}(e)} \tag{30b}$$

$$\frac{\text{IVInl} \quad \texttt{inl}_{\tau_2}(e_1)\ \texttt{indet} \qquad \cdot\,;\Delta \vdash \texttt{inl}_{\tau_2}(e_1) : \tau \qquad e_1' \in \texttt{values}(e_1)}{\texttt{inl}_{\tau_2}(e_1') \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))} \tag{30c}$$

$$\frac{\text{IVInr} \quad \texttt{inr}_{\tau_1}(e_2)\ \texttt{indet} \qquad \cdot\,;\Delta \vdash \texttt{inr}_{\tau_1}(e_2) : \tau \qquad e_2' \in \texttt{values}(e_2)}{\texttt{inr}_{\tau_1}(e_2') \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))} \tag{30d}$$

$$\frac{\text{IVPair} \quad (e_1, e_2)\ \texttt{indet} \qquad \cdot\,;\Delta \vdash (e_1, e_2) : \tau \qquad e_1' \in \texttt{values}(e_1) \qquad e_2' \in \texttt{values}(e_2)}{(e_1', e_2') \in \texttt{values}((e_1, e_2))} \tag{30e}$$

Lemma 4.0.2. *If e $\texttt{indet}$ and $\cdot\,;\Delta \vdash e : \tau$ and $\dot{\xi} : \tau$ and $e \not\models^{\dagger}_{?} \dot{\xi}$ then $e' \not\models^{\dagger}_{?} \dot{\xi}$ whenever $e' \in \texttt{values}(e)$.*

Proof.

(1) e $\texttt{indet}$ by assumption

(2) $\cdot\,;\Delta \vdash e : \tau$ by assumption

(3) $\dot{\xi} : \tau$ by assumption

(4) $e \not\models^{\dagger}_{?} \dot{\xi}$ by assumption

By rule induction over Rules (10) on (3).

Case (10a).

(5) $\dot{\xi} = \top$ by assumption

(6) $e \models \top$ by Rule (16a)

(7) $e \models^{\dagger}_{?} \top$ by Rule (19b) on (6)

Contradicts (4).

Case (1b).

(5) $\dot{\xi} = ?$ by assumption

(6) $e \models_{?} ?$ by Rule (18a)

(7) $e \models^{\dagger}_{?} ?$ by Rule (19a) on (6)

Contradicts (4).

Case (10c).

(5) $\dot{\xi} = \underline{n}$ by assumption

(6) $\tau = \texttt{num}$ by assumption

(7) $\underline{n}\ \texttt{refutable}_{?}$ by Rule (12a)

By rule induction over Rules (26) on (1).

Case (26a).

(8) $e = (\!|\,|\!)^u$ by assumption

(9) $(\!|\,|\!)^u\ \texttt{notintro}$ by Rule (28a)

(10) $(\!|\,|\!)^u \models_{?} \underline{n}$ by Rule (18b) on (9) and (7)

(11) $(\!|\,|\!)^u \models^{\dagger}_{?} \underline{n}$ by Rule (19a) on (10)

Contradicts (4).

Case (26b).

(8) $e = (\!|e_1|\!)^u$ by assumption

(9) $(\!|e_1|\!)^u\ \texttt{notintro}$ by Rule (28b)

(10) $(\!|e_1|\!)^u \models_{?} \underline{n}$ by Rule (18b) on (9) and (7)

(11) $(\!|e_1|\!)^u \models^{\dagger}_{?} \underline{n}$ by Rule (19a) on (10)

Contradicts (4).

Case (26c).

(8)	$e = e_1(e_2)$	by assumption
(9)	$e_1(e_2)$ $\texttt{notintro}$	by Rule (28c)
(10)	$e_1(e_2) \models_? \underline{n}$	by Rule (18b) on (9) and (7)
(11)	$e_1(e_2) \models_?^\dagger \underline{n}$	by Rule (19a) on (10)

Contradicts (4).

Case (26g).

(8)	$e = \texttt{prl}(e_1)$	by assumption
(9)	$\texttt{prl}(e_1)$ $\texttt{notintro}$	by Rule (28e)
(10)	$\texttt{prl}(e_1) \models_? \underline{n}$	by Rule (18b) on (9) and (7)
(11)	$\texttt{prl}(e_1) \models_?^\dagger \underline{n}$	by Rule (19a) on (10)

Contradicts (4).

Case (26h).

(8)	$e = \texttt{prr}(e_1)$	by assumption
(9)	$\texttt{prr}(e_1)$ $\texttt{notintro}$	by Rule (28f)
(10)	$\texttt{prr}(e_1) \models_? \underline{n}$	by Rule (18b) on (9) and (7)
(11)	$\texttt{prr}(e_1) \models_?^\dagger \underline{n}$	by Rule (19a) on (10)

Contradicts (4).

Case (26k).

(8)	$e = \texttt{match}(e_1)\{\hat{rs}\}$	by assumption
(9)	$\texttt{match}(e_1)\{\hat{rs}\}$ $\texttt{notintro}$	by Rule (28d)
(10)	$\texttt{match}(e_1)\{\hat{rs}\} \models_? \underline{n}$	by Rule (18b) on (9) and (7)
(11)	$\texttt{match}(e_1)\{\hat{rs}\} \models_?^\dagger \underline{n}$	by Rule (19a) on (10)

Contradicts (4).

Case (26d), (26e), (26f).

(8)	$e = (e_1, e_2)$	by assumption

By rule induction over Rules (21) on (2), no rule applies due to syntactic contradiction.

Case (26i).

(8)	$e = \texttt{inl}_{\tau_2}(e_1)$	by assumption

By rule induction over Rules (21) on (2), no rule applies due to syntactic contradiction.

Case (26j).

(8) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

By rule induction over Rules (21) on (2), no rule applies due to syntactic contradiction.

Case (10g).

(5) $\dot{\xi} = \mathtt{inl}(\dot{\xi}_1)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $\dot{\xi}_1 : \tau_1$ by assumption

(8) $\mathtt{inl}(\dot{\xi}_1)\ \mathtt{refutable}_?$ by Rule (12b)

By rule induction over Rules (26) on (1).

Case (26a).

(9) $e = (\!|\!)^u$ by assumption

(10) $(\!|\!)^u\ \mathtt{notintro}$ by Rule (28a)

(11) $(\!|\!)^u \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (18b) on (10) and (8)

(12) $(\!|\!)^u \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (19a) on (11)

Contradicts (4).

Case (26b).

(9) $e = (\!|e_1|\!)^u$ by assumption

(10) $(\!|e_1|\!)^u\ \mathtt{notintro}$ by Rule (28b)

(11) $(\!|e_1|\!)^u \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (18b) on (10) and (8)

(12) $(\!|e_1|\!)^u \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (19a) on (11)

Contradicts (4).

Case (26c).

(9) $e = e_1(e_2)$ by assumption

(10) $e_1(e_2)\ \mathtt{notintro}$ by Rule (28c)

(11) $e_1(e_2) \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (18b) on (10) and (8)

(12) $e_1(e_2) \models_?^\dagger \mathtt{inl}(\dot{\xi}_1)$ by Rule (19a) on (11)

Contradicts (4).

Case (26g).

(9) $e = \mathtt{prl}(e_1)$ by assumption

(10) $\mathtt{prl}(e_1)\ \mathtt{notintro}$ by Rule (28e)

(11) $\mathtt{prl}(e_1) \models_? \mathtt{inl}(\dot{\xi}_1)$ by Rule (18b) on (10) and (8)

(12) $\texttt{prl}(e_1) \models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Rule (19a) on (11)

Contradicts (4).

Case (26h).

(9) $e = \texttt{prr}(e_1)$ by assumption

(10) $\texttt{prr}(e_1)\ \texttt{notintro}$ by Rule (28f)

(11) $\texttt{prr}(e_1) \models_{?} \texttt{inl}(\dot{\xi}_1)$ by Rule (18b) on (10) and (8)

(12) $\texttt{prr}(e_1) \models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Rule (19a) on (11)

Contradicts (4).

Case (26k).

(9) $e = \texttt{match}(e_1)\{\hat{rs}\}$ by assumption

(10) $\texttt{match}(e_1)\{\hat{rs}\}\ \texttt{notintro}$ by Rule (28d)

(11) $\texttt{match}(e_1)\{\hat{rs}\} \models_{?} \texttt{inl}(\dot{\xi}_1)$ by Rule (18b) on (10) and (8)

(12) $\texttt{match}(e_1)\{\hat{rs}\} \models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Rule (19a) on (11)

Contradicts (4).

Case (26d), (26e), (26f).

(9) $e = (e_1, e_2)$ by assumption

By rule induction over Rules (26) on (1), no rule applies due to syntactic contradiction.

Case (26i).

(9) $e = \texttt{inl}_{\tau_2'}(e_1)$ by assumption

(10) $e_1\ \texttt{indet}$ by assumption

By rule induction over Rules (21) on (2), only one rule applies.

Case (21j).

(11) $\tau_2' = \tau_2$ by assumption

(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by Lemma 2.0.11 on (4)

(14) if $e_1' \in \texttt{values}(e_1)$ then $e_1' \not\models^{\dagger}_{?} \dot{\xi}_1$
by IH on (10) and (12) and (7) and (13)

To show if $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ then $e' \not\models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$, we assume $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$.

(15) $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ by assumption

By rule induction over Rules (30) on (15).

Case (30a).

(16) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(16) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.7

Case (30c).

(16) $e' = \texttt{inl}_{\tau_2}(e'_1)$ by assumption

(17) $e'_1 \in \texttt{values}(e_1)$ by assumption

(18) $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by (14) on (17)

(19) $\texttt{inl}_{\tau_2}(e'_1) \not\models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Lemma 2.0.11 on (18)

Case (26j).

(9) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

To show if $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$ then $e' \not\models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$, we assume $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$.

(10) $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$ by assumption

By rule induction over Rules (30) on (10).

Case (30a).

(11) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(11) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.8

Case (30d).

(11) $e' = \texttt{inr}_{\tau_1}(e'_2)$ by assumption

(12) $\texttt{inr}_{\tau_1}(e'_2) \not\models^{\dagger}_{?} \texttt{inl}(\dot{\xi}_1)$ by Lemma 1.0.22

Case (10h).

(5) $\dot{\xi} = \texttt{inr}(\dot{\xi}_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $\dot{\xi}_2 : \tau_2$ by assumption

(8) $\texttt{inr}(\dot{\xi}_2)\ \texttt{refutable}_?$ by Rule (12c)

By rule induction over Rules (26) on (1).

Case (26a).

(9) $e = (\!|\!)^u$ by assumption

(10) $(\!|\!)^u\ \texttt{notintro}$ by Rule (28a)

(11)	$(\!\|\|\!)^u \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (18b) on (10) and (8)
(12)	$(\!\|\|\!)^u \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (19a) on (11)

Contradicts (4).

Case (26b).

(9)	$e = (\!\|e_1\|\!)^u$	by assumption
(10)	$(\!\|e_1\|\!)^u$ $\mathtt{notintro}$	by Rule (28b)
(11)	$(\!\|e_1\|\!)^u \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (18b) on (10) and (8)
(12)	$(\!\|e_1\|\!)^u \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (19a) on (11)

Contradicts (4).

Case (26c).

(9)	$e = e_1(e_2)$	by assumption
(10)	$e_1(e_2)$ $\mathtt{notintro}$	by Rule (28c)
(11)	$e_1(e_2) \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (18b) on (10) and (8)
(12)	$e_1(e_2) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (19a) on (11)

Contradicts (4).

Case (26g).

(9)	$e = \mathtt{prl}(e_1)$	by assumption
(10)	$\mathtt{prl}(e_1)$ $\mathtt{notintro}$	by Rule (28e)
(11)	$\mathtt{prl}(e_1) \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (18b) on (10) and (8)
(12)	$\mathtt{prl}(e_1) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (19a) on (11)

Contradicts (4).

Case (26h).

(9)	$e = \mathtt{prr}(e_1)$	by assumption
(10)	$\mathtt{prr}(e_1)$ $\mathtt{notintro}$	by Rule (28f)
(11)	$\mathtt{prr}(e_1) \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (18b) on (10) and (8)
(12)	$\mathtt{prr}(e_1) \models_?^\dagger \mathtt{inr}(\dot{\xi}_2)$	by Rule (19a) on (11)

Contradicts (4).

Case (26k).

(9)	$e = \mathtt{match}(e_1)\{\hat{rs}\}$	by assumption
(10)	$\mathtt{match}(e_1)\{\hat{rs}\}$ $\mathtt{notintro}$	by Rule (28d)
(11)	$\mathtt{match}(e_1)\{\hat{rs}\} \models_? \mathtt{inr}(\dot{\xi}_2)$	by Rule (18b) on (10) and (8)

(12) $\texttt{match}(e_1)\{\hat{rs}\} \models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Rule (19a) on (11)

Contradicts (4).

Case (26d), (26e), (26f).

(9) $e = (e_1, e_2)$ by assumption

By rule induction over Rules (26) on (1), no rule applies due to syntactic contradiction.

Case (26i).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

To show if $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ then $e' \not\models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$, we assume $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$.

(10) $e' \in \texttt{values}(\texttt{inl}_{\tau_2}(e_1))$ by assumption

By rule induction over Rules (30) on (10).

Case (30a).

(11) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(11) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.7

Case (30c).

(11) $e' = \texttt{inl}_{\tau_2}(e'_1)$ by assumption

(12) $\texttt{inl}_{\tau_2}(e'_1) \not\models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$ by Lemma 1.0.21

Case (26j).

(9) $e = \texttt{inr}_{\tau'_1}(e_2)$ by assumption

(10) $e_2\ \texttt{indet}$ by assumption

By rule induction over Rules (21) on (2), only one rule applies.

Case (21k).

(11) $\tau'_1 = \tau_1$ by assumption

(12) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(13) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 2.0.11 on (4)

(14) if $e'_2 \in \texttt{values}(e_2)$ then $e'_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by IH on (10) and (12) and (7) and (13)

To show if $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$ then $e' \not\models^{\dagger}_{?} \texttt{inr}(\dot{\xi}_2)$, we assume $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$.

(15) $e' \in \texttt{values}(\texttt{inr}_{\tau_1}(e_2))$ by assumption

By rule induction over Rules (30) on (15).

Case (30a).

(16) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(16) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.8

Case (30d).

(16) $e' = \texttt{inr}_{\tau_1}(e_2')$ by assumption

(17) $e_2' \in \texttt{values}(e_2)$ by assumption

(18) $e_2' \not\models_?^\dagger \dot{\xi}_2$ by (14) on (17)

(19) $\texttt{inr}_{\tau_1}(e_2') \not\models_?^\dagger \texttt{inr}(\dot{\xi}_2)$ by Lemma 2.0.12 on (18)

Case (10i).

(5) $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2)$ by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ by assumption

(7) $\dot{\xi}_1 : \tau_1$ by assumption

(8) $\dot{\xi}_2 : \tau_2$ by assumption

By rule induction over Rules (26) on (1).

Case (26a), (26b), (26c), (26g), (26h), (26k).

(9) $e = (\!|\!)^u, (\!|e_1|\!)^u, e_1(e_2), \texttt{prl}(e_1), \texttt{prr}(e_1), \texttt{match}(e_1)\{\hat{rs}\}$ by assumption

(10) $e\ \texttt{notintro}$ by Rules (28)

(11) $\texttt{prl}(e)\ \texttt{notintro}$ by Rule (28e)

(12) $\texttt{prr}(e)\ \texttt{notintro}$ by Rule (28f)

(13) $\texttt{prl}(e)\ \texttt{indet}$ by Rule (26g) on (1)

(14) $\texttt{prr}(e)\ \texttt{indet}$ by Rule (26h) on (1)

(15) $\cdot\,; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (21h) on (2)

(16) $\cdot\,; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (21i) on (2)

By case analysis on the result of $\mathit{satisfyormay}(\texttt{prl}(e), \dot{\xi}_1)$.

Case true.

(17) $\mathit{satisfyormay}(\texttt{prl}(e), \dot{\xi}_1) = \text{true}$ by assumption

(18) $\texttt{prl}(e) \models_?^\dagger \dot{\xi}_1$ by Lemma 1.0.4 on (17)

By case analysis on the result of $\mathit{satisfyormay}(\texttt{prr}(e), \dot{\xi}_2)$.

Case true.

(19) $\mathit{satisfyormay}(\mathtt{prr}(e), \dot{\xi}_2) = \text{true}$
by assumption

(20) $\mathtt{prr}(e) \models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.4 on (19)

By rule induction over Rules (19) on (18).

Case (19b)**.**

(21) $\mathtt{prl}(e) \models \dot{\xi}_1$ by asssumption

By rule induction over Rules (19) on (20).

Case (19b)**.**

(22) $\mathtt{prr}(e) \models \dot{\xi}_2$ by asssumption

(23) $e \models (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (16j) on (10) and (21) and (22)

(24) $e \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19b) on (23)

Contradicts (4).

Case (19a)**.**

(22) $\mathtt{prr}(e) \models_{?} \dot{\xi}_2$ by assumption

(23) $\dot{\xi}_2$ $\mathtt{refutable}_{?}$ by Lemma 1.0.17 on (12) and (22)

(24) $(\dot{\xi}_1, \dot{\xi}_2)$ $\mathtt{refutable}_{?}$ by Rule (12e) on (23)

(25) $e \models_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (10) and (24)

(26) $e \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (25)

Case (19a)**.**

(21) $\mathtt{prl}(e) \models_{?} \dot{\xi}_1$ by assumption

(22) $\dot{\xi}_1$ $\mathtt{refutable}_{?}$ by Lemma 1.0.17 on (11) and (21)

(23) $(\dot{\xi}_1, \dot{\xi}_2)$ $\mathtt{refutable}_{?}$ by Rule (12d) on (22)

(24) $e \models_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (18b) on (10) and (23)

(25) $e \models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Rule (19a) on (24)

Case false**.**

(19) $\mathit{satisfyormay}(\mathtt{prr}(e), \dot{\xi}_2) = \text{false}$
by assumption

(20) $\mathtt{prr}(e) \models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 1.0.4 on (19)

(21) if $e'_2 \in \mathtt{values}(\mathtt{prr}(e))$ then $e'_2 \not\models^{\dagger}_{?} \dot{\xi}_2$
by IH on (14) and (16) and (8) and (20)

To show if $e' \in \mathtt{values}(e)$ then $e' \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \mathtt{values}(e)$.

(22) $e' \in \texttt{values}(e)$ by assumption

By rule induction over Rules (30) on (22), only two rules apply.

Case (30a)**.**

(23) $e\ \texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b)**.**

(23) $e'\ \texttt{val}$ by assumption

(24) $\cdot\,; \Delta \vdash e' : (\tau_1 \times \tau_2)$ by assumption

By rule induction over Rules (25) on (23).

Case (25a)**.**

(25) $e' = \underline{n}$ by assumption

By rule induction over Rules (21) on (24), no rule applies due to syntactic contradiction.

Case (25b)**.**

(25) $e' = (\lambda x : \tau'.e'_1)$ by assumption

By rule induction over Rules (21) on (24), no rule applies due to syntactic contradiction.

Case (25c)**.**

(25) $e' = (e'_1, e'_2)$ by assumption

(26) $e'_2\ \texttt{val}$ by assumption

By rule induction over Rules (21) on (24), only one rule applies.

Case (21g)**.**

(27) $\cdot\,; \Delta \vdash e'_2 : \tau_2$ by assumption

(28) $e'_2 \in \texttt{values}(\texttt{prr}(e))$ by Rule (30b) on (12) and (16) and (26) and (27)

(29) $e'_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by (21) on (28)

(30) $(e'_1, e'_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (27)

Case (25d)**.**

(25) $e' = \texttt{inl}_{\tau_2}(e'_1)$ by assumption

By rule induction over Rules (21) on (24), no rule applies due to syntactic contradiction.

Case (25e)**.**

(25) $e' = \texttt{inr}_{\tau_1}(e'_2)$ by assumption

By rule induction over Rules (21) on (24), no rule applies due to syntactic contradiction.

Case false**.**

(17) $\mathit{satisfyormay}(\texttt{prl}(e), \dot{\xi}_1) = \text{false}$ by assumption

(18) $\texttt{prl}(e) \not\models^{\dagger}_{?} \dot{\xi}_1$ by Lemma 1.0.4 on (17)

(19) if $e'_1 \in \texttt{values}(\texttt{prl}(e))$ then $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by IH on (13) and (15) and (7) and (18)

To show if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}(e)$.

(20) $e' \in \texttt{values}(e)$ by assumption

By rule induction over Rules (30) on (20), only two rules apply.

Case (30a)**.**

(21) e $\texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b)**.**

(21) e' $\texttt{val}$ by assumption

(22) $\cdot\,; \Delta \vdash e' : (\tau_1 \times \tau_2)$ by assumption

By rule induction over Rules (25) on (21).

Case (25a)**.**

(23) $e' = \underline{n}$ by assumption

By rule induction over Rules (21) on (22), no rule applies due to syntactic contradiction.

Case (25b)**.**

(23) $e' = (\lambda x : \tau'.e'_1)$ by assumption

By rule induction over Rules (21) on (22), no rule applies due to syntactic contradiction.

Case (25c)**.**

(23) $e' = (e'_1, e'_2)$ by assumption

(24) e'_1 $\texttt{val}$ by assumption

By rule induction over Rules (21) on (22), only one rule applies.

Case (21g)**.**

(25) $\cdot\,; \Delta \vdash e'_1 : \tau_1$ by assumption

(26) $e'_1 \in \texttt{values}(\texttt{prl}(e))$ by Rule (30b) on (11) and (15) and (24) and (25)

(27) $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by (19) on (26)

(28) $(e'_1, e'_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (27)

Case (25d)**.**

(23) $e' = \texttt{inl}_{\tau_2}(e'_1)$ by assumption

By rule induction over Rules (21) on (22), no rule applies due to syntactic contradiction.

Case (25e).

(23) $e' = \mathtt{inr}_{\tau_1}(e'_2)$ by assumption

By rule induction over Rules (21) on (22), no rule applies due to syntactic contradiction.

Case (26d).

(9) $e = (e_1, e_2)$ by assumption

(10) e_1 $\mathtt{indet}$ by assumption

(11) e_2 $\mathtt{val}$ by assumption

(12) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ or $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 2.0.13 on (4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$.

(13) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by assumption

By rule induction over Rules (21) on (2), only one rule applies.

Case (21g).

(14) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(15) if $e'_1 \in \mathtt{values}(e_1)$ then $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by IH on (10) and (14) and (7) and (13)

To show that if $e' \in \mathtt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \mathtt{values}((e_1, e_2))$.

(16) $e' \in \mathtt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (30) on (16).

Case (30a).

(17) (e_1, e_2) $\mathtt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(17) (e_1, e_2) $\mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.9.

Case (30e).

(17) $e' = (e'_1, e'_2)$ by assumption

(18) $e'_1 \in \mathtt{values}(e_1)$ by assumption

(19) $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by (15) on (18)

(20) $(e'_1, e'_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (19)

Case $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$.

(13) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(14) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (30) on (14).

Case (30a).

(15) (e_1, e_2) $\texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(15) (e_1, e_2) $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.9.

Case (30e).

(15) $e' = (e_1', e_2')$ by assumption

(16) $e_2' \in \texttt{values}(e_2)$ by assumption

By rule induction over Rules (30) on (16).

Case (30a).

(17) $e_2' = e_2$ by assumption

(18) $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by (17) and (13)

(19) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (18)

Case (30b).

(17) e_2 $\texttt{notintro}$ by assumption

Contradicts (11) by Lemma 4.0.12.

Case (30c), (30d), (30e).

(17) e_2 $\texttt{indet}$ by assumption

Contradicts (11) by Lemma 4.0.11.

Case (26e).

(9) $e = (e_1, e_2)$ by assumption

(10) e_1 $\texttt{val}$ by assumption

(11) e_2 $\texttt{indet}$ by assumption

(12) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ or $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 2.0.13 on (4)

By case analysis on the disjunction in (12).

Case $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$.

(13) $e_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by assumption

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(14) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (30) on (14).

Case (30a).

(15) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(15) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (30e).

(15) $e' = (e'_1, e'_2)$ by assumption

(16) $e'_1 \in \texttt{values}(e_1)$ by assumption

By rule induction over Rules (30) on (16).

Case (30a).

(17) $e'_1 = e_1$ by assumption

(18) $e'_1 \not\models^{\dagger}_{?} \dot{\xi}_1$ by (17) and (13)

(19) $(e'_1, e'_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (18)

Case (30b).

(17) e_1 `notintro` by assumption

Contradicts (10) by Lemma 4.0.12.

Case (30c), (30d), (30e).

(17) e_1 `indet` by assumption

Contradicts (10) by Lemma 4.0.11.

Case $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$.

(13) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

By rule induction over Rules (21) on (2), only one rule applies.

Case (21g).

(14) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(15) if $e'_2 \in \texttt{values}(e_2)$ then $e'_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by IH on (11) and (14) and (8) and (13)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(16) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (30) on (16).

Case (30a).

(17) (e_1, e_2) `val` by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(17) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (30e).

(17) $e' = (e'_1, e'_2)$ by assumption

(18) $e'_2 \in \texttt{values}(e_2)$ by assumption

(19) $e'_2 \not\models^\dagger_? \dot{\xi}_2$ by (15) on (18)

(20) $(e'_1, e'_2) \not\models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (19)

Case (26f)**.**

(9) $e = (e_1, e_2)$ by assumption

(10) e_1 $\texttt{indet}$ by assumption

(11) e_2 $\texttt{indet}$ by assumption

(12) $e_1 \not\models^\dagger_? \dot{\xi}_1$ or $e_2 \not\models^\dagger_? \dot{\xi}_2$ by Lemma 2.0.13 on (4)

By rule induction over Rules (21) on (2), only one rule applies.

Case (21g)**.**

(13) $\cdot\, ; \Delta \vdash e_1 : \tau_1$ by assumption

(14) $\cdot\, ; \Delta \vdash e_2 : \tau_2$ by assumption

By case analysis on the disjunction in (12).

Case $e_1 \not\models^\dagger_? \dot{\xi}_1$**.**

(15) $e_1 \not\models^\dagger_? \dot{\xi}_1$ by assumption

(16) if $e'_1 \in \texttt{values}(e_1)$ then $e'_1 \not\models^\dagger_? \dot{\xi}_1$ by IH on (10) and (13) and (7) and (15)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(17) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (30) on (17).

Case (30a)**.**

(18) (e_1, e_2) $\texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b)**.**

(18) (e_1, e_2) $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.9.

Case (30e)**.**

(18) $e' = (e'_1, e'_2)$ by assumption

(19) $e'_1 \in \texttt{values}(e_1)$ by assumption

(20) $e'_1 \not\models^\dagger_? \dot{\xi}_1$ by (16) on (19)

(21) $(e'_1, e'_2) \not\models^\dagger_? (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (20)

Case $e_2 \not\models^\dagger_? \dot{\xi}_2$**.**

(15) $e_2 \not\models^{\dagger}_{?} \dot{\xi}_2$ by assumption

(16) if $e_2' \in \texttt{values}(e_2)$ then $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by IH on (11) and (14) and (8) and (15)

To show that if $e' \in \texttt{values}((e_1, e_2))$ then $(e_1, e_2) \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$, we assume $e' \in \texttt{values}((e_1, e_2))$.

(17) $e' \in \texttt{values}((e_1, e_2))$ by assumption

By rule induction over Rules (30) on (17).

Case (30a).

(18) (e_1, e_2) $\texttt{val}$ by assumption

Contradicts (1) by Lemma 4.0.11.

Case (30b).

(18) (e_1, e_2) $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.9.

Case (30e).

(18) $e' = (e_1', e_2')$ by assumption

(19) $e_2' \in \texttt{values}(e_2)$ by assumption

(20) $e_2' \not\models^{\dagger}_{?} \dot{\xi}_2$ by (16) on (19)

(21) $(e_1', e_2') \not\models^{\dagger}_{?} (\dot{\xi}_1, \dot{\xi}_2)$ by Lemma 2.0.13 on (20)

Case (26i).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

By rule induction over Rules (21) on (2), no rule applies due to syntactic contradiction.

Case (26j).

(9) $e = \texttt{inr}_{\tau_1'}(e_2)$ by assumption

By rule induction over Rules (21) on (2), no rule applies due to syntactic contradiction.

Case (10f).

(5) $\dot{\xi} = \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(6) $\dot{\xi}_1 : \tau_1$ by assumption

(7) $\dot{\xi}_2 : \tau_2$ by assumption

(8) $e \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ by assumption

(9) $e \not\models^{\dagger}_{?} \dot{\xi}_1$ by Lemma 2.0.10 on (8)

(10) $e \not\models^{\dagger}_{?} \dot{\xi}_2$ by Lemma 2.0.10 on (8)

(11) if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}_1$ — by IH on (1) and (2) and (6) and (9)

(12) if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}_2$ — by IH on (1) and (2) and (7) and (10)

To show that if $e' \in \texttt{values}(e)$ then $e' \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$, we assume $e' \in \texttt{values}(e)$.

(13) $e' \in \texttt{values}(e)$ — by assumption

(14) $e' \not\models^{\dagger}_{?} \dot{\xi}_1$ — by (11) on (13)

(15) $e' \not\models^{\dagger}_{?} \dot{\xi}_2$ — by (12) on (13)

(16) $e' \not\models^{\dagger}_{?} \dot{\xi}_1 \vee \dot{\xi}_2$ — by Lemma 2.0.10 on (14) and (15)

□

$\boxed{\theta : \Gamma}$ θ is of type Γ

$$\text{STEmpty}\quad \frac{}{\emptyset : \cdot} \tag{31a}$$

$$\text{STExtend}\quad \frac{\theta : \Gamma_\theta \qquad \Gamma\,;\Delta \vdash e : \tau}{\theta, x/e : \Gamma_\theta, x : \tau} \tag{31b}$$

$\boxed{p\ \texttt{refutable}_?}$ p is refutable

$$\text{RNum}\quad \frac{}{\underline{n}\ \texttt{refutable}_?} \tag{32a}$$

$$\text{REHole}\quad \frac{}{(\!|\,|\!)^w\ \texttt{refutable}_?} \tag{32b}$$

$$\text{RHole}\quad \frac{}{(\!|p|\!)^w\ \texttt{refutable}_?} \tag{32c}$$

$$\text{RInl}\quad \frac{}{\texttt{inl}(p)\ \texttt{refutable}_?} \tag{32d}$$

$$\text{RInr}\quad \frac{}{\texttt{inr}(p)\ \texttt{refutable}_?} \tag{32e}$$

$$\text{RPairL}\quad \frac{p_1\ \texttt{refutable}_?}{(p_1, p_2)\ \texttt{refutable}_?} \tag{32f}$$

$$\frac{p_2 \; \texttt{refutable}_?}{(p_1, p_2) \; \texttt{refutable}_?} \; \text{RPairR} \tag{32g}$$

$\boxed{e \rhd p \dashV \theta}$ $\quad e$ matches p, emitting θ

$$\frac{}{e \rhd x \dashV e/x} \; \text{MVar} \tag{33a}$$

$$\frac{}{e \rhd \_ \dashV \cdot} \; \text{MWild} \tag{33b}$$

$$\frac{}{\underline{n} \rhd \underline{n} \dashV \cdot} \; \text{MNum} \tag{33c}$$

$$\frac{e_1 \rhd p_1 \dashV \theta_1 \qquad e_2 \rhd p_2 \dashV \theta_2}{(e_1, e_2) \rhd (p_1, p_2) \dashV \theta_1 \uplus \theta_2} \; \text{MPair} \tag{33d}$$

$$\frac{e \rhd p \dashV \theta}{\texttt{inl}_\tau(e) \rhd \texttt{inl}(p) \dashV \theta} \; \text{MInl} \tag{33e}$$

$$\frac{e \rhd p \dashV \theta}{\texttt{inr}_\tau(e) \rhd \texttt{inr}(p) \dashV \theta} \; \text{MInr} \tag{33f}$$

$$\frac{e \; \texttt{notintro} \qquad \texttt{prl}(e) \rhd p_1 \dashV \theta_1 \qquad \texttt{prr}(e) \rhd p_2 \dashV \theta_2}{e \rhd (p_1, p_2) \dashV \theta_1 \uplus \theta_2} \; \text{MNotIntroPair} \tag{33g}$$

$\boxed{e \; ? \; p}$ $\quad e$ may match p

$$\frac{}{e \; ? \; (\!|\,|\!)^w} \; \text{MMEHole} \tag{34a}$$

$$\frac{}{e \; ? \; (\!|p|\!)^w} \; \text{MMHole} \tag{34b}$$

$$\frac{e \; \texttt{notintro} \qquad p \; \texttt{refutable}_?}{e \; ? \; p} \; \text{MMNotIntro} \tag{34c}$$

$$\frac{e_1 \; ? \; p_1 \qquad e_2 \rhd p_2 \dashV \theta_2}{(e_1, e_2) \; ? \; (p_1, p_2)} \; \text{MMPairL} \tag{34d}$$

$$\frac{e_1 \rhd p_1 \dashV \theta_1 \qquad e_2 \; ? \; p_2}{(e_1, e_2) \; ? \; (p_1, p_2)} \; \text{MMPairR} \tag{34e}$$

$$\dfrac{\overset{\text{MMPair}}{e_1 \mathrel{?} p_1 \qquad e_2 \mathrel{?} p_2}}{(e_1, e_2) \mathrel{?} (p_1, p_2)} \tag{34f}$$

$$\dfrac{\overset{\text{MMInl}}{e \mathrel{?} p}}{\texttt{inl}_\tau(e) \mathrel{?} \texttt{inl}(p)} \tag{34g}$$

$$\dfrac{\overset{\text{MMInr}}{e \mathrel{?} p}}{\texttt{inr}_\tau(e) \mathrel{?} \texttt{inr}(p)} \tag{34h}$$

$\boxed{e \perp p}$ e does not match p

$$\dfrac{\overset{\text{NMNum}}{n_1 \neq n_2}}{\underline{n_1} \perp \underline{n_2}} \tag{35a}$$

$$\dfrac{\overset{\text{NMPairL}}{e_1 \perp p_1}}{(e_1, e_2) \perp (p_1, p_2)} \tag{35b}$$

$$\dfrac{\overset{\text{NMPairR}}{e_2 \perp p_2}}{(e_1, e_2) \perp (p_1, p_2)} \tag{35c}$$

$$\dfrac{\text{NMConfL}}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)} \tag{35d}$$

$$\dfrac{\text{NMConfR}}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{35e}$$

$$\dfrac{\overset{\text{NMInl}}{e \perp p}}{\texttt{inr}_\tau(e) \perp \texttt{inl}(p)} \tag{35f}$$

$$\dfrac{\overset{\text{NMInr}}{e \perp p}}{\texttt{inl}_\tau(e) \perp \texttt{inr}(p)} \tag{35g}$$

$\boxed{e \mapsto e'}$ e takes a step to e'

$$\dfrac{\overset{\text{ITHole}}{e \mapsto e'}}{(\!|e|\!)^u \mapsto (\!|e'|\!)^u} \tag{36a}$$

$$\dfrac{\overset{\text{ITApFun}}{e_1 \mapsto e_1'}}{e_1(e_2) \mapsto e_1'(e_2)} \tag{36b}$$

$$\dfrac{\overset{\text{ITApArg}}{e_1\ \texttt{val} \qquad e_2 \mapsto e_2'}}{e_1(e_2) \mapsto e_1(e_2')} \tag{36c}$$

$$\frac{e_2 \; \texttt{val}}{(\lambda x : \tau . e_1)(e_2) \mapsto [e_2/x]e_1} \; \text{ITAP} \tag{36d}$$

$$\frac{e_1 \mapsto e_1'}{(e_1, e_2) \mapsto (e_1', e_2)} \; \text{ITPairL} \tag{36e}$$

$$\frac{e_1 \; \texttt{val} \qquad e_2 \mapsto e_2'}{(e_1, e_2) \mapsto (e_1, e_2')} \; \text{ITPairR} \tag{36f}$$

$$\frac{(e_1, e_2) \; \texttt{final}}{\texttt{prl}((e_1, e_2)) \mapsto e_1} \; \text{ITPrl} \tag{36g}$$

$$\frac{(e_1, e_2) \; \texttt{final}}{\texttt{prr}((e_1, e_2)) \mapsto e_2} \; \text{ITPrr} \tag{36h}$$

$$\frac{e \mapsto e'}{\texttt{inl}_\tau(e) \mapsto \texttt{inl}_\tau(e')} \; \text{ITInl} \tag{36i}$$

$$\frac{e \mapsto e'}{\texttt{inr}_\tau(e) \mapsto \texttt{inr}_\tau(e')} \; \text{ITInr} \tag{36j}$$

$$\frac{e \mapsto e'}{\texttt{match}(e)\{\hat{rs}\} \mapsto \texttt{match}(e')\{\hat{rs}\}} \; \text{ITExpMatch} \tag{36k}$$

$$\frac{e \; \texttt{final} \qquad e \rhd p_r \dashv\!\mid \theta}{\texttt{match}(e)\{rs_{pre} \mid (p_r \Rightarrow e_r) \mid rs_{post}\} \mapsto [\theta](e_r)} \; \text{ITSuccMatch} \tag{36l}$$

$$\frac{e \; \texttt{final} \qquad e \perp p_r}{\texttt{match}(e)\{rs \mid (p_r \Rightarrow e_r) \mid (r' \mid rs')\} \mapsto \texttt{match}(e)\{(rs \mid (p_r \Rightarrow e_r) \mid \cdot)^\diamond \mid r' \mid rs'\}} \; \text{ITFailMatch} \tag{36m}$$

Lemma 4.0.3. *If* $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{final}$ *then* e_1 $\texttt{final}$.

Proof. By rule induction over Rules (27) on $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{final}$.

Case (27a).

(17) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{val}$ — by assumption

By rule induction over Rules (25) on (17), only one case applies.

Case (25d).

(18) e_1 `val` by assumption

(19) e_1 `final` by Rule (27a) on (18)

Case (27b).

(17) $\texttt{inl}_{\tau_2}(e_1)$ `indet` by assumption

By rule induction over Rules (26) on (17), only one case applies.

Case (26i).

(18) e_1 `indet` by assumption

(19) e_1 `final` by Rule (27b) on (18)

□

Lemma 4.0.4. *If* $\texttt{inr}_{\tau_1}(e_2)$ `final` *then* e_2 `final`*.*

Proof. By rule induction over Rules (27) on $\texttt{inr}_{\tau_1}(e_2)$ `final`.

Case (27a).

(1) $\texttt{inr}_{\tau_1}(e_2)$ `val` by assumption

By rule induction over Rules (25) on (1), only one case applies.

Case (25d).

(2) e_2 `val` by assumption

(3) e_2 `final` by Rule (27a) on (2)

Case (27b).

(1) $\texttt{inr}_{\tau_1}(e_2)$ `indet` by assumption

By rule induction over Rules (26) on (1), only one case applies.

Case (26i).

(2) e_2 `indet` by assumption

(3) e_2 `final` by Rule (27b) on (2)

□

Lemma 4.0.5. *If* (e_1, e_2) `final` *then* e_1 `final` *and* e_2 `final`*.*

Proof. By rule induction over Rules (27) on (e_1, e_2) `final`.

Case (27a).

(1) (e_1, e_2) `val` by assumption

By rule induction over Rules (25) on (1), only one case applies.

Case (25c).

(2) e_1 `val`	by assumption
(3) e_2 `val`	by assumption
(4) e_1 `final`	by Rule (27a) on (2)
(5) e_2 `final`	by Rule (27a) on (3)

Case (27b).

(1) (e_1, e_2) `indet`	by assumption

By rule induction over Rules (26) on (1), only three cases apply.

Case (26d).

(2) e_1 `indet`	by assumption
(3) e_2 `val`	by assumption
(4) e_1 `final`	by Rule (27b) on (2)
(5) e_1 `final`	by Rule (27a) on (3)

Case (26e).

(2) e_1 `val`	by assumption
(3) e_2 `indet`	by assumption
(4) e_1 `final`	by Rule (27a) on (2)
(5) e_1 `final`	by Rule (27b) on (3)

Case (26f).

(2) e_1 `indet`	by assumption
(3) e_2 `indet`	by assumption
(4) e_1 `final`	by Rule (27b) on (2)
(5) e_1 `final`	by Rule (27b) on (3)

□

Lemma 4.0.6. *There doesn't exist* $\underline{n}$ *such that* $\underline{n}$ `notintro`*.*

Proof. By rule induction over Rules (28) on $\underline{n}$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 4.0.7. *There doesn't exist* $\texttt{inl}_\tau(e)$ *such that* $\texttt{inl}_\tau(e)$ `notintro`*.*

Proof. By rule induction over Rules (28) on $\texttt{inl}_\tau(e)$ `notintro`, no case applies due to syntactic contradiction. □

Lemma 4.0.8. *There doesn't exist* $\texttt{inr}_\tau(e)$ *such that* $\texttt{inr}_\tau(e)$ `notintro`*.*

Proof. By rule induction over Rules (28) on $\texttt{inr}_\tau(e)$ $\texttt{notintro}$, no case applies due to syntactic contradiction. $\square$

Lemma 4.0.9. *There doesn't exist* (e_1, e_2) *such that* (e_1, e_2) $\texttt{notintro}$.

Proof. By rule induction over Rules (28) on (e_1, e_2) $\texttt{notintro}$, no case applies due to syntactic contradiction. $\square$

Lemma 4.0.10. *If* e $\texttt{final}$ *and* e $\texttt{notintro}$ *then* e $\texttt{indet}$.

Proof Sketch. By rule induction over Rules (28) on e $\texttt{notintro}$, for each case, by rule induction over Rules (25) on e $\texttt{val}$ and we notice that e $\texttt{val}$ is not derivable. By rule induction over Rules (27) on e $\texttt{final}$, Rule (27a) result in a contradiction with the fact that e $\texttt{val}$ is not derivable while Rule (27b) tells us e $\texttt{indet}$. $\square$

Lemma 4.0.11. *There doesn't exist such an expression* e *such that both* e $\texttt{val}$ *and* e $\texttt{indet}$.

Lemma 4.0.12. *There doesn't exist such an expression* e *such that both* e $\texttt{val}$ *and* e $\texttt{notintro}$.

Lemma 4.0.13 (Finality)**.** *There doesn't exist such an expression* e *such that both* e $\texttt{final}$ *and* $e \mapsto e'$ *for some* e'

Proof. Assume there exists such an e such that both e $\texttt{final}$ and $e \mapsto e'$ for some e' then proof by contradiction.

By rule induction over Rules (27) and Rules (36), *i.e.*, over Rules (25) and Rules (36) and over Rules (26) and Rules (36) respectively. The proof can be done by straightforward observation of syntactic contradictions. $\square$

Lemma 4.0.14 (Matching Determinism)**.** *If* e $\texttt{final}$ *and* $\cdot\,;\Delta_e \vdash e : \tau$ *and* $p : \tau[\xi] \dashv\!\!\mid \Gamma\,;\Delta$ *then exactly one of the following holds*

1. $e \rhd p \dashv\!\!\mid \theta$ *for some* θ
2. $e \,?\, p$
3. $e \perp p$

Proof.

(1)	e $\texttt{final}$	by assumption
(2)	$\cdot\,;\Delta_e \vdash e : \tau$	by assumption
(3)	$p : \tau[\xi] \dashv\!\!\mid \Gamma\,;\Delta$	by assumption

By rule induction over Rules (22) on (3), we would show one conclusion is derivable while the other two are not.

Case (22a)**.**

(4) $p = x$ by assumption

(5) $e \rhd x \dashv\!\mid e/x$ by Rule (33a)

Assume $e \mathrel{?} x$. By rule induction over Rules (34) on it, only one case applies.

Case (34c).

(6) x `refutable`$_?$ by assumption

By rule induction over Rules (32) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \mathrel{?} x}$ by contradiction

Assume $e \perp x$. By rule induction over Rules (35) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp x}$ by contradiction

Case (22b).

(4) $p = \_$ by assumption

(5) $e \rhd \_ \dashv\!\mid \cdot$ by Rule (33b)

Assume $e \mathrel{?} \_$. By rule induction over Rules (34) on it, only one case applies.

Case (34c).

(6) $\_$ `refutable`$_?$ by assumption

By rule induction over Rules (32) on (6), no case applies due to syntactic contradiction.

(7) $\cancel{e \mathrel{?} \_}$ by contradiction

Assume $e \perp \_$. By rule induction over Rules (35) on it, no case applies due to syntactic contradiction.

(8) $\cancel{e \perp \_}$ by contradiction

Case (22c).

(4) $p = (\!|\,|\!)^w$ by assumption

(5) $e \mathrel{?} (\!|\,|\!)^w$ by Rule (34a)

Assume $e \rhd (\!|\!)^w \dashv\!\mid \theta$ for some θ. By rule induction over Rules (34) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd (\!|\!)^w \dashv\!\mid \theta}$ by contradiction

Assume $e \perp (\!|\!)^w$. By rule induction over Rules (35) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp (\!|\!)^w}$ by contradiction

Case (22d).

(4) $p = (\!|p_0|\!)^w$ by assumption

(5) $e \,?\, (\!|p_0|\!)^w$ by Rule (34b)

Assume $e \rhd (\!|p_0|\!)^w \dashv\!\mid \theta$ for some θ. By rule induction over Rules (34) on it, no case applies due to syntactic contradiction.

(6) $\cancel{e \rhd (\!|p_0|\!)^w \dashv\!\mid \theta}$ by contradiction

Assume $e \perp (\!|p_0|\!)^w$. By rule induction over Rules (35) on it, no case applies due to syntactic contradiction.

(7) $\cancel{e \perp (\!|p_0|\!)^w}$ by contradiction

Case (22e).

(4) $p = \underline{n_2}$ by assumption

(5) $\tau = \texttt{num}$ by assumption

(6) $\xi = \underline{n_2}$ by assumption

(7) $\underline{n_2}\ \texttt{refutable}_?$ by Rule (32a)

By rule induction over Rules (21) on (2), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(10) $e \,?\, \underline{n_2}$ by Rule (18b) on (7) and (9)

Assume $e \rhd \underline{n_2} \dashv\!\mid \theta$ for some θ. By rule induction over it, no case applies due to syntactic contradiction.

(11) $\cancel{e \rhd \underline{n_2} \dashv\!\mid \theta}$ by contradiction

Assume $e \perp \underline{n_2}$. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \perp \underline{n_2}}$ by contradiction

Case (21d).

(8) $e = \underline{n_1}$

Assume $\underline{n_1} \mathbin{?} \underline{n_2}$. By rule induction over Rules (34) on it, only two cases apply.

Case (34c).

(9) $\underline{n_1}$ `notintro` by assumption

Contradicts Lemma 4.0.6.

(10) $\cancel{\underline{n_1} \mathbin{?} \underline{n_2}}$ by contradiction

By case analysis on whether $n_1 = n_2$.

Case $n_1 = n_2$.

(11) $n_1 = n_2$ by assumption

(12) $\underline{n_1} \rhd \underline{n_2} \dashV \cdot$ by Rule (33c)

Assume $\underline{n_1} \perp \underline{n_2}$. By rule induction over Rules (35) on it, only one case applies.

Case (35a).

(13) $n_1 \neq n_2$ by assumption

Contradicts (11).

(14) $\cancel{\underline{n_1} \perp \underline{n_2}}$ by contradiction

Case $n_1 \neq n_2$.

(11) $n_1 \neq n_2$ by assumption

(12) $\underline{n_1} \perp \underline{n_2}$ by Rule (35a) on (11)

Assume $\underline{n_1} \rhd \underline{n_2} \dashV \theta$ for some θ. By rule induction over Rules (33) on it, no case applies due to syntactic contradiction.

(13) $\cancel{\underline{n_1} \rhd \underline{n_2} \dashV \theta}$ by contradiction

Case (22f).

(4) $p = \mathtt{inl}(p_1)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \mathtt{inl}(\xi_1)$ by assumption

(7) $p_1 : \tau_1[\xi_1] \dashV \Gamma \,; \Delta$ by assumption

(8) $\mathtt{inl}(p_1)$ $\mathtt{refutable}_?$ by Rule (32d)

By rule induction over Rules (21) on (2), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(9) $e = (\!|\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(11) $e\,?\,\texttt{inl}(p_1)$ by Rule (18b) on (8) and (10)

Assume $e \rhd \texttt{inl}(p_1) \dashv\!\vdash \theta_1$ for some θ_1. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inl}(p_1) \dashv\!\vdash \theta_1}$ by contradiction

Assume $e \perp \texttt{inl}(p_1)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inl}(p_1)}$ by contradiction

Case (21j).

(9) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(10) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$ by assumption

(11) e_1 $\texttt{final}$ by Lemma 4.0.3 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ for some θ_1, $e_1\,?\,p_1$, and $e_1 \perp p_1$ holds.
By case analysis on which one holds.

Case $e_1 \rhd p_1 \dashv\!\vdash \theta_1$.

(12) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ by assumption

(13) $\cancel{e_1\,?\,p_1}$ by assumption

(14) $\cancel{e_1 \perp p_1}$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vdash \theta_1$ by Rule (33e) on (12)

Assume $\texttt{inl}_{\tau_2}(e_1)\,?\,\texttt{inl}(p_1)$. By rule induction over Rules (34) on it, only two cases apply.

Case (34c).

(16) $\texttt{inl}_{\tau_2}(e_1)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (34g).

(16) $e_1\,?\,p_1$ by assumption

Contradicts (13).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1)\,?\,\texttt{inl}(p_1)}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (35) on it, only one case applies.

Case (35f).

(18) $e_1 \perp p_1$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ by contradiction

Case $e_1 \mathrel{?} p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(13) $e_1 \mathrel{?} p_1$ by assumption

(14) $\cancel{e_1 \perp p_1}$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \mathrel{?} \texttt{inl}(p_1)$ by Rule (34g) on (13)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (33) on it, only one case applies.

Case (33e).

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$. By rule induction over Rules (35) on it, only one case applies.

Case (35f).

(18) $e_1 \perp p_1$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)}$ by contradiction

Case $e_1 \perp p_1$.

(12) $\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(13) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(14) $e_1 \perp p_1$ by assumption

(15) $\texttt{inl}_{\tau_2}(e_1) \perp \texttt{inl}(p_1)$ by Rule (35f) on (14)

Assume $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta$ for some θ. By rule induction over Rules (33) on it, only one case applies.

Case (33e).

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!\vert\, \theta}$ by contradiction

Assume $\texttt{inl}_{\tau_2}(e_1) \mathrel{?} \texttt{inl}(p_1)$. By rule induction over Rules (34) on it, only two cases apply.

Case (34c).

(18) $\texttt{inl}_{\tau_2}(e_1)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (34g).

(18) $e_1 \,?\, p_1$ by assumption

Contradicts (13).

(19) $\cancel{\texttt{inl}_{\tau_2}(e_1) \,?\, \texttt{inl}(p_1)}$ by contradiction

Case (22g).

(4) $p = \texttt{inr}(p_2)$ by assumption

(5) $\tau = (\tau_1 + \tau_2)$ by assumption

(6) $\xi = \texttt{inr}(\xi_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\!\mid \Gamma \,;\, \Delta$ by assumption

(8) $\texttt{inr}(p_2)\ \texttt{refutable}_?$ by Rule (32e)

By rule induction over Rules (21) on (2), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(9) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(10) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(11) $e \,?\, \texttt{inr}(p_2)$ by Rule (18b) on (8) and (10)

Assume $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta_2$ for some θ_2. By rule induction over it, no case applies due to syntactic contradiction.

(12) $\cancel{e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta_2}$ by contradiction

Assume $e \perp \texttt{inr}(p_2)$. By rule induction over it, no case applies due to syntactic contradiction.

(13) $\cancel{e \perp \texttt{inr}(p_2)}$ by contradiction

Case (21k).

(9) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(10) $\cdot \,;\, \Delta_e \vdash e_2 : \tau_2$ by assumption

(11) $e_2\ \texttt{final}$ by Lemma 4.0.4 on (1)

By inductive hypothesis on (10) and (11) and (7), exactly one of $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ for some θ_2, $e_2 \,?\, p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which one holds.

Case $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$.

(12) $e_2 \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

(13) $\cancel{e_2 \,?\, p_2}$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta_2$ by Rule (33f) on (12)

Assume $\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)$. By rule induction over Rules (34) on it, only two cases apply.

Case (34c).

(16) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.8.

Case (34h).

(16) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (13).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$. By rule induction over Rules (35) on it, only one case applies.

Case (35g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)}$ by contradiction

Case $e_2 \mathrel{?} p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\vdash \theta}$ by assumption

(13) $e_2 \mathrel{?} p_2$ by assumption

(14) $\cancel{e_2 \perp p_2}$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)$ by Rule (34h) on (13)

Assume $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\vdash \theta$ for some θ. By rule induction over Rules (33) on it, only one case applies.

Case (33f).

(16) $e_2 \rhd p_2 \dashv\!\vdash \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\vdash \theta}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$. By rule induction over Rules (35) on it, only one case applies.

Case (35g).

(18) $e_2 \perp p_2$ by assumption

Contradicts (14).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)}$ by contradiction

Case $e_2 \perp p_2$.

(12) $\cancel{e_2 \rhd p_2 \dashv\!\vdash \theta}$ by assumption

(13) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(14) $e_2 \perp p_2$ by assumption

(15) $\texttt{inr}_{\tau_1}(e_2) \perp \texttt{inr}(p_2)$ by Rule (35g) on (14)

Assume $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\mid \theta$ for some θ. By rule induction over Rules (33) on it, only one case applies.

Case (33f)**.**

(16) $e_2 \rhd p_2 \dashv\!\mid \theta$ by assumption

Contradicts (12).

(17) $\cancel{\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\mid \theta}$ by contradiction

Assume $\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)$. By rule induction over Rules (34) on it, only two cases apply.

Case (34c)**.**

(18) $\texttt{inr}_{\tau_1}(e_2)\ \texttt{notintro}$ by assumption

Contradicts Lemma 4.0.8.

Case (34h)**.**

(18) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (13).

(19) $\cancel{\texttt{inr}_{\tau_1}(e_2) \mathrel{?} \texttt{inr}(p_2)}$ by contradiction

Case (22h)**.**

(4) $p = (p_1, p_2)$ by assumption

(5) $\tau = (\tau_1 \times \tau_2)$ by assumption

(6) $\xi = (\xi_1, \xi_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\!\mid \Gamma_1 ; \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\!\mid \Gamma_2 ; \Delta_2$ by assumption

By rule induction over Rules (21) on (2), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m)**.**

(11) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(13) $e\ \texttt{indet}$ by Lemma 4.0.10 on (1) and (12)

(14) $\texttt{prl}(e)\ \texttt{indet}$ by Rule (26g) on (13)

(15) $\texttt{prl}(e)\ \texttt{final}$ by Rule (27b) on (14)

(16) $\texttt{prr}(e)\ \texttt{indet}$ by Rule (26h) on (13)

(17)	$\texttt{prr}(e)\ \texttt{final}$	by Rule (27b) on (16)
(18)	$\cdot\,; \Delta \vdash \texttt{prl}(e) : \tau_1$	by Rule (21h) on (2)
(19)	$\cdot\,; \Delta \vdash \texttt{prr}(e) : \tau_2$	by Rule (21i) on (2)

Assume $e \perp (p_1, p_2)$. By rule induction on it, no case applies due to syntactic contradiction.

(20)	$\cancel{e \perp (p_1, p_2)}$	by contradiction

By inductive hypothesis on (15) and (18) and (9), exactly one of $\texttt{prl}(e) \rhd p_1 \dashv\mid \theta_1$, $\texttt{prl}(e)\ ?\ p_1$, and $\texttt{prl}(e) \perp p_1$ holds.
By inductive hypothesis on (17) and (19) and (10), exactly one of $\texttt{prr}(e) \rhd p_2 \dashv\mid \theta_2$, $\texttt{prr}(e)\ ?\ p_2$, and $\texttt{prr}(e) \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2. Note that we have already shown $\cancel{e \perp (p_1, p_2)}$.

Case $\texttt{prl}(e) \rhd p_1 \dashv\mid \theta_1, \texttt{prr}(e) \rhd p_2 \dashv\mid \theta_2$**.**

(21)	$\texttt{prl}(e) \rhd p_1 \dashv\mid \theta_1$	by assumption
(22)	$\cancel{\texttt{prl}(e)\ ?\ p_1}$	by assumption
(23)	$\cancel{\texttt{prl}(e) \perp p_1}$	by assumption
(24)	$\texttt{prr}(e) \rhd p_2 \dashv\mid \theta_2$	by assumption
(25)	$\cancel{\texttt{prr}(e)\ ?\ p_2}$	by assumption
(26)	$\cancel{\texttt{prr}(e) \perp p_2}$	by assumption
(27)	$e \rhd (p_1, p_2) \dashv\mid \theta_1 \uplus \theta_2$	by Rule (33g) on (12) and (21) and (24)

Assume $e\,?\,(p_1, p_2)$. By rule induction over Rules (34) on it, only one case applies.

Case (34c)**.**

(28)	$(p_1, p_2)\ \texttt{refutable}_?$	by assumption

By rule induction over Rules (32), only two cases apply.

Case (32f)**.**

(29)	$p_1\ \texttt{refutable}_?$	by assumption
(30)	$\texttt{prl}(e)\ \texttt{notintro}$	by Rule (28e)
(31)	$\texttt{prl}(e)\ ?\ p_1$	by Rule (34c) on (29) and (30)

Contradicts (22).

Case (32g)**.**

(29)	$p_2\ \texttt{refutable}_?$	by assumption
(30)	$\texttt{prr}(e)\ \texttt{notintro}$	by Rule (28f)
(31)	$\texttt{prl}(e)\ ?\ p_1$	by Rule (34c) on (29) and (30)

Contradicts (22).

(32)	$\cancel{e\ ?\ (p_1, p_2)}$	by contradiction

Case $\texttt{prl}(e) \rhd p_1 \dashv\!\mid \theta_1, \texttt{prr}(e) \mathrel{?} p_2$.

(21) $\texttt{prl}(e) \rhd p_1 \dashv\!\mid \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\mid \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\mid \theta$. By rule induction over Rules (33), only one case applies.

Case (33g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prr}(e) \rhd p_2 \dashv\!\mid \theta_2$ by assumption

Contradicts (24).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\mid \theta}$ by contradiction

By rule induction over Rules (34) on (25), the following cases apply.

Case (34a),(34b).

(30) $p_2 = (\!|\!)^w, (\!|p|\!)^w$ by assumption

(31) $p_2\ \texttt{refutable}_?$ by Rule (32b) and Rule (32c)

(32) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (32g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (34c) on (12) and (32)

Case (34c).

(30) $p_2\ \texttt{refutable}_?$ by assumption

(31) $(p_1, p_2)\ \texttt{refutable}_?$ by Rule (32g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (34c) on (12) and (31)

Case $\texttt{prl}(e) \rhd p_1 \dashv\!\mid \theta_1, \texttt{prr}(e) \perp p_2$.

(21) $\texttt{prl}(e) \rhd p_1 \dashv\!\mid \theta_1$ by assumption

(22) $\cancel{\texttt{prl}(e) \mathrel{?} p_1}$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\mid \theta_2}$ by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption

(26) $\texttt{prr}(e) \perp p_2$ by assumption

By rule induction over Rules (35) on (26), no case applies due to syntactic contradiction.

Therefore, vacuously true.

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \rhd p_2 \dashv\!\mid \theta_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2$ by assumption

(25) $\cancel{\texttt{prr}(e) \mathrel{?} p_2}$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (33), only one case applies.

Case (33g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashv\!\!\mid \theta}$ by contradiction

By rule induction over Rules (34) on (22), the following cases apply.

Case (34a),(34b).

(30) $p_1 = (\!|\!)^w, (\!|p|\!)^w$ by assumption

(31) p_1 $\texttt{refutable}_?$ by Rule (32b) and Rule (32c)

(32) (p_1, p_2) $\texttt{refutable}_?$ by Rule (32g) on (31)

(33) $e \mathrel{?} (p_1, p_2)$ by Rule (34c) on (12) and (32)

Case (34c).

(30) p_1 $\texttt{refutable}_?$ by assumption

(31) (p_1, p_2) $\texttt{refutable}_?$ by Rule (32g) on (30)

(32) $e \mathrel{?} (p_1, p_2)$ by Rule (34c) on (12) and (31)

Case $\texttt{prl}(e) \mathrel{?} p_1, \texttt{prr}(e) \mathrel{?} p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1}$ by assumption

(22) $\texttt{prl}(e) \mathrel{?} p_1$ by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\!\mid \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathrel{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

Assume $e \rhd (p_1, p_2) \dashv\!\!\mid \theta$. By rule induction over Rules (33), only one case applies.

Case (33g).

(27) $\theta = \theta_1 \uplus \theta_2$ by assumption

(28) $\texttt{prl}(e) \rhd p_1 \dashv\!\!\mid \theta_1$ by assumption

Contradicts (21).

(29) $\cancel{e \rhd (p_1, p_2) \dashV \theta}$ — by contradiction

By rule induction over Rules (34) on (22), the following cases apply.

Case (34a),(34b)**.**

(30) $p_1 = (\!|\,|\!)^w, (\!|p|\!)^w$ — by assumption

(31) $p_1\ \texttt{refutable}_?$ — by Rule (32b) and Rule (32c)

(32) $(p_1, p_2)\ \texttt{refutable}_?$ — by Rule (32g) on (31)

(33) $e\ ?\ (p_1, p_2)$ — by Rule (34c) on (12) and (32)

Case (34c)**.**

(30) $p_1\ \texttt{refutable}_?$ — by assumption

(31) $(p_1, p_2)\ \texttt{refutable}_?$ — by Rule (32g) on (30)

(32) $e\ ?\ (p_1, p_2)$ — by Rule (34c) on (12) and (31)

Case $\texttt{prl}(e)\ ?\ p_1, \texttt{prr}(e) \perp p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ — by assumption

(22) $\texttt{prl}(e)\ ?\ p_1$ — by assumption

(23) $\cancel{\texttt{prl}(e) \perp p_1}$ — by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashV \theta_2}$ — by assumption

(25) $\cancel{\texttt{prr}(e)\ ?\ p_2}$ — by assumption

(26) $\texttt{prr}(e) \perp p_2$ — by assumption

By rule induction over Rules (35) on (26), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \rhd p_2 \dashV \theta_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ — by assumption

(22) $\cancel{\texttt{prl}(e)\ ?\ p_1}$ — by assumption

(23) $\texttt{prl}(e) \perp p_1$ — by assumption

(24) $\texttt{prr}(e) \rhd p_2 \dashV \theta_2$ — by assumption

(25) $\cancel{\texttt{prr}(e)\ ?\ p_2}$ — by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ — by assumption

By rule induction over Rules (35) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e)\ ?\ p_2$**.**

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashV \theta_1}$ — by assumption

(22) $\cancel{\texttt{prl}(e)\ ?\ p_1}$ — by assumption

(23) $\texttt{prl}(e) \perp p_1$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathbin{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (35) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case $\texttt{prl}(e) \perp p_1, \texttt{prr}(e) \perp p_2$.

(21) $\cancel{\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1}$ by assumption

(22) $\cancel{\texttt{prl}(e) \mathbin{?} p_1}$ by assumption

(23) $\texttt{prl}(e) \perp p_1$ by assumption

(24) $\cancel{\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2}$ by assumption

(25) $\texttt{prr}(e) \mathbin{?} p_2$ by assumption

(26) $\cancel{\texttt{prr}(e) \perp p_2}$ by assumption

By rule induction over Rules (35) on (23), no case applies due to syntactic contradiction.
Therefore, vacuously true.

Case (21g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta \vdash e_2 : \tau_2$ by assumption

(14) e_1 $\texttt{final}$ by Lemma 4.0.5 on (1)

(15) e_2 $\texttt{final}$ by Lemma 4.0.5 on (1)

By inductive hypothesis on (9) and (12) and (14), exactly one of $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$, $e_1 \mathbin{?} p_1$, and $e_1 \perp p_1$ holds.
By inductive hypothesis on (10) and (13) and (15), exactly one of $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$, $e_2 \mathbin{?} p_2$, and $e_2 \perp p_2$ holds.
By case analysis on which conclusion holds for p_1 and p_2.

Case $e_1 \rhd p_1 \dashv\!\vert\, \theta_1, e_2 \rhd p_2 \dashv\!\vert\, \theta_2$.

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by assumption

(17) $\cancel{e_1 \mathbin{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

(20) $\cancel{e_2 \mathbin{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2$ by Rule (33d) on (16) and (19)

Assume $(e_1, e_2) \mathbin{?} (p_1, p_2)$. By rule induction over Rules (34) on it, only four cases apply.

Case (34c).

(23) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (34d)**.**

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

Case (34e)**.**

(23) $e_2 \mathrel{?} p_2$ by assumption

Contradicts (20).

Case (34f)**.**

(23) $e_1 \mathrel{?} p_1$ by assumption

Contradicts (17).

(24) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (35) on it, only two cases apply.

Case (35b)**.**

(25) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (35c)**.**

(25) $e_2 \perp p_2$ by assumption

Contradicts (21).

(26) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \rhd p_1 \dashv\!\vdash \theta_1, e_2 \mathrel{?} p_2$**.**

(16) $e_1 \rhd p_1 \dashv\!\vdash \theta_1$ by assumption

(17) $\cancel{e_1 \mathrel{?} p_1}$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\vdash \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (34e) on (16) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vdash \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vdash \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (35) on it, only two cases apply.

Case (35b).

(26) $e_1 \perp p_1$ — by assumption

Contradicts (18).

Case (35c).

(26) $e_2 \perp p_2$ — by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ — by contradiction

Case $e_1 \rhd p_1 \dashv\!\vert\, \theta_1, e_2 \perp p_2$.

(16) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ — by assumption

(17) $\cancel{e_1\ ?\ p_1}$ — by assumption

(18) $\cancel{e_1 \perp p_1}$ — by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta_2}$ — by assumption

(20) $\cancel{e_2\ ?\ p_2}$ — by assumption

(21) $e_2 \perp p_2$ — by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ — by Rule (35c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ — by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta}$ — by contradiction

Assume $(e_1, e_2)\ ?\ (p_1, p_2)$. By rule induction over Rules (34) on it, only four cases apply.

Case (34c).

(26) (e_1, e_2) `notintro` — by assumption

Contradicts Lemma 4.0.9.

Case (34d).

(26) $e_1\ ?\ p_1$ — by assumption

Contradicts (17).

Case (34e).

(26) $e_2\ ?\ p_2$ — by assumption

Contradicts (20).

Case (34f).

(26) $e_1\ ?\ p_1$ — by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \mathrel{?} (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \rhd p_2 \dashv\mid \theta_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\mid \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $e_2 \rhd p_2 \dashv\mid \theta_2$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (34d) on (17) and (19)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\mid \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\mid \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\mid \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (35) on it, only two cases apply.

Case (35b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (35c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \mathrel{?} p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\mid \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\mid \theta_2}$ by assumption

(20) $e_2 \mathrel{?} p_2$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \mathrel{?} (p_1, p_2)$ by Rule (34f) on (17) and (20)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\mid \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\Vert \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\Vert \theta}$ by contradiction

Assume $(e_1, e_2) \perp (p_1, p_2)$. By rule induction over Rules (35) on it, only two cases apply.

Case (35b)**.**

(26) $e_1 \perp p_1$ by assumption

Contradicts (18).

Case (35c)**.**

(26) $e_2 \perp p_2$ by assumption

Contradicts (21).

(27) $\cancel{(e_1, e_2) \perp (p_1, p_2)}$ by contradiction

Case $e_1 \mathrel{?} p_1, e_2 \perp p_2$**.**

(16) $\cancel{e_1 \rhd p_1 \dashv\Vert \theta_1}$ by assumption

(17) $e_1 \mathrel{?} p_1$ by assumption

(18) $\cancel{e_1 \perp p_1}$ by assumption

(19) $\cancel{e_2 \rhd p_2 \dashv\Vert \theta_2}$ by assumption

(20) $\cancel{e_2 \mathrel{?} p_2}$ by assumption

(21) $e_2 \perp p_2$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (35c) on (21)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\Vert \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\Vert \theta_2$ by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\Vert \theta}$ by contradiction

Assume $(e_1, e_2) \mathrel{?} (p_1, p_2)$. By rule induction over Rules (34) on it, only four cases apply.

Case (34c)**.**

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (34d)**.**

(26) $e_2 \rhd p_2 \dashv\Vert \theta_2$ by assumption

Contradicts (19).

Case (34e).

(26) $e_2 \,?\, p_2$ by assumption

Contradicts (20).

Case (34f).

(26) $e_2 \,?\, p_2$ by assumption

Contradicts (20).

(27) $\cancel{(e_1, e_2) \,?\, (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \rhd p_2 \dashv\!\!| \, \theta_2$.

(16) $\cancel{e_1 \rhd p_1 \dashv\!\!| \, \theta_1}$ by assumption

(17) $\cancel{e_1 \,?\, p_1}$ by assumption

(18) $e_1 \perp p_1$ by assumption

(19) $e_2 \rhd p_2 \dashv\!\!| \, \theta_2$ by assumption

(20) $\cancel{e_2 \,?\, p_2}$ by assumption

(21) $\cancel{e_2 \perp p_2}$ by assumption

(22) $(e_1, e_2) \perp (p_1, p_2)$ by Rule (35b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \, \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d).

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_1 \rhd p_1 \dashv\!\!| \, \theta_1$ by assumption

Contradicts (16).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\!| \, \theta}$ by contradiction

Assume $(e_1, e_2) \,?\, (p_1, p_2)$. By rule induction over Rules (34) on it, only four cases apply.

Case (34c).

(26) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (34d).

(26) $e_1 \,?\, p_1$ by assumption

Contradicts (17).

Case (34e).

(26) $e_2 \,?\, p_2$ by assumption

Contradicts (20).

Case (34f).

(26) $e_1 \,?\, p_1$ by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2) \,?\, (p_1, p_2)}$ by contradiction

Case $e_1 \perp p_1, e_2 \mathbin{?} p_2$.

(16)	$\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$	by assumption
(17)	$\cancel{e_1 \mathbin{?} p_1}$	by assumption
(18)	$e_1 \perp p_1$	by assumption
(19)	$\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta_2}$	by assumption
(20)	$e_2 \mathbin{?} p_2$	by assumption
(21)	$\cancel{e_2 \perp p_2}$	by assumption
(22)	$(e_1, e_2) \perp (p_1, p_2)$	by Rule (35b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d).

(23)	$\theta = \theta_1 \uplus \theta_2$	
(24)	$e_2 \rhd p_2 \dashv\!\vert\, \theta_2$	by assumption

Contradicts (19).

(25)	$\cancel{(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta}$	by contradiction

Assume $(e_1, e_2) \mathbin{?} (p_1, p_2)$. By rule induction over Rules (34) on it, only four cases apply.

Case (34c).

(26)	(e_1, e_2) `notintro`	by assumption

Contradicts Lemma 4.0.9.

Case (34d).

(26)	$e_2 \rhd p_2 \dashv\!\vert\, \theta_2$	by assumption

Contradicts (19).

Case (34e).

(26)	$e_1 \rhd p_1 \dashv\!\vert\, \theta_1$	by assumption

Contradicts (16).

Case (34f).

(26)	$e_1 \mathbin{?} p_1$	by assumption

Contradicts (17).

(27)	$\cancel{(e_1, e_2) \mathbin{?} (p_1, p_2)}$	by contradiction

Case $e_1 \perp p_1, e_2 \perp p_2$.

(16)	$\cancel{e_1 \rhd p_1 \dashv\!\vert\, \theta_1}$	by assumption
(17)	$\cancel{e_1 \mathbin{?} p_1}$	by assumption
(18)	$e_1 \perp p_1$	by assumption
(19)	$\cancel{e_2 \rhd p_2 \dashv\!\vert\, \theta_2}$	by assumption
(20)	$e_2 \mathbin{?} p_2$	by assumption
(21)	$\cancel{e_2 \perp p_2}$	by assumption
(22)	$(e_1, e_2) \perp (p_1, p_2)$	by Rule (35b) on (18)

Assume $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\vert\, \theta$. By rule induction over Rules (33) on it, only one case applies.

Case (33d)**.**

(23) $\theta = \theta_1 \uplus \theta_2$

(24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ — by assumption

Contradicts (19).

(25) $\cancel{(e_1, e_2) \rhd (p_1, p_2)} \dashv\!\vert\, \theta$ — by contradiction

Assume $(e_1, e_2)\ ?\ (p_1, p_2)$. By rule induction over Rules (34) on it, only four cases apply.

Case (34c)**.**

(26) (e_1, e_2) `notintro` — by assumption

Contradicts Lemma 4.0.9.

Case (34d)**.**

(26) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ — by assumption

Contradicts (19).

Case (34e)**.**

(26) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ — by assumption

Contradicts (16).

Case (34f)**.**

(26) $e_1\ ?\ p_1$ — by assumption

Contradicts (17).

(27) $\cancel{(e_1, e_2)\ ?\ (p_1, p_2)}$ — by contradiction

$\square$

Lemma 4.0.15 (Matching Coherence of Constraint)**.** *Suppose that* $\cdot\,; \Delta_e \vdash e : \tau$ *and* e `final` *and* $p : \tau[\xi] \dashv\!\vert\, \Gamma\,; \Delta$. *Then we have*

1. $e \models \xi$ *iff* $e \rhd p \dashv\!\vert\, \theta$

2. $e \models_? \xi$ *iff* $e\ ?\ p$

3. $e \not\models_?^\dagger \xi$ *iff* $e \perp p$

Proof.

(1) $\cdot\,; \Delta_e \vdash e : \tau$ — by assumption

(2) e `final` — by assumption

(3) $p : \tau[\xi] \dashv\!\vert\, \Gamma\,; \Delta$ — by assumption

Given Lemma 3.0.1, Theorem 2.1, and Lemma 4.0.14, it is sufficient to prove

1. $e \models \xi$ iff $e \rhd p \dashv\!\vert\, \theta$

2. $e \models_? \xi$ iff $e \mathrel{?} p$

By rule induction over Rules (22) on (3).

Case (22a)**.**

(4) $p = x$ — by assumption

(5) $\xi = \top$ — by assumption

1. Prove $e \models \top$ implies $e \rhd x \dashv\vdash \theta$ for some θ.

   (6) $e \rhd x \dashv\vdash e/x$ — by Rule (33a)

2. Prove $e \rhd x \dashv\vdash \theta$ implies $e \models \top$.

   (6) $e \models \top$ — by Rule (16a)

3. Prove $e \models_? \top$ implies $e \mathrel{?} x$.

   (6) $e \not\models_? \top$ — by Lemma 2.0.3

   Vacuously true.

4. Prove $e \mathrel{?} x$ implies $e \models_? \top$.

   By rule induction over Rules (34), we notice that either, $e?x$ is in syntactic contradiction with all the cases, or the premise x `refutable`$_?$ is not derivable. Hence, $e \mathrel{?} x$ are not derivable. And thus vacuously true.

Case (22b)**.**

(4) $p = \_$ — by assumption

(5) $\xi = \top$ — by assumption

1. Prove $e \models \top$ implies $e \rhd \_ \dashv\vdash \theta$ for some θ.

   (6) $e \rhd \_ \dashv\vdash \cdot$ — by Rule (33a)

2. Prove $e \rhd \_ \dashv\vdash \theta$ implies $e \models \top$.

   (6) $e \models \top$ — by Rule (16a)

3. Prove $e \models_? \top$ implies $e \mathrel{?} \_$.

   (6) $e \not\models_? \top$ — by Lemma 2.0.3

   Vacuously true.

4. Prove $e \mathrel{?} \_$ implies $e \models_? \xi$.

   By rule induction over Rules (34), we notice that either, $e?\_$ is in syntactic contradiction with all the cases, or the premise $\_$ `refutable`$_?$ is not derivable. Hence, $e \mathrel{?} \_$ are not derivable. And thus vacuously true.

Case (22c)**.**

(4) $p = \llparenthesis\rrparenthesis^w$ — by assumption

(5) $\xi = ?$ — by assumption

(6) $\overline{\xi} = ?$ — by Definition 11

1. Prove $e \models ?$ implies $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\vert\, \theta$ for some θ.

   (7) $e \not\models ?$ — by Rule (33a)

   Vacuously true.

2. Prove $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\vert\, \theta$ implies $e \models ?$.
   By rule induction over Rules (33), we notice that $e \rhd \llparenthesis\rrparenthesis^w \dashv\!\vert\, \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? ?$ implies $e \,?\, \llparenthesis\rrparenthesis^w$.

   (7) $e \,?\, \llparenthesis\rrparenthesis^w$ — by Rule (34a)

4. Prove $e \,?\, \llparenthesis\rrparenthesis^w$ implies $e \models_? ?$.

   (7) $e \models_? ?$ — by Rule (18a)

Case (22d)**.**

(4) $p = \llparenthesis p_0 \rrparenthesis^w$ — by assumption

(5) $\xi = ?$ — by assumption

1. Prove $e \models ?$ implies $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\!\vert\, \theta$ for some θ.

   (6) $e \not\models ?$ — by Rule (33a)

   Vacuously true.

2. Prove $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\!\vert\, \theta$ implies $e \models ?$.
   By rule induction over Rules (33), we notice that $e \rhd \llparenthesis p_0 \rrparenthesis^w \dashv\!\vert\, \theta$ is in syntactic contradiction with all the cases, hence not derivable. And thus vacuously true.

3. Prove $e \models_? ?$ implies $e \,?\, \llparenthesis p_0 \rrparenthesis^w$.

   (6) $e \,?\, \llparenthesis p_0 \rrparenthesis^w$ — by Rule (34b)

4. Prove $e \,?\, \llparenthesis p_0 \rrparenthesis^w$ implies $e \models_? ?$.

   (6) $e \models_? ?$ — by Rule (18a)

Case (22e)**.**

(4) $p = \underline{n}$ — by assumption

(5) $\xi = \underline{n}$ — by assumption

1. Prove $e \models \underline{n}$ implies $e \rhd \underline{n} \dashv\vdash \theta$ for some θ.

   (6) $e \models \underline{n}$ — by assumption

   By rule induction over Rules (16) on (6), only one case applies.

   Case (16b)**.**

   (7) $e = \underline{n}$ — by assumption

   (8) $\underline{n} \rhd \underline{n} \dashv\vdash \cdot$ — by Rule (33c)

2. Prove $e \rhd \underline{n} \dashv\vdash \theta$ implies $e \models \underline{n}$.

   (6) $e \rhd \underline{n} \dashv\vdash \theta$ — by assumption

   By rule induction over Rules (33) on (6), only one case applies.

   Case (33c)**.**

   (7) $e = \underline{n}$ — by assumption

   (8) $\theta = \cdot$ — by assumption

   (9) $\underline{n} \models \underline{n}$ — by Rule (16b)

3. Prove $e \models_? \underline{n}$ implies $e \,?\, \underline{n}$.

   (6) $e \models_? \underline{n}$ — by assumption

   By rule induction over Rules (18) on (6), only one case applies.

   Case (18b)**.**

   (7) e `notintro` — by assumption

   (8) $\underline{n}$ $\texttt{refutable}_?$ — by Rule (32a)

   (9) $e \,?\, \underline{n}$ — by Rule (34c) on (7) and (8)

4. Prove $e \,?\, \underline{n}$ implies $e \models_? \underline{n}$.

   (6) $e \,?\, \underline{n}$ — by assumption

   By rule induction over Rules (34) on (6), only one case applies.

   Case (34c)**.**

   (7) e `notintro` — by assumption

   (8) $\underline{n}$ $\texttt{refutable}_?$ — by Rule (12a)

   (9) $e \models_? \underline{n}$ — by Rule (18) on (7) and (8)

Case (22f)**.**

(4) $p = \texttt{inl}(p_1)$ — by assumption

(5) $\xi = \texttt{inl}(\xi_1)$ — by assumption

(6) $\tau = (\tau_1 + \tau_2)$ — by assumption

(7) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma \,;\, \Delta$ — by assumption

By rule induction over Rules (21) on (1), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(8) $e = (\!|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) $e\ \texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

1. Prove $e \models \texttt{inl}(\xi_1)$ implies $e \rhd \texttt{inl}(p_1) \dashv\!| \theta$ for some θ. By rule induction over Rules (16) on $e \models \texttt{inl}(\xi_1)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \texttt{inl}(p_1) \dashv\!| \theta$ implies $e \models \texttt{inl}(\xi_1)$. By rule induction over Rules (33) on $e \rhd \texttt{inl}(p_1) \dashv\!| \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \texttt{inl}(\xi_1)$ implies $e\ ?\ \texttt{inl}(p_1)$.
   (10) $\texttt{inl}(p_1)\ \texttt{refutable}_?$ by Rule (32d)
   (11) $e\ ?\ \texttt{inl}(p_1)$ by Rule (34c) on (9) and (10)
4. Prove $e\ ?\ \texttt{inl}(p_1)$ implies $e \models_? \texttt{inl}(\xi_1)$.
   (10) $\texttt{inl}(\xi_1)\ \texttt{refutable}_?$ by Rule (12b)
   (11) $e \models_? \texttt{inl}(\xi_1)$ by Rule (18b) on (9) and (10)

Case (21j).

(8) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(9) $\cdot\, ; \Delta_e \vdash e_1 : \tau_1$ by assumption

(10) $e_1\ \texttt{final}$ by Lemma 4.0.3 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!| \theta$ for some θ

(12) $e_1 \models_? \xi_1$ iff $e_1\ ?\ p_1$

1. Prove $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ implies $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!| \theta$ for some θ.
   (13) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$ by assumption
   By rule induction over Rules (16) on (13), only one case applies.
   Case (16g).
   (14) $e_1 \models \xi_1$ by assumption
   (15) $e_1 \rhd p_1 \dashv\!| \theta_1$ for some θ_1 by (11) on (14)
   (16) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!| \theta_1$ by Rule (33e) on (15)
2. Prove $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!| \theta$ implies $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\xi_1)$.
   (13) $\texttt{inl}_{\tau_2}(e_1) \rhd \texttt{inl}(p_1) \dashv\!| \theta$ by assumption
   By rule induction over Rules (33) on (13), only one case applies.

Case (33e).

(14) $e_1 \rhd p_1 \dashv\!\!\mid \theta$ by assumption

(15) $e_1 \models \xi_1$ by (11) on (14)

(16) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\xi_1)$ by Rule (16g) on (15)

3. Prove $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inl}(\xi_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$.

(13) $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inl}(\xi_1)$ by assumption

By rule induction over Rules (18) on (13), only two cases apply.

Case (18b).

(14) $\mathtt{inl}_{\tau_2}(e_1)$ $\mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (18e).

(14) $e_1 \models_{?} \xi_1$ by assumption

(15) $e_1 \, ? \, p_1$ by (12) on (14)

(16) $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$ by Rule (34g) on (15)

4. Prove $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$ implies $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inl}(\xi_1)$.

(13) $\mathtt{inl}_{\tau_2}(e_1) \, ? \, \mathtt{inl}(p_1)$ by assumption

By rule induction over Rules (34) on (13), only two cases apply.

Case (34c).

(14) $\mathtt{inl}_{\tau_2}(e_1)$ $\mathtt{notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (34g).

(14) $e_1 \, ? \, p_1$ by assumption

(15) $e_1 \models_{?} \xi_1$ by (12) on (14)

(16) $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inl}(\xi_1)$ by Rule (18e) on (15)

Case (22g).

(4) $p = \mathtt{inr}(p_2)$ by assumption

(5) $\xi = \mathtt{inr}(\xi_2)$ by assumption

(6) $\tau = (\tau_1 + \tau_2)$ by assumption

(7) $p_2 : \tau_2[\xi_2] \dashv\!\!\mid \Gamma \, ; \Delta$ by assumption

By rule induction over Rules (21) on (1), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(8) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \mathtt{prl}(e_0), \mathtt{prr}(e_0), \mathtt{match}(e_0)\{\hat{rs}\}$ by assumption

(9) e $\mathtt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

1. Prove $e \models \texttt{inr}(\xi_2)$ implies $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ for some θ. By rule induction over Rules (16) on $e \models \texttt{inr}(\xi_2)$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
2. Prove $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ implies $e \models \texttt{inr}(\xi_2)$. By rule induction over Rules (33) on $e \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$, no case applies due to syntactic contradiction.
   Therefore, vacuously true.
3. Prove $e \models_? \texttt{inr}(\xi_2)$ implies $e \,?\, \texttt{inr}(p_2)$.

   (10) $\texttt{inr}(p_2)$ $\texttt{refutable}_?$ — by Rule (32e)

   (11) $e \,?\, \texttt{inr}(p_2)$ — by Rule (34c) on (9) and (10)
4. Prove $e \,?\, \texttt{inr}(p_2)$ implies $e \models_? \texttt{inr}(\xi_2)$.

   (10) $\texttt{inr}(\xi_2)$ $\texttt{refutable}_?$ — by Rule (12c)

   (11) $e \models_? \texttt{inr}(\xi_2)$ — by Rule (18b) on (9) and (10)

Case (21k).

(8) $e = \texttt{inr}_{\tau_1}(e_2)$ — by assumption

(9) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ — by assumption

(10) e_2 $\texttt{final}$ — by Lemma 4.0.3 on (2)

By inductive hypothesis on (10) and (9) and (7).

(11) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\!\mid \theta$ for some θ

(12) $e_2 \models_? \xi_2$ iff $e_2 \,?\, p_2$

1. Prove $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ implies $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ for some θ.

   (13) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ — by assumption

   By rule induction over Rules (16) on (13), only one case applies.

   Case (16g).

   (14) $e_2 \models \xi_2$ — by assumption

   (15) $e_2 \rhd p_2 \dashv\!\!\mid \theta_1$ for some θ_1 — by (11) on (14)

   (16) $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta_1$ — by Rule (33e) on (15)
2. Prove $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ implies $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$.

   (13) $\texttt{inr}_{\tau_1}(e_2) \rhd \texttt{inr}(p_2) \dashv\!\!\mid \theta$ — by assumption

   By rule induction over Rules (33) on (13), only one case applies.

   Case (33e).

   (14) $e_2 \rhd p_2 \dashv\!\!\mid \theta$ — by assumption

   (15) $e_2 \models \xi_2$ — by (11) on (14)

   (16) $\texttt{inr}_{\tau_1}(e_2) \models \texttt{inr}(\xi_2)$ — by Rule (16g) on (15)
3. Prove $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$ implies $\texttt{inr}_{\tau_1}(e_2) \,?\, \texttt{inr}(p_2)$.

(13) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$ by assumption

By rule induction over Rules (18) on (13), only two cases apply.

Case (18b).

(14) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (18e).

(14) $e_2 \models_? \xi_2$ by assumption

(15) $e_2 \, ? \, p_2$ by (12) on (14)

(16) $\texttt{inr}_{\tau_1}(e_2) \, ? \, \texttt{inr}(p_2)$ by Rule (34g) on (15)

4. Prove $\texttt{inr}_{\tau_1}(e_2) \, ? \, \texttt{inr}(p_2)$ implies $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$.

(13) $\texttt{inr}_{\tau_1}(e_2) \, ? \, \texttt{inr}(p_2)$ by assumption

By rule induction over Rules (34) on (13), only two cases apply.

Case (34c).

(14) $\texttt{inr}_{\tau_1}(e_2)$ $\texttt{notintro}$ by assumption

Contradicts Lemma 4.0.7.

Case (34g).

(14) $e_2 \, ? \, p_2$ by assumption

(15) $e_2 \models_? \xi_2$ by (12) on (14)

(16) $\texttt{inr}_{\tau_1}(e_2) \models_? \texttt{inr}(\xi_2)$ by Rule (18e) on (15)

Case (22h).

(4) $p = (p_1, p_2)$ by assumption

(5) $\xi = (\xi_1, \xi_2)$ by assumption

(6) $\tau = (\tau_1 \times \tau_2)$ by assumption

(7) $\Gamma = \Gamma_1 \uplus \Gamma_2$ by assumption

(8) $\Delta = \Delta_1 \uplus \Delta_2$ by assumption

(9) $p_1 : \tau_1[\xi_1] \dashv\vdash \Gamma_1 \, ; \Delta_1$ by assumption

(10) $p_2 : \tau_2[\xi_2] \dashv\vdash \Gamma_2 \, ; \Delta_2$ by assumption

By rule induction over Rules (21) on (1), the following cases apply.

Case (21b),(21c),(21f),(21h),(21i),(21l),(21m).

(11) $e = (\!|\,|\!)^u, (\!|e_0|\!)^u, e_1(e_2), \texttt{prl}(e_0), \texttt{prr}(e_0), \texttt{match}(e_0)\{\hat{rs}\}$ by assumption

(12) e $\texttt{notintro}$ by Rule (28a),(28b),(28c),(28d),(28e),(28f)

(13) e $\texttt{indet}$ by Lemma 4.0.10 on (2) and (12)

(14) $\texttt{prl}(e)$ $\texttt{indet}$ by Rule (26g) on (13)

(15) $\texttt{prl}(e)$ $\texttt{final}$ by Rule (27b) on (14)

(16) $\texttt{prr}(e)\ \texttt{indet}$ by Rule (26h) on (13)
(17) $\texttt{prr}(e)\ \texttt{final}$ by Rule (27b) on (16)
(18) $\cdot\,; \Delta \vdash \texttt{prl}(e) : \tau_1$ by Rule (21h) on (1)
(19) $\cdot\,; \Delta \vdash \texttt{prr}(e) : \tau_2$ by Rule (21i) on (1)

By inductive hypothesis on (9) and (18) and (15) and by inductive hypothesis on (10) and (19) and (17).

(20) $\texttt{prl}(e) \models \xi_1$ iff $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ for some θ_1
(21) $\texttt{prl}(e) \models_? \xi_1$ iff $\texttt{prl}(e)\ ?\ p_1$
(22) $\texttt{prr}(e) \models \xi_2$ iff $\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2$ for some θ_2
(23) $\texttt{prr}(e) \models_? \xi_2$ iff $\texttt{prr}(e)\ ?\ p_2$

1. Prove $e \models (\xi_1, \xi_2)$ implies $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ for some θ.
   (24) $e \models (\xi_1, \xi_2)$ by assumption
   By rule induction over Rules (16) on (24), only one case applies.
   Case (16j).
   (25) $\texttt{prl}(e) \models \xi_1$ by assumption
   (26) $\texttt{prr}(e) \models \xi_2$ by assumption
   (27) $\texttt{prl}(e) \rhd p_1 \dashv\!\vert\, \theta_1$ by (20) on (25)
   (28) $\texttt{prr}(e) \rhd p_2 \dashv\!\vert\, \theta_2$ by (22) on (26)
   (29) $e \rhd (p_1, p_2) \dashv\!\vert\, \theta_1 \uplus \theta_2$ by Rule (33g) on (12) and (27) and (28)
2. Prove $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ implies $e \models (\xi_1, \xi_2)$.
   (24) $e \rhd (p_1, p_2) \dashv\!\vert\, \theta$ by assumption
   By rule induction over Rules (33) on (24), only one case applies.
   Case (33g).
   (25) $\theta = \theta_1 \uplus \theta_2$ by assumption
   (26) $\texttt{prl}(e) \rhd \xi_1 \dashv\!\vert\, \theta_1$ by assumption
   (27) $\texttt{prr}(e) \rhd \xi_2 \dashv\!\vert\, \theta_2$ by assumption
   (28) $\texttt{prl}(e) \models \xi_1$ by (20) on (26)
   (29) $\texttt{prr}(e) \models \xi_2$ by (22) on (27)
   (30) $e \models (\xi_1, \xi_2)$ by Rule (16j) on (12) and (28) and (29)
3. Prove $e \models_? (\xi_1, \xi_2)$ implies $e\ ?\ (p_1, p_2)$.
   (24) $e \models_? (\xi_1, \xi_2)$ by assumption
   By rule induction over Rules (18) on (24), only one case applies.
   Case (18b).
   (25) $(\xi_1, \xi_2)\ \texttt{refutable}_?$ by assumption
   By rule induction over Rules (12) on (25), only two cases apply.
   Case (12d).

(26) ξ_1 $\texttt{refutable}_?$ by assumption

(27) $\texttt{prl}(e)$ $\texttt{notintro}$ by Rule (28e)

(28) $\texttt{prl}(e) \models_? \xi_1$ by Rule (18b) on (26) and (27)

(29) $\texttt{prl}(e)\ ?\ p_1$ by (21) on (28)

By rule induction over Rules (34) on (29), only three cases apply.

Case (34a),(34b)**.**

(30) $p_1 = (\!|\!)^w, (\!|p_0|\!)^w$ by assumption

(31) p_1 $\texttt{refutable}_?$ by Rule (32b) and Rule (32c)

(32) (p_1, p_2) $\texttt{refutable}_?$ by Rule (32f) on (31)

(33) $e\ ?\ (p_1, p_2)$ by Rule (34c) on (12) and (32)

Case (34c)**.**

(30) p_1 $\texttt{refutable}_?$ by assumption

(31) (p_1, p_2) $\texttt{refutable}_?$ by Rule (32f) on (30)

(32) $e\ ?\ (p_1, p_2)$ by Rule (34c) on (12) and (31)

Case (12e)**.**

(26) ξ_2 $\texttt{refutable}_?$ by assumption

(27) $\texttt{prr}(e)$ $\texttt{notintro}$ by Rule (28e)

(28) $\texttt{prr}(e) \models_? \xi_2$ by Rule (18b) on (26) and (27)

(29) $\texttt{prr}(e)\ ?\ p_2$ by (23) on (28)

By rule induction over Rules (34) on (29), only three cases apply.

Case (34a),(34b)**.**

(30) $p_2 = (\!|\!)^w, (\!|p_0|\!)^w$ by assumption

(31) p_2 $\texttt{refutable}_?$ by Rule (32b) and Rule (32c)

(32) (p_1, p_2) $\texttt{refutable}_?$ by Rule (32g) on (31)

(33) $e\ ?\ (p_1, p_2)$ by Rule (34c) on (12) and (32)

Case (34c)**.**

(30) p_2 $\texttt{refutable}_?$ by assumption

(31) (p_1, p_2) $\texttt{refutable}_?$ by Rule (32g) on (30)

(32) $e\ ?\ (p_1, p_2)$ by Rule (34c) on (12) and (31)

4. Prove $e\ ?\ (p_1, p_2)$ implies $e \models_? (\xi_1, \xi_2)$.

(24) $e\ ?\ (p_1, p_2)$ by assumption

By rule induction over Rules (34) on (24), only one case applies.

Case (34c).

(25) (p_1, p_2) `refutable`$_?$ by assumption

By rule induction over Rules (32) on (25), only two cases apply.

Case (32f).

(26) p_1 `refutable`$_?$ by assumption

(27) `prl`(e) `notintro` by Rule (28e)

(28) `prl`(e) $?\, p_1$ by Rule (34c) on (26) and (27)

(29) `prl`$(e) \models_? \xi_1$ by (21) on (28)

By rule induction over Rules (18) on (29), only three cases apply.

Case (18a).

(30) $\xi_1 = ?$ by assumption

(31) ξ_1 `refutable`$_?$ by Rule (2b)

(32) (ξ_1, ξ_2) `refutable`$_?$ by Rule (12d) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (32)

Case (18b).

(30) ξ_1 `refutable`$_?$ by assumption

(31) (ξ_1, ξ_2) `refutable`$_?$ by Rule (12d) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (31)

Case (32g).

(26) p_2 `refutable`$_?$ by assumption

(27) `prr`(e) `notintro` by Rule (28e)

(28) `prr`(e) $?\, p_2$ by Rule (34c) on (26) and (27)

(29) `prr`$(e) \models_? \xi_2$ by (23) on (28)

By rule induction over Rules (18) on (29), only three cases apply.

Case (18a).

(30) $\xi_2 = ?$ by assumption

(31) ξ_2 `refutable`$_?$ by Rule (2b)

(32) (ξ_1, ξ_2) `refutable`$_?$ by Rule (12e) on (31)

(33) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (32)

Case (18b).

(30) ξ_2 `refutable`$_?$ by assumption

(31) (ξ_1, ξ_2) `refutable`$_?$ by Rule (12e) on (30)

(32) $e \models_? (\xi_1, \xi_2)$ by Rule (18b) on (12) and (31)

Case (21g).

(11) $e = (e_1, e_2)$ by assumption

(12) $\cdot\,; \Delta_e \vdash e_1 : \tau_1$ by assumption

(13) $\cdot\,; \Delta_e \vdash e_2 : \tau_2$ by assumption

(14) e_1 `final` by Lemma 4.0.5 on (2)

(15) e_2 `final` by Lemma 4.0.5 on (2)

By inductive hypothesis on (14) and (12) and (9) and by inductive hypothesis on (15) and (13) and (10).

(16) $e_1 \models \xi_1$ iff $e_1 \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1

(17) $e_1 \models_? \xi_1$ iff $e_1 \;?\; p_1$

(18) $e_2 \models \xi_2$ iff $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2

(19) $e_2 \models_? \xi_2$ iff $e_2 \;?\; p_2$

1. Prove $(e_1, e_2) \models (\xi_1, \xi_2)$ implies $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$ for some θ.

   (20) $(e_1, e_2) \models (\xi_1, \xi_2)$ by assumption

   By rule induction over Rules (16) on (20), only two cases apply.

   Case (16i).

   (21) $e_1 \models \xi_1$ by assumption

   (22) $e_2 \models \xi_2$ by assumption

   (23) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1 by (16) on (21)

   (24) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2 by (18) on (22)

   (25) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta_1 \uplus \theta_2$ by Rule (33d) on (23) and (24)

   Case (16j).

   (21) (e_1, e_2) `notintro` by assumption

   Contradicts Lemma 4.0.9.

2. Prove $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$ implies $(e_1, e_2) \models (\xi_1, \xi_2)$.

   (20) $(e_1, e_2) \rhd (p_1, p_2) \dashv\!\mid \theta$ by assumption

   By rule induction over Rules (33) on (20), only two cases apply.

   Case (33d).

   (21) $e_1 \rhd p_1 \dashv\!\mid \theta_1$ for some θ_1 by assumption

   (22) $e_2 \rhd p_2 \dashv\!\mid \theta_2$ for some θ_2 by assumption

   (23) $e_1 \models \xi_1$ by (16) on (21)

   (24) $e_2 \models \xi_2$ by (18) on (22)

   (25) $(e_1, e_2) \models (\xi_1, \xi_2)$ by Rule (16i) on (23) and (24)

   Case (33g).

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

3. Prove $(e_1, e_2) \models_? (\xi_1, \xi_2)$ implies $(e_1, e_2)\,?\,(p_1, p_2)$.

(20) $(e_1, e_2) \models_? (\xi_1, \xi_2)$ by assumption

By rule induction over Rules (18) on (20), only four cases apply.

Case (18b)**.**

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (18g)**.**

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models \xi_2$ by assumption

(23) $e_1\,?\,p_1$ by (17) on (21)

(24) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by (18) on (22)

(25) $(e_1, e_2)\,?\,(p_1, p_2)$ by Rule (34d) on (23) and (24)

Case (18h)**.**

(21) $e_1 \models \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1 \rhd p_1 \dashv\!\vert\, \theta_1$ by (16) on (21)

(24) $e_2\,?\,p_2$ by (19) on (22)

(25) $(e_1, e_2)\,?\,(p_1, p_2)$ by Rule (34e) on (23) and (24)

Case (18i)**.**

(21) $e_1 \models_? \xi_1$ by assumption

(22) $e_2 \models_? \xi_2$ by assumption

(23) $e_1\,?\,p_1$ by (17) on (21)

(24) $e_2\,?\,p_2$ by (19) on (22)

(25) $(e_1, e_2)\,?\,(p_1, p_2)$ by Rule (34f) on (23) and (24)

4. Prove $(e_1, e_2)\,?\,(p_1, p_2)$ implies $(e_1, e_2) \models_? (\xi_1, \xi_2)$.

(20) $(e_1, e_2)\,?\,(p_1, p_2)$ by assumption

By rule induction over Rules (34) on (20), only four cases apply.

Case (34c)**.**

(21) (e_1, e_2) `notintro` by assumption

Contradicts Lemma 4.0.9.

Case (34d)**.**

(21) $e_1\,?\,p_1$ by assumption

(22) $e_2 \rhd p_2 \dashv\!\vert\, \theta_2$ by assumption

(23) $e_1 \models_? \xi_1$ by (17) on (21)

(24) $e_2 \models \xi_2$ by (18) on (22)

(25) $(e_1, e_2) ? (p_1, p_2)$ — by Rule (18g) on (23) and (24)

Case (34e).

(21) $e_1 \rhd p_1 \dashV \theta_1$ — by assumption
(22) $e_2 ? p_2$ — by assumption
(23) $e_1 \models \xi_1$ — by (16) on (21)
(24) $e_2 \models_? \xi_2$ — by (19) on (22)
(25) $(e_1, e_2) ? (p_1, p_2)$ — by Rule (18h) on (23) and (24)

Case (34f).

(21) $e_1 ? p_1$ — by assumption
(22) $e_2 ? p_2$ — by assumption
(23) $e_1 \models_? \xi_1$ — by (17) on (21)
(24) $e_2 \models_? \xi_2$ — by (19) on (22)
(25) $(e_1, e_2) ? (p_1, p_2)$ — by Rule (18i) on (23) and (24)

□

5 Preservation and Progress

Theorem 5.1 (Preservation). *If* $\cdot\,; \Delta \vdash e : \tau$ *and* $e \mapsto e'$ *then* $\cdot\,; \Delta \vdash e' : \tau$

Proof. By rule induction over Rules (21) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (21l).

(1) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ — by assumption
(2) $\mathtt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto e'$ — by assumption
(3) $\cdot\,; \Delta \vdash e_1 : \tau_1$ — by assumption
(4) $\cdot\,; \Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ — by assumption
(5) $\top \models_?^\dagger \xi$ — by assumption

By rule induction over Rules (36) on (2).

Case (36k).

(6) $e' = \mathtt{match}(e_1')\{\cdot \mid r \mid rs\}$ — by assumption
(7) $e_1 \mapsto e_1'$ — by assumption
(8) $\cdot\,; \Delta \vdash e_1' : \tau_1$ — by IH on (3) and (7)
(9) $\cdot\,; \Delta \vdash \mathtt{match}(e_1')\{\cdot \mid r \mid rs\} : \tau$ — by Rule (21l) on (8) and (4) and (5)

Case (36l).

(6) $r = p_r \Rightarrow e_r$ by assumption

(7) $e' = [\theta](e_r)$ by assumption

(8) $e_1 \rhd p_r \dashv\!\!\mid \theta$ by assumption

By rule induction over Rules (24) on (4).

Case (24a).

(9) $\xi = \xi_r$ by assumption

(10) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(11) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (10)

(12) $\Gamma_r\,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (23a) on (10)

(13) $\theta : \Gamma_r$ by Lemma 3.0.7 on (3) and (11) and (8)

(14) $\cdot\,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (12) and (13)

Case (24b).

(9) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(10) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(11) $p_r : \tau_1[\xi_r] \dashv\!\!\mid \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (10)

(12) $\Gamma_r\,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (23a) on (10)

(13) $\theta : \Gamma_r$ by Lemma 3.0.7 on (3) and (11) and (8)

(14) $\cdot\,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (12) and (13)

Case (36m).

(6) $rs = r' \mid rs'$ by assumption

(7) $e' = \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by assumption

(8) e_1 $\mathtt{final}$ by assumption

(9) $e_1 \perp p_r$ by assumption

By rule induction over Rules (24) on (4).

Case (24a). Syntactic contradiction of rs.

Case (24b).

(10) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(11) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(12) $\cdot\,;\Delta \vdash [\bot \vee \xi_r](r' \mid rs') : \tau_1[\xi_{rs}] \Rightarrow \tau$
by assumption

(13) $\xi_r \not\models \bot$ by assumption

(14) $p_r : \tau_1[\xi_r] \dashv\!\mid \Gamma_r \,; \Delta_r$ by Inversion of Rule (23a) on (11)

(15) $\Gamma_r \,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (23a) on (11)

(16) $\cdot\,;\Delta \vdash [\bot](p_r \Rightarrow e_r \mid \cdot) : \tau_1[\xi_r] \Rightarrow \tau$
by Rule (24a) on (11) and (13)

(17) $e_1 \not\models^{\dagger}_{?} \xi_r$ by Lemma 4.0.15 on (3) and (8) and (14) and (9)

(18) $\cdot\,;\Delta \vdash \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\} : \tau$
by Rule (21m) on (3) and (8) and (16) and (12) and (17) and (5)

Case (21m)**.**

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\,;\Delta \vdash \mathtt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\mathtt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} \mapsto e'$ by assumption

(4) $\cdot\,;\Delta \vdash e_1 : \tau_1$ by assumption

(5) $e_1\ \mathtt{final}$ by assumption

(6) $\cdot\,;\Delta \vdash [\bot]rs_{pre} : \tau_1[\xi_{pre}] \Rightarrow \tau$ by assumption

(7) $\cdot\,;\Delta \vdash [\bot \vee \xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$
by assumption

(8) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(9) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (36) on (3).

Case (36k)**.**

(10) $e' = \mathtt{match}(e_1')\{rs_{pre} \mid r \mid rs_{post}\}$ by assumption

(11) $e_1 \mapsto e_1'$ by assumption

By Lemma 4.0.13, (11) contradicts (5).

Case (36l)**.**

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $e' = [\theta](e_r)$ by assumption

(12) $e_1 \rhd p_r \dashv\!\mid \theta$ by assumption

By rule induction over Rules (24) on (7).

Case (24a).

(13) $\xi_{rest} = \xi_r$ by assumption

(14) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv\!\!\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (14)

(16) $\Gamma_r\,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (23a) on (14)

(17) $\theta : \Gamma_r$ by Lemma 3.0.7 on (4) and (15) and (12)

(18) $\cdot\,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (16) and (17)

Case (24b).

(13) $\xi_{rest} = \xi_r \vee \xi_{rs}$ by assumption

(14) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(15) $p_r : \tau_1[\xi_r] \dashv\!\!\vdash \Gamma_r\,; \Delta_r$ by assumption

(16) $\Gamma_r\,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by assumption

(17) $\theta : \Gamma_r$ by Lemma 3.0.7 on (4) and (15) and (12)

(18) $\cdot\,; \Delta \uplus \Delta_r \vdash [\theta](e_r) : \tau$ by Lemma 3.0.6 on (16) and (17)

Case (36m).

(10) $r = p_r \Rightarrow e_r$ by assumption

(11) $rs_{post} = r' \mid rs'$ by assumption

(12) $e' = \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\}$ by assumption

(13) $e_1 \perp p_r$ by assumption

By rule induction over Rules (24) on (7).

Case (24a). Syntactic contradiction of rs_{post}.

Case (24b).

(14) $\xi_{rest} = \xi_r \vee \xi_{post}$ by assumption

(15) $\cdot\,; \Delta \vdash (p_r \Rightarrow e_r) : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(16) $\cdot\,; \Delta \vdash [\bot \vee \xi_{pre} \vee \xi_r](r' \mid rs') : \tau_1[\xi_{post}] \Rightarrow \tau$ by assumption

(17) $\xi_r \not\models \xi_{pre}$ by assumption

(18) $p_r : \tau_1[\xi_r] \dashv\!\!\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (15)

(19) $\Gamma_r\,; \Delta \uplus \Delta_r \vdash e_r : \tau$ by Inversion of Rule (23a) on (15)

(20) $\xi_r : \tau_1$ by Lemma 3.0.2 on (15)

(21) $\xi_{pre} : \tau_1$ by Lemma 3.0.3 on (6)

(22) $\xi_r \not\models \bot \vee \xi_{pre}$ by Lemma 2.0.6 on (20) and (21) and (17)

(23) $\cdot\,; \Delta \vdash [\bot](rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} : \tau_1[\xi_{pre} \vee \xi_r] \Rightarrow \tau$ by Lemma 3.0.4 on (6) and (15) and (22)

(24) $e_1 \not\models_?^{\dagger} \xi_r$ by Lemma 4.0.15 on (4) and (5) and (18) and (13)

(25) $e_1 \not\models_?^{\dagger} \xi_{pre} \vee \xi_r$ by Lemma 2.0.7 on (8) and (24)

(26) $\cdot\,; \Delta \vdash \texttt{match}(e_1)\{(rs_{pre} \mid p_r \Rightarrow e_r \mid \cdot)^{\diamond} \mid r' \mid rs'\} : \tau$ by Rule (21m) on (4) and (5) and (23) and (16) and (25) and (9)

□

Theorem 5.2 (Progress)**.** *If* $\cdot\,; \Delta \vdash e : \tau$ *then either* e `final` *or* $e \mapsto e'$ *for some* e'*.*

Proof. By rule induction over Rules (21) on typing judgment of e. For simplicity, we only consider two cases for match expressions here.

Case (21l)**.**

(1) $\cdot\,; \Delta \vdash \texttt{match}(e_1)\{\cdot \mid r \mid rs\} : \tau$ by assumption

(2) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(3) $\cdot\,; \Delta \vdash [\bot](r \mid rs) : \tau_1[\xi] \Rightarrow \tau$ by assumption

(4) $\top \models_?^{\dagger} \xi$ by assumption

By IH on (2).

Case Scrutinee takes a step**.**

(5) $e_1 \mapsto e_1'$ by assumption

(6) $\texttt{match}(e_1)\{\cdot \mid r \mid rs\} \mapsto \texttt{match}(e_1')\{\cdot \mid r \mid rs\}$ by Rule (36k) on (5)

Case Scrutinee is final**.**

(5) e_1 `final` by assumption

By rule induction over Rules (24) on (3).

Case (24a)**.**

(6) $rs = \cdot$ by assumption

(7) $\xi = \xi_r$ by assumption

(8) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (23a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\!\vert\, \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (8)

(11) $e_1 \models_?^\dagger \xi_r$ by Corollary 2.1.1 on (5) and (4)

By rule induction over Rules (19) on (11).

Case (19a).

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \;?\; p_r$ by Lemma 4.0.15 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{indet}$ by Rule (26k) on (5) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{final}$ by Rule (27b) on (14)

Case (19b).

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\!\vert\, \theta$ by Lemma 4.0.15 on (2) and (5) and (10) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (36l) on (5) and (13)

Case (24b).

(6) $rs = r' \mid rs'$ by assumption

(7) $\xi = \xi_r \vee \xi_{rs}$ by assumption

(8) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(9) $r = p_r \Rightarrow e_r$ by Inversion of Rule (23a) on (8)

(10) $p_r : \tau_1[\xi_r] \dashv\!\vert\, \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (8)

By Lemma 4.0.14 on (2) and (5) and (10).

Case Scrutinee matches pattern.

(11) $e_1 \rhd p_r \dashv\!\vert\, \theta$ by assumption

(12) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\} \mapsto [\theta](e_r)$ by Rule (36l) on (5) and (11)

Case Scrutinee may matches pattern.

(11) $e_1 \mathrel{?} p_r$ by assumption

(12) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}$ $\mathtt{indet}$ by Rule (26k) on (5) and (11)

(13) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs\}$ $\mathtt{final}$ by Rule (27b) on (12)

Case Scrutinee doesn't matche pattern.

(11) $e_1 \perp p_r$ by assumption

(12) $\mathtt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs')\}$ $\mapsto \mathtt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'\}$ by Rule (36m) on (5) and (11)

Case (21m).

(1) $rs_{pre} = r_{pre} \mid rs'_{pre}$ by assumption

(2) $\cdot\,; \Delta \vdash \mathtt{match}(e_1)\{rs_{pre} \mid r \mid rs_{post}\} : \tau$ by assumption

(3) $\cdot\,; \Delta \vdash e_1 : \tau_1$ by assumption

(4) e_1 $\mathtt{final}$ by assumption

(5) $\cdot\,; \Delta \vdash [\xi_{pre}](r \mid rs_{post}) : \tau_1[\xi_{rest}] \Rightarrow \tau$ by assumption

(6) $e_1 \not\models^{\dagger}_{?} \xi_{pre}$ by assumption

(7) $\top \models^{\dagger}_{?} \xi_{pre} \vee \xi_{rest}$ by assumption

By rule induction over Rules (24) on (5).

Case (24a).

(5) $rs_{post} = \cdot$ by assumption

(6) $\xi_{rest} = \xi_r$ by assumption

(7) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(8) $r = p_r \Rightarrow e_r$ by Inversion of Rule (23a) on (7)

(9) $p_r : \tau_1[\xi_r] \dashv\!\vdash \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (7)

(10) $e_1 \models^{\dagger}_{?} \xi_{pre} \vee \xi_r$ by Corollary 2.1.1 on (4) and (7)

(11) $e_1 \models^{\dagger}_{?} \xi_r$ by Lemma 2.0.8 on (10) and (6)

By rule induction over Rules (19) on (11).

Case (19a).

(12) $e_1 \models_? \xi_r$ by assumption

(13) $e_1 \mathrel{?} p_r$ by Lemma 4.0.15 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{indet}$ by Rule (26k) on (4) and (13)

(15) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\}\ \texttt{final}$ by Rule (27b) on (14)

Case (19b)**.**

(12) $e_1 \models \xi_r$ by assumption

(13) $e_1 \rhd p_r \dashv\!\mid \theta$ by Lemma 4.0.15 on (3) and (4) and (9) and (12)

(14) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid \cdot\} \mapsto [\theta](e_r)$ by Rule (36l) on (4) and (13)

Case (24b)**.**

(5) $rs_{post} = r' \mid rs'_{post}$ by assumption

(6) $\cdot\,; \Delta \vdash r : \tau_1[\xi_r] \Rightarrow \tau$ by assumption

(7) $r = p_r \Rightarrow e_r$ by Inversion of Rule (23a) on (6)

(8) $p_r : \tau_1[\xi_r] \dashv\!\mid \Gamma_r\,; \Delta_r$ by Inversion of Rule (23a) on (6)

By Lemma 4.0.14 on (3) and (4) and (8).

Case Scrutinee matches pattern**.**

(9) $e_1 \rhd p_r \dashv\!\mid \theta$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\} \mapsto [\theta](e_r)$ by Rule (36l) on (4) and (9)

Case Scrutinee may matches pattern**.**

(9) $e_1 \mathrel{?} p_r$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}\ \texttt{indet}$ by Rule (26k) on (4) and (9)

(11) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid rs_{post}\}\ \texttt{final}$ by Rule (27b) on (10)

Case Scrutinee doesn't matche pattern**.**

(9) $e_1 \perp p_r$ by assumption

(10) $\texttt{match}(e_1)\{\cdot \mid p_r \Rightarrow e_r \mid (r' \mid rs'_{post})\}$
$\mapsto \texttt{match}(e_1)\{(p_r \Rightarrow e_r \mid \cdot) \mid r' \mid rs'_{post}\}$
by Rule (36m) on (4) and (9)

$\square$

6 Decidability

$\boxed{\Xi\ \texttt{incon}}$ A finite set of constraints, Ξ, is inconsistent

CINCTruth
$$\frac{\Xi\ \texttt{incon}}{\Xi, \top\ \texttt{incon}} \tag{37a}$$

CINCFalsity
$$\frac{}{\Xi, \bot\ \texttt{incon}} \tag{37b}$$

CINCNum
$$\frac{n_1 \neq n_2}{\Xi, \underline{n_1}, \underline{n_2}\ \texttt{incon}} \tag{37c}$$

CINCNotNum
$$\frac{}{\Xi, \underline{n}, \cancel{\underline{n}}\ \texttt{incon}} \tag{37d}$$

CINCAnd
$$\frac{\Xi, \xi_1, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \wedge \xi_2\ \texttt{incon}} \tag{37e}$$

CINCOr
$$\frac{\Xi, \xi_1\ \texttt{incon} \qquad \Xi, \xi_2\ \texttt{incon}}{\Xi, \xi_1 \vee \xi_2\ \texttt{incon}} \tag{37f}$$

CINCInj
$$\frac{}{\Xi, \texttt{inl}(\xi_1), \texttt{inr}(\xi_2)\ \texttt{incon}} \tag{37g}$$

CINCInl
$$\frac{\Xi\ \texttt{incon}}{\texttt{inl}(\Xi)\ \texttt{incon}} \tag{37h}$$

CINCInr
$$\frac{\Xi\ \texttt{incon}}{\texttt{inr}(\Xi)\ \texttt{incon}} \tag{37i}$$

CINCPairL
$$\frac{\Xi_1\ \texttt{incon}}{(\Xi_1, \Xi_2)\ \texttt{incon}} \tag{37j}$$

CINCPairR
$$\frac{\Xi_2\ \texttt{incon}}{(\Xi_1, \Xi_2)\ \texttt{incon}} \tag{37k}$$

Lemma 6.0.1 (Decidability of Inconsistency)**.** *Suppose $\dot{\top}(\xi) = \xi$. It is decidable whether ξ `incon`.*

Lemma 6.0.2 (Inconsistency and Entailment of Constraint)**.** *Suppose that $\dot{\top}(\xi) = \xi$. Then $\overline{\xi}$ `incon` iff $\top \models \xi$*

Lemma 6.0.3. *If $e \models \xi$ then $e \models \dot{\top}(\xi)$*

Proof. By rule induction over Rules (16), it is obvious to see that $\dot{\top}(\xi) = \xi$. □

Lemma 6.0.4. *If $e \models_? \xi$ then $e \models_?^\dagger \dot{\top}(\xi)$.*

Proof.

(11) $e \models_? \xi$ — by assumption

By Rule Induction over Rules (18) on (11).

Case (18a)**.**

(12) $\xi = ?$ — by assumption

(13) $e \models \top$ — by Rule (16a)

(14) $e \models_?^\dagger \top$ — by Rule (19b) on (13)

Case (18b)**.**

(12) e `notintro` — by assumption

(13) ξ $\texttt{refutable}_?$ — by assumption

By Lemma 2.0.15 on (12) and (13) and case analysis on its conclusion. By rule induction over Rules (12).

Case $\dot{\top}(\xi)$ $\texttt{refutable}_?$**.**

(14) $\dot{\top}(\xi)$ $\texttt{refutable}_?$ — by assumption

(15) $e \models_? \dot{\top}(\xi)$ — by Rule (18b) on (12) and (14)

(16) $e \models_?^\dagger \dot{\top}(\xi)$ — by Rule (19b) on (15)

Case $e \models \dot{\top}(\xi)$**.**

(14) $e \models \dot{\top}(\xi)$ — by assumption

(15) $e \models_?^\dagger \top$ — by Rule (19b) on (14)

Case (18c)**.**

(12) $\xi = \xi_1 \vee \xi_2$ — by assumption

(13) $e \models_? \xi_1$ — by assumption

(14)	$e \models^{\dagger}_{?} \dot{\top}(\xi_1)$	by IH on (13)
(15)	$e \models^{\dagger}_{?} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 2.0.10 on (14)

Case (18d).

(12)	$\xi = \xi_1 \vee \xi_2$	by assumption
(13)	$e \models_{?} \xi_2$	by assumption
(14)	$e \models^{\dagger}_{?} \dot{\top}(\xi_2)$	by IH on (13)
(15)	$e \models^{\dagger}_{?} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Lemma 2.0.10 on (14)

Case (18e).

(12)	$e = \mathtt{inl}_{\tau_2}(e_1)$	by assumption
(13)	$\xi = \mathtt{inl}(\xi_1)$	by assumption
(14)	$e_1 \models_{?} \xi_1$	by assumption
(15)	$e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$	by IH on (14)
(16)	$\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\dot{\top}(\xi_1))$	by Lemma 2.0.11 on (15)

Case (18f).

(12)	$e = \mathtt{inr}_{\tau_1}(e_2)$	by assumption
(13)	$\xi = \mathtt{inr}(\xi_2)$	by assumption
(14)	$e_2 \models_{?} \xi_2$	by assumption
(15)	$e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$	by IH on (14)
(16)	$\mathtt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \mathtt{inr}(\dot{\top}(\xi_2))$	by Lemma 2.0.12 on (15)

Case (18g).

(12)	$e = (e_1, e_2)$	by assumption
(13)	$\xi = (\xi_1, \xi_2)$	by assumption
(14)	$e_1 \models_{?} \xi_1$	by assumption
(15)	$e_2 \models \xi_2$	by assumption
(16)	$e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$	by IH on (14)

(17) $e_2 \models \dot{\top}(\xi_2)$ by Lemma 6.0.3 on (15)

(18) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19b) on (17)

(19) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Lemma 2.0.13 on (16) and (18)

Case (18h)**.**

(12) $e = (e_1, e_2)$ by assumption

(13) $\xi = (\xi_1, \xi_2)$ by assumption

(14) $e_1 \models \xi_1$ by assumption

(15) $e_2 \models_{?} \xi_2$ by assumption

(16) $e_1 \models \dot{\top}(\xi_1)$ by Lemma 6.0.3 on (14)

(17) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19b) on (16)

(18) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by IH on (15)

(19) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Lemma 2.0.13 on (17) and (18)

Case (18i)**.**

(12) $e = (e_1, e_2)$ by assumption

(13) $\xi = (\xi_1, \xi_2)$ by assumption

(14) $e_1 \models_{?} \xi_1$ by assumption

(15) $e_2 \models_{?} \xi_2$ by assumption

(16) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by IH on (14)

(17) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by IH on (15)

(18) $(e_1, e_2) \models^{\dagger}_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Lemma 2.0.13 on (16) and (17)

□

Lemma 6.0.5. *$e \models^{\dagger}_{?} \xi$ iff $e \models^{\dagger}_{?} \dot{\top}(\xi)$*

Proof. 1. Sufficiency:

(1) $e \models^{\dagger}_{?} \xi$ by assumption

By rule induction over Rules (19) on (1)

Case (19b).

(2) $e \models \xi$ by assumption

(3) $e \models \dot{\top}(\xi)$ by Lemma 6.0.3 on (2)

(4) $e \models_{?}^{\dagger} \dot{\top}(\xi)$ by Rule (19b) on (3)

Case (19a).

(2) $e \models_{?} \xi$ by assumption

(3) $e \models_{?}^{\dagger} \dot{\top}(\xi)$ by Lemma 6.0.4 on (2)

2. Necessity:

(1) $e \models_{?}^{\dagger} \dot{\top}(\xi)$ by assumption

By structural induction on ξ,

Case $\xi = \top, \bot, \underline{n}, \underline{\not n}$.

(2) $e \models_{?}^{\dagger} \xi$ by (1) and Definition 14

Case $\xi = ?$.

(2) $e \models_{?} ?$ by Rule (18a)

(3) $e \models_{?}^{\dagger} ?$ by Rule (19a) on (2)

Case $\xi = \xi_1 \vee \xi_2$.

(2) $\dot{\top}(\xi_1 \vee \xi_2) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Definition 14

By rule induction over Rules (19) on (1),

Case (19b).

(3) $e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (16) on (3) and two cases apply,

Case (16e).

(4) $e \models \dot{\top}(\xi_1)$ by assumption

(5) $e \models_{?}^{\dagger} \dot{\top}(\xi_1)$ by Rule (19b) on (4)

(6) $e \models_{?}^{\dagger} \xi_1$ by IH on (5)

(7) $e \models_{?}^{\dagger} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case (16f).

(4) $e \models \dot{\top}(\xi_2)$ by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19b) on (4)

(6) $e \models^{\dagger}_{?} \xi_2$ by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case (19a).

(3) $e \models_{?} \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by assumption

By rule induction over Rules (18) on (3) and two cases apply,

Case (18c).

(4) $e \models_{?} \dot{\top}(\xi_1)$ by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19a) on (4)

(6) $e \models^{\dagger}_{?} \xi_1$ by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case (18d).

(4) $e \models_{?} \dot{\top}(\xi_2)$ by assumption

(5) $e \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19a) on (4)

(6) $e \models^{\dagger}_{?} \xi_2$ by IH on (5)

(7) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (6)

Case $\xi = \mathtt{inl}(\xi_1)$.

(2) $e = \mathtt{inl}_{\tau_2}(e_1)$ by assumption

(3) $\dot{\top}(\xi) = \mathtt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (19) on (1),

Case (19b).

(4) $\mathtt{inl}_{\tau_2}(e_1) \models \mathtt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (16) and only one case applies,

Case (16g).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19b) on (5)

(7) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (6)

(8) $\mathtt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \mathtt{inl}(\xi_1)$ by Lemma 2.0.11 on (7)

Case (19a).

(4) $\mathtt{inl}_{\tau_2}(e_1) \models_{?} \mathtt{inl}(\dot{\top}(\xi_1))$ by assumption

By rule induction over Rules (18) and only one case applies,

Case (18e).

(5) $e_1 \models_? \dot{\top}(\xi_1)$ by assumption

(6) $e_1 \models_?^\dagger \dot{\top}(\xi_1)$ by Rule (19a) on (5)

(7) $e_1 \models_?^\dagger \xi_1$ by IH on (6)

(8) $\mathtt{inl}_{\tau_2}(e_1) \models_?^\dagger \mathtt{inl}(\xi_1)$ by Lemma 2.0.11 on (7)

Case $\xi = \mathtt{inr}(\xi_2)$.

(2) $e = \mathtt{inr}_{\tau_1}(e_2)$ by assumption

(3) $\dot{\top}(\xi) = \mathtt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (19) on (1),

Case (19b).

(4) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (16) and only one case applies,

Case (16h).

(5) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (19b) on (5)

(7) $e_2 \models_?^\dagger \xi_2$ by IH on (6)

(8) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Lemma 2.0.12 on (7)

Case (19a).

(4) $\mathtt{inr}_{\tau_1}(e_2) \models_? \mathtt{inr}(\dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (18) and only one case applies,

Case (18f).

(5) $e_2 \models_? \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models_?^\dagger \dot{\top}(\xi_2)$ by Rule (19a) on (5)

(7) $e_2 \models_?^\dagger \xi_2$ by IH on (6)

(8) $\mathtt{inr}_{\tau_1}(e_2) \models_?^\dagger \mathtt{inr}(\xi_2)$ by Lemma 2.0.12 on (7)

Case $\xi = (\xi_1, \xi_2)$.

(2) $e = (e_1, e_2)$ by assumption

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by Definition 14

By rule induction over Rules (19) on (1),

Case (19b).

(4) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (16) on (4) and only one case applies,

Case (16i).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

Case (19a).

(4) $(e_1, e_2) \models_{?} (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by assumption

By rule induction over Rules (18) on (4) and three cases apply,

Case (18g).

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19b) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

Case (18h).

(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19b) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

Case (18i).

(5) $e_1 \models_{?} \dot{\top}(\xi_1)$ by assumption

(6) $e_2 \models_{?} \dot{\top}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \dot{\top}(\xi_1)$ by Rule (19a) on (5)

(8) $e_2 \models^{\dagger}_{?} \dot{\top}(\xi_2)$ by Rule (19a) on (6)

(9) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (7)

(10) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (8)

(11) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (9) and (10)

□

Lemma 6.0.6. *Assume* $\dot{\top}(\xi) = \xi$*. Then* $\top \models^{\dagger}_{?} \xi$ *iff* $\top \models \xi$*.*

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

2. Necessity:

□

Theorem 6.1. $\top \models^{\dagger}_{?} \xi$ *iff* $\top \models \dot{\top}(\xi)$*.*

Lemma 6.1.1. *Assume that* e `val`*. Then* $e \models^{\dagger}_{?} \xi$ *iff* $e \models \dot{\top}(\xi)$

Proof.

(1) e `val` — by assumption

We prove sufficiency and necessity separately.

1. Sufficiency:

   (2) $e \models^{\dagger}_{?} \xi$ — by assumption

   By rule induction over Rules (19) on (2).

   Case (19b)**.**

   (3) $e \models \xi$ — by assumption
   (4) $e \models \dot{\top}(\xi)$ — by Lemma 6.0.3 on (3)

   Case (19a)**.**

   (3) $e \models_{?} \xi$ — by assumption

   By rule induction over Rules (18) on (3).

   Case (18a)**.**

   (4) $\xi = ?$ — by assumption
   (5) $e \models \dot{\top}(\xi)$ — by Rule (16a) and Definition 14

   Case (18b)**.**

   (4) e `notintro` — by assumption

   By rule induction over Rules (28) on (4), for each case, by rule induction over Rules (25) on (1), no case applies due to syntactic contradiction.

   Case (18c)**.**

   (4) $\xi = \xi_1 \vee \xi_2$ — by assumption
   (5) $e \models_{?} \xi_1$ — by assumption

(6)	$\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Definition 14
(7)	$e \models^{\dagger}_{?} \xi_1$	by Rule (19a) on (5)
(8)	$e \models \dot{\top}(\xi_1)$	by IH on (7)
(9)	$e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (16e) on (8)

Case (18d)**.**

(4)	$\xi = \xi_1 \vee \xi_2$	by assumption
(5)	$e \models_{?} \xi_2$	by assumption
(6)	$\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Definition 14
(7)	$e \models^{\dagger}_{?} \xi_2$	by Rule (19a) on (5)
(8)	$e \models \dot{\top}(\xi_2)$	by IH on (7)
(9)	$e \models \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$	by Rule (16f) on (8)

Case (18e)**.**

(4)	$\xi = \texttt{inl}(\xi_1)$	by assumption
(5)	$e \models_{?} \xi_1$	by assumption
(6)	$\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$	by Definition 14
(7)	$e \models^{\dagger}_{?} \xi_1$	by Rule (19a) on (5)
(8)	$e \models \dot{\top}(\xi_1)$	by IH on (7)
(9)	$e \models \texttt{inl}(\dot{\top}(\xi_1))$	by Rule (16g) on (8)

Case (18f)**.**

(4)	$\xi = \texttt{inr}(\xi_2)$	by assumption
(5)	$e \models_{?} \xi_2$	by assumption
(6)	$\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$	by Definition 14
(7)	$e \models^{\dagger}_{?} \xi_2$	by Rule (19a) on (5)
(8)	$e \models \dot{\top}(\xi_2)$	by IH on (7)
(9)	$e \models \texttt{inr}(\dot{\top}(\xi_2))$	by Rule (16h) on (8)

Case (18g)**.**

(4)	$e = (e_1, e_2)$	by assumption
(5)	$\xi = (\xi_1, \xi_2)$	by assumption
(6)	$e_1 \models_{?} \xi_1$	by assumption
(7)	$e_2 \models \xi_2$	by assumption
(8)	$\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Definition 14
(9)	$e_1 \models^{\dagger}_{?} \xi_1$	by Rule (19a) on (6)
(10)	$e_2 \models^{\dagger}_{?} \xi_2$	by Rule (19b) on (7)
(11)	$e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12)	$e_2 \models \dot{\top}(\xi_2)$	by IH on (10)

(13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (16i) on (11) and (12)

Case (18h)**.**

(4) $e = (e_1, e_2)$	by assumption
(5) $\xi = (\xi_1, \xi_2)$	by assumption
(6) $e_1 \models \xi_1$	by assumption
(7) $e_2 \models_? \xi_2$	by assumption
(8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Definition 14
(9) $e_1 \models^{\dagger}_{?} \xi_1$	by Rule (19b) on (6)
(10) $e_2 \models^{\dagger}_{?} \xi_2$	by Rule (19a) on (7)
(11) $e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12) $e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (16i) on (11) and (12)

Case (18i)**.**

(4) $e = (e_1, e_2)$	by assumption
(5) $\xi = (\xi_1, \xi_2)$	by assumption
(6) $e_1 \models_? \xi_1$	by assumption
(7) $e_2 \models_? \xi_2$	by assumption
(8) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Definition 14
(9) $e_1 \models^{\dagger}_{?} \xi_1$	by Rule (19a) on (6)
(10) $e_2 \models^{\dagger}_{?} \xi_2$	by Rule (19a) on (7)
(11) $e_1 \models \dot{\top}(\xi_1)$	by IH on (9)
(12) $e_2 \models \dot{\top}(\xi_2)$	by IH on (10)
(13) $(e_1, e_2) \models (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$	by Rule (16i) on (11) and (12)

2. Necessity:

(2) $e \models \dot{\top}(\xi)$	by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \underline{\not n}$**.**

(3) $\xi = \dot{\top}(\xi)$	by Definition 14
(4) $e \models^{\dagger}_{?} \xi$	by Rule (19b) on (2)

Case $\xi = ?$**.**

(3) $e \models_? ?$	by Rule (18a)

(4) $e \models^{\dagger}_{?} ?$ by Rule (19a) on (3)

Case $\xi = \xi_1 \wedge \xi_2$.

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \wedge \dot{\top}(\xi_2)$ by Definition 14

By rule induction over Rules (16) on (2), only one case applies.

Case (16d).

(4) $e \models \dot{\top}(\xi_1)$ by assumption
(5) $e \models \dot{\top}(\xi_2)$ by assumption
(6) $e \models^{\dagger}_{?} \xi_1$ by IH on (4)
(7) $e \models^{\dagger}_{?} \xi_2$ by IH on (5)
(8) $e \models \xi_1 \wedge \xi_2$ by Lemma 2.0.9 on (6) and (7)

Case $\xi = \xi_1 \vee \xi_2$.

(3) $\dot{\top}(\xi) = \dot{\top}(\xi_1) \vee \dot{\top}(\xi_2)$ by Definition 14

By rule induction over Rules (16) on (2) and only two cases apply.

Case (16e).

(4) $e \models \dot{\top}(\xi_1)$ by assumption
(5) $e \models^{\dagger}_{?} \xi_1$ by IH on (4)
(6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (5)

Case (16f).

(4) $e \models \dot{\top}(\xi_2)$ by assumption
(5) $e \models^{\dagger}_{?} \xi_2$ by IH on (4)
(6) $e \models^{\dagger}_{?} \xi_1 \vee \xi_2$ by Lemma 2.0.10 on (5)

Case $\xi = \texttt{inl}(\xi_1)$.

(3) $\dot{\top}(\xi) = \texttt{inl}(\dot{\top}(\xi_1))$ by Definition 14

By rule induction over Rules (16) on (2) and only one case applies.

Case (16g).

(4) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption
(5) $e_1 \models \dot{\top}(\xi_1)$ by assumption
(6) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (5)
(7) $\texttt{inl}_{\tau_2}(e_1) \models^{\dagger}_{?} \texttt{inl}(\xi_1)$ by Lemma 2.0.11 on (6)

Case $\xi = \texttt{inr}(\xi_2)$.

(3) $\dot{\top}(\xi) = \texttt{inr}(\dot{\top}(\xi_2))$ by Definition 14

By rule induction over Rules (16) on (2) and only one case applies.

Case (16h).

(4) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(5) $e_2 \models \dot{\top}(\xi_2)$ by assumption

(6) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (5)

(7) $\texttt{inr}_{\tau_1}(e_2) \models^{\dagger}_{?} \texttt{inr}(\xi_2)$ by Lemma 2.0.12 on (6)

Case $\xi = (\xi_1, \xi_2)$.

(3) $\dot{\top}(\xi) = (\dot{\top}(\xi_1), \dot{\top}(\xi_2))$ by Definition 14

By rule induction over Rules (16) on (2) and only one case applies.

Case (16i).

(4) $e = (e_1, e_2)$ by assumption

(5) $e_1 \models \dot{\bot}(\xi_1)$ by assumption

(6) $e_2 \models \dot{\bot}(\xi_2)$ by assumption

(7) $e_1 \models^{\dagger}_{?} \xi_1$ by IH on (5)

(8) $e_2 \models^{\dagger}_{?} \xi_2$ by IH on (6)

(9) $(e_1, e_2) \models^{\dagger}_{?} (\xi_1, \xi_2)$ by Lemma 2.0.13 on (7) and (8)

□

Lemma 6.1.2. $e \models \xi$ *iff* $e \models \dot{\bot}(\xi)$

Proof. We prove sufficiency and necessity separately.

1. Sufficiency:

(1) $e \models \xi$ by assumption

By rule induction over Rules (16) on (1).

Case (16a).

(2) $\xi = \top$ by assumption

(3) $e \models \dot{\bot}(\top)$ by (1) and Definition 15

Case (16b).

(2) $\xi = \underline{n}$ by assumption

(3) $e \models \dot{\bot}(\underline{n})$ by (1) and Definition 15

Case (16c).

(2) $\xi = \underline{\not{n}}$	by assumption
(3) $e \models \dot{\bot}(\underline{\not{n}})$	by (1) and Definition 15

Case (16d).

(2) $\xi = \xi_1 \wedge \xi_2$	by assumption
(3) $e \models \xi_1$	by assumption
(4) $e \models \xi_2$	by assumption
(5) $e \models \dot{\bot}(\xi_1)$	by IH on (3)
(6) $e \models \dot{\bot}(\xi_2)$	by IH on (4)
(7) $e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$	by Rule (16d) on (5) and (6)
(8) $e \models \dot{\bot}(\xi_1 \wedge \xi_2)$	by (7) and Definition 15

Case (16e).

(2) $\xi = \xi_1 \vee \xi_2$	by assumption
(3) $e \models \xi_1$	by assumption
(4) $e \models \dot{\bot}(\xi_1)$	by IH on (3)
(5) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (16e) on (4)
(6) $e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 15

Case (16f).

(2) $\xi = \xi_1 \vee \xi_2$	by assumption
(3) $e \models \xi_2$	by assumption
(4) $e \models \dot{\bot}(\xi_2)$	by IH on (3)
(5) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$	by Rule (16f) on (4)
(6) $e \models \dot{\bot}(\xi_1 \vee \xi_2)$	by (5) and Definition 15

Case (16g).

(2) $e = \texttt{inl}_{\tau_2}(e_1)$	by assumption
(3) $\xi = \texttt{inl}(\xi_1)$	by assumption
(4) $e_1 \models \xi_1$	by assumption
(5) $e_1 \models \dot{\bot}(\xi_1)$	by IH on (4)
(6) $\texttt{inl}_{\tau_2}(e_1) \models \texttt{inl}(\dot{\bot}(\xi_1))$	by Rule (16g) on (5)
(7) $\texttt{inl}_{\tau_2}(e_1) \models \dot{\bot}(\texttt{inl}(\xi_1))$	by (6) and Definition 15

Case (16h).

(2) $e = \texttt{inr}_{\tau_1}(e_2)$	by assumption

(3) $\xi = \mathtt{inr}(\xi_2)$	by assumption
(4) $e_2 \models \xi_2$	by assumption
(5) $e_2 \models \dot{\bot}(\xi_2)$	by IH on (4)
(6) $\mathtt{inr}_{\tau_1}(e_2) \models \mathtt{inr}(\dot{\bot}(\xi_2))$	by Rule (16h) on (5)
(7) $\mathtt{inr}_{\tau_1}(e_2) \models \dot{\bot}(\mathtt{inr}(\xi_2))$	by (6) and Definition 15

Case (16i)**.**

(2) $e = (e_1, e_2)$	by assumption
(3) $\xi = (\xi_1, \xi_2)$	by assumption
(4) $e_1 \models \xi_1$	by assumption
(5) $e_2 \models \xi_2$	by assumption
(6) $e_1 \models \dot{\bot}(\xi_1)$	by IH on (4)
(7) $e_2 \models \dot{\bot}(\xi_2)$	by IH on (5)
(8) $(e_1, e_2) \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$	by Rule (16i) on (6) and (7)
(9) $(e_1, e_2) \models \dot{\bot}((\xi_1, \xi_2))$	by (8) and Definition 15

2. Necessity:

(1) $e \models \dot{\bot}(\xi)$	by assumption

By structural induction on ξ.

Case $\xi = \top, \bot, \underline{n}, \cancel{\underline{n}}$**.**

(2) $e \models \xi$	by (1) and Definition 15

Case $\xi = ?$**.**

(2) $e \models \bot$	by (1) and Definition 15
(3) $e \not\models \bot$	by Lemma 2.0.1

(3) contradicts (2).

Case $\xi = \xi_1 \wedge \xi_2$**.**

(2) $e \models \dot{\bot}(\xi_1) \wedge \dot{\bot}(\xi_2)$	by (1) and Definition 15

By rule induction over Rules (16) on (2) and only case applies.

Case (16d)**.**

(3) $e \models \dot{\bot}(\xi_1)$	by assumption
(4) $e \models \dot{\bot}(\xi_2)$	by assumption
(5) $e \models \xi_1$	by IH on (3)

(6) $e \models \xi_2$ by IH on (4)

(7) $e \models \xi_1 \wedge \xi_2$ by Rule (16d) on (5) and (6)

Case $\xi = \xi_1 \vee \xi_2$.

(2) $e \models \dot{\bot}(\xi_1) \vee \dot{\bot}(\xi_2)$ by (1) and Definition 15

By rule induction over Rules (16) on (2) and only two cases apply.

Case (16e).

(3) $e \models \dot{\bot}(\xi_1)$ by assumption

(4) $e \models \xi_1$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (16e) on (4)

Case (16f).

(3) $e \models \dot{\bot}(\xi_2)$ by assumption

(4) $e \models \xi_2$ by IH on (3)

(5) $e \models \xi_1 \vee \xi_2$ by Rule (16f) on (4)

Case $\xi = \texttt{inl}(\xi_1)$.

(2) $e \models \texttt{inl}(\dot{\bot}(\xi_1))$ by (1) and Definition 15

By rule induction over Rules (16) on (2) and only one case applies.

Case (16g).

(3) $e = \texttt{inl}_{\tau_2}(e_1)$ by assumption

(4) $e_1 \models \dot{\bot}(\xi_1)$ by assumption

(5) $e_1 \models \xi_1$ by IH on (4)

(6) $e \models \texttt{inl}(\xi_1)$ by Rule (16g) on (5)

Case $\xi = \texttt{inr}(\xi_2)$.

(2) $e \models \texttt{inr}(\dot{\bot}(\xi_2))$ by (1) and Definition 15

By rule induction over Rules (16) on (2) and only one case applies.

Case (16h).

(3) $e = \texttt{inr}_{\tau_1}(e_2)$ by assumption

(4) $e_2 \models \dot{\bot}(\xi_2)$ by assumption

(5) $e_2 \models \xi_2$ by IH on (4)

(6) $e \models \texttt{inr}(\xi_2)$ by Rule (16h) on (5)

Case $\xi = (\xi_1, \xi_2)$.

(2) $e \models (\dot{\bot}(\xi_1), \dot{\bot}(\xi_2))$ by (1) and Definition 15

By rule induction over Rules (16) on (2) and only case applies.

Case (16i)**.**

(3) $e = (e_1, e_2)$	by assumption
(4) $e_1 \models \dot{\bot}(\xi_1)$	by assumption
(5) $e_2 \models \dot{\bot}(\xi_2)$	by assumption
(6) $e_1 \models \xi_1$	by IH on (4)
(7) $e_2 \models \xi_2$	by IH on (5)
(8) $e \models (\xi_1, \xi_2)$	by Rule (16i) on (6) and (7)

□

Lemma 6.1.3. *Assume* e `val` *and* $\dot{\top}(\xi) = \xi$. *Then* $e \not\models \xi$ *iff* $e \models \overline{\xi}$.

Theorem 6.2. $\xi_r \models \xi_{rs}$ *iff* $\top \models \overline{\dot{\top}(\xi_r)} \vee \dot{\bot}(\xi_{rs})$.