Appendix A. Evaluation of the Inverse Laplace Transform

Pollitz (2003) proposed a method for an inverse Laplace transform in which the Laplace transform $\tilde{f}(s)$ is represented as a weighted sum of functions $[s(s + s_0\alpha_j)]^{-1}$ over a number of collocation points $\{\alpha_j\}$, plus a term proportional to s^-1, i.e.,

$$\tilde{f}(s) = \left(\sum_{i=1}^{N} C_i \frac{1}{s(s + s_0\alpha_i)} \right) + C_{(N+1)} \frac{1}{s_0 s} \tag{A-1}$$

where s_0 is a reference Laplace transform parameter. In the time domain this yields a superposition of decaying exponentials representing post-seismic motions and a step function that represents a static offset, i.e.,

$$f(t) = L^{-1}[\tilde{f}(s)] = \left(\sum_{i=1}^{N} C_i \frac{1 - \exp\left[-s_0\alpha_i t\right]}{s_0\alpha_i} \right) + C_{(N+1)} \frac{H(t)}{s_0} \tag{A-2}$$

where $H(t)$ is the Heaviside step function. The weights are determined by sampling $\tilde{f}(s)$ at a set of sample points $\{s_j$, j=1, J$\}$, i.e., obtaining sample values $\{\tilde{d}(s_j)$, j=1, J$\}$, then defining an inverse problem for the C_i such that eqn A-1 optimally represents $\tilde{d}(s_j)$. Pollitz (2003) employed $N = 7$ collocation points and $J = 12$ sample points in order to represent $\tilde{f}(s)$. That resulted in estimates of $f(t)$ in the time domain very accurate up to $t \sim 150/s_0$.

Pollitz (2003) chose the $\{\alpha_j\}$ through a process of trial and error, based on the ability to replicate test functions, each defined by a decaying exponential with decay time $s_j \leq s_0$. Here we revise the approach to extend the validity up to $t \sim 1000/s_0$, employing a grid search for the $\{\alpha_j\}$ based on minimization of a misfit function in the time domain involving a sum over a set of test functions.

To fit the set of observations $\{\tilde{d}(s_j)$, j=1, J$\}$ with a function $\tilde{f}(s)$ of the form of eqn A-1, we define a cost function

$$C = \sum_{j=1}^{J} \left| \tilde{d}(s_j) \; - \; \tilde{f}(s_j) \right|^2 \tag{A-3}$$

The solution for the $\{C_i\}$ is obtained by minimizing C and can be written in the form

$$C_i = \sum_{j=1}^{J} \tilde{d}(s_j) \left[\left(\sum_{l=1}^{N} A_{il}^{-1} \frac{1}{s_j(s_j + s_l)} \right) + A_{i(N+1)}^{-1} \frac{1}{s_j s_0} \right] \tag{A-4}$$

where A_{il}^{-1} is the il-component of the inverse of the matrix $\mathbf{A}$ with elements

$$A_{il} = \begin{cases} \sum_{j=1}^{J} \frac{1}{s_j(s_j+\alpha_i s_0)} \frac{1}{s_j(s_j+\alpha_l s_0)} & i \leq N, l \leq N \\ \sum_{j=1}^{J} \frac{1}{s_j(s_j+\alpha_i s_0)} \frac{1}{s_j s_0} & i \leq N, l = N+1 \\ \sum_{j=1}^{J} \frac{1}{s_j(s_j+\alpha_l s_0)} \frac{1}{s_j s_0} & i = N+1, l \leq N \\ \sum_{j=1}^{J} \left[\frac{1}{s_j s_0} \right]^2 & i = N+1, l = N+1 \end{cases} \tag{A-5}$$

We now employ a set of trial functions $\tilde{d}(s) = \tilde{g}_k(s)$, where

$$\tilde{g}_k(s) = \begin{cases} \frac{1}{s(s+\beta_k s_0)} & k = 1, K \\ \frac{1}{s s_0} & k = K+1 \end{cases} \tag{A-6}$$

Let $C_i^{(k)}$ be the solution of eqn A-4 for the kth trial function, and define the associated reconstruction of the original function as

$$\tilde{f}_k(s) = \left(\sum_{i=1}^{N} C_i^{(k)} \frac{1}{s(s + s_0\alpha_i)} \right) + C_{(N+1)}^{(k)} \frac{1}{s_0 s} \tag{A-7}$$

With the aim of designing an inverse Laplace transform valid from time 0 to $1000/s_0$, we finally define a misfit function in the time domain

$$M = \sum_{k=1}^{K} \int_0^{1000/s_0} \left[f_k(t) \, - \, g_k(t) \right]^2 \, dt \tag{A-8}$$

In intended applications, s_0 is the reciprocal of the minimum material decay time of the viscoelastic model (which may correspond to a Kelvin element or a Maxwell element of a Burgers body).

The misfit function will depend upon the choices of several sets of parameters, particularly the sample points $\{s_j$, j=1, J$\}$. Our strategy for designing an optimal inverse Laplace transform is to select fixed sets of collocation points $\{\alpha_j\}$ and decay parameters of the test functions $\{\beta_j\}$ and to vary the values of sample points such that M is minimized. Specifically, we define $N = 11$ and

$$\alpha_1, \cdots, \alpha_{11} = 2, 1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{30}, \frac{1}{100}, \frac{1}{300}, \frac{1}{1000}, \frac{1}{3000}, \frac{1}{10000} \tag{A-9}$$

and define $K = 9$ and

$$\beta_1, \cdots, \beta_9 = 0.005, 0.01, 0.05, 0.2, 0.3, 0.5, 0.7, 0.85, 1.0 \tag{A-10}$$

With these fixed sets, we determine $J = 28$ sample points through a global search to minimize M using a simulated annealing procedure. This yields the $\{s_j$, j=1, 28$\}$ and the 12×12 matrix inverse $\mathbf{A}^{-1}$. In the EXAMPLES-VISCO3D folder, these quantities are contained in the file 'INVLAPL.PARAM'. Subroutine LAPL (in SOURCE/visco3dsubs.f) then evaluates the inverse Laplace transform using eqns A-2 and A-4.

Figures A-1, A-2, and A-3 show the fit of eqn A-2 to test functions

$$d(t) = [1 - \exp(-t/\tau)]\,\frac{\tau}{\tau_0} \qquad \text{(A-11)}$$

for different ratios of τ/τ_0.

Reference

Pollitz F.F. , 2003. Postseismic relaxation theory on a laterally heterogeneous viscoelastic model, Geophys. J. Int., 155, 57–78.

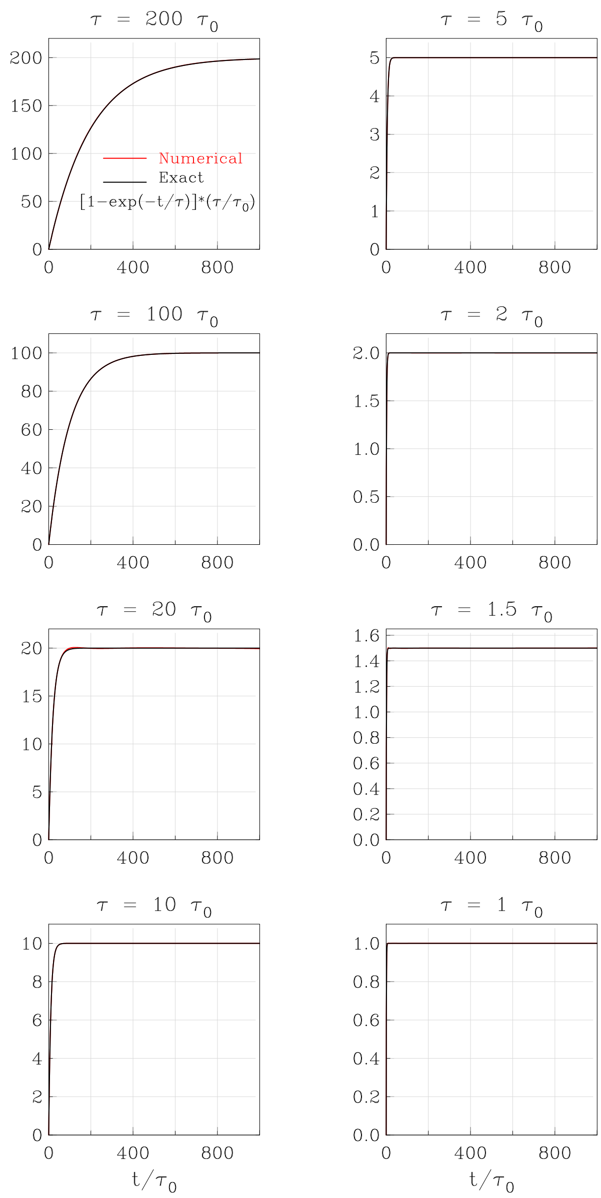

Figure A-1: Fit of eqn A-2 to exponentially-decaying test functions of the form of eqn A-11.

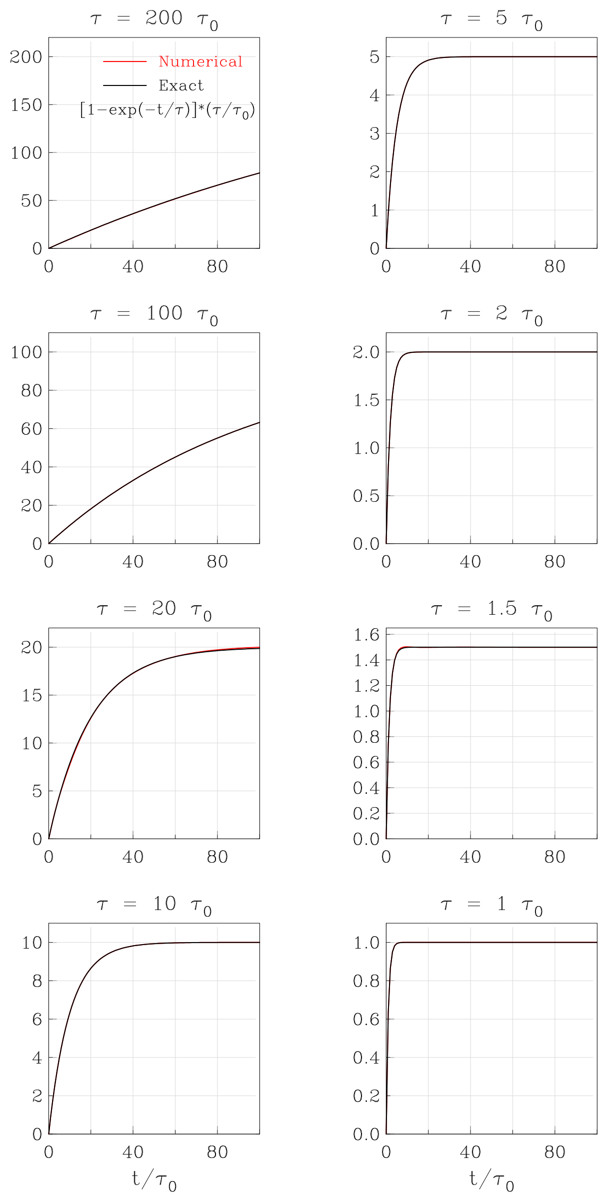

Figure A-2: Same as Figure A-1 with a different scale of the x-axis.

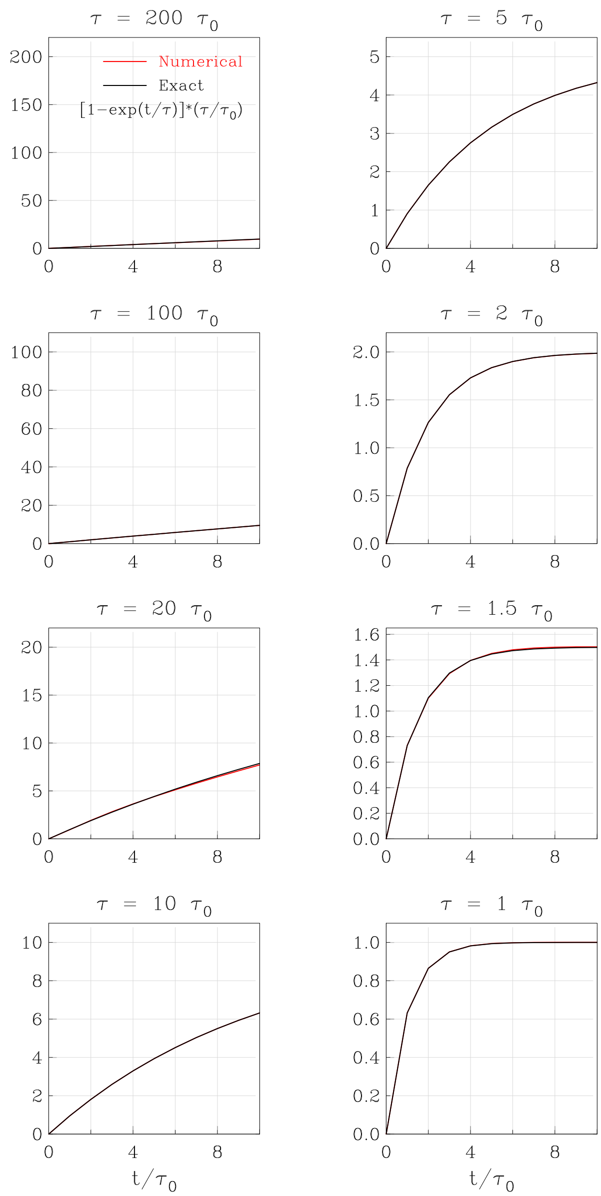

Figure A-3: Same as Figure A-1 with a different scale of the x-axis.