

# SHIFTED QUASI-SYMMETRIC FUNCTIONS AND THE HOPF ALGEBRA OF PEAK FUNCTIONS

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**SUMMARY.** In his work on  $P$ -partitions, Stembridge defined the algebra of peak functions  $\Pi$ , which is both a subalgebra and a retraction of the algebra of quasi-symmetric functions. We show that  $\Pi$  is closed under coproduct, and therefore a Hopf algebra, and describe the kernel of the retraction. Billey and Haiman, in their work on Schubert polynomials, also defined a new class of quasi-symmetric functions — shifted quasi-symmetric functions — and we show that  $\Pi$  is strictly contained in the linear span  $\Xi$  of shifted quasi-symmetric functions. We show that  $\Xi$  is a coalgebra, and compute the rank of the  $n$ th graded component.

**RÉSUMÉ.** Dans ses travaux sur les  $P$ -partitions, Stembridge définit l'algèbre  $\Pi$  des fonctions de pics. Cette algèbre peut être vue comme une sous-algèbre ou un quotient de l'algèbre des fonctions quasi-symétriques. Nous montrons ici que  $\Pi$  est fermée sous le coproduit, et est donc une algèbre de Hopf. Nous décrivons aussi le noyau du quotient ci-dessus. D'autre part, dans leurs travaux sur les polynômes de Schubert, Billey et Haiman ont défini une nouvelle classe de fonctions quasi-symétriques: les fonctions quasi-symétrique décalé. Nous montrons que  $\Pi$  est strictement contenue dans l'espace linéaire  $\Xi$  des fonctions quasi-symétrique gauchis. Puis nous montrons que  $\Xi$  est une coalgèbre et calculons les dimensions des composantes de degré  $n$ .

## 1. INTRODUCTION

Schur  $Q$  functions first arose in the study of projective representations of  $S_n$  [8]. Since then they have appeared in variety of contexts including the representations of Lie superalgebras [9] and cohomology classes dual to Schubert cycles in isotropic Grassmannians [4, 7]. While studying the duality between skew Schur  $P$  and  $Q$  functions and their connection to the Schubert calculus of isotropic flag manifolds, we were led to their quasi-symmetric analogues: the *peak functions* of Stembridge [10]. We show that the *linear span of peak functions is a Hopf algebra* (Theorem 2.2). We also show that these peak functions are contained in the strictly larger set of *shifted quasi-symmetric functions* (Theorem 3.6) introduced by Billey and Haiman [1]. We remark that the quasi-symmetric functions here are not any apparent specialization of the quasi-symmetric  $q$ -analogues of Hivert [3].

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From extensive calculations, we believe that the set of all shifted quasi-symmetric functions form a Hopf algebra, but at present we can only show that:

*The set of all shifted quasi-symmetric functions forms a graded coalgebra whose  $n$ th graded component has rank  $\pi_n$ , where  $\pi_n$  is given by the recurrence*

$$\pi_n = \pi_{n-1} + \pi_{n-2} + \pi_{n-4},$$

*with initial conditions  $\pi_1 = 1$ ,  $\pi_2 = 1$ ,  $\pi_3 = 2$ ,  $\pi_4 = 4$ .*

We shall prove this result (Theorems 3.2 and 4.3) and in addition shall establish some other properties of these functions.

A composition  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$  of a positive integer  $n$  is an ordered list of positive integers whose sum is  $n$ . We denote this by  $\alpha \models n$ . We call the integers  $\alpha_i$  the *components* of  $\alpha$ , and denote the number of components in  $\alpha$  by  $k(\alpha)$ . There exists a natural one-to-one correspondence between compositions of  $n$  and subsets of  $[n - 1]$ . If  $A = \{a_1, a_2, \dots, a_{k-1}\} \subset [n - 1]$ , where  $a_1 < a_2 < \dots < a_{k-1}$ , then  $A$  corresponds to the composition,  $\alpha = [a_1 - a_0, a_2 - a_1, \dots, a_k - a_{k-1}]$ , where  $a_0 = 0$  and  $a_k = n$ . For ease of notation, we shall denote the set corresponding to a given composition  $\alpha$  by  $I(\alpha)$ . For compositions  $\alpha$  and  $\beta$  we say that  $\alpha$  is a *refinement* of  $\beta$  if  $I(\beta) \subset I(\alpha)$ , and denote this by  $\alpha \preccurlyeq \beta$ .

For any composition  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$  we denote by  $M_\alpha$  the *monomial quasi-symmetric function* [2]

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}.$$

We define  $M_0 = 1$ , where 0 denotes the unique empty composition of 0. We denote by  $F_\alpha$  the *fundamental quasi-symmetric function* [2]

$$F_\alpha = \sum_{\alpha \preccurlyeq \beta} M_\beta.$$

**Definition 1.1.** *For any subset  $A \subset [n - 1]$ , let  $A + 1$  be the subset of  $\{2, \dots, n\}$  formed from  $A$  by adding 1 to each element of  $A$ . Let  $\alpha \models n$ . Then we define*

$$\theta_\alpha = \sum_{\substack{\beta \\ I(\alpha) \subset I(\beta) \cup I(\beta) + 1}}^{n} 2^{k(\beta)} M_\beta.$$

This is the natural extension of the definition of peak functions given in [10].

**Example 1.2.** *We shall often omit the brackets that surround the components of a composition.*

*If  $\alpha = 21$ , then  $I(\alpha) = \{2\}$ , and  $I(\alpha) + 1 = \{3\}$ . Hence*

$$\theta_{21} = 4M_{21} + 4M_{12} + 8M_{111}.$$

Let  $\Sigma^n$  be the  $\mathbb{Z}$ -module of quasi-symmetric functions spanned by  $\{M_\alpha\}_{\alpha \vdash n}$  and let  $\Sigma = \bigoplus_{n \geq 0} \Sigma^n$  be the graded  $\mathbb{Z}$ -algebra of quasi-symmetric functions. This is a Hopf algebra [5] with coproduct given by

$$\Delta(M_\alpha) = \sum_{\alpha=\beta \cdot \gamma} M_\beta \otimes M_\gamma,$$

where  $\beta \cdot \gamma$  is the concatenation of compositions  $\beta$  and  $\gamma$ .

**Example 1.3.**  $\Delta(M_{32}) = 1 \otimes M_{32} + M_3 \otimes M_2 + M_{32} \otimes 1$ .

We compute the coproduct of the functions  $\theta_\alpha$ .

**Lemma 1.4.** *For any composition  $\alpha \models n$  we have that*

$$(1) \quad \Delta(\theta_\alpha) = \sum \theta_{\varepsilon \cdot a} \otimes \theta_{\phi(b \cdot \zeta)}$$

where the sum is over all ways of writing  $\alpha$  as  $\varepsilon \cdot (a+b) \cdot \zeta$ , that is, the concatenation of compositions  $\varepsilon$  and  $\zeta$ , and a component of  $\alpha$  written as the sum of numbers  $a, b \geq 0$ . Also  $\phi(b \cdot \zeta) = [1 + \zeta_1, \zeta_2, \dots]$  if  $b = 1$  and  $b \cdot \zeta$  otherwise.

We shall use this result to show that certain subsets of functions  $\theta_\alpha$  span coalgebras (Theorems 2.2 and 3.2).

*Proof.* Definition 1.1 is equivalent to

$$\theta_\alpha = \sum_{\substack{\beta \\ \beta^* \dashv \alpha}}^{n \atop \beta \vdash n} 2^{k(\beta)} M_\beta,$$

where  $\beta^*$  is the refinement of  $\beta$  obtained by replacing all components  $\beta_i > 1$ , for  $i > 1$ , by  $[1, \beta_i - 1]$ . Thus the LHS of equation (1) is equal to

$$(2) \quad \sum_{\substack{\beta \\ \beta^* \dashv \alpha \\ \beta = \gamma \cdot \delta}}^{n \atop \beta \vdash n} 2^{k(\beta)} M_\gamma \otimes M_\delta = \sum_{\substack{\gamma \cdot \delta \\ (\gamma \cdot \delta)^* \dashv \alpha}}^{n \atop (\gamma \cdot \delta) \vdash n} 2^{k(\gamma)} M_\gamma \otimes 2^{k(\delta)} M_\delta.$$

Let  $2^{k(\gamma)} M_\gamma \otimes 2^{k(\delta)} M_\delta$  be a term of this sum, with  $\gamma \models m$ . This term can only appear in one summand on the RHS of equation (1), namely  $\theta_{\varepsilon \cdot a} \otimes \theta_{\phi(b \cdot \zeta)}$  with  $\varepsilon \cdot a \models m$ . To show that it does indeed appear, we need to prove that  $\gamma^* \preccurlyeq \varepsilon \cdot a$  and  $\delta^* \preccurlyeq \phi(b \cdot \zeta)$ . Let  $\delta^{**}$  be the refinement of  $\delta^*$  obtained by replacing the part  $\delta_1$  by  $[1, \delta_1 - 1]$  if  $\delta_1 > 1$ . We have that

$$\gamma^* \cdot \delta^{**} = (\gamma \cdot \delta)^* \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta,$$

which implies that  $\gamma^* \preccurlyeq \varepsilon \cdot a$ , and  $\delta^{**} \preccurlyeq b \cdot \zeta \preccurlyeq \phi(b \cdot \zeta)$ .

If  $\delta_1 = 1$  then  $\delta^* = \delta^{**} \preccurlyeq \phi(b \cdot \zeta)$ . However, if  $\delta_1 > 1$  then there are two possible cases: either  $\delta_1 \leq b$ , or  $b = 1$  and  $\delta_1 - 1 \leq \zeta_1$ . In the former case  $\delta^* \preccurlyeq b \cdot \zeta = \phi(b \cdot \zeta)$ , while in the latter,  $\delta_1 \preccurlyeq 1 + \zeta_1$ , whence  $\delta^* \preccurlyeq [1 + \zeta_1, \zeta_2, \dots] = \phi(b \cdot \zeta)$ .

Conversely, let  $2^{k(\gamma)}M_\gamma \otimes 2^{k(\delta)}M_\delta$  be a term belonging to a tensor  $\theta_{\varepsilon \cdot a} \otimes \theta_{\phi(b \cdot \zeta)}$  on the RHS of equation (1). To show that it appears in equation (2) we must prove that  $(\gamma \cdot \delta)^* \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta$ . We have that  $\gamma^* \preccurlyeq \varepsilon \cdot a$  and  $\delta^* \preccurlyeq \phi(b \cdot \zeta)$ , which imply that

$$(\gamma \cdot \delta)^* = \gamma^* \cdot \delta^{**} \preccurlyeq \gamma^* \cdot \delta^* \preccurlyeq \varepsilon \cdot a \cdot \phi(b \cdot \zeta).$$

If  $b > 1$  then

$$(\gamma \cdot \delta)^* \preccurlyeq \varepsilon \cdot a \cdot \phi(b \cdot \zeta) = \varepsilon \cdot a \cdot b \cdot \zeta \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta.$$

If  $b = 1$  then  $\delta^* \preccurlyeq \phi(b \cdot \zeta) = [1 + \zeta_1, \zeta_2, \dots]$  implies that

$$\delta^{**} = [1, \dots] \preccurlyeq [1, \zeta_1, \dots] = b \cdot \zeta.$$

Therefore,

$$(\gamma \cdot \delta)^* = \gamma^* \cdot \delta^{**} \preccurlyeq \varepsilon \cdot a \cdot b \cdot \zeta \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta$$

as desired.  $\square$

## 2. THE PEAK HOPF ALGEBRA

**Definition 2.1.** For any composition  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$  we say that  $\theta_\alpha$  is a peak function if  $\alpha_i = 1 \Rightarrow i = k$ .

Observe that if  $\theta_\alpha$  is a peak function and  $\alpha \models n$ , then  $I(\alpha) \subset \{2, \dots, n - 1\}$  such that no two  $i$  in  $I(\alpha)$  are consecutive.

Let  $\Pi^n$  be the  $\mathbb{Z}$ -module spanned by all peak functions  $\theta_\alpha$ ,  $\alpha \models n$ , and let  $\Pi = \bigoplus_{n \geq 0} \Pi^n$ . This was studied by Stembridge [10] who showed that the peak functions are F-positive, are closed under product, and form a basis for  $\Pi$ , and so the rank of  $\Pi^n$  is the  $n$ th Fibonacci number. In addition we also know the following about the *algebra of peaks*,  $\Pi$ .

**Theorem 2.2.**  $\Pi$  is closed under coproduct.

*Proof.* If all components of a composition  $\alpha$ , except perhaps the last, are greater than 1, then the same is true for all compositions  $\varepsilon \cdot a$  and  $\phi(b \cdot \zeta)$  appearing in the RHS of equation (1).  $\square$

Let  $\Theta$  be the  $\mathbb{Z}$ -linear map from  $\Sigma$  to  $\Pi$  defined by  $\Theta(F_\alpha) = \theta_{\Lambda(\alpha)}$ , where  $\Lambda(\alpha)$  is the composition formed from  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$  by adding together adjacent components  $\alpha_i, \alpha_{i+1}, \dots, \alpha_{i+j}$  where  $\alpha_{i+l} = 1$  for  $l = 0, \dots, j-1$ , and either  $\alpha_{i+j} \neq 1$ , or  $i + j = k$ .

**Example 2.3.** If  $\alpha = 31125111$  then  $\Lambda(\alpha) = 3453$ .

Stembridge [10] showed that  $\Theta : \Sigma \rightarrow \Pi$  is a graded surjective ring homomorphism, and was an analogue of the retraction from the algebra of symmetric functions to Schur  $Q$  functions. It is clear from our proof above that this morphism is in fact a Hopf homomorphism. We can describe the kernel of  $\Theta$  as follows.

**Lemma 2.4.** *The non-zero differences  $F_\alpha - F_{\Lambda(\alpha)}$  form a basis of the kernel of  $\Theta$ .*

*Proof.* Each difference  $F_\alpha - F_{\Lambda(\alpha)}$  is in the kernel of  $\Theta$  as  $\Theta(F_\alpha - F_{\Lambda(\alpha)}) = 0$  since  $\Lambda(\Lambda(\alpha)) = \Lambda(\alpha)$ . In addition, the non-zero differences are linearly independent as they have different leading terms. Letting  $f_n$  denote the  $n$ th Fibonacci number, there are  $2^{n-1} - f_n$  such differences, and since

$$\begin{aligned}\dim \ker \Theta &= \dim \Sigma^n - \dim \Pi^n \\ &= 2^{n-1} - f_n,\end{aligned}$$

our result follows.  $\square$

### 3. THE COALGEBRA OF SHIFTED QUASI-SYMMETRIC FUNCTIONS

**Definition 3.1.** *For any composition  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k] \models n$  we say that  $\theta_\alpha$  is a shifted quasi-symmetric function (sq-s-function) if  $n \leq 1$  or  $\alpha_1 > 1$ .*

Observe that if  $\theta_\alpha$  is an sqs-function and  $\alpha \models n$ , then  $I(\alpha) \subset \{2, \dots, n-1\}$ .

For integers  $n \geq 0$ , let  $\Xi^n$  be the  $\mathbb{Z}$ -module spanned by all sqs-functions  $\theta_\alpha$ ,  $\alpha \models n$ , and let  $\Xi = \bigoplus_{n \geq 0} \Xi^n$ .

**Theorem 3.2.**  $\Xi$  is closed under coproduct.

*Proof.* If the first component of a composition  $\alpha$  is greater than 1, then the same is true for all compositions  $\varepsilon \cdot a$  and  $\phi(b \cdot \zeta)$  appearing in the RHS of equation (1).  $\square$

Unlike peak functions [10], sqs-functions are not  $F$ -positive since

$$\theta_{211} = F_{22} + F_{112} + 2F_{121} + F_{211} - F_{1111}.$$

**Definition 3.3.** *For any composition,  $\alpha \models n$ , we define the complement  $\alpha^c$  of  $\alpha$  to be the composition for which  $I(\alpha^c) = (I(\alpha))^c$ , the set complement of  $I(\alpha)$  in  $[n-1]$ . We define the graph  $G(\alpha)$  of  $\alpha$  to be the graph obtained from*



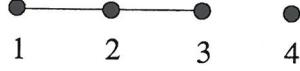
by removing the edge  $(i, i+1)$  if and only if  $i \in I(\alpha)$ .

Observe that  $G(\alpha^c)$  contains the edge  $(i, i+1)$  if and only if this edge is not contained in  $G(\alpha)$ . These graphs will be used later to simplify the proof of Theorem 3.6.

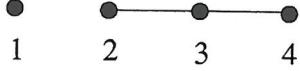
Let a *word* of length  $n$  be any  $n$ -tuple,  $w_1 w_2 \dots w_n$ , and let a *binary word* of length  $n$  be a word  $w_1 w_2 \dots w_n$  such that  $w_i \in \{0, 1\}$  for all  $i$ . For  $2 \leq i \leq n-1$ , let us denote by  $3^{(i)}$  the composition  $[1^{i-2}, 3, 1^{n-i-1}]$  of  $n$ . For some subset  $S \subset \{2, \dots, n-1\}$ , let us denote by  $\bigwedge_{i \in S} 3^{(i)}$  the composition of  $n$  for which  $G(\bigwedge_{i \in S} 3^{(i)})$  has an edge

between vertices  $i$  and  $i + 1$  if and only if an edge exists between vertices  $i$  and  $i + 1$  in  $G(3^{(i)})$  for some  $i \in S$ .

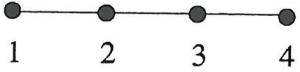
**Example 3.4.** Let  $S = \{2, 3\} \subset [3]$ . Then  $G(3^{(2)})$  is



and  $G(3^{(3)})$  is



hence  $G(\bigwedge_{i \in S} 3^{(i)})$  is



so  $\bigwedge_{i \in S} 3^{(i)}$  is the composition 4.

**Definition 3.5.** [1] Let  $\alpha$  be a composition of  $n$ . Let  $\mathcal{A}(I(\alpha))$  denote the set of all sequences  $j_1 \leq j_2 \leq \dots \leq j_n$  in  $\mathbb{N}$  such that we do not have  $j_{i-1} = j_i = j_{i+1}$  for any  $i \in I(\alpha)$ . The shifted quasi-symmetric function  $\theta_\alpha^{BH}$  is given by

$$\theta_\alpha^{BH} = \sum_{\substack{J = (j_1, \dots, j_n) \\ j_1 \leq \dots \leq j_n \\ J \in \mathcal{A}(I(\alpha))}} 2^{|j|} x_{j_1} \dots x_{j_n},$$

where  $|j|$  denotes the number of distinct values  $j_i$  in  $J$ .

**Theorem 3.6.** For any sqs-function  $\theta_\alpha$  we have that  $\theta_\alpha = \theta_\alpha^{BH}$ .

*Proof.* For each  $i \in I(\alpha) \subset [n - 1]$ ,  $j_{i-1} = j_i = j_{i+1}$  is forbidden in any monomial

$$x_{j_1} x_{j_2} \dots x_{j_i} \dots x_{j_n}$$

appearing as a summand of the function  $\theta_\alpha^{BH}$ . This is equivalent to saying that  $M_\beta$  is a summand of  $\theta_\alpha^{BH}$  if and only if  $G(3^{(i)}) \not\subset G(\beta)$  for all  $i \in I(\alpha)$ . Therefore at least one of  $i - 1$  or  $i$  must be the largest label of a vertex in a connected component in  $G(\beta)$ .

Now when going from compositions of  $n$  to subsets of  $[n - 1]$  we can do so using our graphs,  $G$ . All we have to do is list the label of the vertex that is the largest in each connect component, not listing  $n$ . We call these vertices the *end-points*. We are now in a position to prove the equivalence of Definitions 1.1 and 3.5 for sqs-functions.

The powers of 2 agree so we need only show that the indices of summation do too. To see this, take any sqs-function  $\theta_\alpha$  and let  $i \in I(\alpha)$ . Then  $M_\beta$  is a summand in  $\theta_\alpha^{BH}$  if at least one of  $i - 1$  or  $i$  is an end-point in  $G(\beta)$ . Therefore  $i$  or  $i - 1$  belongs to  $I(\beta)$ , and  $M_\beta$  is a summand of  $\theta_\alpha$ . Conversely, if  $M_\beta$  is a summand of  $\theta_\alpha$ , then this implies that for each  $i \in I(\alpha)$ , we have that  $i - 1$  or  $i$  belongs to  $I(\beta)$ , so one of  $i - 1$  or  $i$  is an end-point in  $G(\beta)$ , so  $M_\beta$  is a summand of  $\theta_\alpha^{BH}$ .  $\square$

#### 4. A BASIS FOR $\Xi$

**Definition 4.1.** Let  $\theta_\alpha$  be an sqs-function and  $\alpha \models n$ . We define an internal peak  $i \in I(\alpha)$  such that  $i - 1, i + 1 \notin I(\alpha)$ , and  $i \in \{3, \dots, n - 2\}$ .

**Remark** Observe that the occurrence of an internal peak in the  $i$ th position in  $I(\alpha) = \{w_1, w_2, \dots\}$ , where  $w_1 < w_2 < \dots$ , is equivalent to having two components of  $\alpha$ , say  $\alpha_i, \alpha_{i+1}$  such that  $\alpha_{i+1} \geq 2$ , and  $\alpha_i \geq 2$  if  $i \neq 1$ , or  $\alpha_i \geq 3$  if  $i = 1$ .

We can now describe the basis of  $\Xi$  as follows.

**Theorem 4.2.** The coalgebra  $\Xi$  has a basis consisting of all sqs-functions  $\theta_\alpha$  where  $I(\alpha)$  contains no internal peak.

We sketch the proof of Theorem 4.2 later.

**Theorem 4.3.** The rank of  $\Xi^n$  is given by the recurrence

$$\pi_n = \pi_{n-1} + \pi_{n-2} + \pi_{n-4},$$

with initial conditions  $\pi_1 = 1, \pi_2 = 1, \pi_3 = 2, \pi_4 = 4$ .

This recurrence was suggested by a superseeker query [6].

*Proof.* By direct calculation we obtain that  $\pi_1 = 1, \pi_2 = 1, \pi_3 = 2$ , and  $\pi_4 = 4$ .

To obtain our recurrence, we observe that for each sqs-function,  $\theta_\alpha$  where  $\alpha \models n$ , we can encode  $I(\alpha)$  as a binary word of length  $n - 2$ , by placing a 1 in position  $i - 1$  if  $i$  is contained in  $I(\alpha)$ , and 0 otherwise. By this one-to-one correspondence we see that  $I(\alpha)$  contains no internal peak if its corresponding binary word does not contain 010 as a subword.

We therefore count binary words of length  $n$  that avoid the subword 010. Appending either 1 or 0 to such a binary word of length  $n - 1$  gives one of length  $n$ , provided that we have not created the subword 010 in the last three positions. Let  $a_n, b_n, c_n$ , and  $d_n$  enumerate those binary words of length  $n - 2$  that avoid the subword 010 and end in, respectively 00, 01, 10, and 11. We then obtain the following 4 simultaneous recursions.

$$a_n = a_{n-1} + c_{n-1}, \quad b_n = a_{n-1} + c_{n-1}, \quad c_n = d_{n-1}, \quad d_n = b_{n-1} + d_{n-1}.$$

Clearly the number of  $I(\alpha)$ s in  $[n - 1]$  with no internal peaks is given by

$$\pi_n = a_n + b_n + c_n + d_n,$$

However by substituting in our recurrences we obtain

$$\begin{aligned}
\pi_n &= a_n + b_n + c_n + d_n \\
&= 2a_{n-1} + b_{n-1} + 2c_{n-1} + 2d_{n-1} \\
&= \pi_{n-1} + a_{n-1} + c_{n-1} + d_{n-1} \\
&= \pi_{n-1} + a_{n-2} + b_{n-2} + c_{n-2} + 2d_{n-2} \\
&= \pi_{n-1} + \pi_{n-2} + d_{n-2} \\
&= \pi_{n-1} + \pi_{n-2} + b_{n-3} + d_{n-3} \\
&= \pi_{n-1} + \pi_{n-2} + a_{n-4} + b_{n-4} + c_{n-4} + d_{n-4} \\
&= \pi_{n-1} + \pi_{n-2} + \pi_{n-4}.
\end{aligned}$$

□

We say that  $M_\beta$  is a maximal term of  $\theta_\alpha$  if for any  $\gamma$  higher in the partial order of compositions  $M_\gamma$  is not a summand of  $\theta_\alpha$ . The following lemma is stated without proof.

**Lemma 4.4.** *Let  $\theta_\alpha$  be an sqs-function. Consider the collection  $S$  of all possible sets derived from  $I(\alpha)$  by adding either  $i-1$  or  $i+1$  to  $I(\alpha)$  for all internal peaks  $i \in I(\alpha)$ . If  $M_\beta$  is a maximal term of  $\theta_\alpha$ , then  $\beta$  is derived from*

$$\bigwedge_{\substack{i \in (I(\tilde{\alpha}))^c \\ I(\tilde{\alpha}) \in S}} 3^{(i)}$$

by adding adjacent components equal to 1 together to give a component equal to 2 as often as possible.

**Lemma 4.5.** *Let  $\theta_\alpha$  be an sqs-function, and let  $I(\alpha)$  have an internal peak in the  $j$ th position, then we have the following linear relation*

$$\begin{aligned}
\theta_\alpha &= \theta_{[\alpha_1, \dots, \alpha_j-1, 1, \alpha_{j+1}, \dots, \alpha_k]} + \theta_{[\alpha_1, \dots, \alpha_j, 1, \alpha_{j+1}-1, \dots, \alpha_k]} \\
&\quad - \theta_{[\alpha_1, \dots, \alpha_j-1, 1, 1, \alpha_{j+1}-1, \dots, \alpha_k]}.
\end{aligned}$$

*Proof.* By Definition 3.5 we have that the leading terms of  $\theta_\alpha$  determine the other summands that belong to  $\theta_\alpha$ . Hence by Lemma 4.4 it follows that the summands of  $\theta_\alpha$  will be the union of the summands of  $\theta_{[\alpha_1, \dots, \alpha_j-1, 1, \alpha_{j+1}, \dots, \alpha_k]}$  and  $\theta_{[\alpha_1, \dots, \alpha_j, 1, \alpha_{j+1}-1, \dots, \alpha_k]}$ . However, those summands that appear in both will be duplicated. By definition these will be the summands of  $\theta_{[\alpha_1, \dots, \alpha_j-1, 1, 1, \alpha_{j+1}-1, \dots, \alpha_k]}$ , and the result follows. □

*Sketch of proof of Theorem 4.2.* From our relation in Lemma 4.5, it follows that any  $\theta_\alpha$  can be rewritten as a linear combination of functions  $\theta_{\tilde{\alpha}}$ , where  $I(\tilde{\alpha})$  contains no internal peaks. In addition, by Lemma 4.4 and definition 3.5 we have that the set of all sqs-functions  $\theta_\alpha$  where  $I(\alpha)$  contains no internal peaks is linearly independent and thus form a basis for  $\Xi$ . □

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