$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \; ; \quad a_{12} = a_{21}$$

symmetric matrix ⇒ real eigenvalues
and a full set of orthogonal real eigenvectors

$$R := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

eigenvectors

we want to compute

$$\boxed{A = R D R^T, \text{ where } D := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$$

$$D = R^T A R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

off-diagonal elements $= a_{12}(\cos^2\theta - \sin^2\theta) + (a_{22} - a_{11})\cos\theta\sin\theta = 0$

$$\Rightarrow \begin{pmatrix} a_{12} \, , & \frac{a_{22}-a_{11}}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix} = 0$$

$d \cdot \left(\frac{a_{22}-a_{11}}{2}, a_{12} \right)$ is orthogonal to $\left(a_{12}, \frac{a_{22}-a_{11}}{2} \right)$

$$\Rightarrow \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix} = d \cdot \begin{pmatrix} \frac{a_{22}-a_{11}}{2} \\ a_{12} \end{pmatrix},$$

d can have any sign, but the norm constraint is to be satisfied

$$\Rightarrow \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix} = \pm \begin{pmatrix} \frac{a_{22}-a_{11}}{2} \\ a_{12} \end{pmatrix} \cdot \frac{1}{\sqrt{\left(\frac{a_{22}-a_{11}}{2}\right)^2 + a_{12}^2}} \qquad (1)$$

⚠ numerical subtlety: make sure to avoid overflows when computing the denominator

we need to compute $\cos\theta$ and $\sin\theta$:

$$\cos\theta = \frac{\sin 2\theta}{2\sin\theta}, \qquad \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \qquad (2)$$

⚠ numerical subtlety: if $\theta \approx 0$ then $\sin\theta \approx 0$ (bad division) and $\cos 2\theta \approx 1$ (bad subtraction)

since (1) offers us a degree of freedom, let us use it:
the sign in (1) is chosen s.t. $\cos 2\theta \leq 0$. this way, $\sin\theta \geq 1/\sqrt{2}$.

$$\lambda_1 = a_{22}\sin^2\theta + a_{12}\sin 2\theta + a_{11}\cos^2\theta$$

$$\lambda_2 = a_{11}\sin^2\theta - a_{12}\sin 2\theta + a_{22}\cos^2\theta$$

⚠ whenever sorting λ_i, make sure that the eigenvectors form a right-hand pair (negate one of the vectors if swapping the eigenvalues)