How to enjoy a mathematical discussion with your laptop

Filippo Alberto Edoardo Nuccio
Mortarino Majno di Capriglio

Université Jean Monnet Saint-Étienne – France

Computer-verified proofs: 48 hours in Rome – January 24st 2024

Thank you

Your support powers our independent journalism

The Guardian

News | **Opinion** | **Sport** | **Culture** | **Lifestyle** | More

Fashion Food Recipes Love & sex Health & fitness Home & garden Women Men Family Travel Money

Dining across the divide

Can breaking bread together help bridge political differences?

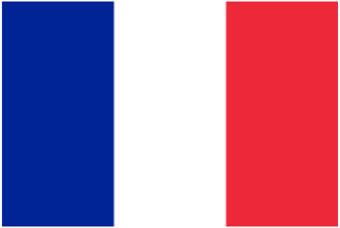
7 December 2023

Dining across the divide: ‘I enjoyed his company. But afterwards thought, hang on a moment … ’

Would a Labour-voting gardener who has worked as a life model and a Tory who’s eyeing up Reform UK find common ground?

1:30 PM

What is `Lean`?



Un assistant de preuve

(= *a proof assistant*)

A theorem prover

```
/-- **Cauchy integral formula**: if `f` is continuous on a
punctured closed disc of radius `R`, is differentiable at all but countably many points of the
interior of this disc, and has a limit `y` at the center of the disc, then the integral
$\oint_{‖z-c‖=R} \frac{f(z)}{z-c}\,dz$ is equal to `2πiy`. -/
theorem Cauchy_formula {c : ℂ}
    {R : ℝ} (h0 : 0 < R) {f : ℂ → ℂ} {y : ℂ} {s : Set ℂ} (hs : s.Countable)
    (hc : ContinuousOn f (closedBall c R \ {c}))
    (hd : ∀ z ∈ (ball c R \ {c}) \ s, DifferentiableAt ℂ f z) (hy : Tendsto f (𝓝[{c}ᶜ] c) (𝓝 y))
    (∮ z in C(c, R), (z - c)⁻¹ • f z) = (2 * π * I : ℂ) • y := by
  rw [← sub_eq_zero, ← norm_le_zero_iff]
  refine' le_of_forall_le_of_dense fun ε ε0 => _
  obtain ⟨δ, δ0, hδ⟩ : ∃ δ > (0 : ℝ), ∀ z ∈ closedBall c δ \ {c}, dist (f z) y < ε / (2 * π)
  exact ((nhdsWithin_hasBasis nhds_basis_closedBall _).tendsto_iff nhds_basis_ball).1 hy _
    (div_pos ε0 Real.two_pi_pos)
  obtain ⟨r, hr0, hrδ, hrR⟩ : ∃ r, 0 < r ∧ r ≤ δ ∧ r ≤ R :=
    ⟨min δ R, lt_min δ0 h0, min_le_left _ _, min_le_right _ _⟩
  have hsub : closedBall c R \ ball c r ⊆ closedBall c R \ {c} :=     You, seconds ago • Uncommitt
    diff_subset_diff_right (singleton_subset_iff.2 <| mem_ball_self hr0)
  have hsub' : ball c R \ closedBall c r ⊆ ball c R \ {c} :=
    diff_subset_diff_right (singleton_subset_iff.2 <| mem_closedBall_self hr0.le)
  have hzne : ∀ z ∈ sphere c r, z ≠ c := fun z hz =>
    ne_of_mem_of_not_mem hz fun h => hr0.ne' <| dist_self c ▸ Eq.symm h
  /- The integral `∮ z in C(c, r), f z / (z - c)` does not depend on `0 < r ≤ R` and tends to
    `2πIy` as `r → 0`. -/
  calc
    ‖(∮ z in C(c, R), (z - c)⁻¹ • f z) - (2 * ↑π * I) • y‖ =
        ‖(∮ z in C(c, r), (z - c)⁻¹ • f z) - ∮ z in C(c, r), (z - c)⁻¹ • y‖ := by
      congr 2
      · exact circleIntegral_sub_center_inv_smul_eq_of_differentiable_on_annulus_off_countable hr0
          hrR hs (hc.mono hsub) fun z hz => hd z ⟨hsub' hz.1, hz.2⟩
      · simp only [circleIntegral.integral_smul_const, ne_eq, hr0.ne', not_false_eq_true,
          circleIntegral.integral_sub_center_inv]
    _ = ‖∮ z in C(c, r), (z - c)⁻¹ • (f z - y)‖ := by
      simp only [smul_sub]
      have hc' : ContinuousOn (fun z => (z - c)⁻¹) (sphere c r) :=
        (continuousOn_id.sub continuousOn_const).inv₀ fun z hz => sub_ne_zero.2 <| hzne _ hz
      rw [circleIntegral.integral_sub] <;> refine' (hc'.smul _).circleIntegrable hr0.le
      · exact hc.mono <| subset_inter
```

```
▼ Colloquium.lean:49:65
 ▼ Tactic state
  1 goal
  c : ℂ
  R : ℝ
  h0 : 0 < R
  f : ℂ → ℂ
  y : ℂ
  s : Set ℂ
  hs : Set.Countable s
  hc : ContinuousOn f (closedBall c R \ {c})
  hd : ∀ z ∈ (ball c R \ {c}) \ s, DifferentiableAt ℂ f z
  hy : Tendsto f (𝓝[≠] c) (𝓝 y)
  ε : ℝ
  ε0 : 0 < ε
  δ : ℝ
  δ0 : δ > 0
  hδ : ∀ z ∈ closedBall c δ \ {c}, dist (f z) y < ε / (2 * π)
  r : ℝ
  hr0 : 0 < r
  hrδ : r ≤ δ
  hrR : r ≤ R
  hsub : closedBall c R \ ball c r ⊆ closedBall c R \ {c}
  ⊢ ‖(∮ (z : ℂ) in C(c, R), (z - c)⁻¹ • f z) - (2 * ↑π * I) • y‖ ≤
▶ All Messages (1)
```

A game inspired by P. Massot:

You are studying Diophantus' *Arithmetica*: you don't understand some detail in the proof that

$$m = 4n + 3 \Longrightarrow m \text{ is not the sum of two squares.}$$

A game inspired by P. Massot:

You are studying Diophantus' *Arithmetica*: you don't understand some detail in the proof that

$$m = 4n + 3 \Longrightarrow m \text{ is not the sum of two squares.}$$

Year	Location	
A. D. 265	Alexandria, Egypt	Everywhere else
	Ask Diophantus	???

A game inspired by P. Massot:

You are studying Diophantus' *Arithmetica*: you don't understand some detail in the proof that

$$m = 4n + 3 \Longrightarrow m \text{ is not the sum of two squares.}$$

Year	Location	
A. D. 265	Alexandria, Egypt	Everywhere else
	Ask Diophantus	???
A. D. 500	Alexandria, Egypt	Everywhere else
	Find someone whose grand-grand-grand-grandparents met Diophantus	???

A game inspired by P. Massot:

You are studying Diophantus' *Arithmetica*: you don't understand some detail in the proof that

$$m = 4n + 3 \Longrightarrow m \text{ is not the sum of two squares.}$$

Year	Location	
A. D. 265	Alexandria, Egypt	Everywhere else
	Ask Diophantus	???
A. D. 500	Alexandria, Egypt	Everywhere else
	Find someone whose grand-grand-grand-grandparents met Diophantus	???
A. D. 1910	Cambridge, UK	Everywhere else
	Read 379 pages of Principia Mathematica to be sure that $1+1=2$	Go to the local bookshop

A game inspired by P. Massot:

You are studying Diophantus' *Arithmetica*: you don't understand some detail in the proof that

$$m = 4n + 3 \Longrightarrow m \text{ is not the sum of two squares.}$$

Year	Location	
A. D. 265	Alexandria, Egypt	Everywhere else
	Ask Diophantus	???
A. D. 500	Alexandria, Egypt	Everywhere else
	Find someone whose grand-grand-grand-grandparents met Diophantus	???
A. D. 1910	Cambridge, UK	Everywhere else
	Read 379 pages of Principia Mathematica to be sure that $1 + 1 = 2$	Go to the local bookshop
A. D. 2024	*Any better?*	

Liquid Tensor Experiment

Posted on December 5th 2020

I [Peter Scholze] want to propose a challenge: Formalise the proof of the following theorem.

Theorem (Clausen–S.)

Let $0 < p' < p \leq 1$ be real numbers, let S be a profinite set, and let V be a p-Banach space. Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S. Then

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$.

Liquid Tensor Experiment

Posted on December 5th 2020

I [Peter Scholze] want to propose a challenge: Formalise the proof of the following theorem.

Theorem (Clausen–S.)

Let $0 < p' < p \leq 1$ *be real numbers, let* S *be a profinite set, and let* V *be a* p*-Banach space. Let* $\mathcal{M}_{p'}(S)$ *be the space of* p'*-measures on* S*. Then*

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$*.*

Why do I want a formalisation?

Liquid Tensor Experiment

Posted on December 5th 2020

- ...
- [...] In the end, we were able to get an argument pinned down on paper, but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts.
- [...] I think the theorem is of utmost foundational importance, so being 99.9% sure is not enough.
- I have occasionally been able to be very persuasive even with wrong arguments.

Liquid Tensor Experiment

Posted on December 5th 2020

- ...
- [...] In the end, we were able to get an argument pinned down on paper, but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts.
- [...] I think the theorem is of utmost foundational importance, so being 99.9% sure is not enough.
- I have occasionally been able to be very persuasive even with wrong arguments.

Completed on July 14th 2022

with **J. Commelin, A. Topaz**, R. Barton, A. Best, R. Brasca, K. Buzzard, Y. Dillies, F. van Doorn, F. Glöckle, M. Himmel, H. Macbeth, P. Massot, B. Mehta, S. Morrison, F. N., J. Riou, D. Testa, A. Yang

Some formalised results

- In mathlib: a lot of algebra, analysis, topology, . . .

Some formalised results

- In mathlib: a lot of algebra, analysis, topology, . . .
- The definition of perfectoid spaces, *October 2019*

  K. Buzzard, J. Commelin, P. Massot

Some formalised results

- In mathlib: a lot of algebra, analysis, topology, ...
- The definition of perfectoid spaces, *October 2019*

  K. Buzzard, J. Commelin, P. Massot
- The Liquid Tensor Experiment, *July 2022*

Some formalised results

- In mathlib: a lot of algebra, analysis, topology, ...
- The definition of perfectoid spaces, *October 2019*

  K. Buzzard, J. Commelin, P. Massot
- The Liquid Tensor Experiment, *July 2022*
- The sphere eversion theorem, *November 2022*

  P. Massot, O. Nash, F. van Doorn

Some formalised results

- In mathlib: a lot of algebra, analysis, topology, . . .
- The definition of perfectoid spaces, *October 2019*

  K. Buzzard, J. Commelin, P. Massot
- The Liquid Tensor Experiment, *July 2022*
- The sphere eversion theorem, *November 2022*

  P. Massot, O. Nash, F. van Doorn
- Fermat's Last Theorem for regular primes, *December 2023*

  R. Brasca, A. Best, C. Birkbeck, E. Rodriguez, R. Van de Velde, A. Yang

Some formalised results

- In mathlib: a lot of algebra, analysis, topology, . . .
- The definition of perfectoid spaces, *October 2019*

  K. Buzzard, J. Commelin, P. Massot
- The Liquid Tensor Experiment, *July 2022*
- The sphere eversion theorem, *November 2022*

  P. Massot, O. Nash, F. van Doorn
- Fermat's Last Theorem for regular primes, *December 2023*

  R. Brasca, A. Best, C. Birkbeck, E. Rodriguez, R. Van de Velde, A. Yang
- More and more papers formalizing mathematics in Lean

Want to get involved?

Want to get involved?

JFK Ask not what Lean can do for you — ask what you can do for Lean.

Want to get involved?

JFK Ask not what Lean can do for you — ask what you can do for Lean.

- There is a Zulip thread where we discuss, we ask questions, we have fun.

Want to get involved?

JFK Ask not what Lean can do for you — ask what you can do for Lean.

- There is a Zulip thread where we discuss, we ask questions, we have fun.
- Enjoy 48 hours in Rome chatting about computer-verified proofs!

Thank you