Determinants

Areas, Volumes and Cross Products

DEFINITION: (i) The **determinant of** 1×1 **matrix** is its sole entry.
(ii)The **determinant of a** 2×2 **matrix** is given by

$$det(A) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2, \qquad \text{where } A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

(iii) The **determinant of a** 3×3 **matrix** is given by

$$det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

DEFINITION: If $\mathbf{a} = [a_1, a_2, a_3], \mathbf{b} = [b_1, b_2, b_3] \in \mathbb{R}^3$ the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right] = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Note: $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

THEOREM: (i) If a parallelogram is determined by two nonzero vectors, $\mathbf{a}=[a_1, a_2]$ and $\mathbf{b}=[b_1, b_2]$ in $\mathbb{R}^2$, then its area is given by

$$Area = |a_1b_2 - a_2b_1| = \left|det\left(\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right)\right| = \left|\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right|$$

(ii) If a parallelogram is determined by two nonzero vectors, $\mathbf{a}=[a_1, a_2, a_3]$ and $\mathbf{b}=[b_1, b_2, b_3]$ in $\mathbb{R}^3$, then its area is given by $\|\mathbf{a} \times \mathbf{b}\|$.

(iii) If a parallelepiped is determined by three nonzero vectors $\mathbf{a}=[a_1, a_2, a_3]$, $\mathbf{b}=[b_1, b_2, b_3]$ and $\mathbf{c}=[c_1, c_2, c_3]$ in $\mathbb{R}^3$, then the volume of the box is given by

$$Volume = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

The Determinant of a Square Matrix

DEFINITION: The determinant of 1×1, 2×2 and 3×3 matrices are defined. Let $n>1$ and suppose that the determinant of $(n-1)\times(n-1)$ matrices is defined. Let $A=[a_{ij}]$ be an $n\times n$ matrix.

1. The **minor** matrix A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by removing the ith row and jth column of A.

2. The **cofactor** of a_{ij} of A is

$$a'_{ij}=(-1)^{i+j}det(A_{ij})=(-1)^{i+j}|A_{ij}|$$

3. The determinant of A is

$$\begin{aligned}
det(A) &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= a_{11}a'_{11}+a_{12}a'_{12}+\cdots+a_{1n}a'_{1n} \\
&= \sum_{i=1}^{n} a_{1i}a'_{1i}=\sum_{i=1}^{n}(-1)^{(i+1)}a_{1i}|A_{1i}| \qquad \text{expansion along first row} \\
&= \sum_{i=1}^{n} a_{ji}a'_{ji}=\sum_{i=1}^{n}(-1)^{(i+j)}a_{ji}|A_{ji}| \qquad \text{expansion along } j\text{th row} \\
&= \sum_{i=1}^{n} a_{ij}a'_{ij}=\sum_{i=1}^{n}(-1)^{(i+j)}a_{ij}|A_{ij}| \quad \text{expansion along } j\text{th column}
\end{aligned}$$

THEOREM: Let A, C be $n \times n$ matrices. Then:

1. $det(A) = det(A^T)$.

2. $A \; {\scriptstyle R_i \longleftrightarrow R_j} \; B$, then $det(B) = -det(A)$

3. $A \; {\scriptstyle R_i \longrightarrow rR_i} \; B$, then $det(B) = r \; det(A)$

4. $A \; {\scriptstyle R_i \longrightarrow R_i + rR_j} \; B$, then $det(B) = det(A)$

5. If A contains proportional rows (or columns), then $det(A) = 0$.

6. A is invertible $\iff$ $det(A) \neq 0$.

7. $det(AB) = det(A)det(B)$.

8. If A is invertible, then $det(A^{-1}) = \dfrac{1}{det(A)}$.

9. If A is a triangular matrix, then $det(A)$ is the product of all of its entries along the main diagonal.

Cramer's Rule

THEOREM: If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns and $det(A) \neq 0$, then the unique solution $\mathbf{x} = [x_1, x_2, \cdots, x_n]$ is of the form

$$x_k = \frac{det(B_k)}{det(A)} \quad \text{for } k = 1, \cdots n$$

where B_k is the matrix A with the k^{th} column replaced by the column vector $\mathbf{b}$.

EXAMPLE **0.0.1** *Use Cramer's Rule to solve the system:*

$$\begin{array}{lllllll} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & - & x_2 & & & = & 11 \\ & & x_2 & + & 4x_3 & = & 3 \end{array}$$

DEFINITION: Let A be an $n \times n$ matrix.

1. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and j^{th} column of A.

2. The **cofactor** of a_{ij} of A is

$$a'_{ij} = (-1)^{i+j} det(A_{ij}) = (-1)^{i+j} |A_{ij}|$$

3. Let $A' = [\ a'_{ij}\ ]$ be the matrix with ij^{th} entry the ij^{th} cofactor of A. Then the **adjoint of** A denoted by $adj(A)$ is the $n \times n$ matrix

$$adj(A) = (A')^T$$

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THEOREM: If A is an $n \times n$ matrix then

$$(adj(A))\, A = A\, (adj(A)) = det(A) I$$

Corollary: If A is an $n \times n$ matrix and $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(A)} adj(A).$$