

TOPOLOGY OF INTERSECTIONS OF SCHUBERT  
CELLS AND HECKE ALGEBRA

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**Abstract.** We consider intersections of Schubert cells  $B\alpha \cdot B$  and  $\sigma B\sigma^{-1}\beta \cdot B$  in the space of complete flags  $F = SL/B$ , where  $B$  denotes the Borel subgroup of upper triangular matrices, while  $\alpha, \beta$  and  $\sigma$  belong to the Weyl group  $W$  (coinciding with the symmetric group). We obtain a special decomposition of  $F$  which subdivides all  $B\alpha \cdot B \cap \sigma B\sigma^{-1}\beta \cdot B$  into strata of a simple form and which is in its turn subdivided by decomposition into small cells, see [GS]. It enables us to establish a new geometrical meaning of the structure constants for the corresponding Hecke algebra and in particular of the so-called  $R$ -polynomials used in Kazhdan-Lusztig theory. Structure constants of the Hecke algebra appear to be the alternating sums of the Hodge numbers for the mixed Hodge structure in the cohomology with compact supports of the above intersections. We derive a new efficient combinatorial algorithm calculating the  $R$ -polynomials and structure constants in general.

MAIN RESULTS

Intersections of pairs and, more generally, of  $k$ -tuples of Schubert cells each belonging to its own Schubert cell decompositions of a flag space appeared in many articles, see e.g. [BB], [KL1], [KL2], [De], [GS]. Topological properties of such intersections are of particular importance in representation theory. Intersections of some special interesting arrangements of Schubert cells are directly related to the problem of representability of matroids, see [GS] and §1. Most likely, for somewhat general class of arrangements of Schubert cells their intersections and complements to such intersections do not have any good topological properties (opposite to the case of arrangements of linear subspaces where there exists a developed theory of analogous topological spaces, see e.g. [OT]). Even the problem of nonemptiness of such intersections in the complex flag varieties is very hard. However, in the important case of pairs of Schubert cells topology of their intersections although still quite complicated appears to be more accessible. Namely, one can obtain a special decomposition of such intersections into products of algebraic tori and linear subspaces of different dimensions generalizing the standard Schubert cell decomposition. These strata in their turn are intersections of more than two Schubert cells originating from the initial pair. Such decomposition enables us to calculate (algorithmically) additive topological characteristics of considered intersections such as their Euler  $E^{p,q}$ -characteristics, see [DK]. Generally speaking, the considered decomposition does not stratify all pairwise intersections of Schubert cells, i.e. the closure of a stratum is not always the union of strata of lower dimensions. Still, for nice intersections, (i.e. for those for which the above mentioned decomposition actually gives a stratification) the analysis of the corresponding Leray spectral sequences leads to an interesting conjecture about the purity of the mixed Hodge structure in the cohomology and thus about the Betti numbers of such intersections.

In order to formulate the main results let us recall and introduce some notions.

Let  $F_n = SL_n/B_n$  denote the space of complete flags in  $C^n$ . Each compete flag  $f$  can be interpreted both as a Borel subgroup and as a sequence of enclosed subspaces of all dimensions from 0 to  $n$ . The Schubert cell decomposition  $D_f$  of  $F_n$  relative to  $f$  consists of cells formed by all flags having some given set of dimensions of intersections with subspaces of  $f$ . Thus we have a family of Schubert cell decompositions parameterized by  $F_n$ . Cells of any such decomposition are in 1-1-correspondence with permutations on  $n$  elements.

1. **DEFINITION.** For any  $k$ -tuple of flags  $f, g, h, \dots$  in  $F_n$  we introduce the  $k$ -tuple Schubert decomposition  $D_{f,g,h,\dots}$  consisting of all nonempty intersections of  $k$ -tuples of cells one taken from each decomposition  $D_f, D_g, \dots$ , etc.

Generally speaking a  $k$ -tuple Schubert decomposition is not a stratification of the space of complete flags and its strata can have very complicated topology.

**2. DEFINITION.** Given some set of linear subspaces denote by their  $\cap +$ -completion the set of intersections and sums of all possible subsets of these subspaces. By an  $\cap +$ -completion of a pair of flags denote the  $\cap +$ -completion of the set of all subspaces constituting these flags.

**3. IMPORTANT DEFINITION.** A refined double decomposition  $\bar{D}_{f,g}$  of the space of complete flags relative to a given pair of flags  $(f,g)$  is a decomposition into strata formed by all flags with some fixed dimensions of intersections with all subspaces from the  $\cap +$ -completion of  $(f,g)$ . Strata of this decomposition will be called refined double strata.

**4. REMARK.** Refined double decomposition coincides with some special  $k$ -tuple decomposition.

**5. REMARK.** Refined double decomposition  $\bar{D}_{f,g}$  subdivides the standard decompositions  $D_f$ ,  $D_g$  and the double decomposition  $D_{f,g}$ , i.e. each Schubert cell with respect to  $f$  or  $g$  as well as their pairwise intersections consist of some number of refined double strata.

**THEOREM A.** Each refined double stratum is biholomorphically equivalent to a product of a complex torus by a linear space.

Given a system of coordinates in  $C^n$  a flag is called coordinate if all its subspaces are spanned by coordinate vectors. Each coordinate flag is obviously identified with a permutation of coordinates. The coordinate flag is called standard if it is identified with the unit permutation, i.e. for all  $i$  its  $i$ -dimensional subspace is spanned by the first  $i$  coordinate vectors. For any two flags  $f$  and  $g$  in  $F_n$  one can always choose a system of coordinates such that  $f$  will be the standard coordinate flag and  $g$  will be some coordinate flag given by a permutation  $\sigma$ .

**6. DEFINITION.** Pairs  $(f,g)$  such that in a suitable system of coordinates  $f$  is the standard flag and  $g$  is the coordinate flag identified with  $\sigma$  are called pairs in the relative position  $\sigma$ .

Now we enumerate (algorithmically) all strata of  $\bar{D}_{f,g}$  using the permutation  $\sigma$ . By a decreasing subsequence in a permutation  $\sigma = (i_1, \dots, i_n)$  we certainly understand a subsequence  $i_{j_1}, i_{j_2}, \dots, i_{j_k}$  such that  $j_1 < j_2 < \dots < j_k$  and  $i_{j_1} > i_{j_2} > \dots > i_{j_k}$ .

**7. DEFINITION.** A cyclic shift of  $\sigma$  about a decreasing subsequence  $i_{j_1}, i_{j_2}, \dots, i_{j_k}$  is a transformation sending  $i_{j_1}$  onto  $i_{j_k}$ ,  $i_{j_2}$  onto  $i_{j_1}, \dots$ , and  $i_{j_k}$  onto  $i_{j_{k-1}}$  and preserving the rest of the elements. (If subsequence consists of just one element then the transformation is identical.)

We also assign two numbers to a cyclic shift of  $\sigma$  about some decreasing subsequence.

**8. DEFINITION.** The reduced length of a decreasing subsequence is equal to the number of the elements in decreasing subsequence minus one. The domination of a decreasing subsequence is equal to the number of elements in  $\sigma$  for which there exists at least one element from decreasing subsequence which stands further in  $\sigma$  and is strictly bigger.

**EXAMPLE.** Consider  $\sigma = 6723451$  and its decreasing subsequence 731. Then the cyclic shift of  $\sigma$  about 731 is 6321457. The reduced length is 2 and the number of dominated elements is also 2, namely, the element 6 is dominated by 7 from decreasing subsequence and the element 2 is dominated by 3.

Fixing some permutation  $\sigma$  on  $n$  elements let us apply to it the following  $n$ -step procedure.

#### THE MAIN ALGORITHM.

The 1st step. Find all decreasing subsequences in  $\sigma$ . Applying to  $\sigma$  cyclic shifts about each of these decreasing subsequences obtain the set of resulting permutations. In each of these permutations block the first (and the biggest) element in the applied decreasing subsequence. (To block just means that this element will be ignored on all subsequent steps of the algorithm.)

The  $i$ th step. For each permutation obtained on the previous step apply the same procedure

that was applied to  $\sigma$  on the 1st step. Namely, find all its decreasing subsequences (disregarding all blocked elements). Make cyclic shifts about each of these decreasing subsequences and finally in each of the obtained permutations block the 1st element of the decreasing subsequence applied on this step.

**9. REMARK.** The algorithm stops exactly after  $n$  steps since in each permutation obtained after  $i$  steps we get exactly  $i$  elements blocked, (i.e. on the  $i+1$ th step of algorithm we actually work with permutations on  $n-i$  elements). Moreover, each permutation with at least one unblocked element contains at least one decreasing subsequence (probably consisting of just one element).

The whole procedure for  $\sigma = 321$  is illustrated on Fig.1. Bold numbers are nonblocked while underlined numbers are blocked and numbers on edges present decreasing subsequences.

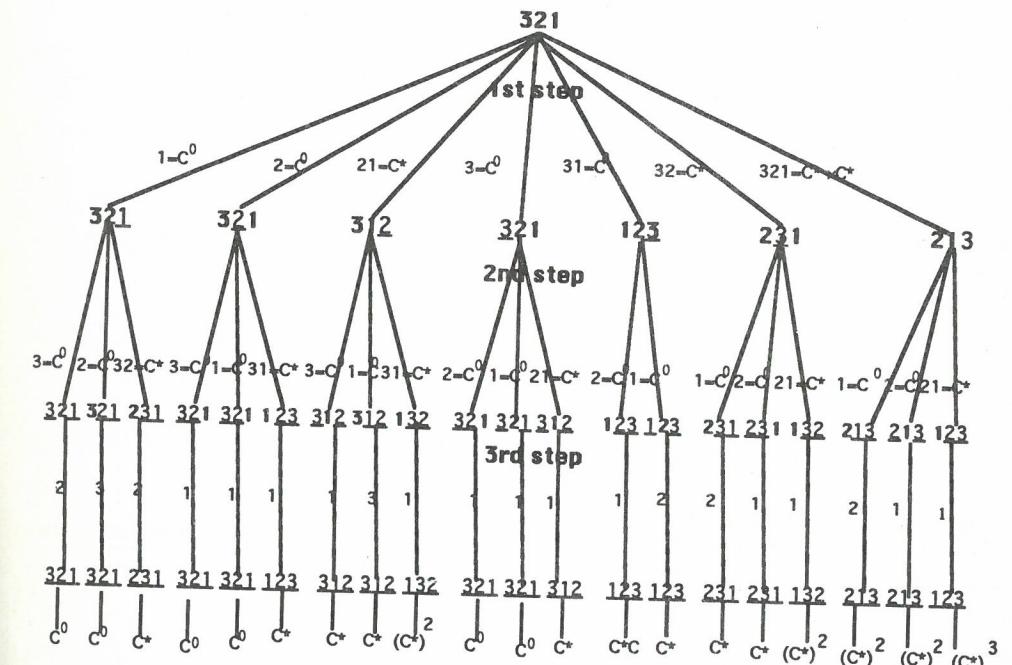


Fig.1. Illustration of the main algorithm in the case  $\sigma=321$ .

**10. DEFINITION.** By a chain of permutations we mean a sequence of  $n+1$  permutations starting with  $\sigma$  and such that each consequent permutation is obtained from the preceding one as the result of a cyclic shift on the corresponding step of the above procedure.

For example, there are 20 chains shown above on Fig.1. They are just paths in the represented tree starting from the top element  $\sigma$  and going down to the bottom.

**11. DEFINITION.** Assign to each chain two numbers, namely its total length equal to the sum of reduced lengths of all permutations involved and its total domination equal to the sum of all dominations. (When we calculate domination in a permutation with blocked elements we just disregard them completely.)

**THEOREM B.** If a pair of flags are in the relative position  $\sigma$  then strata of the refined double decomposition relative to this pair are in 1-1-correspondence with the chains in the above procedure starting with  $\sigma$ . The stratum corresponding to a given chain is a product

of a complex torus of dimension equal to the total length of this chain by a linear space of dimension equal to its total domination.

EXAMPLE. The structure of each refined double stratum is given in the bottom line on figure 1.

Till the end of this section we assume that an arbitrary pair of flags  $(f, g)$  is already transformed into the standard coordinate pair  $(1, \sigma)$  (The element 1 denotes the standard coordinate flag and  $\sigma$  the flag corresponding to the permutation  $\sigma$ ).

For any two permutations  $\alpha$  and  $\beta$  denote by  $C_{1,\alpha}$  the Schubert cell consisting of all flags which are in relative position  $\alpha$  with respect to 1 and by  $C_{\sigma,\beta}$  the Schubert cell of all flags in relative position  $\beta$  with respect to  $\sigma$ . (In the standard notations  $C_{1,\alpha} = \mathcal{B}\alpha \cdot \mathcal{B}$  and  $C_{\sigma,\beta} = \sigma^{-1}\mathcal{B}\sigma\beta \cdot \mathcal{B}$ .)

By definition the cell  $C_{1,\alpha}$  belongs to decomposition  $D_1$ , the cell  $C_{\sigma,\beta}$  belongs to decomposition  $D_\sigma$  and their intersection to the double decomposition  $D_{1,\sigma}$ .

Since by definition the refined double decomposition  $\bar{D}_{1,\sigma}$  subdivides decompositions  $D_1$ ,  $D_\sigma$  and  $D_{1,\sigma}$  we now describe all refined double strata included in the Schubert cells  $C_{1,\alpha}$ ,  $C_{\sigma,\beta}$  and their intersection.

Let us assign to a chain of permutations the following two new permutations.

12. DEFINITION. The first permutation of a chain is a sequence of successively blocked elements, i.e. its  $i$ th entry is the element blocked on  $i$ th step of procedure. The second permutation of a chain is the sequence of positions on which the successively blocked elements stand, i.e. its  $i$ th element is the number of position on which the  $i$ th blocked element stands.

EXAMPLE, see Fig.1. For the chain  $321 \rightarrow 123 \rightarrow 123 \rightarrow 123$  the first permutation is 321 and the second permutation is 321. For the chain  $321 \rightarrow 312 \rightarrow 132 \rightarrow 132$  the first permutation is 231 and the second permutation is 321. For the chain  $321 \rightarrow 321 \rightarrow 312 \rightarrow 312$  the first permutation is 321 and the second permutation is 132.

THEOREM C. A stratum from  $\bar{D}_{1,\sigma}$  belongs to the Schubert cell  $C_{1,\alpha}$  if and only if the first permutation of its chain coincides with  $\alpha$ ; a stratum from  $\bar{D}_{1,\sigma}$  belongs to the Schubert cell  $C_{\sigma,\beta}$  if and only if the second permutation of its chain coincides with  $\beta$ . Therefore a stratum belongs to the intersection  $C_{1,\alpha} \cap C_{\sigma,\beta}$  if and only if its first permutation is  $\alpha$  and its second permutation is  $\beta$ .

This theorem enables to modify slightly the described algorithm in order to obtain refined double decompositions of Schubert cells  $C_{1,\alpha}$ ,  $C_{\sigma,\beta}$  or their intersection.

#### MODIFICATIONS OF THE MAIN ALGORITHM

THE 1ST MODIFICATION. In order to obtain decomposition of  $C_{1,\alpha}$  one must consider on the  $i$ th step for  $i = 1, \dots, n$  only decreasing subsequences starting at the  $i$ th element of the permutation  $\alpha$ .

THE 2ND MODIFICATION. In order to obtain decomposition of  $C_{\sigma,\beta}$  one must consider on the  $i$ th step for  $i = 1, \dots, n$  only decreasing subsequences ending at the position which number is equal to the  $i$ th element in the permutation  $\beta$ .

THE 3RD MODIFICATION. Finally, in order to obtain decomposition of  $C_{1,\alpha} \cap C_{\sigma,\beta}$  one must consider on the  $i$ th step for  $i = 1, \dots, n$  only decreasing subsequences starting at the  $i$ th element of permutation  $\alpha$  and ending at the position which number is equal to the  $i$ th element in the permutation  $\beta$ . See corresponding three examples on Fig. 2-3.

A similar process was proposed by Francesco Brenti [Br] in the case  $\sigma = \omega_0$ .

The following remark is valid for all versions of the main algorithm, i.e. for the refined double decomposition of the whole space of complete flags, some Schubert cell or some pairwise decomposition of Schubert cells.

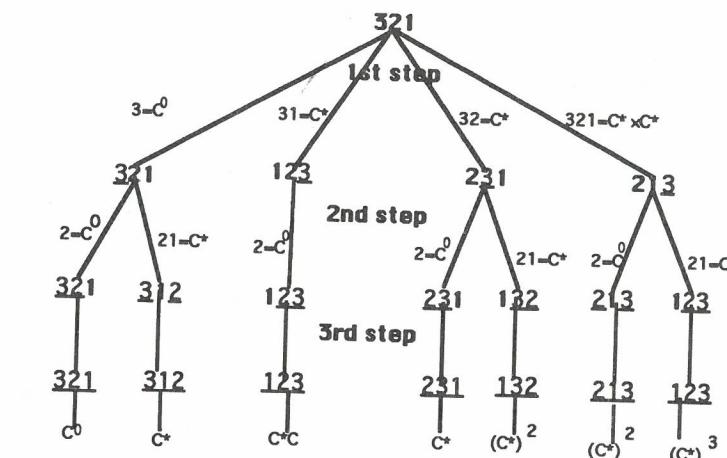


Fig.2. Modified algorithm in the case  $\sigma=\alpha=321$ .

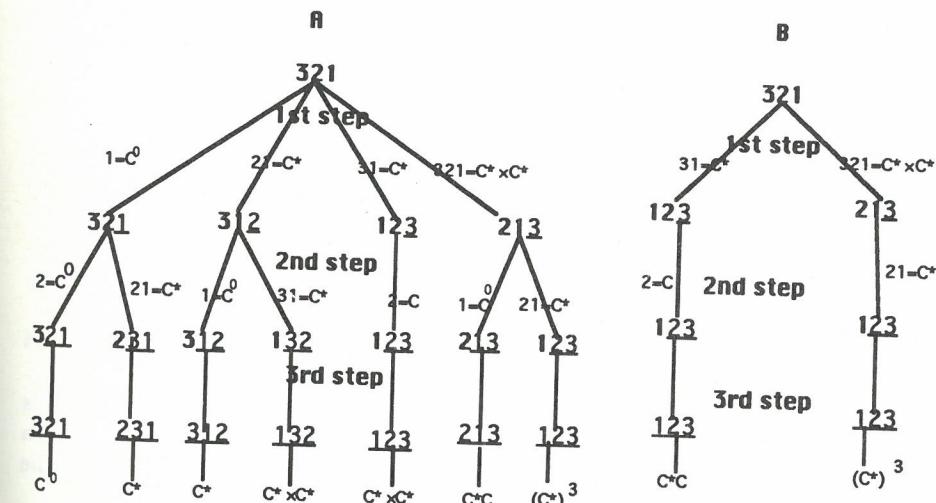


Fig.3. Modified algorithm in the cases  $\sigma=\beta=321$  and  $\sigma=\alpha=\beta=321$ .

13. REMARK. Each step of the above algorithm is geometrically interpreted as projection of flags from the considered set on a linear subspace of the corresponding codimension. This means in particular that sets of restricted chains, i.e. those chains which start at a permutation with blocked elements obtained after some  $i$ th step present decomposition of  $F_{n-i}$ , (some Schubert cell in  $F_{n-i}$  or some pairwise intersection of Schubert cells in  $F_{n-i}$  depending on the corresponding modification of the algorithm). This feature will enable us later to obtain

inductive formulas for structure constants in Hecke algebras different from the standard ones, compare [KL1-2], [De] and [Br]. To be more precise we introduce the following operation on permutations with blocked elements. Consider some permutation on  $n$  elements with  $i$  blocked entries.

14. DEFINITION. The reduction of the blocked part from a given permutation is the operation which forms the new permutation on  $n - i$  elements in the following way. We exclude all blocked elements and subtract from each nonblocked element the number of all blocked elements which are less.

EXAMPLE. The reduction of the blocked part from the permutation 7563241 gives 4312.

15. PROPOSITION. I) The set of all restricted chains, i.e. chains starting at some permutation  $\tilde{\sigma}$  obtained after  $i$  steps of the algorithm (and thus containing  $i$  blocked elements) geometrically presents:

- (1) for the main algorithm - a refined double decomposition of  $F_{n-i}$  relative to the pair  $(1, \tilde{\sigma}')$ , where  $\tilde{\sigma}'$  is obtained from  $\tilde{\sigma}$  by reduction of all blocked elements;
- (2) for the 1st modification - a refined double decomposition of the Schubert cell  $C_{1,\alpha'}$  in  $F_{n-i}$  where  $\alpha'$  is obtained by reduction of the first  $i$  elements from  $\alpha$ ;
- (3) for the 2nd modification - a refined double decomposition of the Schubert cell  $C_{\tilde{\sigma}',\beta'}$  in  $F_{n-i}$ , where  $\tilde{\sigma}'$  is the same as above and  $\beta'$  is obtained by reduction of the first  $i$  elements from  $\beta$ ;
- (4) for the 3rd modification of the algorithm a refined double decomposition of  $C_{1,\alpha'} \cap C_{\tilde{\sigma}',\beta'}$ .

II) Moreover, the geometrical meaning of  $i$  step of any variant of the algorithm is decomposition of the initial object ( $F_n$ ,  $C_{1,\alpha}$ ,  $C_{\sigma,\beta}$  or  $C_{1,\alpha} \cap C_{\sigma,\beta}$ ) into a disjoint union of products of analogous objects in  $F_{n-i}$  enumerated by the set of all permutations  $\tilde{\sigma}$  obtained after the  $i$ th step by  $(C^*)^{l(\tilde{\sigma})} \times C^{d(\tilde{\sigma})}$ . Here  $l(\tilde{\sigma})$  is equal to the sum of reduced lengths of all permutations in the chain starting from  $\sigma$  and ending at  $\tilde{\sigma}$  and  $d(\tilde{\sigma})$  is the sum of all dominations in this chain.

EXAMPLE. The set of all reduced chains passing through any permutation obtained after the first step except 123 on Fig.1. presents refined decomposition of the space of complete flags  $F_2 = \mathbf{CP}^1$  relative to  $\sigma = 21$  into three strata, namely two points and  $\mathbf{C}^*$ . Chains passing through 123 (or  $\sigma = 12$ ) present the standard Schubert cell decomposition of  $F_2$  into a point and  $\mathbf{C}$ .

Now we list some special combinatorial and topological properties of strata included in some pairwise intersection of Schubert cells.

THEOREM D. Refined double strata included in any nonempty intersection of Schubert cells enjoy the following additional properties.

a) The sum of (complex) dimension of a torus and doubled dimension of a linear space equals  $l_{ng}(\alpha) + l_{ng}(\beta) - l_{ng}(\sigma)$ , (where  $l_{ng}$  is the usual length of a permutation) independently on a choice of nonempty stratum in a given  $C_{1,\alpha} \cap C_{\sigma,\beta}$ , in particular all strata of the same dimension have the same form.

b) There are no gaps in (complex) dimension, i.e. there exist strata of all intermediate (complex) dimensions between minimal and maximal.

EXAMPLE. The only nontrivial intersection  $C_{1,w_0} \cap C_{w_0,w_0}$  (where  $w_0 = 321$ ) consists of two refined double strata  $(\mathbf{C}^*)^3$  and  $\mathbf{C}^* \times \mathbf{C}$ , see fig.3 B.

The next result describes "adjacency" of strata in  $C_{1,\alpha} \cap C_{\sigma,\beta}$ , i.e. enumerates strata in  $C_{1,\alpha} \cap C_{\sigma,\beta}$  which have nonempty intersection with the closure of some given stratum. Since refined double decomposition is in general not a stratification this is not the standard notion of adjacency. Consider two ordered  $k$ -tuples of permutations.

16. DEFINITION. The second  $k$ -tuple is called less or equal than the first  $k$ -tuple if each

permutation of the second  $k$ -tuple is less or equal in the Bruhat order than the corresponding permutation of the first  $k$ -tuple, see the notion of the Bruhat order in e.g. [Hu]. The above partial order on the set of all refined double strata (or their chains of permutations) in  $C_{1,\alpha} \cap C_{\sigma,\beta}$  is called the adjacency partial order.

By theorem C each nonempty refined double stratum from  $\bar{D}_{1,\sigma}$  is contained in the only pairwise intersection  $C_{1,\alpha} \cap C_{\sigma,\beta}$ , that is  $\alpha$  equals to the first and  $\beta$  equals to the second permutation of the chain of the considered stratum.

THEOREM E. The closure of a given refined double stratum in its corresponding pairwise intersection of Schubert cells belongs (but in general does not coincide with) to the union of all refined double strata included in the same pairwise intersection such that their chains of permutations are less or equal in the adjacency partial order than the chain of the considered stratum.

See an example of decomposition and "adjacency" on Fig.4.

This theorem enables to construct rather easily examples of  $C_{1,\alpha} \cap C_{\sigma,\beta}$  refined double decompositions of which are not stratifications. In particular, refined double decomposition of  $C_{1234,4231} \cap C_{4231,4231}$  consists of three strata, namely  $\mathbf{C}^*(\mathbf{C}^*)^2$  and two copies of  $(\mathbf{C}^*)^3 \mathbf{C}$ , see fig.7. In the notations of fig.7. the closure of the stratum  $C = (\mathbf{C}^*)^3 \mathbf{C}$  is nonempty in the stratum  $B = (\mathbf{C}^*)^3 \mathbf{C}$ , moreover the closure of  $C = (\mathbf{C}^*)^3 \mathbf{C}$  does not contain the whole stratum  $A = \mathbf{C}^*(\mathbf{C}^*)^2$ .

17. REMARK. The above results hold in complete generality for the spaces of complete flags over any algebraically closed field or  $\mathbb{R}$ . In this case  $\mathbf{C}^*$  must be substituted by the multiplicative and  $\mathbf{C}$  by the additive group of the field, compare [Cu3].

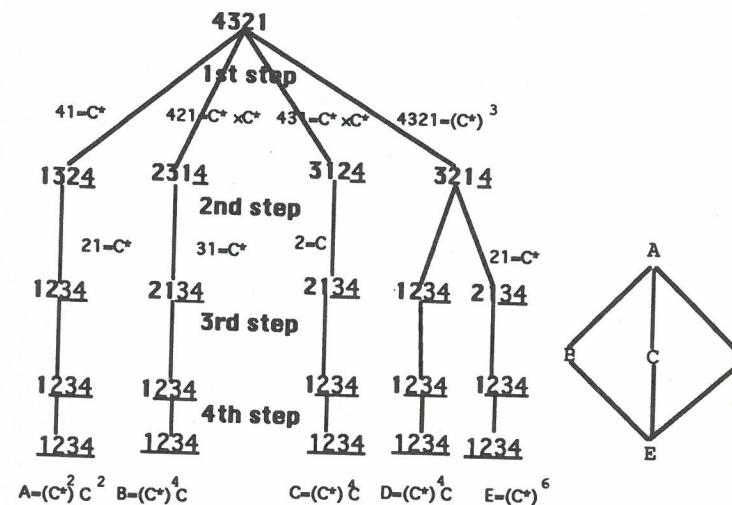


Fig.4. Modified algorithm in the case  $\alpha=\sigma=4321$  and "adjacency" of strata in decomposition.

Let us briefly recall several notions from complex algebraic geometry. We assume that readers are familiar with the Hodge decomposition in the cohomology of smooth compact

Kaehlerian varieties (see for example [GH]), further referred to as the *pure Hodge structure*.

According to [Dl1-3], in the cohomology (with compact supports) of an arbitrary complex quasiprojective variety  $V$  there exists (and is uniquely defined) a so-called *mixed Hodge structure*. It consists of two filtrations  $W$  and  $F$  in  $H^*(V, \mathbb{C})$  (in  $H_c^*(V, \mathbb{C})$ , respectively);  $W$  is an increasing rational filtration determined by the topology of  $V$ , while  $F$  is a decreasing filtration defined by the analytic structure of  $V$  which defines the pure Hodge structure on any quotient  $W_i/W_{i-1}$ ; both filtrations are respected by arbitrary algebraic maps.

Using the notion of the mixed Hodge structure in, say,  $H_c^*(V, \mathbb{C})$ , one can define the corresponding *Hodge numbers*  $h_k^{p,q}$  by the following formula:

$$h_k^{p,q} = \dim \frac{F_p(W_{p+q}(H_c^k(V, \mathbb{C}))/W_{p+q-1}(H_c^k(V, \mathbb{C})))}{F_{p-1}(W_{p+q}(H_c^k(V, \mathbb{C}))/W_{p+q-1}(H_c^k(V, \mathbb{C})))}.$$

**COROLLARY F (OF THEOREMS B AND E)** The Hodge numbers  $h_k^{p,q}$  of any intersection  $C_{1,\alpha} \cap C_{\sigma,\beta}$  can be positive only if  $p = q$ .

Similarly to the notion of the usual Euler characteristics of  $V$ , one can define the series of generalized Euler characteristics depending on  $p$  and  $q$  by

$$\chi^{p,q} = \sum_k (-1)^k h_k^{p,q},$$

and form their generating function, called the  *$E^{p,q}$ -polynomial* or just the  *$E$ -polynomial* of  $V$ ,

$$E_V(u, v) = \sum_{p,q} \chi^{p,q} u^p v^q.$$

The following crucial property of  $E_V$  follows from the additivity of generalized Euler characteristics for the cohomology with compact supports.

**18. LEMMA** (see e.g. [Du]). *If a quasiprojective complex variety  $V$  is represented as the disjoint union of quasiprojective subvarieties,  $V = \cup_i V_i$ , then*

$$E_V(u, v) = \sum_i E_{V_i}(u, v).$$

#### COROLLARY OF THEOREM B.

$$E_\sigma^{\alpha,\beta} = E_{C_{1,\alpha} \cap C_{\sigma,\beta}}(u, v) = \sum_{ch \in CH} z^{d(ch)} (z-1)^{l(ch)},$$

where  $z = uv$ ;  $CH$  denotes the set of chains of all strata included in  $C_{1,\alpha} \cap C_{\sigma,\beta}$ ,  $d(ch)$  and  $l(ch)$  are the total domination and the total length of a given chain respectively, compare [De].

The same expression can be rewritten as an inductive formula using the above remark on the geometrical meaning of our algorithm. More precisely, let  $SUB$  denote the set of all decreasing subsequences in  $\sigma$  starting at the element  $\alpha_1$  and ending at the position with the number  $\beta_1$ , (i.e. those subsequences which are used on the first step of construction the refined double decomposition of  $C_{1,\alpha} \cap C_{\sigma,\beta}$ ). For any decreasing subsequence  $sub \in SUB$  let  $l(sub)$  and  $d(sub)$  denote its reduced length and domination respectively. Finally, let  $\alpha'$  and  $\beta'$  denote the results of reduction of the first elements  $\alpha_1$  and  $\beta_1$  from  $\alpha$  and  $\beta$  respectively;  $\sigma(sub)$  denote the result of the cyclic shift of  $\sigma$  about  $sub$  and reduction of the first element

of  $sub$ , (see description of the algorithm and its modifications above). Then, by proposition 15

$$E_\sigma^{\alpha,\beta} = E_{C_{1,\alpha} \cap C_{\sigma,\beta}}(u, v) = \sum_{sub \in SUB} z^{d(sub)} (z-1)^{l(sub)} E_{C_{1,\alpha'} \cap C_{\sigma(sub), \beta'}}.$$

**EXAMPLE.**  $E_{(C_{1284,4321} \cap C_{4321,4321})} = (z-1)E_{(C_{128,321} \cap C_{132,321})} + (z-1)^2 E_{(C_{128,321} \cap C_{281,321})} + (z-1)^2 E_{(C_{128,321} \cap C_{312,321})} + (z-1)^3 E_{(C_{128,321} \cap C_{321,321})}.$

Consider now the Leray spectral sequence converging to the cohomology of  $C_{1,\alpha} \cap C_{\sigma,\beta}$  with compact supports which is associated with the refined double decomposition of the pairwise intersection of Schubert cells or more precisely with the corresponding filtration of  $C_{1,\alpha} \cap C_{\sigma,\beta}$  by union of all strata of dimension less or equal than some given value. Recall that its first page contains the cohomology with compact supports of the differences between consequent terms of filtration and the differential  $d_1$  is induced by the long exact sequence of triples. In the case when the initial decomposition is a stratification the cohomology with compact supports of the differences between the  $i$ th and  $i-1$ st terms of filtration coincides with the direct sum of cohomology with compact supports of all strata of dimension  $i$ .

**19. DEFINITION.** We call a pairwise intersection of Schubert cells  $C_{1,\alpha} \cap C_{\sigma,\beta}$  nice if the refined double decomposition  $\bar{D}_{1,\alpha}$  gives its stratification.

**THEOREM G.** The Leray spectral sequence of the refined double decomposition of any nice pairwise intersection degenerates at the second page.

**CONJECTURE H.** For any nice pairwise intersection of Schubert cells the rows of Leray spectral sequence form acyclic (with the exception of the highest dimension) complexes.

**HYPOTHETICAL COROLLARY I.** The mixed Hodge structure of any nice intersection  $C_{1,\alpha} \cap C_{\sigma,\beta}$  is pure and nonvanishing Hodge numbers are equal (up to  $\pm 1$ ) to the corresponding coefficients of the  *$E$ -polynomial*, namely  $h_i^{i,i} = (-1)^i \chi_c^{i,i}$ .

**CONJECTURE J.** If  $RC_{1,\alpha} \cap RC_{\sigma,\beta}$  is a nonempty intersection of Schubert cells over  $\mathbb{R}$  and  $C_{1,\alpha} \cap C_{\sigma,\beta}$  is its complexification then the actions of differentials in the corresponding Leray spectral sequences are concordant, i.e. respected by complex conjugation.

**HYPOTHETICAL COROLLARY K.** The intersection  $RC_{1,\alpha} \cap RC_{\sigma,\beta}$  enjoys the so-called *M*-property, i.e. the sum of Betti numbers with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients of  $RC_{1,\alpha} \cap RC_{\sigma,\beta}$  coincides with that of  $C_{1,\alpha} \cap C_{\sigma,\beta}$ , (compare e.g. [SS]).

Finally, let us recall the notion of the Hecke algebra  $H$  in its simplest version as a  $\mathbb{C}$ -algebra depending on a complex parameter  $q$  and given by a standard set of generators and relations.

The basis of  $H$  consists of elements  $T_\omega$ ,  $\omega$  is any permutation on  $n$  elements, and multiplication rules are as follows:

- (1)  $T_\omega T_{\omega'} = T_{\omega\omega'}$  if  $\lambda(\omega\omega') = \lambda(\omega) + \lambda(\omega')$ , where  $\lambda(\omega)$  is the length function equal to the number of inversions in  $\omega$ ;
- (2)  $T_{s_i} T_{s_i} = (z-1)T_{s_i} + zT_1$ .

If  $z$  is some power of prime number then  $H$  could be interpreted as the algebra of functions on  $GL_n$  over the finite field of  $z$  elements which are constant on double cosets  $Bx\bar{B}$  with multiplication given by convolution, see [Cu1-2]. (Analogously for generic  $z$  the algebra  $H$  could be identified with the set of sheaves constant on Schubert cells with multiplication given by sheaf convolution, see e.g. [So].)

Denote by  $c_{w_1, w_2}^{w_3}$  the structure constant of Hecke algebra in the expansion  $T_{w_1} T_{w_2} = \sum_{w_3 \in S_n} c_{w_1, w_2}^{w_3}(z) T_{w_3}$ .

The structure constants of  $H$  are polynomials on  $z$  counting the number of points in the intersection of Schubert cells over finite fields if  $z$  is a power of prime, see [Cu1].

**THEOREM L.** The set of structure constants for the Hecke algebra coincide with the set of  *$E$ -polynomials* for the pairwise intersections of cells of the two Schubert decompositions of

$F_n$ . Namely,

$$E_\sigma^{\alpha, \beta}(z) \equiv c_{\alpha, \beta^{-1}}^\sigma(z)$$

for any  $\alpha, \beta, \sigma \in S_n$ .

The above inductive formula of the  $E$ -polynomials of  $C_{1,\alpha} \cap C_{\sigma,\beta}$  can obviously be rewritten as the inductive formula for the structure constants.

An analogous family of decompositions was introduced 10 years earlier by V.V.Deodhar in the case of intersections  $C_{1,\alpha} \cap C_{w_0, \beta}$ , where  $w_0$  denotes the longest element in a arbitrary finite Coxeter system, see [De1-2]. These decompositions were extended to all intersections  $C_{1,\alpha} \cap C_{\sigma,\beta}$  by C.Curtis in [Cu3]. Decompositions depend on a reduced presentation of the element  $\alpha$  into product of simple reflections and different choices of such presentation lead to different decompositions. Combinatorial data coding strata in approach of Deodhar-Curtis is similar to ours but more lengthy and complicated. We are convinced that the refined double decomposition coincides with one of decompositions suggested by V.V.Deodhar for some particular choice of reduced presentation. If  $\sigma = w_0$  is the longest permutation and we consider its standard presentation  $w_0 = s_1 s_2 \dots s_n s_1 s_2 \dots s_{n-1} \dots s_1 s_2 s_1$  then we are able to find exact correspondence between more lengthy Deodhar's coding of strata and our chains. Moreover, refined double decomposition is not the only geometrical way to refine double decompositions. These other decompositions of geometrical origin probably coincide with Deodhar's decompositions corresponding to various choices of reduced presentations.

The starting point of this study was an attempt to calculate the cohomology of pairwise intersections of Schubert cells of the maximal dimension (see [SV]). Francesco Brenti has attracted our attention to the fact that the properties of the  $E$ -polynomials calculated in these polynomials from our decomposition theorem [SV] and the results of Deodhar [De1]. Later Brenti [Br] has extended our combinatorial construction to a more general case, and succeeded in proving the coincidence of the  $E$ - and the  $R$ -polynomials in a pure combinatorial way. Here we prove analogous results in a more general setting by using geometrical arguments.

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