**SET THEORY**

* **Set** is a collection of objects of any sort.
* Also called as class, collection or aggregate.
* **Eg:**

Set of numbers: {1,2,3,4}

Set of Alphabet: {a,b,c}

* We form a set with a common behaviour but its not compulsory.

A: {apple,a,52}

↓

Members, objects, elements

* Any object belonging to a set is called a member or an element of set.
* **Well Defined Set:**
* A set is said to be well defined if it is possible to determine, by means of certain rules, whether any given object is a member of set.

Well Defined → Membership clause

* N: {2,4,6,8,10}

Clause: Even numbers between 1 to 10

◦ 2 is a number of set N.

2 Є N

↓

belongs to

◦ 5 is not a member of set N.

5 ∉ N

↓

doesn’t belong to

* M= {1,2,3,4………}

↓

Infinite set

* A= {a,b,c…….z}

↓

Many elements

* **Set Builder Notation {x|P(x)}:**
* A set can be defined by a predicate
* **Eg:**

S= {x|x is an even positive integer}

S= {x|x is a chocolate}

* **Cardinality |S| or n(S):**
* The number of elements in the set is called as the cardinality of set.
* **Eg:**

◦ A = {1,2,5,9,20}

|A| or n(A) = 5

◦ B = {a,b,c………z}

|B| = 26

◦ S = {a, b, {m,n}, z}

|S| = 4

* In the above set S, has cardinality 4 because {m,n} is treated as single element.

a Є S

{m,n} Є S

z Є S

m ∉ S

n ∉ S

m Є {m,n}

* **Subset (⊆):**
* Let A and B be any two sets. If every element of A is an element of B, then A is called a subset of B.

A ⊆ B

↓

A subset of B

|  |
| --- |
| A ⊆ B≡Ɐx(xЄA→xЄB) |

* **Ex:**

A = {1,2,3,4,5} B⊆A

B = {1,2,3} C⊆A

C = {3,4} D⊄A

D = {5,6} E⊄A

E = {7} F⊆A

F = {3}

* We need to be careful while writing subset and member of a set.

M = {a,b,c} N⊆M

N = {a} a⊄N aЄN

X ✓

{a}⊆N {a}⊆M

✓ ✓

* **Eg:**

P = {{a,b},c,d}

{a,b}⊄P {{a,b}}⊆P

{a,b}ЄP

* **Properties:**
* Reflexive: A⊆A

✓ Every set is subset of itself.

* Symmetric: A⊆B ⇒ B⊆A

X

* Transitive: A⊆B, B⊆C ⇒ A⊆C
* **Equal Set(=):**
* Two sets A and B are said to be equal iff A⊆B and B⊆A.

|  |
| --- |
| A=B≡Ɐx(xЄA⇔xЄB)  ≡A⊆BɅ B⊆A |

* **Eg:**

{1,2,3} = {1,2,2,3,3}

{1,2,4} = {4,2,1}

* Order doesn’t matter
* Repetition doesn’t matter.
* {1,{2,3}} ≠ {1,2,3}
* {{1}} ≠ {1}
* {1}Є{{1}} {1}⊄{{1}}
* **Proper Subset(⊂):**
* A set ‘A’ is called a proper subset of a set ‘B’ if A⊆B and A≠B.
* B should have atleast one element more than A.

|  |
| --- |
| A⊂B ≡ Ǝx(x∉AɅxЄB)  ≡ A⊄BɅA≠B |

* **Properties:**

|  |  |  |
| --- | --- | --- |
|  | Equal | Proper |
| Reflexive | ✓ | x |
| Symmetric | ✓ | x |
| Transitive | ✓ | ✓ |

* **Universal Set(U):**
* A set is called universal set if it includes every set under discussion.

|  |
| --- |
| U={x|P(X)V¬P(x)} |

**Ex:**

A: {1,2,3} B: {4,5}

U = {1,2,3,4,5,6,7}

* **Null Set or Empty Set ∅ or {}:**
* A set which doesn’t contain any element is called an empty set or a null set.

|  |
| --- |
| ∅ = {x|P(x)Ʌ¬P(x)} |

* **Things to Note:**
* ∅ means no element

Cardinality – 0

* {∅} is a set with null element

Cardinality – 1

* ∅ ≠ {∅}

∅ Є {∅}

∅ ⊆ {∅}

* ∅ ⊆ A

Null set is subset of any set

* **Power Set P(A) or 2A:**
* For a set A, a collection of all subsets of A is called the power set if A.

|  |
| --- |
| P(A)=2A={x|x⊆A} |

* **Eg:**

A = {1,2,3}

P(A) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}}

* The cardinality of power set is 2|A|.
* ∅ ⊆ P(A)

∅ Є P(A)

A Є P(A)

* **Venn Diagrams:**
* A Venn diagram is a schematic representation of a set.
* We use a rectangle for universal set and a circular for set.
* Shaded region are the elements.

Bbbb

U A

* **Set Operations:**
* **Union(AUB)**

For any two sets A and B, the union of A and B, written as AUB, is the set of all elements which are members of the Set A or Set B or both.

|  |
| --- |
| AUB={x|xЄA V xЄB} |

**Eg:**

A = {1,2,3}

B = {3,4}

AUB = {1,2,3,4}

|  |
| --- |
| U  A B |

AUB

## Intersection(A∩B)

## The intersection of any two sets A and B, is the set consisting of all the elements which belong to both A and B.

|  |
| --- |
| A∩B={x|xЄA Ʌ xЄB} |

## Eg:

## A = {1,2,3}

## B = {3,4}

## A∩B = {3}

|  |
| --- |
| A B |

A**∩**B

**Note:**

If A**∩B=**∅, then A and B are called disjoint.

* **Difference(A/B) or (A-B)**

The difference of two sets is the set consisting of all elements of A which are not elements of B.

|  |
| --- |
| A-B={x|xЄA Ʌ x∉B} |

## 

## Eg:

## A = {1,2,3}

## B = {3,4}

## A-B = {1,2}

|  |
| --- |
| A B |

A-B

* **Complement(Ac):**

The complement of set A, is the set of elements which are not in A.

|  |
| --- |
| Ac=U-A={x|xЄA U x∉B} |

## 

**Eg:**

A = {1,2,3}

B = {1,2,3,4,5,6}

Ac = {4,5,6}

|  |
| --- |
|  |

Ac

* **Symmetric Difference(AB):**

Let A and B be any two sets.

The Symmetric difference of A and B is:

AB = (A-B)U(B-A)

|  |
| --- |
| AB={x|(xЄA V xЄB} Ʌ x∉(A**∩**B)} |

**Eg:**

A = {1,2,3}

B = {3,4}

AB = {1,2,4}

|  |
| --- |
| A B |

AB

* **Ordered Pairs**
* An ordered pair consists of two objects in a given fixed order.
* It is not a set.
* The ordering of objects is important.
* (2,3) ≠ (3,2)
* **Cartesian Product(AxB)**
* The set of all ordered pairs such that first member of the ordered pair is an element of A and the second member is an element of B is called the Cartesian Product of A and B.

|  |
| --- |
| AxB={(x,y)|xЄA Ʌ yЄB} Ʌ |

**Eg:**

A = {1,2}

B = {a,b,c}

AxB = {(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)}

* The cardinality of product is |A|x|B|.
* **Digital Logic Notation:**
* AUB A+B (or)
* A∩B AB (and)
* Ac A’ (invert)
* A-B AB’
* AB AꚚB(ExOR)
* **Idendities:**
* AU∅ = A
* A∩∅ = ∅
* A-∅ = A
* A = A
* **Idempotent:**
* AUA = A
* A∩A = A
* A-A =
* A =
* **Commutative:**
* AUB = BUA
* A∩B = B∩A
* A-B ≠ B-A
* A = B
* AxB ≠ BxA
* **Associative:**
* (AUB)UC = AU(BUC)
* (A∩B)∩C = A∩(B∩C)
* (A-B)-C ≠ A-(B-C)
* (AB)C = A(BC)
* (AxB)XC ≠ Ax(BxC)
* **Distributive:**
* A∩(BUC) = (A∩B)U(A∩C)
* A∩(BC) = (A∩B)(A∩C)
* Ax(BUC) = (AxB)U(AxC)
* **Absorption:**
* AU(A∩B) = A
* A∩(AUB) = A
* **Demorgan’s:**
* (AUB)c = Ac∩Bc
* (A∩B)c = AcUBc

**1)Let p(s) denote the power set of a set S which of the following is always true?**

**a) p(p(S)) = p(S)**

**b) p(S)∩p(p(S)) = {}**

**c) p(S)∩S = p(S)**

**d) S∉p(S) (Gate 2000)**

**A)** Lets take example

S = {1}

p(S) = {, {1}}

p(p(S)) = {, {, {{1}}, {{1}}

By solving with options:

p(S)∩p(p(S)) = {}

So option(b) is correct.

**2) The number of elements in the power set p(S) of the set S = {{**}, 1, {2,3}}

**a) 2**

**b) 4**

**c) 8**

**d) None Gate 1995**

**A)** S = {{}, 1, {2,3}}

|S| = 3

|P(S)| = 2|s|

= 23

= 8

3) Let E, F and G be finite sets.

X = (E∩F)-(F∩G) and  
 Y = (E-(E∩G))-[E-F]

**Which of the following is true?**

1. **x<y**
2. **x>y**
3. **x=y**
4. **x-y**≠ and y-x≠ Gate 2006
5. X=EF-FG

=EF(FG)’

=EF(F’+G’)

=EFF’+EFG’

=EFG’

Y=(E-EG)-(E-F)

=E(EG)’-EF’

=E(E’+G’)-EF’

=EG’-EF’

=EG’[EF’]’

=EG’[E’+F’]

=EFG’

⇒ X=Y

X Y

**=**

**4) If P,Q,R are subsets of universal set U then**

**(P∩Q∩R)U(PC∩Q∩R)UQCURC is?**

**a)QCURC**

**b)PUQCURC**

**c)PCUQCUR**

**d)U Gate 2008**

**A)** PQR+P’QR+Q’+R’

(P+P’)QR+Q’+R’

QR+Q’+R’

(Q+Q’)(R+Q’)+R’

R+Q’+R’

= 1→Universal Set

⇒option(d) is correct

**5) In a class of 200 students 125 have taken programming language, 85 took data structures, 65 have taken computer organization, 50 took programming and data structures, 35 took data structures and computer organization, 30 took programming and computer organization, 15 took all three, How many have not taken any course?**

**a)15**

**b)20**

**c)25**

**d)30 Gate 2004**

**A)** U=200

n(P)=125

n(D)=85

n(C)=65

n(P∩D)=50

n(D∩C)=35

n(P∩C)=30

n(P∩D∩C)=15

n(PUDUC)=n(P)+n(D)+n(C)-n(P∩D)-n(D∩C)-n(P∩C)+n(P∩D∩C)

=125+85+65-50-35-30+15

=175

n[PUDUC]C=U-n[PUDUC]

=200-175

=25

⇒option(c) is correct.

**RELATIONS**

* In general sense, relation gives a connection between objects.

**Eg:**

Ram is a father of Ravi

5 is greater than 3

* Relations can be set between two or more objects. In this chapter we confine to two objects and its called a **binary relation**. And we represent it with an **ordered pair.**

**Eg:**

Father(F)={(Ram,Ravi),(Surya,Vrinda),(Raju,Pawan)}

(Ram,Ravi)→Ram is father of Ravi

(Surya,Vrinda)→Surya is father of Vrinda

(x,y)→x is father of y

(Ram,Ravi)ЄF

* **Generalization**

(x,y)ЄR, xRy

↓

x is in relation R to y

R={(x,y)|x is in relation to y}

* **Domain(D) & Range(R)**

(x,y)ЄR, where x=set of x is Domain

y=set of y is Range

**Eg:**

S = {(a,1),(b,2),(λ,p),(Ram,µ)}

D(S) = {a,b,λ,Ram}

R(S) = {1,2,p,µ}

* **Cartesian Product & Relation**
* For two sets A and B, the pairing of every element of A with every of B gives cartesian product.
* If we pair up few elements of A with few elements of B with some logic it gives a relation.

|  |
| --- |
| relation⊆Cartesian Product |

R⊆AxB

* If A=B, then R⊆AxA then we can say it as R is a relation in A.
* R=AxA R=

↓ ↓

Universal relation Void relation

* A relation is a set of ordered pairs, so it is possible to apply usual operation sets to relations as well.
* x(RUS)y = xRy U xSy
* x(R∩S)y = xRy ∩ xSy
* x(R-S)y = xRy Ʌ xSy
* x(Rc)y = xRy
* If set A has n elements

|  |
| --- |
| No of relations = |

* **Representation**

→Matrix

→Graph

* Representation of relation gives a visual interpretation of data and inturn makes it easy to identify properties of relation.

**Eg:**

A = {a,b,c}

R = {(a,a), (b,a), (a,c), (c,a)}

* **Matrix Representation**

a b c

* If relation exists, we write the matrix element as 1 else 0.
* **Graph Representation**

aRa

bRa

aRc cRa

* Here elements are taken as nodes and relation as line with arrow.
* **Properties of Binary Relation**
* Reflexive: A binary relation R in a set X is reflexive if, for every xЄX.

(x,x)ЄR

|  |
| --- |
| Reflexive = Ɏx(xЄR→xRx) |

**Eg:** < and > are reflexive

* Irreflexive: A relation R in a set X is irreflexive if, for every xЄX.

(x,x)∉R

|  |
| --- |
| Irreflexive = Ɏx(xЄR→xRx) |

**Eg:** < and > are irreflexive

* Note that sometime a relation is neither reflexive nor irreflexive.

**Eg:** A = {1,2,3}

R = {(1,1),(1,2),(3,1)}

→Not Reflexive – as (2,2),(3,3) is not there

→Not Irreflexive – as (1,1) exists

* **Symmetric:**

A relation R in a set X is symmetric if, for ever x and y in X, whenever xRy, then yRx.

|  |
| --- |
| Symmetric = ɎxɎy{xЄX Ʌ yЄY Ʌ xRy → yRx} |

**Eg:**

= is symmetric

‘cousin of’ is symmetric

<

< Non Symmetric

brother of

* **Antisymmetric:**

A relation R in a set X is antisymmetric if, for ever x and y in x, whenever xRy implies yRx unless x=y.

|  |
| --- |
| Antisymmetric = ɎxɎy{xЄX Ʌ yЄY Ʌ (xRy → yRx} V (xRy) → yRx Ʌ x=y) |

**Eg:**

=

< Anti symmetric

<

* Note that it is possible to have a relation which is both symmetric and antisymmetric.

**Eg:** =

* **Asymmetric:**

A relation R in a set X is asymmetric if, for ever x and y in x, whenever xRy implies yRx even when x=y.

|  |
| --- |
| Asymmetric = ɎxɎy{xЄX Ʌ yЄY Ʌ xRy → yRx} |

* In antisymmetric self loops (i.e., reflexive) is allowed but in asymmetric there are no self loops it is strictly unidirectional.
* **Transitive:**

A relation R in a set X is transitive if, for every x, y and z in X, whenever xRy and yRz, then xRz.

|  |
| --- |
| Transitive = ɎxɎyɎz{xЄX Ʌ yЄY Ʌ zЄZ Ʌ xRy Ʌ yRz→xRz} |

**Eg:**

=

< Transitive

<

⊆

* **Identification**
* Reflexive:

a b c

Diagonal is 1

Self Loops

* Symmetric:

a b c

Symmetric Matrix

* **Common Relations**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Reflexive | Irreflexive | Symmetric | Antisymmetric | Transitive |
| >  < | x | ✓ | x | ✓ | ✓ |
| >  < | ✓ | x | x | ✓ | ✓ |
| = | ✓ | X | ✓ | ✓ | ✓ |
| ⊂ | x | ✓ | x | ✓ | ✓ |

**Note:**

* If a relation is symmetric and transitive, it doesn’t guarantee reflexive.

**Eg:**

1)→Not reflexive x

→Symmetric ✓

→Transitive ✓

2) A={1,2,3}

R={(1,2), (2,1), (1,1), (2,2)}

→ Not reflexive x [as (3,3) is not there]

→Symmetric ✓

→Transitive ✓

* If a relation is irreflexive and symmetric it cannot be transitive.
* For a set ‘A’ with n elements then number of relations are:-

reflexive - -1

symmetric - =

anti symmetric - 2n x = 2n x

* **Cover**

For a set S and A = {A1,A2,…….An}is a subset of S and

I = S

Then Set A is called a covering of S and A1,A2,…….An are said to cover S.

* **Partition**
* A partition is a covering of set S, where Ai are mutually disjoint.
* A1,A2,…….An are called the blocks of partition.

**Eg:**

S = {a,b,c} →cover

A = {{a,b}, {b,c}} →Not cover not partition x

B = {{a}, {a,c}} →partition

C = {{a}, {b,c}} →partition

D = {{a,b,c}} →partition

E = {{a}, {b}, {c}} →partition

F = {{a}, {a,b}, {a,c}} →cover

B is neither partition nor covering because in the union set element ‘b’ is missing.

* Every partition is a covering.
* Partition

→largest-|S|

→smallest-1 (itself)

* **Equivalence Relation(≡)**
* A relation R in a set X is called an equivalence relation if it is:
* Reflexive
* Symmetric
* Transitive
* If R is an equivalence relation in a set X, then D(R), the domain of R is ‘X’ itself.

**Eg:**

1. = on Real Numbers
2. A=B on sets

* Graphs make it easy to test for equivalence

→ Reflexive

→Symmetric

→Transitive

**Note:**

* Every equivalence relation on a set generates a unique partition of the set.
* For every partition we can define an equivalence relation.
* **Congruence Relation**

This is a special type and frequently used equivalence relation.

R = {(x,y)|x-y is divisible by m}

↓

x≡y(mod m)

**Eg:**

A = {1,2,3,………7}

R = {(x,y)|x-y divisible by 3}

**Eg classes:**

[1]R={1,4,7}

[2]R={2,5}

[3]R={3,6}

* **Compatibility Relation(~)**
* A relation ‘R’ in X is said to be a compatibility relation if it is:

→Reflexive

→Symmetric

* All equivalence relations are compatibility relations.

**Eg:**

X={ball,bed,dog,cat,egg}

R={(x,y)|x,yЄX Ʌ xRy if x and y contain some common

letters}

* A compatibility relation defines a covering which may not be a partition.

ball bed

egg dog

cat

A={{ball,bed,cat},{bed,dog,egg}}

* **Maximal compatibility block**
* A subset A⊆X is called maximal compatibility block if any element of A is compatible to every other element of A and no element of X-A is compatible to all elements of A.
* In a graph, the maximal compatibility block is the largest complete polygon.

**Eg:**

MCB: {1,2,3,4}

{2,5}

{3,6}

{5,6}

1. 2

6

5

4 3

* **Composition**
* R = {(1,2),(3,4),(2,2)}

S = {(4,2),(2,5),(3,1)}

R◦S = {(1,5),(3,2),(2,5)}

↓

Composition

* Composition is associative

R◦S◦P = R◦(S◦P) = R◦S◦P

* **Converse()**
* If R from X to Y then from Y to X is its converse.
* **Transitive Closure**
* Includes the missing elements of relation to make its transitive.

**Eg:**

A={a,b,c}

R={(a,b),(b,c),(c,a)}

Closure of R = {(a,b),(b,c),(c,a),(a,c),(a,a),(b,a),(c,c)}

1. **The binary relation R={(1,1),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)} on set**

**A={1,2,3,4} is?**

1. **Reflexive, Symmetric and Transitive**
2. **Neither reflexive, nor irreflexive but transitive**
3. **Irreflexive, Symmetric and transitive**
4. **Irreflexive and Anti symmetric Gate 1998**

**A)**

* 4 has no loops so not reflexive
* 1,2,3 have self loops not irreflexive
* Transitive property is satisfied
* Option(b) is correct.

1. **Suppose A={a,b,c,d} and P is the partition of A where P={{a,b,c},{d}}**
2. **List the ordered pairs of the equivalence relation induced by P**
3. **Draw the graph of the above equivalence relation.(Gate 1998)**

R={(a,a),(b,b),(c,c),(a,b),(a,c),(b,a),(b,c),(c,a),(c,b),(d,d)}

1. **A relation R is defined on the set of integers as xRy iff (x+y) is even. Which of the following statements are true?**
2. **R is not an equivalence relation.**
3. **R is an equivalence relation having 1 equivalent classes.**
4. **R is an equivalent relation having 2 equivalent classes.**
5. **R is an equivalent relation having 3 equivalent classes.**

**Gate 2000**

1. I={0,±1,±2…………..}

R={(1,1),(1,3),(1,5)…. (odd,odd)

(2,2),(2,4),…………..} (even,even)

Classes

[odd]={±1,±3……}

[even]={0,±2,±4……}

* It forms an equivalence relation with 2 classes
* option(c) is correct

1. **The number of equivalence relations on the set {1,2,3,4} is**
2. **15**
3. **16**
4. **24**
5. **4 Gate 1997**
6. N = 4C4+4C3+4C2+4C1

= 1+4+6+4

= 15

**FUNCTIONS**

* Function is a particular class of a relation.
* **Def**

Let X and Y be nay two sets. A relation f from X to Y is called a function if for every xЄX there is an unique yЄY such that (x,y)ЄF.

* **Condition**
* Domain of f must be complete set ‘X’ not a subset
* Unique Image

f : x→y

Where x = domain

→=mapping or function

y=codomain

**Eg:**

f : {(x,x2)|xЄR} where f = function

g : {(x2,x)|xЄR} where g = not function

as (x,x2) and (x2,x) are two outcomes for one element

* **Number of functions**

f = x→y

The number of function yx

* **Types**
* one-one(Injection)

A mapping f:x→y is called one to one if distinct elements of x are mapped into distinct elements of y.

The number of one-one functions is yPX

* Onto(Surjection)

A mapping of f:x→y is called onto if the range is codomain.

Number of function =

nm-nC1(n-1)m+nC2(n-2)m……..(-1)n-1 nCn-11m

* One-One and Onto(Bijection)

A mapping f:x→y is called bijection if it is both one-one and onto.

* **Composition of Functions**

Let f:x→y and g:y→z be two functions then the composition relation g◦f = g[f(x)]

* **Inverse Function**
* If f:x→y and f is bijection then f-1exists which is f-1: y→x.
* f-1◦f = Ix
* f◦f-1 = Iy
* If f(x)=x3 then f-1(x)=x1/3

1. **How many one to one functions are there from a set A with n elements onto itself? Gate 1987**
2. Number of one-one = yPx

= nPn

= n!

1. **How many onto functions are there from an n-elemet set to a 2-element set? Gate 2012**
2. Nm-nC1(n-1)m+nC2(n-2)m+…………(-1)n-1 nCn-11m

Where m=n and n=2

⇒2n-2C1(2-1)n+2C2(2-2)n

⇒2n-2

1. **Let S denote the set of all functions f:{0,1}4→{0,1}. Denote y ‘N’ the number of functions from S to the set {0,1}. The value of log2log2N is? Gate 2014**
2. f:{0,1}4→{0,1}→2 elements

↓

16 elements

S = function of f

= nm

= 216

N=Number of function

S→{0,1} → 2 elements

↓

216 elements

= nm

=

Therefore, log2log2N = log2log2

= log2216

= 16

1. **If g(x)=1-x and h(x)=, then is**
2. **Gate 2015**
3. =

=

=

=

=

=

Answer doesn’t match option so lets test options.

⇒ = = =

⇒option(a) is correct

**5)Let f:R🡪R and f(x)=2x+3 is f invertible, if yes what is the inverse?**

**A)** f(x) is bijection

as for each x there is only one unique y and every image has a pre- image

f(x) = 2x+3

y = 2x+3

y-3 = 2x

x =

⇒f-1=

**GROUPS**

* **Semi Group:**

An algebra system (s,\*) is called a semigroup if the binary operation \* follows:

* Closure property: [x\*yЄS, Ɏx,yЄS]
* Associative property: [(x\*y)\*z = x\*(y\*z) Ɏx,y,zЄS]

**Eg:**

(z,+)→Integers on addition

(z,x)→Integers on multiplication

(V\*,◦)→alphabet on concatenation

x (z,-1)→Integer subtraction is not a semigroup as it fails on associativity.

(x-y)-z≠x-(y-z)

* **Monoid:**

An algebra system (m,\*) is called a monoid if the binary operation \* follows:

* Closure property
* Associative property
* Has unique ‘e’ identity element such that

ɎxЄM,e\*x = x\*e = x

**Eg:**

(z,+) e=0

(z,x) e=1

(V\*,◦) e=

**Note**

* Every monoid is a semigroup.
* (z+,+)→This is not a monoid as identity element ‘0’ is not there but it’s a semi group.
* **Sub semigroup:**

Let (s,\*) be a semigroup, if T⊆S and closed on operation then (T,\*) is a sub semigroup.

**Eg:**

(z,+) Integers semigroup on addition

(E,+) Even numbers is sub semigroup as E⊆Z.

* **Sub Monoid:**

Let (m,\*) be a monoid, if T⊆S and closed on \* and eЄT, then (T,\*) is called a sub monioid.

* **Group:**

A group (G,\*) is an algebraic system in which the binary operation \* on G satisfies four conditions:

* Closure property
* Associative property
* Has unique ‘e’ identity element
* Each element has inverse element

x-1\*x = x\*x-1 = e

**Eg:**

(z,+) e=0

for x=1, x-1=-1

x=2, x-1=-2

x=3, x-1=-3

x (z,x)→This is not a group as for multiplication a-1 = and ‘0’ has no inverse element as is not defined.

x (V\*,◦)→Concatenation of strings is not a group as there is no inverse element.

**Note:**

* Every group is a monoid and semigroup.
* Order of group |G| is the number of elements of G, when G is finite.
* Composition tables gives a simplified representation of group.

|G| = 1 <{e},\*>

|G| = 2 <{e,a},\*>

|  |  |  |
| --- | --- | --- |
| \* | e | a |
| e | e | a |
| a | a | e |

|G| = 3 <{e,a,b},\*>

|  |  |  |  |
| --- | --- | --- | --- |
| \* | e | a | b |
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

|G| = 4 <{e,a,b,c},\*>

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | e | a | b | c |
| a | a | b | c | e |
| b | b | c | e | a |
| c | c | e | a | b |

* **Abelian Group:**
* A group (G,\*) in which the operation \* is commutative is called an abelian group.

**Eg:**

(z,+)

(R-{0},X)

x (M,\*)→Non Singular Matrix, multiplication is a group but not Abelian as A.B≠B.A

✓ (Zm,+4)→Z modulo m under addition is Abelian group

**Eg:**

(z4,+4)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| +4 | 0 | 1 | 2 | 3 |
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

* The set of {1,2,….p-1} forms an abelian group under multiplication where p is prime number.
* **Cyclic Group**
* A group (G,\*) is said to be cyclic if there exists an element aЄG such that every element of G can be written as some power of a i.e., an for some integer n.

an=a\*a\*a………..n times

**Eg:**

(z,+) generator=1

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| +4 | 0 | 1 | 2 | 3 |
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

0n→0+0=0

1n→1+1+…………..12

2+1……………13

3+1………….14

1 is a generator with power(n)=4

2n→2+2=0

3n→3+3+…………..32

2+3…………33

1+3………34

3 is a generator with power(n)=4

**Note:**

* Cyclic group is an abelian group.
* Both x and x-1 are generators.
* To find number of generators of order n:

1. Change ‘n’ to prime factors

n =

1. For each prime factor

∆1=-

1. Generators = ∆1\*∆2………..∆n

**Eg:**

n = 63

= 32\*71

∆1 =32-31=6

∆2 =71-70=6

G(n) = ∆1\*∆2

= 6\*6

= 36

1. **The following is the incomplete operation table of a 4-element group**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | **e** | **a** | **b** | **c** |
| a | **a** | **b** | **c** | **e** |
| b |  |  |  |  |
| c |  |  |  |  |

**The last row of the table is**

1. **caeb**
2. **cbae**
3. **cbea**
4. **ceab Gate 2004**
5. First fill the first column

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | e | a | b | c |
| a | a | b | c | e |
| b | b |  |  |  |
| c | c |  |  |  |

⇓

a\*c=e ⇒ c\*a=e

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | e | a | b | c |
| a | a | b | c | e |
| b | b |  |  |  |
| c | c | e |  |  |

Each element in a row or column must be unique

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | e | a | b | c |
| a | a | b | c | e |
| b | b | c |  |  |
| c | c | e |  |  |

Last column cannot have another ‘e’ so ‘e’ has to be at b\*b

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | e | a | b | c |
| a | a | b | c | e |
| b | b | c | e |  |
| c | c | e |  |  |

Now we can guess the other

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | e | a | b | c |
| e | e | a | b | c |
| a | a | b | c | e |
| b | b | c | e | a |
| c | c | e | a | b |

Therefore the answer is option(d) ceab

1. **The set {1,2,4,7,8,11,13,14} is a group under multiplication modulo 15. The inverse of 4 and 7 are respectively**
2. **3 and 13**
3. **2 and 11**
4. **4 and 13**
5. **8 and 14 Gate 2005**
6. It is easy to go by option
7. 3X154 = 12

13X157 = 1

1. 2X154 = 8

11X157 = 2

1. 4X154 = 1

13X157 = 1 ✓

1. 8X154 = 2

14X157 = 8

The correct option is (c)

1. **Some group (G,◦) is known to be abelian. Then, which one of the following if true for G?**
2. **G=g-1 for every gЄG**
3. **G=g2 for every gЄG**
4. **(g◦h)2=g2◦h2 for every g,hЄG**
5. **G is of finite order Gate 1994**
6. G is abelian so g◦h=h◦g

(g◦h)2= (g◦h)◦(g◦h)

= g◦[h◦g]◦h

= g◦[g◦h]◦h

= (g◦g)◦(h◦h)

= g2◦h2

⇒option(c) is correct

1. **Let A be the set of all non singular matrices over real numbers and let \* be the matrix multiplication operator then**
2. **A is closed under \* but <A,\*> is not semigroup**
3. **<A,\*> is a semigroup but not a monoid**
4. **<A,\*> is a monoid but not a group**
5. **<A,\*> is a group but not an abelian group.**
6. Non singular matrices form a group but not abelian as

A.B ≠ B.A

⇒option(d) is correct

1. **For the composition table of a cyclic group shown below**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | a | b | c | d |
| a | **a** | **b** | **c** | **d** |
| b | **b** | **a** | **d** | **c** |
| c | **c** | **d** | **b** | **a** |
| d | **d** | **c** | **a** | **b** |

**Which of the following choices is correct?**

1. **a,b are generators**
2. **b,c are generators**
3. **c,d are generators**
4. **d,a are generators**
5. an=a\*a=a

bn=b\*b…..

a\*b…..

b\*b….

cn=c\*c…..

b\*c….

d\*c……

a ✓

dn=d\*d……

b\*d….

c\*d…..

a ✓

⇒option(c) is correct

* **Sub Group**

Let <G,\*> be a group and S⊆G be such that it satisfies the following conditions:

* eЄS, where e is identity of < G,\*>
* ɎaЄS, a-1ЄS
* Ɏa,bЄS, a\*bЄS

Then <S,\*> is called a subgroup of <G,\*>

**Eg:**

<z6,+6> = {0,1,2,3,4,5}

Subgroups

A = {0}

trivial B = {0,3} proper

C = {0,2,4}

D = {0,1,2,3,4,5}

**Note:**

* For any group <G,\*>, <{e},\*>, <G,\*> are trivial sub groups and all other groups are proper subgroups.
* **Langrange’s Theoram**

The order of each subgroup of a finite group is a divisor of the order of the group.

⇒<Z6,+6> can have subgroups of order either

Divisor’s(6)=1,2,3,6

* **Cosets**

Let <H,\*> be a subgroup of <G,\*>. For any aЄG, the set aH defined by aH={a\*h|hЄH} is called the left coset of H in G when aЄG.

**Eg:**

<Z6,+6>={0,1,2,3,4,5}

H = {0,3}

→For ‘0’ H0={0,3}

→For ‘1’ H1={1,4}

→For ‘2’ H2={2,5}

→For ‘3’ H3={0,3}

→For ‘4’ H4={1,4}

→For ‘5’ H5={2,5}

{Hn} = {{0,3}.{1,4},{2,5}}

↓

Set of cosets is a partition of set G.

* **Normal Subgroups**

A subgroup <H,\*> of <G,\*> is called a normal subgroup if for any aЄG, aH=Ha

**Note:**

* Every subgroup of an abelian group is normal.
* The trivial subgroups are also normal.

**Extra Concepts**

* **Ring**

An algebraic system <s,+,.> is called a ring if the binary operation + and . on s satisfy

1. <s,+> is abelian group
2. <s,.> is semigroup
3. <+,.> are distributive

**Eg:**  Set of integers, real, rational even, complex over addition and multiplication form Rings.

**Note:**

<S,+,.>

↓

→It commutative then commutative ring

→Monoid then Ring with Identity

→<S-{0},.> then Ring without division of zero.

* **Integral Domain**

A commutative ring <S,+,.> with identity and without divisors of zero is called an integral domain.

**Eg:**

<Z7,+7,X7> is integral domain

<Z6,+6,X6> is not as 3X62=0

**Note:**

Zp forms integral domain where p is prime.

* **Field**

A commutative ring <S,+,.> which has more than one element such that every non-zero element of S has s multiplicative inverse in S is called a field.

**Eg:**

Real, Rational, complex numbers forms fields.

* Integer are not fields but integral domains.

**Note:**

Every finite integral domain is a field.

**PARTIAL ORDER AND LATTICES**

* **Partial Ordering(P,<)**

A binary relation R in a set P is called a partial order relation iff R is

→reflexive

→Anti symmetric

→Transitive

**Eg:**

1. Less than or equal to
2. Greater than or equal to
3. Subset over powerset
4. Integral Multiple(divider)
5. Lex:cograhic ordering

* **Total Ordering**

If for every x,yЄP we have either x<y or y<x then < is total ordering <P,<> in called Toset or chain.

* Hasse digrams make it easy to represent poset.

1. **Let x={2,3,6,12,24,36} and the relation < be such that x<y if x divides y. Draw the Hasse diagram of <x,<>**

24 36

12

**6**

1. **3**

Hasse digrams make it easy to represent poset.

* **Least and Greatest Member:**
* If there exists an element yЄP such that y<x for all xЄP, then y is called Least member.
* Least member if it exists, is unique.
* Generally, Least member is denoted by 0 and greatest by 1.
* **Minimal and Maximal Member:**
* An element yЄP is called a minimal member of P relative to a partial ordering < if for no ‘x’ЄP is x<y.
* Minimal member need not be unique.
* Distinct minimal member are in comparable.

**Eg:**

B→Greatest, Maximal {B}

A→Least, Minimal {A}

C→Greatest, Maximal {C}

No Least, Minimal{A,B}

R S

Maximal{R,S}, No Greatest

P Q Minimal{P,Q}, No least

* **Lower and Upper Bound:**
* Let <P,<> be a partially ordered set and let A⊆P.
* Any element xЄP is an upper bound for A if for all aЄA, a<x.
* **Least Upper Bound and Greatest Lower Bound:**
* Let (P,<) be a partially ordered set and let A⊆P.
* An element xЄP is a least upper bound(LUB) or supremum(sup), for A if x is an upperbound for A and x<y where y is any upperpound for A.
* LUB if exists, is unique.

**Eg:**

A={a,b,c} <P(A), ⊆>

{a,b} {a,c} {b,c}

{a} {b} {c}

∅

**→M⊆P(A)** A

{a,b}

{b}

→Upper bounds=A

LUB=A

→Lower bounds={b},∅

GLB={b}

**→N⊆P(A)**

{a,b}

{b}

→Upper bounds={a,b},A

LUB={a,b}

→Lower bounds={b},∅

GLB={b}

→**Q⊆P(A)**

{b,c}

{b} {c}

→Upper bounds={b,c},A

LUB={b,c}

→Lower bounds=∅

GLB=∅

**Eg:**

24 36

12

6

2 3

Y = 6

2 3

→Upper bounds(Y)=6,12,24,36

LUB(Y)=6

→Lower bounds={Y} - None

GLB=None

**Q)Let X={2,3,6,12,24}. Let S be the partial order defined y x<y if x divides y. The number of edges in the Hasse diagram of (X,<) is?**

**a) 3**

**b)4**

**c)9**

**d)None of above Gate 1996**

**A)** 24

12

6

2 3

Therefore number of edges is four option(b)

* **Lattice**
* A lattice is a partially ordered set (L,<) in which every pair of elements a,bЄL has a greatest lower bound and a least upper bound.

LUB(a,b) = a+b join

GLB(a,b) = a.b meet

**Common Lattices**

1. **b)**

1. **c) e)**

**f) g)**

* **Non Lattices**

No GLB No GLB, LUB No LUB

**2)Consider the following Hasse diagrams**

1. **(2) (3) (4)**

**Which of the above represent a Lattice?**

**a) (1) and (3) only**

**b) (2) and (4) only**

**c) (4) only**

**d) (1), (2) and (3) only Gate 2008**

**A)**

No GLb, LUB No LUB

Therefore, (1) and (3) only is the correct answer

* **Properties Of Lattice:**
* Idempotent

a.a=a, a+a=a

↓ ↓

meet join

* Commutative

a.b=b.a, a+b=b+a

* Associative

(a.b).c=a.(b.c)

(a+b)+c=a+(b+c)

* Absorption

A.(A+B)=A

A+(AB)=A

* Distributive Inequality

a.(b+c)>(a.b)+(a.c)

a+(bc)<(a+b).(a+c)

* **Some Special Lattices**
* **Complete**
* A lattice is called complete if each of its non empty subsets has a LUB and GLB.
* Every finite lattice must be complete.
* **Complement**
* In a bounded lattice an element bЄL is called a complement of an element aЄL if

a.b=0

a+b=1

**Eg:** 1

x1 x2

0

Complement of x1 is x2

1

x1 x2 x3

0

Complement of x1 are x2, x3

x2 are x1, x3

1

x1 x2

0 x3

Complement of x1 are x2, x3

1

P Q R

A B C

0

Complement of P is C

A is R

* **Distributive Lattice**

In distribute lattice meet an join follow distributive property.

a.(b+c)=ab+ac

a+(bc)=(a+b)(a+c)

**Eg:**

* **Non Distributive**

**Note:**

* Every chain is a distributive lattice
* The direct product of any two distributive lattice is a distributive lattice.
* **Boolean Algebra**

A Boolean algebra is a bounded, complemented distributive lattice.

**Q) The inclusion of which of the following sets into S={{1,2}, {1,2,3}, {1,2,5}, {1,2,4}, {1,2,3,4,5}] is necessary and sufficient to make S a complete lattice under the partial order defined by set containment?**

**a) {1}**

**b) {1}, {2,3}**

**c) {1}, {1,3}**

**d) {1}, {1,3}, {1,2,3,4}, {1,2,3,5} Gate 2004**

**A)**  {1,2,3,4,5}

{1,2,3} {1,2,4} {1,3,5}

GLB

missing

{1,2}

{1,2,3,4,5}

{1,2,3} {1,2,4} {1,3,5}

{1,2}

{1}

Option(a) is the answer

**Q) The complements of the element ‘a’ in the lattice shown in figure is (are) ?**

**1**

**b**

**a d c e**

1. **Gate 1998**
2. a.d=0 a+d=1 ✓

**a.**b=0 a+b=1 ✓

a.c=0 a+c=1 ✓

a.e=0 a=e=1 ✓

⇒’a’ complements are b,c,d,e.