2D Divergence theorem:

$$\iint \operatorname{div}\vec F\, dA = \oint \vec F\cdot\vec n\, ds$$

normalized!

$$\nabla\varphi = \begin{bmatrix} \partial\varphi/\partial x \\ \partial\varphi/\partial y \end{bmatrix}$$

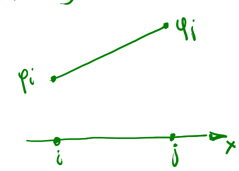
$$\operatorname{div}\begin{bmatrix}\varphi\\0\end{bmatrix} = \frac{\partial\varphi}{\partial x} + 0$$

$$\operatorname{div}\begin{bmatrix}0\\\varphi\end{bmatrix} = 0 + \frac{\partial\varphi}{\partial y}$$

$$\iint \frac{\partial\varphi}{\partial x}\, dA = \underbrace{\iint \operatorname{div}\begin{bmatrix}\varphi\\0\end{bmatrix} dA = \int \begin{bmatrix}\varphi\\0\end{bmatrix}\cdot\vec n\, ds}_{\text{div theorem}} = \int \varphi\cdot n_x\, ds$$

$$\iint \frac{\partial\varphi}{\partial y}\, dA = \iint \operatorname{div}\begin{pmatrix}0\\\varphi\end{pmatrix} dA = \int \varphi\, n_y\, dS$$

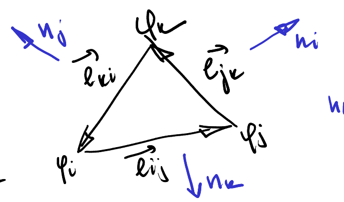
the integral of a linear function over a segment is the average of the vertices



$$\varphi(x) = \varphi_i + (\varphi_j - \varphi_i)\,\frac{x-i}{j-i}$$

$$\int_i^j \varphi(x)\,dx = \frac{\varphi_i+\varphi_j}{2}\cdot(j-i)$$

Let us discretize:



normals are normalized!!!!

the gradient is constant over the triangle, we have 3 samples $\varphi_i\ \varphi_j\ \varphi_k$

Since the gradient is constant, we have

$$(\nabla\varphi)_{ijk} = \begin{pmatrix} \left(\frac{\partial\varphi}{\partial x}\right)_{ijk} \\ \left(\frac{\partial\varphi}{\partial y}\right)_{ijk} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial\varphi}{\partial x}\right)_{ijk}\cdot \frac{\iint 1\,dA}{A_{ijk}} \\ \left(\frac{\partial\varphi}{\partial y}\right)_{ijk}\cdot \frac{\iint 1\,da}{A_{ijk}} \end{pmatrix} = \begin{pmatrix} \frac{\iint \left(\frac{\partial\varphi}{\partial x}\right)_{ijk} dA}{A_{ijk}} \\ \frac{\iint \left(\frac{\partial\varphi}{\partial y}\right)_{ijk} dA}{A_{ijk}} \end{pmatrix} = \begin{pmatrix} \frac{\int \varphi\cdot n_x\, dS}{A_{ijk}} \\ \frac{\int \varphi\cdot n_y\, dS}{A_{ijk}} \end{pmatrix} =$$

n is piecewise constant, and φ is linear.

$$\Rightarrow (\nabla\varphi)_{ijk} = \frac{\vec n_i\cdot\left(\frac{\varphi_k+\varphi_j}{2}\right)\cdot\|\vec e_{jk}\| + \vec n_j\cdot\left(\frac{\varphi_i+\varphi_k}{2}\right)\|\vec e_{ki}\| + n_k\left(\frac{\varphi_i+\varphi_j}{2}\right)\cdot\|\vec e_{ij}\|}{A_{ijk}} =$$

$$= \frac{\varphi_i\left(\vec n_j\|\vec e_{ki}\| + \vec n_k\|\vec e_{ij}\|\right) + \varphi_j\left(\vec n_i\|\vec e_{jk}\| + \vec n_k\|\vec e_{ij}\|\right) + \varphi_k\left(\vec n_i\|\vec e_{jk}\| + \vec n_j\|\vec e_{ki}\|\right)}{2A_{ijk}} =$$

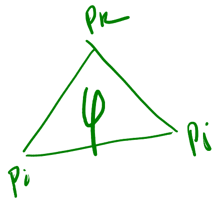
$$= -\frac{1}{2A_{ijk}}\left(\varphi_i\cdot\vec n_i\|\vec e_{jk}\| + \varphi_j\cdot\vec n_j\|\vec e_{ki}\| + \varphi_k\vec n_k\|\vec e_{ij}\|\right)$$

note that $\vec n_i\cdot\|\vec e_{jk}\| + \vec n_j\|\vec e_{ki}\| + \vec n_k\|\vec e_{ij}\| = \vec 0$ (Gauss theorem)

3D

The integral of a linear function over a triangle is the average of the vertices

any point of the triangle can be described as



$$p(u,v) = (1-u-v)\,p_i + u\,p_j + v\,p_k$$

a linear function φ s.t. $\varphi(p_i)=\varphi_i$ $\varphi(p_j)=\varphi_j$ $\varphi(p_k)=\varphi_k$

can be written as $\varphi(u,v) = (1-u-v)\,\varphi_i + u\,\varphi_j + v\,\varphi_k$

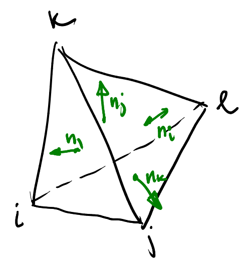
$$\int_\Delta \varphi = 2A_{ijk}\int_0^1\int_0^{1-u}\left[(1-u-v)\varphi_i + u\varphi_j + v\varphi_k\right]dv\,du = \frac{A_{ijk}}{3}\left(\varphi_i+\varphi_j+\varphi_k\right)$$

Divergence theorem:

$$\iiint \operatorname{div}\vec F\, dV = \oiint \vec F\cdot\vec n\, dA$$

normalized!

$$(\nabla\varphi)_{ijkl} = \frac{1}{V_{ijkl}}\begin{pmatrix}\iint \varphi\cdot n_x\,dA \\ \iint \varphi\cdot n_y\,dA \\ \iint \varphi\cdot n_z\,dA\end{pmatrix} =$$



Same naming convention: $\vec n_i$ is the normalized normal to the facet opposite to the vertex i

(φ is linear, n is piecewise constant)

$$= \frac{1}{V_{ijkl}}\left(\vec n_i\cdot\frac{(\varphi_j+\varphi_k+\varphi_l)}{3}\cdot A_{jkl} + \vec n_j\cdot\frac{(\varphi_i+\varphi_k+\varphi_l)}{3}\cdot A_{ikl} + \vec n_k\cdot\frac{(\varphi_i+\varphi_j+\varphi_l)}{3}\cdot A_{ijl} + \vec n_l\cdot\frac{(\varphi_i+\varphi_j+\varphi_k)}{3}\cdot A_{ijk}\right)$$

(Gauss still applies)

$$= -\frac{1}{3V_{ijkl}}\left(\varphi_i\,\vec n_i\,A_{jkl} + \varphi_j\,\vec n_j\,A_{ikl} + \varphi_k\,\vec n_k\,A_{ijl} + \varphi_l\,\vec n_l\,A_{ijk}\right)$$