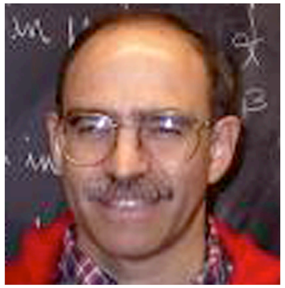
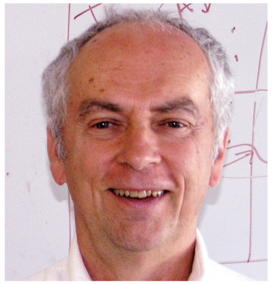
Advances in Applied Mathematics **28**, 145–168 (2002)

The Specification of 2-trees

Tom Fowler Ira Gessel Gilbert Labelle and Pierre Leroux





Nous présentons de nouvelles équations fonctionnelles pour certaines classes de 2-arbres, incluant un théorème de dissymétrie. Nous en déduisons diverses séries génératrices associées à ces espèces. Nous obtenons ainsi des formules énumératives pour les 2-arbres non-étiquetés qui sont plus explicites que les résultats connus jusquà présent. De plus le comportement asymptotique de ces structures est établi.

The species a of 2-trees

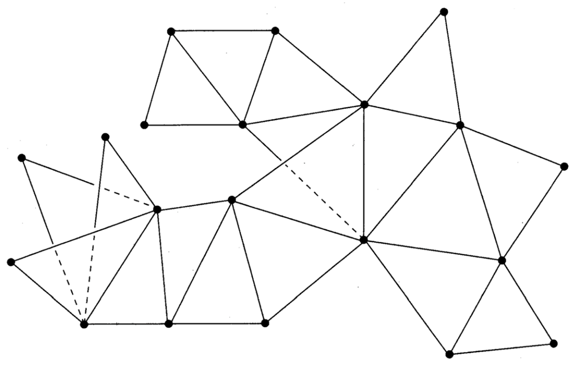


FIG. a-structure

The species $B = a^{\to}$

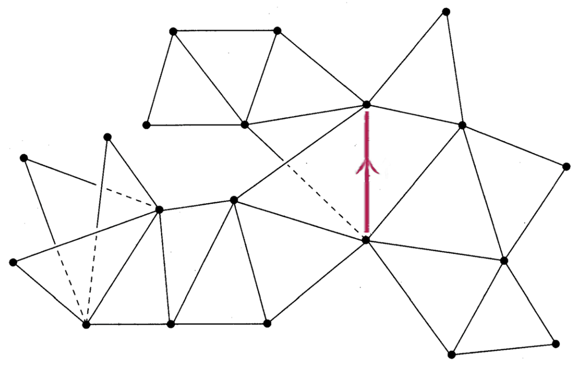


FIG. $a^{\to}$-structure

Theorem 1. $$B = E(XB^2)$$

Corollary 2. *The species $B = a^{\to}$ of oriented-edge rooted 2-trees satisfies*

$$B = \sqrt{\frac{A(2X)}{2X}},$$

where A is the species of rooted trees.

COROLLARY 3. *For the species $B = a^{\rightarrow}$ of oriented-edge rooted 2-trees, we have*

$$a_n^{\rightarrow} = (2n+1)^{n-1}$$

and

$$a_{n_1,n_2,\ldots}^{\rightarrow} = \prod_{i=1}^{\infty} \left(1 + 2\sum_{d|i} dn_d\right)^{n_i-1} \left(1 + 2\sum_{\substack{d|i \\ d<i}} dn_d\right).$$

Moreover, the numbers $b_n = \tilde{a}_n^{\rightarrow}$ satisfy the recurrence

$$b_n = \frac{1}{n} \sum_{\substack{1\le k\le n \\ i+j+1|k}} (i+j+1)b_i b_j b_{n-k}, \qquad b_0 = 1.$$

THEOREM 5 (Dissymmetry theorem for 2-trees). *There is an isomorphism of species*

$$a^{-} + a^{\triangle} = a + a^{\overset{\triangle}{-}}.$$

$$a = a^- + a^{\Delta} - a^{\underline{\Delta}}$$

PROPOSITION 16. *We have the following expressions for the three main series associated with the species a of 2-trees in terms of the species $a^{\rightarrow}$ of oriented-edge rooted 2-trees,*

$$a(x) = \frac{1}{2}(a^{\rightarrow}(x) + e^x) + \frac{x}{3}(1 - (a^{\rightarrow}(x))^3),$$

$$\tilde{a}(x) = \frac{1}{2}\left(\tilde{a}^{\rightarrow}(x) + \exp\left(\sum_{i\geq 1}\frac{1}{2i}\left(2x^i\tilde{a}^{\rightarrow}(x^{2i}) + x^{2i}(\tilde{a}^{\rightarrow}(x^{2i}))^2 - x^{2i}\tilde{a}^{\rightarrow}(x^{4i})\right)\right)\right)$$
$$+\frac{x}{3}(\tilde{a}^{\rightarrow}(x^3) - (\tilde{a}^{\rightarrow}(x))^3),$$

$$Z_a = \frac{1}{2}\left(Z_{a^{\rightarrow}} + \exp\left(\sum_{i\geq 1}\frac{1}{2i}\left(2x_i(Z_{a^{\rightarrow}})_{2i} + x_{2i}(Z_{a^{\rightarrow}})^2_{2i} - x_{2i}(Z_{a^{\rightarrow}})_{4i}\right)\right)\right)$$
$$+\frac{x_1}{3}((Z_{a^{\rightarrow}})_3 - (Z_{a^{\rightarrow}})^3).$$

ETC Explicit expressions, asymptotics, ... $\tilde{a}_n \sim \alpha_2 \beta^n n^{-5/2}$

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Theoretical Computer Science 307 (2003) 277–302

Theoretical Computer Science

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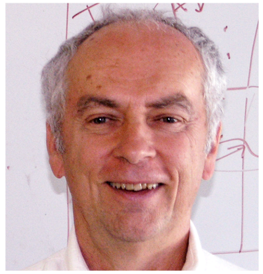
Two bijective proofs for the arborescent form of the Good–Lagrange formula and some applications to colored rooted trees and cacti

Michel Bousquet, Cedric Chauve, Gilbert Labelle, Pierre Leroux



Abstract

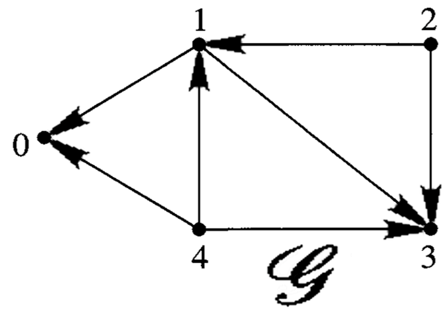
Goulden and Kulkarni (J. Combin. Theory Ser. A 80 (2) (1997) 295) give a bijective proof of an arborescent form of the Good–Lagrange multivariable inversion formula. This formula was first stated explicitly by Bender and Richmond (Electron. J. Combin. 5 (1) (1998) 4pp) but is implicit in Goulden and Kulkarni (1997). In this paper, we propose two new simple bijective proofs of this formula and we illustrate the interest of these proofs by applying them to the enumeration and random generation of colored rooted trees and rooted m-ary cacti.

Implicit and explicit forms of Good–Lagrange formula

$$A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x})), \qquad \textit{for} \quad i = 1, \ldots, m.$$

$$[\mathbf{x}^{\mathbf{n}}] \frac{F(\mathbf{A}(\mathbf{x}))}{\det\left(\delta_{i,j} - x_i(\partial R_i/\partial x_j)(\mathbf{A}(\mathbf{x}))\right)_{m\times m}} = [\mathbf{x}^{\mathbf{n}}] F(\mathbf{x}) \mathbf{R}(\mathbf{x})^{\mathbf{n}},$$

$$[\mathbf{x}^{\mathbf{n}}] F(\mathbf{A}(\mathbf{x})) = [\mathbf{x}^{\mathbf{n}}] F(\mathbf{x}) \mathbf{R}(\mathbf{x})^{\mathbf{n}} \det\left(\delta_{i,j} - \frac{x_i}{R_i(\mathbf{x})} \frac{\partial R_i(\mathbf{x})}{\partial x_j}\right)_{m\times m}.$$



$$d^+(0) = 0$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathscr{G}} = \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_4} f_0(\mathbf{x})\right) \left(\frac{\partial}{\partial x_2} \frac{\partial}{\partial x_4} f_1(\mathbf{x})\right) f_2(\mathbf{x}) \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_4} f_3(\mathbf{x})\right) f_4(\mathbf{x})$$

Arborescent Good–Lagrange formula

$$A_i(\mathbf{x}) = x_i R_i(\mathbf{A}(\mathbf{x})), \qquad \textit{for} \quad i = 1, \ldots, m.$$

$$[\mathbf{x^n}]F(\mathbf{A}(\mathbf{x})) = \left(\prod_{i=1}^{m} \frac{1}{n_i}\right) [\mathbf{x^{n-1}}] \sum_{\mathscr{T} \in T_m} \frac{\partial(F(\mathbf{x}), R_1(\mathbf{x})^{n_1}, \ldots, R_m(\mathbf{x})^{n_m})}{\partial \mathscr{T}},$$

T_m: the set of trees $\mathscr{T}$ on the set $V = \{0, 1, \ldots, m\}$,

rooted at 0, and where all the edges are directed towards the root.

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Theoretical Computer Science 307 (2003) 337–363

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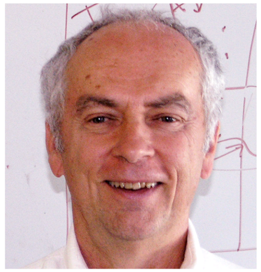
A classification of plane and planar 2-trees

G. Labelle, C. Lamathe, P. Leroux



Abstract

We present new functional equations for the species of plane and of planar (in the sense of Harary and Palmer, Graphical Enumeration, Academic Press, New York, 1973) 2-trees and some associated pointed species. We then deduce the explicit molecular expansion of these species, i.e. a classification of their structures according to their stabilizers. Therein result explicit formulas in terms of Catalan numbers for their associated generating series, including the asymmetry index series. This work is related to the enumeration of polyene hydrocarbons of molecular formula C_nH_{n+2}.

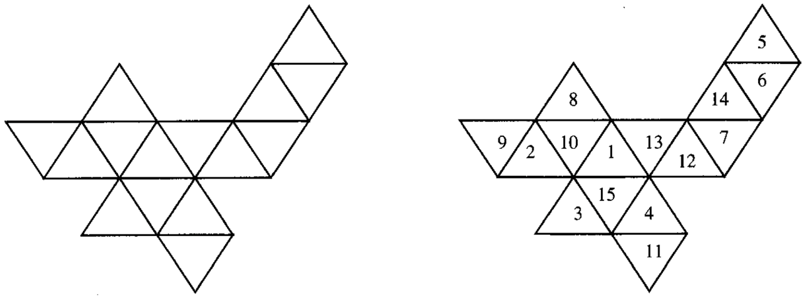


Fig. 1. An unlabelled plane 2-tree and one of its labellings.

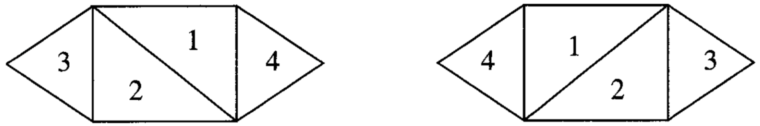


Fig. 2. Two different plane 2-trees, one planar 2-tree.

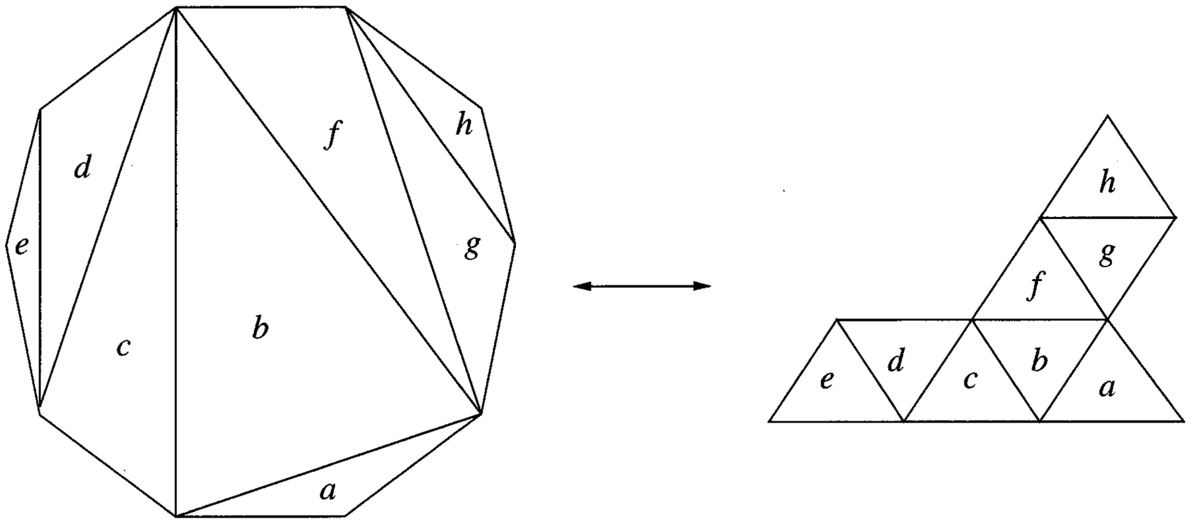


Fig. 3. Correspondence between triangulations of a polygon and plane 2-trees.

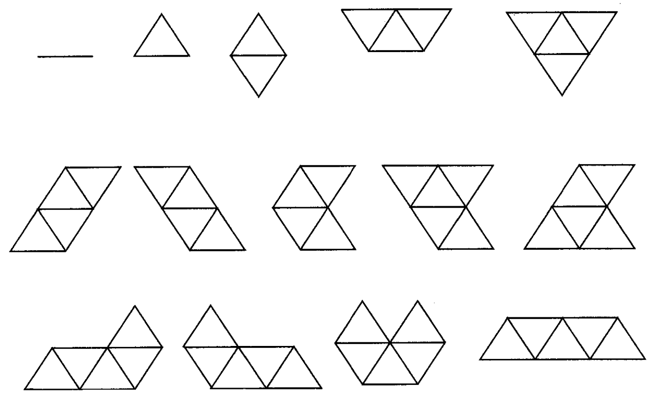


Fig. 4. First terms of the molecular expansion of the species $\boldsymbol{a}_\pi$ of plane 2-trees.

$$\boldsymbol{a}_\pi = \boldsymbol{a}_\pi(X) = 1 + X + E_2(X) + X^3 + XC_3(X) + 2E_2(X^2) + X^4 + 6X^5 + \cdots,$$

Theorem 7. *The molecular expansion of the species $\boldsymbol{a}_\pi$ of plane 2-trees is given by*

$$\boldsymbol{a}_\pi = \boldsymbol{a}_\pi(X) = 1 + X + \sum_{k\geqslant 2} b_k X^k + \sum_{k\geqslant 1} c_k E_2(X^k) + \sum_{k\geqslant 1} d_k XC_3(X^k),$$

where

$$b_k = \tfrac{2}{3}\,\mathbf{c}_k - \tfrac{1}{6}\,\mathbf{c}_{k+1} - \tfrac{1}{2}\,\mathbf{c}_{k/2} - \tfrac{1}{3}\,\mathbf{c}_{(k-1)/3}, \qquad c_k = d_k = \mathbf{c}_k,$$

$$\mathbf{c}_n = [1/(n+1)]\binom{2n}{n} \quad (\textit{Catalan numbers}).$$

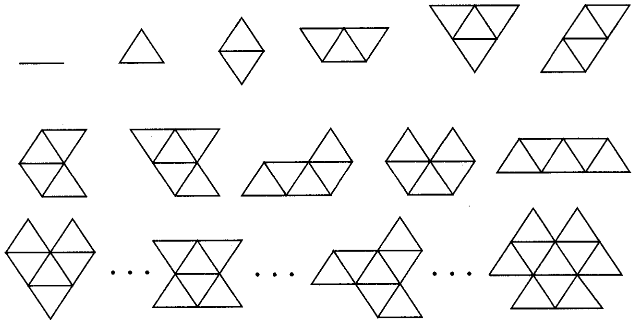


Fig. 5. First terms of the molecular expansion of the species $\boldsymbol{a}_{\mathrm{p}}$ of planar 2-trees.

$$\boldsymbol{a}_{\mathrm{p}} = \boldsymbol{a}_{\mathrm{p}}(X) = 1 + X + E_2(X) + XE_2(X) + XE_3(X) + 2E_2(X^2) + 2X^5 \ + 2XE_2(X^2) + X^2E_2(X^2) + \cdots + P_4^{\mathrm{bic}}(X,X) + \cdots \ + XC_3(X^2) + \cdots + XP_6^{\mathrm{bic}}(X,X) + \cdots.$$

Theorem 12. *The molecular expansion of the species $\boldsymbol{a}_{\mathrm{p}}$ of planar 2-trees is given by the following formula*:

$$\boldsymbol{a}_{\mathrm{p}}(X) = 1 + \sum_{k\geqslant 1} a_k^1 X^k + \sum_{k\geqslant 1} a_k^2 E_2(X^k) + \sum_{k\geqslant 1} a_k^3 XE_2(X^k) + \sum_{k\geqslant 2} a_k^4 X^2 E_2(X^k) + \sum_{k\geqslant 2} a_k^5 XC_3(X^k) + \sum_{k\geqslant 0} a_k^6 P_4^{\mathrm{bic}}(X,X^k) + \sum_{k\geqslant 0} a_k^7 XP_6^{\mathrm{bic}}(X,X^k),$$

where
$$a_k^1 = -\tfrac{1}{12}\,\mathbf{c}_{k+1} + \tfrac{1}{3}\,\mathbf{c}_k - \tfrac{3}{4}\,\mathbf{c}_{k/2} - \tfrac{1}{2}\,\mathbf{c}_{(k-1)/2} - \tfrac{1}{6}\,\mathbf{c}_{(k-1)/3} + \tfrac{1}{2}\,\mathbf{c}_{(k-2)/4} + \tfrac{1}{2}\,\mathbf{c}_{(k-4)/6},$$

$$a_k^2 = \mathbf{c}_k - \mathbf{c}_{(k-1)/2}, \qquad a_k^3 = a_k^6 = a_k^7 = \mathbf{c}_k, \qquad a_k^4 = \tfrac{1}{2}\,(\mathbf{c}_{k+1} - \mathbf{c}_{k/2}) - \mathbf{c}_{(k-1)/3}, \qquad a_k^5 = \tfrac{1}{2}\,(\mathbf{c}_k - \mathbf{c}_{(k-1)/2}).$$

ETC Cycle index series, assymmetry index series, enumeration, ...

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Journal of Combinatorial Theory, Series A 106 (2004) 193–219

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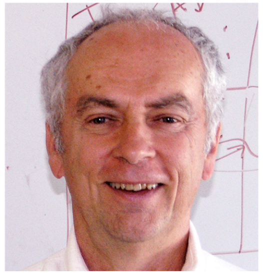
Labelled and unlabelled enumeration of k-gonal 2-trees

Gilbert Labelle, Cédric Lamathe, Pierre Leroux





Abstract

In this paper, we generalize 2-trees by replacing triangles by quadrilaterals, pentagons or k-sided polygons (k-gons), where $k \geqslant 3$ is given. This generalization, to k-gonal 2-trees, is natural and is closely related, in the planar case, to some specializations of the cell-growth problem. Our goal is the labelled and unlabelled enumeration of k-gonal 2-trees according to the number n of k-gons. We give explicit formulas in the labelled case, and, in the unlabelled case, recursive and asymptotic formulas.

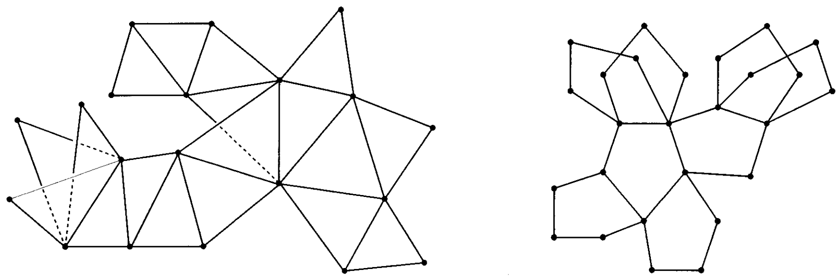
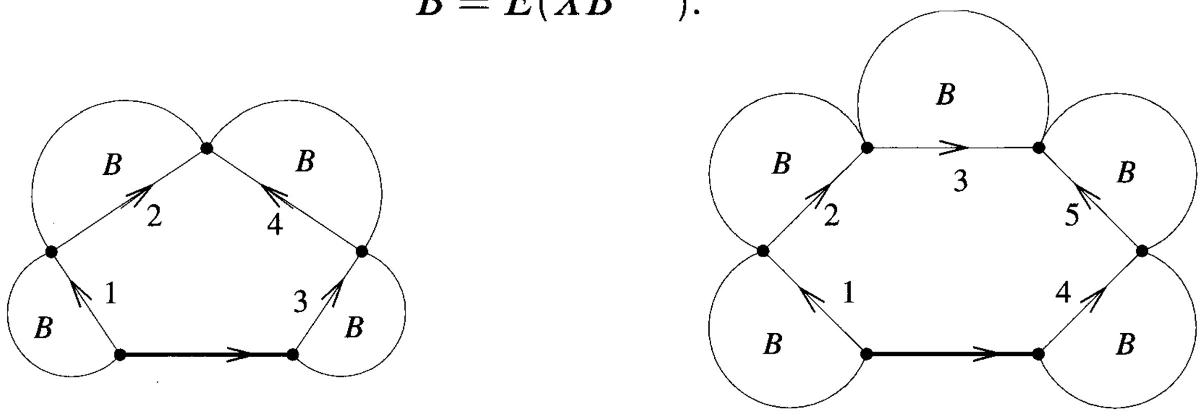


Fig. 1. k-gonal 2-trees with $k = 3$ and $k = 5$.

Theorem 2. *The species $B = a^{\rightarrow}$ of oriented-edge rooted k-gonal 2-trees satisfies the following functional equation* (*isomorphism*):

$$B = E(XB^{k-1}).$$



k odd

Proposition 4. *If k is odd, the number a_n of labelled k-gonal 2-trees on n k-gons is given by*

$$a_n = \frac{1}{2}(m^{n-2}+1), \quad n \geqslant 2,$$

where $m=(k-1)n+1$ is the number of edges.

Corollary 3. *For $k \geqslant 3$, odd, the number $\tilde{a}_n$ of unlabelled k-gonal 2-trees over n k-gons, satisfy the following recurrence*

$$\tilde{a}_n = \frac{1}{2n}\sum_{j=1}^{n}\left(\sum_{l|j} l\omega_l\right)\left(\tilde{a}_{n-j} - \frac{1}{2}\tilde{a}_{o,n-j}\right) + \frac{1}{2}\tilde{a}_{o,n}, \quad \tilde{a}_0 = 1,$$

where

$$\omega_n = 2b_{\frac{n-1}{2}}^{\left(\frac{k-1}{2}\right)} + b_{\frac{n-2}{2}}^{(k-1)} - b_{\frac{n-2}{4}}^{\left(\frac{k-1}{2}\right)}, \qquad b_l^{(m)} = [x^l]\tilde{B}^m(x).$$

k even

Proposition 9. *If k is even, the number a_n of labelled k-gonal 2-trees on n k-gons is given by*

$$a_n = \frac{1}{2}\left(m^{n-2} + (n+1)^{n-2}\right), \quad n \geqslant 2,$$

where $m = (k-1)n + 1$ is the number of edges.

Corollary 5. *Let k be an even integer, $k \geqslant 4$. Then the number of unlabelled k-gonal 2-trees over n k-gons is given by*

$$\tilde{a}_n = \frac{1}{2}\tilde{a}_{o,n} + \frac{1}{2}\alpha_n + \frac{1}{4} b_{\frac{n-1}{2}}^{\left(\frac{k}{2}\right)} - \frac{1}{4}\sum_{i+j=n-1} \alpha_i^{(2)} \cdot b_{\frac{j}{2}}^{\left(\frac{k-2}{2}\right)},$$

where

$$b_l^{(m)} = [x^l]\tilde{B}^m(x), \quad \alpha_i^{(2)} = [x^i]\tilde{a}_S^2(x).$$

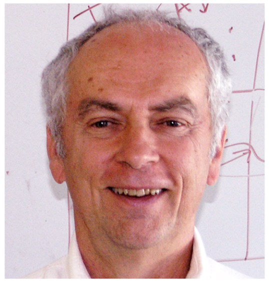
ETC Asymptotic formulas, generalized Otter's constants, ...

Annales des Sciencces Mathématiques du Québec, 29 (2005), no.2, 215-236.

Dénombrement des 2-arbres k-gonaux selon la taille et le périmètre

G. Labelle, C. Lamathe, P. Leroux





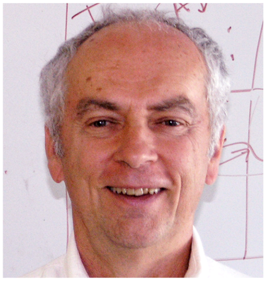
RÉSUMÉ : *Dans cet article, nous nous intéressons au dénombrement étiqueté et non étiqueté des 2-arbres k-gonaux par rapport à leur taille et à leur périmètre, à savoir, le nombre de k-gones et d'arêtes externes (de degré un) du 2-arbre, respectivement. Cette famille de 2-arbres est une variante des 2-arbres (libres) dans laquelle on a remplacé les triangles par des polygones à k-côtés, appelés k-gones. On s'attache à donner des formules énumératives explicites dans le cas étiqueté et des formules de récurrence ou des séries génératrices explicites pour le cas non étiqueté.*

PU.M.A. (Pure Mathematics and Applications) Vol. 16 (2005), No. 3, 1-20.

The structure and labelled enumeration of $K_{3,3}$-subdivision-free projective-planar graphs

Andrei Gagarin, Gilbert Labelle and Pierre Leroux





Abstract

We consider the class $\mathcal{F}$ of 2-connected non-planar $K_{3,3}$-subdivision-free graphs that are embeddable in the projective plane. We show that these graphs admit a unique decomposition as a graph K_5 (the *core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. A method to enumerate these graphs in the labelled case is described. Moreover, we enumerate the homeomorphically irreducible graphs in $\mathcal{F}$ and homeomorphically irreducible 2-connected planar graphs. Particular use is made of two-pole (directed) series-parallel networks. We also show that the number m of edges of graphs in $\mathcal{F}$ satisfies the bound $m \leq 3n - 6$, for $n \geq 6$ vertices.

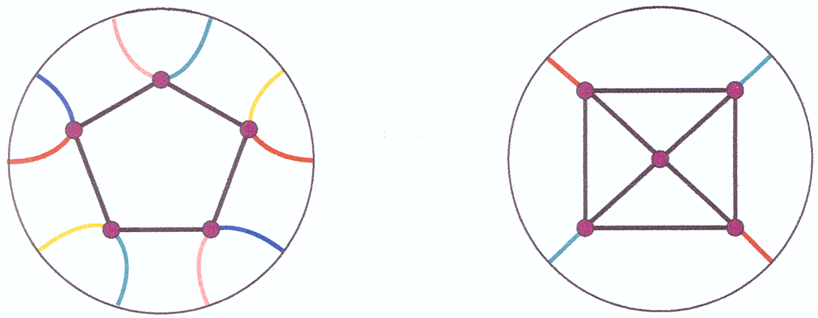
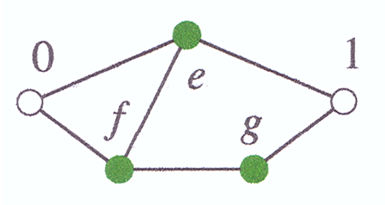


Figure 1: *Embeddings of K_5 in the projective plane.*

A *two-pole network* (or more simply, a *network*) is a connected graph N with two distinguished vertices 0 and 1, such that the graph $N \cup 01$ is 2-connected.

A network N is *strongly planar* if the graph $N \cup 01$ is planar.

Denote by $\mathcal{N}_P$ the class of strongly planar networks.



Theorem 1 *The class $\mathcal{F}$ of 2-connected non-planar projective-planar $K_{3,3}$-free graphs can be expressed as a canonical composition*

$$\mathcal{F} = K_5 \uparrow \mathcal{N}_P.$$

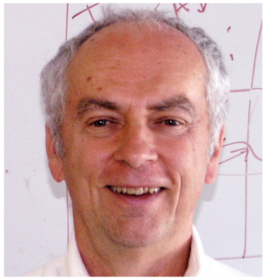
ETC Enumeration of these graphs and related graphs

Discrete Mathematics, 307 (2007) 2993-3005.

The structure of $K_{3,3}$-subdivision-free toroidal graphs

Andrei Gagarin, Gilbert Labelle and Pierre Leroux





Abstract

We consider the class $\mathcal{T}$ of 2-connected non-planar $K_{3,3}$-subdivision-free graphs that are embeddable in the torus. We show that any graph in $\mathcal{T}$ admits a unique decomposition as a basic toroidal graph (the *toroidal core*) where the edges are replaced by two-pole networks constructed from 2-connected planar graphs. Toroidal cores can be enumerated, using matching polynomials of cycle graphs. As a result, we enumerate labelled graphs in $\mathcal{T}$ having minimum vertex degree two or three, according to their number of vertices and edges. We also show that the number m of edges of graphs in $\mathcal{T}$ satisfies the bound $m \leq 3n - 6$, for $n \geq 6$ vertices.

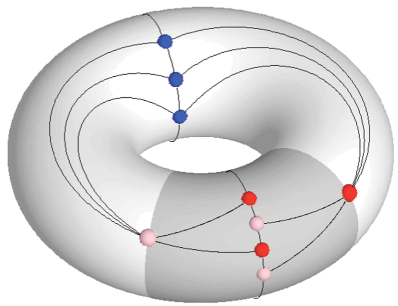
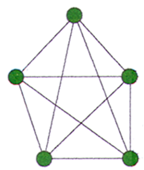
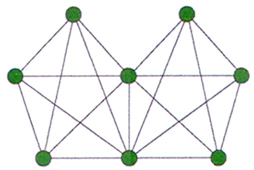


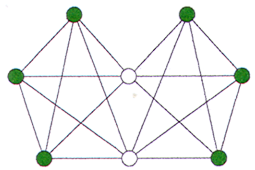
Figure 1: Graph embedded on the torus



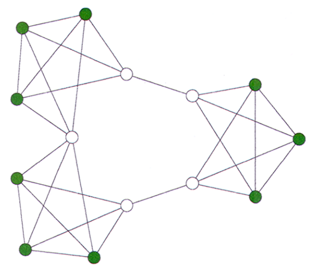
K_5-graph



M-graph



M^*-graph



$\mathcal{H}$: the class of toroidal crowns

Theorem 2 *The class $\mathcal{T}$ of 2-connected non-planar $K_{3,3}$-free toroidal graphs is characterized by*

$$\mathcal{T} = \mathcal{T}_{\mathrm{C}} \uparrow \mathcal{N}_P, \qquad \mathcal{T}_{\mathrm{C}} = K_5 + M + M^* + \mathcal{H},$$

the composition being canonical.

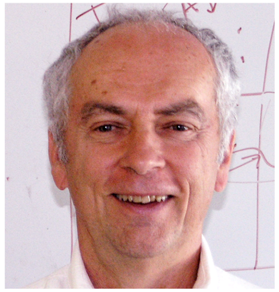
ETC we enumerate labelled graphs in $\mathcal{T}$ having minimum vertex degree two or three, ...

Advances in Applied Mathematics 39 (2007), 51-75.

Counting unlabelled toroidal graphs with no $K_{3,3}$-subdivisions

Andrei Gagarin, Gilbert Labelle and Pierre Leroux





Abstract

We provide a description of unlabelled enumeration techniques, with complete proofs, for graphs that can be canonically obtained by substituting 2-pole networks for the edges of core graphs. Using structure theorems for toroidal and projective-planar graphs containing no $K_{3,3}$-subdivisions, we apply these techniques to obtain their unlabelled enumeration.

$$\mathcal{F} = K_5 \uparrow \mathcal{N}_P,$$ (*2-connected, non-planar, $K_{3,3}$-free and projective-planar graphs*)

$$\mathcal{T} = \mathcal{T}_C \uparrow \mathcal{N}_P,$$ (*2-connected non-planar $K_{3,3}$-free toroidal graphs*)

$$(\mathcal{G} \uparrow \mathcal{N})(x, y) = \mathcal{G}(x, \mathcal{N}(x, y))$$ (labelled case)

$$(\mathcal{G} \uparrow \mathcal{N})\tilde{}(x, y) = \text{?}$$ (unlabelled case)

Walsh series $W_{\mathcal{G}}(\mathbf{a}; \mathbf{b}; \mathbf{c})$, $W_{\mathcal{N}}^{+}(\mathbf{a}; \mathbf{b}; \mathbf{c})$ and $W_{\mathcal{N}}^{-}(\mathbf{a}; \mathbf{b}; \mathbf{c})$

Theorem 3 *Let $\mathcal{G}$ be a species of graphs and $\mathcal{N}$ be a symmetric species of networks. Then the Walsh index series of the species $\mathcal{G} \uparrow \mathcal{N}$ is given by*

$$W_{\mathcal{G}\uparrow\mathcal{N}}(\mathbf{a}; \mathbf{b}; \mathbf{c}) = W_{\mathcal{G}}(a_1, a_2, \ldots; (W_{\mathcal{N}}^{+})_1, (W_{\mathcal{N}}^{+})_2, \ldots; (W_{\mathcal{N}}^{-})_1, (W_{\mathcal{N}}^{-})_2, \ldots).$$

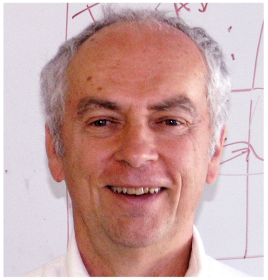
$$(\mathcal{G} \uparrow \mathcal{N})\tilde{}(x, y) = W_{\mathcal{G}}(x, x^2, \ldots; \widetilde{\mathcal{N}}(x, y), \widetilde{\mathcal{N}}(x^2, y^2), \ldots; \widetilde{\mathcal{N}}_\tau(x, y), \widetilde{\mathcal{N}}_\tau(x^2, y^2), \ldots).$$

ETC Complete details about the theory of Walsh series, explicit computations and tables.

Séminaire lotharingien de combinatoire, Lucelle, France (2004), vol. 54

Graph weights arising from Mayer's theory of cluster integrals

G. Labelle, P. Leroux and M. G. Ducharme



A notre très bon ami Xavier Viennot


Abstract

We study graph weights (i.e. graph invariants) which arise naturally in Mayer's theory of cluster integrals in the context of a non-ideal gas. Various choices of the interaction potential between two particles yield various graph weights $w(g)$. For example, in the case of the Gaussian interaction, the so-called Second Mayer weight $w(c)$ of a connected graph c is closely related to the graph complexity, i.e. the number of spanning trees, of c. We give special attention to the Second Mayer weight $w(c)$ which arises from the hard-core continuum gas in one dimension. This weight is a signed volume of a convex polytope $\mathcal{P}(c)$ naturally associated with c. Among our results are the values $w(c)$ for all 2-connected graphs c of size at most 6, in Appendix B, and explicit formulas for three infinite families: complete graphs, (unoriented) cycles and complete graphs minus an edge.

The grand canonical partition function

$$Z(V,T,N) = \frac{1}{N!\lambda^{dN}} \int_V \cdots \int_V \exp\left(-\beta \sum_{i<j} \varphi(|\vec{x}_i - \vec{x}_j|)\right) d\vec{x}_1 \ldots d\vec{x}_N$$

Mayer's idea

$$1 + f_{ij} = \exp\left(-\beta\varphi(|\vec{x_i} - \vec{x_j}|)\right)$$

$$Z(V,N,T) = \frac{1}{N!\lambda^{dN}} \sum_{g\in\mathcal{G}[N]} W(g)$$

$$W(g) = \int_{V^N} \prod_{\{i,j\}\in g} f_{ij} d\vec{x_1} \cdots d\vec{x_N} \qquad (\text{ the weight of a graph } g\,)$$

$$Z_{\mathrm{gr}}(V,T,z) = \sum_{N=0}^{\infty} Z(V,N,T)(\lambda^d z)^N$$

$$= \mathcal{G}_W(z)$$

(the exponential generating series of graphs weighted by the function W)

Pressure

$$\frac{P}{kT} = \frac{1}{V} \log Z_{\mathrm{gr}}(V,T,z) = \frac{1}{V} \mathcal{C}_W(z)$$

The thermodynamic limit $w(c)$ of a connected graph c

$$w(c) \;=\; \lim_{V\to\infty}\frac{1}{V}W(c) \;=\; \lim_{V\to\infty}\frac{1}{V}\int_{V^N}\prod_{\{i,j\}\in c} f_{ij}\,d\vec{x}_1\ldots d\vec{x}_N$$

Proposition 3 *If the function $f:[0,\infty)\to\mathbb{R}$ is integrable and bounded and if*

$$\int_0^\infty r^{d-1}|f(r)|\,dr < \infty,$$

(for example if $|f(r)| = \mathrm{O}(1/r^{d+\epsilon})$, $r\to\infty$), then the limit exists and

$$w(c) = \int_{(\mathbb{R}^d)^{N-1}}\prod_{\{i,j\}\in c;\ \vec{x}_N=\vec{0}} f_{ij}\,d\vec{x}_1\ldots d\vec{x}_{N-1}.$$

Proposition 5 *The second Mayer weight w is* block-multiplicative. *More precisely, for any connected graph c whose blocks are $b_1, b_2, \ldots, b_m$, we have*

$$w(c) = w(b_1)w(b_2)\ldots w(b_m).$$

Gaussian models

$$f_{ij} = -\exp(-\alpha\|\vec{x}_i - \vec{x}_j\|^2), \qquad w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \gamma(c)^{-\frac{d}{2}}$$

$$f_{ij} = -\exp\left(-\alpha y_i y_j \|\vec{x}_i - \vec{x}_j\|^2\right), \qquad w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \left(\sum_{t \in T(c)} y_1^{d_t(1)} y_2^{d_t(2)} \dots y_n^{d_t(n)}\right)^{-\frac{d}{2}}$$

$$f_{ij} = -\exp\left(-w_{i,j} \|\vec{x}_i - \vec{x}_j\|^2\right), \qquad w(c) = (-1)^{e(c)} (\pi)^{\frac{d(n-1)}{2}} \left(\sum_{t \in T(c)} \prod_{\{i,j\} \in t} w_{i,j}\right)^{-\frac{d}{2}}$$

The hard-core continuum gas in one dimension

$$1 + f_{ij} = \chi(|x_i - x_j| \geq 1) \quad \Leftrightarrow \quad f_{ij} = -\chi(|x_i - x_j| < 1)$$

$$\begin{aligned}
w(K_N) &= (-1)^{\binom{N}{2}} N \\
w(C_N) &= (-1)^N \frac{2^N}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^N dt \\
&= \frac{(-1)^N}{(N-1)!} \sum_{i=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^i \binom{N}{i} (N-2i)^{N-1} \\
&\sim (-2)^N \left(\frac{3}{2\pi N}\right)^{\frac{1}{2}} \left(1 - \frac{3}{20N} - \frac{13}{1120N^2} + \dots\right)
\end{aligned}$$

The hard-core continuum gas in one dimension (continued)

In general,

$$w(c) = (-1)^{e(c)} \mathrm{Vol}(\mathcal{P}(c))$$

$$\mathcal{P}(c) = \{X \in \mathbb{R}^N \mid x_N = 0 \text{ and } |x_i - x_j| \leq 1 \ \forall \{i,j\} \in c\}\,, \qquad \textit{convex polytope}$$

The vertices of $\mathcal{P}(c)$ have integer coordinates

$$\mathrm{Vol}(\mathcal{P}(c)) = \nu(c)/(N-1)! \qquad \nu(c) \ \textbf{integer}$$

We have techniques to compute $\nu(c)$

(Ehrhart polynomials, fractional representations of simplicial subpolytopes, ...)

We have a table of $\nu(c)$ for all 2-connected graphs c of size at most 6

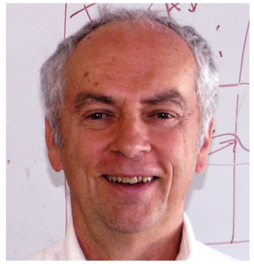
ETC

Proceedings FPSAC07, Nankai University, Tianjin, Chine

A classification of outerplanar K-gonal 2-trees

Martin Ducharme, Gilbert Labelle, Cédric Lamathe et Pierre Leroux



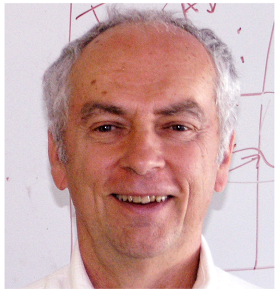
Abstract. We give in this work the molecular expansion of the species of outerplanar K-gonal 2-trees, extending previous work on ordinary ($K = 3$) outerplanar 2-trees. This is equivalent to a classification of these graphs according to their symmetries (automorphism groups). We give explicit formulas for all coefficients occuring in this expansion.

Soumis pour publication

Two-connected graphs with prescribed three-connected components

Andrei Gagarin, Gilbert Labelle, Pierre Leroux, and Timothy Walsh






Abstract

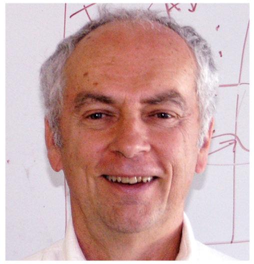
We adapt the classical 3-decomposition of any 2-connected graph to the case of simple graphs (no loops or multiple edges). By analogy with the block-cutpoint tree of a connected graph, we deduce from this decomposition a bicolored tree $\mathrm{tc}(g)$ associated with any 2-connected graph g, whose white vertices are the *3-components* of g (3-connected components or polygons) and whose black vertices are bonds linking together these 3-components, arising from separating pairs of vertices of g. Two fundamental relationships on graphs and networks follow from this construction. The first one is a dissymmetry theorem which leads to the expression of the class $\mathcal{B} = \mathcal{B}(\mathcal{F})$ of 2-connected graphs, all of whose 3-connected components belong to a given class $\mathcal{F}$ of 3-connected graphs, in terms of various rootings of $\mathcal{B}$. The second one is a functional equation which characterizes the corresponding class $\mathcal{R} = \mathcal{R}(\mathcal{F})$ of two-pole networks all of whose 3-connected components are in $\mathcal{F}$. All the rootings of $\mathcal{B}$ are then expressed in terms of $\mathcal{F}$ and $\mathcal{R}$. There follow corresponding identities for all the associated series, in particular the edge index series. Numerous enumerative consequences are discussed.

En préparation 2008

Mayer and Ree-Hoover weights of infinite families of 2-connected graphs

Amel Kaouche, Gilbert Labelle, Cédric Lamathe et Pierre Leroux



We study graph weights (i.e., graph invariants) which arise naturally in Mayer's theory and Ree-Hoover's theory of virial expansions in the context of a non-ideal gas. We give special attention to the Second Mayer weight $w_M(c)$ and the Ree-Hoover weight $w_{RH}(c)$ of a 2-connected graph c which arise from the hard-core continuum gas in one dimension. These weights are signed volumes of a convex polytope naturally associated with the graph c. Among our results are the values of Mayer's weight and Ree-Hoover's weight for all 2-connected graphs b of size at most 8, and explicit formulas for certain infinite families.

En préparation

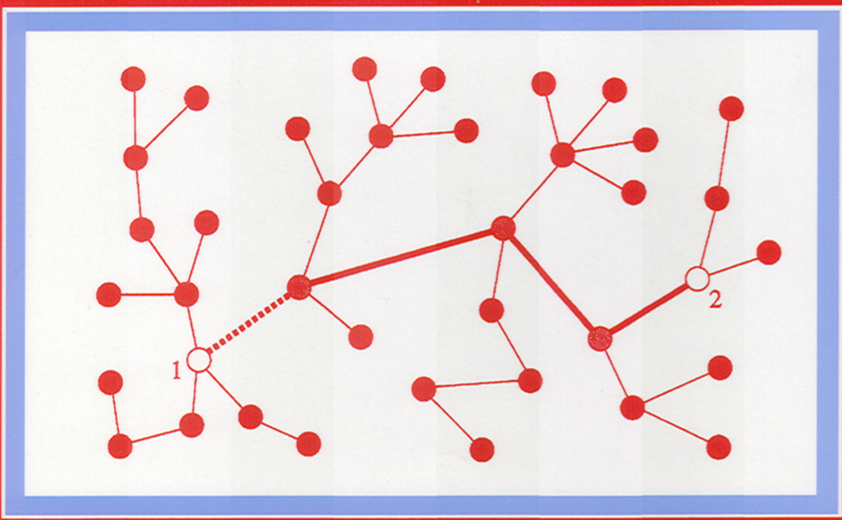
Enumerating combinatorial structures equipped with a list of commuting automorphisms.

Miguel Méndez, Gilbert Labelle and Pierre Leroux

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 67

COMBINATORIAL SPECIES AND TREE-LIKE STRUCTURES

F. BERGERON,
G. LABELLE,
P. LEROUX



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