The convergence of the probability towards 0.5 is very fast. It is thus possible to generate large random polyominoes effectively with our algorithm. However, a large random polyomino tends to look like a rectangle standing on a corner (see also [Ben74]). This is not surprising, since the interpretation of a random 0-1 sequence as a left-up walk will asymptotically yield a diagonal.

References

[Ben74] E. Bender: Convex n-ominoes *Discr. Math. 8 (1974) 219 – 226*

[Ber81] C. Berge, C. C. Chen, V. Chvatal and C. S. Seow: Combinatorial properties of polyominoes *Combinatorica 1 (1981) 217 – 224*

[Con90] J. H. Conway and J. C. Lagrias: Tiling with polyominoes and combinatorial group theory *J. Comb. Th. A53 (1990) 183 – 208*

[Del84] M. P. Delest and G. Viennot: Algebraic languages and polyominoes enumeration *Theor. Comp. Sci. 34 (1984) 169 – 206*

[Del91] M. Delest: Polyominoes and animals: Some recent results *J. Math. Chem. 8 (1991) 3 – 18*

[Gol89] S. Golomb: Polyominoes while tile rectangles *J. Comb. Th. A51 (1989) 117 – 124*

[Gutt88] A. J. Guttmann: The number of convex polygons on the square and honeycomb lattices *J. Phys. A21 (1988) 467 – 474*

[Lin88] K. Y. Lin and S. J. Chang: Rigorous results for the number of convex polygons on the square and honeycomb lattices *J. Phys. A21 (1988) 2635 – 2642*

[Mar91] G. E. Martin: Polyominoes: A Guide to Puzzles and Problems in Tiling *Mathematical Association of America Spectrum Series 1991*

[Pol69] G. Pólya: On the number of certain lattice polygons *J. Comb. Th. 6 (1969) 102 – 105*

Quantum Letter-Place Algebra

Rosa Q. Huang[1]
Math. Dept., VPI&SU, Blacksburg, VA 24061
and
James J. Zhang[2]
Math. Dept., Univ. of Michigan, Ann Arbor, MI 48109

Abstract

A quantum analogue of the supersymmetric letter-place algebra of Rota and his school is presented. Although it is significantly different than the ordinary letter-place algebra, it still possesses the most important combinatorial structure: standard quantum left (resp. right) bitableaux form a linear basis of the quantum letter-place algebra. The quantum straightening formula holds for quantum letter-place algebras.

0. Introduction

In [DKR] and [GRS], Rota and his school systematically developed the theory of the supersymmetric letter-place algebra. Although the algebraic structure of the letter-place algebra is isomorphic to the algebra generated by all minors of a generic supersymmetric matrix, the invention of the so-called letters and places distinguishes the letter-place algebra with its broad applicability and rich combinatorial structure. For this reason, this algebra has been widely applied, and proved to be an effective algebraic-combinatorial tool so far, to different areas like classical invariant theory ([GRS], [KR], [Hu1]), representation theory ([BPT], [BT]), resolutions of certain algebras and modules ([AR], [BR]), projective geometry ([RS], [Wh]), rigidity theory ([WW]), etc.

In the present paper we use Manin's approach of quantum groups to develop a quantum analogue of supersymmetric letter-place algebra. Generalizing Manin's definition of quantum general linear supergroups, we define (supersymmetric) quantum letter-place algebra $Super[L|P]_q$ by requiring that both the left co-representation T_l from the quantum letter algebra $super[L]_q$ to $Super[L|P]_q \otimes Super[P]_q$ and the right co-representation T_r from quantum place algebra $Super[P]_q$ to $Super[L]_q \otimes Super[L|P]_q$ are algebra homomorphisms. In the case that $L = P$, the super bialgebra $Super[L|L]_q$ coincides with the quantum general linear super(semi-)group E_q defined in [Ma2]. Hence $Super[L|P]_q$ can be also viewed as a supersymmetric analogue of the quantum linear semi-groups $M_n(q)$. Again, as an algebra, $Super[L|P]_q$ is isomorphic to the algebra generated by all (left or right) quantum minors of a generic quantum supersymmetric matrix. However this quantum letter-place algebra is significantly different than the ordinary letter-place algebra. For examples, one has to distinguish between "left" and "right" quantum biproducts, although it turns out that they only differ by a scalar multiple, and only left-sided (resp. right-sided) Laplace expansion holds for left (resp. right) quantum biproducts (Proposition 4). No clear relation has been

[1]The research of R. Huang is supported in part by an NSA grant.
[2]The research of J. Zhang supported in part by an NSF grant.

found so far between the products of (left or right) quantum biproducts in different orders and, the worst of all, the exchange identity [GRS, Prop 10], which is one of the sharpest formula in the ordinary letter-place algebra, no longer holds and has to be replaced by a much weaker identity (Lemma 9). Using this identity, the identities in the definition of quantum flag scheme (see [TT]) can be verified immediately. More important, we can show that the quantum letter-place algebra still possesses the marvelous combinatorial structure: standard quantum left (resp. right) bitableaux form a linear basis of the quantum letter-place algebra (Theorem 8), which is the well-known straightening formula in the classical case. However, the actual algorithm of expressing a given quantum (left or right) bitableau as a linear combination of standard ones becomes much more mysterious due to noncommutativity of the algebra.

The quantization accomplished here is not just an analogue of its own interest. Instead, it suggests various directions for further developments by using the supersymmetric quantum letter-place algebra, such as: to study the invariants of quantum general linear groups acting on quantum spaces ([Ma1]); to understand the structure of the quantum letter-place algebra as a module of quantum general linear Lie super algebra, which will first requires the full understanding of the action of differential operators of the form $y\frac{\partial}{\partial x}$ (they are also called polarization operators) on the quantum letter-place algebra; to study the combinatorics of the quantized enveloping algebra of general linear Lie super algebra.

The paper is organized as follows. In Section 1 and 2, we define supersymmetric quantum algebra $Super[L]_q$ and supersymmetric quantum letter-place algebra $Super[L|P]_q$. Some basic properties and explicit relations of the algebra $Super[L|P]_q$ are given. In Section 3, we define left and right quantum biproducts and prove the corresponding Laplace expansions. Relationship between left and right biproducts is studied. In Section 4, we state and prove the quantum straightening formula for $Super[L|P]_q$. A weaker form of exchange identity is also given for the quantum case.

We would like to thank the referee for very useful suggestions and comments and thank Naihuan Jing for several enlightening conversations on the subject.

1. Supersymmetric Quantum Algebras

Throughout this paper K will be a fixed field with any characteristic, and algebras will be super algebras, namely, $\mathbf{Z}_2$-graded associative K-algebras $A = A_0 \oplus A_1$ with identity 1. We say non-zero elements in A_0 are **positive** and non-zero elements in A_1 are **negative**. The identity element 1 has $\mathbf{Z}_2$-degree 0 and hence it is positive. All algebra homomorphisms from one ($\mathbf{Z}_2$-graded) algebra to another will be $\mathbf{Z}_2$-graded homomorphisms. Given two $\mathbf{Z}_2$-graded algebras R and S, the usual tensor product of algebras R and S is also a $\mathbf{Z}_2$-graded algebra. However the multiplication of the usual tensor product of algebras will not be used in this paper. Instead, we are going to use the multiplication with sign as follows. Let $R\otimes S$ be the tensor product of R and S as K-vector spaces. The multiplication of **super tensor product** $R\otimes S$ is defined by $(a\otimes b)\cdot(c\otimes d) = (-1)^{|b||c|}ac\otimes bd$ for all $\mathbf{Z}_2$-homogeneous elements $a, c \in R$ and $b, d \in S$. (The multiplication of the usual tensor product of two algebras has the above form without the sign $(-1)^{|b||c|}$.) The super tensor product $R\otimes S$ is an associative $\mathbf{Z}_2$-graded algebra. For all homogeneous elements a and b, the $\mathbf{Z}_2$-degree $|a\otimes b|$ of $a\otimes b$ is $|a|+|b|$ modulo 2. From now on, $\otimes$ means super tensor product.

Let $L = L^+ \cup L^-$ be a $\mathbf{Z}_2$-graded set with a linear order $<$. We have $|a| = 0$ for all $a \in L^+$ and $|a| = 1$ for all $a \in L^-$. Elements in L^+ are called **positive variables** and elements in L^- **negative variables**, and sometimes we indicate a positive variable by a^+ and a negative variable by a^-. The **tensor algebra** (it is also called free algebra) $Tens[L]$, generated by the elements of L, is a $\mathbf{Z}_2$-graded algebra with $|a_1\cdots a_n| = |a_1| + \cdots + |a_n|$ modulo 2; any $\mathbf{Z}_2$-graded algebra is a homomorphic image of a tensor algebra $Tens[L]$ with a suitable choice of L^+ and L^-. Let q be a non-zero element in K. The **supersymmetric quantum algebra** generated by L (with a linear order $<$), denoted by $Super[L]_q$, is defined to be the quotient associative algebra of $Tens[L]$ subject to the following relations:

(i) $ba = (-1)^{|a||b|}q^{-1}ab$, whenever $a < b$,

(ii) $a^-a^- = 0$.

Remark. The supersymmetric quantum algebra $Super[L]_{q^{-1}}$ coincides with "quantum superspace" A_q (with all $q_{ij} = q$) defined in [Ma2, page 136].

Let $Mon(L)$ be the set of words, namely, monomials on L. Given a word $u \in Mon(L)$, let $length(u)$ be the number of variables occurred in u, and let $|u|$ be the $\mathbf{Z}_2$-degree of u, which is the number of negative variables occurred in u modulo 2. For two words $u, v \in Mon(L)$, one can easily check that $uv = (-1)^{|u||v|}q^{-i(uv)+i(vu)}vu$, where $i(u)$ denotes the number of inversions in the word u, namely, if $u = a_1a_2\cdots a_k$, then $i(u)$ is the number of pairs (a_i, a_j) such that $a_i > a_j$ and $i < j$.

The **coproduct** Δ, which is a linear (but not algebraic) operator from $Super[L]_q$ to the vector space $Super[L]_q \otimes Super[L]_q$, is defined such that for any monomial $u \in Mon(L)$,

(i) $\Delta u = 0$ if some negative variable occurs more than once in u; otherwise

(ii) $\Delta u = \sum_{(v,w)} k_{vw} v\otimes w$, where the sum ranges over the partitions (v, w) of u as multisets and each coefficient k_{vw} satisfies

$$u = k_{vw}vw \tag{1}$$

in $Super[L]_q$, where $k_{vw} \in K$. For example, if $a^- < b^-$ and $c^+ < d^+$, then

$$\Delta a^-b^- = 1\otimes a^-b^- + a^-\otimes b^- - qb^-\otimes a^- + a^-b^-\otimes 1.$$

For convenience, we will use the Sweedler notation $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$ to denote the sum $\sum_{(v,w)} k_{vw}v \otimes w$. It can be verified directly that Δ is well-defined and it satisfies the coassociativity law: $(id\otimes\Delta)\circ\Delta = (\Delta\otimes id)\circ\Delta$ as linear maps from $Super[L]_q$ to $Super[L]_q^{\otimes 3}$.

Remark. The coproduct defined here cannot be made into an algebra homomorphism, since

$$\Delta(a^+a^+) = 1 \otimes a^+a^+ + a^+ \otimes a^+ + a^+a^+ \otimes 1$$

instead of having $2a^+\otimes a^+$ in the middle term.

2. Quantum Letter-Place Algebras

Let $L = L^+ \cup L^-$ and $P = P^+ \cup P^-$ be two linearly ordered $\mathbf{Z}_2$-graded finite sets. Elements in L are called **letters** and elements in P **places**. In this case we call $Super[L]_q$ (resp. $Super[P]_q$) **supersymmetric quantum letter** (resp. **place**) **algebra**. Let $Tens[L|P]$ denote the tensor algebra generated by all $\mathbf{Z}_2$-graded elements $(a|\alpha)$ where $a \in L$, $\alpha \in P$ and $|(a|\alpha)| = |a| + |\alpha|$ modulo 2. Given a monomial $m = (a|\alpha)(b|\beta)\cdots(c|\gamma)$, the $\mathbf{Z}_2$-degree of m is given by

$$|m| \equiv |a| + |\alpha| + |b| + |\beta| + \cdots + |c| + |\gamma| \pmod 2.$$

To keep notations simple, we usually write $|(a|\alpha)|$ as $|a\alpha|$.

Let $Tens[L|P] \otimes Super[P]_q$ and $Super[L]_q \otimes Tens[L|P]$ denote the super tensor products. Then the multiplications of these two algebras satisfy:

$$(m \otimes \mu) \cdot (m' \otimes \mu') = (-1)^{|\mu||m'|} mm' \otimes \mu\mu',$$

$$(u \otimes m) \cdot (u' \otimes m') = (-1)^{|m||u'|} uu' \otimes mm',$$

where m, m'; μ, μ'; u, u' are $\mathbf{Z}_2$-homogeneous elements in $Tens[L|P]$, $Super[P]_q$, $Super[L]_q$, respectively. Next, we define the maps

$$T_l : L \to Tens[L|P] \otimes Super[P]_q,$$
$$T_r : P \to Super[L]_q \otimes Tens[L|P]$$

by $T_l(a) = \sum_{\alpha \in P} (a|\alpha) \otimes \alpha$ and $T_r(\alpha) = \sum_{a \in L} a \otimes (a|\alpha)$. The **supersymmetric quantum letter-place algebra**, denoted by $Super[L|P]_q$, is defined to be the quotient $\mathbf{Z}_2$-graded associative K-algebra of $Tens[L|P]$ with minimal relations such that the map T_l can be extended to an algebra homomorphism

$$T_l : Super[L]_q \to Super[L|P]_q \otimes Super[P]_q$$

and the map T_r can be extended to an algebra homomorphism

$$T_r : Super[P]_q \longrightarrow Super[L]_q \otimes Super[L|P]_q.$$

The algebra homomorphisms T_l and T_r are called a **left co-representation** and a **right co-representation** of $Super[L|P]_q$ respectively. For convenience, we will keep the same notation for an element in $Tens[L|P]$ and its image in $Super[L|P]_q$ under the canonical map.

Proposition 1 *The supersymmetric quantum letter-place algebra $Super[L|P]_q$ is the quo-*

tient algebra of $Tens[L|P]$ subject to the following relations

$$
\begin{aligned}
&(R1) \quad (a^-|\alpha^+)^2 = 0\\
&(R2) \quad (a^+|\alpha^-)^2 = 0\\
&(R3) \quad (a^-|\beta)(a^-|\alpha) = (-1)^{|a\beta||a\alpha|} q (a^-|\alpha)(a^-|\beta) \text{ for all } \alpha < \beta\\
&(R4) \quad (a^+|\beta)(a^+|\alpha) = (-1)^{|a\beta||a\alpha|} q^{-1} (a^+|\alpha)(a^+|\beta) \text{ for all } \alpha < \beta\\
&(R5) \quad (b|\alpha^-)(a|\alpha^-) = (-1)^{|b\alpha||a\alpha|} q (a|\alpha^-)(b|\alpha^-) \text{ for all } a < b\\
&(R6) \quad (b|\alpha^+)(a|\alpha^+) = (-1)^{|b\alpha||a\alpha|} q^{-1} (a|\alpha^+)(b|\alpha^+) \text{ for all } a < b\\
&(R7) \quad (b|\alpha)(a|\beta) = (-1)^{|b\alpha||a\beta|} (a|\beta)(b|\alpha) \text{ for all } a < b, \alpha < \beta\\
&(R8) \quad (b|\beta)(a|\alpha) - (-1)^{|b\beta||a\alpha|} (a|\alpha)(b|\beta)\\
&\qquad = (-1)^{|b\beta||a| + |b||\beta|} (q^{-1} - q)(a|\beta)(b|\alpha) \text{ for all } a < b, \alpha < \beta.
\end{aligned}
$$

One observes that the sign rules in the above relations agree with the sign rules in the supersymmetric letter-place algebra defined in [GRS] .

Remarks. (1) If $L = P$, then $Super[L|L]_{q^{-1}}$ coincides with the "quantum linear super semi-group" E_q (with all $q_{ij} = q$) defined in [Ma2, page 136]. As a consequence, if $L = L^- = P = P^-$, then $Super[L|L]_q$ is isomorphic to the quantum semi-group $M_n(q)$; if $L = L^+ = P = P^+$, then $Super[L|L]_q$ is isomorphic to the quantum semi-group $M_n(q^{-1})$.
(2) The parameter q can be replaced by $\mathbf{q}$ which is a set $\{q_{ab} = \pm q | \forall a < b \text{ in } L\} \bigcup \{q_{\alpha\beta} = \pm q | \forall \alpha < \beta \text{ in } P\}$ where the signs of q_{ab} and $q_{\alpha\beta}$ are independent. Then we define the quantum letter algebra $Super[L]_{\mathbf{q}}$ to be generated by elements of L subject to the following relations:
(i) $ba = (-1)^{|a||b|} q_{ab}^{-1} ab$, where $a < b$ in L,
(ii) $a^- a^- = 0$.
Similarly the quantum place algebra $Super[P]_{\mathbf{q}}$ and quantum letter-place algebra $Super[L|P]_{\mathbf{q}}$ are defined. All results which hold for $Super[L|P]_q$ in the paper hold for $Super[L|P]_{\mathbf{q}}$.
(3) The base field K can be replaced by any commutative ring with a sub-field containing q, and everything still works.
(4) Given any element $\alpha \in P^+$, the map $a \longrightarrow (a|\alpha)$ defines a unique injective algebra homomorphism from $Super[L]_q$ to $Super[L|P]_q$. Given any element $a \in L^+$, the map $\alpha \longrightarrow (a|\alpha)$ defines a unique injective algebra homomorphism from $Super[P]_q$ to $Super[L|P]_q$. Consequently, if $P = P^+ = \{\alpha\}$, then $Super[L]_q \cong Super[L|P]_q$; if $L = L^+ = \{a\}$, then $Super[P]_q \cong Super[L|P]_q$.

Suppose now $L = P$. One can prove that there is a unique algebra homomorphism Λ from $Super[L|L]_q$ to $Super[L|L]_q \otimes Super[L|L]_q$ defined by $\Lambda((a|c)) = x_{(a|c)} = \sum_b (a|b) \otimes (b|c)$.

Proposition 2 [Ma2, Theorem3.2] *The algebra $Super[L|L]_q$ is a super bialgebra* with co-multiplication Λ defined above and counit ε defined by $\varepsilon((a|c)) = \delta_{ac}$, where $\delta_{ac} = 0$ if $a \neq c$ and $\delta_{aa} = 1$.*

*: Super bialgebra is a $\mathbf{Z}_2$-graded algebra satisfying all axioms of bialgebra where super tensor product is used instead of usual tensor product.

Remark. There are some study of $E_q = Super[L|L]_{q^{-1}}$ in [Ma2, Section 4.3]. The algebra E_q is not a Hopf super algebra, but the natural Hopf super algebra associated to E_q is the Hopf envelope H_q of E_q (see [Ma2]).

The supersymmetric quantum letter-place algebra $Super[L|P]_q$ is a continuous deformation of the supersymmetric letter-place algebra $Super[L|P]$ with the parameter q. If $q = 1$, then $Super[L|P]_1 = Super[L|P]$. It is well-known that the set of all ordered monomials (including 1), in which no negative variable appears more than once, in $Mon(L|P)$ forms a K-linear basis of $Super[L|P]$. This statement is also true for $Super[L|P]_q$ for all non-zero $q \in K$. Let us recall the ordering "<" on $Mon(L|P)$:

$$(a|\alpha) < (b|\beta) \text{ if and only if either } a < b \text{ or } a = b \text{ and } \alpha < \beta.$$

A monomial $(a_1|\alpha_1)\cdots(a_n|\alpha_n)$ is called **ordered** or **non-decreasing** if $(a_i|\alpha_i) \leq (a_{i+1}|\alpha_{i+1})$ for all $i = 1, \cdots, n-1$.

Theorem 3 *The set of ordered monomials, in which no negative variable appears more than once, in $Mon(L|P)$ is a K-linear basis of $Super[L|P]_q$ for all non-zero $q \in K$.*

3. Left and Right Quantum Biproducts

In this section we are going to use the left and the right co-representations T_l and T_r to define left and right supersymmetric quantum biproducts. Given a word $u = a_1a_2\cdots a_k \in Mon(L)$, let $N(u) = \sum_{i<j}|a_i||a_j|$. One can see that $N(u)$ only depends on the **content** of u, which is the multiset of the elements occurred in u; in fact, $N(u) = \frac{1}{2}l(l-1)$ where l is the number of negative variables in u. Given a word $\mu \in Mon(P)$, $N(\mu)$ is defined in the same way. We define bilinear maps:

$$(\cdot|\cdot)_r : Super[L]_{q^{-1}} \otimes Super[P]_q \to Super[L|P]_q,$$

$$(\cdot|\cdot)_l : Super[L]_q \otimes Super[P]_{q^{-1}} \to Super[L|P]_q,$$

such that

$$T_r(\mu) = \sum(-1)^{N(u)} u \otimes (u|\mu)_r, \tag{2}$$

$$T_l(u) = \sum(-1)^{N(\mu)} (u|\mu)_l \otimes \mu, \tag{3}$$

where the sum in (2) ranges over the words u in $Mon(L)$ of different contents, namely, over a monomial basis of $Super[L]_q$. In another word, we take one and only one word u from each collection of words of the same content to form the summands in (2). For example, the sum in (2) can range over all ordered non-zero monomials $a_1a_2\cdots a_k$ in $Mon(L)$, where $a_1 \leq a_2 \leq \cdots \leq a_k$. The sum in (3) is similarly defined. If no negative variable occurs more than once in u or in μ, then $(u|\mu)_l$ and $(u|\mu)_r$ are determined by (2) and (3). If some negative variable occurs more than once in u or in μ, we set

$$(u|\mu)_r = (u|\mu)_l = 0. \tag{4}$$

For example, let $L = \{a^+, b^+\}$ and $P = \{\alpha^+, \beta^+\}$ with $a^+ < b^+$ and $\alpha^+ < \beta^+$. Then

$$\begin{aligned}
T_r(\alpha\beta) &= T_r(\alpha)T_r(\beta) \\
&= (a \otimes (a|\alpha) + b \otimes (b|\alpha)) \cdot (a \otimes (a|\beta) + b \otimes (b|\beta)) \\
&= a^2 \otimes (a|\alpha)(a|\beta) + ab \otimes (a|\alpha)(b|\beta) + ba \otimes (b|\alpha)(a|\beta) + b^2 \otimes (b|\alpha)(b|\beta) \\
&= a^2 \otimes (a|\alpha)(a|\beta) + ab \otimes [(a|\alpha)(b|\beta) + q^{-1}(b|\alpha)(a|\beta)] + b^2 \otimes (b|\alpha)(b|\beta) \\
&= a^2 \otimes (a|\alpha)(a|\beta) + ba \otimes [(b|\alpha)(a|\beta) + q(a|\alpha)(b|\beta)] + b^2 \otimes (b|\alpha)(b|\beta).
\end{aligned}$$

Therefore by definition,

$$\begin{aligned}
(aa|\alpha\beta)_r &= (a|\alpha)(a|\beta), \\
(bb|\alpha\beta)_r &= (b|\alpha)(b|\beta), \\
(ab|\alpha\beta)_r &= (a|\alpha)(b|\beta) + q^{-1}(b|\alpha)(a|\beta), \\
(ba|\alpha\beta)_r &= (b|\alpha)(a|\beta) + q(a|\alpha)(b|\beta).
\end{aligned}$$

We call $(u|\mu)_r$ and $(u|\mu)_l$ a **right** and a **left supersymmetric quantum biproduct** respectively. Obviously, $(u|\mu)_r = (u|\mu)_l = 0$ if $length(u) \neq length(\mu)$.

Remark. It is clear that $T_r(\mu\alpha^-\alpha^-\nu) = 0$ implies $(u|\mu\alpha^-\alpha^-\nu)_r = 0$ for all $u \in Mon(L)$. However, we can not get $(ua^-a^-v|\mu)_r = 0$ by using the same idea; instead, we have to set $(ua^-a^-v|\mu)_r = 0$.

Proposition 4 *Supersymmetric quantum biproducts have the following Laplace expansions:*

$$(u|\mu\nu)_r = \sum_u (-1)^{|\mu||u_{(2)}|}(u_{(1)}|\mu)_r \cdot (u_{(2)}|\nu)_r, \tag{5}$$

$$(uv|\mu)_l = \sum_\mu (-1)^{|\mu_{(1)}||v|}(u|\mu_{(1)})_l \cdot (v|\mu_{(2)})_l, \tag{6}$$

where $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$ and $\Delta\mu = \sum_\mu \mu_{(1)} \otimes \mu_{(2)}$ are the coproducts defined on $Super[L]_{q^{-1}}$ and $Super[P]_{q^{-1}}$ respectively.

Remark. Laplace expansions for classical quantum groups $GL_n(q)$ are known (see for example [TT] and [PW]). For right or left supersymmetric quantum biproducts, there are only one-sided Laplace expansions. In another word, a formula like $(uv|\mu)_r = \sum_\mu (-1)^{|\mu_{(1)}||v|}(u|\mu_{(1)})_r \cdot (v|\mu_{(2)})_r$ cannot be made true, whether $\Delta\mu = \sum_\mu \mu_{(1)} \otimes \mu_{(2)}$ is considered as a coproduct defined on $Super[P]_q$ or on $Super[P]_{q^{-1}}$. For example, $(a^+a^+|\alpha^+\beta^+)_r = (a|\alpha)(a|\beta)$, it equals neither $(a|\alpha)(a|\beta) + q(a|\beta)(a|\alpha)$ nor $(a|\alpha)(a|\beta) + q^{-1}(a|\beta)(a|\alpha)$.

As a corollary, we have obtained explicit formulas for quantum biproducts:

Proposition 5 *Let $a_i \in L$ and $\alpha_i \in P$, $i = 1, 2, \cdots, n$. We have*

$$(a_1a_2\cdots a_n|\alpha_1\alpha_2\cdots\alpha_n)_r = \sum_\sigma (-1)^{\sum_{i>j}|a_{\sigma_i}||\alpha_j|} k_\sigma(q)(a_{\sigma_1}|\alpha_1)(a_{\sigma_2}|\alpha_2)\cdots(a_{\sigma_n}|\alpha_n) \tag{7}$$

if $a_1a_2\cdots a_n \neq 0$ *in* $Super[L]_{q^{-1}}$*, and*

$$(a_1a_2\cdots a_n|\alpha_1\alpha_2\cdots\alpha_n)_l = \sum_\sigma (-1)^{\sum_{i>j}|a_i||\alpha_{\sigma_j}|} l_\sigma(q)(a_1|\alpha_{\sigma_1})(a_2|\alpha_{\sigma_2})\cdots(a_n|\alpha_{\sigma_n}) \tag{8}$$

if $\alpha_1\alpha_2\cdots\alpha_n \neq 0$ *in* $Super[P]_{q^{-1}}$*; where the sums in* (7) *and* (8) *range over the permutations of the multisets* $a_1a_2\cdots a_n$ *and* $\alpha_1\alpha_2\cdots\alpha_n$*, respectively; and where the coefficients* $k_\sigma(q)$ *and* $l_\sigma(q)$ *are defined such that*

$$a_1a_2\cdots a_n = k_\sigma(q)a_{\sigma_1}a_{\sigma_2}\cdots a_{\sigma_n} \text{ in } Super[L]_{q^{-1}},$$

$$\alpha_1\alpha_2\cdots\alpha_n = l_\sigma(q)\alpha_{\sigma_1}\alpha_{\sigma_2}\cdots\alpha_{\sigma_n} \text{ in } Super[P]_{q^{-1}}.$$

■

Remark. If $a_1a_2\cdots a_n$ and $\alpha_1\alpha_2\cdots\alpha_n$ have repetitions of negative variables, we may take $k_\sigma(q)$ and $l_\sigma(q)$ to be zero for every σ. Hence identities (7) and (8) still hold.

Examples.

1. If $a_1^- < a_2^- < \cdots < a_n^-$, then

$$(a_1^-a_2^-\cdots a_n^-|\alpha_1^-\alpha_2^-\cdots\alpha_n^-)_r = (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma\in S_n} (-q)^{-i(\sigma)}(a_{\sigma_1}|\alpha_1)(a_{\sigma_2}|\alpha_2)\cdots(a_{\sigma_n}|\alpha_n),$$

where $i(\sigma)$ denotes the number of inversions in the permutation σ. This is called right quantum determinant.

2. If $\alpha_1^+ < \alpha_2^+ < \cdots < \alpha_n^+$, then

$$(a_1^+a_2^+\cdots a_n^+|\alpha_1^+\alpha_2^+\cdots\alpha_n^+)_l = \sum_{\sigma\in S_n} q^{-i(\sigma)}(a_1|\alpha_{\sigma_1})(a_2|\alpha_{\sigma_2})\cdots(a_n|\alpha_{\sigma_n}).$$

This is called left quantum permanent.

3. If $a_1^+ < a_2^+ < \cdots < a_n^+$, then

$$(a_1^+a_2^+\cdots a_n^+|\alpha_1^-\alpha_2^-\cdots\alpha_n^-)_r = \sum_{\sigma\in S_n} q^{-i(\sigma)}(a_{\sigma_1}|\alpha_1)(a_{\sigma_2}|\alpha_2)\cdots(a_{\sigma_n}|\alpha_n).$$

4. If $a^- < b^-$, then

$$(a^-b^-|\alpha^+\alpha^+)_r = (a^-|\alpha^+)(b^-|\alpha^+) - q^{-1}(b^-|\alpha^+)(a^-|\alpha^+)$$
$$= (1+q^{-2})(a^-|\alpha^+)(b^-|\alpha^+).$$

On the other hand $(a^-b^-|\alpha^+\alpha^+)_l = (a^-|\alpha^+)(b^-|\alpha^+)$. Hence

$$(a^-b^-|\alpha^+\alpha^+)_r = (1+q^{-2})(a^-b^-|\alpha^+\alpha^+)_l.$$

5. We have $(a^+a^+|\alpha^+\alpha^+)_r = (a^+a^+|\alpha^+\alpha^+)_l = (a^+|\alpha^+)(a^+|\alpha^+)$.

Remark. In the case that $L = P$, we can define quantum bideterminant of $Super[L|L]_q$, and it is not group-like element unless $L = L^-$.

In general a left supersymmetric quantum biproduct is not equal to the corresponding right one. However, they only differ a scalar multiple. Let us fix some notations first. For a positive integer n, let

$$\begin{aligned}
[n]_q &= 1+q+q^2+\cdots+q^{n-1},\\
[n]_q! &= [n]_q[n-1]_q\cdots[1]_q,\\
\begin{bmatrix} n \\ n_1n_2\cdots n_i \end{bmatrix}_q &= \frac{[n]_q!}{[n_1]_q![n_2]_q!\cdots[n_i]_q!}
\end{aligned}$$

where $n_1+n_2+\cdots+n_i = n$. **From now on we assume that** $[n]_q \neq 0$ **for all** $n > 0$.

Theorem 6 *Let* $a_1 < a_2 < \cdots < a_s$ *and* $\alpha_1 < \alpha_2 < \cdots < \alpha_t$*. Given positive integers* $n, n_1, n_2, \cdots, n_s$ *and* $m_1, m_2, \cdots, m_t$ *such that* $n = \sum_i n_i = \sum_i m_i$*, we have*

$$\begin{aligned}
&\begin{bmatrix} n \\ n_1n_2\cdots n_s \end{bmatrix}_{q^{-2}} (a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}|\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_t^{m_t})_l\\
= &\begin{bmatrix} n \\ m_1m_2\cdots m_t \end{bmatrix}_{q^{-2}} (a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}|\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_t^{m_t})_r.
\end{aligned}$$

Corollary 7 *If* $a_1 < a_2 < \cdots < a_n$ *and* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$*, then*

$$(a_1a_2\cdots a_n|\alpha_1\alpha_2\cdots\alpha_n)_l = (a_1a_2\cdots a_n|\alpha_1\alpha_2\cdots\alpha_n)_r.$$

■

To simplify notations, we define

$$(u) = [n_1]_{q^{-2}}![n_2]_{q^{-2}}!\cdots[n_s]_{q^{-2}}!$$

for any word u in $Mon(L)$ of content $a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}$. A word $u = ab\cdots c$ is called **non-decreasing** or **ordered** if $a \leq b \leq \cdots \leq c$. Then Theorem 6 can be stated as:

$$\frac{1}{(u)}(u|\mu)_l = \frac{1}{(\mu)}(u|\mu)_r \tag{9}$$

for all non-decreasing words $u \in Mon(L)$ and $\mu \in Mon(P)$. This is the form to be used in the next section.

4. Quantum straightening formula

In this section we will prove the straightening formula for supersymmetric quantum letter-place algebra $Super[L|P]_q$. First let us recall some basic concepts. A **tableau** of shape λ on L (resp. P) is obtained by filling the squares of λ with letters of L (resp. places of P). The **content** of a tableau T, denoted by $cont(T)$, is the multiset of the elements appeared in T. The **row sequence** of a tableau is the word obtained by lining up its rows. A tableau on L or on P is called **standard** if

(i) all the rows and columns are non-decreasing,

(ii) no negative letter or place occurs more than once in any row,

(iii) no positive letter or place occurs more than once in any column.

Let

$$T = \begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{matrix} \quad \text{and} \quad T^\dagger = \begin{matrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{matrix}$$

be tableaux on L and on P of the same shape, where u_i and μ_i are words in $Mon(L)$ and $Mon(P)$ respectively. We define the **left quantum bitableau** $(T|T^\dagger)_l$ as follows:

$$(T|T^\dagger)_l = \left(\begin{array}{c|c} u_1 & \mu_1 \\ u_2 & \mu_2 \\ \vdots & \vdots \\ u_k & \mu_k \end{array} \right)_l := (-1)^{\sum_{i>j} |u_i||\mu_j|} (u_1|\mu_1)_l (u_2|\mu_2)_l \cdots (u_k|\mu_k)_l.$$

The **right quantum bitableau** $(T|T^\dagger)_r$ is defined similarly.

Given a quantum bitableau $(T|T^\dagger)_r$ or $(T|T^\dagger)_l$, its **shape** is the common shape of T and $T^\dagger$; its **content**, denoted by $cont(T|T^\dagger)$, is the union of $cont(T)$ and $cont(T^\dagger)$; it is called **standard** if both T and $T^\dagger$ are standard tableaux. We are now ready to state the main theorem of this paper, which is called the quantum straightening formula or the standard basis theorem for $Super[L|P]_q$. Again we assume that $[n]_q \neq 0$ for all $n > 0$.

Theorem 8 (Quantum Straightening Formula) *Both standard right bitableaux and standard left bitableaux form a linear basis of $Super[L|P]_q$. Moreover, if*

$$(T|T^\dagger)_l = \sum a_{SS^\dagger}(S|S^\dagger)_l, \qquad a_{SS^\dagger} \neq 0,$$

where each $(S|S^\dagger)_l$ is standard, then (i) $cont(S|S^\dagger)_l = cont(T|T^\dagger)_l$, (ii) the shape of $(S|S^\dagger)_l$ is greater than or equal to the shape of $(T|T^\dagger)_l$ in lexicographic order, (iii) if $(S|S^\dagger)$ and $(T|T^\dagger)$ have the same shape, then the row sequence of S followed by the row sequence of $S^\dagger$ is less than the row sequence of T followed by the row sequence of $T^\dagger$ in lexicographic order. The same statement is true for left bitableaux.

We will prove the theorem for left bitableaux. The proof consists of two parts:

(I) The set of left standard bitableaux spans $Super[L|P]_q$ as a vector space.

(II) Left standard bitableaux are linearly independent.

Part (I) needs two lemmas below.

The first one serves the similar role as the exchange identity in [GRS, Prop 10], although it is in a much weaker form.

Lemma 9 (Exchange Identity) *Let u, v, w and μ, ν, ω be non-decreasing words. Then*

$$\sum_v \frac{1}{(u)(v_{(1)})(v_{(2)})(w)} \left(\begin{array}{c|c} uv_{(1)} & \mu \\ v_{(2)}w & \nu \end{array} \right)_l = \sum_{\mu,\nu} \frac{1}{(\mu_{(1)})(\mu_{(2)})(\nu_{(1)})(\nu_{(2)})} \left(\begin{array}{c|c} u & \mu_{(1)} \\ v & \mu_{(2)}\nu_{(1)} \\ w & \nu_{(2)} \end{array} \right)_r \tag{10}$$

and

$$\sum_\nu \frac{1}{(\mu)(\nu_{(1)})(\nu_{(2)})(w)} \left(\begin{array}{c|c} u & \mu\nu_{(1)} \\ v & \nu_{(2)}\omega \end{array} \right)_r = \sum_{u,v} \frac{1}{(u_{(1)})(u_{(2)})(v_{(1)})(v_{(2)})} \left(\begin{array}{c|c} u_{(1)} & \mu \\ u_{(2)}v_{(1)} & \nu \\ v_{(2)} & \omega \end{array} \right)_l \tag{11}$$

where $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$ and $\Delta\mu = \sum_\mu \mu_{(1)} \otimes \mu_{(2)}$ denote the coproducts defined on $Super[L]_{q^{-1}}$ and $Super[P]_{q^{-1}}$ respectively such that the components $u_{(1)}$, $u_{(2)}$, $\mu_{(1)}$, $\mu_{(2)}$ are non-decreasing words (with suitable coefficients).

Remarks. (1) We have to require in this lemma that $u_{(1)}$, $u_{(2)}$, etc are non-decreasing words; otherwise, the expressions in (10) and (11) are not well defined.

(2) If $L = L^-$ and $P = P^-$, then we may assume that all words in the exchange identity have no repetitions of negative variables; then "the classical quantum exchange identity" can be stated as follows.

$$\sum_v \left(\begin{array}{c|c} uv_{(1)} & \mu \\ v_{(2)}w & \nu \end{array} \right) = \sum_{\mu,\nu} \left(\begin{array}{c|c} u & \mu_{(1)} \\ v & \mu_{(2)}\nu_{(1)} \\ w & \nu_{(2)} \end{array} \right)$$

and

$$\sum_\nu \left(\begin{array}{c|c} u & \mu\nu_{(1)} \\ v & \nu_{(2)}\omega \end{array} \right) = \sum_{u,v} \left(\begin{array}{c|c} u_{(1)} & \mu \\ u_{(2)}v_{(1)} & \nu \\ v_{(2)} & \omega \end{array} \right).$$

(3) The identities in the definition of quantum flag scheme, namely, quantum shape-algebra (see [TT, Section 3]) can be verified immediately for $f_q(i_1, \cdots, i_s) := (1, \cdots, s|i_1, \cdots, i_s)$ by using the above classical quantum exchange identity. For example, Young symmetry relations (the identity (3.2c) [TT, page 20]) is equivalent to

$$0 = \sum_i \left(\begin{array}{c|c} 1 \cdots t & i_{(1)} \\ 1 \cdots s & i_{(2)}j \end{array} \right)$$

where $1 \leq r \leq s \leq t$, $i = i_1 \cdots i_{t+r}$ and $j = j_1 \cdots j_{s-r}$ are increasing words; and the commutation relations (the identity (3.2d) [TT, page 20]) is equivalent to

$$\left(\begin{array}{c|c} 1 \cdots r & j \\ 1 \cdots s & i \end{array} \right) = (-q)^{-r(s-r)} \sum_i \left(\begin{array}{c|c} 1 \cdots s & ji_{(1)} \\ 1 \cdots r & i_{(2)} \end{array} \right)$$

where $1 \leq r \leq s$, $i = i_1 \cdots i_s$ and $j = j_1 \cdots j_r$ are increasing words. Both displayed identities are special cases of classical quantum exchange identity.

Next lemma deals with the commutativity of quantum biproducts. In general, it is far from true that $(u|\mu)(v|\nu) = (v|\nu)(u|\mu)$ where $(u|\mu)$ and $(v|\nu)$ are either left or right quantum biproducts. One of the major difficulty of the whole subject is that quantum biproducts, left or right, have very little commutative relation.

Lemma 10 *Let u, v be words on L and μ, ν be words on P such that $length(u) = length(\mu)$ and $length(v) = length(\nu)$. Then the element $(u|\mu)_l(v|\nu)_l$ in $Super[L|P]_q$ can be written as a linear combination of elements of the forms $(w|\omega)_l m$ where $length(w) = length(\omega) \geq length(v) = length(\nu)$ and m is a monomial in $Super[L|P]_q$.* ∎

We now go back to prove Theorem 8.

Proof of Part (I). Given two tableaux S and T of the same content, we say S is less than T, denoted by $S < T$, if the shape of S is less than or equal to the shape of T in lexicographic order, and in the latter case the row sequence of S is greater than the row sequence of T in lexicographic order. For example,

$$\begin{matrix} 2\,1\,1 \\ 3\,2\,1 \end{matrix} < \begin{matrix} 1\,2\,1 \\ 3\,2\,1 \end{matrix} < \begin{matrix} 2\,1\,1\,3 \\ 2 \end{matrix}.$$

Given bitableaux $(T|T^\dagger)_l$ and $(S|S^\dagger)_l$ of the same content, we say $(T|T^\dagger)_l$ is less than $(S|S^\dagger)_l$, denoted $(T|T^\dagger)_l < (S|S^\dagger)_l$, if $T < S$, or $T = S$ and $T^\dagger < S^\dagger$. The biggest bitableau of a given content is of the form $(u|\mu)$, where u and μ are non-decreasing words. Given an arbitrary bitableau $(T|T^\dagger)_l$, suppose any bitableau $(E|E^\dagger)_l > (T|T^\dagger)_l$ is a linear combination of the standard ones satisfying the conditions (i), (ii), (iii) in Theorem 8. We claim that so is $(T|T^\dagger)$. The proof proceeds as follows. First we may assume that the rows of T and $T^\dagger$ are non-decreasing and no negative letter or place appears more than once in a row. Suppose T is not standard. (Similar discussions will work if $T^\dagger$ is not standard.) Consider a violation pair $b_i > a_i$; or $b_i = a_i, |b_i| = |a_i| = 0$ between two consecutive rows of T :

$$\begin{matrix} b_1 b_2 \cdots b_i \cdots b_s \\ a_1 a_2 \cdots a_i \cdots a_t, \end{matrix} \qquad s \geq t.$$

Let

$$(T|T^\dagger)_l = \pm (T_1|T_1^\dagger)_l \left(\begin{matrix} b_1 b_2 \cdots b_i \cdots b_s \\ a_1 a_2 \cdots a_i \cdots a_t \end{matrix} \middle| \begin{matrix} \mu \\ \nu \end{matrix} \right)_l (T_2|T_2^\dagger)_l.$$

Applying the exchange identity (10), by letting $u = b_1 b_2 \cdots b_{i-1}, v = a_1 a_2 \cdots a_i b_i \cdots b_s, w = a_{i+1} a_{i+2} \cdots a_t$, we get

$$\pm (T_1|T_1^\dagger)_l \sum_\sigma k_\sigma(q) \left(\begin{matrix} b_1 \cdots b_{i-1} \sigma(b_i) \sigma(b_{i+1}) \cdots \sigma(b_s) \\ \sigma(a_1) \cdots \sigma(a_{i-1}) \sigma(a_i) a_{i+1} \cdots a_t \end{matrix} \middle| \begin{matrix} \mu \\ \nu \end{matrix} \right)_l (T_2|T_2^\dagger)_l$$

$$= \pm (T_1|T_1^\dagger)_l \sum_{\mu,\nu} \frac{1}{(\mu_{(1)})(\mu_{(2)})(\nu_{(1)})(\nu_{(2)})} \left(\begin{matrix} b_1 b_2 \cdots b_{i-1} \\ a_1 \cdots a_i b_i \cdots b_s \\ a_{i+1} \cdots a_t \end{matrix} \middle| \begin{matrix} \mu_{(1)} \\ \mu_{(2)} \nu_{(1)} \\ \nu_{(2)} \end{matrix} \right)_r (T_2|T_2^\dagger)_l \quad (12)$$

where $k_\sigma(q)$ are suitable coefficients and the sum on the left-hand side ranges over permutations σ of the multiset $\{a_1, \cdots, a_i, b_i, \cdots, b_s\}$ such that both $\sigma(a_1) \cdots \sigma(a_i)$ and $\sigma(b_{i+1}) \cdots \sigma(b_s)$ are non-decreasing words. Notice that $(T|T^\dagger)$ appears (with the coefficient $k_{id}(q)$) on the left-hand side once and only once when $\sigma = id$. Since $a_1 \leq \cdots \leq a_i \leq b_i \leq \cdots \leq b_s$, all other terms on the left-hand side have smaller row sequence in lexicographic order. Therefore, they can be expressed as linear combinations of standard bitableaux satisfying the conditions in Theorem 8. Now applying Lemma 10 to the terms on the right-hand side of (12), by letting $u = b_1 b_2 \cdots b_{i-1}$ and $v = a_1 \cdots a_i b_i \cdots b_s$, they can be written as a linear combination of elements of the form $(T_1|T_1^\dagger)_l (w|\omega)_l m$ where $length(w) = length(\omega) \geq length(v) = s+1$ and m is a monomial in $Super[L|P]_q$. Therefore, by induction the right-hand side of (12) is a linear combination of standard bitableaux satisfying the conditions in Theorem 8, since the shape of $(T_1|T_1^\dagger)_l (w|\omega)_l m$ (when put in a tableau form) is longer than the shape of $(T|T^\dagger)_l$. In conclusion, $(T|T^\dagger)_l$ is a linear combination of the standard bitableaux satisfying the conditions (i), (ii), (iii) in Theorem 8.

Proof of Part (II). By Theorem 3 in Section 2, $Super[L|P]_q = \bigoplus_{M,M'} V_{MM'}$ and $V_{MM'}$ is a K-subvector space spanned by the ordered monomials

$$(a_1|\alpha_1)(a_2|\alpha_2) \cdots (a_k|\alpha_k) \qquad (13)$$

of letter content M and place content M' where (i) $a_1 \leq a_2 \leq \cdots \leq a_k$, (ii) if $a_i = a_{i+1}$ for some i, then $\alpha_i \leq \alpha_{i+1}$, and (iii) if $a_i = a_{i+1}$ and $\alpha_i = \alpha_{i+1}$ for some i, then $|(a|\alpha)| = 0$. On the other hand, it was just proved that standard left (resp. right) quantum bitableaux $(S|S^\dagger)_l$ (resp. $(S|S^\dagger)_r$) with $cont(S) = M$ and $cont(S^\dagger) = M'$ form a spanning set of $V_{MM'}$. So in order to prove the linear independence, it is sufficient to know that the cardinality of the set of monomials in (13) is the same as the cardinality of the set of the pairs $(S, S^\dagger)$, where $S, S^\dagger$ are standard and $cont(S) = M$ and $cont(S^\dagger) = M'$. This combinatorial result is proved directly in [BSV], where a supersymmetric Schensted correspondence was constructed. ∎

Remark. In the quantum straightening formula proved in Theorem 8, one has to use the lexicographic order for shapes instead of the dominance order; while in [GRS], then dominance order is used. Hence the straightening formula for $Super[L|P]_q$ is slightly weaker than the one for $Super[L|P]$. The very reason for this is that the exchange identity (Lemma 9) is weaker than the exchange identity in [GRS, Prop 10].

References

[AR] Anick, D. & Rota, G.-C., *Higher-order syzygies for the bracket algebra and for the ring of coordinates of the Grassmannian*, Proc. Natl. Acad. Sci. USA, **88** (1991).

[Be] Bergman, G. *The Diamond Lemma for ring theory*, Adv. in Math. **29** (1978), 178-218.

[BR] Buchsbaum, D.A. & Rota, G.-C., *Projective resolutions of Weyl modules*, Proc. Natl. Acad. Sci. USA, to appear.

[BSV] Bonetti, F., Senato, D. & Venezia, A, *La corrispondenza di Robinson-Schensted per la fourfold algebra*, Bollettino U.M.I. (7) **2**-B (1988), 541-554.

[BPT] Brini, A., Palareti, A. & Teolis, A., *Gordan-Capelli series in superalgebras*, Proc. Natl. Acad. Sci. USA, **85** (1988), 1330-1333.
[BT] Brini, A. & Teolis, A., *Capelli bitableaux and* **Z**-*forms of general linear Lie superalgebras*, Proc. Natl., Acad., Sci. USA, **87** (1990), pp. 56-60.
[DKR] Désarménien, J., Kung, J. P. S. & Rota, G.-C., *Invariant Theory, Young Bitableaux and combinatorics,* Adv. in Math. **27** (1978), 63-92.
[DRS] Dubilet, P., Rota, G.-C. & Stein, J. A., *On the foundations of combinatorial theory IX: Combinatorial method in invariant theory,* Stud. Appl. Math **53** (1976), 185-216.
[GRS] Grosshans, F. D. & Rota, G.-C. & Stein, J. A., Invariant Theory and Superalgebras, Regional Conference series in Mathematics, Number 69, 1985.
[KR] Kung, J. P. S. & Rota G.-C., *The invariant theory of binary forms,* Bull. Amer. Math. Soc. **10** (1984), 27-85.
[HRS] Huang, R., Rota, G.-C. & Stein, J. A., *Supersymmetric algebra, supersymmetric space, and invariant theory,* Annali Scuola Normale Superiore (Volume dedicated to L. Radicati), (1989) Pisa.
[Hu1] Huang, R., *Invariants of sets of linear varieties,* Proc. of Natl. Acad. of Sci., USA, **88**(1990) 4557-4560.
[Hu2] Huang, R., *Invariants of sets of lines in Projective 3-space,* J. Algebra, **143**(1991), 208-218.
[Ma1] Manin, Yu, *Quantum groups and non-commutative geometry,* Les publications du Centre de Recherches Mathematiques, Universite de Montreal, 1988.
[Ma2] Manin, Yu, Topics in Noncommutative Geometry, Princeton University Press, 1991.
[PW] Parshall, B. & Wang, J.P., *Quantum general linear groups,* Memoirs of the Amer. Math. Soc., Nomber 439, 1991.
[RS] Rota, G.-C. & Stein, J. A., *Applications of Cayley algebras,* Colloquio Internazionale sulle Teorie Comninatoriee, Tomo II, Accad. Naz. Lincei, Rome, (1976) 71-97.
[St] Stanley, R. Enumerative Combinatorics (Volume I), The Wadsworth & Brooks/Cole Mathematics Series, 1986.
[SW] Sturmfels, B. & White, N. L., *Gröbner bases and invariant theory,* Adv. in Math. **76** (1989), 245-259.
[TT] Taft E. & Towber J., *Quantum Deformation of Flag Schemes and Grassmann Schemes. I. A q-Deformation of the Shape-Algebra for GL(n),* Journal of Algebra, **142**, 1-36 (1991).
[Wh] White, N. L., *Multilinear Cayley factorization,* to appear in J. of Symb. Comp.
[WW] White, N. L. & Whitely, W., *The algebraic geometry of stresses in frameworks,* SIAM J. Algebraic and Discrete Methods, **4** (1983), pp. 481-511.
[Wo] Woronowicz, S. L., *Tannaka-Krein duality for compact matrix pseudogroups, Twisted SU(N) groups* , Inv. Math. **93** (1988), 35-76.

THE CONJECTURE OF STANLEY FOR SYMMETRIC MAGIC SQUARES

Rong-Qing Jia
Department of Mathematics
University of Alberta
Edmonton, Canada T6G 2G1

In this paper we announce a proof of Stanley's conjecture for symmetric magic squares (see [Sta76], [Sta83, p.40] and [Sta86, p.262]). The solution of the conjecture is a nice application of multivariate spline theory to combinatorics.

1. Introduction

An $m \times m$ matrix with non-negative integer entries is called *a magic r-square of order m* if every row and column sums to $r \in \mathbb{N}$ where $\mathbb{N}$ is the set of non-negative integers. Let $H_m(r)$ denote the number of all magic r-squares of order m. For instance, $H_1(r) = 1$ and $H_2(r) = r + 1$. It seems that MacMahon [Mac15, §407] first computed $H_3(r)$:

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Guided by this evidence, Anand, Dumir and Gupta [ADG66] conjectured that $H_m(r)$ is a polynomial in r of degree $(m-1)^2$. Their conjecture was confirmed by Stanley in [Sta73].

An $m \times m$ *symmetric* matrix with non-negative integer entries is called *a symmetric magic r-square of order m* if every row (and hence every column) sums to r. Let $S_m(r)$ denote the number of all symmetric magic r-squares of order m. Carlitz [Car66] calculated $S_m(r)$ for $m \le 4$ and found that $S_m(r)$ are not polynomials in r for $m = 3$ and 4; rather, $S_m(2r)$ and $S_m(2r+1)$ are polynomials in r ($m = 3$ and 4). He conjectured that this is the case for all m. His conjecture was solved by Stanley in [Sta73]. Later, Stanley [Sta76] obtained the following result.

Theorem 1. *Let $m \ge 1$, and let $S_m(r)$ be the number of symmetric magic r-squares of order m. Then $S_m(r) = P_m(r) + (-1)^r Q_m(r)$ for all $r \in \mathbb{N}$, where $P_m(r)$ and $Q_m(r)$ are polynomials in r with $\deg P_m = \binom{m}{2}$. Moreover,*

$$\deg Q_m \le \binom{m-1}{2} - 1 \text{ if } m \text{ is odd}; \quad \deg Q_m \le \binom{m-2}{2} - 1 \text{ if } m \text{ is even.} \tag{1.1}$$