Background

The topics discussed in this chapter are not entirely new to students taking this course. You have already studied many of these topics in earlier courses or are expected to know them from your previous training. Even so, this background material deserves a review because it is so pervasive in the area of signals and systems. Investing a little time in such a review will pay big dividends later. Furthermore, this material is useful not only for this course but also for several courses that follow. It will also be helpful as reference material in your future professional career.

B.1 Complex Numbers

Complex numbers are an extension of ordinary numbers and are an integral part of the modern number system. Complex numbers, particularly **imaginary numbers**, sometimes seem mysterious and unreal. This feeling of unreality derives from their unfamiliarity and novelty rather than their supposed nonexistence! Mathematicians blundered in calling these numbers "imaginary," for the term immediately prejudices perception. Had these numbers been called by some other name, they would have become demystified long ago, just as irrational numbers or negative numbers were. Many futile attempts have been made to ascribe some physical meaning to imaginary numbers. However, this effort is needless. In mathematics we assign symbols and operations any meaning we wish as long as internal consistency is maintained. A healthier approach would have been to define a symbol i (with any term but "imaginary"), which has a property $i^2 = -1$. The history of mathematics is full of entities which were unfamiliar and held in abhorrence until familiarity made them acceptable. This fact will become clear from the following historical note.

B.1-1 A Historical Note

Among early people the number system consisted only of natural numbers (positive integers) needed to count the number of children, cattle, and quivers of arrows. These people had no need for fractions. Whoever heard of two and one-half children or three and one-fourth cows!

However, with the advent of agriculture, people needed to measure continuously varying quantities, such as the length of a field, the weight of a quantity of butter, and so on. The number system, therefore, was extended to include fractions. The ancient Egyptians and Babylonians knew how to handle fractions, but **Pythagoras** discovered that some numbers (like the diagonal of a unit square) could not be expressed as a whole number or a fraction. Pythagoras, a number mystic, who regarded numbers as the essence and principle of all things in the universe, was so appalled at his discovery that he swore his followers to secrecy and imposed a death penalty for divulging this secret.[1] These numbers, however, were included in the number system by the time of Descartes, and they are now known as **irrational numbers**.

Until recently, **negative numbers** were not a part of the number system. The concept of negative numbers must have appeared absurd to early man. However, the medieval Hindus had a clear understanding of the significance of positive and negative numbers.[2,3] They were also the first to recognize the existence of absolute negative quantities.[4] The works of **Bhaskar** (1114-1185) on arithmetic (*Līlāvatī*) and algebra (*Bījaganit*) not only use the decimal system but also give rules for dealing with negative quantities. Bhaskar recognized that positive numbers have two square roots.[5] Much later, in Europe, the banking system that arose in Florence and Venice during the late Renaissance (fifteenth century) is credited with developing a crude form of negative numbers. The seemingly absurd subtraction of 7 from 5 seemed reasonable when bankers began to allow their clients to draw seven gold ducats while their deposit stood at five. All that was necessary for this purpose was to write the difference, 2, on the debit side of a ledger.[6]

Thus the number system was once again broadened (generalized) to include negative numbers. The acceptance of negative numbers made it possible to solve equations such as $x+5=0$, which had no solution before. Yet for equations such as $x^2+1=0$, leading to $x^2=-1$, the solution could not be found in the real number system. It was therefore necessary to define a completely new kind of number with its square equal to -1. During the time of Descartes and Newton, imaginary (or complex) numbers came to be accepted as part of the number system, but they were still regarded as algebraic fiction. The Swiss mathematician **Leonhard Euler** introduced the notation i (for **imaginary**) around 1777 to represent $\sqrt{-1}$. Electrical engineers use the notation j instead of i to avoid confusion with the notation i often used for electrical current. Thus

$$j^2 = -1$$

and

$$\sqrt{-1} = \pm j$$

This notation allows us to determine the square root of any negative number. For example,

$$\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = \pm 2j$$

When imaginary numbers are included in the number system, the resulting numbers are called **complex numbers**.

Origins of Complex Numbers

Ironically (and contrary to popular belief), it was not the solution of a quadratic equation, such as $x^2+1=0$, but a cubic equation with real roots that made

Gerolamo Cardano (left) and Karl Friedrich Gauss (right).

imaginary numbers plausible and acceptable to early mathematicians. They could dismiss $\sqrt{-1}$ as pure nonsense when it appeared as a solution to $x^2 + 1 = 0$ because this equation has no real solution. But in 1545, **Gerolamo Cardano** of Milan published *Ars Magna* (*The Great Art*), the most important algebraic work of the Renaissance. In this book he gave a method of solving a general cubic equation in which a root of a negative number appeared in an intermediate step. According to his method, the solution to a third-order equation†

$$x^3 + ax + b = 0$$

is given by

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

For example, to find a solution of $x^3 + 6x - 20 = 0$, we substitute $a = 6$, $b = -20$ in the above equation to obtain

$$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = \sqrt[3]{20.392} - \sqrt[3]{0.392} = 2$$

We can readily verify that 2 is indeed a solution of $x^3 + 6x - 20 = 0$. But when Cardano tried to solve the equation $x^3 - 15x - 4 = 0$ by this formula, his solution

†This equation is known as the *depressed cubic* equation. A general cubic equation

$$y^3 + py^2 + qy + r = 0$$

can always be reduced to a depressed cubic form by substituting $y = x - \frac{p}{3}$. Therefore any general cubic equation can be solved if we know the solution to the depressed cubic. The depressed cubic was independently solved, first by **Scipione del Ferro** (1465-1526) and then by **Niccolo Fontana** (1499-1557). The latter is better known in the history of mathematics as **Tartaglia** ("Stammerer"). Cardano learned the secret of the depressed cubic solution from Tartaglia. He then showed that by using the substitution $y = x - \frac{p}{3}$, a general cubic is reduced to a depressed cubic.

was

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What was Cardano to make of this equation in the year 1545? In those days negative numbers were themselves suspect, and a square root of a negative number was doubly preposterous! Today we know that

$$(2 \pm j)^3 = 2 \pm j11 = 2 \pm \sqrt{-121}$$

Therefore, Cardano's formula gives

$$x = (2 + j) + (2 - j) = 4$$

We can readily verify that $x = 4$ is indeed a solution of $x^3 - 15x - 4 = 0$. Cardano tried to explain halfheartedly the presence of $\sqrt{-121}$ but ultimately dismissed the whole enterprise as being "as subtle as it is useless." A generation later, however, **Raphael Bombelli** (1526-1573), after examining Cardano's results, proposed acceptance of imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solution. In other words, while we begin and end with real numbers, we seem compelled to move into an unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this proposal seemed incredibly strange.[7] Yet they could not dismiss the idea of imaginary numbers so easily because this concept yielded the real solution of an equation. It took two more centuries for the full importance of complex numbers to become evident in the works of Euler, Gauss, and Cauchy. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra.[7]

In 1799, the German mathematician **Karl Friedrich Gauss**, at a ripe age of 22, proved the fundamental theorem of algebra, namely that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less. Gauss was also one of the first to give a coherent account of complex numbers and to interpret them as points in a complex plane. It is he who introduced the term *complex numbers* and paved the way for general and systematic use of complex numbers. The number system was once again broadened or generalized to include imaginary numbers. Ordinary (or real) numbers became a special case of generalized (or complex) numbers.

The utility of complex numbers can be understood readily by an analogy with two neighboring countries X and Y, as illustrated in Fig. B.1. If we want to travel from City a to City b (both in Country X), the shortest route is through Country Y, although the journey begins and ends in Country X. We may, if we desire, perform this journey by an alternate route that lies exclusively in X, but this alternate route is longer. In mathematics we have a similar situation with real numbers (Country X) and complex numbers (Country Y). All real-world problems must start with real numbers, and all the final results must also be in real numbers. But the derivation of results is considerably simplified by using complex numbers as an intermediary. It is also possible to solve all real-world problems by an alternate method, using real numbers exclusively, but such procedure would increase the work needlessly.

B.1-2 Algebra of Complex Numbers

A complex number $(a,\, b)$ or $a + jb$ can be represented graphically by a point whose Cartesian coordinates are $(a,\, b)$ in a complex plane (Fig. B.2). Let us denote this complex number by z so that

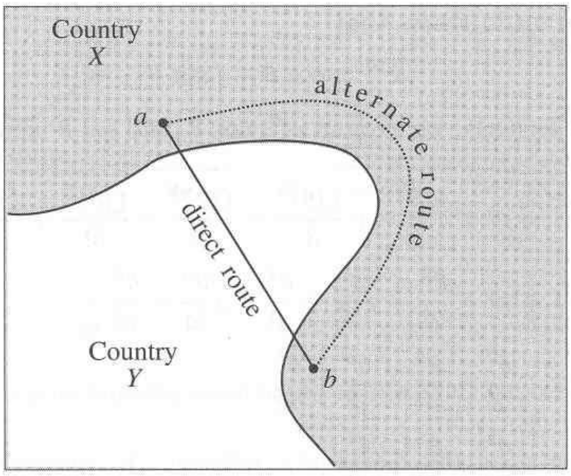

Fig. B.1 Use of complex numbers can reduce the work.

$$z = a + jb \tag{B.1}$$

The numbers a and b (the abscissa and the ordinate) of z are the **real part** and the **imaginary part**, respectively, of z. They are also expressed as

$$\text{Re } z = a$$

$$\text{Im } z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r \cos \theta$$

$$b = r \sin \theta$$

and

$$\begin{aligned} z = a + jb &= r \cos \theta + jr \sin \theta \\ &= r(\cos \theta + j \sin \theta) \end{aligned} \tag{B.2}$$

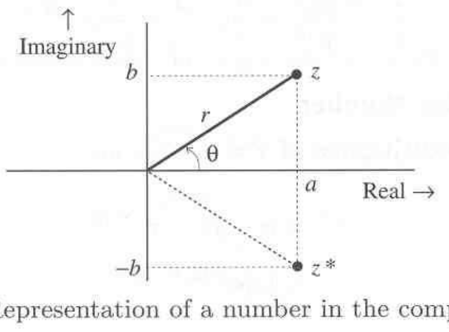

Fig. B.2 Representation of a number in the complex plane.

The **Euler formula** states that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

To prove the Euler formula, we expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$ using a Maclaurin series

$$\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}\cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots
\end{aligned}$$

Hence, it follows that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.3}$$

Using (B.3) in (B.2) yields

$$\begin{aligned}
z &= a + jb \\
&= re^{j\theta}
\end{aligned} \tag{B.4}$$

Thus, a complex number can be expressed in Cartesian form $a + jb$ or polar form $re^{j\theta}$ with

$$a = r\cos\theta, \qquad b = r\sin\theta \tag{B.5}$$

and

$$r = \sqrt{a^2 + b^2}, \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{B.6}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the **magnitude** (or **absolute value**) of z and is denoted by $|z|$. Similarly θ is called the angle of z and is denoted by $\angle z$. Therefore

$$|z| = r, \qquad \angle z = \theta$$

and

$$z = |z|e^{j\angle z} \tag{B.7}$$

Also

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z} \tag{B.8}$$

Conjugate of a Complex Number

We define z^*, the **conjugate** of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} \tag{B.9a}$$

$$= |z|e^{-j\angle z} \tag{B.9b}$$

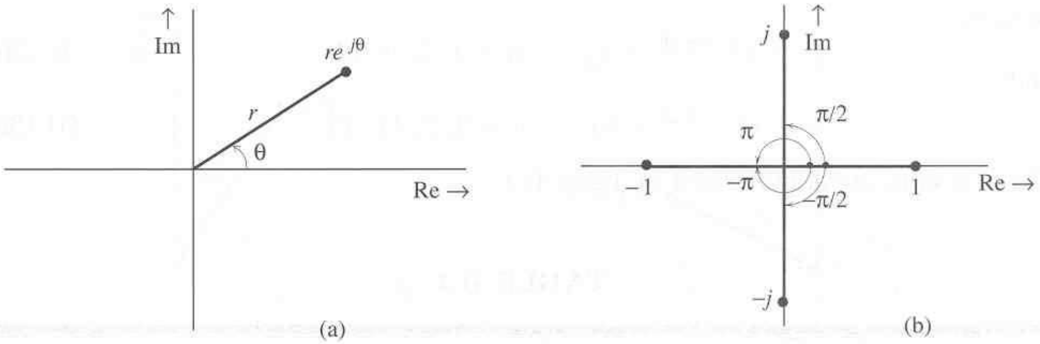

Fig. B.3 Understanding some useful identities in terms of $re^{j\theta}$.

The graphical representation of a number z and its conjugate z^* is depicted in Fig. B.2. Observe that z^* is a mirror image of z about the horizontal axis. *To find the conjugate of any number, we need only to replace j by $-j$ in that number* (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z \tag{B.10a}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2 \tag{B.10b}$$

Understanding Some Useful Identities

In a complex plane, $re^{j\theta}$ represents a point at a distance r from the origin and at an angle θ with the horizontal axis, as shown in Fig. B.3a. For example, the number -1 is at a unit distance from the origin and has an angle π or $-\pi$ (in fact, any odd multiple of $\pm\pi$), as seen from Fig. B.3b. Therefore,

$$1e^{\pm j\pi} = -1$$

In fact,

$$e^{\pm jn\pi} = -1 \qquad n \text{ odd integer} \tag{B.11}$$

The number 1, on the other hand, is also at a unit distance from the origin, but has an angle 2π (in fact, $\pm 2n\pi$ for any integral value of n). Therefore,

$$e^{\pm j2n\pi} = 1 \qquad n \text{ integer} \tag{B.12}$$

The number j is at unit distance from the origin and its angle is $\pi/2$ (see Fig. B.3b). Therefore,

$$e^{j\pi/2} = j$$

Similarly,

$$e^{-j\pi/2} = -j$$

Thus

$$e^{\pm j\pi/2} = \pm j \tag{B.13a}$$

In fact,

$$e^{\pm jn\pi/2} = \pm j \qquad n = 1,\, 5,\, 9,\, 13,\, \cdots \tag{B.13b}$$

and

$$e^{\pm jn\pi/2} = \mp j \qquad n = 3,\, 7,\, 11,\, 15,\, \cdots \tag{B.13c}$$

These results are summarized in Table B.1.

TABLE B.1

r	θ	$re^{j\theta}$
1	0	$e^{j0} = 1$
1	$\pm\pi$	$e^{\pm j\pi} = -1$
1	$\pm n\pi$	$e^{\pm jn\pi} = -1$ $\quad n$ odd integer
1	$\pm 2\pi$	$e^{\pm j2\pi} = 1$
1	$\pm 2n\pi$	$e^{\pm j2n\pi} = 1$ $\quad n$ integer
1	$\pm\pi/2$	$e^{\pm j\pi/2} = \pm j$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \pm j$ $\quad n = 1, 5, 9, 13, \ldots$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \mp j$ $\quad n = 3, 7, 11, 15, \ldots$

This discussion shows the usefulness of the graphic picture of $re^{j\theta}$. This picture is also helpful in several other applications. For example, to determine the limit of $e^{(\alpha+j\omega)t}$ as $t \to \infty$, we note that

$$e^{(\alpha+j\omega)t} = e^{\alpha t} e^{j\omega t}$$

Now the magnitude of $e^{j\omega t}$ is unity regardless of the value of ω or t because $e^{j\omega t} = re^{j\theta}$ with $r = 1$. Therefore, $e^{\alpha t}$ determines the behavior of $e^{(\alpha+j\omega)t}$ as $t \to \infty$ and

$$\lim_{t\to\infty} e^{(\alpha+j\omega)t} = \lim_{t\to\infty} e^{\alpha t} e^{j\omega t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases} \tag{B.14}$$

In future discussions you will find it very useful to remember $re^{j\theta}$ as a number at a distance r from the origin and at an angle θ with the horizontal axis of the complex plane.

A Warning About Using Electronic Calculators in Computing Angles

From the Cartesian form $a + jb$ we can readily compute the polar form $re^{j\theta}$ [see Eq. (B.6)]. Electronic calculators provide ready conversion of rectangular into polar and vice versa. However, if a calculator computes an angle of a complex number using an inverse trigonometric function $\theta = \tan^{-1}(b/a)$, proper attention must be paid to the quadrant in which the number is located. For instance, θ corresponding to the number $-2 - j3$ is $\tan^{-1}(\frac{-3}{-2})$. This result is not the same as $\tan^{-1}(\frac{3}{2})$. The former is $-123.7°$, whereas the latter is $56.3°$. An electronic calculator cannot make this distinction and can give a correct answer only for angles in the first and

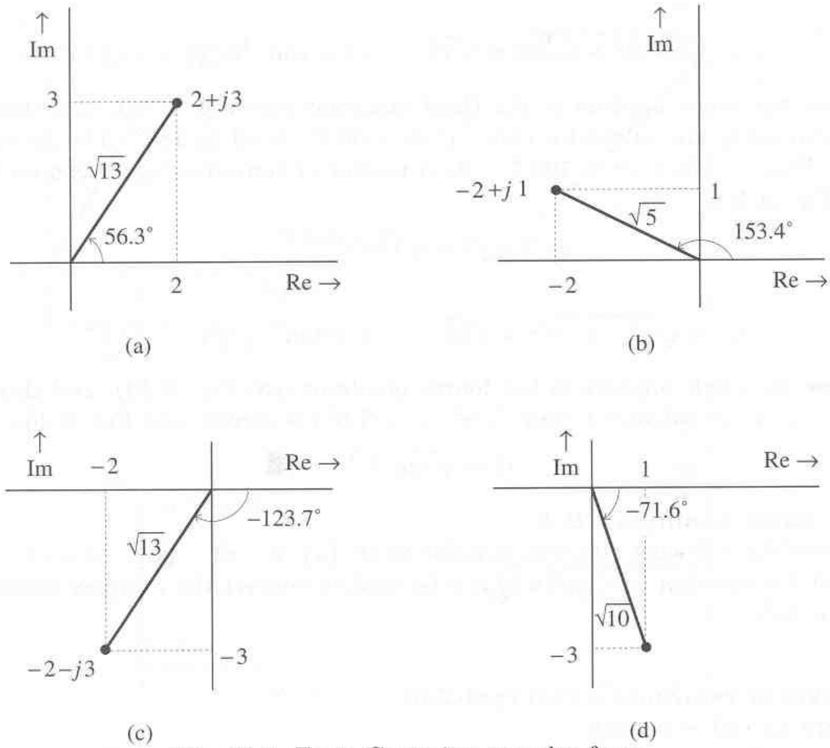

Fig. B.4 From Cartesian to polar form.

fourth quadrants. It will read $\tan^{-1}(\frac{-3}{-2})$ as $\tan^{-1}(\frac{3}{2})$, which is clearly wrong. In computing inverse trigonometric functions, if the angle appears in the second or third quadrant, the answer of the calculator is off by 180°. The correct answer is obtained by adding or subtracting 180° to the value found with the calculator (either adding or subtracting yields the correct answer). For this reason it is advisable to draw the point in the complex plane and determine the quadrant in which the point lies. This issue will be clarified by the following examples.

■ **Example B.1**

Express the following numbers in polar form:

(a) $2 + j3$ **(b)** $-2 + j1$ **(c)** $-2 - j3$ **(d)** $1 - j3$

(a)

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{3}{2}\right) = 56.3^\circ$$

In this case the number is in the first quadrant, and a calculator will give the correct value of 56.3°. Therefore, (see Fig. B.4a)

$$2 + j3 = \sqrt{13}\, e^{j56.3^\circ}$$

(b)

$$|z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \qquad \angle z = \tan^{-1}\left(\tfrac{1}{-2}\right) = 153.4^\circ$$

In this case the angle is in the second quadrant (see Fig. B.4b), and therefore the answer given by the calculator ($\tan^{-1}(\frac{1}{-2}) = -26.6^\circ$) is off by 180°. The correct answer is $(-26.6 \pm 180)^\circ = 153.4^\circ$ or -206.6°. Both values are correct because they represent the same angle. As a matter of convenience, we choose an angle whose numerical value is less than 180°, which in this case is 153.4°. Therefore,

$$-2 + j1 = \sqrt{5} e^{j153.4^\circ}$$

(c)

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \qquad \angle z = \tan^{-1}(\tfrac{-3}{-2}) = -123.7^\circ$$

In this case the angle appears in the third quadrant (see Fig. B.4c), and therefore the answer obtained by the calculator ($\tan^{-1}(\frac{-3}{-2}) = 56.3^\circ$) is off by 180°. The correct answer is $(56.3 \pm 180)^\circ = 236.3^\circ$ or -123.7°. As a matter of convenience, we choose the latter and (see Fig. B.4c)

$$-2 - j3 = \sqrt{13}e^{-j123.7^\circ}$$

(d)

$$|z| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \qquad \angle z = \tan^{-1}(\tfrac{-3}{1}) = -71.6^\circ$$

In this case the angle appears in the fourth quadrant (see Fig. B.4d), and therefore the answer given by the calculator ($\tan^{-1}(\frac{-3}{1}) = -71.6^\circ$) is correct (see Fig. B.4d).

$$1 - j3 = \sqrt{10}e^{-j71.6^\circ} \quad \blacksquare$$

⊙ **Computer Example CB.1**

Express the following numbers in polar form: **(a)** $2 + j3$ **(b)** $-2 + j1$

MATLAB function *cart2pol(a,b)* can be used to convert the complex number $a + jb$ to its polar form.

(a)

```
[Zangle_in_rad,Zmag]=cart2pol(2,3)
Zangle_in_rad = 0.9828
Zmag =3.6056
Zangle_in_deg=Zangle_in_rad*(180/pi)
Zangle_in_deg=56.31
```

Therefore

$$z = 2 + j3 = 3.6056e^{j56.31^\circ}$$

(b)

```
[Zangle_in_rad,Zmag]=cart2pol(-2,1)
Zangle_in_rad = 2.6779
Zmag =2.2361
Zangle_in_deg=Zangle_in_rad*(180/pi)
Zangle_in_deg=153.4349
```

Therefore

$$z = -2 + j1 = 2.2361e^{j153.4349^\circ}$$

Note that MATLAB automatically takes care of the quadrant in which the complex number lies. ⊙

■ **Example B.2**

Represent the following numbers in the complex plane and express them in Cartesian form: **(a)** $2e^{j\pi/3}$ **(b)** $4e^{-j3\pi/4}$ **(c)** $2e^{j\pi/2}$ **(d)** $3e^{-j3\pi}$ **(e)** $2e^{j4\pi}$ **(f)** $2e^{-j4\pi}$.

(a) $2e^{j\pi/3} = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 1 + j\sqrt{3}$ (see Fig. B.5a)

(b) $4e^{-j3\pi/4} = 4\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right) = -2\sqrt{2} - j2\sqrt{2}$ (see Fig. B.5b)

(c) $2e^{j\pi/2} = 2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 2(0 + j1) = j2$ (see Fig. B.5c)

(d) $3e^{-j3\pi} = 3(\cos 3\pi - j\sin 3\pi) = 3(-1 + j0) = -3$ (see Fig. B.5d)

(e) $2e^{j4\pi} = 2(\cos 4\pi + j\sin 4\pi) = 2(1 + j0) = 2$ (see Fig. B.5e)

(f) $2e^{-j4\pi} = 2(\cos 4\pi - j\sin 4\pi) = 2(1 - j0) = 2$ (see Fig. B.5f) ■

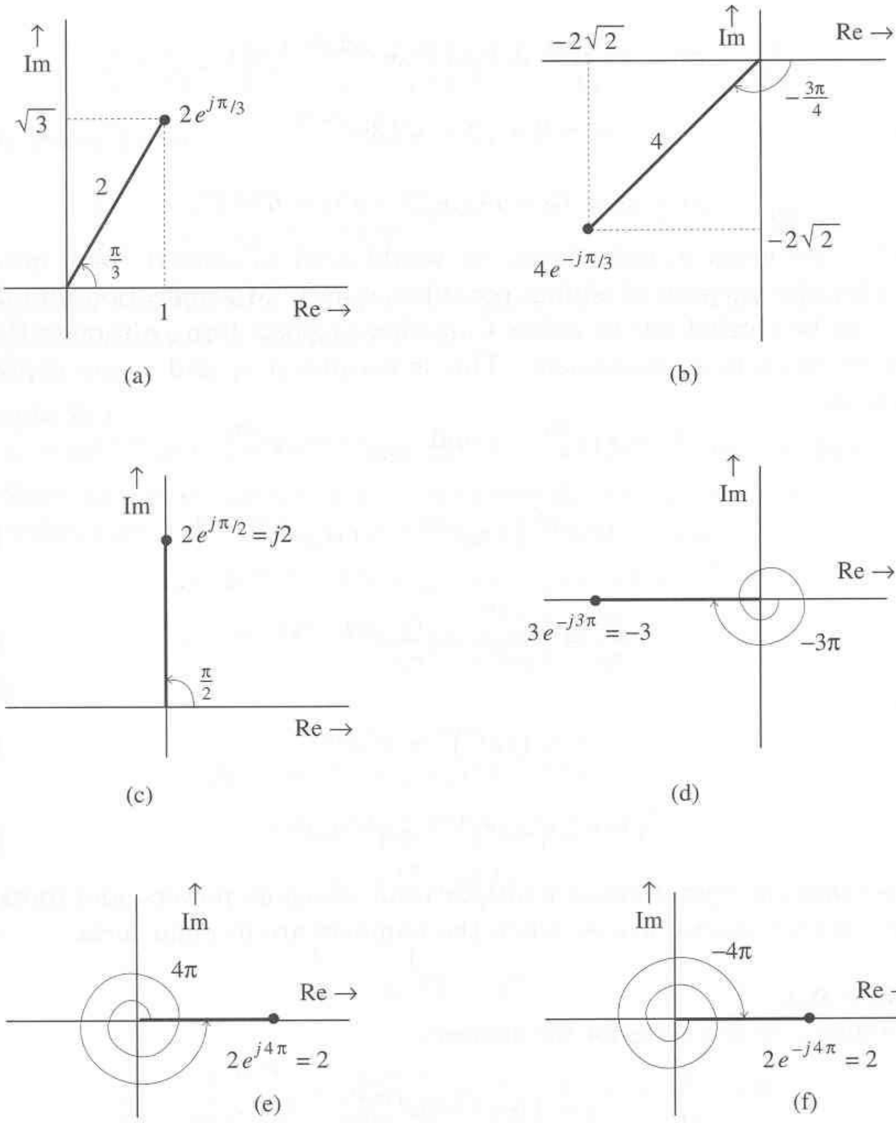

Fig. B.5 From polar to Cartesian form.

Computer Example CB.2

Represent $4e^{-j\frac{3\pi}{4}}$ in Cartesian form.

MATLAB function pol2cart(θ, r) converts the complex number $re^{j\theta}$ to Cartesian form.

[Zreal,Zimag]=pol2cart(-3*pi/4,4)
```
Zreal=-2.8284
Zimag=-2.8284
```
Therefore

$$4e^{-j\frac{3\pi}{4}} = -2.8284 - j2.8284$$

Arithmetical Operations, Powers, and Roots of Complex Numbers

To perform addition and subtraction, complex numbers should be expressed in Cartesian form. Thus, if

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

and

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

then

$$z_1 + z_2 = (3 + j4) + (2 + j3) = 5 + j7$$

If z_1 and z_2 are given in polar form, we would need to convert them into Cartesian form for the purpose of adding (or subtracting). Multiplication and division, however, can be carried out in either Cartesian or polar form, although the latter proves to be much more convenient. This is because if z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{j\theta_2}$$

then

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)} \tag{B.15a}$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \tag{B.15b}$$

Moreover,

$$z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta} \tag{B.15c}$$

and

$$z^{1/n} = \left(re^{j\theta}\right)^{1/n} = r^{1/n} e^{j\theta/n} \tag{B.15d}$$

This shows that the operations of multiplication, division, powers, and roots can be carried out with remarkable ease when the numbers are in polar form.

■ **Example B.3**

Determine $z_1 z_2$ and z_1/z_2 for the numbers

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

We shall solve this problem in both polar and Cartesian forms.

Multiplication: Cartesian Form

$$z_1 z_2 = (3 + j4)(2 + j3) = (6 - 12) + j(8 + 9) = -6 + j17$$

Multiplication: Polar Form

$$z_1 z_2 = \left(5e^{j53.1^\circ}\right)\left(\sqrt{13}e^{j56.3^\circ}\right) = 5\sqrt{13}e^{j109.4^\circ}$$

Division: Cartesian Form

$$\frac{z_1}{z_2} = \frac{3 + j4}{2 + j3}$$

In order to eliminate the complex number in the denominator, we multiply both the numerator and the denominator of the right-hand side by $2 - j3$, the denominator's conjugate. This yields

$$\frac{z_1}{z_2} = \frac{(3+j4)(2-j3)}{(2+j3)(2-j3)} = \frac{18-j1}{2^2+3^2} = \frac{18-j1}{13} = \frac{18}{13} - j\frac{1}{13}$$

Division: Polar Form

$$\frac{z_1}{z_2} = \frac{5e^{j53.1^\circ}}{\sqrt{13}e^{j56.3^\circ}} = \frac{5}{\sqrt{13}}e^{j(53.1^\circ-56.3^\circ)} = \frac{5}{\sqrt{13}}e^{-j3.2^\circ} \quad \blacksquare$$

It is clear from this example that multiplication and division are easier to accomplish in polar form than in Cartesian form.

■ **Example B.4**

For $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$, find **(a)** $2z_1 - z_2$ **(b)** $\frac{1}{z_1}$ **(c)** $\frac{z_1}{z_2^2}$ **(d)** $\sqrt[3]{z_2}$

(a) Since subtraction cannot be performed directly in polar form, we convert z_1 and z_2 to Cartesian form:

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}\right) = \sqrt{2} + j\sqrt{2}$$
$$z_2 = 8e^{j\pi/3} = 8\left(\cos\tfrac{\pi}{3} + j\sin\tfrac{\pi}{3}\right) = 4 + j4\sqrt{3}$$

Therefore,

$$\begin{aligned} 2z_1 - z_2 &= 2(\sqrt{2} + j\sqrt{2}) - (4 + j4\sqrt{3}) \\ &= (2\sqrt{2} - 4) + j(2\sqrt{2} - 4\sqrt{3}) \\ &= -1.17 - j4.1 \end{aligned}$$

(b)

$$\frac{1}{z_1} = \frac{1}{2e^{j\pi/4}} = \frac{1}{2}e^{-j\pi/4}$$

(c)

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\frac{\pi}{4}-\frac{2\pi}{3})} = \frac{1}{32}e^{-j\frac{5\pi}{12}}$$

(d)

$$\sqrt[3]{z_2} = z_2^{1/3} = \left(8e^{j\pi/3}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(e^{j\pi/3}\right)^{1/3} = 2e^{j\pi/9} \quad \blacksquare$$

⊙ **Computer Example CB.3**

Determine z_1z_2 and z_1/z_2 if $z_1 = 3 + j4$ and $z_2 = 2 + j3$

Multiplication and division: Cartesian Form

```
z1=3+j*4; z2=2+j*3;
z1z2=z1*z2
z1z2=-6.000+17.0000i
z1_over_z2=z1/z2
z1_over_z2=1.3486-0.0769i
```

Therefore

$(3 + j4)(2 + j3) = -6 + j17$ and $(3 + j4)/(2 + j3) = 1.3486 - 0.0769$ ⊙

Example B.5

Consider $F(\omega)$, a complex function of a real variable ω:

$$F(\omega) = \frac{2 + j\omega}{3 + j4\omega} \tag{B.16a}$$

(a) Express $F(\omega)$ in Cartesian form, and find its real and imaginary parts. **(b)** Express $F(\omega)$ in polar form, and find its magnitude $|F(\omega)|$ and angle $\angle F(\omega)$.

(a) To obtain the real and imaginary parts of $F(\omega)$, we must eliminate imaginary terms in the denominator of $F(\omega)$. This is readily done by multiplying both the numerator and denominator of $F(\omega)$ by $3 - j4\omega$, the conjugate of the denominator $3 + j4\omega$ so that

$$F(\omega) = \frac{(2 + j\omega)(3 - j4\omega)}{(3 + j4\omega)(3 - j4\omega)} = \frac{(6 + 4\omega^2) - j5\omega}{9 + 16\omega^2} = \frac{6 + 4\omega^2}{9 + 16\omega^2} - j\frac{5\omega}{9 + \omega^2} \tag{B.16b}$$

This is the Cartesian form of $F(\omega)$. Clearly the real and imaginary parts $F_r(\omega)$ and $F_i(\omega)$ are given by

$$F_r(\omega) = \frac{6 + 4\omega^2}{9 + 16\omega^2}, \qquad F_i(\omega) = \frac{-5\omega}{9 + 16\omega^2}$$

(b)

$$\begin{aligned} F(\omega) = \frac{2 + j\omega}{3 + j4\omega} &= \frac{\sqrt{4 + \omega^2}\, e^{j \tan^{-1}\left(\frac{\omega}{2}\right)}}{\sqrt{9 + 16\omega^2}\, e^{j \tan^{-1}\left(\frac{4\omega}{3}\right)}} \\ &= \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}\, e^{j\left[\tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)\right]} \end{aligned} \tag{B.16c}$$

This is the polar representation of $F(\omega)$. Observe that

$$|F(\omega)| = \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}, \qquad \angle F(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right) \tag{B.17}$$

■