RIGIDIFICATION AND THE COHERENT NERVE FOR ENRICHED QUASICATEGORIES

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ABSTRACT. We introduce, for $\mathcal{C}$ a regular Cartesian Reedy category a model category whose fibrant objects are an analogue of quasicategories enriched in simplicial presheaves on C. We then develop a coherent realization and nerve for this model structure and demonstrate using an enriched version of the necklaces of Dugger and Spivak that our model category is Quillen-equivalent to the category of categories enriched in simplicial presheaves on $\mathcal{C}$. We then show that for any Cartesian-closed left-Bousfield localization of the category of simplicial presheaves on $\mathcal{C}$, the coherent nerve and realization descend to a Quillen equivalence on the localizations of these model categories. As an application, we demonstrate a version of Yoneda's lemma for these enriched quasicategories.

INTRODUCTION

In his thesis, David Oury introduced machinery to give a novel proof that his constructed model structure on Θ_2-sets is cartesian-monoidal closed. Charles Rezk constructed a model structure on Θ_2-spaces that is known to be Quillen biequivalent to Oury's model structure. However, Rezk's construction allows for enrichment in a larger class of model categories, namely cartesian-closed model categories whose underlying categories are simplicial presheaves on a small category $\mathcal{C}$ satisfying some tame restrictions.

Bergner and Rezk also showed by means of a zig-zag of Quillen equivalences that Θ_n-spaces model the same homotopy theory as $\mathrm{Psh}_\Delta(\Theta_{n-1})$-enriched categories. Because the equivalence is indirect, however, many of the ideas from Lurie's work on $(\infty, 1)$-categories cannot be adapted in a straightforward manner. In order to rectify this, we construct a generalized version of the coherent realization and nerve using Θ_n-sets (or more generally $\Theta[\mathcal{C}]$-sets), and we demonstrate that this adjunction is a Quillen equivalence using an enriched version of Dugger and Spivak's calculus of necklaces. In fact, our result is strictly stronger than the result of Rezk and Bergner because it allows us to account for the cases $\Theta = \Theta_\omega$ as well as prove the equivalence when $\mathcal{C} \neq \Theta_n$, which the Rezk-Bergner approach could not handle, since one of the categories appearing in the zig-zag (the height-n analogue of Segal categories) does not make sense for general $\mathcal{C}$.

Taking all of these results together allows us to define suitably enriched versions of the Yoneda embedding, which is a significant result that as a corollary allows for substantial development of the theory of weak ω-categories (that is, (∞, ∞)-categories). In particular, we can define weighted pseudolimits and pseudocolimits in terms of representability of presheaves, and these can be used to define the other universal constructions.

In the first section, we apply a general construction to define what we call *labeled simplicial sets* with respect to a monoidal category $\mathcal{V}$. We then specialize to the case where $\mathcal{V}$ is the Cartesian-monoidal category of presheaves of sets on a small category $\mathcal{C}$, which we additionally require to be a special kind of Reedy category that axiomatizes a form of the Eilenberg-Zilber shuffle decomposition

for products of simplicial sets. We then define $\Theta[\mathcal{C}]$ to be the full subcategory of the labeled simplicial sets whose underlying simplicial sets are simplices and whose edges are all labeled by representable presheaves on $\mathcal{C}$. We define the category of $\mathcal{C}$-*cellular sets* to be the category of presheaves of sets on this category.

In the second section, we define an intertwining functor that allows us to work with an important class of objects in the category of $\mathcal{C}$-cellular sets, called *sober* cellular sets. The maps into these objects can be decomposed into maps on underlying simplicial sets together with compatible families of maps on labels.

In the third section, we give a construction of the homotopy-coherent realization for $\mathcal{C}$-cellular sets using the category of labeled simplices. Interestingly, this functor takes values in the category of categories enriched in simplicial presheaves on $\mathcal{C}$, and taking $\mathcal{C}$ to be the terminal category, we recover the classical coherent realization. The right adjoint of this functor is called the homotopy-coherent nerve.

In the fourth section, we give a way to compute the Hom-objects of the coherent realization of a $\mathcal{C}$-cellular set using a modified version of the calculus of necklaces following Dugger and Spivak [DS11a]. Our version is somewhat more difficult to work with, but the proofs are largely the same.

In the fifth section, we show that Hom-objects of the coherent realization can be computed as the nerves of ordinary categories naturally for each c in $\mathcal{C}$.

In the sixth section, we introduce the Horizontal Joyal model structure on $\mathcal{C}$-cellular sets and state some important facts about it, though we defer the proofs to the appendices.

In the seventh section, we give a combinatorial proof that the coherent realization of a horizontal inner anodyne is a trivial cofibration of categories enriched in simplicial presheaves on $\mathcal{C}$. We then show that the coherent nerve preserves fibrations between fibrant objects, thereby demonstrating that the adjunction is a Quillen adjunction.

In the eighth section, we give a definition of an enriched version of a Dugger-Spivak gadget, and we demonstrate that such gadgets give another way to compute the Hom objects for the coherent realization.

In the ninth section, we give another way to compute the Hom objects, this time using cosimplicial resolutions of the objects $[1](c)$. We introduce four distinguished such cosimplicial resolutions, which all have their own different uses.

In the tenth section, we prove one of the main theorems of the paper, demonstrating that the counit of the coherent nerve-realization adjunction is a weak equivalence for all fibrant categories enriched in simplicial presheaves on $\mathcal{C}$ and also demonstrate that for any horizontally fibrant C-cellular set, there is a natural zig-zag of weak equivalences between the different models of the mapping object between two objects.

In the eleventh section, we prove that the coherent realization and nerve in fact define a Quillen equivalence.

In the twelfth section, we introduce localizations on both sides of the Quillen equivalence with respect to a Cartesian-closed left-Bousfield localization of the category of simplicial presheaves on $\mathcal{C}$. We then demonstrate that the coherent realization and nerve remain a Quillen equivalence after these localizations.

In the thirteenth section, we prove that an enriched form of the Yoneda embedding and Yoneda's lemma hold. As an example, we demonstrate how representability can help us define weighted limits and colimits.

1. THE WREATH PRODUCT WITH Δ

In this section, we will consider a slightly more general definition of the wreath product with Δ, as in Oury's thesis. Segal observed long ago that a monoidal category is classified precisely by a pseudofunctor $M_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ such that $M_0 = *$ is the terminal category and the maps $M_n \to (M_1)^n$ induced by the inclusion of the spine $Sp[n] \hookrightarrow \Delta^n$ are all isomorphisms. This brings us to our first definition:

Definition 1.1. Suppose $\mathcal{V}$ is a monoidal category. Then we construct a Grothendieck fibration

$$\Delta \int \mathcal{V} \to \Delta$$

by applying the Grothendieck construction to the functor

$$\mathcal{V}_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Cat}$$

classifying $\mathcal{V}$. We call the total space of this fibration the *wreath product* of Δ with $\mathcal{V}$.

Recall that the 2-pseudofunctor $\mathbf{CAT}(\Delta^{\mathrm{op}}, \mathbf{Cat})(\cdot, \cdot)$ is defined by sending a pair of pseudofunctors $F_\bullet, G_\bullet$ to the category whose objects are pseudonatural transformations $F_\bullet \Rightarrow G_\bullet$ and whose morphisms are modifications. If $G_\bullet$ is a pseudofunctor, we define the functor

$$h_{G_\bullet} = \mathbf{CAT}(\Delta^{\mathrm{op}}, \mathbf{Cat})(\cdot, G_\bullet) : \mathbf{CAT}(\Delta^{\mathrm{op}}, \mathbf{Cat}) \to \mathbf{CAT}$$

Recall also that there is a fully-faithful embedding

$$\iota_* : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}) \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{CAT}),$$

obtained from $\iota : \mathrm{Set} \hookrightarrow \mathbf{Cat}$.

Then by composition, we obtain a functor

$$h_{\mathcal{V}_\bullet} \circ \iota_* : \widehat{\Delta}^{\mathrm{op}} \to \mathbf{CAT}.$$

Then we define the fibration

$$\widehat{\Delta} \int \mathcal{V} \to \widehat{\Delta}$$

to be the Grothendieck construction of $h_{\mathcal{V}_\bullet} \circ \iota_*$. The total space of this fibration is called the category of $\mathcal{V}$-*labeled simplicial sets*.

Proposition 1.2. *The pullback of the fibration*

$$\widehat{\Delta} \int V \to \widehat{\Delta}$$

along the Yoneda embedding $\Delta \hookrightarrow \widehat{\Delta}$ is exactly the fibration

$$\Delta \int \mathcal{V} \to \Delta,$$

and therefore, the induced map

$$\Delta \int \mathcal{V} \hookrightarrow \widehat{\Delta} \int \mathcal{V}$$

is a fully faithful embedding.

Proof. Notice that by the bicategorical Yoneda lemma there is a natural equivalence

$$h_{\mathcal{V}_\bullet} \circ \iota_*(\Delta^n) \simeq \mathcal{V}_n,$$

so $\mathcal{V}_\bullet$ is naturally equivalent to the restriction of $h_\mathcal{V} \circ \iota_*$ along the Yoneda embedding, so it follows that $\Delta \int \mathcal{V} \to \Delta$ is equivalent to the pullback of $\widehat{\Delta} \int \mathcal{V} \to \widehat{\Delta}$ along a fully faithful embedding and therefore the evident map

$$\Delta \int \mathcal{V} \to \widehat{\Delta} \int \mathcal{V}$$

is also fully faithful. $\square$

For the purposes of this paper, we do not need this level of generality. We specialize as follows:

Definition 1.3. A small regular skeletal Reedy category $\mathcal{C}$ (see [Cis06, 8.2.3]) is called a *regular Cartesian Reedy category* if it satisfies two conditions:

   (CR1) The class of regular presheaves on $\mathcal{C}$ (see [Cis06, 8.2]) is closed under finite products.
   (CR2) If I is a finite set and $c \to \prod_{i \in I} c_i$ is a nondegenerate section, then $\dim c \leq \sum_{i \in I} \dim c_i$.

Note 1.4. The axioms for regular Cartesian Reedy categories imply that $\mathcal{C}$ is a Reedy multicategory in the sense of [BR11]. It also asserts a weak form of the Eilenberg-Zilber shuffle decomposition. There may be a way to prove (CR2) from (CR1), but we were unable to do so.

In the sequel, we assume that $\mathcal{V} = \widehat{\mathcal{C}}$ is the category of presheaves of sets on a small regular Cartesian Reedy category $\mathcal{C}$ admitting a terminal object. Then we give the following definition:

Definition 1.5. For any regular Cartesian Reedy category $\mathcal{C}$ admitting a terminal object, we define the category of $\mathcal{C}$-cells

$$\Theta[\mathcal{C}] \subseteq \Delta \int \widehat{\mathcal{C}}$$

to be the full subcategory spanned by the objects of the form $[n](h_{c_1}, \cdots, h_{c_n})$ for $c_1, \cdots, c_n \in \mathcal{C}$, and where $h_\bullet$ denotes the Yoneda embedding.

Remark 1.6. The requirement that $\mathcal{C}$ have a terminal object is a technical condition that ensures that Δ embeds fully and faithfully in $\Theta[\mathcal{C}]$. The condition that $\mathcal{C}$ is regular Cartesian Reedy is probably not necessary, but it will ensure later on that the generating cofibrations of the injective model structure on simplicial presheaves $\mathrm{Psh}_\Delta(\mathcal{C})$ admit a very simple description.

Remark 1.7. When $\mathcal{C}$ has a terminal object, the inclusion $* \to \mathcal{C}$ is a fully faithful right-adjoint. The construction Θ preserves fully faithful right-adjoints, so we have a right-adjoint functor

$$\Delta = \Theta[*] \hookrightarrow \Theta[\mathcal{C}].$$

Passing to presheaf categories, this functor also extends to a colimit-preserving functor

$$\mathscr{H} : \widehat{\Delta} \hookrightarrow \widehat{\Theta[\mathcal{C}]}$$

In particular, on representables we have that

$$\mathscr{H}(\Delta^n) = [n](*, \ldots, *).$$

Therefore, we have an adjoint triple,

$$\widehat{\Theta[\mathcal{C}]} \underset{\underset{\mathscr{N}}{\xrightarrow{\bot}}}{\overset{\overset{\pi}{\underset{\bot}{\xrightarrow{\hspace{1cm}}}}}{\underset{\mathscr{H}}{\xleftarrow{\bot}}}} \widehat{\Delta}$$

Definition 1.8. We define a special cosimplicial object in $\widehat{\Theta[\mathcal{C}]}$ by the formula

$$E^\bullet = \mathscr{H}(\mathrm{cosk}_0 \Delta^\bullet).$$

This cosimplicial object will be a cosimplicial resolution of a point, once we define our model structures.

2. The generalized intertwiner and $\widehat{\Delta} \int \widehat{\mathcal{C}}$

Rezk introduced an intertwining functor by means of an explicit construction, but Oury gave an even more powerful version, which we recall here:

Definition 2.1. Recall that we have a fully-faithful embedding

$$L : \Theta[\mathcal{C}] \hookrightarrow \Delta \int \widehat{\mathcal{C}} \hookrightarrow \widehat{\Delta} \int \widehat{\mathcal{C}}.$$

We define the *intertwiner* to be the functor

$$\square : \widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Theta[\mathcal{C}]}$$

by the formula

$$(S, \Omega) \mapsto S \square \Omega = \mathrm{Hom}_{\widehat{\Delta} \int \widehat{\mathcal{C}}}(L(\cdot), (S, \Omega)).$$

Note 2.2. The restriction of the intertwiner to $\Delta \int \widehat{\mathcal{C}}$ is exactly the intertwiner of Rezk. When we apply the intertwiner to an object belonging to the full subcategory $\Delta \int \widehat{\mathcal{C}}$, that is, $(S, \Omega) = [n](A_1, \ldots, A_n)$, we will switch to Rezk's notation, namely

$$V[n](A_1, \ldots, A_n) \overset{\mathrm{def}}{=} S \square \Omega$$

Definition 2.3. An object (S, Ω) of $\widehat{\Delta} \int \widehat{\mathcal{C}}$ is called *normalized* if the image of Ω_1 does not contain the empty presheaf on $\mathcal{C}$.

Proposition 2.4. *The restriction of the intertwiner to the full subcategory of normalized objects in $\widehat{\Delta} \int \widehat{\mathcal{C}}$ is fully faithful.*

Proof. Recall before we begin that a map $(S, \Omega) \to (S', \Omega')$ is given by a morphism of simplicial sets $f : S \to S'$ and a natural modification $\zeta : \Omega \to \Omega' \circ f$.

In order to prove fullness, let $\gamma : S \square \Omega \to S' \square \Omega'$ be a map in $\widehat{\Theta[\mathcal{C}]}$. We notice that $\mathrm{Hom}([n](\emptyset, \ldots, \emptyset), (S, \Omega))$ is naturally isomorphic to S_n, and proceed by diagram chase. Since by assumption (S, Ω) is normalized, every map $[n](\emptyset, \ldots, \emptyset) \to (S, \Omega)$ factors uniquely through at least one map $[n](c_1, \ldots, c_n) \to (S, \Omega)$.

Choosing such a factorization, the natural transformation γ sends this to a map

$$[n](c_1, \ldots, c_n) \to (S', \Omega'),$$

and finally, precomposing this map with the unique map $[n](\emptyset, \ldots, \emptyset) \to [n](c_1, \ldots, c_n)$, we obtain a map $[n](\emptyset, \ldots, \emptyset) \to (S', \Omega')$. Taking these together gives a map $S_n \to S'_n$, naturally in n.

Now assume that $S' = S$ and that the map induced by γ is the identity. Then notice that a map

$$[n](c_1, \ldots, c_n) \to (S, \Omega)$$

is completely determined by its action on the degree n part, but this amounts to picking an n-simplex of S together with its labeling $(A_1, \ldots, A_n)$, and a map $(c_1, \ldots, c_n) \to (A_1, \ldots, A_n)$. Then the natural transformation gives a natural map $((A_1)_{c_1}, \ldots, (A_n)_{c_n}) \to ((A'_1)_{c_1}, \ldots, (A'_n)_{c_n})$, taking the naturality in n and the c_i, these together determine a natural modification $\Omega \to \Omega'$.

To see faithfulness, notice that the construction in the proof of fullness defines a left-inverse to the definition of the map on morphisms defined by the intertwiner. $\square$

Definition 2.5. We call a presheaf of sets on $\Theta[\mathcal{C}]$ a $\mathcal{C}$-*cellular set*.

Note 2.6. Although the case when $\mathcal{C} = \Theta_{n-1}$ (respectively $\mathcal{C} = \Theta = \Theta_\omega$) are not strictly the focus of this paper, note that $\Theta[\Theta_{n-1}] = \Theta_n$ (respectively $\Theta[\Theta] = \Theta$). In these cases, we call presheaves of sets on $\Theta[\mathcal{C}]$ n-*cellular sets* (respectively, *cellular sets*).

Definition 2.7. We say that a $\mathcal{C}$-cellular set X is *sober* if it is the image of a normalized object of $\widehat{\Delta} \int \widehat{\mathcal{C}}$. If $f : X \to Y$ is the image under the intertwiner of a cartesian map of normalized labeled simplicial sets, we call f *cartesian*.

Proposition 2.8. *All representable $\mathcal{C}$-cellular sets are sober.*

Proof. By construction. $\qquad\square$

Proposition 2.9. *The class of sober $\mathcal{C}$-cellular sets is closed under cartesian product.*

Proof. From the construction of the intertwiner, we see that if $\widehat{\Delta} \int \widehat{\mathcal{C}}$ has all cartesian products, then the intertwiner preserves them, since

$$\mathrm{Hom}_{\widehat{\Delta} \int \widehat{\mathcal{C}}}(L(\cdot), (S,\Omega) \times (S',\Omega')) = \mathrm{Hom}_{\widehat{\Delta} \int \widehat{\mathcal{C}}}(L(\cdot), (S,\Omega)) \times \mathrm{Hom}_{\widehat{\Delta} \int \widehat{\mathcal{C}}}(L(\cdot), (S',\Omega')).$$

Then we define the cartesian product of (S,Ω) and (S',Ω') by the formula

$$S \times S' \xrightarrow{\Omega \times \Omega'} \widehat{\mathcal{C}}_\bullet \times \widehat{\mathcal{C}}_\bullet \xrightarrow{\times} \widehat{\mathcal{C}}_\bullet.$$

It is clear that this satisfies the universal property of the product. $\qquad\square$

Proposition 2.10. *The functor associated with the fibration $\Theta[\mathcal{C}] \to \Delta$ induces an adjunction*

$$\widehat{\Theta[\mathcal{C}]} \underset{\mathscr{H}}{\overset{\pi}{\rightleftarrows}} \widehat{\Delta},$$

as we saw earlier. If $X = S\square\Omega$ is sober, and $f : S' \to S$ is a map of simplicial sets, then the image of the cartesian lift $\tilde{f} : (S', f^*(\Omega)) \to (S,\Omega)$ under the intertwiner is exactly the pullback of $\mathscr{H}(f)$ along the component at X of the unit of the adjunction $\mu_X : X \to \mathscr{H}\pi X = \mathscr{H}S$.

Proof. By inspection of the definition of $S\square\Omega$, we can see that $\pi(S\square\Omega) = S$.

We can see that $\mathscr{H}$ factors as $\square \circ \mathfrak{t}$, where $\mathfrak{t}$ is the right-adjoint to the projection $\widehat{\Delta} \int \widehat{\mathcal{C}} \to \widehat{\Delta}$, which exists by explicit computation as the functor sending the simplicial set S to the object $(S, \Omega_\mathfrak{t})$ where $\Omega_\mathfrak{t}$ is the labeling sending all simplices of S to the terminal presheaf on $\mathcal{C}$. We can see that the pullback of (S,Ω) along $f : S' \to S$ satisfies the universal property of the fibre product of the unit map $(S,\Omega) \to \mathfrak{t}(S)$ with the map $\mathfrak{t}(f)$, so such pullbacks exist in $\widehat{\Delta} \int \widehat{\mathcal{C}}$ and are obviously preserved by $\square$, which by construction preserves whatever limits exist. The proposition follows immediately from these two observations. $\qquad\square$

3. The coherent realization for $\Theta[\mathcal{C}]$

There is an adjoint pair that should be familiar to those who have experience with $(\infty, 1)$-categories, namely the coherent realization and nerve pair,

$$\widehat{\Theta[*]} \underset{\mathfrak{N}_\Delta}{\overset{\mathfrak{C}_\Delta}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(*)},$$

where we have obvious isomorphisms $\Theta[*] \cong \Delta$ and $\mathrm{Psh}_\Delta(*) \cong \widehat{\Delta}$

An important theorem early in *Higher Topos Theory* tells us that this adjunction gives a Quillen equivalence when the lefthand side is equipped with the Joyal model structure and the righthand side is equipped with the Bergner model structure. The goal of this section is to show that for any $\mathcal{C}$ with the aforementioned properties, we can construct an analogous adjunction:

$$\widehat{\Theta[\mathcal{C}]} \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$$

We will extensively abuse notation in what follows by identifying a simplicial set with its associated constant simplicial presheaf on $\mathcal{C}$ and identifying a presheaf on $\mathcal{C}$ with its associated discrete simplicial presheaf.

Definition 3.1. We define a construction on objects

$$Q : \Delta \int \widehat{\mathcal{C}} \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}.$$

Suppose $[n](X_1, \ldots, X_n)$ is any object of $\Delta \int \widehat{\mathcal{C}}$. Then we define $Q([n](X_1, \ldots, X_n)$ as follows:
- The objects are the vertices $\{0, \ldots, n\}$
- The Hom-object

$$\mathrm{Hom}(i,j) = \begin{cases} \emptyset & \text{for } i > j \\ c\Delta^0 & \text{for } i = j \\ X_{i+1} \times \Delta^1 \times \cdots \times \Delta^1 X_j & \text{for } i < j \end{cases}$$

- The associative composition law, $\mathrm{Hom}(i,j) \times \mathrm{Hom}(j,k) \to \mathrm{Hom}(j,k)$ which is the inclusion on the bottom face with respect to j:

$$X_{i+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_j \times \{1\} \times X_{j+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_k$$
$$\downarrow$$
$$X_{i+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_j \times \Delta^1 \times X_{j+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_k$$

Proposition 3.2. *The construction Q is functorial.*

Proof. Recall that a map

$$[n](X_1, \ldots, X_n) \to [m](Y_1, \ldots Y_m)$$

in $\Delta \int \widehat{\mathcal{C}}$ is given by a pair $(\gamma, \mathbf{f})$, where $\gamma : [n] \to [m]$ is a map of simplices together with a family of maps

$$\mathbf{f} = \left(f_i : X_i \to \prod_{j=\gamma(i-1)}^{\gamma(i)} Y_j \right)_{i=1}^n .$$

If for $0 < i \leq n$, we have $\gamma(i-1) = \gamma(i)$, we can see easily that γ factors through the codegeneracy map

$$[n](X_1, \ldots, X_n) \to [n-1](X_1, \ldots, \widehat{X_i}, \ldots, X_n).$$

Applying this factorization repeatedly, we factor $(\gamma, \mathbf{f})$ as a codegeneracy followed by a map (γ', f') such that γ' is the inclusion of a coface $[n'] \hookrightarrow [m]$.

Since $\Delta \int \widehat{\mathcal{C}}$ is fibred over Δ, we may take the cartesian lift of γ', which is the map

$$\overline{\gamma'} = (\gamma', \mathbf{id}) : [n'] \left(\prod_{j=\gamma'(0)}^{\gamma'(1)} Y_j, \ldots, \prod_{j=\gamma'(n'-1)}^{\gamma'(n')} Y_j \right) \hookrightarrow [m](Y_1, \ldots, Y_m).$$

By cartesianness, we have a unique factorization of $(\gamma', \mathbf{f}')$ by this map, yielding a map

$$(\mathrm{id}, \overline{f'}) : [n'](X_1', \ldots, X_{n'}') \to [n'] \left(\prod_{j=\gamma'(0)}^{\gamma'(1)} Y_j, \ldots, \prod_{j=\gamma'(n'-1)}^{\gamma'(n')} Y_j \right).$$

Then to prove the proposition, we need to show functoriality in three cases:

- If the map $(\gamma, \mathbf{f})$ is a codegeneracy of codimension 1, suppose $\gamma = \sigma^i : [n+1] \to [n]$ for $0 \le i \le n$. Then $Q((\sigma^i, \mathbf{id}))_{ab} : \mathrm{Hom}(a,b) \to \mathrm{Hom}(\sigma^i(a), \sigma^i(b))$ is defined on the homs as follows:

$$Q(\sigma^i)_{ab} = \begin{cases} \mathrm{id} \times \min \circ \tau_{X_i} \times \mathrm{id} & \text{if } a < i \le b \\ \mathrm{id} & \text{otherwise} \end{cases}$$

where $\min : \Delta^1 \times \Delta^1 \to \Delta^1$ is induced by the map of posets sending $(x,y) \mapsto \min(x,y)$ and $\tau_{X_i} \to *$ is the terminal map. Specifically, in the case where $a < i \le b$, the map is given by the composite:

$$X_{a+1} \times \Delta^1 \times \cdots \times \Delta^1 \times X_i \times \Delta^1 \times \cdots \times \Delta^1 \times X_b$$
$$\downarrow$$
$$X_{a+1} \times \Delta^1 \times \cdots \times \Delta^1 \times * \times \Delta^1 \times \cdots \times \Delta^1 \times X_b$$
$$\|$$
$$X_{a+1} \times \Delta^1 \times \cdots \times \Delta^1 \times \Delta^1 \times \cdots \times \Delta^1 \times X_b$$
$$\downarrow$$
$$X_{a+1} \times \Delta^1 \times \cdots \times \Delta^1 \times \cdots \times \Delta^1 \times X_b$$

  If $i = 0$ or $i = n$, we consider $\Delta^0 = *$ to be $\{0\}$.
- If the map $(\gamma, \mathbf{f})$ is a pure coface of codimension 1, we have two subcases: If it is an outer coface, the map is just the obvious inclusion. If it is an inner coface, it has a term that looks like $X_i \times X_{i+1}$, and this is included in all of the Hom objects as $X_i \times \{0\} \times X_{i+1}$
- If the map $(\gamma, \mathbf{f})$ is such that $\gamma = \mathrm{id}$, since each of the Hom objects is given as a product of the X_i with Δ^1 of the same length, just map them by $f_a \times \Delta^1 \times \cdots \times \Delta^1 \times f_{b-1}$ using the functoriality of the cartesian product.

It is an easy exercise to see that this assignment is functorial and completely analogous to the unenriched case. $\qquad\square$

Finally, we come to the form of this functor that we will be using:

Definition 3.3. Let $\mathfrak{C}$ be the composite

$$\Theta[\mathcal{C}] \hookrightarrow \Delta \int \widehat{\mathcal{C}} \xrightarrow{Q} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}.$$

Since $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ is cocomplete, there exists a colimit-preserving extension to $\widehat{\Theta[\mathcal{C}]}$, the *homotopy-coherent realization*, which by abuse of notation, we also call $\mathfrak{C}$. It is the left adjoint in an adjunction

$$\mathfrak{C} : \widehat{\Theta[\mathcal{C}]} \rightleftarrows \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})} : \mathfrak{N},$$

wherein the right adjoint is called the *homotopy-coherent nerve*.

4. Enriched necklaces and the coherent realization

Necklaces were introduced by Dugger and Spivak in order to understand the mapping objects $\mathrm{Hom}_{\mathfrak{C}(X)}$ and give a much easier proof than Lurie's that the homotopy-coherent realization and nerve form a Quillen equivalence. In this section, we will introduce an enriched version of necklaces that will serve the same purpose.

Definition 4.1. A *pre-necklace* is a sober $\mathcal{C}$-cellular set whose projection to $\widehat{\Delta}$ is a simplicial necklace in the sense of Dugger and Spivak.

Given a pre-necklace T, we define the *shape of* T to be its associated simplicial set $\pi(T)$.

Suppose T is a pre-necklace such that its projection is the simplicial necklace $\Delta^{m_1} \vee \cdots \vee \Delta^{m_k}$. Then we say that T is a *necklace* if the pullback of T along each bead inclusion $\Delta^{m_i} \hookrightarrow \Delta^{m_1} \vee \cdots \vee \Delta^{m_k}$ is representable for each $1 \leq i \leq k$.

We consider every necklace T as bi-pointed by its initial and terminal vertex, which we will write as α and ω respectively. A *morphism of necklaces* is a morphism of $f : T \to T'$ of $\widehat{\Theta[\mathcal{C}]}_{*,*}$ between necklaces such that $f(\alpha_T) = \alpha_{T'}$ and $f(\omega_T) = \omega_{T'}$. We define the category $\mathcal{N}ec$ to be the full subcategory $\widehat{\Theta[\mathcal{C}]}_{*,*}$ consisting of the necklaces and morphisms of necklaces between them.

We define the sets V_T (resp. J_T) of *vertices of* T (resp. *joints of* T) to be the sets of vertices and joints of the underlying simplicial necklace.

Similarly, for a pair of vertices a, b of T with $a \leq b$ we define $V_T(a,b)$ to be the subset of all vertices i such that $a \leq i \leq b$. We define $J_T(a,b)$ to be $\{a,b\} \cup (V_T(a,b) \cap J_T)$.

Given $a, b \in V_T$, there is a full simplicial subset $\pi(T)(a,b) \subseteq \pi(T)$ consisting of the simplicial set of simplices σ of $\pi(T)$ for which all vertices of σ lie in $V_T(a,b)$. We define $T(a,b)$ to be the pullback of T along the inclusion $\pi(T)(a,b) \hookrightarrow \pi(T)$. It is clear from this definition that $V_T(a,b) = V_{T(a,b)}$ and $J_T(a,b) = J_{T(a,b)}$.

Following Dugger and Spivak, we define a construction as follows: Given a $\mathcal{C}$-cellular set X with two vertices $x, y \in X_0$, we obtain a functor

$$\mathcal{E}_X(x,y) : (\mathcal{N}ec \downarrow X_{x,y}) \to \mathrm{Psh}_\Delta(\mathcal{C})$$

defined by the rule

$$(T \to X_{x,y}) \mapsto \mathfrak{C}(T)(\alpha_T, \omega_T).$$

We define

$$E_X(x,y) = \mathrm{colim}(\mathcal{E}(x,y)),$$

which by the universal property of colimits admits a universal map

$$E_X(x,y) \to \mathfrak{C}(X)(x,y).$$

We can see that there is an associative composition operation

$$E_X(x,y) \times E_X(y,z) \to E_X(x,z)$$

inherited from the operation of wedge-concatenation of necklaces

$$\mathcal{E}_X(x,y) \times \mathcal{E}_X(y,z) \to \mathcal{E}_X(x,z).$$

This makes E_X into a $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched category equipped with a functor $E_X \to \mathfrak{C}(X)$.

Proposition 4.2. *For any $\mathcal{C}$-cellular set X, the induced map $E_X \to \mathfrak{C}(X)$ is an isomorphism.*

Proof. Following along closely with the proof of [DS11a, Proposition 4.3], we consider the following commutative diagram:

$$
\begin{array}{ccc}
\left(\displaystyle\mathrm{colim}_{[t]\in(\Theta[\mathcal{C}]\downarrow X)} E_{[t]} \right)(x,y) & \longrightarrow & E_X(x,y) \\
{\scriptstyle \cong}\big\downarrow & & \big\downarrow \\
\left(\displaystyle\mathrm{colim}_{[t]\in(\Theta[\mathcal{C}]\downarrow X)} \mathfrak{C}([t]) \right)(x,y) & =\!=\!= & \mathfrak{C}(X)(x,y)
\end{array}
$$

.

The bottom horizontal equality is by definition, and the left vertical map is an isomorphism because $E_{[t]} \cong \mathfrak{C}([t])$ for all representables, since they are all necklaces and therefore are terminal in their respective diagrams defining E. It follows that the top horizontal map is injective, and it suffices therefore to show that it is surjective. Choose any representable $\xi : \Delta^n \times c \to E_X(x,y)$. Since $\Delta^n \times [c]$ is representable and $E_X(x,y)$ is a colimit, it follows that the map $\Delta^n \times c \to E_X(x,y)$ factors through some

$$f : \mathfrak{C}(T)(\alpha,\omega) \to \mathfrak{C}(X)(x,y)$$

and is represented therefore represented by the data of such a factorization. Consider the commutative diagram diagram:

$$
\begin{array}{ccc}
\left(\operatorname*{colim}_{[t]\in(\Theta[\mathcal{C}]\downarrow T)} \mathfrak{C}([t]) \right)(\alpha,\omega) & \longrightarrow & \mathfrak{C}(T)(\alpha,\omega) \\
\uparrow & & \uparrow \\
\left(\operatorname*{colim}_{[t]\in(\Theta[\mathcal{C}]\downarrow T)} E_{[t]} \right)(x,y) & \longrightarrow & E_T(\alpha,\omega) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle E_f} \\
\left(\operatorname*{colim}_{[t]\in(\Theta[\mathcal{C}]\downarrow X)} E_{[t]} \right)(x,y) & \longrightarrow & E_X(x,y)
\end{array}
$$

.

Therefore, it suffices to show that the middle horizontal map is surjective, since if this is the case, we can chase ξ back to an element of $\left(\operatorname{colim}_{[t]\in(\Theta[\mathcal{C}]\downarrow X)} E_{[t]} \right)(x,y)$, which proves surjectivity. But the top horizontal map is an isomorphism by the definition of $\mathfrak{C}$, and both top row vertical maps are isomorphisms, again because the diagrams over which $E_{[t]}$ and E_T are colimits over have terminal objects, namely the necklaces $[t]$ and T themselves. $\qquad\square$

This is not the end of the story. This colimit is still very complicated, and we must simplify it further. In particular, we will show that $\mathfrak{C}(X)(x,y)_c$ can be represented as a colimit of contractible spaces functorially in c. This will play an important role in obtaining appropriate analogues of the other models for $\mathfrak{C}$ from [DS11a]. In order to continue down this road, we need the following definition:

Definition 4.3. We say that a necklace T is *of uniform type* $c \in \mathcal{C}$ if the pullback of T along each bead inclusion $\Delta^{m_i} \hookrightarrow \Delta^{m_1} \vee \cdots \vee \Delta^{m_k}$ is the representable $\mathcal{C}$-cellular set associated with $[m_i](c,\ldots,c)$. If T is any simplicial necklace, we denote by $T\{c\}$ the necklace of type c of the same underlying simplicial shape. We define the category $\mathcal{N}ec_c$ to be the full subcategory of the category $\mathcal{N}ec$ spanned by the necklaces of uniform type c.

Definition 4.4. We define the subcategory

$$\mathcal{N}ec_c^{\mathbf{sp}} \subseteq \mathcal{N}ec_c$$

to be the wide subcategory whose morphisms are are *special*, which are maps that factor as the composite of a pure codegeneracy followed by a map whose restriction to each edge of the spine of the domain is a diagonal $c \xrightarrow{id^k} c^k$ of the appropriate arity.

We begin by giving the following construction: Given a $\mathcal{C}$-cellular set X together with two vertices $x, y \in X_0$, we give a functor

$$\mathcal{E}_{X,c}(x,y) : (\mathcal{N}ec_c^{\mathbf{sp}} \downarrow X_{x,y}) \to \widehat{\Delta}$$

defined by the rule
$$T \mapsto \mathfrak{C}_\Delta(\pi(T))(\alpha, \omega),$$
where $\mathfrak{C}_\Delta$ denotes the ordinary coherent realization of a simplicial set.

We then define a simplicial set
$$E_{X,c}(x,y) = \operatorname{colim} \mathcal{E}_{X,c}.$$
We note that by concatenation of necklaces of uniform type c, we obtain an associative composition law
$$E_{X,c}(x,y) \times E_{X,c}(y,z) \to E_{X,c}(x,z).$$
We will see in what follows that $E_{X,c}$ is naturally isomorphic to $\mathfrak{C}(X)_c$.

Lemma 4.5. *If T is a necklace of uniform type c and T' is any necklace, then every morphism of necklaces $f : T \to T'$ factors uniquely as the composite of a special map $T \to T'\{c\}$ and a map $f_* : T'\{c\} \to T'$ that projects to the identity map in $\widehat{\Delta}$.*

Proof. We reduce immediately to the case where the map on underlying simplicial necklaces is injective, using the Eilenberg-Zilber property for necklaces. We can also assume that T' is representable of the form $[n](c_1, \ldots, c_n)$, since given an injective map of simplicial necklaces $\pi(T) \to \pi(T')$, every bead of $\pi(T)$ lands in exactly one bead of $\pi(T')$.

Then we look at the action of f on each edge e of the spine of T. Notice that if f maps an edge e of the spine of $\pi(T)$ to the edge $i < j$, we obtain a map
$$c \to \prod_{k=i+1}^{j} c_k,$$
which by the universal property of the product, corresponds to a family of maps $(f_k : c \to c_k)_{k=i+1}^{j}$. Since f must map the spine of $\pi(T)$ to a directed path from $0 \to n$, taken together, we obtain maps
$$(f_k : c \to c_k)_{k=1}^{n}.$$
These data together with the identity map $\operatorname{id} : [n] \to [n]$ specify precisely a map
$$[n](c, \ldots, c) \to [n](c_1, \ldots, c_n).$$
We have the obvious map $T \to [n](c, \ldots, c)$ where each edge of the spine is assigned the appropriate diagonal map, and this composes with the new map $[n](c, \ldots, c) \to [n](c_1, \ldots, c_n)$ to give the original map. This factorization is clearly unique. $\square$

Proposition 4.6. *If T is a necklace, then $E_{T,c}(\alpha, \omega)$ is canonically isomorphic to $\mathfrak{C}(T)(\alpha, \omega)_c$*

Proof. By the lemma, we see that there is a discrete full cofinal subcategory of $(\mathcal{N}ec_c^{\mathbf{sp}} \downarrow T)$ spanned by the maps $T\{c\} \to T$, so it suffices to show that $\mathfrak{C}(T)(\alpha, \omega)_c$ is a disjoint union of copies of $\mathfrak{C}_\Delta(\pi(T\{c\}))(\alpha, \omega)$ indexed by the maps $T\{c\} \to T$ that project to the identity, but this follows by an easy direct computation of $\mathfrak{C}(T)(\alpha, \omega)_c$, which we give for the case $T = [n](c_1, \ldots, c_n)$ as
$$\operatorname{Hom}(c, c_1) \times \Delta^1 \times \cdots \times \Delta^1 \times \operatorname{Hom}(c, c_n).$$
For a more general necklace of shape $\Delta^{m_1} \vee \cdots \vee \Delta^{m_k}$, it is the same, but omitting the appropriate Δ^1 terms. $\square$

These propositions give us what we need to prove the aforementioned reduction:

Proposition 4.7. *For any $\mathcal{C}$-cellular set X, we have natural isomorphisms of $\operatorname{Psh}_\Delta(\mathcal{C})$-enriched categories, $E_{X,\bullet} \cong E_X \cong \mathfrak{C}(X)$.*

Proof. We begin by naming the natural inclusion

$$\iota_c : (\mathcal{N}ec^{\mathbf{sp}}_c \downarrow X_{x,y}) \hookrightarrow (\mathcal{N}ec \downarrow X_{x,y})$$

Then we compute:

$$E_{X,c}(x,y) = \operatorname{colim}_{(\mathcal{N}ec^{\mathbf{sp}}_c \downarrow X_{x,y})} \mathcal{E}_{X,c}(x,y)$$
$$= \operatorname{Lan}_{\mathrm{pt}} \mathcal{E}_{X,c}(x,y)$$

where pt denotes the terminal functor

$$= \operatorname{Lan}_{\mathrm{pt} \circ \iota_c} \mathcal{E}_{X,c}(x,y)$$
$$\cong \operatorname{Lan}_{\mathrm{pt}} \left(\operatorname{Lan}_{\iota_c} \mathcal{E}_{X,c}(x,y)\right)$$
$$= \operatorname{colim}_{(\mathcal{N}ec \downarrow X_{x,y})} \left(\operatorname{Lan}_{\iota_c} \mathcal{E}_{X,c}(x,y)\right),$$

but by the formula for pointwise Left Kan extensions,

$$\cong \operatorname{colim}_{(\mathcal{N}ec \downarrow X_{x,y})} \left(\operatorname{colim}_{(\mathcal{N}ec^{\mathbf{sp}}_c \downarrow T)} \mathcal{E}_{T,c}(\alpha,\omega)\right)$$
$$= \operatorname{colim}_{(\mathcal{N}ec \downarrow X_{x,y})} \mathfrak{C}(T)(\alpha,\omega)_c$$
$$= E_X(x,y)_c$$
$$\cong \mathfrak{C}(X)(x,y)_c,$$

which proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Homotopical models for $\mathfrak{C}$

In their paper [DS11a], Dugger and Spivak make use of another model for $\mathfrak{C}_\Delta$, which they call $\mathfrak{C}^{\mathcal{N}ec}$, but which we will denote by $\mathfrak{C}^{\mathcal{N}ec}_\Delta$. They show that this functor is related by a zig-zag of weak equivalences to $\mathfrak{C}_\Delta$. Although it is not a left-adjoint, it is highly computable and easy to understand because its mapping spaces are always just the nerves of ordinary categories.

We will define a version of $\mathfrak{C}^{\mathcal{N}ec}$ for $\Theta[\mathcal{C}]$-sets and show that it too is related by a zig-zag of natural weak equivalences of Psh_Δ-enriched categories to $\mathfrak{C}$. Following Dugger and Spivak, we also construct a third model $\mathfrak{C}^{\mathrm{hoc}}$ modeled by taking the homotopy-colimit instead of the ordinary colimit that we showed defines $\mathcal{C}$.

Definition 5.1. The *necklace realization* $\mathfrak{C}^{\mathcal{N}ec}(X)$ of a $\mathcal{C}$-cellular set X is defined to be the $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched category whose set of objects is the set of vertices of X and whose mapping objects are simplicial presheaves on $\mathcal{C}$ defined by the rule:

$$c \mapsto \mathfrak{C}^{\mathcal{N}ec}(x,y)_c = N((\mathcal{N}ec^{\mathbf{sp}}_c \downarrow X_{x,y})).$$

As usual, the composition

$$\mathfrak{C}^{\mathcal{N}ec}(X)(x,y) \times \mathfrak{C}^{\mathcal{N}ec}(X)(y,z) \to \mathfrak{C}^{\mathcal{N}ec}(X)(x,z)$$

is obtained by concatenation of uniform necklaces.

The *homotopy colimit realization* is defined similarly to the ordinary $\mathfrak{C}$, but instead of an using an ordinary colimit, we define

$$\mathfrak{C}^{\mathcal{H}oc}(X)(x,y)_c = \operatorname{hocolim} \mathcal{E}_{X,c}(x,y).$$

Dugger and Spivak use a very specific model of the homotopy colimit of a diagram in simplicial sets, and it works perfectly here as well. By [DS11a, Remark 5.1], we note that the homotopy colimit of a diagram $F : D \to \widehat{\Delta}$ can be modeled as the diagonal simplicial set of the bisimplicial set whose k, ℓ simplices are given by pairs

$$(\sigma : [n] \to D; x \in F(\sigma(0))_\ell).$$

Using this model we can see that the nerve of a category is isomorphic to this model of the homotopy colimit of the constant diagram at Δ^0.

In the case of $\mathfrak{C}^{\mathcal{H}oc}$, we can see immediately that there is a unique natural transformation

$$\mathcal{E}_{X,c}(x,y) \to \mathrm{pt},$$

and this induces a map on homotopy colimits. Moreover, since $\mathcal{E}_{X,c}(x,y)(T) = \mathfrak{C}_\Delta(\pi(T))(\alpha,\omega)$ and since $\pi(T)$ is a simplicial necklace, $\mathfrak{C}_\Delta(\pi(T))(\alpha,\omega)$ is weakly contractible. Therefore, the induced map on homotopy-colimits is a weak equivalence of simplicial sets. This shows that the natural map

$$\mathfrak{C}^{\mathcal{H}oc} \to \mathfrak{C}^{\mathcal{N}ec}$$

is a weak equivalence.

Then we need to show that $E_{X,c}$ is a homotopy colimit:

Theorem 5.2. *The natural map*

$$\mathfrak{C}_c^{\mathcal{H}oc} \to \mathfrak{C}_c$$

is a natural weak equivalence.

Proof. See [DS11a, 4.4, 4.10, 5.2]. Their proof works exactly the same way as in our case. What they show is that the ℓth row of the bisimplicial set of pairs

$$(\sigma : [n] \to (\mathcal{N}ec_c^{\mathbf{sp}} \downarrow X_{x,y}); \zeta \in \mathfrak{C}_\Delta(\pi\sigma(0))_\ell)$$

is homotopy-discrete, which means that the homotopy and ordinary colimit agree. Our indexing category is just a disjoint union of copies of their indexing category, so if theirs is homotopy-discrete, so is ours. $\qquad\square$

6. The horizontal Joyal model structure

We define a Cisinski model structure on $\widehat{\Theta[\mathcal{C}]}$ and state several results that we will need in the sequel:

Definition 6.1. There is a Cisinski model structure called the *horizontal Joyal model structure* on $\widehat{\Theta[\mathcal{C}]}$ where the separating interval is given by

$$E^1 = \mathcal{H}(\mathrm{cosk}_0 \Delta^1),$$

and the set of generating anodynes is given by

$$\mathcal{J} = \{\square_n^\lrcorner(\lambda_k^n, \delta^{c_1}, \dots, \delta^{c_n}) : 0 < k < n \text{ and } c_1, \dots, c_n \in \mathrm{Ob}\,\mathcal{C}\},$$

where $\lambda_k^n : \Lambda_k^n \hookrightarrow \Delta^n$ is the simplicial horn inclusion, and where $\delta^c : \partial c \hookrightarrow c$ is the inclusion of the boundary of c (recall that $\mathcal{C}$ was taken to be a regular Cartesian Reedy category, so this makes sense).

We call $\mathrm{Cell}(\mathcal{J})$ the class of *horizontal inner anodynes*, and we call $\mathrm{rlp}(\mathcal{J})$ the class of *horizontal inner fibrations*.

Remark 6.2. The precise definition and construction of the corner-intertwiner $\square_n^\lrcorner$ is deferred to Appendix A.1, but in this particular case, we can compute it by hand in terms of the intertwiner to be

$$V_{\Lambda_k^n}(c_1, \dots, c_n) \cup \left(\bigcup_{i=1}^{n} V[n](c_1, \dots, \partial c_i \dots, c_n) \right) \hookrightarrow [n](c_1, \dots, c_n),$$

where $V_{\Lambda_k^n}(c_1, \dots, c_n)$ is the pullback of $[n](c_1, \dots, c_n)$ by the inclusion $\Lambda_k^n \hookrightarrow \Delta^n$ (whenever $K \subseteq \Delta^n$, we can apply this formula to compute the corner tensor).

Definition 6.3. We call an object with the right lifting property with respect to $\mathcal{J}$ a *formal $\mathcal{C}$-quasicategory.*

The following results are stated here without proof. All proofs are heavily inspired by [Our10] and provided in full in the Appendices A.2, A.3, and A.6.

Proposition 6.4. *The class of all monomorphisms of $\widehat{\Theta[\mathcal{C}]}$ is exactly* $\mathrm{Cell}(\mathcal{M})$, *where*
$$\mathcal{M} = \{\Box_n^{\lrcorner}(\delta^n, \delta^{c_1}, \ldots, \delta^{c_n}) : n \geq 0 \text{ and } c_1, \ldots, c_n \in \mathrm{Ob}\,\mathcal{C}\}.$$

Proposition 6.5. *For any inner anodyne inclusion* $\iota : K \hookrightarrow \Delta^n$ *and any family* $f_1, \ldots, f_n$ *of monomorphisms of* $\widehat{\mathcal{C}}$, *the map*
$$\Box_n^{\lrcorner}(\iota, f_1, \ldots, f_n)$$
is horizontal inner anodyne.

Theorem 6.6. *The horizontal Joyal model structure is Cartesian-closed, and in particular,*
$$\mathrm{Cell}(\mathcal{M}) \times^{\lrcorner} \mathrm{Cell}(\mathcal{J}) \subseteq \mathrm{Cell}(\mathcal{J}).$$

Theorem 6.7. *A horizontal inner fibration between formal* $\mathcal{C}$-*quasicategories is a fibration for the horizontal Joyal model structure if and only if it has the right lifting property with respect to the map* $\Delta^0 \hookrightarrow E^1$. *In particular, the formal* $\mathcal{C}$-*quasicategories are exactly the fibrant objects for the horizontal Joyal model structure.*

Proposition 6.8. *Given a necklace* T, *there is a unique embedding* $\iota_T : T \hookrightarrow \Delta[T]$, *where* $\Delta[T]$ *is the unique representable whose spine is exactly the spine of* T. *This map sends the spine of* T *isomorphically onto the spine of* $\Delta[T]$ *and is a horizontal inner anodyne.*

7. Quillen functoriality

In this section, we show that the adjunction
$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \overset{\mathfrak{C}}{\underset{\mathfrak{N}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$$
is a Quillen pair. We begin with the following observation:

Proposition 7.1. *For any* $n > 0$, *let* $K \subseteq \{1, \ldots, n-1\}$ *and define*
$$\Lambda_K^n = \bigcup_{i \notin K} \partial_i \Delta^n,$$
and let
$$\lambda_K^n : \Lambda_K^n \hookrightarrow \Delta^n$$
denote the inclusion map. Then
$$\mathfrak{C}(\Box_n^{\lrcorner}(\lambda_K^n, \delta^{c_1}, \ldots, \delta^{c_n}))(i, j)$$
is an isomorphism whenever $i \neq 0$ *or* $j \neq n$. *Moreover, the map*
$$\mathfrak{C}(\Box_n^{\lrcorner}(\lambda_K^n, \delta^{c_1}, \ldots, \delta^{c_n}))(0, n)$$
is exactly
$$\delta^{c_1} \times^{\lrcorner} h_K^1 \times^{\lrcorner} \cdots \times^{\lrcorner} h_K^{n-1} \times^{\lrcorner} \delta^{c_n},$$
where
$$h_K^k = \begin{cases} \lambda_1^1 \ \textit{if } k \in K \\ \delta^1 \ \textit{otherwise} \end{cases}.$$

Proof. Let X denote the domain of $\square_n^{\lrcorner}(\lambda_K^n, \delta^{c_1}, \ldots, \delta^{c_n})$. If $f : T \to [n](c_1, \ldots, c_n)_{i,j}$ is a bi-pointed map from a necklace T, with $i \neq 0$, then f factors through the inclusion of the subobject $[n - 1](c_2, \ldots, c_n) \subseteq V_{\Lambda_K^n}(c_1, \ldots, c_n)$, so $\mathfrak{C}(X)(i, j) = \mathfrak{C}([n](c_1, \ldots, c_n))$. The case where $j \neq n$ follows by symmetry.

The second part comes from the observation that when $K = \{1, \ldots, n - 1\}$,

$$\mathfrak{C}(V_{\Lambda_K^n}(c_1, \ldots, c_n))(0, n) = \bigcup_{i=1}^{n-1} c_1 \times \Gamma_i^1 \times \cdots \times \Gamma_i^{n-1} \times c_n,$$

where

$$\Gamma_i^{\ell} = \begin{cases} \Lambda_1^1 \text{ for } \ell = i \\ \Delta^1 \text{ otherwise} \end{cases}.$$

To see this, notice that Λ_K^n is the union of the two outer faces, and attaching them along their common face gives a colimit in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$ where $\mathfrak{C}(V_{\Lambda_K^n}(c_1, \ldots, c_n)(0, n)$ is freely generated by compositions

$$\mathfrak{C}([n - 1](c_1, \ldots, c_{n-1}))(0, \ell) \times \{1\} \times \mathfrak{C}([n - 1](c_2, \ldots, c_n))(\ell, n).$$

For when K is otherwise, each additional inner face gives the factor

$$\mathfrak{C}([n - 1](c_1, \ldots, c_{n-1}))(0, \ell) \times \{0\} \times \mathfrak{C}([n - 1](c_2, \ldots, c_n))(\ell, n),$$

so in general,

$$\mathfrak{C}(V_{\Lambda_K^n}(c_1, \ldots, c_n))(0, n) = \bigcup_{i=1}^{n-1} c_1 \times \Gamma_{i,K}^1 \times \cdots \times \Gamma_{i,K}^{n-1} \times c_n,$$

where

$$\Gamma_{i,K}^{\ell} = \begin{cases} \partial \Delta^1 \text{ for } \ell = i \text{ and } i \in K \\ \Lambda_1^1 \text{ for } \ell = i \text{ and } i \notin K \\ \Delta^1 \text{ otherwise} \end{cases}.$$

Each factor

$$V[n](c_1, \ldots, \partial c_j, \ldots, c_n)$$

contributes

$$\mathfrak{C}(V[n](c_1, \ldots, \partial c_j, \ldots, c_n))(0, n) = c_1 \times \Delta^1 \times \cdots \times \Delta^1 \times \partial c_j \times \Delta^1 \times \cdots \times \Delta^1 \times c_n,$$

and taking the union of all of the factors gives exactly the domain of the inclusion

$$\delta^{c_1} \times^{\lrcorner} h_K^1 \times^{\lrcorner} \cdots \times^{\lrcorner} h_K^{n-1} \times^{\lrcorner} \delta^{c_n}.$$

$\square$

Proposition 7.2. *The functor $\mathfrak{C}$ sends monomorphisms to cofibrations and horizontal inner anodynes to trivial cofibrations.*

Proof. Let

$$\mathbf{2} : \mathrm{Psh}_\Delta(\mathcal{C}) \to \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}$$

be the functor sending a simplicial presheaf X to the enriched category with objects $\{0, 1\}$ with $\mathbf{2}(X)(0,0) = \mathbf{2}(X)(1,1) = *$, $\mathbf{2}(X)(1,0) = \emptyset$, and $\mathbf{2}(X)(0,1) = X$.

When $K = \emptyset$, $\lambda_K^n = \delta^n$, so the lemma tells us that

$$\mathfrak{C}(\square_n^{\lrcorner}(\delta^n, \delta^{c_1}, \ldots, \delta^{c_n}))$$

is a pushout of the map

$$\mathbf{2}(\delta^{c_1} \times^{\lrcorner} \delta^1 \times^{\lrcorner} \cdots \times^{\lrcorner} \delta^1 \times^{\lrcorner} \delta^{c_n}),$$

which is a cofibration, which proves the claim.

Similarly, when K is a singleton, $\lambda_K^n = \lambda_k^n$ is the inclusion of an inner horn, so

$$\mathfrak{C}(\square_n^{\lrcorner}(\lambda_k^n, \delta^{c_1}, \dots, \delta^{c_n})$$

is the pushout of the map

$$\mathbf{2}(\delta^{c_1} \times^{\lrcorner} h_k^1 \times^{\lrcorner} \cdots \times^{\lrcorner} h_k^{n-1} \times^{\lrcorner} \delta^{c_n}),$$

where $h_k^k = \lambda_1^1$. This is a corner map where one factor is a trivial cofibration (because it is Kan anodyne), and therefore its image under $\mathbf{2}$ is a trivial cofibration. Since the pushout of a trivial cofibration is a trivial cofibration, we are done. $\square$

Corollary 7.3. *The coherent nerve of a fibrant $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched category is a formal $\mathcal{C}$-quasicategory.*

Lemma 7.4. *The object $\mathfrak{C}(E^n)$ is weakly contractible for all n.*

Proof. We notice immediately that $\mathfrak{C}(E^n)(i,j)_\bullet$ is a constant simplicial presheaf for all i, j, so it suffices to show that $\mathfrak{C}(E^n)(i,j)_*$ is contractible for all i, j, but then it follows immediately from the classical case. $\square$

Proposition 7.5. *The coherent nerve sends fibrations between fibrant $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories to fibrations for the horizontal Joyal model structure.*

Proof. Given a fibration between two fibrant $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched categories, $p : \mathcal{D} \to \mathcal{D}'$, we see immediately that the coherent nerve takes this fibration to a horizontal inner fibration between formal $\mathcal{C}$-quasicategories by Proposition 7.2. To show that it is a fibration for the horizontal Joyal model structure, it suffices by Theorem 6.7 to show that it has the right lifting property with respect to the inclusion $e : \Delta^0 \hookrightarrow E^1$. By Proposition 7.2, we see that $\mathfrak{C}$ takes the monomorphism e to a cofibration, and by the previous lemma, we see that $\mathfrak{C}(e)$ is a weak equivalence. It follows that $\mathfrak{N}(p)$ is a fibration for the horizontal Joyal model structure. $\square$

Corollary 7.6. *The adjunction*

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \overset{\mathfrak{C}}{\underset{\mathfrak{N}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$$

is a Quillen pair.

Proof. If $\mathfrak{C}$ takes cofibrations to cofibrations, and $\mathfrak{N}$ takes fibrations between fibrant objects to fibrations between fibrant objects, then the adjunction is a Quillen pair, but this is exactly what we proved in this section. $\square$

8. Enriched gadgets

The general theory of gadgets developed in [DS11a] is difficult to adapt to the enriched setting, and we give a less-than-ideal generalization in the sequel:

Definition 8.1. A *gadget* of rank n is a functor

$$G : \mathcal{C} \to \widehat{\Theta[\mathcal{C}]}_{*,*}$$

such that there exists a c-indexed simplicial presheaf $S_G^\bullet$ and a natural zig-zag of weak homotopy equivalences of simplicial presheaves

$$\mathfrak{C}(G(c))(\alpha, \omega) \overset{\sim}{\leftarrow} S_G^c \overset{\sim}{\rightarrow} c^n$$

for all $c \in \mathcal{C}$ (where c^n denotes the nth cartesian power of the representable), where naturality implies that for any $f : c \to d$ in $\mathcal{C}$, the diagram

$$
\begin{array}{ccccc}
\mathfrak{C}(G(c))(\alpha, \omega) & \longleftarrow & S_G^c & \longrightarrow & c^n \\
\scriptstyle{\mathfrak{C}(G(f))(\alpha,\omega)} \downarrow & & \scriptstyle{s_G^f} \downarrow & & \downarrow \scriptstyle{f^n} \\
\mathfrak{C}(G(d))(\alpha, \omega) & \longleftarrow & S_G^d & \longrightarrow & d^n
\end{array}
$$

commutes.

Remark 8.2. We can see from the definition that every simplicial necklace T defines a gadget sending c in $\mathcal{C}$ to the uniform necklace $T\{c\}$ of type c. In what follows, by abuse of notation, we will use T to denote both the underlying simplicial necklace as well as this gadget.

Unlike in the simplicial case, we have seen that we cannot simply get away with looking at full subcategories, so we have to be careful about morphisms.

Definition 8.3. Let T be a simplicial necklace. Then for a gadget G of rank n, we define a *special morphism* $f : T \to G$ to be a natural transformation such that for each c in $\mathcal{C}$, the image of the induced map

$$\mathfrak{C}_\Delta(\pi(T))(\alpha, \omega) \to \mathfrak{C}(G)(\alpha, \omega)_c$$

lands in the connected component corresponding to the nth diagonal $(\mathrm{id}_c)^n$.

More generally, given a pair of gadgets G, G', we define a *special morphism* $\phi : G \to G'$ to be a natural transformation such that given any simplicial necklace T and any special morphism $f : T \to G$, the induced map $\phi \circ f : T \to G'$ is special.

Remark 8.4. If T and T' are two simplicial necklaces, the component at c of a special morphism $T \to T'$ is precisely a special map between uniform necklaces of type c.

Definition 8.5. We define *a category of gadgets* $\mathcal{G}$ to be a subcategory of the category of all gadgets and special maps containing all necklaces and all special morphisms $T \to G$ where T is a necklace and G is in $\mathcal{G}$. We say that the category of gadgets is *closed under wedges* if it is closed under concatenation of gadgets.

We define $\mathcal{G}_c$ to be the image of $\mathcal{G}$ under evaluation at $c \in \mathcal{C}$.

Definition 8.6. Given a $\mathcal{C}$-cellular set X, two vertices x, y in X_0 and a category of gadgets $\mathcal{G}$, we define a simplicial presheaf on $\mathcal{C}$ by the formula

$$\mathfrak{C}^{\mathcal{G}}(X)(x, y)_c \overset{\text{def}}{=} N(\mathcal{G}_c \downarrow X_{x,y}).$$

When $\mathcal{G}$ is closed under wedges, we can define the $\mathrm{Psh}_\Delta(\mathcal{C})$-enriched category $\mathfrak{C}^{\mathcal{G}}(X)$ to be the category whose objects are the vertices of X and whose Hom-objects are

$$\mathfrak{C}^{\mathcal{G}}(X)(x, y)..$$

This defines an enriched category by taking the composition operation to be concatenation of gadgets, which works since $\mathcal{G}$ is closed under wedges.

Proposition 8.7. *Given a $\mathcal{C}$-cellular set X and two vertices x, y of X and a category of gadgets $\mathcal{G}$ the map*

$$N(\mathcal{N}ec_c^{\mathbf{sp}} \downarrow X_{x,y}) \hookrightarrow N(\mathcal{G}_c \downarrow X_{x,y})$$

is a weak homotopy equivalence.

Proof. By Quillen's theorem A, it suffices to look at the overcategories $(\mathcal{N}ec_c^{\mathbf{sp}} \downarrow G(c))$ along the inclusion $\mathcal{N}ec_c^{\mathbf{sp}} \hookrightarrow \mathcal{G}_c$ for all G in $\mathcal{G}$ and show that their nerves are contractible, but these overcategories correspond on-the-nose to the subcategories classifying the diagonal component of $\mathfrak{C}^{\mathcal{N}ec}(G)(\alpha, \omega)$, which is contractible because G is a gadget. $\square$

Corollary 8.8. *The constructions $\mathfrak{C}^{\mathcal{N}ec}$ and $\mathfrak{C}^{\mathcal{G}}$ are naturally weakly equivalent when $\mathcal{G}$ is closed under wedges.*

Proof. The map between the two constructions is the identity on objects and induces Hom-wise weak equivalences of simplicial presheaves. $\square$

9. The Hom by cosimplicial resolution

For every object $[1](c)$ in $\Theta[\mathcal{C}]$, we introduce four canonical cosimplicial resolutions, which we can use to define simplicial presheaves that represent the mapping space between two vertices of a $\Theta[C]$-set.

First, we define the functor

$$(\bullet)^{\triangleright}(c), \text{ resp. } (\bullet)^{\triangleleft}(c) : \Delta \to \Theta[\mathcal{C}],$$

which sends

$$[n] \mapsto [n+1](*, \ldots, *, c), \text{ resp. } [n] \mapsto [n+1](c, *, \ldots, *).$$

We see immediately that there are natural embeddings

$$([n])^{\triangleright, c} \hookrightarrow \Delta^n \times [1](c) \hookleftarrow ([n])^{\triangleleft, c}$$
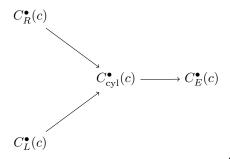
where each map embeds along the respective outer shuffle. We also have an obvious natural embedding

$$\mathcal{H}(\Delta^n) \times [1](c) \hookrightarrow E^n \times [1](c).$$

Then we define the following four cosimplicial objects:

$$C_{\text{cyl}}^{\bullet}(c) \overset{\text{def}}{=} \text{colim}\left(V[1](\emptyset) \leftarrow \Delta^{\bullet} \times V[1](\emptyset) \hookrightarrow \mathcal{H}(\Delta^{\bullet}) \times [1](c)\right)$$

$$C_E^{\bullet}(c) \overset{\text{def}}{=} \text{colim}\left(V[1](\emptyset) \leftarrow E^{\bullet} \times V[1](\emptyset) \hookrightarrow E^{\bullet} \times [1](c)\right)$$

$$C_R^{\bullet}(c) \overset{\text{def}}{=} \text{colim}\left(* \leftarrow \Delta^{\bullet} \xrightarrow{\partial^{\bullet}} (\bullet)^{\triangleright}(c)\right)$$

$$C_L^{\bullet}(c) \overset{\text{def}}{=} \text{colim}\left(* \leftarrow \Delta^{\bullet} \xrightarrow{\partial^0} (\bullet)^{\triangleleft}(c)\right).$$

These cosimplicial objects fit in a natural diagram

induced by the inclusions we described above.

Proposition 9.1. *Each of the cosimplicial objects described above is a Reedy-cofibrant object of bi-pointed presheaves on $\Theta[\mathcal{C}]$, and each is objectwise horizontal-Joyal equivalent to the constant cosim-plicial object $[1](c)$. That is to say, each of these cosimplicial objects is a cosimplicial resolution of $[1](c)$ in $\widehat{\Theta[\mathcal{C}]}_{*,*}$.*

Proof. That they are Reedy-cofibrant is obvious, and it is also obvious that $C_E^\bullet(c)$ is objectwise horizontal-Joyal equivalent to $[1](c)$. The proof that the others are horizontal-Joyal equivalent will be covered in Appendix A.5 by showing that each of the projection maps admits a horizontal inner-anodyne section. $\square$

Corollary 9.2. *Each of the functors $C_{\mathrm{cyl}}^n(\bullet), C_R^n(\bullet), C_L^n(\bullet), C_E^n(\bullet)$ defines a rank-1 gadget for each $n \geq 0$.*

Proof. Since $\mathfrak{C}$ is left-Quillen, it preserves weak equivalences between cofibrant objects in $\widehat{\Theta[\mathcal{C}]}$. It follows that since for each $n \geq 0$ we have that $C_-^n(\bullet) \to [1](\bullet)$ is a weak equivalence, then

$$\mathfrak{C}(C_-^n(\bullet))(\alpha, \omega) \to \mathfrak{C}([1](\bullet))(\alpha, \omega) = \bullet$$

is a weak equivalence, so we have the natural zig-zag we require. $\square$

Definition 9.3. Given a bipointed $\mathcal{C}$-cellular set $X_{x,y}$ we define the *mapping object* from x to y to the simplicial presheaf obtained by taking homotopy function complexes

$$\mathrm{Map}_X(x,y)_c \stackrel{\mathrm{def}}{=} h\widehat{\Theta[\mathcal{C}]}_{*,*}([1](c), X).$$

We define a slightly modified version for special maps.

Definition 9.4. If G is a gadget, let $\mathcal{G}$ denote the category of all gadgets with special maps between them. Then we define the *special mapping object* to be

$$\mathrm{Map}_G^{\mathbf{sp}}(\alpha, \omega)_c = \mathcal{G}(C_R^\bullet(c), G(c)).$$

Proposition 9.5. *Given a necklace gadget T, the special mapping object*

$$\mathrm{Map}_T^{\mathbf{sp}}(\alpha, \omega)$$

is contractible.

Proof. Since $T(c) \hookrightarrow \Delta[T](c)$ is a horizontal inner-anodyne and $\Delta[T](c)$ is fibrant, we can compute $\mathrm{Map}_{T(c)}^{\mathbf{sp}}(\alpha, \omega)$ by the formula

$$\mathrm{Map}_T^{\mathbf{sp}}(\alpha, \omega)_{k,c} = \mathcal{G}(C_R^k(c), \Delta[T](c))$$

but every map $C_R^k(c) \to \Delta[T](c)$ factors through the map $C_R^n(c) \to [1](c)$, and the only special map $[1](c) \to \Delta[T](c)$ is the one that maps c into $c \times \cdots \times c$ via the diagonal. $\square$

10. COMPARING $\mathfrak{C}(X)(x,y)$ WITH $\mathrm{Map}_X(x,y)$

We begin by defining a special category of gadgets $\mathcal{Y}$, which is the full subcategory of the category of all gadgets whose objects are those gadgets G such that $\mathrm{Map}_G^{\mathbf{sp}}(\alpha, \omega)$ is contractible.

In particular, by Proposition 9.5, we see that every necklace gadget belongs to this category, so it is indeed a category of gadgets. We define a full subcategory $\mathcal{Y}_{\mathrm{f}} \subseteq \mathcal{Y}$ to be the full subcategory of $\mathcal{Y}$ spanned by the gadgets G such that $G(c)$ is fibrant for all $c \in \mathrm{Ob}\,\mathcal{C}$.

Let

$$C^\bullet(c) \xrightarrow{\sim} R^\bullet(c) \xrightarrow{\sim} [1](c)$$

be a factorization into a Reedy trivial cofibration followed by a fibration, which is also a Reedy trivial fibration since we are factoring a Reedy equivalence.

The following proposition follows [DS11b, 5.2] almost exactly word for word.

Proposition 10.1. *If X is formal $\mathcal{C}$-quasicategory, and x, y are two vertices of x, there is a commutative diagram*

$$\mathfrak{C}^{\mathcal{N}ec}(X)(x,y)_c \xrightarrow{\quad\sim\quad} \mathfrak{C}^{\mathcal{Y}}(X)(x,y)_c \xleftarrow{\quad\sim\quad} \mathfrak{C}^{\mathcal{Y}_f}(X)(x,y)_c$$

$$N\left(\Delta \downarrow \widehat{\Theta[\mathcal{C}]}_{*,*}(C^\bullet(c), X_{x,y})\right) \xleftarrow{\quad\sim\quad} N\left(\Delta \downarrow \widehat{\Theta[\mathcal{C}]}_{*,*}(R^\bullet(c), X_{x,y})\right)$$

$$.$$

in which all of the maps are weak equivalences.

Proof. First, we already know that the map

$$\mathfrak{C}^{\mathcal{N}ec}(X)(x,y)_c \to \mathfrak{C}^{\mathcal{Y}}(X)(x,y)_c$$

is a weak equivalence by Proposition 8.7. The map

$$\mathfrak{C}^{\mathcal{Y}_f}(X)(x,y)_c \hookrightarrow \mathfrak{C}^{\mathcal{Y}}(X)(x,y)_c$$

is the image under the nerve of the functor

$$j : (\mathcal{Y}_f(c) \downarrow X_{x,y}) \hookrightarrow (\mathcal{Y}(c) \downarrow X_{x,y}).$$

We will show that it is a weak homotopy equivalence as follows: Let $Z \mapsto \mathscr{F}(Z)$ denote a functorial fibrant replacement of Z in the horizontal Joyal model structure. Then since X is fibrant, there exists a map $\mathscr{F}(X) \to X$ retracting the inclusion $X \hookrightarrow \mathscr{F}(X)$. Using this fact, we define a functor

$$F : (\mathcal{Y}(c) \downarrow X_{x,y}) \hookrightarrow (\mathcal{Y}_f(c) \downarrow X_{x,y})$$

sending

$$Y(c) \to X_{x,y} \mapsto \mathscr{F}Y(c) \to \mathscr{F}(X) \to X.$$

This works because

$$\mathrm{Map}^{\mathbf{sp}}_{Y(c)}(\alpha, \omega) \to \mathrm{Map}^{\mathbf{sp}}_{\mathscr{F}(Y(c))}(\alpha, \omega)$$

and

$$\mathfrak{C}(Y(c))(\alpha, \omega) \to \mathfrak{C}(\mathscr{F}(Y(c)))(\alpha, \omega)$$

are weak equivalences, in the first instance because the formation of the special mapping space was homotopy-invariant, and in the second instance because $\mathfrak{C}$ is left-Quillen. Then we see that Fj and jF both admit natural transformations back to the appropriate identity functors, which proves that they induce a weak homotopy equivalence on nerves.

The righthand vertical map comes from applying the nerve to the functor

$$f : \left(\Delta \downarrow \widehat{\Theta[\mathcal{C}]}_{*,*}(R^\bullet(c), X_{x,y})\right) \to (\mathcal{Y}_f(c) \downarrow X_{x,y})$$

defined by the rule

$$([n], R^n(c) \to X_{x,y}) \mapsto (R^n(c), R^n(c) \to X_{x,y}).$$

To show that this functor induces a weak equivalence on nerves, we apply Quillen's theorem A. Notice that for an object $y = (Y(c), Y(c) \to X_{x,y})$, the comma category $(f \downarrow y)$ is precisely

$$(\Delta \downarrow \mathcal{G}(c)(R^\bullet(c), Y(c))) = \left(\Delta \downarrow \mathrm{Map}^{\mathbf{sp}}_{Y(c)}(\alpha, \omega)\right).$$

But by [DS11b], the nerve of the category of elements of a simplicial set is weakly equivalent to that simplicial set, and since $\mathrm{Map}^{\mathbf{sp}}_{Y(c)}(\alpha, \omega)$ was assumed to be contractible, the result follows.

To see that the bottom map is an equivalence, it follows simply because $C^\bullet(c) \to R^\bullet(c)$ is a Reedy trivial cofibration, so

$$\widehat{\Theta[\mathcal{C}]}_{*,*}(R^\bullet(c), X_{x,y}) \to \widehat{\Theta[\mathcal{C}]}_{*,*}(C^\bullet(c), X_{x,y})$$

is a weak equivalence. Therefore, again by [DS11b], the nerve of the category of elements of simplicial sets preserves weak equivalences. □

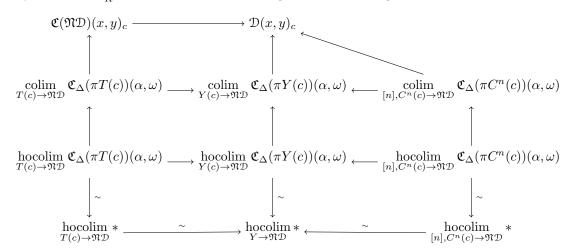**Corollary 10.2.** *For any formal $\mathcal{C}$-quasicategory X and any pair of vertices x, y, there is a natural zig-zag of weak equivalences between $\mathrm{Map}_X(x, y)$ and $\mathfrak{C}(X)(x, y)$.*

Theorem 10.3. *For any fibrant $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched category $\mathcal{D}$, the counit map*

$$\epsilon_\mathcal{D} : \mathfrak{C}(\mathfrak{N}\mathcal{D}) \to \mathcal{D}$$

is a weak equivalence of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched categories.

Proof. Let $\mathcal{C}^\bullet = C_R^\bullet$. Then consider the following commutative diagram:

The two horizontal maps in the bottom row are weak equivalences by the previous proposition and the fact that $\mathfrak{N}$ takes fibrant objects to fibrant objects. The indicated vertical maps are also weak equivalences because π takes gadgets to simplicial gadgets, which have the property that $\mathfrak{C}_\Delta(G)(\alpha, \omega)$ is weakly contractible. By 3-for-2, it follows that the horizontal maps in the third row are all weak equivalences. We reduce this diagram to a smaller diagram

in which the left vertical map is a weak equivalence by Theorem 5.2. By 3-for-2, we can see that it suffices to show that the map

$$\gamma_c : \operatorname*{hocolim}_{[n], C^n(c) \to \mathfrak{N}\mathcal{D}} \mathfrak{C}_\Delta(\pi C^n(c))(\alpha, \omega) \to \operatorname*{colim}_{[n], C^n(c) \to \mathfrak{N}\mathcal{D}} \mathfrak{C}_\Delta(\pi C^n(c))(\alpha, \omega) \to \mathcal{D}(x, y), c$$

is a weak equivalence. To do this, notice that

$$\widehat{\Theta[\mathcal{C}]}_{*,*}(C^n(c), \mathfrak{N}\mathcal{D}_{x,y}) \cong \big(\partial[1] \downarrow \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})}\big)(\mathfrak{C}(C^n(c)_{\alpha,\omega}), \mathcal{D}_{x,y})$$
$$\cong \mathrm{Psh}_\Delta(\mathcal{C})(\mathfrak{C}(C^n(c))(\alpha, \omega), \mathcal{D}(x,y)).$$

As in the proof of [DS11b, Proposition 5.8], we define the cosimplicial simplicial set $Q^\bullet = \mathfrak{C}_\Delta(\pi C^\bullet)(\alpha, \omega)$, which is obviously isomorphic to the $Q^\bullet$ defined in [DS11b]. Moreover, by direct computation, we see that

$$\mathfrak{C}(C^\bullet(c))(\alpha, \omega) \cong Q^\bullet \times c.$$

Then we see immediately that

$$\mathrm{Psh}_\Delta(\mathcal{C})(\mathfrak{C}(C^n(c))(\alpha, \omega), \mathcal{D}(x,y)) \cong \mathrm{Psh}_\Delta(\mathcal{C})(Q^n \times c, \mathcal{D}(x,y)) \cong \widehat{\Delta}(Q^n, \mathcal{D}(x,y)_c),$$

so γ_c is precisely the map obtained by composing

$$\left(\operatorname*{hocolim}_{[n], Q^n \to \mathcal{D}(x,y)_c} Q^n\right) \to \left(\operatorname*{colim}_{[n], Q^n \to \mathcal{D}(x,y)_c} Q^n\right) \to \mathcal{D}(x,y)_c.$$

The result then follows immediately by application of [DS11b, Lemma 5.9]. $\square$

Remark 10.4. This result is even stronger than it first appears, because it implies that the counit map is a weak equivalence for fibrant categories enriched in any cartesian-closed left-Bousfield localization of $\mathrm{Psh}_\Delta(\mathcal{C})_{inj}$. It reduces proving comparison theorems for such localizations to showing that $\mathfrak{C}$ is a left-Quillen functor (something we already know for the horizontal Joyal model structure by Corollary 7.6) and reflects weak equivalences.

11. The horizontal comparison theorem

Dugger and Spivak introduce a definition of a Dwyer-Kan equivalence as a stepping stone to proving the comparison theorem. They use the definition of DK-equivalence as an intermediate step to proving that $\mathfrak{C}_\Delta$ is homotopy-conservative. We give an analogous definition as follows:

Definition 11.1. A map $f : X \to Y$ of presheaves on $\Theta[\mathcal{C}]$ is called a *horizontal Dwyer-Kan equivalence* if the following two properties hold:
- The induced map

$$f_* : \mathrm{Ho}(\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}}(*, X) \to \mathrm{Ho}(\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}}(*, Y)$$

  is bijective, and
- For any two vertices $x, x' \in X_0$, the induced map

$$\mathrm{Map}_X(x, x') \to \mathrm{Map}_Y(f(x), f(x'))$$

  is a weak equivalence of simplicial presheaves on $\mathcal{C}$.

Proposition 11.2. *A map $f : X \to Y$ of presheaves on $\Theta[\mathcal{C}]$ is a horizontal weak equivalence if and only if it is a horizontal Dwyer-Kan equivalence.*

Proof. It is clear that any horizontal Joyal equivalence is automatically horizontally Dwyer-Kan since our constructions are all homotopy-invariant, so we prove that all horizontal Dwyer-Kan equivalences are horizontal Joyal equivalences. We notice immediately that if X and Y are fibrant, the horizontal Dwyer-Kan condition implies that the associated map $\mathfrak{Q}(f) : \mathfrak{Q}(X) \to \mathfrak{Q}(Y)$ between complete $\Theta[\mathcal{C}]$ Segal spaces is an equivalence, where

$$\mathfrak{Q} : \widehat{\Theta[\mathcal{C}]} \to \widehat{\Theta[\mathcal{C}] \times \Delta}$$

is defined by the rule

$$\mathcal{Q}(X)_{[n](c_1,\ldots,c_n),m} \overset{\text{def}}{=\!=} \mathrm{Hom}([n](c_1,\ldots,c_n) \times E^m, X).$$

Since $\mathcal{Q}$ is the right adjoint of a Quillen equivalence by Appendix A.4, a map f between fibrant objects is a weak equivalence if and only if its image under $\mathcal{Q}$ is. Therefore, the claim holds for X and Y fibrant.

In general, given a horizontal Dwyer-Kan equivalence $f : X \to Y$ where X and Y are no longer assumed to be fibrant, we can take a fibrant replacement $\tilde{Y}$ of Y such that $Y \to Y'$ is a trivial cofibration for the horizontal Joyal model structure. Then we can also factor $X \to Y \to \tilde{Y}$ into a trivial Joyal cofibration $X \to \tilde{X}$ followed by a fibration $\tilde{X} \to \tilde{Y}$. But notice now that the condition of being horizontally DK-equivalent is homotopy invariant, so the map $\tilde{X} \to \tilde{Y}$ is also a horizontal DK-equivalence. Since $\tilde{Y}$ is fibrant and $\tilde{X} \to \tilde{Y}$ is a horizontal Joyal fibration, this is a horizontal Joyal equivalence. Then by 3-for-2 we see that f is also a horizontal Joyal equivalence, which concludes the proof. $\square$

Proposition 11.3. *A map $f : X \to Y$ of presheaves on $\Theta[\mathcal{C}]$ is a horizontal Joyal equivalence if and only if $\mathfrak{C}(f)$ is a weak equivalence of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$-enriched categories.*

Proof. We only need to check one direction, since the other direction is immediate by the fact that $\mathfrak{C}$ is left-Quillen. Assume $f : X \to Y$ has the property that $\mathfrak{C}(f)$ is an equivalence. Then as in the previous proposition, we can reduce to the case where X and Y are fibrant, but in this case, we know from Corollary 10.2 that $\mathfrak{C}(X)(x,x')$ is connected by a natural zig-zag of weak equivalences to $\mathrm{Map}_X(x,y)$, so if the map $\mathfrak{C}(X)(x,x') \to \mathfrak{C}(Y)(f(x),f(x'))$ is a weak equivalence, it follows that the map $\mathrm{Map}_X(x,x') \to \mathrm{Map}_Y(f(x),f(x'))$ is also a weak equivalence.

Then it suffices to show that when $\mathfrak{C}(f)$ is an equivalence, the induced map on sets of homotopy classes

$$[*,X]_{E^1} \to [*,Y]_{E^1}$$

is a bijection. Notice that

$$[*,X]_{E^1} \cong \pi_0 \widehat{\Theta[\mathcal{C}]}(E^\bullet, X),$$

and since $E^n = \mathcal{H}\,\mathrm{cosk}_0\,\Delta^n$. By abuse of notation, we also denote the simplicial set $\mathrm{cosk}_0\,\Delta^n$ by E^n. We noticed earlier that $\mathcal{H}$ has a right adjoint, which we now denote by $\mathcal{N}$. Using this, we can rewrite the question as asking for the induced map to give a bijection

$$\pi_0 \widehat{\Delta}(E^n, \mathcal{N}X) \to \pi_0 \widehat{\Delta}(E^n, \mathcal{N}X),$$

which is the same as giving a bijection

$$[\Delta^0, \mathcal{N}X]_{E^1} \to [\Delta^0, \mathcal{N}X]_{E^1}$$

.

Notice also that the data classifying an equivalence in $\mathfrak{C}(X)$ all factor through the simplicial category $\mathfrak{C}(X)_{*_\mathcal{C}}$ obtained by evaluating each of the Hom objects at the terminal object $*_\mathcal{C}$ of $\mathcal{C}$. It is an easy exercise to see that

$$\mathfrak{C}(X)_{*_C} \cong \mathfrak{C}_\Delta(\mathcal{N}X),$$

but since $\mathcal{N}X$ is quite clearly a quasicategory, the claim follows immediately from the ordinary case. This implies that f is a horizontal Dwyer-Kan equivalence, and therefore by the previous proposition, a horizontal Joyal equivalence, which concludes the proof. $\square$

Theorem 11.4. *The Quillen pair*

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \overset{\mathfrak{C}}{\underset{\mathfrak{N}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}}$$

is a Quillen equivalence.

Proof. All we have left to show is that the derived unit of the adjunction,

$$X \to \mathfrak{N}\mathfrak{C}(X) \to \mathfrak{N}\mathcal{D}$$

is a weak equivalence for all presheaves X on $\Theta[\mathcal{C}]$, where $\mathfrak{C}(X) \to \mathcal{D}$ is a weak equivalence and $\mathcal{D}$ is fibrant. However, by the previous proposition, we see that it suffices to show that the map

$$\mathfrak{C}(X) \to \mathfrak{C}\mathfrak{N}\mathfrak{C}(X) \to \mathfrak{C}\mathfrak{N}\mathcal{D}$$

is a weak equivalence. We obtain a naturality square from the counit

$$
\begin{array}{ccc}
\mathfrak{C}\mathfrak{N}\mathfrak{C}(X) & \longrightarrow & \mathfrak{C}\mathfrak{N}\mathcal{D} \\
\downarrow & & \downarrow {\scriptstyle\sim} \\
\mathfrak{C}(X) & \xrightarrow{\ \sim\ } & \mathcal{D}
\end{array}
,
$$

in which the indicated arrows are equivalences (for the bottom horizontal, this was by choice, and for the righthand vertical, it comes from Theorem 10.3. But if we precompose with the unit map $\mathfrak{C}\eta_X : \mathfrak{C}(X) \to \mathfrak{C}\mathfrak{N}\mathfrak{C}(X)$, the lefthand arrow becomes the identity by the triangle identities, which proves the claim by 3-for-2. $\square$

12. THE $(\mathcal{C}, setS)$-ENRICHED MODEL STRUCTURE

While our presentation of the horizontal Joyal model structure comes mainly from David Oury's thesis [Our10], what follows is independent, making use of the resolution technology we developed in the previous section to give a simple and satisfying story. Suppose $\mathcal{M} = (\mathcal{C}, \mathscr{S})$ is a Cartesian presentation in the sense of Rezk, where $\mathscr{S}$ is a set of monomorphisms of $\mathrm{Psh}_\Delta(\mathcal{C})$ such that the left-Bousfield localization of $\mathrm{Psh}_\Delta(\mathcal{C})_{\mathrm{inj}}$ at $\mathscr{S}$ is a Cartesian model category.

Recall that we had a number of functorial cosimplicial objects

$$C^\bullet_{(-)}(\bullet) : \Delta \times \mathcal{C} \to \widehat{\Theta[\mathcal{C}]}_{*,*},$$

such that $C^\bullet_{(-)}(\bullet)$ was a cosimplicial resolution of $[1](\bullet)$, which is a Reedy cofibrant diagram $\mathcal{C} \to \widehat{\Theta[\mathcal{C}]}_{*,*}$. Since $\widehat{\Theta[\mathcal{C}]}_{*,*}$ is cocomplete, $C^\bullet_{(-)}(\bullet)$ extends to a cocontinuous functor

$$\Sigma : \mathrm{Psh}_\Delta(\mathcal{C}) \to \widehat{\Theta[\mathcal{C}]}_{*,*}.$$

Proposition 12.1. *The functor $\Sigma_{(-)}$ is left-Quillen when $\widehat{\Theta[\mathcal{C}]}_{*,*}$ is equipped with the horizontal Joyal model structure.*

Proof. It clearly preserves cofibrations, so it suffices to show that its right adjoint preserves fibrations between fibrant objects. However, this is clear, since the right adjoint sends a bi-pointed formal $\mathcal{C}$ quasicategory $X_{x,y}$ to $\mathrm{Map}_X(x,y)$, which we saw sends horizontal Joyal fibrations to injective fibrations of simplicial presheaves on $\mathcal{C}$. $\square$

Corollary 12.2. *The functor $\Sigma_{(-)}$ is independent up-to-homotopy of choice of resolution $C^\bullet_{(-)}(\bullet)$.*

Proof. Since simplicial presheaves are always canonically the homotopy-colimit of their representables, and since left-Quillen functors send homotopy-colimits to homotopy-colimits, it suffices to show that $\Sigma_{(-)}(\Delta^n \times c)$ is independent up-to-homotopy. But this is clear since all $C^\bullet_{(-)}(\bullet)$ are connected by natural zig-zags of natural weak equivalences, since they are all cosimplicial resolutions of the same functor $[1](\bullet)$. $\square$

We can therefore, without any worry, denote $\Sigma_{(-)}$ simply by Σ. Then we define the following model structure:

Definition 12.3. If $\mathcal{M} = (\mathcal{C}, \mathscr{S})$ is a Cartesian presentation, we define the model category $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$ to be the left-Bousfield localization of $\widehat{\Theta[\mathcal{C}]}_{\text{hJoyal}}$ at the set $\Sigma(\mathscr{S})$, where we call the fibrant objects $\mathcal{M}$-*enriched quasicategories* or simply $\mathcal{M}$-*quasicategories*.

Proposition 12.4. *Let $\mathscr{B}$ denote the set of simplicial boundary inclusions. Then a formal $\mathcal{C}$-quasicategory is an $\mathcal{M}$-quasicategory if and only if it has the right lifting property with respect to*

$$\Sigma(\mathscr{B} \times^{\lrcorner} \mathscr{S}).$$

Proof. Let $E : \widehat{\Delta} \to \widehat{\Theta[\mathcal{C}]}$ be the left Kan extension of the cosimplicial object $E^\bullet$. We know by the construction of the left-Bousfield localization of Cisinski model categories that a formal $\mathcal{C}$-quasicategory X is $\Sigma(\mathscr{S})$-local if and only if it has the right-lifting property with respect to $E(\mathscr{B}) \times^{\lrcorner} \Sigma(\mathscr{S})$, but this occurs only when for every $s : A \to B$ in $\mathscr{S}$, the map $X^{\Sigma(f)} : X^{\Sigma(B)} \to X^{\Sigma(A)}$ has the right lifting property with respect to $E(\mathscr{B})$.

By we claim that by adjunction, this happens if and only if $\mathcal{G}(X^{\Sigma(f)})$ is a trivial fibration, where $\mathcal{G}$ is the functor sending a formal $\mathcal{C}$-quasicategory X to the Kan core of the underlying quasicategory $\mathcal{N}(X)$. To see this, notice that since all of the maps $f \in \mathscr{S}$ are monic, and since Σ preserves cofibrations, it follows that the maps $X^{\Sigma(f)}$ are all horizontal Joyal fibrations because the horizontal Joyal model structure is Cartesian. Then, since $\mathcal{N}$ sends horizontal Joyal fibrations between formal $\mathcal{C}$-quasicategories to Joyal fibrations of quasicategories, and since the Kan core of a Joyal fibration between quasicategories is a Kan fibration of Kan complexes, it suffices to show that the Kan fibration $\mathcal{G}(X^f)$ is a trivial fibration as we claimed.

Notice that $\mathcal{G}(X^\bullet)$ is exactly the simplicial mapping space $\text{hMap}^{\Delta}(\bullet, X)$. For any two vertices x, y of X, it suffices therefore to show that $\text{hMap}^{\Delta}_{*,*}(\Sigma(f), X_{x,y})$ is a trivial fibration, since a Kan fibration is a trivial fibration if and only if it has contractible fibres. But $\text{hMap}_{*,*}(\Sigma(f), X_{x,y})$ is exactly $\text{hMap}^{\Delta}(f, \text{Map}_X(x, y))$, which is a trivial fibration if and only if $\text{Map}_X(x, y)$ has the right lifting property with respect to $b \times^{\lrcorner} f$ where b is a simplicial boundary inclusion, which proves the proposition. $\qquad\square$

Corollary 12.5. *A formal $\mathcal{C}$-quasicategory X is an $\mathcal{M}$-quasicategory if and only if $\text{Map}_X(x, y)$ is $\mathscr{S}$-local for all pairs of vertices x, y in X.*

Theorem 12.6. *For any cartesian presentation $\mathcal{M} = (\mathcal{C}, \mathscr{S})$, the model category $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}}$ is Cartesian-closed.*

Proof. This is exactly [Rez10, Proposition 8.5]. $\qquad\square$

In what follows, let Σ be Σ_R.

Proposition 12.7. *The pair $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}} \overset{\mathfrak{C}}{\underset{\mathfrak{N}}{\rightleftarrows}} \mathbf{Cat}_{\text{Psh}_\Delta(\mathcal{C})_{\mathscr{S}}}$ is a Quillen pair.*

Proof. It suffices to show that $\mathfrak{N}$ preserves fibrant objects by the properties of the left-Bousfield localization. Since the coherent nerve of any fibrant $\text{Psh}_\Delta(\mathcal{C})$-enriched category $\mathcal{D}$ is already a formal $\mathcal{C}$-quasicategory, it suffices to show that $\mathfrak{N}\mathcal{D}$ has the right-lifting property with respect to $\Sigma(\mathscr{B} \times^{\lrcorner} \mathscr{S})$. This will be true so long as the maps $\mathfrak{C}(\Sigma(\mathscr{B} \times^{\lrcorner} \mathscr{S}))$ are all weak equivalences. To see this, let $\mathbf{2}(A)$ for any simplicial presheaf A on $\mathcal{C}$ denote the $\text{Psh}_\Delta(\mathcal{C})$-enriched category whose objects are $\{0, 1\}$ and where

$$\mathbf{2}(A)(x, y) = \begin{cases} * & \text{if } x = y \\ A & \text{if } x < y \\ \emptyset & \text{otherwise} \end{cases} .$$

For all $n \geq 0$ and $c \in \mathcal{C}$, there is a natural weak equivalence

$$\mathfrak{C}(\Sigma(\Delta^n \times c))(\alpha, \omega) \cong Q^n \times c \xrightarrow{\sim} \Delta^n \times c \cong \mathbf{2}(c \times \Delta^n)(0, 1).$$

Following [Lur09, Proposition 2.2.2.7], We define a realization

$$|\bullet|_Q : \mathrm{Psh}_\Delta(\mathcal{C}) \to \mathrm{Psh}_\Delta(\mathcal{C})$$

by left Kan extension of the functor $\Delta^n \times c \mapsto Q^n \times c$ along the Yoneda embedding. Let $\mathcal{A}$ denote the class of simplicial presheaves A on $\mathcal{C}$ such that the map

$$|A|_Q \to A$$

is an injective equivalence. This class is closed under filtered colimits, since injective weak equivalences are closed under filtered colimits, so it suffices to consider the case where A has finitely many nondegenerate representable cells $[n] \times c$. Since Δ and $\mathcal{C}$ are regular Cartesian Reedy, so is their product by [Cis06, 8.2.7], and the boundary of a representable cell is given by the formula

$$\partial(\Delta^n \times c) = \partial\Delta^n \times c \cup \Delta^n \times \partial c.$$

We work by induction on Reedy dimension and number of cells. If $A = \emptyset$, we are done, since the map in question is the identity. Otherwise, suppose

$$A = A' \coprod_{\partial(\Delta^n \times c)} \Delta^n \times c.$$

This is a homotopy pushout since $\partial(\Delta^n \times c) \to \Delta^n \times c$ is an injective cofibration. Similarly,

$$|A|_Q = |A'| \coprod_{|\partial(\Delta^n \times c)|_Q} |\Delta^n \times c|$$

is also a homotopy-pushout since $|\bullet|_Q$ preserves monomorphisms. Then we see that the map

$$|\Delta^n \times c|_Q = Q^n \times c \to \Delta^n \times c$$

is already a weak equivalence since $Q^n \to \Delta^n$ is a weak equivalence and the injective model structure is cartesian. The map

$$|\partial(\Delta^n \times c)|_Q \to \partial(\Delta^n \times c)$$

is a weak equivalence by the induction hypothesis, since the Reedy dimension of $\partial(\Delta^n \times c)$ is less than the dimension of $\Delta^n \times c$. Finally, we see that

$$|A'|_Q \to A'$$

is a weak equivalence since A' has one fewer nondegenerate cell than A and is therefore also covered in the induction hypothesis.

Therefore, the natural map

$$\mathfrak{C}(\Sigma(A)) \cong \mathbf{2}(|A|_Q) \xrightarrow{\sim} \mathbf{2}(A)$$

is a weak equivalence in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(C)_{\mathrm{inj}}}$ for all simplicial presheaves A on $\mathcal{C}$. From this, it follows that since $\mathbf{2}(b \times^{\lrcorner} f)$ is an $\mathcal{M}$-equivalence for any $f \in \mathscr{S}$, and since we have a natural equivalence of arrows

$$\mathfrak{C}(\Sigma(b \times^{\lrcorner} f)) \xrightarrow{\sim} \mathbf{2}(b \times^{\lrcorner} f),$$

then by 3-for-2, $\mathfrak{C}(\Sigma(b \times^{\lrcorner} f)$ is a weak equivalence, which proves left-Quillen functoriality. $\square$

Theorem 12.8. *The Quillen pair* $\widehat{\Theta[\mathcal{C}]}_{\mathcal{M}} \overset{\mathfrak{C}}{\underset{\mathfrak{N}}{\rightleftarrows}} \mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}}}$ *is a Quillen equivalence.*

Proof. It suffices to show that $\mathfrak{C}$ is homotopy-conservative, so let $f : X \to Y$ be a map in $\widehat{\Theta[\mathbb{C}]}$ such that $\mathfrak{C}(f)$ is an equivalence in $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathbb{C})_{\mathcal{M}}}$. Using the same argument as in Section 11, we reduce to the case where $f : X \to Y$ is a map between $\mathcal{M}$-quasicategories. Since $\mathcal{M}$-quasicategories are also formal $\mathbb{C}$-quasicategories, we can apply Corollary 10.2 to obtain a natural zig-zag of weak equivalences between $\mathrm{Map}_X(x, y)$ and $\mathfrak{C}(X)(x, y)$ for any pair of vertices x, y of X. By 3-for-2 and since

$$\mathfrak{C}(X)(x, y) \to \mathfrak{C}(Y)(fx, fy)$$

was assumed to be an $\mathcal{M}$-equivalence, we see that the map $\mathrm{Map}_X(x, y) \to \mathrm{Map}_Y(fx, fy)$ must also be an $\mathcal{M}$-equivalence. In fact, since both $\mathrm{Map}_X(x, y)$ and $\mathrm{Map}_Y(fx, fy)$ are local, this map is actually an equivalence for $\mathbf{Cat}_{\mathrm{Psh}_\Delta(\mathbb{C})_{\mathrm{inj}}}$. The argument showing that f is bijective on iso-components is the same as in the proof of Proposition 11.3 by passing to the underlying quasicategory. Therefore, it follows that f is a horizontal Dwyer-Kan equivalence, which concludes the proof. $\qquad\square$

13. The Yoneda embedding and Yoneda's lemma

We need the following easy lemma:

Lemma 13.1. *There is a natural isomorphism* $\mathfrak{C}(X^{\mathrm{op}}) \cong \mathfrak{C}(X)^{\mathrm{op}}$.

Proof. It suffices to check on representables, and this is left as an easy exercise to the reader. $\qquad\square$

Before we give a construction of the Yoneda embedding and a proof of Yoneda's lemma for $\mathcal{M}$-quasicategories, we need two lemmas from [Lur09]. We fix a cartesian presentation $\mathcal{M} = (\mathbb{C}, \mathcal{S})$ for the remainder of this section.

Proposition 13.2. [Lur09, 4.2.4.4] *Let* $X \in \widehat{\Theta[\mathbb{C}]}$ *be a cellular* $\mathbb{C}$*-set,* $\mathcal{D}$ *a small* $\mathrm{Psh}_\Delta(\mathbb{C})$*-enriched category, and let* $\mathfrak{C}(X) \to \mathcal{D}$ *be an equivalence of* $\mathrm{Psh}_\Delta(\mathbb{C})_{\mathcal{S}}$*-enriched categories. Suppose* $\mathbf{A}$ *is a* $\mathrm{Psh}_\Delta(\mathbb{C})_{\mathcal{S}}$*-enriched model category, and let* $\mathcal{U}$ *be* $\mathcal{D}$*-chunk (see* [Lur09, A.3.4.9] *for the definition). Then the induced map*

$$\mathfrak{N}((\mathcal{U}^{\mathcal{D}})^\circ) \to \mathfrak{N}(\mathcal{U}^\circ)^X$$

is an equivalence of $\mathcal{M}$*-quasicategories.*

Proof. Although we have altered the statement slightly, the only result used in the proof in [Lur09] that doesn't hold for all excellent monoidal model categories is [Lur09, 2.2.5.1], but the analogue of this is exactly Theorem 12.8. $\qquad\square$

Proposition 13.3. [Lur09, 4.2.4.7] *Let* $\mathcal{I}$ *be a fibrant* $\mathrm{Psh}_\Delta(\mathbb{C})_{\mathcal{S}}$*-enriched category,* X *an object of* $\widehat{\Theta[\mathbb{C}]}$*, and* $p : \mathfrak{N} \to X$ *be any map. Then we can find the following:*

- *A fibrant* $\mathrm{Psh}_\Delta(\mathbb{C})_{\mathcal{S}}$*-enriched category* $\mathcal{D}$.
- *An enriched functor* $P : \mathcal{I} \to \mathcal{D}$.
- *A map* $j : X \to \mathfrak{N}(\mathcal{D})$ *that is a weak equivalence in* $\widehat{\Theta[\mathbb{C}]}_{\mathcal{M}}$.
- *An equivalence between* $j \circ p$ *and* $\mathfrak{N}(P)$ *as objects of the* $\mathcal{M}$*-quasicategory* $\mathfrak{N}(\mathcal{D})^{\mathfrak{N}(\mathcal{I})}$.

Proof. No change to the proof of [Lur09, 4.2.4.7] is needed. $\qquad\square$

We begin by constructing the Yoneda embedding:

Definition 13.4. Let X be a $\mathbb{C}$-cellular set, and let $\Phi : \mathfrak{C}(X) \xrightarrow{\sim} \mathcal{D}$ be an $\mathcal{M}$-enriched fibrant replacement. Since $\mathcal{D}$ is fibrant, the functor

$$\mathrm{Hom}_{\mathcal{D}} : \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Psh}_\Delta(\mathbb{C})$$

factors through the full subcategory $\mathrm{Psh}_\Delta(\mathcal{C})^\circ$. By [Lur09, Corollary A.3.4.14], this gives rise up to homotopy to a universal map

$$J_\mathcal{D} : \mathcal{D} \to \left(\mathrm{Psh}_\Delta(\mathcal{C})^{\mathcal{D}^{\mathrm{op}}}\right)^\circ_{\mathrm{proj}}$$

that is fully faithful up to homotopy, since it is homotopic to the enriched Yoneda embedding. Then we have a map

$$\mathfrak{C}(X \times X^{\mathrm{op}}) \xrightarrow{\mathfrak{C}(p_1) \times \mathfrak{C}(p_2)} \mathfrak{C}(X) \times \mathfrak{C}(X)^{\mathrm{op}} \xrightarrow{\Phi \times \Phi^{\mathrm{op}}} \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_\mathcal{D}}, \mathrm{Psh}_\Delta(\mathcal{C})^\circ_\mathscr{S})$$

which yields by adjunction

$$X \to \mathfrak{N}(\mathrm{Psh}_\Delta(\mathcal{C})^\circ_\mathscr{S})^{X^{\mathrm{op}}}.$$

We denote $\mathfrak{N}(\mathrm{Psh}_\Delta(\mathcal{C})^\circ_\mathscr{S})$ simply by $\mathcal{M}$, and finally, we obtain *the Yoneda embedding* for $\mathcal{M}$-quasicategories:

$$j : X \to \mathcal{M}^{X^{\mathrm{op}}}.$$

Let $\mathcal{P}(X)$ denote the large $\mathcal{M}$-quasicategory $\mathcal{M}^{X^{\mathrm{op}}}$, and let $\mathcal{M}^+$ denote the coherent nerve of the huge enriched category of not-necessarily-small $\mathscr{S}$-local injectively fibrant simplicial presheaves on $\mathcal{C}$.

Definition 13.5. We say that a functor $F : X^{\mathrm{op}} \to \mathcal{M}$ is *representable* if the object it classifies belongs to the essential image of the Yoneda embedding $j : X \to \mathcal{P}(X)$. If $x : * \to X$ is a vertex of X, we denote the associated representable functor by h_x.

Proposition 13.6 (Yoneda embedding). [Lur09, 5.1.3.1] *The Yoneda embedding is fully faithful.*

Proof. First, let $\Phi : \mathfrak{C}(X^{\mathrm{op}}) \xrightarrow{\sim} \mathcal{D}$ be a fibrant replacement. We have an equivalence by Propisition 13.2

$$j'' : \mathfrak{N}\left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S})^\mathcal{D}_{\mathrm{proj}}\right)^\circ\right) \xrightarrow{\sim} \mathcal{M}^{\mathfrak{N}(\mathcal{D})},$$

and since $\mathcal{M}$ is fibrant and $\Psi : X^{\mathrm{op}} \to \mathfrak{N}\mathcal{D}$, the adjunct of Φ is an equivalence between cofibrant objects, it follows that the induced map $\mathcal{M}^{\mathfrak{N}\mathcal{D}} \to \mathcal{P}(X)$ is an equivalence between fibrant objects, since the model structure is Cartesian. Since X is cofibrant, it follows that there is a map $h : X \to \mathcal{M}^{\mathfrak{N}\mathcal{D}}$ such that the composite

$$X \xrightarrow{h} \mathcal{M}^{\mathfrak{N}\mathcal{D}} \xrightarrow{\mathcal{M}^\Psi} \mathcal{P}(X)$$

is equivalent to j, and again, since $\mathfrak{N}\left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S})^\mathcal{D}_{\mathrm{proj}}\right)^\circ\right)$ is fibrant and j'' is a weak equivalence between fibrant objects, we can find a map

$$j' : X \to \mathfrak{N}\left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S})^\mathcal{D}_{\mathrm{proj}}\right)^\circ\right)$$

such that the composite

$$X \xrightarrow{j'} \mathfrak{N}\left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S})^\mathcal{D}_{\mathrm{proj}}\right)^\circ\right) \xrightarrow{j''} \mathcal{M}^{\mathfrak{N}\mathcal{D}} \xrightarrow{\sim} \mathcal{P}(X)$$

is equivalent to j. It suffices to show that the map j' is fully faithful. Let

$$J : \mathfrak{C}(X) \to \left((\mathrm{Psh}_\Delta(\mathcal{C})_\mathscr{S})^\mathcal{D}_{\mathrm{proj}}\right)^\circ$$

be the adjunct of j'. Then we see that J is equivalent to the composite $J_{\mathcal{D}^{\mathrm{op}}} \circ \Phi^{\mathrm{op}}$, where $J_{\mathcal{D}^{\mathrm{op}}}$ is fully-faithful and Φ^{op} is an equivalence, which concludes the proof. □

Proposition 13.7 (Yoneda's Lemma). [Lur09, 5.5.2.1] *Let X be a small $\mathcal{C}$-cellular set, and let $f : X^{\mathrm{op}} \to \mathcal{M}$ be an object of $\mathcal{P}(X)$. Then let $F : \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{M}^+$ be the functor represented by f. Then the composite*

$$X^{\mathrm{op}} \xrightarrow{j_X^{\mathrm{op}}} \mathcal{P}(X)^{\mathrm{op}} \xrightarrow{F} \mathcal{M}^+$$

is equivalent to f.

Proof. By Proposition 13.3, we can choose a small fibrant $\mathcal{M}$-enriched category $\mathcal{D}$ and an equivalence $\Phi : X^{\mathrm{op}} \to \mathfrak{N}(\mathcal{D})$ such that $f \sim \mathfrak{N}(f') \circ \Phi$ for some $f' : \mathcal{D} \to \mathrm{Psh}_\Delta(\mathcal{C})^\circ_{\mathscr{S}}$. Without loss of generality, we can assume that f' is a projectively cofibrant diagram. Using Proposition 13.2, we have an equivalence of $\mathcal{M}$-quasicategories

$$\Psi : \mathfrak{N}\left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}})^{\mathcal{D}}_{\mathrm{proj}}\right)^\circ\right) \xrightarrow{\sim} \mathcal{P}(X).$$

We observe that $F \circ \Psi^{\mathrm{op}}$ can be identified with the coherent nerve of the the map

$$G : \left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}})^{\mathcal{D}}_{\mathrm{proj}}\right)^\circ\right)^{\mathrm{op}} \to \left(\mathrm{Psh}_\Delta(\mathcal{C})^+_{\mathscr{S}}\right)^\circ$$

represented by f'. The Yoneda embedding factors through Ψ by the adjunct of the composite

$$j' : \mathfrak{C}(X) \xrightarrow{\Phi^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}} \hookrightarrow \left((\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}})^{\mathcal{D}}_{\mathrm{proj}}\right)^\circ,$$

so it follows that $F \circ j^{\mathrm{op}}$ can be identified with the adjunct of

$$\mathfrak{C}(X)^{\mathrm{op}} \xrightarrow{(j')^{\mathrm{op}}} \left(\left((\mathrm{Psh}_\Delta(\mathcal{C})_{\mathscr{S}})^{\mathcal{D}}_{\mathrm{proj}}\right)^\circ\right)^{\mathrm{op}} \xrightarrow{G} \left(\mathrm{Psh}_\Delta(\mathcal{C})^+_{\mathscr{S}}\right)^\circ.$$

This composite is equal to the f' we started with, so its coherent nerve is equivalent to f. $\square$

14. Weighted limits and colimits

We fix a Cartesian presentation $\mathcal{M} = (\mathcal{C}, \mathscr{S})$. Our presentation here follows the one given in [na18].

Definition 14.1. Let D be a small $\mathcal{C}$-cellular set, that is, an object of $\widehat{\Theta[\mathcal{C}]}$, and suppose we have a diagram $f : D \to \mathcal{M}$ and a *weight* $W : D \to \mathcal{M}$. If X is an $\mathcal{M}$-quasicategory and $f : D \to X$ is a diagram, we define the map $h_f : X^{\mathrm{op}} \to D$ to be the adjunct of the composite

$$D \to X \xrightarrow{j} \mathcal{M}^{X^{\mathrm{op}}},$$

and we define the *weight representation* $h^W : \mathcal{M}^D \to \mathcal{M}^+$ be the map co-representing W as an object of $\mathcal{M}^D$. Then if the composite $h^W \circ h_f : X^{\mathrm{op}} \to \mathcal{M}^+$ is representable, we call its representing object *the limit of f weighted by W*, denoted by $\lim^W f$.

Dually, coweights for colimits are diagrams $D^{\mathrm{op}} \to \mathcal{M}$, and their coweighting functors are their associated corepresentable functors $\mathcal{M}^{D^{\mathrm{op}}} \to \mathcal{M}^+$. Given a diagram $f : D \to X$, the *colimit of f weighted by W* is defined to be an object of X that corepresents the weighted limit of f^{op}, which is denoted by $\mathrm{colim}^W f$.

Proposition 14.2. *The large $\mathcal{M}$-quasicategory $\mathcal{M}$ has all small weighted limits and colimits.*

Proof. Given $f : D \to \mathcal{M}$ and a weight $W : D \to \mathcal{M}$, it is straightforward to see that a representing object for the weighted limit is exactly

$$\mathrm{Map}_{\mathcal{M}^D}(W, f),$$

by unwinding the definitions. $\square$

For now, we don't have much to say for this particular application. If other definitions are proposed for weighted limits and colimits, they should be equivalent to this one.

15. Examples

The only examples we really care about are the cases where $\mathcal{C} = \Theta_n$ for $0 \le n \le \omega$ and where $\mathcal{S}$ is the set of generating anodynes for the model structure on weak n-categories. We invite the reader to consider other applications. We expect that a simple application would be to consider the left-Bousfield localization of spaces at homology equivalences, but we aren't certain if this is a Cartesian model structure.

Also note that our definitions of weighted limits and colimits do not work for computing lax and oplax weighted limits and colimits in weak ω-categories (taking $\mathcal{C}$ to be $\Theta = \Theta_\omega$ and $\mathcal{S}$ to be the union of all of the generating weak equivalences for the Rezk model structure). The problem is that the lax Gray tensor product has not yet been shown to be homotopy-invariant (in particular, a left-Quillen bifunctor), so the function complexes of lax and oplax natural transformations are themselves not yet known to be homotopy invariant. This is why we encourage people in the future to refer to weighted limits and colimits in these cases as *weighted pseudolimits* and *weighted pseudocolimits* respectively.

Appendix A.1. The corner tensor construction

The overwhelming majority of this section is due to Oury, although we had to redo some of the proofs, since they contained mistakes. Following [Our10, 3.1] we define the corner tensor, a vast generalization of the corner product.

Definition A.1.1. Let $\mathcal{V}$ be a cocomplete symmetric monoidal closed category, and let all categories, functors, and natural transformations in what follows to be $\mathcal{V}$-enriched. Suppose we have a category $\mathcal{T}$ and an n-ary functor

$$\wedge : \mathcal{T}^{\otimes n} \to \mathcal{T}$$

be a functor. Let $(\mathcal{A}_i)_{i=1}^n$ and $\mathcal{D}$ be categories admitting enough colimits such that all tensors with Hom_T exist and coends over $\mathcal{T}$ exist. Let

$$\nabla_i : \mathcal{A}_i \otimes \mathcal{A}_i \to \mathcal{A}, \qquad \nabla : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}, \qquad \square : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \to \mathcal{D}$$

be functors. Then we define the following functors:

$$\nabla_i^\lrcorner : \mathcal{A}_i^{\mathcal{T}} \otimes \mathcal{A}_i^{\mathcal{T}} \to \mathcal{A}^{\mathcal{T}}, \qquad \nabla^\lrcorner : \mathcal{D}^{\mathcal{T}} \otimes \mathcal{D}^{\mathcal{T}} \to \mathcal{D}^{\mathcal{T}}, \qquad \square^\lrcorner : \mathcal{A}_1^{\mathcal{T}} \otimes \cdots \otimes \mathcal{A}_n^{\mathcal{T}} \to \mathcal{D}^{\mathcal{T}}$$

by the Day convolution, for example,

$$\square^\lrcorner(M_1, \ldots, M_n)t = \int^{u_1, \ldots, u_n} \mathcal{T}(\wedge(u_1, \ldots, u_n), t) \otimes \square(M_1(u_1), \ldots, M_n(u_n)).$$

Lemma A.1.2. *The functor $\square^\lrcorner$ preserves all colimits preserved by $\square$ in each variable and by $\otimes$ in its second variable.*

Proof. By coend manipulation. $\square$

We specialize now to the case where $\mathcal{V}$ is just the category of sets and where $\mathcal{T} = [1]$ is the 1-simplex. The functor $\wedge : [1]^n \to [1]$ is given by taking the infimum.

Definition A.1.3. Given $\square : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{D}$, where each category appearing is cocomplete, we define the *corner tensor* $\square^\lrcorner : \mathcal{A}_1^{[1]} \times \cdots \times \mathcal{A}_n^{[1]} \to \mathcal{D}^{[1]}$ by the formula

$$\square^\lrcorner(f_1, \ldots, f_n)(t) = \int^{u_1, \ldots, u_n} [1](u_1 \wedge \cdots \wedge u_n, t) \cdot \square(f_1(u_1), \ldots, f_n(u_n)).$$

If $(g, h) : f_i \to f_i'$ is a commutative square, let

$$(g^\lrcorner, h) : \square^\lrcorner(f_1, \ldots, f_i, \ldots, f_n) \to \square^\lrcorner(f_1, \ldots, f_i', \ldots, f_n)$$

be the induced commutative square.

Lemma A.1.4. *If any $f_i : A_i \to B_i$ is an identity map, then $\square^{\lrcorner}(f_1, \ldots, f_n) = \mathrm{id}$*

Proof. Assume for simplicity that $i = 1$. Then the result follows by setting

$$U(s,t) = \int^{u_1,\ldots,u_n} ([1](u_1, s) \times [1](u_2 \wedge \cdots \wedge u_n, t)) \cdot \square(f_1(u_1), f_2(u_2), \ldots, f_n(u_n)).$$

Notice that by Yoneda reduction,

$$\int^{s,t} [1](s \wedge t, x) \times [1](u_1, s) \times [1](u_2 \wedge \cdots \wedge u_n, t) = [1](u_1 \wedge u_2 \wedge \cdots \wedge u_n, x),$$

so we have that

$$\square^{\lrcorner}(f_1, \ldots, f_n)x = \int^{s,t} [1](s \wedge t, x)U(s,t),$$

which exhibits

$$\square^{\lrcorner}(f_1, \ldots, f_n)0 = U(0,1) \coprod_{U(0,0)} U(1,0),$$

but $U(0,1) = \square(B_1, \ldots, B_n)$, but the map $U(0,0) \to (U(1,0)$ is the identity by cofinality, so the map $U(0,1) \to \square^{\lrcorner}(f_1, \ldots, f_n)0$ is the pushout of an identity map, and the map $U(0,1) \to \square^{\lrcorner}(f_1, \ldots, f_n)1$ is also the identity, so it follows that the map $\square^{\lrcorner}(f_1, \ldots, f_n)$ must also be the identity. $\square$

Lemma A.1.5. *Suppose $\square$ preserves pushouts in the first argument and $(g, h) : f_1 \to f_1'$ is a pushout square in $\mathcal{A}_1$, then the induced square*

$$(g^{\lrcorner}, h) : \square^{\lrcorner}(f_1, \ldots, f_n) \to \square^{\lrcorner}(f_1', \ldots, \ldots, f_n)$$

is a pushout square.

Proof. Let Q denote $\square^{\lrcorner}(f_1, \ldots, f_n)(0)$ and $Q' = \square^{\lrcorner}(f_1', \ldots, f_n)$, with $g^{\lrcorner} : Q \to Q'$. Let $(\alpha, \alpha') : \square(g, B_2, \ldots, B_n) \to g^{\lrcorner}$ be the induced commutative square. Then we can factor the square

$$\square(f_1, B_2, \ldots, B_n), \square(f_1', B_2, \ldots, B_n)) = (g^{\lrcorner}, h) \circ (\alpha, \alpha'),$$

where the outer square is a pushout square by the fact that $\square$ preserves colimits in the first argument. Then to prove the lemma, it suffices to prove the claim that (α, α') is a pushout square. Then we define

$$U(s,t) = \int^{u_1,\ldots,u_n} ([1](u_1, s) \times [1](u_2 \wedge \cdots \wedge u_n, t)) \cdot \square(f_1(u_1), f_2(u_2), \ldots, f_n(u_n)).$$

and

$$U'(s,t) = \int^{u_1,\ldots,u_n} ([1](u_1, s) \times [1](u_2 \wedge \cdots \wedge u_n, t)) \cdot \square(f_1'(u_1), f_2(u_2), \ldots, f_n(u_n)).$$

Then notice that $\alpha : \square(A_1, B_2, \ldots, B_n) \to Q$ is the pushout of $u : U(0,0) \to U(1,0)$ and $\alpha' : \square(A_1', B_2, \ldots, B_n) \to Q$ is the pushout of $u' : U'(0,0) \to U'(1,0)$. So we have a commutative cube

$$u \to \alpha \to \alpha' = u \to u' \to \alpha',$$

where $u \to \alpha$ and $u' \to \alpha'$ are pushouts. Then by the pasting law for pushouts, it suffices to show that the map $u \to u'$ is a pushout. By cofinality, we can see that the map $u : U(0,0) \to U(1,0)$ is induced by the natural family of maps

$$\square(f_1, f_2(u_2), \ldots, f_n(u_n)) : \square(A_1, f_2(u_2), \ldots, f_n(u_n)) \to \square(B_1, f_2(u_2), \ldots, f_n(u_n)),$$

that $u' : U'(0,0) \to U'(1,0)$ is induced by the natural family of maps

$$\square(f_1', f_2(u_2), \ldots, f_n(u_n)) : \square(A_1', f_2(u_2), \ldots, f_n(u_n)) \to \square(B_1', f_2(u_2), \ldots, f_n(u_n)),$$

that $U(0,0) \to U'(0,0)$ is induced by the natural family of maps

$$\square(g, f_2(u_2), \ldots, f_n(u_n)) : \square(A_1, f_2(u_2), \ldots, f_n(u_n)) \to \square(A_1', f_2(u_2), \ldots, f_n(u_n)),$$

and that $U(1,0) \to U'(1,0)$ is induced by the natural family of maps

$$\square(h, f_2(u_2), \ldots, f_n(u_n)) : \square(B_1, f_2(u_2), \ldots, f_n(u_n)) \to \square(B_1', f_2(u_2), \ldots, f_n(u_n)).$$

But these form a natural pushout square, which implies that $u \to u'$ is a pushout because $\square$ preserves colimits in the first argument and the pushout moves outside of the coend. This proves the claim that $\alpha \to \alpha'$ is a pushout, which proves the lemma. $\qquad\square$

Note A.1.6. The above holds for $i \neq 1$ as well, but for simplicity of notation, we proved it only in the case $i = 1$.

Corollary A.1.7. *If $\square$ preserves pushouts in the ith argument, then if f_i is a pushout of f_i', it follows that $\square^{\lrcorner}(f_1, \ldots, f_i, \ldots, f_n)$ is a pushout of $\square^{\lrcorner}(f_1, \ldots, f_i, \ldots, f_n)$.*

Lemma A.1.8. *Suppose $(u, v) : h \to g$ is a pushout square in $\mathcal{A}^1$, and suppose that $(g, \mathrm{id}) : p \to q$ is a commutative square. Given family of maps $(\mathbf{f} = f_i \in \mathcal{A}_i^{[1]})_{i=2}^n$, let $Q_{\bullet, \mathbf{f}}$ denote the evaluation of $\square^{\lrcorner}(\bullet, \mathbf{f})$ at 0, that is, the domain. $g^{\lrcorner} : Q_{p, \mathbf{f}} \to Q_{q, \mathbf{f}}$ is a pushout of $\square^{\lrcorner}(h, \mathbf{f})$.*

Proof. The data give us commutative cube

$$h \xrightarrow{(h, \mathrm{id})} \mathrm{id} \xrightarrow{(v, qv)} q = h \xrightarrow{(u, qv)} p \xrightarrow{(g, \mathrm{id})} q,$$

and we can see it is a pushout in $\mathcal{A}_1^{[1]}$, since both its front and back faces are pushouts. Then applying $\square^{\lrcorner}(\bullet, \mathbf{f})$, its front and back faces remain pushouts, since $\square^{\lrcorner}$ preserves all colimits in each argument preserved by $\square$. Then the map $g^{\lrcorner} : Q_{p, \mathbf{f}} \to Q_{q, \mathbf{f}}$ is a pushout of the map $h^{\lrcorner} : Q_{h, \mathbf{f}} \to Q_{\mathrm{id}, \mathbf{f}}$, but by Lemma A.1.4, we see that $\square^{\lrcorner}(\mathrm{id}, \mathbf{f}) = \mathrm{id}$, so by commutativity, it follows that $h^{\lrcorner} = \square^{\lrcorner}(h, \mathbf{f})$, and therefore, $g^{\lrcorner}$ is a pushout of $\square^{\lrcorner}(h, \mathbf{f})$. $\qquad\square$

Assume in the sequel that $\square$ preserves connected colimits in each argument.

Lemma A.1.9 ([Our10, Lemma 3.10]). *Let $(\mathcal{J}_i)_{i=1}^n$ be a family of sets of morphisms of each $\mathcal{A}_i$. Then*

$$\square^{\lrcorner}(\mathcal{J}_1, \ldots, \mathrm{Cell}(\mathcal{J}_i), \ldots, \mathcal{J}_n) \subseteq \mathrm{Cell}(\square^{\lrcorner}(\mathcal{J}_1, \ldots, \mathrm{Cell}(\mathcal{J}_i), \ldots, \mathcal{J}_n)).$$

Proof. Fix some $1 \leq k \leq n$, and let $f_i : X_i \to Y_i \in \mathcal{J}_i$ for each $i \neq k$. Let f_k be a map in $\mathcal{A}_k$ that belongs to $\mathrm{Cell}(\mathcal{J}_k$. Then this gives the data of a cocontinous diagram $D : \alpha \to \mathcal{A}_k$ for an ordinal α with colimit C and structure maps $\phi_i : D(i) \to C$ such that $f_k = \phi_0$. Additionally, for any $i < \alpha$, the maps $g_i : D(i) \to D(i+1)$ are pushouts of maps belonging to $\mathcal{J}_k$. Let $D^+ : \alpha + 1 \to \mathcal{A}_k$ be the extension of D sending α to C and such that $D^+(i \to \alpha) = \phi_i$ for $i < \alpha$. Define $D^2 : \alpha + 1 \to \mathcal{A}_k^{[1]}$ by the rule $i \mapsto \phi_i$ for $i < \alpha$ and $\alpha \mapsto \mathrm{id}_C$ and sending $i \to i+1$ to the commutative square $(g_i, \mathrm{id}_c) : \phi_i \to \phi_i + 1$ and sending the map $i \to \alpha$ to the commutative square $(\phi_i, \mathrm{id}_C) : \phi_i \to \mathrm{id}_C$. Then the colimit of D^2 is id_C.

Denote $\square^{\lrcorner}(f_1, \ldots, \bullet, \ldots, f_n)$ by $\square(\mathbf{f}, \bullet)$ and the domain by by $Q_{\mathbf{f}, \bullet}$. Then since $\square^{\lrcorner}$ preserves connected colimits, it follows that

$$\mathrm{colim} \, \square^{\lrcorner}(\mathbf{f}, D^2) = \square^{\lrcorner}(\mathbf{f}, \mathrm{id}_C),$$

and so we have the structure maps of the colimiting cocone

$$\square^{\lrcorner}(\mathbf{f}, (\phi_i, \mathrm{id}_C)) : \square^{\lrcorner}(\mathbf{f}, \phi_i) \to \square^{\lrcorner}(\mathbf{f}, \mathrm{id}_C),$$

which exhibit (ϕ_0, id_C) as the transfinite composite of the diagram $\square^{\lrcorner}(\mathbf{f}, D^2)$. Then notice that

$$\begin{aligned}
\square^{\lrcorner}(f_1, \ldots, f_{k-1}, f_k, f_{k+1}, \ldots, f_n) &= \square^{\lrcorner}(f_1, \ldots, f_{k-1}, \phi_0, f_{k+1}, \ldots, f_n) \\
&= \square^{\lrcorner}(\mathbf{f}, \mathrm{dom}(\phi_0, \mathrm{id}_C)) \\
&= \mathrm{dom}\,\square^{\lrcorner}(\mathbf{f}, (\phi_0, \mathrm{id}_C)) \\
&= \square^{\lrcorner}(\mathbf{f}, \phi_0) \\
&= \square^{\lrcorner}(\mathbf{f}, \mathrm{id}_C) \circ \phi_0^{\lrcorner},
\end{aligned}$$

where $\phi_0^{\lrcorner} : Q_{\mathbf{f}, \phi_0} \to Q_{\mathbf{f}, \mathrm{id}_C}$ is the transfinite composite of the maps $g_i^{\lrcorner} : Q_{\mathbf{f}, \phi_i} \to Q_{\mathbf{f}, \phi_{i+1}}$. By A.1.4, $\square^{\lrcorner}(\mathbf{f}, \mathrm{id}_C) = \mathrm{id}$. Then it suffices to show that the $g_i^{\lrcorner}$ are pushouts of maps belonging to

$$square^{\lrcorner}(\mathcal{J}_1, \ldots, \mathcal{J}_k, \ldots, \mathcal{J}_n).$$

Since each g_i was a pushout of a morphism $h_i \in \mathcal{J}_k$, and since we have a commutative square $(g_i, \mathrm{id}) : \phi_i \to \phi_{i+1}$, we are in the situation of the previous lemma, which implies that each $g_i^{\lrcorner}$ is a pushout of $\square^{\lrcorner}(\mathbf{f}, j_i)$, which proves the proposition. $\square$

Corollary A.1.10. *There is an inclusion*

$$\square^{\lrcorner}(\mathrm{Cell}(\mathcal{J}_1), \ldots, \mathrm{Cell}(\mathcal{J}_n)) \subseteq \mathrm{Cell}(\square^{\lrcorner}(\mathcal{J}_1, \ldots, \mathcal{J}_n)).$$

Proof. Apply the previous lemma n times, using the fact that Cell is idempotent, since it is a closure operator. $\square$

Definition A.1.11. Let Rex_c denote the symmetric sub-multi-category of **Cat** whose objects are the categories admitting all finite colimits and whose k-morphisms are the k-fold functors that preserve finite connected colimits.

Observation A.1.12. The corner tensor construction, sending a multimorphism

$$F : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{D}$$

to its corner tensor

$$F^{\lrcorner} : \mathcal{A}_1^{[1]} \times \cdots \times \mathcal{A}_n^{[1]} \to \mathcal{D}^{[1]}$$

is a morphism of multicategories from Rex_c to itself. In particular, it is functorial with respect to the composition in Rex_c.

APPENDIX A.2. THE REGULAR REEDY STRUCTURE OF $\Theta[\mathcal{C}]$

In this section, we demonstrate that the class of monomorphisms is exactly the class of relative cell complexes for $\mathcal{M}$, which we will show coincides with the set of boundary maps for the regular Reedy category $\Theta[\mathcal{C}]$.

Proposition A.2.1. *The category $\Theta[\mathcal{C}]$ is a regular skeletal Reedy category whenever $\mathcal{C}$ is a regular Cartesian Reedy category. The dimension function of this regular Reedy category is given by*

$$\dim[n](c_1, \ldots, c_n) \stackrel{\mathrm{def}}{=} n + \dim_{\mathcal{C}} c_1 + \cdots + \dim_{\mathcal{C}} c_n.$$

Proof. It follows by [BR11, Proposition 4.4] that $\Theta[\mathcal{C}]$ is normal skeletal Reedy with the desired dimension function (see [Cis06, 8.1]). To prove that $\Theta[\mathcal{C}]$ is regular, it suffices to show that any nondegenerate section

$$(\alpha, \mathbf{f}) : [n](c_1, \ldots, c_n) \to [m](d_1, \ldots, d_m)$$

is monic. If $n > m$, then the map α is a simplicial degeneracy and therefore the map factors through $[m](c_{i_1}, \ldots, c_{i_m}$, which means that $(\alpha, \mathbf{f})$ cannot be nondegenerate.

Then we have two cases, when $n = m$ or $n < m$, and therefore either α is the identity or a simplicial face map. If $n = m$, then clearly $\alpha, \mathbf{f}$ is monic, since each nondegenerate section $c_i \to d_i$ must be monic by the regularity of $\mathcal{C}$.

If α is a composite of outer face maps, then $(\alpha, \mathbf{f})$ lands in $[n](d_1, \ldots, d_n)$ or $[n](d_{m-n}, \ldots, d_m)$, and then from the previous case together with the fact that those inclusions are monic. Otherwise, by induction on dimension, we can assume that α is the inclusion of a codimension 1 inner face map obtained by removing the kth vertex. Then $(\alpha, \mathbf{f})$ factors through the inclusion

$$V[n](d_1, \ldots, d_{k-1} \times d_k, \ldots, d_m) \subseteq [m](d_1, \ldots, d_m)$$

by a map $\mathrm{id}, \mathbf{g}$ where each g_i is a nondegenerate section. Then each g_i must be monic since for $i \neq k$, g_i is a nondegenerate section of a representable, which is monic, and for $i = k$, g_i is a nondegenerate section $c \to d_{k-1} \times d_k$, which is monic by the fact that $\mathcal{C}$ is regular Cartesian. $\square$

Corollary A.2.2. *The boundary $\partial[n](c_1, \ldots, c_n)$ can be computed using the corner-intertwiner (see A.3)*

$$Q = \mathrm{dom}\,\square_n^{\lrcorner}(\delta^n, \delta^{c_1}, \ldots, \delta^{c_n}).$$

Proof. It is clear that $Q \subseteq \partial[n](c_1, \ldots, c_n)$. The calculations in the proof of the proposition prove the converse. $\square$

Corollary A.2.3. *We define the set*

$$\mathcal{M} = \{\partial[x] \to [x] | [x] \in \mathrm{Ob}(\Theta[\mathcal{C}])\}.$$

Then the class $\mathrm{Cell}(\mathcal{M})$ is exactly the the class of monomorphisms of $\widehat{\Theta[\mathcal{C}]}$.

Proof. This is an immediate consequence of [Cis06, Proposition 8.1.37] or [BR11, 4.4]. $\square$

Lemma A.2.4. *The category $\Theta[\mathcal{C}]$ is regular Cartesian Reedy when $\mathcal{C}$ is.*

Proof. $\square$

APPENDIX A.3. THE ANODYNE THEOREM FOR HORIZONTAL INNER ANODYNES

In this section, following [Our10, 3.4.4], we will demonstrate that the horizontal inner anodynes are closed under corner products with monomorphisms. As a corollary of the analysis in this section, we will demonstrate that $\Theta[\mathcal{C}]$ is regular Cartesian Reedy. We make no claim to originality.

Lemma A.3.1. *Given a simplicial set S, the category of labelings of S, that is, the fibre $\left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_S$ is closed under finite Cartesian products.*

Proof. The product of two S-labeled simplicial sets $(S, \Omega), (S, \Omega')$ can be given by the formula

$$S \xrightarrow{\Delta} S \times S \xrightarrow{\Omega \times \Omega'} \widehat{\mathcal{C}}_{\bullet} \times \widehat{\mathcal{C}}_{\bullet} \xrightarrow{\times} \widehat{\mathcal{C}}_{\bullet}.$$

We leave the verification to the reader. $\square$

Definition A.3.2. Given a simplicial set S define the functor

$$H_S : \left(\widehat{\Delta} \downarrow S\right) \times \left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_S \to \widehat{\Delta} \int \widehat{\mathcal{C}}$$

by the rule

$$\left(S' \xrightarrow{f} S, (S, \Omega)\right) \mapsto (S', \Omega \circ f),$$

and we define the *relative intertwiner over S*

$$\square_S \overset{\mathrm{def}}{=} \square \circ H_S.$$

Notice that when $S = \Delta^n$, the fibre decomposes as

$$\left(\widehat{\Delta} \int \widehat{\mathcal{C}} \right)_n \cong \widehat{\mathcal{C}}^n.$$

So we can write

$$\square_n : \left(\widehat{\Delta} \downarrow \Delta^n \right) \times \widehat{\mathcal{C}}^n \to \widehat{\Theta[\mathcal{C}]}.$$

Observation A.3.3. Given a labeled simplicial set (S, Ω), a map of simplicial sets $Y \to S$, and an object $[t] = [n](c_1, \ldots, c_n)$ of $\Theta[\mathcal{C}]$, we can compute $\square_S(f, \Omega)_t$ as follows: For any n-simplex $s \in S_n$, let $(W_{s,i})_{i=1^n}$ be the family of $\mathcal{C}$-sets obtained by evaluation of Ω on s. A map $[t] \to Y \square \Omega \circ f$ is by definition a map $[t] \to (Y, \Omega \circ f)$. Such a map is determined by giving an n-simplex $y \in Y_n$ together with a family of maps

$$(c_i \xrightarrow{\zeta_i} W_{fy,i})_{i=1}^n.$$

Then we can compute

$$\square_S(f, \Omega)_t \cong \coprod_{y \in Y_n} \prod_{i=1}^n W_{fy,i,c_i}.$$

Definition A.3.4. Given a finite family of simplicial sets $\mathbf{S} = (S_i)_{i=1}^n$, we define a functor:

$$H_{\mathbf{S}} : \left(\widehat{\Delta} \downarrow \prod_{i=1}^n S_i \right) \times \prod_{i=1}^n \left(\widehat{\Delta} \int \widehat{\mathcal{C}} \right)_{S_i} \to \widehat{\Delta} \int \widehat{\mathcal{C}}$$

by the rule:

$$\left(S \xrightarrow{\prod_{i=1}^n f_i} \prod_{i=1}^n S_i, (\Omega_i)_{i=1}^n \right) \mapsto \left(S, \prod_{i=1}^n (\Omega_i \circ f_i) \right).$$

As in the previous definition, we define the *relative multi-intertwiner* by the formula

$$\square_{\mathbf{S}} \overset{\text{def}}{=} \square \circ H_{\mathbf{S}}.$$

Remark A.3.5. Notice that given a finite family of labeled simplicial sets $(\mathbf{S}, \mathbf{\Omega}) = (S_i, \Omega_i)_{i=1}^n$ and a family $\mathbf{f} = \left(S \xrightarrow{f_i} S_i \right)_{i=1}^n$,

$$H_{\mathbf{S}}(\mathbf{f}, \mathbf{\Omega}) \cong H_{S_1}(f_1, \Omega_1) \times^S \cdots \times^S H_{S_n}(f_n, \Omega_n),$$

where $\times^S$ denotes the product in $\left(\widehat{\Delta} \int \widehat{\mathcal{C}} \right)_S$.

Observation A.3.6. Let

$$(\mathbf{S}, \mathbf{\Omega}) = (\Delta^{m_i}, \Omega_i)_{i=1}^n$$

be a family of labeled simplices, and let

$$\mathbf{f} = \left(\Delta^r \xrightarrow{f_i} \Delta^{m_i} \right)_{i=1}^n,$$

be a family of maps defining an r-simplex of the product. We may identify the Ω_i with families of $\mathcal{C}$-presheaves $(X_{i,\ell})_{\ell=1}^{m_i}$, so we compute $H_{\Delta^{m_i}}(f_i, \Omega_i)$ as

$$\left(\prod_{k=f_i(j-1)+1}^{f_i(j)} X_{i,k} \right)_{j=1}^r,$$

and therefore, we can compute $H_{\mathbf{S}}(\mathbf{f}, \boldsymbol{\Omega})$ as

$$\left(\prod_{i=1}^{n} \left(\prod_{k=f_i(j-1)+1}^{f_i(j)} X_{i,k} \right) \right)_{j=1}^{r} .$$

Lemma A.3.7. *The relative intertwiner $\Box_S$ preserves colimits in the first variable.*

Proof. Since colimits are computed objectwise in presheaves, it suffices to show that the functor $\Box_S(\bullet, \Omega)_t$ preserves colimits for all $[t] = [n](c_1, \ldots, c_n) \in \Theta[\mathcal{C}]$ and all labels Ω of S. Therefore, it suffices by Observation A.3.3 to show this in the case where $\mathcal{C}$ is the terminal category, since we may fix the family of objects $(c_1, \ldots, c_n)$. For each $s \in S_n$ let $(W_{s,i})_{i=1}^{n}$ be the evaluation of Ω on s. Then given $f : Y \to S$, we have a cartesian square

$$\begin{array}{ccc}
\coprod_{y \in Y_n} \prod_{i=1}^{n} W_{fy,i} & \longrightarrow & Y_n \\
{\scriptstyle \tau^* f} \downarrow & & \downarrow {\scriptstyle f} \\
\coprod_{s \in S_n} \prod_{i=1}^{n} W_{s,i} & \xrightarrow{\ \tau\ } & S_n
\end{array} ,$$

exhibiting $\coprod_{y \in Y_n} \prod_{i=1}^{n} W_{fy,i}$ as the pullback of f along τ, but by the universality of colimits in the category of sets, we are done. $\qquad\square$

Note A.3.8. This is Oury's proof, but this statement can also be seen to follow immediately from 2.10.

Lemma A.3.9. *The relative intertwiner $\Box_n$ preserves connected colimits in each variable.*

Proof. We saw from the previous lemma that it preserves colimits in the first variable, so representing Y as the colimit of its simplices, we immediately reduce to the case where Y is a simplex. But we know in this case that any map $[p] \to [n]$ factors as a degeneracy followed by a face map. In the case that f is a face map, we can compute the pullback of $V[n](X_1, \ldots, X_n)$ along f to be

$$V[p] \left(\prod_{i=f(0)+1}^{f(1)} X_i, \ldots, \prod_{i=f(p-1)}^{f(p)} X_i \right) .$$

By universality of colimits in $\widehat{\mathcal{C}}$, we see that it suffices to show that

$$V[p] (\bullet, \ldots, \bullet) = \Box_p(\mathrm{id}_{\Delta^p}, \bullet, \ldots, \bullet)$$

preserves connected colimits in each variable. In the case where f is a degeneracy map, we can compute the pullback over $[p]$ to be

$$V[p](*, \ldots, X_1, \ldots, *, \ldots, X_n, \cdots *),$$

where we fill in the terminal object of $\widehat{\mathcal{C}}$ in each argument i where $f(i-1) = f(i)$. In this case again, it again suffices to show that $V[p]$ preserves connected colimits in each variable, but this is precisely the content of [Rez10, Proposition 4.5], where the proof proceeds by first showing that if we set $X_k = \emptyset$, then

$$V[p+1+q](A_1, \ldots, A_p, \emptyset, B_1, \ldots, B_q) \cong V[p](A_1, \ldots, A_p) \coprod V[q](B_1, \ldots, B_q),$$

and then exhibiting the obvious parametric right adjoint

$$\left(\left(V[p](A_1, \ldots, A_p) \coprod V[q](B_1, \ldots, B_q) \right) \downarrow \widehat{\Theta[\mathcal{C}]} \right) \to \widehat{\mathcal{C}}.$$

$\square$

Note A.3.10. This proof is substantially easier than Oury's proof, which relies on a long direct computation.

Definition A.3.11. Since the categories $\widehat{\mathcal{C}}$, $\widehat{\Theta[\mathcal{C}]}$, $\widehat{\mathcal{C}}^n$, and $\left(\widehat{\Delta} \downarrow \Delta^n\right)$ are all cocomplete (since they are all presheaf categories), and since the intertwiner preserves connected colimits argument-by-argument, we can use A.1 to define the functor

$$\square_n^{\lrcorner} : \left(\widehat{\Delta} \downarrow \Delta^n\right)^{[1]} \times \underbrace{\widehat{\mathcal{C}}^{[1]} \times \cdots \times \widehat{\mathcal{C}}^{[1]}}_{n \text{ times}} \to \widehat{\Theta[\mathcal{C}]}^{[1]},$$

called the *corner intertwiner*.

More generally, for any finite family of simplices $(\Delta^{m_i})_{i=1}^n$, we can do the same trick and define the *corner-multi-intertwiner*

$$\square_{m_1,\ldots,m_n}^{\lrcorner} : \left(\widehat{\Delta} \downarrow \Delta^{m_1}\right)^{[1]} \times \cdots \times \left(\widehat{\Delta} \downarrow \Delta^{m_n}\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^{m_1} \times \cdots \times \left(\widehat{\mathcal{C}}^{[1]}\right)^{m_n} \to \widehat{\Theta[\mathcal{C}]}^{[1]}.$$

Following [Our10, 3.85], we begin with the following observations:

Observation A.3.12. We saw from the definition of $\square$ and the definition of products in $\widehat{\Delta} \int \widehat{\mathcal{C}}$ that the diagram

$$
\begin{array}{ccc}
\widehat{\Delta} \int \widehat{\mathcal{C}} \times \widehat{\Delta} \int \widehat{\mathcal{C}} & \xrightarrow{\square \times \square} & \widehat{\Theta[\mathcal{C}]} \times \widehat{\Theta[\mathcal{C}]} \\
\downarrow{\scriptstyle \times} & & \downarrow{\scriptstyle \times} \\
\widehat{\Delta} \int \widehat{\mathcal{C}} & \xrightarrow{\square} & \widehat{\Theta[\mathcal{C}]}
\end{array}
$$

commutes. We also computed that the diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^n\right) \times \widehat{\mathcal{C}}^n \times \left(\widehat{\Delta} \downarrow \Delta^m\right) \times \widehat{\mathcal{C}}^m & \xrightarrow{H_n \times H_m} & \widehat{\Delta} \int \widehat{\mathcal{C}} \times \widehat{\Delta} \int \widehat{\mathcal{C}} \\
\downarrow{\scriptstyle \varsigma} & & \\
\left(\widehat{\Delta} \downarrow \Delta^n\right) \times \left(\widehat{\Delta} \downarrow \Delta^m\right) \times \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m & & \downarrow{\scriptstyle \times} \\
\downarrow{\scriptstyle P \times \mathrm{id} \times \mathrm{id}} & & \\
\left(\widehat{\Delta} \downarrow \Delta^n \times \Delta^m\right) \times \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m & \xrightarrow{H_{n,m}} & \widehat{\Delta} \int \widehat{\mathcal{C}}
\end{array}
$$

commutes as well where ς permutes the factors and P is the functor sending a pair of labeled simplicial sets $f : S \to \Delta^n, \Omega$ and $g : S' \to \Delta^m, \Omega'$ over Δ^n and Δ^m respectively to the labeled simplicial set

$$f \times g : S \times S' \to \Delta^n \times \Delta^m, \Omega \times \Omega'$$

over the product. Taking these two diagrams together, we see that the diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^n\right) \times \widehat{\mathcal{C}}^n \times \left(\widehat{\Delta} \downarrow \Delta^m\right) \times \widehat{\mathcal{C}}^m & \xrightarrow{\square_n \times \square_m} & \widehat{\Theta[\mathcal{C}]} \times \widehat{\Theta[\mathcal{C}]} \\
\downarrow{\scriptstyle \varsigma} & & \\
\left(\widehat{\Delta} \downarrow \Delta^n\right) \times \left(\widehat{\Delta} \downarrow \Delta^m\right) \times \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m & & \downarrow{\scriptstyle \times} \\
\downarrow{\scriptstyle P \times \mathrm{id} \times \mathrm{id}} & & \\
\left(\widehat{\Delta} \downarrow \Delta^n \times \Delta^m\right) \times \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m & \xrightarrow{\square_{n,m}} & \widehat{\Theta[\mathcal{C}]}
\end{array}
$$

also commutes.

Then by A.3.9, every functor appearing in this diagram preserves connected colimits in each argument, the intertwiners by the lemma, and the functors P and $\times$, since they are products in presheaf categories and therefore preserve colimits in both arguments. Then by the functoriality of the corner tensor functor A.1.12, we obtain a commutative diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^n\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^n \times \left(\widehat{\Delta} \downarrow \Delta^m\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^m & \xrightarrow{\square_n^{\lrcorner} \times \square_m^{\lrcorner}} & \widehat{\Theta[\mathcal{C}]}^{[1]} \times \widehat{\Theta[\mathcal{C}]}^{[1]} \\
\downarrow{\scriptstyle \varsigma} & & \\
\left(\widehat{\Delta} \downarrow \Delta^n\right)^{[1]} \times \left(\widehat{\Delta} \downarrow \Delta^m\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^n \times \left(\widehat{\mathcal{C}}^{[1]}\right)^m & & \downarrow{\scriptstyle \times^{\lrcorner}} \\
\downarrow{\scriptstyle P^{\lrcorner} \times \mathrm{id} \times \mathrm{id}} & & \\
\left(\widehat{\Delta} \downarrow \Delta^n \times \Delta^m\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^n \times \left(\widehat{\mathcal{C}}^{[1]}\right)^m & \xrightarrow{\square_{n,m}^{\lrcorner}} & \widehat{\Theta[\mathcal{C}]}^{[1]}
\end{array}
$$

also commutes, where $P^{\lrcorner} = \times^{\lrcorner}$.

Observation A.3.13. Consider the corner product of a simplicial inner horn inclusion with a simplicial boundary inclusion

$$
\lambda_j^n \times^{\lrcorner} \delta^m : \Lambda_j^n \times \Delta^m \cup \Delta^n \times \partial \Delta^m \hookrightarrow \Delta^n \times \Delta^m.
$$

Then it is a standard fact of quasicategory theory that we can factor this map as a sequence

$$
\Lambda_j^n \times \Delta^m \cup \Delta^n \times \partial \Delta^m = X_0 \subseteq X_1 \subseteq \cdots \to X_{k-1} \subseteq X_k = \Delta^n \times \Delta^m
$$

where each inclusion $X_{i-1} \hookrightarrow X_i$ is the pushout of an inner horn inclusion $\Lambda_{\ell_i}^{r_i} \to \Delta^{r_i}$ along an inclusion $\Lambda_{\ell_i}^{r_i} \hookrightarrow X_{i-1}$. By the construction of the sequence, each $[r_i] \to X_i \to \Delta^n \times \Delta^m$ is nondegenerate and does not factor through X_{i-1}, so in particular, it does not factor through X_0, and therefore the maps $\alpha_i : \Delta^{r_i} \to \Delta^n$ and $\beta_i : [r_i] \to \Delta^m$ do not factor through Λ_j^n or $\partial \Delta^m$. In particular, the image of α_i is either $\partial_j \Delta^n$ or all of Δ^n, and the image of β_i must be all of Δ^m, so all three maps α_i, β_i, and $\alpha_i \times \beta_i$ send the initial and terminal vertices of Δ^{r_i} to the initial and terminal vertices of Δ^n, Δ^m, and $\Delta^n \times \Delta^m$ respectively.

Observation A.3.14. Let $(\alpha \times \beta) : \Delta^r \to \Delta^n \times \Delta^m$ be an injective map preserving initial and terminal elements. Let $\mathbf{A} = (A_i)_{i=1}^n$ and $\mathbf{B} = (B_i)_{i=1}^m$ be objects of $\widehat{\mathcal{C}}^n$ and $\widehat{\mathcal{C}}^m$ respectively. Let

$$K_{\alpha,\beta} : \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m \to \widehat{\mathcal{C}}^r$$

be the functor defined by the rule

$$(\mathbf{U}, \mathbf{V}) \mapsto \alpha^* \mathbf{U} \times \beta^* \mathbf{V},$$

taking the product of the pullbacks to the fibre over Δ^r. Then we have a diagram:

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m & \xrightarrow{\mathrm{id} \times K_{\alpha,\beta}} & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
\downarrow{\scriptstyle ((\alpha,\beta)) \circ (-) \times \mathrm{id} \times \mathrm{id}} & & \downarrow{\scriptstyle H_r} \\
\left(\widehat{\Delta} \downarrow \Delta^n \times \Delta^m\right) \times \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m & \xrightarrow{H_{n,m}} & \widehat{\Delta} \int \widehat{\mathcal{C}}
\end{array}
$$

To show that the diagram commutes, let $p : X \to \Delta^r$ be a map. Then evaluating $H_r(p, K_{\alpha,\beta}(\mathbf{U}, \mathbf{V}) = H_r(p, \alpha^* \mathbf{U} \times \beta^* \mathbf{V})$ on a simplex $x : \Delta^s \to X$ is

$$
\begin{aligned}
(px)^*(\alpha^* \mathbf{U} \times \beta^* \mathbf{V}) &= (px)^* \alpha^* \mathbf{U} \times (px)^* \beta^* \mathbf{U} \\
&= (\alpha px)^* \mathbf{U} \times (\beta px)^* \mathbf{V} \\
&= H_{n,m}((\alpha px, \beta px), \mathbf{U}, \mathbf{V}) \\
&= H_{n,m} \circ ((\alpha, \beta) \circ \times \mathrm{id} \times \mathrm{id})(px, \mathbf{U}, \mathbf{V}),
\end{aligned}
$$

which demonstrates that the diagram commutes. Let $(t_i)_{i=1}^r$ such that $t_i = \alpha(i) - \alpha(i-1) + \beta(i) - \beta(i-1)$. Note that the sum of the t_i is exactly $n + m$, since α and β preserve initial and terminal objects. We define a functor

$$\tau_i : \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m \to \widehat{\mathcal{C}}^{t_i}$$

by the rule

$$(\mathbf{A}, \mathbf{B}) \mapsto (A_{\alpha(i-1)+1}, \ldots, A_{\alpha(i)}, B_{\beta(i-1)+1}, \ldots, B_{\beta(i)}).$$

Then define

$$\tau : \widehat{\mathcal{C}}^n \times \widehat{\mathcal{C}}^m \to \prod_{i=1}^r \widehat{\mathcal{C}}^{t_i}.$$

It is clear that τ is a permutation of variables and therefore an isomorphism. Then let

$$P_i : \widehat{\mathcal{C}}^{t_i} \to \widehat{\mathcal{C}}$$

be the functor defined by the rule

$$(X_1, \ldots, X_{t_i}) \mapsto X_1 \times \cdots \times X_{t_i}$$

Then the P_i assemble to a map $(P_1, \ldots, P_r)$ such that

$$(P_1, \ldots, P_r) \circ \tau = K_{\alpha,\beta}.$$

Then the diagram

$$\left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m \xrightarrow{\ \mathrm{id}\times\tau\ } \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\prod_{i=1}^r\widehat{\mathcal{C}}^{t_i} \xrightarrow{\ \mathrm{id}\times(P_i)_{i=1}^r\ } \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^r$$

$$\downarrow{\scriptstyle((\alpha,\beta))\circ(-)\times\mathrm{id}\times\mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle H_r}$$

$$\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m \xrightarrow{\qquad\qquad H_{n,m}\qquad\qquad} \widehat{\Delta}\int\widehat{\mathcal{C}}$$

commutes, and therefore, composing the bottom horizontal and right vertical maps with $\square$, we have another commutative diagram

$$\left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m \xrightarrow{\ \mathrm{id}\times\tau\ } \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\prod_{i=1}^r\widehat{\mathcal{C}}^{t_i} \xrightarrow{\ \mathrm{id}\times(P_i)_{i=1}^r\ } \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\widehat{\mathcal{C}}^r$$

$$\downarrow{\scriptstyle((\alpha,\beta))\circ(-)\times\mathrm{id}\times\mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle\square_r}$$

$$\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)\times\widehat{\mathcal{C}}^n\times\widehat{\mathcal{C}}^m \xrightarrow{\qquad\qquad \square_{n,m}\qquad\qquad} \widehat{\Theta[\mathcal{C}]}$$

.

The bottom horizontal and right vertical maps preserve connected colimits, as we have seen. The left vertical map preserves connected colimits because colimits are computed in the domain for comma categories. The map $\prod_{i=1}^r Pi$ preserves colimits in each argument because colimits are universal in toposes. Then applying the corner tensor functor, we have the commutative diagram

$$\left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^n\times\left(\widehat{\mathcal{C}}^{[1]}\right)^m \xrightarrow{\ \mathrm{id}\times\tau\ } \left(\widehat{\Delta}\downarrow\Delta^r\right)^{[1]}\times\prod_{i=1}^r\left(\widehat{\mathcal{C}}^{[1]}\right)^{t_i} \xrightarrow{\ \mathrm{id}\times(P_i^{\lrcorner})\ } \left(\widehat{\Delta}\downarrow\Delta^r\right)\times\left(\widehat{\mathcal{C}}^{[1]}\right)^r$$

$$\downarrow{\scriptstyle((\alpha,\beta))\circ(-)^{\lrcorner}\times\mathrm{id}\times\mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle\square_r^{\lrcorner}}$$

$$\left(\widehat{\Delta}\downarrow\Delta^n\times\Delta^m\right)^{[1]}\times\left(\widehat{\mathcal{C}}^{[1]}\right)^{n+m} \xrightarrow{\qquad\qquad \square_{n,m}^{\lrcorner}\qquad\qquad} \widehat{\Theta[\mathcal{C}]}^{[1]}$$

.

Finally, we reach our destination. As we mentioned before, this entire section is due to David Oury [Our10], and the following theorem is the crescendo:

Theorem A.3.15 (Anodyne Theorem [Our10, 3.88])**.** *The class of horizontal anodynes is closed under corner products with monomorphisms. In particular, if we let*

$$\mathscr{J} = \{\square_n^{\lrcorner}(\lambda_k^n,\delta^{c_1},\ldots,\delta^{c_n})|\ \text{for }n\geq 2, 0 < k < n\}.$$

Then we have

$$\mathscr{M}\times^{\lrcorner}\mathscr{J}\subseteq\mathrm{Cell}(\mathscr{J}).$$

Proof. Let $f_0 : \partial\Delta^n\hookrightarrow\Delta^n$, and let $\mathbf{f} = (f_i)_{i=1}^n$ be a family of boundary inclusions in $\widehat{\mathcal{C}}$. Let $g_0 : \Lambda_k^m\hookrightarrow\Delta^n$ be an inner horn inclusion, and let $\mathbf{g} = (g_i)_{i=1}^n$ be a family of boundary inclusions in $\widehat{\mathcal{C}}$. By A.3.12, we have

$$\square_n^{\lrcorner}(f_0,\ldots,f_n)\times^{\lrcorner}\square_m^{\lrcorner}(g_0,\ldots,g_m)\cong\square_{n,m}^{\lrcorner}(f_0\times^{\lrcorner}g_0, f_1,\ldots,f_n, g_1,\ldots,g_n).$$

By A.3.13, we know that $f_0\times^{\lrcorner}g_0$ can be factored as a finite sequence of pushouts of inner horn inclusions. By A.1.7 and A.1.9, it follows that

$$\square_{n,m}^{\lrcorner}(f_0\times^{\lrcorner}g_0, f_1,\ldots,f_n, g_1,\ldots,g_n)$$

is a finite composites of pushouts of maps

$$\Box_{n,m}^{\lrcorner}(h_i, f_1, \ldots, f_n, g_1, \ldots, g_n)$$

where $(h_i : \Lambda_{\ell_i}^{r_i} \to \Delta^{r_i})_{i=1}^k$ are inner horn inclusions where h_i is a morphism over $\Delta^n \times \Delta^m$. Let h_i' denote this morphism as map in simplicial sets over Δ^{r_i}, and let $(\alpha_i, \beta_i) : \Delta^r \to \Delta^m \times \Delta^n$ be initial and terminal vertex preserving as in A.3.13. Then by A.3.14, we see that

$$\Box_{n,m}^{\lrcorner}(h_i, \mathbf{f}, \mathbf{g}) \cong \Box_{r_i}^{\lrcorner}(h_i', (P_1^{\lrcorner}, \ldots, P_{r_i}^{\lrcorner}) \circ \tau(\mathbf{f}, \mathbf{g})).$$

But the value of the argument in position $1 \le j \le r_i$ is

$$P_j^{\lrcorner} \circ \tau_j(\mathbf{f}, \mathbf{g}) = f_{\alpha_i(j-1)+1} \times^{\lrcorner} \cdots \times^{\lrcorner} f_{\alpha_i(j)} \times^{\lrcorner} g_{\beta_i(j-1)+1} \times \ldots^{\lrcorner} \times^{\lrcorner} g_{\beta_i(j)},$$

which belongs to $\mathrm{Cell}(\mathscr{B})$, where $\mathscr{B}$ is the set of boundary inclusions in $\widehat{\mathcal{C}}$. So we see that

$$\Box_{r_i}^{\lrcorner}(h_i', (P_1^{\lrcorner}, \ldots, P_{r_i}^{\lrcorner} \circ \tau)(\mathbf{f}, \mathbf{g}))$$

is in

$$\{\Box_{r_i}^{\lrcorner}(\mathscr{L}, \mathrm{Cell}(\mathscr{B}), \ldots, \mathrm{Cell}(\mathscr{B}))\},$$

where $\mathscr{L}$ is the set of inner horn inclusions, which by A.1.9 is a subset of

$$\mathrm{Cell}(\{\Box_{r_i}^{\lrcorner}(\mathscr{L}, \mathscr{B}, \ldots, \mathscr{B})\}),$$

which belongs to $\mathrm{Cell}(\mathscr{J})$, since $\{\Box_{r_i}^{\lrcorner}(\mathscr{L}, \mathscr{B}, \ldots, \mathscr{B})\} \subseteq \mathscr{J}$. $\qquad\qquad\square$

Appendix A.4. Comparison with Rezk's complete $\Theta[\mathcal{C}]$-spaces

Since Rezk's complete Segal model structure on $\mathrm{Psh}_\Delta(\Theta[\mathcal{C}])$ is Cartesian, and since $* \hookrightarrow E$ is one of the generators of the localization (see [Rez10]), and since $\Theta[\mathcal{C}]$ is regular skeletal Reedy, it follows by several results of Cisinski [Cis06, Proposition 8.2.9, Theorem 3.4.36, Proposition 2.3.30] that Rezk's localizer for complete $\Theta[\mathcal{C}]$-Segal spaces is the simplicial completion of a localizer on $\Theta[\mathcal{C}]$.

Observation A.4.1. To show that Rezk's localizer is the simplicial completion of the localizer generated by the horizontal inner anodynes, it suffices to show the following two properties hold:

(i) The maps $\Box_n^{\lrcorner}(\lambda_i^n, \delta^{c_1}, \ldots, \delta^{c_n}) \times \Delta^0$ belong to the localizer for complete $\Theta[\mathcal{C}]$-Segal spaces for $n \ge 2$ and $0 < i < n$.

(ii) The Segal maps $\mathrm{Se}[n](c_1, \ldots, c_n) : \mathrm{Sp}[n](c_1, \ldots, c_n) \hookrightarrow [n](c_1, \ldots, c_n)$ are horizontal inner anodyne.

We will make use of the following lemma:

Lemma A.4.2. *For any inner horn inclusion $\Lambda_k^n \hookrightarrow \Delta^n$, and any presheaves $X_1, \ldots, X_n$ on $\mathcal{C}$, the map $V_{\Lambda_k^n}(X_1, \ldots, X_n) \times \Delta^0 \hookrightarrow V[n](X_1, \ldots, X_n) \times \Delta^0$ belongs to the localizer for complete Segal-$\Theta[\mathcal{C}]$-spaces.*

Proof. We will suppress the $\times \Delta^0$ factor denoting discrete simplicial presheaves for the duration of this proof. By [Rez10, 5.2], we know that the maps $\mathrm{Se}[n](X_1, \ldots, X^n)$ are already weak equivalences. Then we proceed following the argument of [JT07, Lemma 3.5]. Notice that trivial cofibrations have the right-cancellation property with respect to monomorphisms. Then we show that since the class of trivial cofibrations contains the class of maps $\mathrm{Se}[n](X_1, \ldots, X_n)$, it also contains the class of maps

$$V_{\partial_0 \Delta^n \cup \partial_n \Delta^n}(X_1, \ldots, X_n) \hookrightarrow V[n](X_1, \ldots, X_n)$$

by induction on n. Notice first that the map

$$V_{\Lambda_1^2}(X_1, X_2) \hookrightarrow V[n](X_1, X_2)$$

is automatically a trivial cofibration, since $\Lambda_1^2 = \mathrm{Sp}[2]$. For the case of $n > 2$, notice that by cancellation, it suffices to show that the maps

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \xrightarrow{i_n} V_{\partial_0 \Delta^n \cup \mathrm{Sp}[n]}(X_1, \ldots, X_n) \xrightarrow{j_n} V_{\partial_0 \Delta^n \cup \partial_n \Delta^n}(X_1, \ldots, X_n)$$

are trivial cofibrations. Then notice that

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \xrightarrow{i_n} V_{\partial_0 \Delta^n \cup \mathrm{Sp}[n]}(X_1, \ldots, X_n)$$

is a pushout of the map

$$V_{\mathrm{Sp}[n-1]}(X_1, \ldots, X_n) \hookrightarrow V_{\partial_0 \Delta^n}(X_1, \ldots, X_n),$$

and is therefore a trivial cofibration. Notice that for $d_0 : [n-1] \to [n]$, $d_0^{-1}(\mathrm{Sp}[n]) = \mathrm{Sp}[n-1]$ and $d_0^{-1}(\partial_n \Delta^n) = \partial_{n-1}\Delta^{n-1}$. Then the square

$$
\begin{array}{ccc}
V_{\mathrm{Sp}[n-1]\cup\partial_{n-1}\Delta^{n-1}}(X_1, \ldots, X_n) & \longrightarrow & V_{\mathrm{Sp}[n-1]\cup\partial_n\Delta^n}(X_1, \ldots, X_n) \\
\Big\downarrow{\scriptstyle k_{n-1}} & & \Big\downarrow{\scriptstyle j_n} \\
V_{\partial_0\Delta^n}(X_1, \ldots, X_n) & \xrightarrow{\ \ d_0' \ \ } & V_{\partial_0\Delta^n\cup\partial_n\Delta^n}(X_1, \ldots, X_n)
\end{array}
$$

is cocartesian, and k_{n-1} is a trivial cofibration by using the cancellation property with the map j_{n-1}. Therefore, it follows that j_n is also a trivial cofibration.

We now prove the lemma: By the cancellation property, it suffices to show that

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda_k^n}(X_1, \ldots, X_n)$$

is a trivial cofibration for $n \geq 2$ and $0 < i < n$. The case $n = 2$ is obvious, so it suffices to show for the case $n > 2$. Given a set $S \subseteq [n]$, let

$$\Lambda_S^n = \bigcup_{i \notin S} \partial_i \Delta^n.$$

We will show that for $n > 2$ and S a nonempty subset of $[1, \ldots, n-1]$, the map

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda_S^n}(X_1, \ldots, X_n)$$

is a trivial cofibration. We argue by induction on n and $s = n - \mathrm{Card}(S)$. If $s = 1$, $\Lambda_S^n = \partial_0\Delta^n \cup \partial_n\Delta^n$, in which case we are done by the previous argument. If $s > 1$, let $T = S \cup \{b\}$ for some $b \in [1, \ldots, n-1] \setminus S$. Then by the inductive hypothesis,

$$V_{\mathrm{Sp}[n]}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda_T^n}(X_1, \ldots, X_n)$$

is a trivial cofibration. Then it suffices to show that

$$V_{\Lambda_T^n}(X_1, \ldots, X_n) \hookrightarrow V_{\Lambda_S^n}(X_1, \ldots, X_n)$$

is a trivial cofibration. We see that the diagram

$$
\begin{array}{ccc}
V_{\Lambda_T^n \cap \partial_b\Delta^n}(X_1, \ldots, X_n) & \longrightarrow & V_{\Lambda_T^n}(X_1, \ldots, X_n) \\
\Big\downarrow & & \Big\downarrow \\
V_{\partial_b\Delta^n}(X_1, \ldots, X_n) & \longrightarrow & V_{\Lambda_S^n}(X_1, \ldots, X_n)
\end{array}
$$

is a pushout, and therefore, it suffices to show that

$$V_{\Lambda_T^n \cap \partial_b\Delta^n}(X_1, \ldots, X_n) \hookrightarrow V_{\partial_b\Delta^n}(X_1, \ldots, X_n)$$

is a trivial cofibration. But this is true by the inductive hypothesis on n. Therefore, we are done. $\square$

Proposition A.4.3. *The map*
$$\square_n^{\lrcorner}(\lambda_k^n, \delta^{c_1}, \ldots, \delta^{c_n}) \times \Delta^0$$
is a trivial cofibration for the model structure on complete $\Theta[\mathcal{C}]$-Segal spaces.

Proof. We again suppress the $\times \Delta^0$ factor. Let $Q = \square_n^{\lrcorner}(\lambda_k^n, \delta^{c_1}, \ldots, \delta^{c_n})$. Evaluation of Q on 0 is the source and evaluation on 1 is the target. We must show that the monomorphism $Q : Q(0) \hookrightarrow Q(1)$ is a trivial cofibration. Notice first that
$$\square_n(\Lambda_k^n, c_1, \ldots, c_n) \hookrightarrow Q(0) \hookrightarrow Q(1) = [n](c_1, \ldots, c_n)$$
is a weak equivalence by the lemma. Then by right-cancellation, it suffices to show that
$$\square_n(\Lambda_k^n, c_1, \ldots, c_n) \hookrightarrow Q(0)$$
is a trivial cofibration. Let
$$U(s,t) = \int^{u_0, \ldots, u_n} ([1](u_0, s) \times [1](u_1 \wedge \cdots \wedge u_n, t)) \cdot \square(\lambda_k^n(u_0), \delta^{c_1}(u_1), \ldots, \delta^{c_n}(u_n)),$$
where evaluation on $u_i \in [1]$ denotes taking the source or target. Then we see by coend reduction that
$$\int^{s,t} [1](s \wedge t, x) \times [1](u_0, s) \times [1](u_1 \wedge \cdots \wedge u_n, t) = [1](u_0 \wedge u_1 \wedge \cdots \wedge u_n, x),$$
so by commutation of coends, we see that
$$Q(x) = \int^{s,t} [1](s \wedge t, x) U(s,t),$$
which proves that
$$Q(0) = U(1,0) \coprod_{U(0,0)} U(0,1),$$
but $U(0,1) = \square_n(\Lambda_k^n, c_1, \ldots, c_n)$, so the map
$$\square_n(\Lambda_k^n, c_1, \ldots, c_n) \hookrightarrow Q(0)$$
is a pushout of $U(0,0) \to U(1,0)$, which we will show is a trivial cofibration. Notice that in $U(0,0)$, everything vanishes when $u_0 = 1$, so we have that
$$U(0,0) \cong \int^{u_1 \ldots, u_n} [1](u_1 \wedge \cdots \wedge u_n, 0) \cdot \square(\lambda_k^n(0), \delta^{c_1}(u_1), \ldots, \delta^{c_n}(u_n)).$$
Notice also that by cofinality, we have that
$$U(1,0) = \int^{u_0, \ldots, u_n} ([1](u_0, 1) \times [1](u_1 \wedge \cdots \wedge u_n, 0)) \cdot \square(\lambda_k^n(u_0), \delta^{c_1}(u_1), \ldots, \delta^{c_n}(u_n)),$$
is isomorphic to
$$\int^{u_1, \ldots, u_n} [1](u_1 \wedge \cdots \wedge u_n, t) \cdot \square(\lambda_k^n(1), \delta^{c_1}(u_1), \ldots, \delta^{c_n}(u_n)).$$
Then the map $U(0,0) \hookrightarrow U(1,0)$ is induced by the natural maps
$$\square(\lambda_k^n(0), \delta^{c_1}(u_1), \ldots, \delta^{c_n}(u_n)) \hookrightarrow \square(\lambda_k^n(1), \delta^{c_1}(u_1), \ldots, \delta^{c_n}(u_n)).$$
But $\lambda_k^n(0) \hookrightarrow \lambda_k^n(1)$ is the inner horn inclusion $\Lambda_k^n \hookrightarrow \Delta^n$, and therefore, by the lemma, these are all trivial cofibrations. But $U(0,0)$ and $U(0,1)$ are clearly homotopy coends, and therefore, the map $U(0,0) \hookrightarrow U(0,1)$ is a monic weak equivalence and therefore a trivial cofibration, which proves the proposition. $\square$

This proves one direction of the theorem; now we prove the converse.

Proposition A.4.4. *The maps*

$$\mathrm{Se}[n](c_1, \ldots, c_n) : \mathrm{Sp}[n](c_1, \ldots, c_n) \hookrightarrow [n](c_1, \ldots, c_n)$$

are horizontal inner anodyne.

Proof. Since the map

$$\lambda^n : \Lambda^n = \partial_0 \Delta^n \cup \partial_n \Delta^n \hookrightarrow \Delta^n$$

is inner anodyne, and since the empty maps $e^{c_i} : \emptyset \hookrightarrow c_i$ are monic, it follows by A.1.9 that the corner-intertwiner

$$\square_n^{\lrcorner}(\lambda^n, e^{c_1}, \ldots, e^{c_n})$$

is horizontal inner anodyne. However, it is easy to see that this map is exactly

$$\square_n(\Lambda^n, c_1, \ldots, c_n) \hookrightarrow [n](c_1, \ldots, c_n).$$

Therefore, it suffices to show that the map

$$\square_n(\mathrm{Sp}[n], c_1, \ldots, c_n) \hookrightarrow \square_n(\Lambda^n, c_1, \ldots, c_n)$$

is a horizontal inner anodyne. We will first show that the map

$$\square_n(\mathrm{Sp}[n] \cup \partial_0 \Delta^n, c_1, \ldots, c_n) \hookrightarrow \square_n(\Lambda^n, c_1, \ldots, c_n)$$

is horizontal inner anodyne. To see this, we proceed by induction on n. This is immediate for $n \leq 2$. Suppose $n > 2$. Then the map

$$\square_n(\mathrm{Sp}[n] \cup \partial_0 \Delta^n, c_1, \ldots, c_n) \hookrightarrow \square_n(\Lambda^n, c_1, \ldots, c_n)$$

is horizontal inner anodyne, as it is a pushout of the map

$$\square_n(\mathrm{Sp}[n-1] \cup \partial_0 \Delta^{n-1}, c_1, \ldots, c_n) \hookrightarrow \square_n(\partial_n \Delta^n, c_1, \ldots, c_n),$$

which is horizontal inner anodyne by the induction hypothesis. Then it suffices to show that

$$\square_n(\mathrm{Sp}[n], c_1, \ldots, c_n) \hookrightarrow \square_n(\mathrm{Sp}[n] \cup \partial_0 \Delta^n, c_1, \ldots, c_n)$$

is horizontal inner anodyne. Again, we proceed by induction on n and notice that this is clear for $n \leq 2$, but we see immediately that

$$\square_n(\mathrm{Sp}[n], c_1, \ldots, c_n) \hookrightarrow \square_n(\mathrm{Sp}[n] \cup \partial_0 \Delta^n, c_1, \ldots, c_n)$$

is a pushout of

$$\square_n(\mathrm{Sp}[n-1], c_1, \ldots, c_n) \hookrightarrow \square_n(\partial_0 \Delta^n, c_1, \ldots, c_n),$$

which is horizontal inner anodyne by the induction hypothesis, which concludes the proof. $\square$

Corollary A.4.5. *The left Kan extension of the functor*

$$Y \times E^{\bullet} : \Theta[\mathcal{C}] \times \Delta \to \widehat{\Theta[\mathcal{C}]}$$

(where Y is the Yoneda embedding) defined by the rule

$$[n](c_1, \ldots, c_n) \times \Delta^m \mapsto [n](c_1, \ldots, c_n) \times E^m$$

induces a Quillen equivalence

$$\mathrm{Psh}_{\Delta}(\Theta[\mathcal{C}])_{\mathrm{CSS}} \underset{\mathrm{Sing}_E}{\overset{\mathrm{Real}_E}{\rightleftarrows}} \widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}}$$

between the model structure for complete $\Theta[\mathcal{C}]$-Segal spaces and the horizontal Joyal model structure, and the left Kan extension of the functor

$$d : \Theta[\mathcal{C}] \to \mathrm{Psh}_{\Delta}(\Theta[\mathcal{C}])$$

defined by the rule

$$[n](c_1, \ldots, c_n) \mapsto [n](c_1, \ldots, c_n) \times \Delta^0$$

induces a Quillen equivalence

$$\widehat{\Theta[\mathcal{C}]}_{\mathrm{hJoyal}} \overset{d_!}{\underset{d^*}{\rightleftarrows}} \mathrm{Psh}_\Delta(\Theta[\mathcal{C}])_{\mathrm{CSS}}.$$

That is to say, the two model categories are Quillen bi-equivalent.

Proof. This is an immediate consequence of [Cis06, Proposition 2.3.27]. $\qquad\square$

APPENDIX A.5. PROOF OF COSIMPLICIAL RESOLUTIONS

In this section, we demonstrate that the cosimplicial objects $C_L^\bullet(c)$ and $C_R^\bullet(c)$ are cosimplicial resolutions of $[1](c)$. The proof for $C_{\mathrm{cyl}}^\bullet(c)$ is similar and can be obtained by applying the scheme for $C_R^\bullet(c)$ to the proof of that case in [DS11b, Proposition 9.4].

Proposition A.5.1. *The cosimplicial objects $C_L^\bullet(c)$ and $C_R^\bullet(c)$ are a cosimplicial resolutions of $[1](c)$.*

Proof. We follow the proof of [DS11b, Lemma 9.3]. We define a filtration on $C_L^n(c)$ as follows:

$$X_0 = [0,1](c),$$
$$X_1 = \bigcup_{1 < i \leq n} [0,1,i](c,*),$$
$$X_2 = \bigcup_{1 < i < j \leq n} [0,1,i,j](*,*,c),$$

and so on. We see that $X_{n-1} = C_L^n(c)$, so it suffices to show that each inclusion $X_{k-1} \subseteq X_k$ is obtained by horizontal inner anodyne attachments. Notice first of all that

$$h_1^\ell : \square_\ell(\Lambda_1^\ell, c, *, \ldots, *) \hookrightarrow \square_\ell(\Delta^\ell, c, *, \ldots, *)$$

is horizontal inner anodyne, since it is exactly

$$\square_k^{\lrcorner}(\lambda_1^\ell, e^c, e^*, \ldots, e^*),$$

where e^x denotes the empty map $\emptyset \to x$. Then choose $1 < i_1 < \cdots < i_k \leq n$ in X_k. Then X_{k-1} contains all of the faces $[0,1,i_1,\ldots,\hat{i}_{k_j},\ldots,i_k](c,*,\ldots,*)$, but it also contains the degenerate face $[1,i_1,\ldots,i_k](*,\ldots,*)$, so therefore the map $X_{k-1} \subseteq X_k$ is obtained by pushing out along copies of h_1^{k+1}, and therefore it is horizontal inner anodyne. Therefore, the composite of the filtration $[1](c) = X_0 \subseteq \cdots \subseteq X_{n-1}$ is horizontal inner anodyne, and by 3-for-2, we see that the retraction $C_L^n(c) \to [1](c)$ is a horizontal Joyal equivalence. The proof for $C_R^\bullet$ follows by symmetry. $\qquad\square$

APPENDIX A.6. RECOGNITION OF HORIZONTAL JOYAL FIBRATIONS

In this section, we will prove the anlogue of Joyal's pseudofibration theorem for the horizontal model structure on $\Theta[\mathcal{C}]$. We will need to set up some definitions.

Definition A.6.1. Recall from Definition A.3.4, we defined the functor $H_\mathbf{S}$ for a finite family of simplicial sets $\mathbf{S}$. Consider the case of the functor $H_{S,n}$, where the family is made up of two simplicial sets Δ^n and some simplicial set S. Then we define the functor

$$\mathcal{H}_{S,n} : \left(\widehat{\Delta} \downarrow S \times \Delta^n\right) \times \left(\widehat{\Delta} \int \widehat{\mathcal{C}}\right)_n \to \widehat{\Delta} \int \widehat{\mathcal{C}},$$

to be the restriction of $H_{S,n}$ to the terminal labeling of S, the unique labeling of S that sends all edges of S to the terminal object of $\widehat{\mathcal{C}}$. Composing $\square$ with $\mathcal{H}_{S,n}$ is denoted by $\square_{S,n}$. For the remainder of this section, we also, by abuse of notation, define

$$\square_S : \left(\widehat{\Delta} \downarrow S\right) \to \widehat{\Theta[\mathcal{C}]}$$

to be the composite of $\square$ with the restriction of H_S to the terminal labeling of S.

Observation A.6.2. Observe that from A.3.9 the functor

$$\square_{S,n} : \left(\widehat{\Delta} \downarrow \Delta^n \times S\right) \times \widehat{\mathcal{C}}^n \to \widehat{\Theta[\mathcal{C}]}$$

preserves connected colimits in each argument and therefore can be corner tensored. By the same argument as A.3.12, given $h : Y \to S$ a map of simplicial sets, we can compute

$$\square_n^\lrcorner(f_0, \ldots, f_n) \times^\lrcorner h = \square_n^\lrcorner(f_0, \ldots, f_n)\square_S^\lrcorner(h)$$
$$= \square_{S,n}^\lrcorner(h \times^\lrcorner f_0, f_1, \ldots, f_n).$$

In what follows, we will use some excellent new results of Danny Stevenson [Ste]. Consider the case of the simplicial set E^1 and the map $e : \Delta^0 \hookrightarrow E^1$.

Theorem A.6.3 ([Ste, Example 5.8]). *The map of simplicial sets*

$$e \times^\lrcorner \delta^n : E^1 \times \partial\Delta^n \cup \Delta^0 \times \Delta^n \hookrightarrow E \times \Delta^n,$$

is an inner anodyne for all $n > 0$.

It follows that we can filter $e \times^\lrcorner \delta^n$ as the transfinite composition of pushouts of inner horn inclusions.

Observation A.6.4. A simplex $(\alpha, \beta) : \Delta^r \to E^1 \times \Delta^n$ is determined by the destination of its vertices. We can assume that β is surjective, since otherwise, the simplex would factor through $E^1 \times \partial\Delta^n$. We label the vertices by (a, i) and (b, i) for $0 \leq i \leq n$, where e is the inclusion of a in E^1. Given a simplex $(\alpha, \beta) : \Delta^r \to E^1 \times \Delta^n$, and a family of presheaves $\mathbf{U} = (U_i)_{i=1}^n$ on $\mathcal{C}$.

$$K_\beta : \widehat{\mathcal{C}}^n \to \widehat{\mathcal{C}}^r$$

to be the functor sending

$$\mathbf{U} \mapsto \beta^*\mathbf{U}.$$

Similar to A.3.14, we have a diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{\mathrm{id} \times K_\beta} & \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
{\scriptstyle ((\alpha,\beta))\circ(-)\times \mathrm{id} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle H_r} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right) \times \widehat{\mathcal{C}}^n & \xrightarrow{\mathcal{H}_{E^1,n}} & \widehat{\Delta}\int\widehat{\mathcal{C}}
\end{array}
$$
,

and the diagram commutes by a direct computation. Set $(t_i = \beta(i) - \beta(i-1))_{i=1}^r$, and let

$$\tau_i : \widehat{\mathcal{C}}^n \to \widehat{\mathcal{C}}^{t_i}$$

be the map defined by sending

$$\mathbf{U} \mapsto (A_{\beta(i-1)+1}, \ldots, A_{\beta(i)}).$$

Then this family of maps defines a map

$$\widehat{\mathcal{C}}^n \to \prod_{i=1}^{r} \widehat{\mathcal{C}}^{t_i}$$

which is a permutation and therefore an isomorphism. Then let

$$P_i : \widehat{\mathcal{C}}^{t_i} \to \widehat{\mathcal{C}}$$

be the map defined by the rule

$$(X_1, \ldots, X_{t_i}) \mapsto X_1 \times \cdots \times X_{t_i}.$$

Then these P_i assemble to a map

$$(P_1, \ldots, P_r) : \prod_{i=1}^{r} \widehat{\mathcal{C}}^{t_i} \to \widehat{\mathcal{C}}^r,$$

such that $(P_1, \ldots, P_r) \circ \tau = K_\beta$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n \xrightarrow{\mathrm{id} \times \tau} \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \prod_{i=1}^{r} \widehat{\mathcal{C}}^{t_i} \xrightarrow{\mathrm{id} \times (P_i)_{i=1}^{r}} \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
\downarrow{\scriptstyle ((\alpha,\beta))\circ(-)\times\mathrm{id}\times\mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle H_r} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right) \times \widehat{\mathcal{C}}^n \xrightarrow{\quad H_{E^1,n} \quad} \widehat{\Delta} \int \widehat{\mathcal{C}}
\end{array}
$$

,

and therefore, composing the bottom horizontal and right vertical maps with $\square$, we have another commutative diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^n \xrightarrow{\mathrm{id} \times \tau} \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \prod_{i=1}^{r} \widehat{\mathcal{C}}^{t_i} \xrightarrow{\mathrm{id} \times (P_i)_{i=1}^{r}} \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \widehat{\mathcal{C}}^r \\
\downarrow{\scriptstyle ((\alpha,\beta))\circ(-)\times\mathrm{id}\times\mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \square_r} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right) \times \widehat{\mathcal{C}}^n \xrightarrow{\quad \square_{E^1,n} \quad} \widehat{\Theta[\mathcal{C}]}
\end{array}
$$

.

We see that each of the arrows in this picture preserves connected colimits argument-by-argument, so applying the corner tensor functor, we obtain a commutative diagram

$$
\begin{array}{ccc}
\left(\widehat{\Delta} \downarrow \Delta^r\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^n \xrightarrow{\mathrm{id} \times \tau} \left(\widehat{\Delta} \downarrow \Delta^r\right)^{[1]} \times \prod_{i=1}^{r} \left(\widehat{\mathcal{C}}^{[1]}\right)^{t_i} \xrightarrow{t_i\,\mathrm{id} \times (P_i^{\lrcorner})} \left(\widehat{\Delta} \downarrow \Delta^r\right) \times \left(\widehat{\mathcal{C}}^{[1]}\right)^r \\
\downarrow{\scriptstyle ((\alpha,\beta))\circ(-)^{\lrcorner}\times\mathrm{id}\times\mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \square_r^{\lrcorner}} \\
\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right)^{[1]} \times \left(\widehat{\mathcal{C}}^{[1]}\right)^n \xrightarrow{\quad \square_{E^1,n}^{\lrcorner} \quad} \widehat{\Theta[\mathcal{C}]}^{[1]}
\end{array}
$$

.

Theorem A.6.5. *Set $e : \Delta^0 \to E^1$. Then for any boundary inclusion*

$$\square_n^{\lrcorner}(\delta^n, \delta^{c_1}, \ldots, \delta^{c_n}),$$

with $n > 0$, the map

$$e \times^{\lrcorner} \square_n^{\lrcorner}(\delta^n, \delta^{c_1}, \ldots, \delta^{c_n})$$

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is a horizontal inner anodyne.

Proof. From A.6.2, we see that

$$e \times^\lrcorner \square_n^\lrcorner(\delta^n, \delta^{c_1}, \dots, \delta^{c_n}) \cong \square_{E^1,n}^\lrcorner(e \times^\lrcorner \delta^n, \delta^{c_1}, \dots, \delta^{c_n},$$

and by A.6.3, we see that $e \times^\lrcorner \delta^n$ can be factored as a transfinite composite of pushouts of inner horn inclusions. By A.1.7 and A.1.9, it follows that

$$\square_{E^1,n}^\lrcorner(e \times^\lrcorner \delta^n, \delta^{c_1}, \dots, \delta^{c_n})$$

is a transfinite composite of pushouts of maps

$$\square_{E^1,n}^\lrcorner(h_i, \delta^{c_1}, \dots, \delta^{c_n}),$$

where the $h_i : \Lambda_{\ell_i}^{r_i} \to \Delta^{r_i}$ are inner horn inclusions and the h_i are maps in $\left(\widehat{\Delta} \downarrow E^1 \times \Delta^n\right)$. Let h_i' denote the same map in the category $\left(\widehat{\Delta} \downarrow \Delta^{r_i}\right)$, and let (α, β) denote the map $\Delta^{r_i} \to E^1 \times \Delta^n$, with β surjective as we saw above. Then by A.6.4,

$$\square_{E^1,n}^\lrcorner(h_i, \delta^{c_1}, \dots, \delta^{c_n}) \cong \square_{r_i}^\lrcorner(h_i', (P_1^\lrcorner, \dots, P_{r_i}^\lrcorner) \circ \tau(\delta^{c_1}, \dots, \delta^{c_n})),$$

but we see that the $P_i^\lrcorner$ are corner products of boundary inclusions as in A.3.15, and therefore, the maps

$$\square_{r_i}^\lrcorner(h_i', (P_1^\lrcorner, \dots, P_{r_i}^\lrcorner) \circ \tau(\delta^{c_1}, \dots, \delta^{c_n}))$$

are all horizontal inner anodyne, demonstrating the claim that

$$e \times^\lrcorner \square_n^\lrcorner(\delta^n, \delta^{c_1}, \dots, \delta^{c_n})$$

is inner anodyne. $\square$

Corollary A.6.6. *The fibrations between fibrant objects in the horizontal Joyal model structure are horizontal inner fibrations having the right lifting property with respect to the map $e : \Delta^0 \hookrightarrow E^1$.*

Proof. A fibration between fibrant objects must have the right lifting property with respect to all horizontal inner anodynes and all maps of the form $e \times^\lrcorner \square_n^\lrcorner(\delta^n, \delta^{c_1}, \dots, \delta^{c_n})$. Since every inner fibration has the right lifting property with respect to all of those maps for $n > 0$, it follows that an inner fibration between fibrant objects need only have the right lifting property with respect to the case where $n = 0$, which is exactly the map e. $\square$

Corollary A.6.7. *The formal $\mathcal{C}$-quasicategories are the fibrant objects of the horizontal Joyal model structure.*

Proof. Since the formal $\mathcal{C}$-quasicategories have the right lifting property with respect to all inner anodynes, it suffices to check that they have the right lifting property with respect to $e : \Delta^0 \hookrightarrow E^1$, but such a lift always exists by choosing the obvious degenerate lift. $\square$

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