# MES AVENTURES MATHÉMATIQUES AVEC PIERRE LEROUX

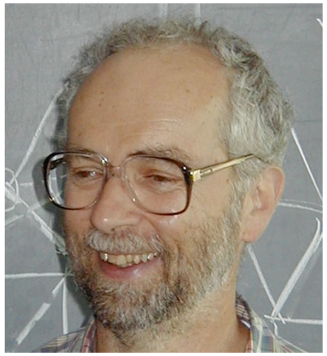
Gilbert Labelle

FPSAC'08 - Valparaiso, Chili

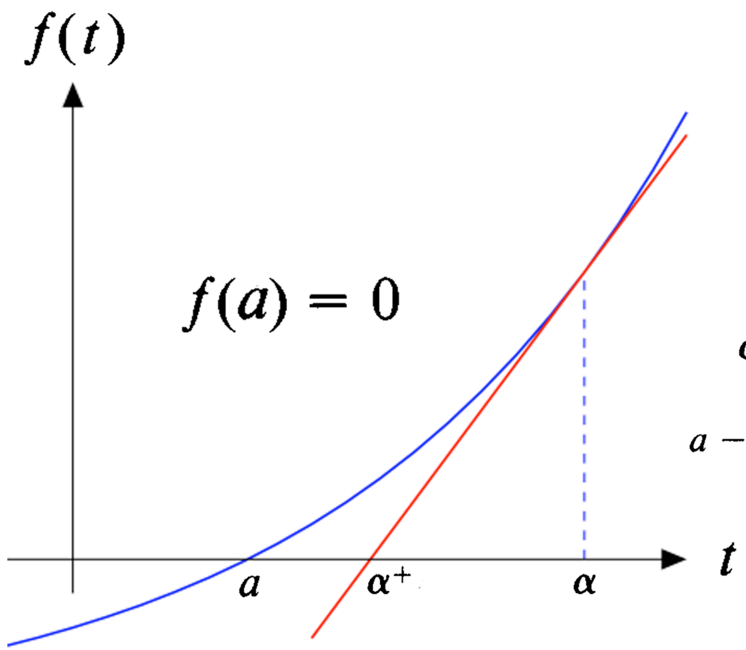
ADVANCES IN APPLIED MATHEMATICS **3**, 407–416 (1982)

# Une approche combinatoire pour l'itération de Newton – Raphson

H. DÉCOSTE, G. LABELLE, ET P. LEROUX



Starting with an approximation $\alpha$ having a contact of order $n$ with the species $A$ of $R$-enriched rooted trees (in the sense of Joyal (*Advances in Math*. **42** (1981), 1–82) and Labelle (*Advances in Math*. **42** (1981), 217–247)), a new approximation $\alpha^+$, having a contact of order $2n+2$ with $A$, is deduced by a purely combinatorial argumentation. This provides a combinatorial setting for the classical Newton–Raphson iterative scheme. A generalization involving contacts of higher orders is also developed.




$$\alpha^+ = \alpha - [f'(\alpha)]^{-1} f(\alpha)$$

$$\alpha_{n+1} = \alpha_n^+$$

$$a - \alpha_{n+1} = O\big((a - \alpha_n)^2\big) \to 0 \qquad \text{si } n \to \infty.$$

**Le Cas des Séries Formelles**

$$c = c(x) = c_1 x + c_2 x^2 + \cdots = x/r(x),$$

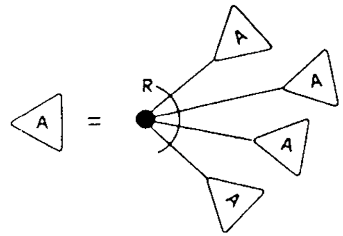
$$f(t) = c(t) - x, \qquad t = t(x)$$

$$f(a) = 0 \quad \textbf{ssi} \quad a = x\,r(a) \quad \textbf{ssi} \quad a = c^{\langle -1 \rangle}(x)$$

$$\alpha^+ = \alpha + \frac{xr(\alpha) - \alpha}{1 - xr'(\alpha)}$$

## L'APPROCHE COMBINATOIRE

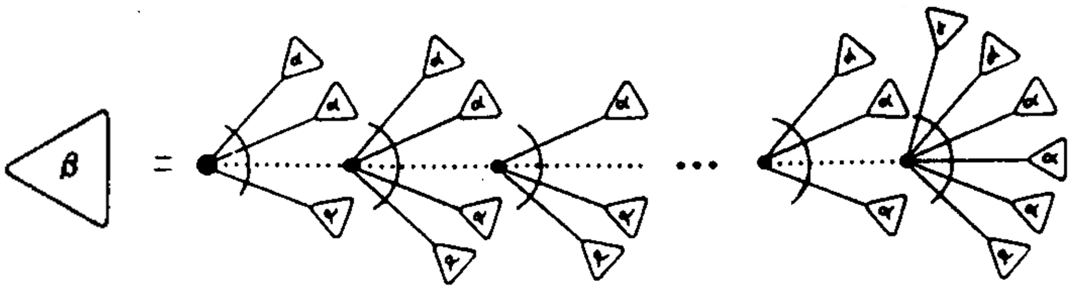
L'espèce $A$ des arborescences $R$-enrichies $\quad A = XR(A)$




PROPOSITION 2. Soit $L$ l'espèce des ordres linéaires et soit $\alpha$ l'espèce des arborescences $R$-enrichies légères (i.e., portées par des cardinalités $\leq n$). Alors l'espèce $\alpha^+$ définie par

$$\alpha^+ = \alpha + L(XR'(\alpha)) \cdot (XR(\alpha) - \alpha) \tag{2.9}$$

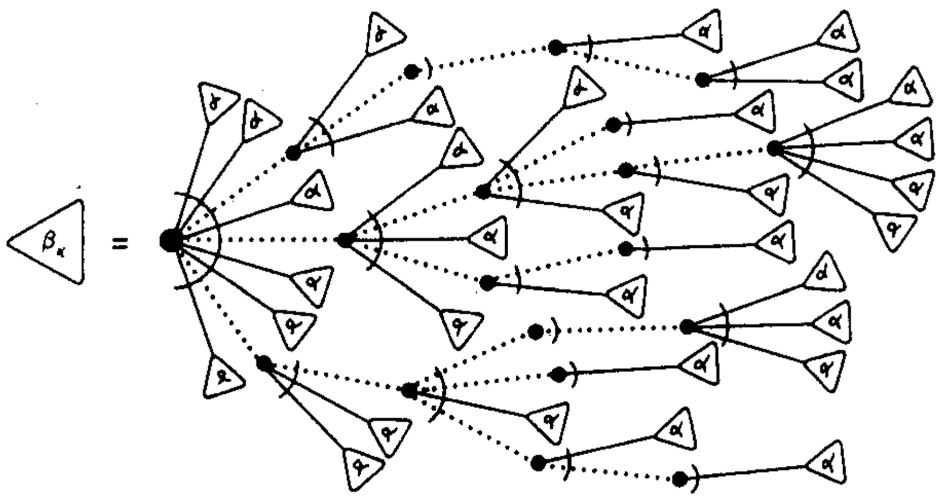
*possède un contact d'ordre $2n+2$ avec l'espèce $A$ des arborescences $R$-enrichies.*



**Proposition 4.** *Soit $\alpha$ l'espèce des arborescences R-enrichies légères (i.e. portées par des cardinalités $\leq n$) et soit $\gamma$ l'espèce définie implicitement par l'équation combinatoire polynomiale*

$$\gamma = (XR(\alpha) - \alpha) + \sum_{i=1}^{k} \frac{XR^{(i)}(\alpha)}{i!}\gamma^i.$$

*Alors l'espèce $\alpha^+$ définie par $\alpha^+ = \alpha + \gamma$ possède un contact d'ordre $(k+1)(n+1)$ avec l'espèce $A$ des arborescences R-enrichies.*

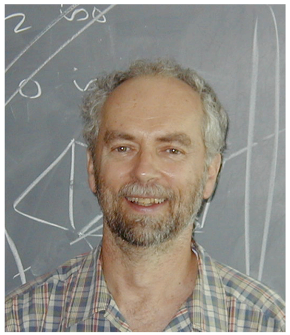



**ETC**

Discrete Applied Mathematics 34 (1991) 49–66
North-Holland

# Computation of the expected number of leaves in a tree having a given automorphism, and related topics

F. Bergeron, G. Labelle and P. Leroux



*Abstract*

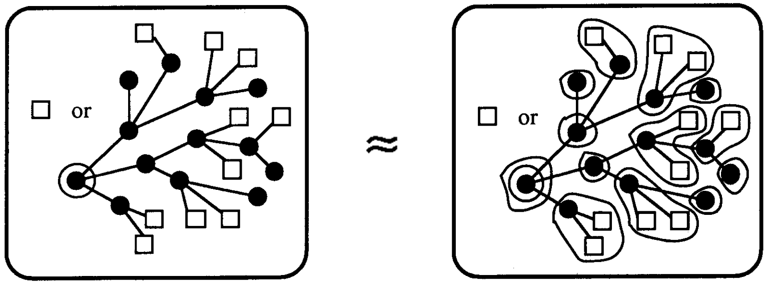
We derive explicit formulas for the expected number of leaves in a random rooted tree that is fixed by a given permutation of the nodes, and similarly for (unrooted) trees and endofunctions. The main tool is the cycle index series of a species. The cases of asymmetric rooted trees and ***R***-enriched trees and rooted trees are also discussed.

- $A = A(X)$ of rooted trees, and
- $B = B(X, Y)$ of rooted trees with internal points of sort $X$ and leaves of sort $Y$.

**Proposition 1.** *The species $A = A(X)$ and $B = B(X, Y)$ are related by the following combinatorial equation*

$$B(X, X + T) = T + A(X \cdot E(T)), \tag{2.3}$$

*where $T$ denotes an auxiliary sort of singletons.*




$B(X, X+T)$ $\qquad$ $T + A(X \cdot E(T))$

$$A_w(X) = B(X, X_t), \qquad T := X_t - X, \qquad A_w(X) = X_t - X + A(X \cdot E(X_t - X)).$$

$$Z_{A_w} = (t-1)x_1 + Z_A\left(x_1 \exp \sum_{k \geq 1} \frac{(t^k - 1)x_k}{k},\ x_2 \exp \sum_{k \geq 1} \frac{(t^{2k} - 1)x_{2k}}{k}, \ldots\right)$$

The number of rooted trees which are fixed by a permutation $\sigma$

$$a_\sigma = \sigma_1^{\sigma_1 - 1} \prod_{k \geq 2} \left\{ (\sigma^k)_1^{\sigma_k} - k\sigma_k (\sigma^k)_1^{\sigma_k - 1} \right\}, \quad \text{where} \quad (\sigma^k)_1 = \sum_{d \mid k} d\sigma_d. \tag{2.2}$$

**Proposition 2.** *Let $U$ be an $n$-set, $\sigma$ be a permutation of $U$ whose cyclic type is $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $P = \{k \mid 1 \leq k \leq n,\ \sigma_k \neq 0\}$. Then the expected number of leaves in a random rooted tree on $U$ for which $\sigma$ is an automorphism is given by* 1 *if* $n = 1$, *and by*

$$\frac{1}{a_\sigma} \sum_{k \in P} k\sigma_k \cdot ((\sigma^k)_1 - k) \cdot a_{\sigma - \delta^k}, \tag{2.8}$$

*if $n \neq 1$ and $a_\sigma \neq 0$, where $a_\sigma$ and $(\sigma^k)_1$ are given by* (2.2) *and $\sigma - \delta^k$ denotes a permutation obtained from $\sigma$ by dropping out one* (*arbitrary*) *$k$-cycle* (*i.e., the cyclic type of $\sigma - \delta^k$ is given by $(\sigma_1, \ldots, \sigma_k - 1, \ldots, \sigma_n)$*).

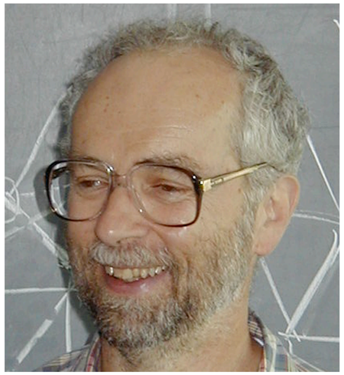
**ETC**

Discrete Mathematics 99 (1992) 31–48
North-Holland

# The functorial composition of species, a forgotten operation

Hélène Décoste, Gilbert Labelle and Pierre Leroux





*Abstract*

In order to study the functorial composition of species, we introduce the auxiliary concepts of *cyclic type* and *fixed points enumerator* of a species. Basic formulas are established and applications are given to the computation of the cycle index series of classes of graphs, pure $m$-complexes, coverings and $m$-ary relations that are structured in various ways.

The functorial composition

$$(F \square G)[U] = F[G[U]], \quad (F \square G)[\sigma] = F[G[\sigma]]$$

- Simple graphs: $\mathcal{G} = \wp \square \wp^{[2]}$
- Directed graphs: $\mathcal{D} = \wp \square (E^{\bullet} \times E^{\bullet})$
- $m$-ary relations: $\mathrm{Rel}^{[m]} = \wp \square (E^{\bullet})^{\times m}$

$$\mathbf{Z}_{F \square G}(x_1, x_2, \ldots) = \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\sigma \in S[n]} \mathrm{fix}\, F[G[\sigma]] x_1^{\sigma_1} x_2^{\sigma_2} \cdots = ?$$

**Proposition 2.2.** *The cyclic type $(\beta_k)_{k=1,2,3,\ldots}$ of an arbitrary species $G$ depends only on the function* fix $G[\sigma] = \beta(\sigma_1, \sigma_2, \sigma_3, \ldots)$. *Indeed we have the formula*

$$\beta_k = (G[\sigma])_k = \frac{1}{k} \sum_{d \mid k} \mu\left(\frac{k}{d}\right) \mathrm{fix}\, G[\sigma^d]. \qquad (2.4)$$

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***Applications*** : Enumeration of
Classes of graphs, pure $m$-complexes, coverings, structured $m$-ary relations, structured words ... .

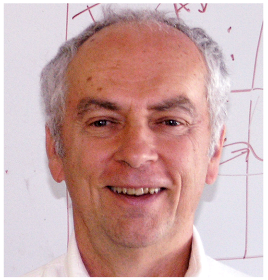
ELSEVIER

Discrete Mathematics 157 (1996) 227–240

DISCRETE MATHEMATICS

# Enumeration of (uni- or bicolored) plane trees according to their degree distribution

Gilbert Labelle, Pierre Leroux



**Résumé**

L'objectif principal de ce texte est de donner des formules explicites pour le nombre de types d'isomorphie d'arbres plans bicolorés ayant une double distribution $(1^{i_1}2^{i_2}\ldots;1^{j_1}2^{j_2}\ldots)$ de degrés donnés à l'avance. Ces arbres sont en étroite relation avec les polynômes de Shabat, c'est-à-dire les polynômes sur $\mathbb{C}$ ayant au plus deux valeurs critiques (cf. Shabat and Zvonkin, 1994). Dans le cas des arbres enracinés (pointés en une feuille), ce problème a été résolu par Tutte en 1964 à l'aide de l'inversion de Lagrange multivariée. Ici la clé de la solution réside dans le théorème de dissymétrie pour les arbres enrichis qui, dans le cas bicoloré, prend une forme particulièrement simple et qui permet de se défaire du pointage. Nous dénombrons également ces arbres dans le cas étiqueté, dans le cas unicoloré, ainsi que lorsque le groupe d'automorphismes, nécessairement cyclique, est d'un ordre $h$ égal à (ou multiple de) un entier $k \geqslant 1$ donné.

As was emphasized by A. Zvonkin during the 5th FPSAC meeting in Florence, these trees are closely related to polynomials $P(z)$ over $\mathbb{C}$ having at most two critical values, called Shabat polynomials.

***Unlabelled bicolored plane trees*** $\cong$ ***Equivalence classes of Shabat polynomials under affine transformations***

The black vertices correspond to the roots of the equation $P(z)=0$,
the white vertices correspond to the roots of the equation $P(z)=1$,
the degrees correspond to multiplicities of these roots.

Double degree distribution $\boldsymbol{i} = (i_1, i_2, \ldots)$, $\boldsymbol{j} = (j_1, j_2, \ldots)$, with $s(\boldsymbol{i}) = i_1 + i_2 + \cdots$ black points and $s(\boldsymbol{j}) = j_1 + j_2 + \cdots$ white points,

the edge count implies the double equality

$$i_1 + 2i_2 + 3i_3 + \cdots \quad = \quad j_1 + 2j_2 + 3j_3 + \cdots \quad = \quad s(\boldsymbol{i}) + s(\boldsymbol{j}) - 1. \tag{0.1}$$

**QUESTION :** How many unlabelled bicolored plane trees have such a double distribution **?**

**Theorem 1.** *Let* $\boldsymbol{i} = (i_1, i_2, \ldots)$ *and* $\boldsymbol{j} = (j_1, j_2, \ldots)$ *be two sequences of integers* $\geqslant 0$, *with* $s(\boldsymbol{i}) = i_1 + i_2 + \cdots < \infty$ *and* $s(\boldsymbol{j}) < \infty$, *satisfying the relations* (0.1). *Then the number* $\tilde{\wp}(\boldsymbol{i}, \boldsymbol{j})$ *of unlabelled bicolored plane trees, whose degree distribution is* $(\boldsymbol{i}, \boldsymbol{j})$, *is given by*

$$\tilde{\wp}(\boldsymbol{i}, \boldsymbol{j}) = \tilde{F}(\boldsymbol{i}, \boldsymbol{j}) + \tilde{F}(\boldsymbol{j}, \boldsymbol{i}) - \tilde{G}(\boldsymbol{i}, \boldsymbol{j}), \tag{0.2}$$

*with*

$$\tilde{F}(\boldsymbol{i}, \boldsymbol{j}) = \frac{1}{n} \sum_{h,d} \phi(d) \binom{n/d}{\boldsymbol{i}/d} \binom{(m-1)/d}{(\boldsymbol{j} - \delta_h)/d}, \tag{0.3}$$

*where* $n = s(\boldsymbol{i})$, $m = s(\boldsymbol{j})$, $\phi$ *denotes the arithmetic Euler* $\phi$*-function, and the sum is taken over all pairs of integers* $h \geqslant 1$, $d \geqslant 1$ *such that* $h \in \operatorname{Supp}(\boldsymbol{j})$ *and* $d \in \operatorname{Div}(h, \boldsymbol{i}, \boldsymbol{j} - \delta_h)$, *and*

$$\tilde{G}(\boldsymbol{i}, \boldsymbol{j}) = \frac{n+m-1}{nm} \binom{n}{\boldsymbol{i}} \binom{m}{\boldsymbol{j}}. \tag{0.4}$$

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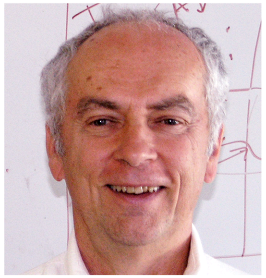
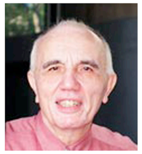
THE ELECTRONIC JOURNAL OF COMBINATORICS 3 (2) (1996), #R12

# An Extension of the Exponential Formula in Enumerative Combinatorics

Gilbert Labelle and Pierre Leroux



En hommage à Dominique Foata, à l'occasion de son soixantième anniversaire.



**Résumé**

Soit $\alpha$ une variable formelle et $F_w$ une espèce de structures pondérée (classe de structures fermée sous les isomorphismes préservant les poids) de la forme $F_w = E(F_w^c)$, où $E$ et $F_w^c$ désignent respectivement l'espèce des *ensembles* et celle des *$F_w$-structures connexes*. En multipliant par $\alpha$ le poids de chaque $F_w^c$-structure, on obtient l'espèce $F_{w(\alpha)} = E(F_{\alpha w}^c)$. Nous introduisons une espèce virtuelle "universelle", $\Lambda^{(\alpha)}$, telle que $F_{w(\alpha)} = \Lambda^{(\alpha)} \circ F_w^+$, où $F_w^+$ désigne l'espèce des $F_w$-structures non-vides. En faisant appel à des propriétés générales de $\Lambda^{(\alpha)}$, nous calculons les diverses séries formelles énumératives $G(x)$, $\widetilde{G}(x)$, $\overline{G}(x)$, $G(x;q)$, $G\langle x;q\rangle$, $Z_G(x_1,x_2,x_3,\ldots)$, $\Gamma_G(x_1,x_2,x_3,\ldots)$, de $G = F_{w(\alpha)}$, en fonction de $F_w$. Comme cas spéciaux des formules que nous développons, on retrouve la formule exponentielle, $F_{w(\alpha)}(x) = \exp(\alpha F_w(x)) = (F_w(x))^\alpha$, les identités cyclotomiques, ainsi que leurs $q$-analogues. L'espèce virtuelle pondérée, $\Lambda^{(\alpha)}$, est, en fait, un nouveau relèvement combinatoire de la fonction $(1+x)^\alpha$.

$$f(x) = \exp(g(x))$$
$$f(x)^\alpha = \exp(\alpha\, g(x))$$

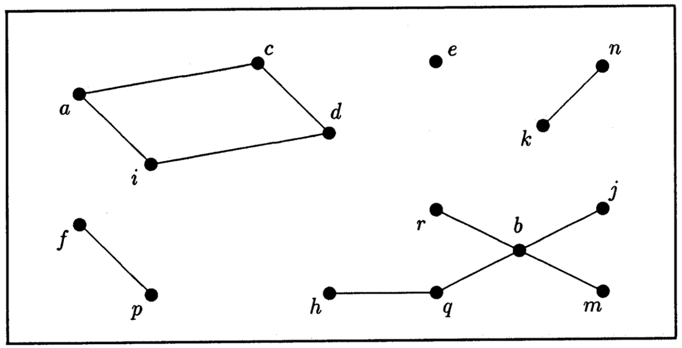



Figure 1: A graph $g$ with $w(g) = y^{11}$ and $w^{(\alpha)}(g) = \alpha^5 y^{11}$

$$F_w = E(F_w^c)$$
$$\downarrow$$
$$F_{w^{(\alpha)}} = E(F_{\alpha w}^c)$$

**Theorem 2.2** *There exists a "universal" virtual weighted species, $\Lambda^{(\alpha)}$, such that*

$$F_{w^{(\alpha)}} = \Lambda^{(\alpha)} \circ F_w^+,$$

*for any species of the form* $F_w = E(F_w^c)$.

$$\Lambda^{(\alpha)}(x) = (1+x)^\alpha,$$

$$Z_{\Lambda^{(\alpha)}} = \prod_{n\geq 1}(1+x_n)^{\lambda_n(\alpha)}, \qquad \widetilde{\Lambda^{(\alpha)}}(x) = \prod_{n\geq 1}(1+x^n)^{\lambda_n(\alpha)}, \qquad \Lambda^{(\alpha)}(x;q) = \prod_{n\geq 1}\left(1+\frac{(1-q)^n}{(1-q^n)}x^n\right)^{\lambda_n(\alpha)},$$

$$\Gamma_{\Lambda^{(\alpha)}} = \prod_{n\geq 1}(1+x_n)^{\gamma_n(\alpha)}, \qquad \overline{\Lambda^{(\alpha)}}(x) = \prod_{n\geq 1}(1+x^n)^{\gamma_n(\alpha)}, \qquad \Lambda^{(\alpha)}\langle x;q\rangle = \prod_{n\geq 1}\left(1+\frac{(1-q)^n}{(1-q^n)}x^n\right)^{\gamma_n(\alpha)},$$

$$\lambda_n(\alpha) = \frac{1}{n}\sum_{d|n}\mu(n/d)\alpha^d, \qquad \gamma_n(\alpha) = -\lambda_n(-\alpha) - \lambda_{n/2}(-\alpha) - \lambda_{n/4}(-\alpha) - \cdots$$

$$\Lambda^{(\alpha)} = E \circ X_\alpha \circ (E^+)^{<-1>} = 1 + X_\alpha - (E_2)_\alpha + (E_2)_{\alpha^2} - (E_3)_\alpha + (XE_2)_\alpha - (XE_2)_{\alpha^2} + (E_3)_{\alpha^3}$$
$$+ (E_2 \circ E_2)_\alpha - (E_4)_\alpha + (XE_3)_\alpha - (X^2E_2)_\alpha$$
$$+ (E_2^2)_{\alpha^2} - (XE_3)_{\alpha^2} + (X^2E_2)_{\alpha^2} - (E_2 \circ E_2)_{\alpha^2} - (E_2^2)_{\alpha^3} + (E_4)_{\alpha^4}$$
$$+ (E_2E_3)_\alpha + (XE_4)_\alpha + (X^3E_2)_\alpha - (X^2E_3)_\alpha - (XE_2^2)_\alpha - (E_5)_\alpha$$
$$+ (E_2E_3)_{\alpha^2} + (X^2E_3)_{\alpha^2} - (XE_4)_{\alpha^2} - (X^3E_2)_{\alpha^2} + (X\ E_2 \circ E_2)_{\alpha^2} - (XE_2^2)_{\alpha^2}$$
$$+ 2\,(XE_2^2)_{\alpha^3} - (E_2E_3)_{\alpha^3} - (X\ E_2 \circ E_2)_{\alpha^3} - (E_2E_3)_{\alpha^4} + (E_5)_{\alpha^5} + \cdots$$

**Corollary 2.3** *Let $F_w$ be a weighted virtual species such that $F_w(0) = 1$ and set $F_{w(\alpha)} = \Lambda^{(\alpha)} \circ F_w^+$. Then*

$$F_{w(\alpha)}(x) = F_w(x)^\alpha,$$

$$Z_{F_{w(\alpha)}}(x_1, x_2, x_3, \ldots) = \prod_{n\geq 1} Z_{F_{w^n}}(x_n, x_{2n}, x_{3n}, \ldots)^{\lambda_n(\alpha)},$$

$$\widetilde{F_{w(\alpha)}}(x) = \prod_{n\geq 1} \widetilde{F_{w^n}}(x^n)^{\lambda_n(\alpha)},$$

$$F_{w(\alpha)}(x; q) = \prod_{n\geq 1} F_{w^n}\left(\frac{(1-q)^n}{(1-q^n)} x^n; q^n\right)^{\lambda_n(\alpha)},$$

$$\Gamma_{F_{w(\alpha)}}(x_1, x_2, x_3, \ldots) = \prod_{n\geq 1} \Gamma_{F_{w^n}}(x_n, x_{2n}, x_{3n}, \ldots)^{\gamma_n(\alpha)},$$

$$\overline{F_{w(\alpha)}}(x) = \prod_{n\geq 1} \overline{F_{w^n}}(x^n)^{\gamma_n(\alpha)},$$

$$F_{w(\alpha)}\langle x; q\rangle = \prod_{n\geq 1} F_{w^n}\left\langle \frac{(1-q)^n}{(1-q^n)} x^n; q^n\right\rangle^{\gamma_n(\alpha)}.$$

**ETC**

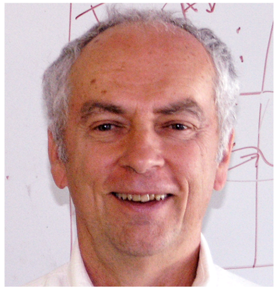
Cyclotomic identities, cocyclotomic identities, $q$-series identities, ...

ADVANCES IN APPLIED MATHEMATICS **21**, 499–546 (1998)

# Cubical Species and Nonassociative Algebras

Gábor Hetyei   Gilbert Labelle   and   Pierre Leroux



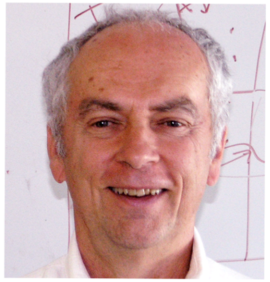


We lay down the foundations of a theory of *cubical species*, as a variant of Joyal's classical theory of species (A. Joyal, *Adv. Math.* **42** (1981), 1–82). Informally, a cubical species associates in a functorial way a set of structures to each hypercube. In this context, the hyperoctahedral groups replace the symmetric groups. We analyze cubical species, molecular cubical species, and basic operations among them, along with explicit examples. We show, in particular, that the cubical product gives rise, in a natural way, to a commutative nonassociative ring of formal power series. We conclude with a detailed analysis of this nonassociative ring.

Journal of Combinatorial Theory, Series A **91**, 15–48 (2000)

# Generalized Binomial Coefficients for Molecular Species

Pierre Auger, Gilbert Labelle, and Pierre Leroux



DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

Let $\xi$ be a complex variable. We associate a polynomial in $\xi$, denoted $\binom{M}{N}_\xi$, to any two molecular species $M=M(X)$ and $N=N(X)$ by means of a binomial-type expansion of the form

$$M(\xi+X)=\sum_N \binom{M}{N}_\xi N(X).$$

In the special case $M(X)=X^m$, the species of linear orders of length $m$, the above formula reduces to the classical binomial expansion

$$(\xi+X)^m=\sum_n \binom{m}{n} \xi^{m-n}X^n.$$

When $\xi=1$, a $M(1+X)$-structure can be interpreted as a partially labelled $M$-structure and $\binom{M}{N}_1$ is a nonnegative integer, denoted $\binom{M}{N}$ for simplicity. We develop some basic properties of these "generalized binomial coefficients" and apply them to study solutions, $\Phi$, of combinatorial equations of the form $M(\Phi)=\Psi$ in the context of $\mathbb{C}$-species, $M$ being molecular and $\Psi$ being a given $\mathbb{C}$-species. This generalizes the study of symmetric square roots (where $M=E_2$, the species of 2-element sets) initiated by P. Bouchard, Y. Chiricota, and G. Labelle in (1995, *Discrete Math.* **139**, 49–56).

$$(1+X)^m = \sum_{n \leqslant m} \binom{m}{n} X^n,$$

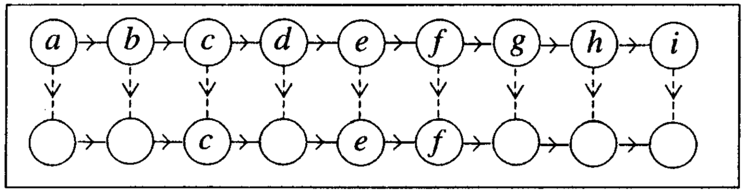



**FIG. 1.** A 3-partially labelled 9-list.

DEFINITION 2.1. Let $M = M(X)$ be any molecular species. The *species of partially labelled M-structures*, denoted $M(1+X)$, is defined by

$$M(1+X) = M(T+X)|_{T:=1}, \tag{2.11}$$

where $T$ is an auxiliary sort of singletons.

## GENERALIZED BINOMIAL COEFFICIENTS

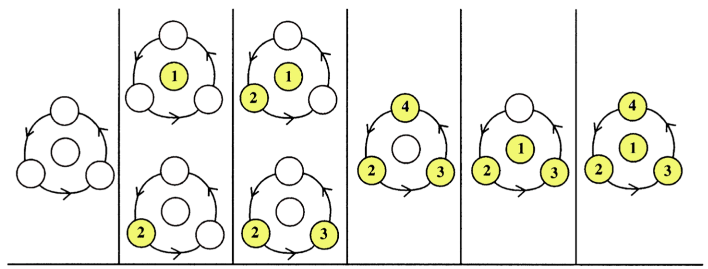
DEFINITION 2.2. The *generalized binomial coefficients*, $\binom{M}{N}$, are the non-negative integers arising from the molecular decompositions

$$M(1+X) = \sum_{N} \binom{M}{N} N(X), \tag{2.14}$$

where $M, N$ run through the molecular species (up to isomorphism of species).

| $M\backslash N$ | $1$ | $X$ | $E_2$ | $X^2$ | $E_3$ | $C_3$ | $XE_2$ | $X^3$ | $E_4$ | $E_4^{\pm}$ | $E_2(E_2)$ | $XE_3$ | $E_2^2$ | $P_4^{\text{bic}}$ | $C_4$ | $XC_3$ | $X^2E_2$ | $E_2(X^2)$ | $X^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1$ | $\boxed{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X$ | $\boxed{1}$ | $\boxed{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_2$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X^2$ | $\boxed{1}$ | $\boxed{2}$ | 0 | $\boxed{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_3$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_3$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $XE_2$ | 1 | 2 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X^3$ | $\boxed{1}$ | $\boxed{3}$ | 0 | $\boxed{3}$ | 0 | 0 | 0 | $\boxed{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_4$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_4^{\pm}$ | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_2(E_2)$ | 1 | 1 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $XE_3$ | 1 | 2 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_2^2$ | 1 | 2 | 2 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_4^{\text{bic}}$ | 1 | 1 | 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $C_4$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $XC_3$ | 1 | 2 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $X^2E_2$ | 1 | 3 | 1 | 3 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $E_2(X^2)$ | 1 | 2 | 2 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $X^4$ | $\boxed{1}$ | $\boxed{4}$ | 0 | $\boxed{6}$ | 0 | 0 | 0 | $\boxed{4}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\boxed{1}$ |

**Example :** $M(X) = XC_3(X)$



$$M(1+X) = 1 + 2X + 2X^2 + C_3(X) + X^3 + XC_3(X)$$

## Classical identities

$$(1+X)^m = \sum_{n \le m} \binom{m}{n} X^n$$

$$\binom{m}{0} = \binom{m}{m} = 1$$

$$\sum_{n \le m} \binom{m}{n} = 2^m$$

$$\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}$$

$$\binom{m_1+m_2}{n} = \sum_{n_1+n_2=n} \binom{m_1}{n_1}\binom{m_2}{n_2}$$

## Generalized identities

$$M(1+X) = \sum_{N} \binom{M}{N} N(X)$$

$$\binom{M}{1} = \binom{M}{M} = 1$$

$$\sum_{N \preceq M} \binom{M}{N} = \frac{1}{|H|} \sum_{h \in H} 2^{c(h)}$$

$$\binom{XM}{N} = \begin{cases} \binom{M}{N} + \binom{M}{P} & \text{if } N = XP \\ \binom{M}{N} & \text{otherwise} \end{cases}$$

$$\binom{M_1 M_2}{N} = \sum_{N_1 N_2 = N} \binom{M_1}{N_1}\binom{M_2}{N_2}$$

**ETC**

Explicit expressions, applications to combinatorial equations :

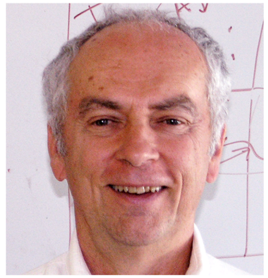
$$E_m(\Phi) = X, \quad E_m^{\pm}(\Phi) = X, \quad C_m(\Phi) = X.$$

Logarithm of the generalized Pascal triangle.

Advances in Applied Mathematics **24,** 22–56 (2000)

# Enumeration of $m$-Ary Cacti

Miklós Bóna, Michel Bousquet, Gilbert Labelle, and Pierre Leroux

The purpose of this paper is to enumerate various classes of cyclically colored $m$-gonal plane cacti, called $m$-ary cacti. This combinatorial problem is motivated by the topological classification of complex polynomials having at most $m$ critical values, studied by Zvonkin and others. We obtain explicit formulae for both labelled and unlabelled $m$-ary cacti, according to (i) the number of polygons, (ii) the vertex-color distribution, (iii) the vertex-degree distribution of each color. We also enumerate $m$-ary cacti according to the order of their automorphism group. Using a generalization of Otter's formula, we express the species of $m$-ary cacti in terms of rooted and of pointed cacti. A variant of the $m$-dimensional Lagrange inversion is then used to enumerate these structures. The method of Liskovets for the enumeration of unrooted planar maps can also be adapted to $m$-ary cacti.

$$\mathbf{n}_1 = (0,7,1,0,1,0,\ldots) = 1^7 2^1 4^1, \qquad \mathbf{n}_2 = (0,7,3,0,0,0,\ldots) = 1^7 2^3,$$

$$\mathbf{n}_3 = (0,8,1,1,0,0,\ldots) = 1^8 2^1 3^1, \qquad \mathbf{n}_4 = (0,9,2,0,0,0,\ldots) = 1^9 2^2,$$

$$n_1 = 9, \quad n_2 = 10, \quad n_3 = 10, \quad n_4 = 11, \quad n = 40, \quad \text{and } p = 13.$$

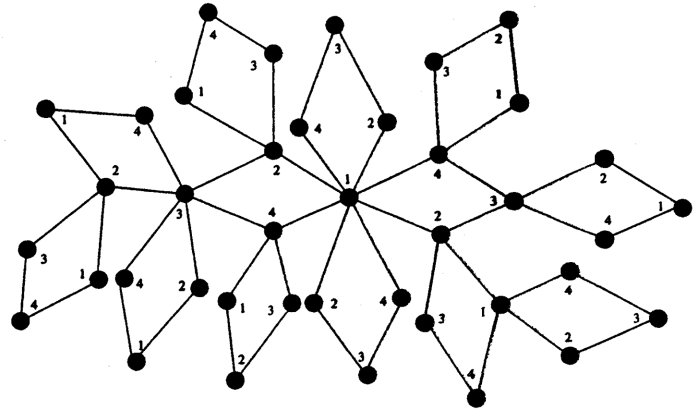


FIG. 1. A quaternary cactus.

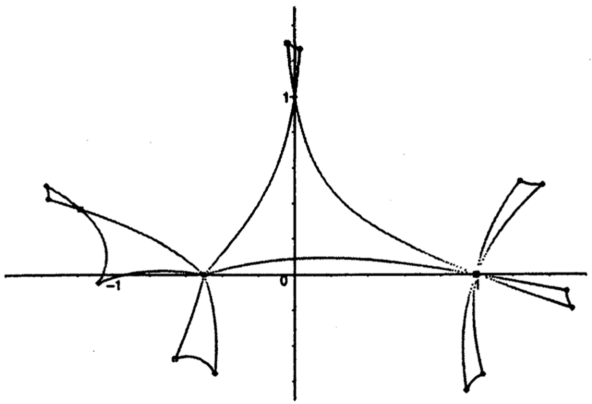


FIG. 2. Cactus associated to a polynomial of degree 8, having three critical values.

We consider the class $\mathscr{K}$ of $m$-ary cacti as an $m$-sort species.

$\mathscr{A}_i$ : $m$-ary cacti, *planted* at a vertex of color $i$,

$\mathscr{K}^{\bullet_i}$: $m$-ary cacti, *pointed* at a vertex of color $i$,

$\mathscr{K}^{\diamond}$: *rooted* $m$-ary cacti, $\hat{\mathscr{A}}_i := \prod_{j \neq i} \mathscr{A}_j$ .

PROPOSITION 1. *We have the following isomorphisms of species, for* $i = 1, \ldots, m$

$$\mathscr{A}_i = X_i L\big(\hat{\mathscr{A}}_i\big), \tag{5}$$

$$\mathscr{K}^{\bullet_i} = X_i\Big(1 + \mathscr{C}\big(\hat{\mathscr{A}}_i\big)\Big), \tag{6}$$

$$\mathscr{K}^{\diamond} = \mathscr{A}_1 \mathscr{A}_2 \cdots \mathscr{A}_m . \tag{7}$$

THEOREM 2 Dissymmetry theorem for $m$-ary cacti. *There is an isomorphism of species*

$$\mathscr{K}^{\bullet_1} + \mathscr{K}^{\bullet_2} + \cdots + \mathscr{K}^{\bullet_m} = \mathscr{K} + (m-1)\mathscr{K}^{\diamond} . \tag{8}$$

**ETC**

Applications to the enumeration of labelled and unlabelled m-ary cacti according to face and degree distributions.

ELSEVIER

Discrete Mathematics 222 (2000) 1–25

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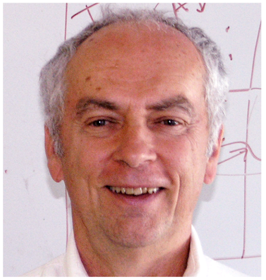
# Enumeration of planar two-face maps

Michel Bousquet, Gilbert Labelle, Pierre Leroux





**Résumé**

Nous dénombrons les cartes planaires (à homéomorphisme préservant l'orientation près) non pointées à deux faces, selon le nombre de sommets et selon la distribution des degrés des sommets et des faces, étiquetées (aux sommets) ou non. Nous abordons d'abord les cartes planes, c'est-à-dire plongées dans le plan, et déduisons ensuite le cas des cartes planaires (ou sphériques), plongées sur la sphère. Une étape cruciale est le dénombrement des cartes planes à deux faces admettant une symétrie antipodale et la méthode de Liskovets est utilisée pour cela. La motivation de cette recherche provient de la classification topologique des fonctions de Belyi.

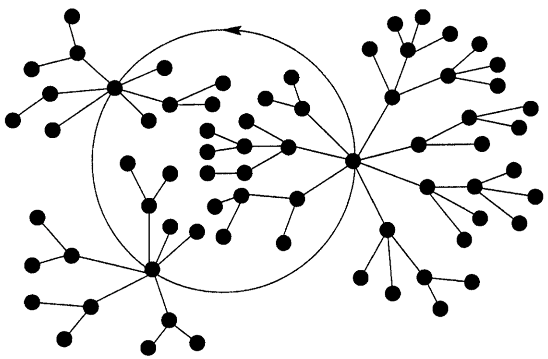


Fig. 2. A two-face plane map.

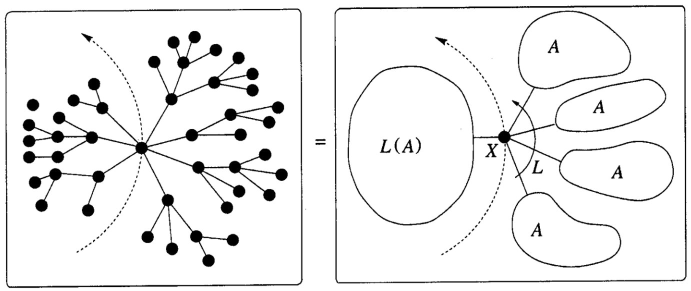



Fig. 3. An $XL^2(A)$-structure.

**Theorem 2.** *The species* $\mathsf{M}$ *of two-face plane maps satisfies the following combinatorial identity*:

$$\mathsf{M} = C(XL^2(A)).$$

**Theorem 3.** *The numbers $|\mathsf{M}_n|$ and $|\tilde{\mathsf{M}}_n|$ of labelled and unlabelled two-face plane maps on $n$ vertices are respectively given by*

$$|\mathsf{M}_n| = \frac{(n-1)!}{2}\left(2^{2n} - \binom{2n}{n}\right) \quad \textit{and} \quad |\tilde{\mathsf{M}}_n| = \frac{1}{2n}\sum_{d|n} \phi\left(\frac{n}{d}\right)\left(2^{2d} - \binom{2d}{d}\right).$$

**Theorem 7.** *Let $\boldsymbol{d}$ satisfy $||\boldsymbol{d}|| = 2|\boldsymbol{d}|$ and $\alpha, \beta > 0$ be two integers having the same parity, where $|\boldsymbol{d}| = (\alpha+\beta)/2 = n$. Then the number $|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}|$ of labelled two-face plane maps on $[n]$ having joint vertex and face degree distributions $\boldsymbol{d}$ and $(\alpha,\beta)$ is given by*

$$|\mathsf{M}_{\boldsymbol{d},(\alpha,\beta)}| = n!H(\boldsymbol{d},(\alpha,\beta)), \tag{44}$$

*and the corresponding number $|\tilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}|$ of unlabelled two-face plane maps is given by*

$$|\tilde{\mathsf{M}}_{\boldsymbol{d},(\alpha,\beta)}| = \sum_{m|(\boldsymbol{d},\alpha,\beta)} \frac{\phi(m)}{m} H\left(\frac{\boldsymbol{d}}{m}, \left(\frac{\alpha}{m}, \frac{\beta}{m}\right)\right) \tag{45}$$

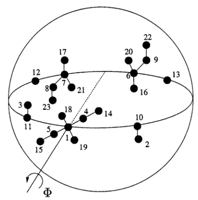
*with*

$$H(\boldsymbol{d},(\alpha,\beta)) = \sum_{\boldsymbol{g},\boldsymbol{h},\boldsymbol{k}} \frac{\Phi(\boldsymbol{h})\Phi(\boldsymbol{k})\Theta(\boldsymbol{g},\boldsymbol{h})}{|\boldsymbol{g}|}\binom{|\boldsymbol{g}|}{\boldsymbol{g}}\binom{|\boldsymbol{h}|}{\boldsymbol{h}}\binom{|\boldsymbol{k}|}{\boldsymbol{k}},$$

*where the sum runs over all $\boldsymbol{g}, \boldsymbol{h}$ and $\boldsymbol{k}$ satisfying conditions* 1–5 *in* (42).

ETC

**Sphere maps**



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Discrete Mathematics 246 (2002) 177–195

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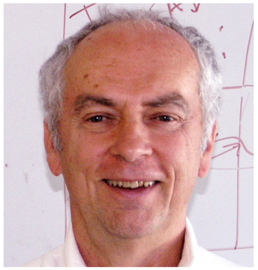
# Stirling numbers interpolation using permutations with forbidden subsequences

G. Labelle, P. Leroux, E. Pergola, R. Pinzani

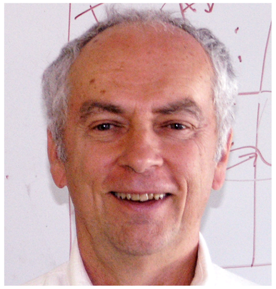
**Résumé**

Nous présentons une famille de suites de nombres qui interpole entre la suite $B_n$ des nombres de Bell et la suite $n!$. Cette famille est définie en termes de permutations à motifs interdits. L'introduction comme paramètre du nombre d'éléments saillants minimums de gauche à droite donne une interpolation plus fine entre les nombres de Stirling de deuxième espèce $S(n,m)$ et de première espèce (sans signe) $c(n,m)$. De plus, un $q$-comptage de ces permutations selon des inversions particulières donne une interpolation entre des variantes des $q$-analogues habituels de ces nombres.

Advances in Applied Mathematics **28**, 302–342 (2002)

# Combinatorial Addition Formulas and Applications

Pierre Auger, Gilbert Labelle, and Pierre Leroux



Nous obtenons des formules d'addition combinatoires, c'est-à-dire des équations de la forme $F(X_1 + X_2 + \cdots + X_k) = \Phi_F(X_1, X_2, \ldots, X_k)$, où $F = F(X)$ est une espèce de structures donnée et $\Phi_F$ est une espèce, dépendant de $F$, sur $k$ sortes de singletons $X_1, X_2, \ldots, X_k$. Nous donnons des formules générales pour les espèces moléculaires $M = X^n/H$ et des résultats plus spécifiques dans le cas des espèces $L_n$, des $n$-listes (listes de longueur $n$), $\mathrm{Cha}_n$, des $n$-chaînes, $E_n$, des $n$-ensembles, $E_n^{\pm}$, des $n$-ensembles orientés, $C_n$, des $n$-cycles (orientés), et $P_n$, des $n$-gones (cycles non orientés). Ces formules sont utiles pour calculer le développement moléculaire d'espèces définies par des équations fonctionnelles. Nous présentons également des applications au calcul des séries indicatrices des cycles ou d'asymétrie, à l'extension de la substitution aux espèces virtuelles (et aux $\mathbb{K}$-espèces), et à l'analyse des coefficients binomiaux généralisés $\binom{M}{N}_k$ pour les espèces moléculaires.

Addition formulas in classical analysis.

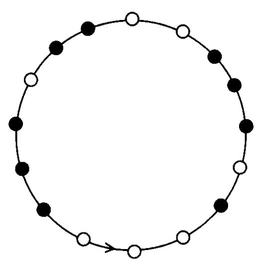
$$e^{x+y} = e^x \cdot e^y, \qquad \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

$$(x+y+z)^n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} x^i y^j z^k.$$

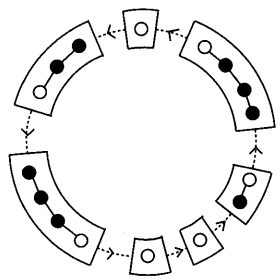
DEFINITION 1.1. Let $F = F(X)$ be a combinatorial species of one sort. A *combinatorial addition formula* for the species $F$ is a combinatorial equation of the form

$$F(X_1 + X_2 + \cdots + X_k) = \Phi_F(X_1, X_2, \ldots, X_k), \tag{1.1}$$

where $\Phi_F$ is a species, depending on $F$, on the $k$ sorts of singletons $X_1$, $X_2, \ldots, X_k$, $k \geq 2$.



$$C(X+Y) = C(X) + C(L(X)Y)$$



Molecular addition formulas : $M = M(X)$ molecular

$$M(X_1 + X_2 + \cdots + X_k) = \sum_N c_{M,N} N(X_1, X_2, \ldots, X_k)$$

Classical multinomial formula : $M(X) = X^n$

$$(X_1 + \cdots + X_k)^n = \sum_{n_1+\cdots+n_k=n} \frac{n!}{n_1! \cdots n_k!} X_1^{n_1} \cdots X_k^{n_k}$$

THEOREM 2.1 (Generalized multinomial expansion). *Let $M = M(X)$ be a molecular species of degree $n$; then*

$$M(X_1 + \cdots + X_k) = \sum_{n_1+\cdots+n_k=n} \sum_{s \in M[n]} \frac{|\mathrm{Aut}(s)_{n_1,\ldots,n_k}|}{n_1! \cdots n_k!} \frac{X_1^{n_1} \cdots X_k^{n_k}}{\mathrm{Aut}(s)_{n_1,\ldots,n_k}},$$

*where*

$$M[n] = \textit{ the set of all } M\textit{-structures on } [n],$$

$$\mathrm{Aut}(s) = \{\sigma \in S_n \mid \sigma \textit{ is an automorphism of } s\} \leq S_n.$$

THEOREM 2.2 ($k$-colored molecular expansion). *Let $M=M(X)=X^n/H$ be a molecular species with $H \leq S_n$. Then*

$$M(X_1+\cdots+X_k)=\sum_{U_1+\cdots+U_k=[n]} \frac{|H_{U_1,\ldots,U_k}|}{|H|}\frac{X_1^{U_1}\cdots X_k^{U_k}}{H_{U_1,\ldots,U_k}},$$

*where $X_1^{U_1}X_2^{U_2}\cdots X_k^{U_k}/H_{U_1,U_2,\ldots,U_k}$ is the molecular species whose set of structures, on $[V_1,V_2,\ldots,V_k]$, is defined to be the set*

$$\{\lambda H_{U_1,U_2,\ldots,U_k} \mid \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k),\ \lambda_i\colon U_i \xrightarrow{\sim} V_i\}.$$

THEOREM 2.3 (Word class expansion). *Let $M=M(X)=X^n/H$ be a molecular species with $H \leq S_n$. Then*

$$M(X_1+\cdots+X_k)=\sum_{n_1+\cdots+n_k=n}\ \sum_{\alpha\in\Omega_{n_1,\ldots,n_k}} \frac{X_1^{n_1}\cdots X_k^{n_k}}{\sigma_\alpha(\mathrm{Aut}_H\,\alpha)\sigma_\alpha^{-1}},$$

*where $\Omega_{n_1,n_2,\ldots,n_k}$ is a set of representatives (for example, lexicographically smallest words) of the $H$-classes of words on $\mathscr{X}=\{1,2,\ldots,k\}$ having exactly $n_i$ occurrences of letter $i$ for $i=1,2,\ldots,k$.*

**ETC**

Examples : $n$-lists, $n$-chains, $n$-sets, $n$-oriented sets, $n$-cycles, $n$-gons, ...

Applications : functional eqns, identities $1-x+y=\prod_{i+j>0}(1-x^iy^j)^{-\alpha_{i,j}}$.