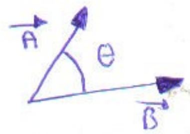
① Ángulo que forman $\vec{A}$ y $\vec{B}$



$\vec{A} = 2\hat{i} + 2\hat{j} + (-\hat{k})$ $\quad |\vec{A}| = A = 3$

$\vec{B} = 6\hat{i} + 3(-\hat{j}) + 2\hat{k}$ $\quad |\vec{B}| = B = 7$

$\vec{A} \cdot \vec{B} = A \cdot B \cdot \cos\theta$ $\qquad \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{A \cdot B} = \dfrac{(2\hat{i}+2\hat{j}-\hat{k})(6\hat{i}-3\hat{j}+2\hat{k})}{21} = \dfrac{12-6-2}{21} =$

$= \dfrac{4}{21}$ $\qquad \theta = \arccos\left(\dfrac{4}{21}\right) = 79^\circ$

② Demostrar que $\vec{A}$, $\vec{B}$ y $\vec{C}$ forman un triángulo rectángulo:

Para que lo formen, al menos dos deben ser perpendiculares, o sea, $\vec{A} \cdot \vec{B} = 0$

(producto escalar de los dos es 0)

$\vec{A} \cdot \vec{B} = A \cdot B \cdot \cos\theta = 0 \Rightarrow$

$\cos\theta = 0 \Rightarrow$

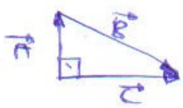
$\theta = 90^\circ$

$\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$ $\qquad \vec{A} \cdot \vec{B} = 14 \neq 0$

$\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$ $\qquad \vec{B} \cdot \vec{C} = -21 \neq 0$

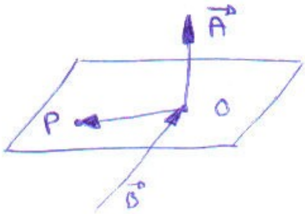
$\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ $\qquad \vec{A} \cdot \vec{C} = 0 \Rightarrow \vec{A} \perp \vec{C}$

c.q.d.



(3) <u>Ecuación del plano $\perp$ al vector $\vec{A} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ que pasa por el extremo de $\vec{B} = \hat{i} + 5\hat{j} + 3\hat{k}$:</u>

- Buscamos un vector contenido en la superf. $\perp$ que pase por el pto. $O = (1,5,3)$, siendo O el extremo del vector $\vec{B}$.



$$\vec{OP} \cdot \vec{A} = 0$$

$$(x-1)\cdot 2 + (y-5)\cdot 3 + (z-3)\cdot 6 = 0$$

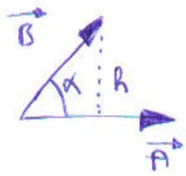
$$2x - 2 + y3 - 15 + 6z - 18 = 0$$

$$2x + 3y + 6z - 35 = 0$$

$$\boxed{2x + 3y + 6z = 35}$$

(4) <u>Demostrar que el área de un paralelogramo de lados $\vec{A}$ y $\vec{B}$ es $|\vec{A} \times \vec{B}|$:</u>

$$|\vec{A} \times \vec{B}| = A \cdot B \cdot \operatorname{sen} \theta$$



$$\operatorname{sen} \alpha = \frac{h}{B}$$

$$h = B \cdot \operatorname{sen} \alpha$$

Área = base x altura

$$\text{Área} = \vec{A} \cdot \underbrace{h}_{h = B \cdot \operatorname{sen} \alpha} = A \cdot B \cdot \operatorname{sen} \alpha$$

c.q.d.

⑤ Demostrar que $|\vec{A} \cdot (\vec{B} \times \vec{C})|$ es igual al vol. de un paralelepípedo de aristas $\vec{A}, \vec{B}$ y $\vec{C}$. Demostrar la igualdad:

$$\text{Volumen} = \text{Área base} \cdot h = |\vec{B} \times \vec{C}| \cdot h = |\vec{B} \times \vec{C}| \cdot A \cdot \operatorname{sen}\alpha =$$

Ejerc ④

$$= (\vec{B} \times \vec{C}) \cdot \vec{A} = |\vec{B} \times \vec{C}| \cdot A \cdot \cos\beta$$

$\operatorname{sen}\alpha = \cos(90° - \alpha) = \cos\beta$

$\beta = 90° - \alpha$

c.q.d.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \vec{A} \cdot \left[(A_y B_z - A_z B_y)\hat{\imath} - (A_x B_z - A_z B_x)\hat{\jmath} + \right.$$

$$\left. + (A_x B_y - A_y B_x)\hat{k} \right] = (A_y B_z - A_z \cdot B_y) C_x \cdot \hat{\imath} - (A_x B_z - A_z B_x) C_y \cdot \hat{\jmath} +$$

$$+ (A_x B_y - A_y B_x) C_z \cdot \hat{k} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} C_x \cdot \hat{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} C_y \cdot \hat{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} C_z \cdot \hat{k} =$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

c.q.d.

⑥ Hallar la ecuación del plano formado por P_1, P_2 y P_3:

$P_1\ (2,-1,1)$

$P_2\ (3,2,-1)$

$P_3\ (-1,3,2)$

$\left[\begin{array}{l}\text{Ec. del plano:}\\ \text{Dos puntos y } \vec{n}\end{array}\right]$

$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

$$\overrightarrow{P_1P_2} = (3-2,\ 2-(-1),\ -1-1) = (1,3,-2)$$

$$\overrightarrow{P_1P_3} = (-1-2,\ 3-(-1),\ 2-1) = (-3,4,1)$$

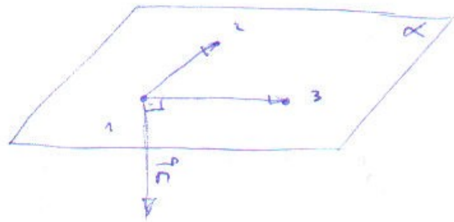
$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 11\hat{\imath} - (-5)\hat{\jmath} + 13\hat{k} = (11,5,13)$$

Plano:

$$(x-2,\ y+1,\ z-1)\cdot(11,5,13) = 0$$

$$11x - 22 + 5y + 5 + 13z - 13 = 0$$

$$11x + 5y + 13z = 30$$



⑦ Hallar $\nabla\phi$ siendo:

a) $\phi = \ln|\vec{r}|$

$\left[\vec{r} = x\cdot\hat{\imath} + y\cdot\hat{\jmath} + z\cdot\hat{k}\right]$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\phi = \ln\sqrt{x^2+y^2+z^2} = \ln\left(x^2+y^2+z^2\right)^{1/2} = \frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$\left[\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}\right]$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2+z^2}\cancel{2}x \qquad \frac{\partial\phi}{\partial y} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2+z^2}\cancel{2}y$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+y^2+z^2}\cancel{2}z$$

$$\nabla\phi = \frac{x}{x^2+y^2+z^2}\,\hat{i} + \frac{y}{x^2+y^2+z^2}\,\hat{j} + \frac{z}{x^2+y^2+z^2}\,\hat{k} = \frac{\vec{r}}{r^2} = \frac{\vec{r}}{|\vec{r}|^2}$$

b) $\phi = \frac{1}{|\vec{r}|}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\phi = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{(x^2+y^2+z^2)^{1/2}} = (x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial\phi}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}x \qquad \frac{\partial\phi}{\partial y} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}y$$

$$\frac{\partial\phi}{\partial z} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}z$$

$$\nabla\phi = \frac{-x}{\sqrt{(x^2+y^2+z^2)^3}}\,\hat{i} + \frac{-y}{\sqrt{(x^2+y^2+z^2)^3}}\,\hat{j} + \frac{-z}{\sqrt{(x^2+y^2+z^2)^3}}\,\hat{k} =$$

$$= \frac{-\vec{r}}{r^3} = \frac{-\vec{r}}{|\vec{r}|^3}$$

8) Ec. del plano tangente a la superficie $2xz^2-3xy-4x=7$ en el punto $(1,-1,2)$

$\boxed{\text{Ec. plano tg: Dos puntos y } \vec{n}}$

$$\vec{n} = \nabla f(x,y,z)$$

$$\nabla f(x,y,z) = (2z^2-3y-4)\,\hat{i} + (-3x)\,\hat{j} + (4xz)\cdot\hat{k}$$

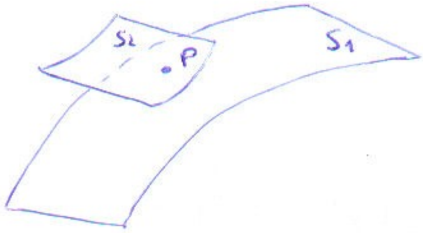
$$\nabla f(1,-1,2) = (2\cdot2^2-3(-1)-4)\,\hat{i} + (-3\cdot1)\,\hat{j} + 4\cdot1\cdot2\cdot\hat{k} =$$

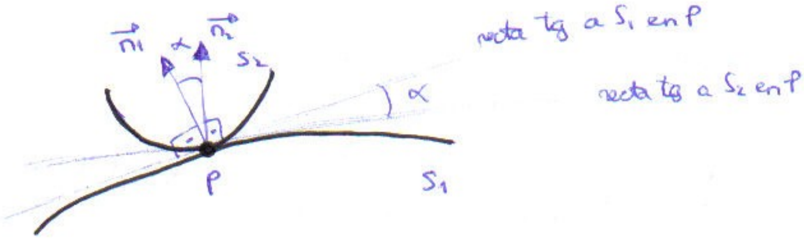
$$= 7\hat{i} - 3\hat{j} + 8\hat{k}$$

$$(x-1,\ y+1,\ z-2)\cdot(7,-3,8) = 0 \qquad (x-1)\cdot7 + (y+1)(-3) + (z-2)\cdot8 = 0$$

9) Hallar el ángulo que forman las superficies $x^2+y^2+z^2=9$ y $z=x^2+y^2-3$ en el punto $(2,-1,2)$:

• El ángulo que forman dos superficies es el mismo que el que forman sus vectores normales:





$$\alpha = \text{ang}(\vec{n_1}, \vec{n_2})$$

$$\vec{n_1} = \nabla f_1 \Big|_{P(2,-1,2)} \qquad f_1: x^2+y^2+z^2-9=0$$

$$\vec{n_2} = \nabla f_2 \Big|_{P(2,-1,2)} \qquad f_2: x^2+y^2-z-3=0$$

$$\nabla f_1 = \frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_1}{\partial y}\hat{j} + \frac{\partial f_1}{\partial z}\hat{k} = 2x\cdot\hat{i} + 2y\cdot\hat{j} + 2z\cdot\hat{k} \Big|_{P(2,-1,2)} =$$

$$= 4\hat{i} - 2\hat{j} + 4\hat{k} = (4,-2,4)$$

$$\nabla f_2 = \frac{\partial f_2}{\partial x}\cdot\hat{i} + \frac{\partial f_2}{\partial y}\hat{j} + \frac{\partial f_2}{\partial z}\hat{k} = 2x\cdot\hat{i} + 2y\cdot\hat{j} - \hat{k} \Big|_{P(2,-1,2)} =$$

$$= 4\hat{i} - 2\hat{j} - \hat{k} = (4,-2,-1)$$

$$\cos(\nabla f_1, \nabla f_2) = \frac{(\nabla f_1)\cdot(\nabla f_2)}{|\nabla f_1|\cdot|\nabla f_2|} = \frac{16+4-4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\alpha = \arccos\left(\frac{8}{3\sqrt{21}}\right) = 54{,}41^\circ$$

(10) Demostrar:

a) $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

$$\nabla\cdot(\vec{A}+\vec{B}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(A_x+B_x, A_y+B_y, A_z+B_z) = \frac{\partial}{\partial x}(A_x+B_x) +$$

$$+\frac{\partial}{\partial y}(A_y+B_y) + \frac{\partial}{\partial z}(A_z+B_z) = \frac{\partial A_x}{\partial x} + \frac{\partial B_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial B_y}{\partial y} + \frac{\partial A_z}{\partial z} +$$

$$+\frac{\partial B_z}{\partial z} = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$ c.q.d.

b) $\nabla\cdot(\phi\cdot\vec{A}) = (\nabla\cdot\phi)\cdot\vec{A} + \phi\cdot(\nabla\vec{A})$

$\phi \equiv \phi(x,y,z)$

$\vec{A} = (A_x, A_y, A_z)$

$$\nabla\cdot(\phi\vec{A}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\phi\cdot(A_x,A_y,A_z)\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(\phi A_x, \phi A_y, \phi A_z) =$$

$$= \frac{\partial(\phi A_x)}{\partial x} + \frac{\partial(\phi A_y)}{\partial y} + \frac{\partial(\phi A_z)}{\partial z} = \frac{\partial\phi}{\partial x}A_x + \frac{\partial A_x}{\partial x}\phi + \frac{\partial \phi}{\partial y}A_y +$$

$$+\frac{\partial A_y}{\partial y}\phi + \frac{\partial\phi}{\partial z}A_z + \frac{\partial A_z}{\partial z}\phi = (\nabla\phi)\cdot\vec{A} + \phi\cdot(\nabla\cdot\vec{A})$$ c.q.d.

(11) Demostrar:

a) $\nabla\times(\nabla\phi) = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial\phi}{\partial y}\right)\hat{i} + \left(\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial\phi}{\partial x}\right)\hat{j} +$$

$$+\left(\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\cdot\frac{\partial\phi}{\partial x}\right)\cdot\hat{k} = \vec{0}$$ c.q.d.

b) $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial y} & \frac{\partial A_z}{\partial z} \end{vmatrix} = \ldots = \vec{0}$$

Si ponemos ∇ multiplicando a la 3ª componente, quedará como el apartado anterior

(12) Sea $\vec{A} = (3x^2 + 6y)\hat{i} - (14yz)\cdot\hat{j} + (20xz^2)\cdot\hat{k}$.

Hallar la integral curvilínea de $\vec{A}$ desde (0,0,0) a (1,1,1) siendo la trayectoria:

a) $x = t$, $y = t^2$, $z = t^3$

$$\left[\int_{(0,0,0)}^{(1,1,1)} \vec{A} \cdot \vec{dl} \qquad \vec{dl} = (dx, dy, dz)\right] \qquad \int_C \vec{A} \cdot \vec{dl} = \int_C A_x \cdot dx + A_y \cdot dy + A_z \cdot dz$$

• Expresamos $A_x, A_y, A_z, dx, dy, dz$ en términos de t:

$$A_x = 3x^2 + 6y = 3t^2 + 6t^2 = 9t^2 \qquad dx = dt$$
$$A_y = -14yz = -14 \cdot t^2 \cdot t^3 = -15t^5 \qquad dy = 2t \cdot dt$$
$$A_z = 20xz^2 = 20t \cdot t^6 = 20t^7 \qquad dz = 3t^2 \cdot dt$$

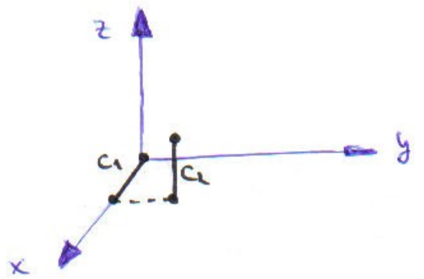
• Cambiamos los límites de integración:

$$c = (0,0,0) \longrightarrow (1,1,1)$$
$$c' = (0,0,0) \longrightarrow (1,1,1)$$
$$(1, 1^2, 1^3)$$
$$t = 0 \longrightarrow t = 1$$

$$\int_C 9t^2\cdot dt - 14t^5\cdot 2t\cdot dt + 20t^7\cdot 3t^2\cdot dt = \int_{C'} (9t^2 - 28t^6 + 60t^4)\cdot dt =$$

$$= \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_{t=0}^{1} = 3 - 4 + 6 = 5$$

b) Las rectas que unen el punto $(0,0,0)$ con $(1,0,0)$, y el $(1,1,0)$ con el $(1,1,1)$:



C_1:

$x = t \quad , dx = dt$

$y = 0 \quad , dy = 0$

$z = 0 \quad , dz = 0$

($x = t$ porque es la que varía en ese momento)

$$x\Big|_0^1 \longrightarrow t\Big|_0^1$$

$$\int_{C_1} \vec{A}\cdot\vec{dl} = \int_{C_1} Ax\cdot dx + \underbrace{Ay\cdot dy}_{0} + \underbrace{Az\cdot dz}_{0} = \int_{C_1} Ax\cdot dx = \int_{C_1} 3t^2\cdot dt = t^3\Big]_{t=0}^{1} = 1$$

$Ax = 3x^2 + 6y^2 = 3t^2$

$dx = dt$

C_2:

$x = 1 \quad , dx = 0$

$y = 1 \quad , dy = 0$

$z = t \quad , dz = dt$

$$z\Big|_0^1 \longrightarrow t\Big|_0^1$$

$$\int_{C_2} \vec{A}\cdot\vec{dl} \quad \int_{C_2} \underbrace{Ax\cdot dx}_{0} + \underbrace{Ay\cdot dy}_{0} + Az\cdot dz = \int_{C_2} Az\cdot dz = \int_{C_2} 20t^2\cdot dt = 20\frac{t^3}{3}\Big]_{t=0}^{1} = \frac{20}{3}$$

$Az = 20xz^2 = 20t^2$

$dz = dt$

$$\int_C \vec{A}\cdot\vec{dl} = \int_{C_1} \vec{A}\cdot\vec{dl_1} + \int_{C_2} \vec{A}\cdot\vec{dl_2} = 1 + \frac{20}{3} = \frac{23}{3}$$

c) La recta que une el $(0,0,0)$ con el $(1,1,1)$:

C:
$x=y=z=t \quad , dx=dy=dz=dt$

$$\int_C \vec{A}\cdot\vec{dl} = \int_{(0,0,0)}^{(1,1,1)} (3x^2+6y)\cdot dx - \int_{(0,0,0)}^{(1,1,1)} (14yz)\cdot dy + \int_{(0,0,0)}^{(1,1,1)} (20xz^2)\cdot dz = \int_{t=0}^{1} (3t^2+6t)\cdot dt -$$

$$-\int_{t=0}^{1} 14t^2\cdot dt + \int_{t=0}^{1} 20t^3\cdot dt = t^3+3t^2\Big]_{t=0}^{1} - 14\frac{t^3}{3}\Big]_{t=0}^{1} + 20\frac{t^4}{4}\Big]_{t=0}^{1} = \frac{13}{3}$$

13) Hallar el trabajo total realizado para desplazar una partícula en un campo de fuerzas dado por $\vec{F}=3xy\cdot\hat{i}-5z\cdot\hat{j}+10x\cdot\hat{k}$ a lo largo de la curva C: $x=t^2+1,\ y=2t^2,\ z=t^3$ desde $t=1$ a $t=2$:

$x=t^2+1 \quad , dx=2t\cdot dt$
$y=2t^2 \quad , dy=4t\cdot dt$
$z=t^3 \quad , dz=3t^2\cdot dt$

$$\int_C \vec{F}\cdot\vec{dl} = \int_C (3xy, -5z, 10x)(dx,dy,dz) =$$

$$= \int_{t=1}^{2} \left(3\cdot(t^2+1)(2t^2), -5t^3, 10(t^2+1)\right)\cdot\left(2t\,dt, 4t\,dt, 3t^2\cdot dt\right) = \int_{t=1}^{2} (6t^5+6t^3)2t\cdot dt +$$

$$+\int_1^2 -5t4t\,dt + \int_1^2 (10t^2+10)\,3t^2\cdot dt = \int_1^2 (12t^6+12t^4)\,dt - \int_1^2 20t^4\cdot dt + \int_1^2 (30t^4+30t^2)\,dt =$$

$$= \int_1^2 (12t^6+22t^4+30t^2)\,dt = \left[12\frac{t^7}{7}+22\frac{t^5}{5}+30\frac{t^3}{3}\right]_{t=1}^{2} = \left(\frac{12\cdot2^7}{7}+\frac{22\cdot2^5}{5}+\frac{30\cdot2^3}{3}\right)-$$

$$-\left(\frac{12\cdot1^7}{7}+\frac{22\cdot1^5}{5}+\frac{30\cdot1^3}{3}\right) = \dots = 303$$

(14) Siendo $\vec{F} = 3xy\cdot\hat{\imath} - y^2\cdot\hat{\jmath}$, hallar $\int_C \vec{F}\cdot\vec{dl}$ a lo largo de la curva C del plano XY de ecuación $y = 2x^2$, desde el punto (0,0) hasta el punto (1,2):

C:

$x = t \qquad dx = dt$

$y = 2x^2 = 2t^2, \quad dy = 4t\,dt$

$(x,y)\Big|_{(0,0)}^{(1,2)} \longrightarrow t\Big|_0^1$

$$\int_C \vec{F}\cdot\vec{dl} = \int_{(0,0)}^{(1,2)} (3xy, -y^2)(dx, dy) = \int_{t=0}^{1} (3\cdot t\cdot 2t^2, -4t^4)(dt, 4t\cdot dt) =$$

$$= \int_{t=0}^{1} 6t^3\cdot dt - 16t^5\cdot dt = \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_{t=0}^{1} = \frac{6}{4} - \frac{16}{6} = \frac{-7}{6}$$