MES AVENTURES MATHÉMATIQUES AVEC PIERRE LEROUX

Gilbert Labelle

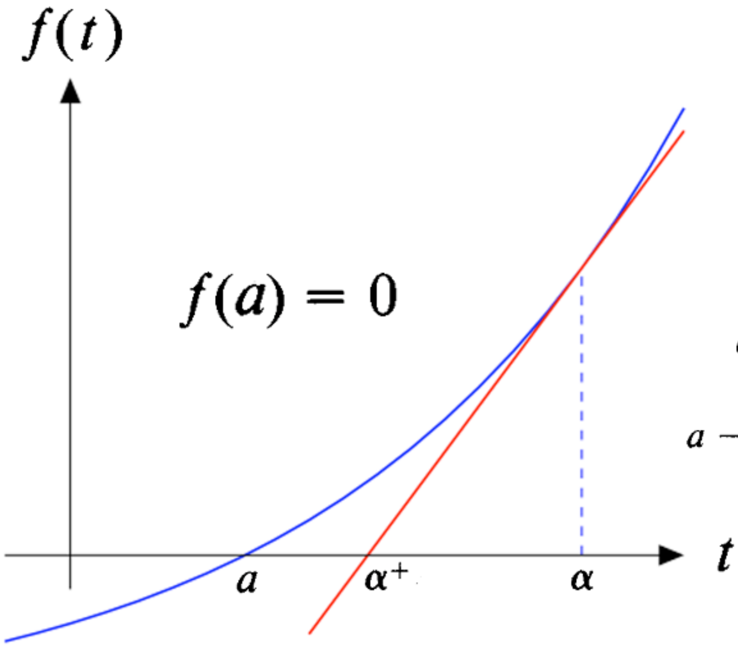
FPSAC'08 - Valparaiso, Chili

ADVANCES IN APPLIED MATHEMATICS **3**, 407–416 (1982)

Une approche combinatoire pour l'itération de Newton – Raphson

H. DÉCOSTE, G. LABELLE, ET P. LEROUX

Starting with an approximation α having a contact of order n with the species A of R-enriched rooted trees (in the sense of Joyal (*Advances in Math*. **42** (1981), 1–82) and Labelle (*Advances in Math*. **42** (1981), 217–247)), a new approximation α^+, having a contact of order $2n+2$ with A, is deduced by a purely combinatorial argumentation. This provides a combinatorial setting for the classical Newton–Raphson iterative scheme. A generalization involving contacts of higher orders is also developed.



$$\alpha^+ = \alpha - [f'(\alpha)]^{-1} f(\alpha)$$

$$\alpha_{n+1} = \alpha_n^+$$

$$a - \alpha_{n+1} = O\big((a - \alpha_n)^2\big) \to 0 \qquad \text{si } n \to \infty.$$

Le Cas des Séries Formelles

$$c = c(x) = c_1 x + c_2 x^2 + \cdots = x/r(x),$$

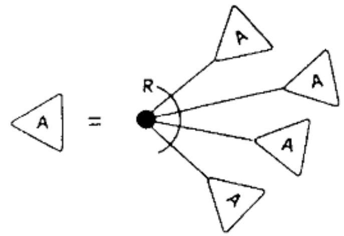
$$f(t) = c(t) - x, \qquad t = t(x)$$

$$f(a) = 0 \quad \textbf{ssi} \quad a = x\,r(a) \quad \textbf{ssi} \quad a = c^{\langle -1 \rangle}(x)$$

$$\alpha^+ = \alpha + \frac{x r(\alpha) - \alpha}{1 - x r'(\alpha)}$$

L'approche Combinatoire

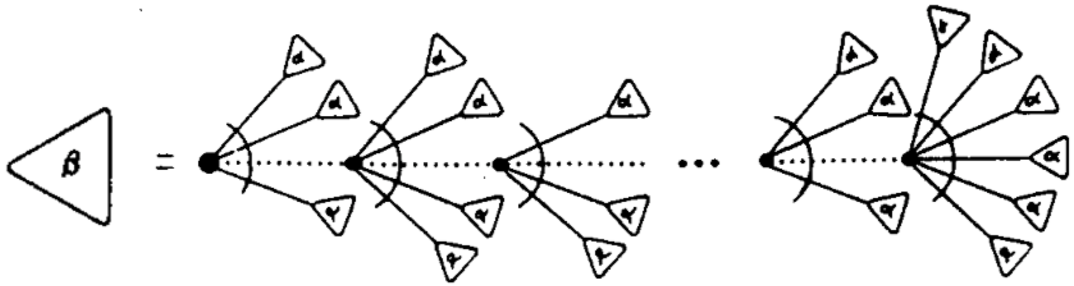
L'espèce A des arborescences R-enrichies $A = XR(A)$



Proposition 2. Soit L l'espèce des ordres linéaires et soit α l'espèce des arborescences R-enrichies légères (i.e., portées par des cardinalités $\leq n$). Alors l'espèce α^+ définie par

$$\alpha^+ = \alpha + L(XR'(\alpha)) \cdot (XR(\alpha) - \alpha) \tag{2.9}$$

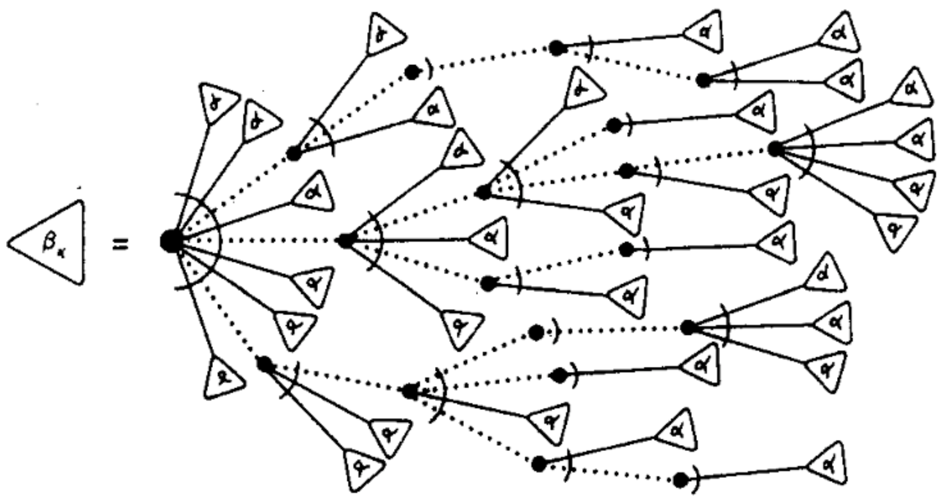
possède un contact d'ordre $2n+2$ *avec l'espèce* A *des arborescences* R*-enrichies.*



Proposition 4. *Soit α l'espèce des arborescences R-enrichies légères (i.e. portées par des cardinalités $\leq n$) et soit γ l'espèce définie implicitement par l'équation combinatoire polynomiale*

$$\gamma = (XR(\alpha) - \alpha) + \sum_{i=1}^{k} \frac{XR^{(i)}(\alpha)}{i!} \gamma^i.$$

Alors l'espèce α^+ définie par $\alpha^+ = \alpha + \gamma$ possède un contact d'ordre $(k+1)(n+1)$ avec l'espèce A des arborescences R-enrichies.



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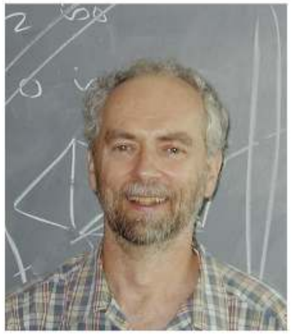
Discrete Applied Mathematics 34 (1991) 49–66
North-Holland

Computation of the expected number of leaves in a tree having a given automorphism, and related topics

F. Bergeron, G. Labelle and P. Leroux



Abstract

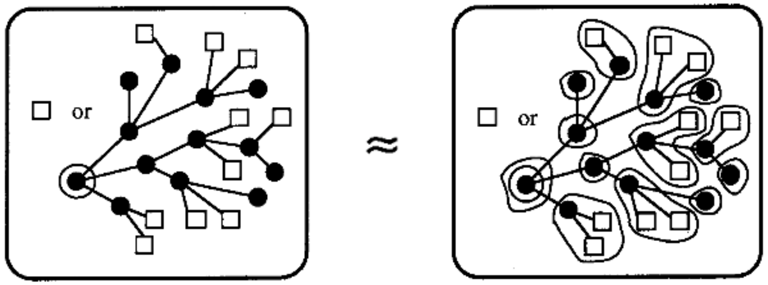
We derive explicit formulas for the expected number of leaves in a random rooted tree that is fixed by a given permutation of the nodes, and similarly for (unrooted) trees and endofunctions. The main tool is the cycle index series of a species. The cases of asymmetric rooted trees and ***R***-enriched trees and rooted trees are also discussed.

- $A = A(X)$ of rooted trees, and
- $B = B(X, Y)$ of rooted trees with internal points of sort X and leaves of sort Y.

Proposition 1. *The species $A = A(X)$ and $B = B(X, Y)$ are related by the following combinatorial equation*

$$B(X, X+T) = T + A(X \cdot E(T)), \tag{2.3}$$

where T denotes an auxiliary sort of singletons.



$B(X, X+T)$ $\qquad$ $T + A(X \cdot E(T))$

$$A_w(X) = B(X, X_t), \qquad T := X_t - X, \qquad A_w(X) = X_t - X + A(X \cdot E(X_t - X)).$$

$$Z_{A_w} = (t-1)x_1 + Z_A\left(x_1 \exp \sum_{k \geq 1} \frac{(t^k - 1)x_k}{k},\ x_2 \exp \sum_{k \geq 1} \frac{(t^{2k} - 1)x_{2k}}{k}, \ldots\right)$$

The number of rooted trees which are fixed by a permutation σ

$$a_\sigma = \sigma_1^{\sigma_1 - 1} \prod_{k \geq 2} \left\{ (\sigma^k)_1^{\sigma_k} - k\sigma_k (\sigma^k)_1^{\sigma_k - 1} \right\}, \quad \text{where} \quad (\sigma^k)_1 = \sum_{d \mid k} d\sigma_d . \tag{2.2}$$

Proposition 2. *Let U be an n-set, σ be a permutation of U whose cyclic type is $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $P = \{k \mid 1 \leq k \leq n,\ \sigma_k \neq 0\}$. Then the expected number of leaves in a random rooted tree on U for which σ is an automorphism is given by* 1 *if $n = 1$, and by*

$$\frac{1}{a_\sigma} \sum_{k \in P} k\sigma_k \cdot ((\sigma^k)_1 - k) \cdot a_{\sigma - \delta^k}, \tag{2.8}$$

if $n \neq 1$ and $a_\sigma \neq 0$, where a_σ and $(\sigma^k)_1$ are given by (2.2) *and $\sigma - \delta^k$ denotes a permutation obtained from σ by dropping out one* (*arbitrary*) *k-cycle* (*i.e., the cyclic type of $\sigma - \delta^k$ is given by $(\sigma_1, \ldots, \sigma_k - 1, \ldots, \sigma_n)$*).

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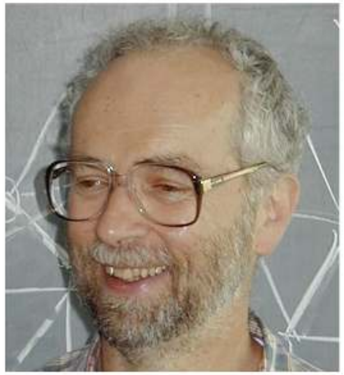
Discrete Mathematics 99 (1992) 31–48
North-Holland

The functorial composition of species, a forgotten operation

Hélène Décoste, Gilbert Labelle and Pierre Leroux



Abstract

In order to study the functorial composition of species, we introduce the auxiliary concepts of *cyclic type* and *fixed points enumerator* of a species. Basic formulas are established and applications are given to the computation of the cycle index series of classes of graphs, pure m-complexes, coverings and m-ary relations that are structured in various ways.

The functorial composition

$$(F \square G)[U] = F[G[U]], \quad (F \square G)[\sigma] = F[G[\sigma]]$$

– Simple graphs: $\mathcal{G} = \wp \square \wp^{[2]}$

– Directed graphs: $\mathcal{D} = \wp \square (E^{\bullet} \times E^{\bullet})$

– m-ary relations: $\mathrm{Rel}^{[m]} = \wp \square (E^{\bullet})^{\times m}$

$$\mathbf{Z}_{F \square G}(x_1, x_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S[n]} \operatorname{fix} F[G[\sigma]] x_1^{\sigma_1} x_2^{\sigma_2} \cdots = \; ?$$

Proposition 2.2. *The cyclic type* $(\beta_k)_{k=1,2,3,\ldots}$ *of an arbitrary species* G *depends only on the function* $\operatorname{fix} G[\sigma] = \beta(\sigma_1, \sigma_2, \sigma_3, \ldots)$*. Indeed we have the formula*

$$\beta_k = (G[\sigma])_k = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) \operatorname{fix} G[\sigma^d]. \tag{2.4}$$

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Applications : Enumeration of
Classes of graphs, pure *m*-complexes, coverings, structured *m*-ary relations, structured words ... .

ELSEVIER

Discrete Mathematics 157 (1996) 227–240

DISCRETE MATHEMATICS

Enumeration of (uni- or bicolored) plane trees according to their degree distribution

Gilbert Labelle, Pierre Leroux

Résumé

L'objectif principal de ce texte est de donner des formules explicites pour le nombre de types d'isomorphie d'arbres plans bicolorés ayant une double distribution $(1^{i_1}2^{i_2}\ldots;1^{j_1}2^{j_2}\ldots)$ de degrés donnés à l'avance. Ces arbres sont en étroite relation avec les polynômes de Shabat, c'est-à-dire les polynômes sur $\mathbb{C}$ ayant au plus deux valeurs critiques (cf. Shabat and Zvonkin, 1994). Dans le cas des arbres enracinés (pointés en une feuille), ce problème a été résolu par Tutte en 1964 à l'aide de l'inversion de Lagrange multivariée. Ici la clé de la solution réside dans le théorème de dissymétrie pour les arbres enrichis qui, dans le cas bicoloré, prend une forme particulièrement simple et qui permet de se défaire du pointage. Nous dénombrons également ces arbres dans le cas étiqueté, dans le cas unicoloré, ainsi que lorsque le groupe d'automorphismes, nécessairement cyclique, est d'un ordre h égal à (ou multiple de) un entier $k\geqslant 1$ donné.

As was emphasized by A. Zvonkin during the 5th FPSAC meeting in Florence, these trees are closely related to polynomials $P(z)$ over $\mathbb{C}$ having at most two critical values, called Shabat polynomials.

Unlabelled bicolored plane trees $\cong$ ***Equivalence classes of Shabat polynomials under affine transformations***

The black vertices correspond to the roots of the equation $P(z)=0$,
the white vertices correspond to the roots of the equation $P(z)=1$,
the degrees correspond to multiplicities of these roots.

Double degree distribution $\boldsymbol{i}=(i_1,i_2,\ldots)$, $\boldsymbol{j}=(j_1,j_2,\ldots)$, with $s(\boldsymbol{i})=i_1+i_2+\cdots$ black points and $s(\boldsymbol{j})=j_1+j_2+\cdots$ white points,

the edge count implies the double equality

$$i_1+2i_2+3i_3+\cdots \;=\; j_1+2j_2+3j_3+\cdots \;=\; s(\boldsymbol{i})+s(\boldsymbol{j})-1. \qquad (0.1)$$

QUESTION : How many unlabelled bicolored plane trees have such a double distribution **?**

Theorem 1. *Let* $\boldsymbol{i} = (i_1, i_2, \dots)$ *and* $\boldsymbol{j} = (j_1, j_2, \dots)$ *be two sequences of integers* $\geqslant 0$, *with* $s(\boldsymbol{i}) = i_1 + i_2 + \cdots < \infty$ *and* $s(\boldsymbol{j}) < \infty$, *satisfying the relations* (0.1). *Then the number* $\tilde{\wp}(\boldsymbol{i}, \boldsymbol{j})$ *of unlabelled bicolored plane trees, whose degree distribution is* $(\boldsymbol{i}, \boldsymbol{j})$, *is given by*

$$\tilde{\wp}(\boldsymbol{i}, \boldsymbol{j}) = \tilde{F}(\boldsymbol{i}, \boldsymbol{j}) + \tilde{F}(\boldsymbol{j}, \boldsymbol{i}) - \tilde{G}(\boldsymbol{i}, \boldsymbol{j}), \tag{0.2}$$

with

$$\tilde{F}(\boldsymbol{i}, \boldsymbol{j}) = \frac{1}{n} \sum_{h,d} \phi(d) \binom{n/d}{\boldsymbol{i}/d} \binom{(m-1)/d}{(\boldsymbol{j} - \delta_h)/d}, \tag{0.3}$$

where $n = s(\boldsymbol{i})$, $m = s(\boldsymbol{j})$, ϕ *denotes the arithmetic Euler* ϕ*-function, and the sum is taken over all pairs of integers* $h \geqslant 1$, $d \geqslant 1$ *such that* $h \in \operatorname{Supp}(\boldsymbol{j})$ *and* $d \in \operatorname{Div}(h, \boldsymbol{i}, \boldsymbol{j} - \delta_h)$, *and*

$$\tilde{G}(\boldsymbol{i}, \boldsymbol{j}) = \frac{n+m-1}{nm} \binom{n}{\boldsymbol{i}} \binom{m}{\boldsymbol{j}}. \tag{0.4}$$

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THE ELECTRONIC JOURNAL OF COMBINATORICS 3 (2) (1996), #R12

An Extension of the Exponential Formula in Enumerative Combinatorics

Gilbert Labelle and Pierre Leroux

En hommage à Dominique Foata, à l'occasion de son soixantième anniversaire.

Résumé

Soit α une variable formelle et F_w une espèce de structures pondérée (classe de structures fermée sous les isomorphismes préservant les poids) de la forme $F_w = E(F_w^c)$, où E et F_w^c désignent respectivement l'espèce des *ensembles* et celle des *F_w-structures connexes*. En multipliant par α le poids de chaque F_w^c-structure, on obtient l'espèce $F_{w^{(\alpha)}} = E(F_{\alpha w}^c)$. Nous introduisons une espèce virtuelle "universelle", $\Lambda^{(\alpha)}$, telle que $F_{w^{(\alpha)}} = \Lambda^{(\alpha)} \circ F_w^+$, où F_w^+ désigne l'espèce des F_w-structures non-vides. En faisant appel à des propriétés générales de $\Lambda^{(\alpha)}$, nous calculons les diverses séries formelles énumératives $G(x)$, $\widetilde{G}(x)$, $\overline{G}(x)$, $G(x;q)$, $G\langle x;q\rangle$, $Z_G(x_1,x_2,x_3,\ldots)$, $\Gamma_G(x_1,x_2,x_3,\ldots)$, de $G = F_{w^{(\alpha)}}$, en fonction de F_w. Comme cas spéciaux des formules que nous développons, on retrouve la formule exponentielle, $F_{w^{(\alpha)}}(x) = \exp(\alpha F_w(x)) = (F_w(x))^\alpha$, les identités cyclotomiques, ainsi que leurs q-analogues. L'espèce virtuelle pondérée, $\Lambda^{(\alpha)}$, est, en fait, un nouveau relèvement combinatoire de la fonction $(1+x)^\alpha$.

$$f(x) = \exp(g(x))$$
$$f(x)^{\alpha} = \exp(\alpha\, g(x))$$

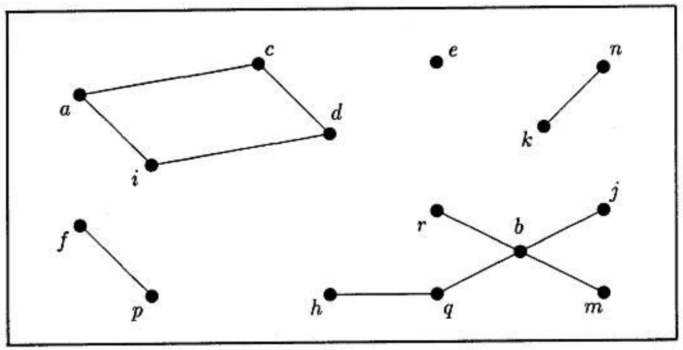


Figure 1: A graph g with $w(g) = y^{11}$ and $w^{(\alpha)}(g) = \alpha^5 y^{11}$

$$\begin{array}{c} F_w = E(F_w^c) \\ \Big\downarrow \\ F_{w^{(\alpha)}} = E(F_{\alpha w}^c) \end{array}$$

Theorem 2.2 *There exists a "universal" virtual weighted species,* $\Lambda^{(\alpha)}$, *such that*

$$F_{w^{(\alpha)}} = \Lambda^{(\alpha)} \circ F_w^{+},$$

for any species of the form $F_w = E(F_w^c)$.

$$\Lambda^{(\alpha)}(x) = (1+x)^{\alpha},$$

$$Z_{\Lambda^{(\alpha)}} = \prod_{n\geq 1} (1+x_n)^{\lambda_n(\alpha)}, \qquad \widetilde{\Lambda^{(\alpha)}}(x) = \prod_{n\geq 1} (1+x^n)^{\lambda_n(\alpha)}, \qquad \Lambda^{(\alpha)}(x;q) = \prod_{n\geq 1} \left(1 + \frac{(1-q)^n}{(1-q^n)}\, x^n\right)^{\lambda_n(\alpha)},$$

$$\Gamma_{\Lambda^{(\alpha)}} = \prod_{n\geq 1} (1+x_n)^{\gamma_n(\alpha)}, \qquad \overline{\Lambda^{(\alpha)}}(x) = \prod_{n\geq 1} (1+x^n)^{\gamma_n(\alpha)}, \qquad \Lambda^{(\alpha)}\langle x;q\rangle = \prod_{n\geq 1} \left(1 + \frac{(1-q)^n}{(1-q^n)}\, x^n\right)^{\gamma_n(\alpha)},$$

$$\lambda_n(\alpha) = \frac{1}{n}\sum_{d|n} \mu(n/d)\alpha^d, \qquad \gamma_n(\alpha) = -\lambda_n(-\alpha) - \lambda_{n/2}(-\alpha) - \lambda_{n/4}(-\alpha) - \cdots$$

$$\begin{aligned}\Lambda^{(\alpha)} = E \circ X_\alpha \circ (E^+)^{<-1>} = {} & 1 + X_\alpha - (E_2)_\alpha + (E_2)_{\alpha^2} - (E_3)_\alpha + (XE_2)_\alpha - (XE_2)_{\alpha^2} + (E_3)_{\alpha^3} \\ & + (E_2 \circ E_2)_\alpha - (E_4)_\alpha + (XE_3)_\alpha - (X^2E_2)_\alpha \\ & + (E_2^2)_{\alpha^2} - (XE_3)_{\alpha^2} + (X^2E_2)_{\alpha^2} - (E_2 \circ E_2)_{\alpha^2} - (E_2^2)_{\alpha^3} + (E_4)_{\alpha^4} \\ & + (E_2E_3)_\alpha + (XE_4)_\alpha + (X^3E_2)_\alpha - (X^2E_3)_\alpha - (XE_2^2)_\alpha - (E_5)_\alpha \\ & + (E_2E_3)_{\alpha^2} + (X^2E_3)_{\alpha^2} - (XE_4)_{\alpha^2} - (X^3E_2)_{\alpha^2} + (X\ E_2 \circ E_2)_{\alpha^2} - (XE_2^2)_{\alpha^2} \\ & + 2\,(XE_2^2)_{\alpha^3} - (E_2E_3)_{\alpha^3} - (X\ E_2 \circ E_2)_{\alpha^3} - (E_2E_3)_{\alpha^4} + (E_5)_{\alpha^5} + \cdots\end{aligned}$$

Corollary 2.3 *Let F_w be a weighted virtual species such that $F_w(0) = 1$ and set $F_{w(\alpha)} = \Lambda^{(\alpha)} \circ F_w^+$. Then*

$$F_{w(\alpha)}(x) = F_w(x)^\alpha,$$

$$\begin{aligned} Z_{F_{w(\alpha)}}(x_1, x_2, x_3, \ldots) &= \prod_{n\geq 1} Z_{F_{w^n}}(x_n, x_{2n}, x_{3n}, \ldots)^{\lambda_n(\alpha)}, \\ \widetilde{F_{w(\alpha)}}(x) &= \prod_{n\geq 1} \widetilde{F_{w^n}}(x^n)^{\lambda_n(\alpha)}, \\ F_{w(\alpha)}(x;q) &= \prod_{n\geq 1} F_{w^n}\left(\frac{(1-q)^n}{(1-q^n)}x^n; q^n\right)^{\lambda_n(\alpha)}, \\ \Gamma_{F_{w(\alpha)}}(x_1, x_2, x_3, \ldots) &= \prod_{n\geq 1} \Gamma_{F_{w^n}}(x_n, x_{2n}, x_{3n}, \ldots)^{\gamma_n(\alpha)}, \\ \overline{F_{w(\alpha)}}(x) &= \prod_{n\geq 1} \overline{F_{w^n}}(x^n)^{\gamma_n(\alpha)}, \\ F_{w(\alpha)}\langle x;q\rangle &= \prod_{n\geq 1} F_{w^n}\left\langle\frac{(1-q)^n}{(1-q^n)}x^n; q^n\right\rangle^{\gamma_n(\alpha)}. \end{aligned}$$

ETC Cyclotomic identities, cocyclotomic identities, q-series identities, ...

ADVANCES IN APPLIED MATHEMATICS **21**, 499–546 (1998)

Cubical Species and Nonassociative Algebras

Gábor Hetyei   Gilbert Labelle   and   Pierre Leroux



We lay down the foundations of a theory of *cubical species*, as a variant of Joyal's classical theory of species (A. Joyal, *Adv. Math.* **42** (1981), 1–82). Informally, a cubical species associates in a functorial way a set of structures to each hypercube. In this context, the hyperoctahedral groups replace the symmetric groups. We analyze cubical species, molecular cubical species, and basic operations among them, along with explicit examples. We show, in particular, that the cubical product gives rise, in a natural way, to a commutative nonassociative ring of formal power series. We conclude with a detailed analysis of this nonassociative ring.

Journal of Combinatorial Theory, Series A **91**, 15–48 (2000)

Generalized Binomial Coefficients for Molecular Species

Pierre Auger, Gilbert Labelle, and Pierre Leroux



DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

Let ξ be a complex variable. We associate a polynomial in ξ, denoted $\binom{M}{N}_\xi$, to any two molecular species $M = M(X)$ and $N = N(X)$ by means of a binomial-type expansion of the form

$$M(\xi + X) = \sum_N \binom{M}{N}_\xi N(X).$$

In the special case $M(X) = X^m$, the species of linear orders of length m, the above formula reduces to the classical binomial expansion

$$(\xi + X)^m = \sum_n \binom{m}{n} \xi^{m-n} X^n.$$

When $\xi = 1$, a $M(1 + X)$-structure can be interpreted as a partially labelled M-structure and $\binom{M}{N}_1$ is a nonnegative integer, denoted $\binom{M}{N}$ for simplicity. We develop some basic properties of these "generalized binomial coefficients" and apply them to study solutions, Φ, of combinatorial equations of the form $M(\Phi) = \Psi$ in the context of $\mathbb{C}$-species, M being molecular and Ψ being a given $\mathbb{C}$-species. This generalizes the study of symmetric square roots (where $M = E_2$, the species of 2-element sets) initiated by P. Bouchard, Y. Chiricota, and G. Labelle in (1995, *Discrete Math.* **139**, 49–56).

$$(1+X)^m = \sum_{n \leqslant m} \binom{m}{n} X^n,$$

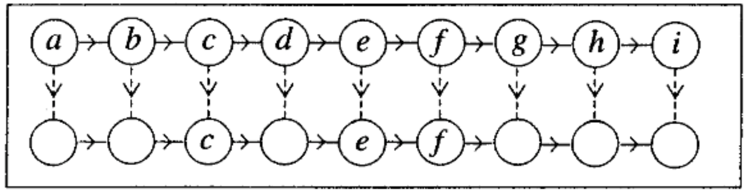


FIG. 1. A 3-partially labelled 9-list.

DEFINITION 2.1. Let $M = M(X)$ be any molecular species. The *species of partially labelled M-structures*, denoted $M(1+X)$, is defined by

$$M(1+X) = M(T+X)|_{T:=1}, \tag{2.11}$$

where T is an auxiliary sort of singletons.

GENERALIZED BINOMIAL COEFFICIENTS

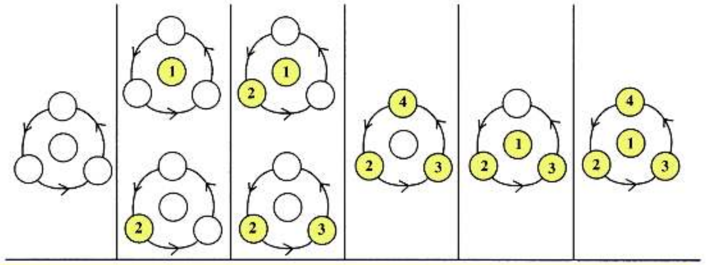
DEFINITION 2.2. The *generalized binomial coefficients*, $\binom{M}{N}$, are the non-negative integers arising from the molecular decompositions

$$M(1+X) = \sum_N \binom{M}{N} N(X), \tag{2.14}$$

where M, N run through the molecular species (up to isomorphism of species).

$M\backslash N$	1	X	E_2	X^2	E_3	C_3	XE_2	X^3	E_4	$E_4^{\pm}$	$E_2(E_2)$	XE_3	E_2^2	P_4^{bic}	C_4	XC_3	X^2E_2	$E_2(X^2)$	X^4
1	$\boxed{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X	$\boxed{1}$	$\boxed{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E_2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X^2	$\boxed{1}$	$\boxed{2}$	0	$\boxed{1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E_3	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C_3	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
XE_2	1	2	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
X^3	$\boxed{1}$	$\boxed{3}$	0	$\boxed{3}$	0	0	0	$\boxed{1}$	0	0	0	0	0	0	0	0	0	0	0
E_4	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$E_4^{\pm}$	1	1	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
$E_2(E_2)$	1	1	2	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
XE_3	1	2	1	1	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0
E_2^2	1	2	2	1	0	0	2	0	0	0	0	0	1	0	0	0	0	0	0
P_4^{bic}	1	1	3	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
C_4	1	1	1	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
XC_3	1	2	0	2	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0
X^2E_2	1	3	1	3	0	0	2	1	0	0	0	0	0	0	0	0	1	0	0
$E_2(X^2)$	1	2	2	2	0	0	0	2	0	0	0	0	0	0	0	0	0	1	0
X^4	$\boxed{1}$	$\boxed{4}$	0	$\boxed{6}$	0	0	0	$\boxed{4}$	0	0	0	0	0	0	0	0	0	0	$\boxed{1}$

Example : $M(X) = XC_3(X)$



$$M(1+X) = 1 + 2X + 2X^2 + C_3(X) + X^3 + XC_3(X)$$

Classical identities

$$(1+X)^m = \sum_{n \le m} \binom{m}{n} X^n$$

$$\begin{aligned}
\binom{m}{0} &= \binom{m}{m} = 1 \\
\sum_{n \le m} \binom{m}{n} &= 2^m \\
\binom{m+1}{n} &= \binom{m}{n} + \binom{m}{n-1} \\
\binom{m_1+m_2}{n} &= \sum_{n_1+n_2=n} \binom{m_1}{n_1}\binom{m_2}{n_2}
\end{aligned}$$

Generalized identities

$$M(1+X) = \sum_{N} \binom{M}{N} N(X)$$

$$\begin{aligned}
\binom{M}{1} &= \binom{M}{M} = 1 \\
\sum_{N \preceq M} \binom{M}{N} &= \frac{1}{|H|} \sum_{h \in H} 2^{c(h)} \\
\binom{XM}{N} &= \begin{cases} \binom{M}{N} + \binom{M}{P} & \text{if } N = XP \\ \binom{M}{N} & \text{otherwise} \end{cases} \\
\binom{M_1 M_2}{N} &= \sum_{N_1 N_2 = N} \binom{M_1}{N_1}\binom{M_2}{N_2}
\end{aligned}$$

ETC

Explicit expressions, applications to combinatorial equations :

$$E_m(\Phi) = X, \quad E_m^{\pm}(\Phi) = X, \quad C_m(\Phi) = X.$$

Logarithm of the generalized Pascal triangle.

Advances in Applied Mathematics **24**, 22–56 (2000)

Enumeration of m-Ary Cacti

Miklós Bóna, Michel Bousquet, Gilbert Labelle, and Pierre Leroux

The purpose of this paper is to enumerate various classes of cyclically colored m-gonal plane cacti, called m-ary cacti. This combinatorial problem is motivated by the topological classification of complex polynomials having at most m critical values, studied by Zvonkin and others. We obtain explicit formulae for both labelled and unlabelled m-ary cacti, according to (i) the number of polygons, (ii) the vertex-color distribution, (iii) the vertex-degree distribution of each color. We also enumerate m-ary cacti according to the order of their automorphism group. Using a generalization of Otter's formula, we express the species of m-ary cacti in terms of rooted and of pointed cacti. A variant of the m-dimensional Lagrange inversion is then used to enumerate these structures. The method of Liskovets for the enumeration of unrooted planar maps can also be adapted to m-ary cacti.

$$\mathbf{n}_1 = (0,7,1,0,1,0,\dots) = 1^7 2^1 4^1, \qquad \mathbf{n}_2 = (0,7,3,0,0,0,\dots) = 1^7 2^3,$$

$$\mathbf{n}_3 = (0,8,1,1,0,0,\dots) = 1^8 2^1 3^1, \qquad \mathbf{n}_4 = (0,9,2,0,0,0,\dots) = 1^9 2^2,$$

$$n_1 = 9, \quad n_2 = 10, \quad n_3 = 10, \quad n_4 = 11, \quad n = 40, \quad \text{and } p = 13.$$

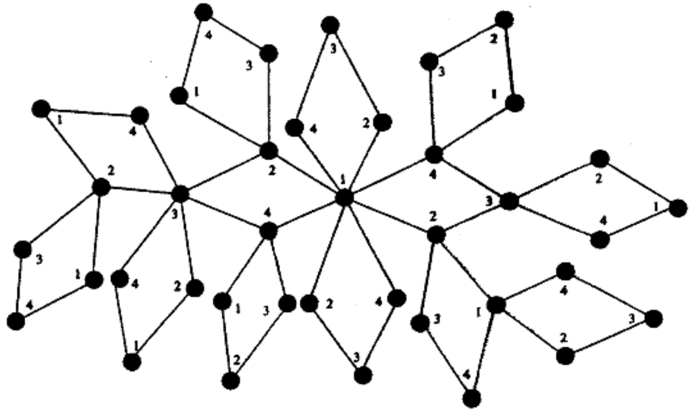


FIG. 1. A quaternary cactus.

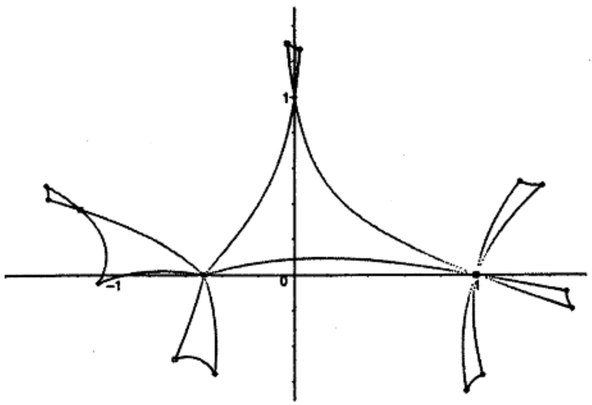


FIG. 2. Cactus associated to a polynomial of degree 8, having three critical values.

We consider the class $\mathscr{K}$ of m-ary cacti as an m-sort species.

$\mathscr{A}_i$: m-ary cacti, *planted* at a vertex of color i,

$\mathscr{K}^{\bullet i}$: m-ary cacti, *pointed* at a vertex of color i,

$\mathscr{K}^{\diamond}$: *rooted* m-ary cacti, $\hat{\mathscr{A}}_i := \prod_{j \neq i} \mathscr{A}_j$.

PROPOSITION 1. *We have the following isomorphisms of species, for* $i = 1, \ldots, m$

$$\mathscr{A}_i = X_i L(\hat{\mathscr{A}}_i), \tag{5}$$

$$\mathscr{K}^{\bullet i} = X_i\big(1 + \mathscr{C}(\hat{\mathscr{A}}_i)\big), \tag{6}$$

$$\mathscr{K}^{\diamond} = \mathscr{A}_1 \mathscr{A}_2 \cdots \mathscr{A}_m . \tag{7}$$

THEOREM 2 Dissymmetry theorem for m-ary cacti. *There is an isomorphism of species*

$$\mathscr{K}^{\bullet 1} + \mathscr{K}^{\bullet 2} + \cdots + \mathscr{K}^{\bullet m} = \mathscr{K} + (m - 1)\mathscr{K}^{\diamond} . \tag{8}$$

ETC Applications to the enumeration of labelled and unlabelled m-ary cacti according to face and degree distributions.

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Discrete Mathematics 222 (2000) 1–25

DISCRETE MATHEMATICS

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Enumeration of planar two-face maps

Michel Bousquet, Gilbert Labelle, Pierre Leroux

Résumé

Nous dénombrons les cartes planaires (à homéomorphisme préservant l'orientation près) non pointées à deux faces, selon le nombre de sommets et selon la distribution des degrés des sommets et des faces, étiquetées (aux sommets) ou non. Nous abordons d'abord les cartes planes, c'est-à-dire plongées dans le plan, et déduisons ensuite le cas des cartes planaires (ou sphériques), plongées sur la sphère. Une étape cruciale est le dénombrement des cartes planes à deux faces admettant une symétrie antipodale et la méthode de Liskovets est utilisée pour cela. La motivation de cette recherche provient de la classification topologique des fonctions de Belyi.

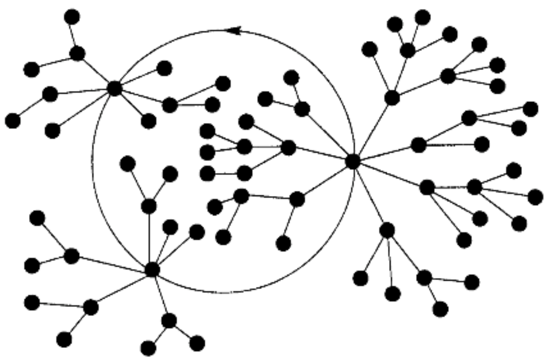


Fig. 2. A two-face plane map.

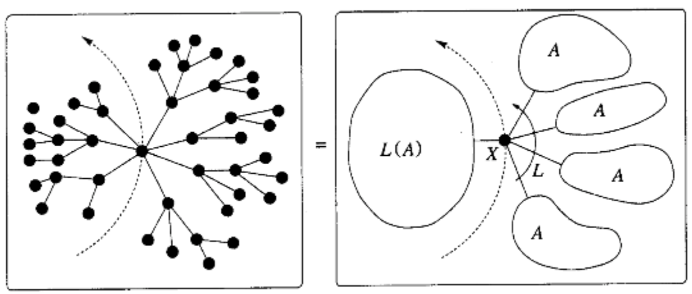


Fig. 3. An $XL^2(A)$-structure.

Theorem 2. *The species* M *of two-face plane maps satisfies the following combinatorial identity*:

$$\mathsf{M} = C(XL^2(A)).$$

Theorem 3. *The numbers* $|\mathsf{M}_n|$ *and* $|\tilde{\mathsf{M}}_n|$ *of labelled and unlabelled two-face plane maps on* n *vertices are respectively given by*

$$|\mathsf{M}_n| = \frac{(n-1)!}{2}\left(2^{2n} - \binom{2n}{n}\right) \quad \text{and} \quad |\tilde{\mathsf{M}}_n| = \frac{1}{2n}\sum_{d|n} \phi\left(\frac{n}{d}\right)\left(2^{2d} - \binom{2d}{d}\right).$$

Theorem 7. *Let* $\boldsymbol{d}$ *satisfy* $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$ *and* $\alpha, \beta > 0$ *be two integers having the same parity, where* $|\boldsymbol{d}| = (\alpha+\beta)/2 = n$. *Then the number* $|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}|$ *of labelled two-face plane maps on* $[n]$ *having joint vertex and face degree distributions* $\boldsymbol{d}$ *and* (α,β) *is given by*

$$|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}| = n!H(\boldsymbol{d},(\alpha,\beta)), \tag{44}$$

and the corresponding number $|\tilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}|$ *of unlabelled two-face plane maps is given by*

$$|\tilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}| = \sum_{m|(\boldsymbol{d},\alpha,\beta)} \frac{\phi(m)}{m} H\left(\frac{\boldsymbol{d}}{m}, \left(\frac{\alpha}{m}, \frac{\beta}{m}\right)\right) \tag{45}$$

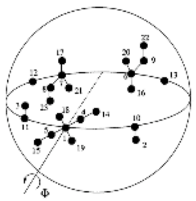
with

$$H(\boldsymbol{d},(\alpha,\beta)) = \sum_{\boldsymbol{g},\boldsymbol{h},\boldsymbol{k}} \frac{\Phi(\boldsymbol{h})\Phi(\boldsymbol{k})\Theta(\boldsymbol{g},\boldsymbol{h})}{|\boldsymbol{g}|}\binom{|\boldsymbol{g}|}{\boldsymbol{g}}\binom{|\boldsymbol{h}|}{\boldsymbol{h}}\binom{|\boldsymbol{k}|}{\boldsymbol{k}},$$

where the sum runs over all $\boldsymbol{g}, \boldsymbol{h}$ *and* $\boldsymbol{k}$ *satisfying conditions* 1–5 *in* (42).

ETC

Sphere maps



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Discrete Mathematics 246 (2002) 177–195

DISCRETE MATHEMATICS

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Stirling numbers interpolation using permutations with forbidden subsequences

G. Labelle, P. Leroux, E. Pergola, R. Pinzani




Résumé

Nous présentons une famille de suites de nombres qui interpole entre la suite B_n des nombres de Bell et la suite $n!$. Cette famille est définie en termes de permutations à motifs interdits. L'introduction comme paramètre du nombre d'éléments saillants minimums de gauche à droite donne une interpolation plus fine entre les nombres de Stirling de deuxième espèce $S(n,m)$ et de première espèce (sans signe) $c(n,m)$. De plus, un q-comptage de ces permutations selon des inversions particulières donne une interpolation entre des variantes des q-analogues habituels de ces nombres.

Advances in Applied Mathematics **28,** 302–342 (2002)

Combinatorial Addition Formulas and Applications

Pierre Auger, Gilbert Labelle, and Pierre Leroux

Nous obtenons des formules d'addition combinatoires, c'est-à-dire des équations de la forme $F(X_1 + X_2 + \cdots + X_k) = \Phi_F(X_1, X_2, \ldots, X_k)$, où $F = F(X)$ est une espèce de structures donnée et Φ_F est une espèce, dépendant de F, sur k sortes de singletons $X_1, X_2, \ldots, X_k$. Nous donnons des formules générales pour les espèces moléculaires $M = X^n/H$ et des résultats plus spécifiques dans le cas des espèces L_n, des n-listes (listes de longueur n), Cha_n, des n-chaînes, E_n, des n-ensembles, $E_n^{\pm}$, des n-ensembles orientés, C_n, des n-cycles (orientés), et P_n, des n-gones (cycles non orientés). Ces formules sont utiles pour calculer le développement moléculaire d'espèces définies par des équations fonctionnelles. Nous présentons également des applications au calcul des séries indicatrices des cycles ou d'asymétrie, à l'extension de la substitution aux espèces virtuelles (et aux $\mathbb{K}$-espèces), et à l'analyse des coefficients binomiaux généralisés $\binom{M}{N}_k$ pour les espèces moléculaires.

Addition formulas in classical analysis.

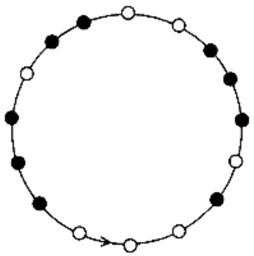
$$e^{x+y} = e^x \cdot e^y, \qquad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

$$(x+y+z)^n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} x^i y^j z^k.$$

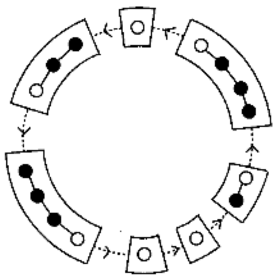
DEFINITION 1.1. Let $F = F(X)$ be a combinatorial species of one sort. A *combinatorial addition formula* for the species F is a combinatorial equation of the form

$$F(X_1 + X_2 + \cdots + X_k) = \Phi_F(X_1, X_2, \ldots, X_k), \tag{1.1}$$

where Φ_F is a species, depending on F, on the k sorts of singletons X_1, $X_2, \ldots, X_k$, $k \geq 2$.



$$C(X+Y) = C(X) + C(L(X)Y)$$



Molecular addition formulas : $M = M(X)$ molecular

$$M(X_1 + X_2 + \cdots + X_k) = \sum_N c_{M,N} N(X_1, X_2, \ldots, X_k)$$

Classical multinomial formula : $M(X) = X^n$

$$(X_1 + \cdots + X_k)^n = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1! \cdots n_k!} X_1^{n_1} \cdots X_k^{n_k}$$

THEOREM 2.1 (Generalized multinomial expansion). *Let $M = M(X)$ be a molecular species of degree n; then*

$$M(X_1 + \cdots + X_k) = \sum_{n_1+\cdots+n_k=n} \sum_{s \in M[n]} \frac{|\mathrm{Aut}(s)_{n_1,\ldots,n_k}|}{n_1! \cdots n_k!} \frac{X_1^{n_1} \cdots X_k^{n_k}}{\mathrm{Aut}(s)_{n_1,\ldots,n_k}},$$

where

$$M[n] = \text{ the set of all } M\text{-structures on } [n],$$

$$\mathrm{Aut}(s) = \{\sigma \in S_n \mid \sigma \text{ is an automorphism of } s\} \leq S_n.$$

THEOREM 2.2 (k-colored molecular expansion). *Let* $M = M(X) = X^n/H$ *be a molecular species with* $H \leq S_n$. *Then*

$$M(X_1 + \cdots + X_k) = \sum_{U_1 + \cdots + U_k = [n]} \frac{|H_{U_1,\ldots,U_k}|}{|H|} \frac{X_1^{U_1} \cdots X_k^{U_k}}{H_{U_1,\ldots,U_k}},$$

where $X_1^{U_1} X_2^{U_2} \cdots X_k^{U_k} / H_{U_1, U_2, \ldots, U_k}$ *is the molecular species whose set of structures, on* $[V_1, V_2, \ldots, V_k]$, *is defined to be the set*

$$\{\lambda H_{U_1, U_2, \ldots, U_k} \mid \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ \lambda_i\colon U_i \xrightarrow{\sim} V_i\}.$$

THEOREM 2.3 (Word class expansion). *Let* $M = M(X) = X^n/H$ *be a molecular species with* $H \leq S_n$. *Then*

$$M(X_1 + \cdots + X_k) = \sum_{n_1 + \cdots + n_k = n} \ \sum_{\alpha \in \Omega_{n_1, \ldots, n_k}} \frac{X_1^{n_1} \cdots X_k^{n_k}}{\sigma_\alpha (\mathrm{Aut}_H\, \alpha) \sigma_\alpha^{-1}},$$

where $\Omega_{n_1, n_2, \ldots, n_k}$ *is a set of representatives (for example, lexicographically smallest words) of the* H*-classes of words on* $\mathscr{X} = \{1, 2, \ldots, k\}$ *having exactly* n_i *occurrences of letter* i *for* $i = 1, 2, \ldots, k$.

ETC

Examples : n-lists, n-chains, n-sets, n-oriented sets, n-cycles, n-gons, ...

Applications : functional eqns, identities $1 - x + y = \prod_{i+j>0} (1 - x^i y^j)^{-\alpha_{i,j}}$.

Advances in Applied Mathematics **28**, 145–168 (2002)

The Specification of 2-trees

Tom Fowler Ira Gessel Gilbert Labelle and Pierre Leroux




Nous présentons de nouvelles équations fonctionnelles pour certaines classes de 2-arbres, incluant un théorème de dissymétrie. Nous en déduisons diverses séries génératrices associées à ces espèces. Nous obtenons ainsi des formules énumératives pour les 2-arbres non-étiquetés qui sont plus explicites que les résultats connus jusquà présent. De plus le comportement asymptotique de ces structures est établi.

The species a of 2-trees

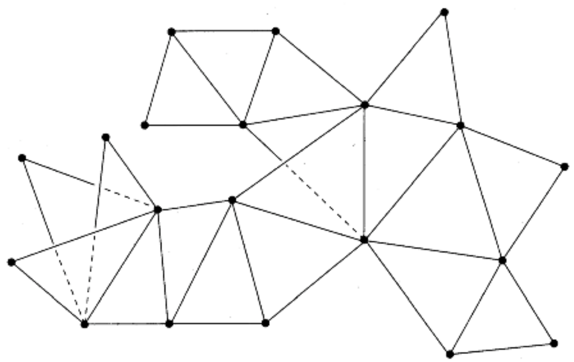


FIG. a-structure

The species $B = a^{\rightarrow}$

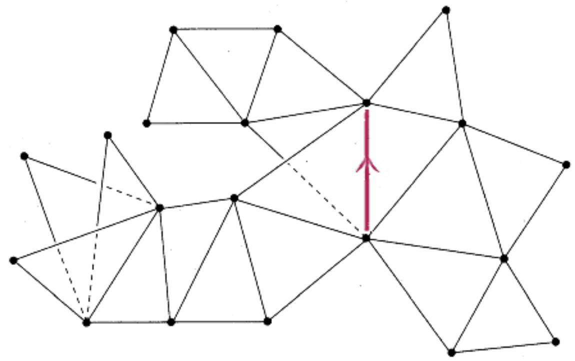


FIG. $a^{\rightarrow}$-structure

THEOREM 1. $$B = E(XB^2)$$

COROLLARY 2. *The species $B = a^{\rightarrow}$ of oriented-edge rooted 2-trees satisfies*

$$B = \sqrt{\frac{A(2X)}{2X}},$$

where A is the species of rooted trees.

COROLLARY 3. *For the species $B = a^{\to}$ of oriented-edge rooted 2-trees, we have*

$$a_n^{\to} = (2n+1)^{n-1}$$

and

$$a_{n_1,n_2,\ldots}^{\to} = \prod_{i=1}^{\infty}\left(1+2\sum_{d\mid i} dn_d\right)^{n_i-1}\left(1+2\sum_{\substack{d\mid i\\ d<i}} dn_d\right).$$

Moreover, the numbers $b_n = \tilde{a}_n^{\to}$ satisfy the recurrence

$$b_n = \frac{1}{n}\sum_{\substack{1\le k\le n\\ i+j+1\mid k}} (i+j+1)b_ib_jb_{n-k}, \qquad b_0 = 1.$$

THEOREM 5 (Dissymmetry theorem for 2-trees). *There is an isomorphism of species*

$$a^{-} + a^{\Delta} = a + a^{\underline{\Delta}}.$$

$$a = a^{-} + a^{\Delta} - a^{\underline{\Delta}}$$

PROPOSITION 16. *We have the following expressions for the three main series associated with the species a of 2-trees in terms of the species $a^{\rightarrow}$ of oriented-edge rooted 2-trees,*

$$a(x) = \frac{1}{2}(a^{\rightarrow}(x) + e^{x}) + \frac{x}{3}(1 - (a^{\rightarrow}(x))^3),$$

$$\tilde{a}(x) = \frac{1}{2}\left(\tilde{a}^{\rightarrow}(x) + \exp\left(\sum_{i\geq 1}\frac{1}{2i}\left(2x^i\tilde{a}^{\rightarrow}(x^{2i}) + x^{2i}(\tilde{a}^{\rightarrow}(x^{2i}))^2 - x^{2i}\tilde{a}^{\rightarrow}(x^{4i})\right)\right)\right)$$
$$+\frac{x}{3}(\tilde{a}^{\rightarrow}(x^3) - (\tilde{a}^{\rightarrow}(x))^3),$$

$$Z_a = \frac{1}{2}\left(Z_{a^{\rightarrow}} + \exp\left(\sum_{i\geq 1}\frac{1}{2i}\left(2x_i(Z_{a^{\rightarrow}})_{2i} + x_{2i}(Z_{a^{\rightarrow}})_{2i}^2 - x_{2i}(Z_{a^{\rightarrow}})_{4i}\right)\right)\right)$$
$$+\frac{x_1}{3}((Z_{a^{\rightarrow}})_3 - (Z_{a^{\rightarrow}})^3).$$

ETC Explicit expressions, asymptotics, ... $\tilde{a}_n \sim \alpha_2 \beta^n n^{-5/2}$

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Theoretical Computer Science 307 (2003) 277–302

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Two bijective proofs for the arborescent form of the Good–Lagrange formula and some applications to colored rooted trees and cacti

Michel Bousquet, Cedric Chauve, Gilbert Labelle, Pierre Leroux

Abstract

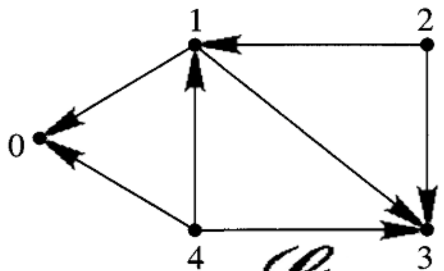
Goulden and Kulkarni (J. Combin. Theory Ser. A 80 (2) (1997) 295) give a bijective proof of an arborescent form of the Good–Lagrange multivariable inversion formula. This formula was first stated explicitly by Bender and Richmond (Electron. J. Combin. 5 (1) (1998) 4pp) but is implicit in Goulden and Kulkarni (1997). In this paper, we propose two new simple bijective proofs of this formula and we illustrate the interest of these proofs by applying them to the enumeration and random generation of colored rooted trees and rooted m-ary cacti.

Implicit and explicit forms of Good–Lagrange formula

$$A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x})), \qquad \mathit{for} \quad i = 1, \ldots, m.$$

$$[\mathbf{x}^{\mathbf{n}}] \frac{F(\mathbf{A}(\mathbf{x}))}{\det\left(\delta_{i,j} - x_i(\partial R_i/\partial x_j)(\mathbf{A}(\mathbf{x}))\right)_{m\times m}} = [\mathbf{x}^{\mathbf{n}}] F(\mathbf{x}) \mathbf{R}(\mathbf{x})^{\mathbf{n}},$$

$$[\mathbf{x}^{\mathbf{n}}] F(\mathbf{A}(\mathbf{x})) = [\mathbf{x}^{\mathbf{n}}] F(\mathbf{x}) \mathbf{R}(\mathbf{x})^{\mathbf{n}} \det\left(\delta_{i,j} - \frac{x_i}{R_i(\mathbf{x})} \frac{\partial R_i(\mathbf{x})}{\partial x_j}\right)_{m\times m}.$$



$\mathscr{G}$

$$d^+(0) = 0$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathscr{G}} = \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_4} f_0(\mathbf{x})\right) \left(\frac{\partial}{\partial x_2} \frac{\partial}{\partial x_4} f_1(\mathbf{x})\right) f_2(\mathbf{x}) \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_4} f_3(\mathbf{x})\right) f_4(\mathbf{x})$$

Arborescent Good–Lagrange formula

$$A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x})), \qquad \textit{for} \quad i = 1, \ldots, m.$$

$$[\mathbf{x}^{\mathbf{n}}]F(\mathbf{A}(\mathbf{x})) = \left(\prod_{i=1}^{m} \frac{1}{n_i}\right) [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{\mathscr{T} \in T_m} \frac{\partial(F(\mathbf{x}), R_1(\mathbf{x})^{n_1}, \ldots, R_m(\mathbf{x})^{n_m})}{\partial \mathscr{T}},$$

T_m: the set of trees $\mathscr{T}$ on the set $V = \{0, 1, \ldots, m\}$,

rooted at 0, and where all the edges are directed towards the root.

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Theoretical Computer Science 307 (2003) 337–363

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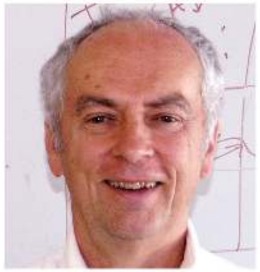
A classification of plane and planar 2-trees

G. Labelle, C. Lamathe, P. Leroux



Abstract

We present new functional equations for the species of plane and of planar (in the sense of Harary and Palmer, Graphical Enumeration, Academic Press, New York, 1973) 2-trees and some associated pointed species. We then deduce the explicit molecular expansion of these species, i.e. a classification of their structures according to their stabilizers. Therein result explicit formulas in terms of Catalan numbers for their associated generating series, including the asymmetry index series. This work is related to the enumeration of polyene hydrocarbons of molecular formula C_nH_{n+2}.

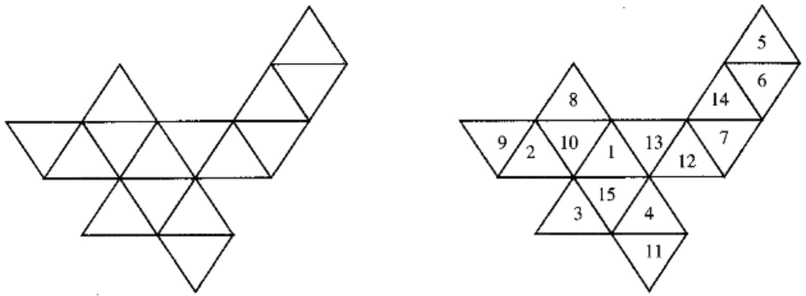


Fig. 1. An unlabelled plane 2-tree and one of its labellings.

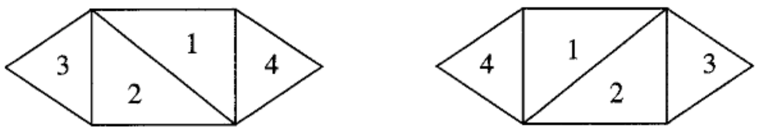


Fig. 2. Two different plane 2-trees, one planar 2-tree.

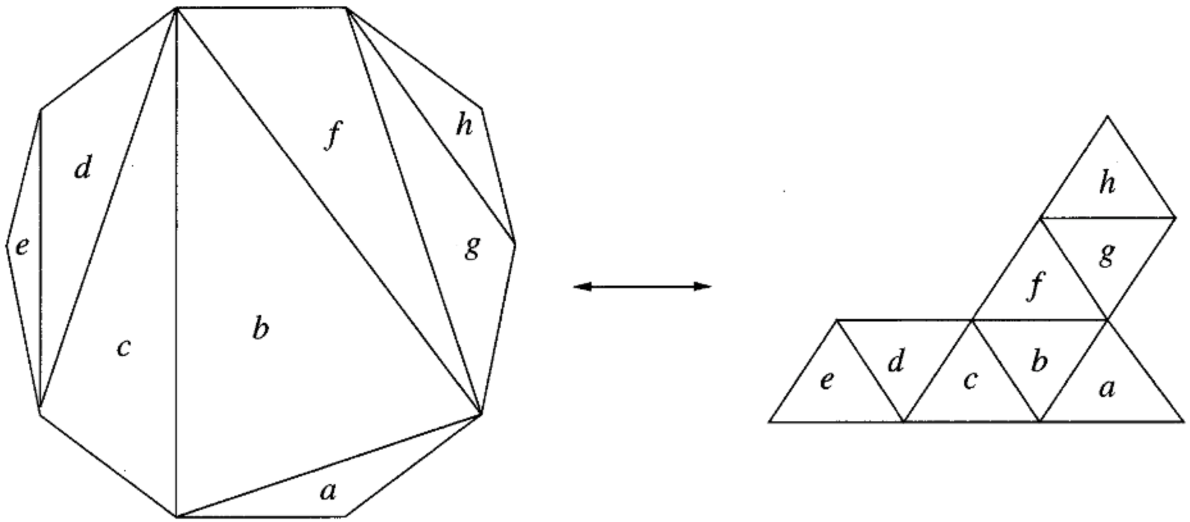


Fig. 3. Correspondence between triangulations of a polygon and plane 2-trees.

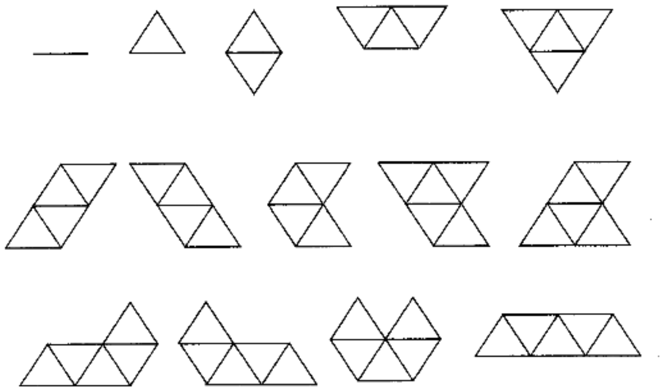


Fig. 4. First terms of the molecular expansion of the species $\boldsymbol{a}_\pi$ of plane 2-trees.

$$\boldsymbol{a}_\pi = \boldsymbol{a}_\pi(X) = 1 + X + E_2(X) + X^3 + XC_3(X) + 2E_2(X^2) + X^4 + 6X^5 + \cdots,$$

Theorem 7. *The molecular expansion of the species $\boldsymbol{a}_\pi$ of plane 2-trees is given by*

$$\boldsymbol{a}_\pi = \boldsymbol{a}_\pi(X) = 1 + X + \sum_{k\geqslant 2} b_k X^k + \sum_{k\geqslant 1} c_k E_2(X^k) + \sum_{k\geqslant 1} d_k XC_3(X^k),$$

where

$$b_k = \tfrac{2}{3}\,\mathbf{c}_k - \tfrac{1}{6}\,\mathbf{c}_{k+1} - \tfrac{1}{2}\,\mathbf{c}_{k/2} - \tfrac{1}{3}\,\mathbf{c}_{(k-1)/3}, \qquad c_k = d_k = \mathbf{c}_k,$$

$$\mathbf{c}_n = [1/(n+1)]\binom{2n}{n} \quad (\textit{Catalan numbers}).$$

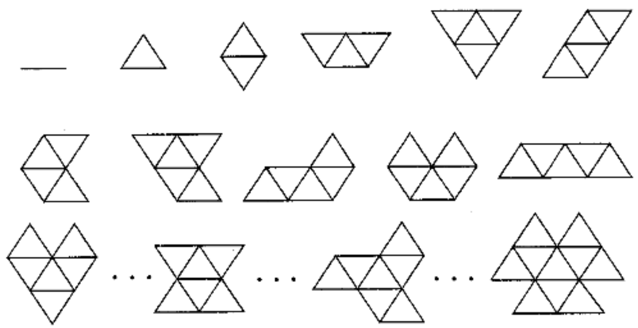


Fig. 5. First terms of the molecular expansion of the species $\boldsymbol{a}_{\mathrm{p}}$ of planar 2-trees.

$$\boldsymbol{a}_{\mathrm{p}} = \boldsymbol{a}_{\mathrm{p}}(X) = 1 + X + E_2(X) + XE_2(X) + XE_3(X) + 2E_2(X^2) + 2X^5 + 2XE_2(X^2) + X^2E_2(X^2) + \cdots + P_4^{\mathrm{bic}}(X,X) + \cdots + XC_3(X^2) + \cdots + XP_6^{\mathrm{bic}}(X,X) + \cdots.$$

Theorem 12. *The molecular expansion of the species $\boldsymbol{a}_{\mathrm{p}}$ of planar 2-trees is given by the following formula*:

$$\boldsymbol{a}_{\mathrm{p}}(X) = 1 + \sum_{k\geqslant 1} a_k^1 X^k + \sum_{k\geqslant 1} a_k^2 E_2(X^k) + \sum_{k\geqslant 1} a_k^3 XE_2(X^k) + \sum_{k\geqslant 2} a_k^4 X^2 E_2(X^k) + \sum_{k\geqslant 2} a_k^5 XC_3(X^k) + \sum_{k\geqslant 0} a_k^6 P_4^{\mathrm{bic}}(X,X^k) + \sum_{k\geqslant 0} a_k^7 XP_6^{\mathrm{bic}}(X,X^k),$$

where

$$a_k^1 = -\tfrac{1}{12}\,\mathbf{c}_{k+1} + \tfrac{1}{3}\,\mathbf{c}_k - \tfrac{3}{4}\,\mathbf{c}_{k/2} - \tfrac{1}{2}\,\mathbf{c}_{(k-1)/2} - \tfrac{1}{6}\,\mathbf{c}_{(k-1)/3} + \tfrac{1}{2}\,\mathbf{c}_{(k-2)/4} + \tfrac{1}{2}\,\mathbf{c}_{(k-4)/6},$$

$$a_k^2 = \mathbf{c}_k - \mathbf{c}_{(k-1)/2}, \qquad a_k^3 = a_k^6 = a_k^7 = \mathbf{c}_k, \qquad a_k^4 = \tfrac{1}{2}\,(\mathbf{c}_{k+1} - \mathbf{c}_{k/2}) - \mathbf{c}_{(k-1)/3}, \qquad a_k^5 = \tfrac{1}{2}\,(\mathbf{c}_k - \mathbf{c}_{(k-1)/2}).$$

ETC Cycle index series, assymmetry index series, enumeration, ...

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Journal of Combinatorial Theory, Series A 106 (2004) 193–219

Journal of Combinatorial Theory

Series A

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Labelled and unlabelled enumeration of k-gonal 2-trees

Gilbert Labelle, Cédric Lamathe, Pierre Leroux



Abstract

In this paper, we generalize 2-trees by replacing triangles by quadrilaterals, pentagons or k-sided polygons (k-gons), where $k \geqslant 3$ is given. This generalization, to k-gonal 2-trees, is natural and is closely related, in the planar case, to some specializations of the cell-growth problem. Our goal is the labelled and unlabelled enumeration of k-gonal 2-trees according to the number n of k-gons. We give explicit formulas in the labelled case, and, in the unlabelled case, recursive and asymptotic formulas.

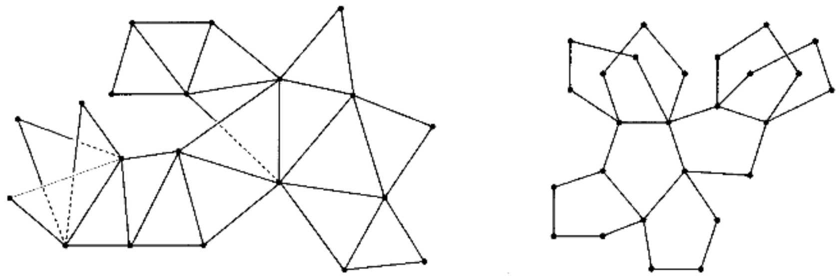
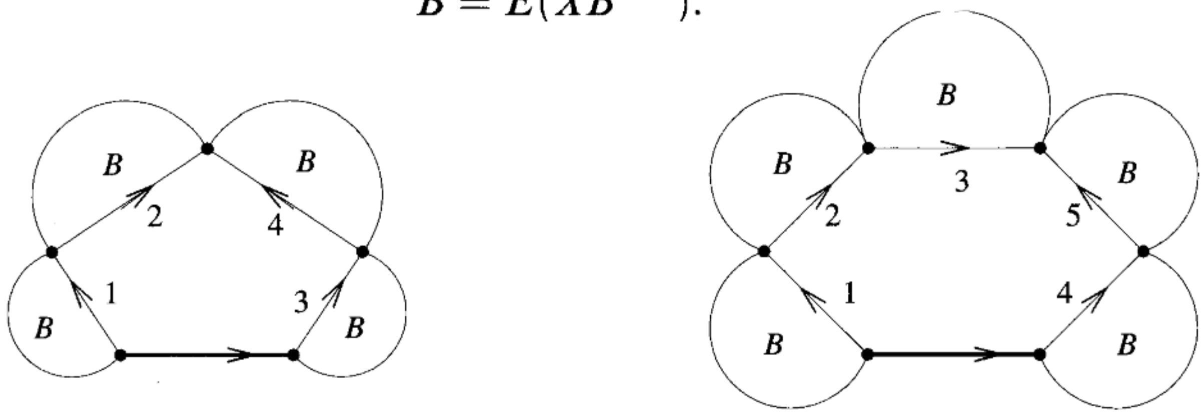


Fig. 1. k-gonal 2-trees with $k = 3$ and $k = 5$.

Theorem 2. *The species $B = a^{\rightarrow}$ of oriented-edge rooted k-gonal 2-trees satisfies the following functional equation* (*isomorphism*):

$$B = E(XB^{k-1}).$$



k odd

Proposition 4. *If k is odd, the number a_n of labelled k-gonal 2-trees on n k-gons is given by*

$$a_n = \frac{1}{2}(m^{n-2}+1), \quad n \geqslant 2,$$

where $m = (k-1)n+1$ is the number of edges.

Corollary 3. *For $k \geqslant 3$, odd, the number $\tilde{a}_n$ of unlabelled k-gonal 2-trees over n k-gons, satisfy the following recurrence*

$$\tilde{a}_n = \frac{1}{2n}\sum_{j=1}^{n}\left(\sum_{l|j} l\omega_l\right)\left(\tilde{a}_{n-j} - \frac{1}{2}\tilde{a}_{o,n-j}\right) + \frac{1}{2}\tilde{a}_{o,n}, \quad \tilde{a}_0 = 1,$$

where

$$\omega_n = 2b_{\frac{n-1}{2}}^{\left(\frac{k-1}{2}\right)} + b_{\frac{n-2}{2}}^{(k-1)} - b_{\frac{n-2}{4}}^{\left(\frac{k-1}{2}\right)}, \qquad b_l^{(m)} = [x^l]\tilde{B}^m(x).$$

k even

Proposition 9. *If k is even, the number a_n of labelled k-gonal 2-trees on n k-gons is given by*

$$a_n = \frac{1}{2}\left(m^{n-2} + (n+1)^{n-2}\right), \quad n \geqslant 2,$$

where $m = (k-1)n + 1$ is the number of edges.

Corollary 5. *Let k be an even integer, $k \geqslant 4$. Then the number of unlabelled k-gonal 2-trees over n k-gons is given by*

$$\tilde{a}_n = \frac{1}{2}\tilde{a}_{o,n} + \frac{1}{2}\alpha_n + \frac{1}{4} b_{\frac{n-1}{2}}^{\left(\frac{k}{2}\right)} - \frac{1}{4} \sum_{i+j=n-1} \alpha_i^{(2)} \cdot b_{\frac{j}{2}}^{\left(\frac{k-2}{2}\right)},$$

where

$$b_l^{(m)} = [x^l]\tilde{B}^m(x), \quad \alpha_i^{(2)} = [x^i]\tilde{a}_{\mathrm{S}}^2(x).$$

ETC Asymptotic formulas, generalized Otter's constants, ...

Annales des Sciencces Mathématiques du Québec, 29 (2005), no.2, 215-236.

Dénombrement des 2-arbres k-gonaux selon la taille et le périmètre

G. Labelle, C. Lamathe, P. Leroux



RÉSUMÉ : *Dans cet article, nous nous intéressons au dénombrement étiqueté et non étiqueté des 2-arbres k-gonaux par rapport à leur taille et à leur périmètre, à savoir, le nombre de k-gones et d'arêtes externes (de degré un) du 2-arbre, respectivement. Cette famille de 2-arbres est une variante des 2-arbres (libres) dans laquelle on a remplacé les triangles par des polygones à k-côtés, appelés k-gones. On s'attache à donner des formules énumératives explicites dans le cas étiqueté et des formules de récurrence ou des séries génératrices explicites pour le cas non étiqueté.*

PU.M.A. (Pure Mathematics and Applications) Vol. 16 (2005), No. 3, 1-20.

The structure and labelled enumeration of $K_{3,3}$-subdivision-free projective-planar graphs

Andrei Gagarin, Gilbert Labelle and Pierre Leroux

Abstract

We consider the class $\mathcal{F}$ of 2-connected non-planar $K_{3,3}$-subdivision-free graphs that are embeddable in the projective plane. We show that these graphs admit a unique decomposition as a graph K_5 (the *core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. A method to enumerate these graphs in the labelled case is described. Moreover, we enumerate the homeomorphically irreducible graphs in $\mathcal{F}$ and homeomorphically irreducible 2-connected planar graphs. Particular use is made of two-pole (directed) series-parallel networks. We also show that the number m of edges of graphs in $\mathcal{F}$ satisfies the bound $m \leq 3n - 6$, for $n \geq 6$ vertices.

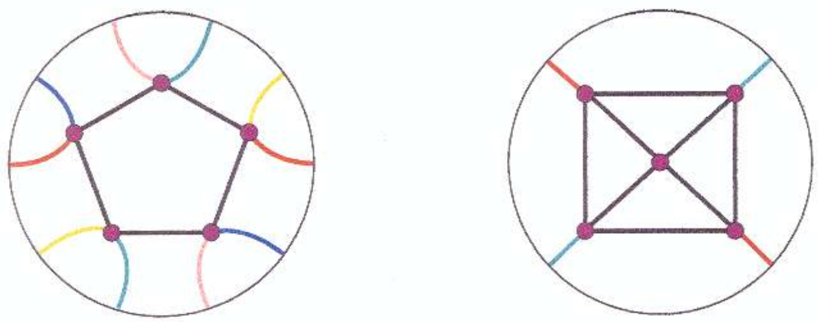
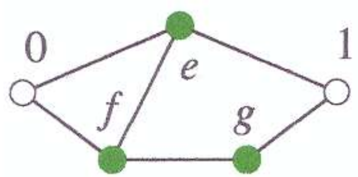


Figure 1: *Embeddings of K_5 in the projective plane.*

A *two-pole network* (or more simply, a *network*) is a connected graph N with two distinguished vertices 0 and 1, such that the graph $N \cup 01$ is 2-connected

A network N is *strongly planar* if the graph $N \cup 01$ is planar.

Denote by $\mathcal{N}_P$ the class of strongly planar networks.



Theorem 1 *The class $\mathcal{F}$ of 2-connected non-planar projective-planar $K_{3,3}$-free graphs can be expressed as a canonical composition*

$$\mathcal{F} = K_5 \uparrow \mathcal{N}_P.$$

ETC Enumeration of these graphs and related graphs

Discrete Mathematics, 307 (2007) 2993-3005.

The structure of $K_{3,3}$-subdivision-free toroidal graphs

Andrei Gagarin, Gilbert Labelle and Pierre Leroux

Abstract

We consider the class $\mathcal{T}$ of 2-connected non-planar $K_{3,3}$-subdivision-free graphs that are embeddable in the torus. We show that any graph in $\mathcal{T}$ admits a unique decomposition as a basic toroidal graph (the *toroidal core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. Toroidal cores can be enumerated, using matching polynomials of cycle graphs. As a result, we enumerate labelled graphs in $\mathcal{T}$ having minimum vertex degree two or three, according to their number of vertices and edges. We also show that the number m of edges of graphs in $\mathcal{T}$ satisfies the bound $m \leq 3n - 6$, for $n \geq 6$ vertices.

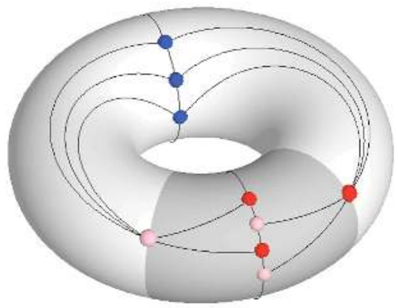
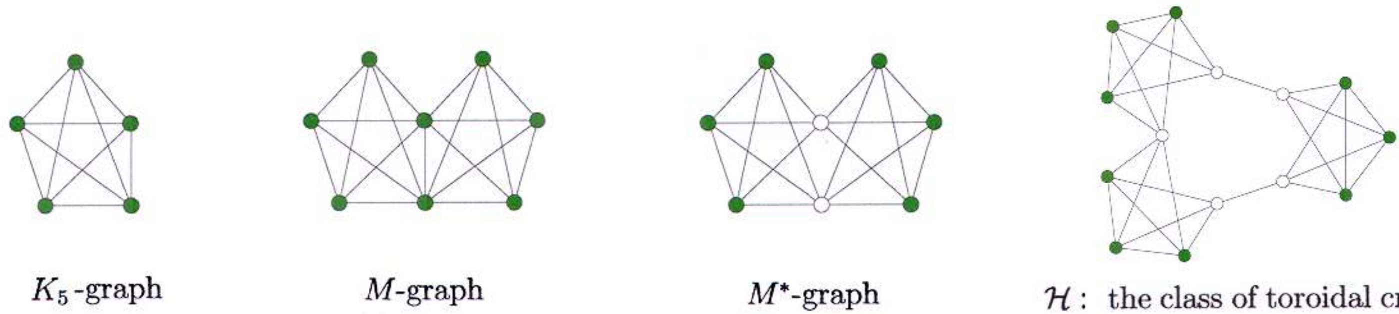


Figure 1: Graph embedded on the torus



Theorem 2 *The class $\mathcal{T}$ of 2-connected non-planar $K_{3,3}$-free toroidal graphs is characterized by*

$$\mathcal{T} = \mathcal{T}_{\mathrm{C}} \uparrow \mathcal{N}_P, \qquad \mathcal{T}_{\mathrm{C}} = K_5 + M + M^* + \mathcal{H},$$

the composition being canonical.

ETC we enumerate labelled graphs in $\mathcal{T}$ having minimum vertex degree two or three, ...

Advances in Applied Mathematics 39 (2007), 51-75.

Counting unlabelled toroidal graphs with no $K_{3,3}$-subdivisions

Andrei Gagarin, Gilbert Labelle and Pierre Leroux



Abstract

We provide a description of unlabelled enumeration techniques, with complete proofs, for graphs that can be canonically obtained by substituting 2-pole networks for the edges of core graphs. Using structure theorems for toroidal and projective-planar graphs containing no $K_{3,3}$-subdivisions, we apply these techniques to obtain their unlabelled enumeration.

$$\mathcal{F} = K_5 \uparrow \mathcal{N}_P,$$ (*2-connected, non-planar, $K_{3,3}$-free and projective-planar graphs*)

$$\mathcal{T} = \mathcal{T}_{\mathrm{C}} \uparrow \mathcal{N}_P,$$ (*2-connected non-planar $K_{3,3}$-free toroidal graphs*)

$$(\mathcal{G} \uparrow \mathcal{N})(x, y) = \mathcal{G}(x, \mathcal{N}(x, y))$$ (labelled case)

$$(\mathcal{G} \uparrow \mathcal{N})\tilde{\ }(x, y) = \textbf{?}$$ (unlabelled case)

Walsh series $W_{\mathcal{G}}(\mathbf{a}; \mathbf{b}; \mathbf{c})$, $W^+_{\mathcal{N}}(\mathbf{a}; \mathbf{b}; \mathbf{c})$ and $W^-_{\mathcal{N}}(\mathbf{a}; \mathbf{b}; \mathbf{c})$

Theorem 3 *Let $\mathcal{G}$ be a species of graphs and $\mathcal{N}$ be a symmetric species of networks. Then the Walsh index series of the species $\mathcal{G} \uparrow \mathcal{N}$ is given by*

$$W_{\mathcal{G}\uparrow\mathcal{N}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{G}}(a_1, a_2, \ldots; (W^+_{\mathcal{N}})_1, (W^+_{\mathcal{N}})_2, \ldots; (W^-_{\mathcal{N}})_1, (W^-_{\mathcal{N}})_2, \ldots).$$

$$(\mathcal{G} \uparrow \mathcal{N})\tilde{\ }(x, y) = W_{\mathcal{G}}(x, x^2, \ldots; \widetilde{\mathcal{N}}(x, y), \widetilde{\mathcal{N}}(x^2, y^2), \ldots; \widetilde{\mathcal{N}}_\tau(x, y), \widetilde{\mathcal{N}}_\tau(x^2, y^2), \ldots).$$

ETC Complete details about the theory of Walsh series, explicit computations and tables.

Séminaire lotharingien de combinatoire, Lucelle, France (2004), vol. 54

Graph weights arising from Mayer's theory of cluster integrals

G. Labelle, P. Leroux and M. G. Ducharme



A notre très bon ami Xavier Viennot

Abstract

We study graph weights (i.e. graph invariants) which arise naturally in Mayer's theory of cluster integrals in the context of a non-ideal gas. Various choices of the interaction potential between two particles yield various graph weights $w(g)$. For example, in the case of the Gaussian interaction, the so-called Second Mayer weight $w(c)$ of a connected graph c is closely related to the graph complexity, i.e. the number of spanning trees, of c. We give special attention to the Second Mayer weight $w(c)$ which arises from the hard-core continuum gas in one dimension. This weight is a signed volume of a convex polytope $\mathcal{P}(c)$ naturally associated with c. Among our results are the values $w(c)$ for all 2-connected graphs c of size at most 6, in Appendix B, and explicit formulas for three infinite families: complete graphs, (unoriented) cycles and complete graphs minus an edge.

The grand canonical partition function

$$Z(V,T,N) = \frac{1}{N!\lambda^{dN}} \int_V \cdots \int_V \exp\left(-\beta \sum_{i<j} \varphi(|\vec{x}_i - \vec{x}_j|)\right) d\vec{x}_1 \ldots d\vec{x}_N$$

Mayer's idea

$$1 + f_{ij} = \exp\left(-\beta\varphi(|\vec{x_i} - \vec{x_j}|)\right)$$

$$Z(V,N,T) = \frac{1}{N!\lambda^{dN}} \sum_{g\in\mathcal{G}[N]} W(g)$$

$$W(g) = \int_{V^N} \prod_{\{i,j\}\in g} f_{ij}\, d\vec{x_1} \cdots d\vec{x_N} \qquad (\text{ the weight of a graph } g\,)$$

$$\begin{aligned} Z_{\mathrm{gr}}(V,T,z) &= \sum_{N=0}^{\infty} Z(V,N,T)(\lambda^d z)^N \\ &= \mathcal{G}_W(z) \end{aligned}$$

(the exponential generating series of graphs weighted by the function W)

Pressure

$$\frac{P}{kT} = \frac{1}{V} \log Z_{\mathrm{gr}}(V,T,z) = \frac{1}{V}\, \mathcal{C}_W(z)$$

The thermodynamic limit $w(c)$ of a connected graph c

$$w(c) \quad = \quad \lim_{V\to\infty} \frac{1}{V} W(c) \quad = \quad \lim_{V\to\infty} \frac{1}{V} \int_{V^N} \prod_{\{i,j\}\in c} f_{ij}\, d\vec{x}_1 \ldots d\vec{x}_N$$

Proposition 3 *If the function $f : [0,\infty) \to \mathbb{R}$ is integrable and bounded and if*

$$\int_0^\infty r^{d-1} |f(r)|\, dr < \infty,$$

(for example if $|f(r)| = \mathrm{O}(1/r^{d+\epsilon})$, $r \to \infty$), then the limit exists and

$$w(c) = \int_{(\mathbb{R}^d)^{N-1}} \prod_{\{i,j\}\in c;\ \vec{x}_N=\vec{0}} f_{ij}\, d\vec{x}_1 \ldots d\vec{x}_{N-1}.$$

Proposition 5 *The second Mayer weight w is* block-multiplicative. *More precisely, for any connected graph c whose blocks are $b_1, b_2, \ldots, b_m$, we have*

$$w(c) = w(b_1) w(b_2) \ldots w(b_m).$$

Gaussian models

$$f_{ij} = -\exp(-\alpha\|\vec{x}_i - \vec{x}_j\|^2), \qquad w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \gamma(c)^{-\frac{d}{2}}$$

$$f_{ij} = -\exp\left(-\alpha y_i y_j \|\vec{x}_i - \vec{x}_j\|^2\right), \qquad w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \left(\sum_{t\in T(c)} y_1^{d_t(1)} y_2^{d_t(2)} \dots y_n^{d_t(n)}\right)^{-\frac{d}{2}}$$

$$f_{ij} = -\exp\left(-w_{i,j} \|\vec{x}_i - \vec{x}_j\|^2\right), \qquad w(c) = (-1)^{e(c)} (\pi)^{\frac{d(n-1)}{2}} \left(\sum_{t\in T(c)} \prod_{\{i,j\}\in t} w_{i,j}\right)^{-\frac{d}{2}}$$

The hard-core continuum gas in one dimension

$$1 + f_{ij} \;=\; \chi(|x_i - x_j| \geq 1) \quad\Leftrightarrow\quad f_{ij} \;=\; -\chi(|x_i - x_j| < 1)$$

$$\begin{aligned}
w(K_N) &= (-1)^{\binom{N}{2}} N \\
w(C_N) &= (-1)^N \frac{2^N}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^N dt \\
&= \frac{(-1)^N}{(N-1)!} \sum_{i=0}^{\lfloor\frac{N-1}{2}\rfloor} (-1)^i \binom{N}{i} (N-2i)^{N-1} \\
&\sim (-2)^N \left(\frac{3}{2\pi N}\right)^{\frac{1}{2}} \left(1 - \frac{3}{20N} - \frac{13}{1120N^2} + \dots\right)
\end{aligned}$$

The hard-core continuum gas in one dimension (continued)

In general,

$$w(c) = (-1)^{e(c)}\mathrm{Vol}(\mathcal{P}(c))$$

$$\mathcal{P}(c) = \left\{ X \in \mathbb{R}^N \mid x_N = 0 \text{ and } |x_i - x_j| \leq 1 \ \forall \{i,j\} \in c \right\}, \qquad \textit{convex polytope}$$

The vertices of $\mathcal{P}(c)$ have integer coordinates

$$\mathrm{Vol}(\mathcal{P}(c)) = \nu(c)/(N-1)! \qquad \nu(c) \ \textbf{integer}$$

We have techniques to compute $\nu(c)$

(Ehrhart polynomials, fractional representations of simplicial subpolytopes, ...)

We have a table of $\nu(c)$ for all 2-connected graphs c of size at most 6

ETC

Proceedings FPSAC07, Nankai University, Tianjin, Chine

A classification of outerplanar K-gonal 2-trees

Martin Ducharme, Gilbert Labelle, Cédric Lamathe et Pierre Leroux



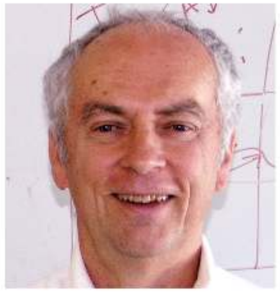

ABSTRACT. We give in this work the molecular expansion of the species of outerplanar K-gonal 2-trees, extending previous work on ordinary ($K = 3$) outerplanar 2-trees. This is equivalent to a classification of these graphs according to their symmetries (automorphism groups). We give explicit formulas for all coefficients occuring in this expansion.

Soumis pour publication

Two-connected graphs with prescribed three-connected components

Andrei Gagarin, Gilbert Labelle, Pierre Leroux, and Timothy Walsh






Abstract

We adapt the classical 3-decomposition of any 2-connected graph to the case of simple graphs (no loops or multiple edges). By analogy with the block-cutpoint tree of a connected graph, we deduce from this decomposition a bicolored tree $\mathrm{tc}(g)$ associated with any 2-connected graph g, whose white vertices are the *3-components* of g (3-connected components or polygons) and whose black vertices are bonds linking together these 3-components, arising from separating pairs of vertices of g. Two fundamental relationships on graphs and networks follow from this construction. The first one is a dissymmetry theorem which leads to the expression of the class $\mathcal{B} = \mathcal{B}(\mathcal{F})$ of 2-connected graphs, all of whose 3-connected components belong to a given class $\mathcal{F}$ of 3-connected graphs, in terms of various rootings of $\mathcal{B}$. The second one is a functional equation which characterizes the corresponding class $\mathcal{R} = \mathcal{R}(\mathcal{F})$ of two-pole networks all of whose 3-connected components are in $\mathcal{F}$. All the rootings of $\mathcal{B}$ are then expressed in terms of $\mathcal{F}$ and $\mathcal{R}$. There follow corresponding identities for all the associated series, in particular the edge index series. Numerous enumerative consequences are discussed.

En préparation 2008

Mayer and Ree-Hoover weights of infinite families of 2-connected graphs

Amel Kaouche, Gilbert Labelle, Cédric Lamathe et Pierre Leroux




We study graph weights (i.e., graph invariants) which arise naturally in Mayer's theory and Ree-Hoover's theory of virial expansions in the context of a non-ideal gas. We give special attention to the Second Mayer weight $w_M(c)$ and the Ree-Hoover weight $w_{RH}(c)$ of a 2-connected graph c which arise from the hard-core continuum gas in one dimension. These weights are signed volumes of a convex polytope naturally associated with the graph c. Among our results are the values of Mayer's weight and Ree-Hoover's weight for all 2-connected graphs b of size at most 8, and explicit formulas for certain infinite families.

En préparation

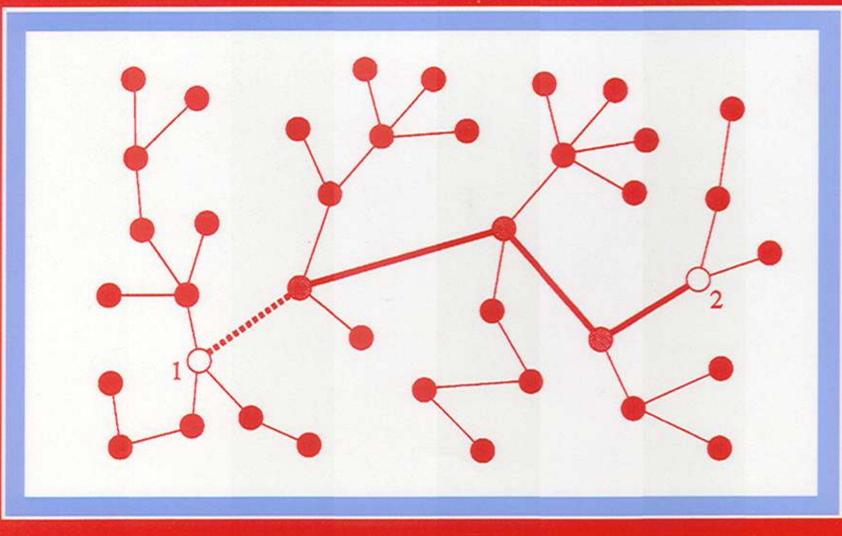
Enumerating combinatorial structures equipped with a list of commuting automorphisms.

Miguel Méndez, Gilbert Labelle and Pierre Leroux

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 67

COMBINATORIAL SPECIES AND TREE-LIKE STRUCTURES

F. BERGERON,
G. LABELLE,
P. LEROUX



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