Formalising modern research mathematics

Kevin Buzzard, Imperial College London

Computer-verified proofs: 48 hours in Rome, 26th January 2024

Before we start

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In my mind, currently the most exciting thing about this area is that five years ago, the third item seemed like science fiction, but now it is happening.

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In an interactive theorem prover, you can *prove*.

For example, you can prove that there are infinitely many prime numbers, or say "let G be a group".

History: calculating vs proving.

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Proving:
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You can guess which application caught on amongst mathematicians.

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This had a *huge* effect on the kind of mathematics which was being done in them.

ITPs in the 21st century.

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To the computer scientists: a breakthrough result (discrete and continuous mathematics being handled by the same system).

To the mathematicians: a completely standard 100 year old result which is in an undergraduate or MSc curriculum.

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Again, a complex theorem about simple mathematical objects.

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So I suggested "definition of a scheme", which took us two months.

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But this was not my "research" – this was just a fun project!

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Schemes were not complex enough.

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We decided to go for it.

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A bunch of Italian number theorists showed up on the Lean chat :-)

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The paper was four pages long :-)

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The challenge: prove the “fundamental theorem of liquid vector spaces”.

This was a theorem of Clausen and Scholze, announced in 2019 (and still not published AFAIK?)

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Commelin decided to gamble.

The Liquid Tensor Experiment

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Commelin, Topaz, myself, Nuccio, and many others got to work.

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If I have internet, let’s take a look at it.

LaTeX

Blueprint for the Liquid Tensor Experiment

error term a.

As preparation for the proof, we have the following results.

Lemma 1.6.5 (Gordan's lemma)✓

Let Λ be a finite free abelian group, and let $\lambda_1, \ldots, \lambda_m \in \Lambda$ be elements. Let $M \subset \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the submonoid $\{x \mid x(\lambda_i) \geq 0 \text{ for all } i = 1, \ldots, m\}$. Then M is finitely generated as monoid.

Proof ▾

This is a standard result. We omit the proof here. It is done in Lean.

Lemma 1.6.6 ✓

Let Λ be a finite free abelian group, let N be a positive integer, and let $\lambda_1, \ldots, \lambda_m \in \Lambda$ be elements. Then there is a finite subset $A \subset \Lambda^\vee$ such that for all $x \in \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ there is some $x' \in A$ such that $x - x' \in N\Lambda^\vee$ and for all $i = 1, \ldots, m$, the numbers $x'(\lambda_i)$ and $(x - x')(\lambda_i)$ have the same sign, i.e. are both nonnegative or both nonpositive.

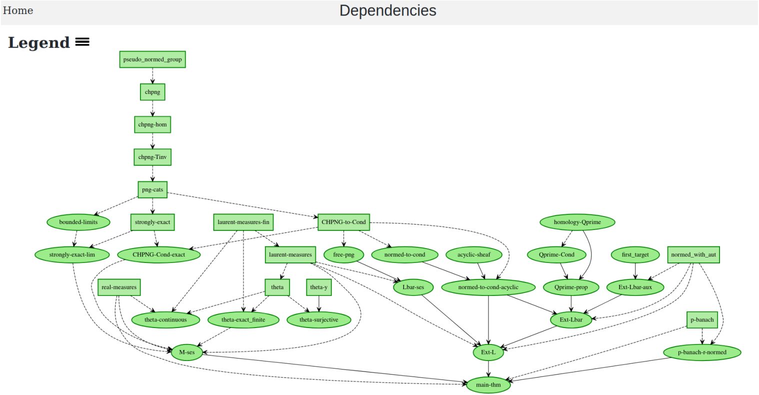
Proof ▾

It suffices to prove the statement for all x such that $\lambda_i(x) \geq 0$ for all i; indeed, applying this variant to all $\pm\lambda_i$, one gets the full statement.

Thus, consider the submonoid $\Lambda_+^\vee \subset \Lambda^\vee$ of all x that pair nonnegatively with all λ_i. This is a finitely generated monoid by Lemma 1.6.5; let $y_1, \ldots, y_M$ be a set of generators. Then we can take for A all sums $n_1 y_1 + \ldots + n_M y_M$ where all $n_j \in \{0, \ldots, N-1\}$.

Lemma 1.6.7 ✓

Graph


Home
Dependencies
Legend
pseudo_normed_group
chpng
chpng-hom
chpng-Tinv
png-cats
bounded-limits
strongly-exact
laurent-measures-fin
CHPNG-to-Cond
homology-Qprime
strongly-exact-lim
CHPNG-Cond-exact
laurent-measures
free-png
normed-to-cond
acyclic-sheaf
Qprime-Cond
first_target
normed_with_aut
real-measures
theta
theta-y
Lbar-ses
normed-to-cond-acyclic
Qprime-prop
Ext-Lbar-aux
theta-continuous
theta-exact_finite
theta-surjective
Ext-Lbar
p-banach
M-ses
Ext-L
p-banach-r-normed
main-thm

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If a node is blue, this means "we need the Lean code for this proof."

If it's green, this means the Lean code is already written.

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If you are a mathematician who knows *nothing* about liquid vector spaces or solid abelian groups, you can still help!

You just find a node which corresponds to a mathematical argument which you understand.

Key fact: The experts do not need to check your proof – the computer checks it for you.

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The next thing I knew, Nature was on the phone asking what the heck was going on.

But the project was not finished: we still had to formalise the five remaining lines in the proof.

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This experience taught the community *many* things.

The art of formalisation

One thing which really hit home for me: this showed us that *formalisation of mathematical definitions is an art*.

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In fact, "how to teach the computer the objects used in modern mathematics" is an *active research area*.

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The homology of $A \to B \to C$ is both a limit and a colimit.

But the experience taught us what we *should* have done.

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And we know it is usable.

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Mehta wrote an appendix to the paper explaining the work.

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But the real bombshells were yet to come.

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You can read about Tao's experience on his blog.

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Conclusions

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- If the prerequisites are *not* in `mathlib`, then you can formalise your result anyway, and *make* `mathlib` *better*.
- For certain topics (parts of analytic number theory, parts of additive combinatorics), the prerequisites are now often there.

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More lessons:

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- *A multi-author project is really good fun*.
- Your line manager might be confused about your work, which might be published in a journal they've never heard of.
- `mathlib` is becoming *enormously powerful* and it is getting better at an *extremely* fast rate.

Fermat's Last Theorem

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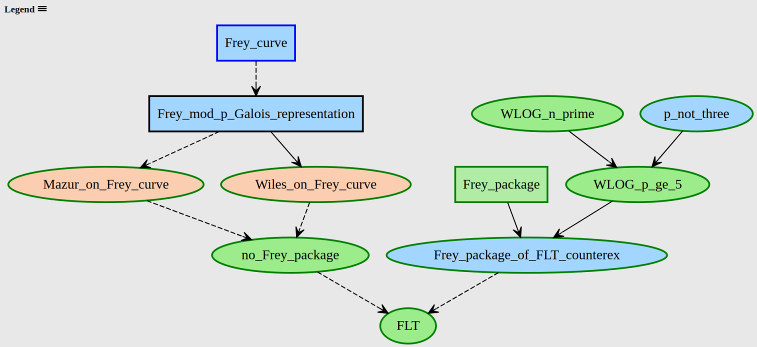
Here is the current state of my (secret) blueprint:

FLT blueprint graph



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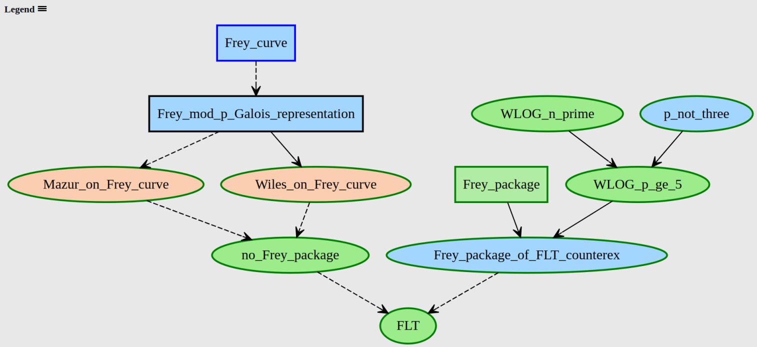

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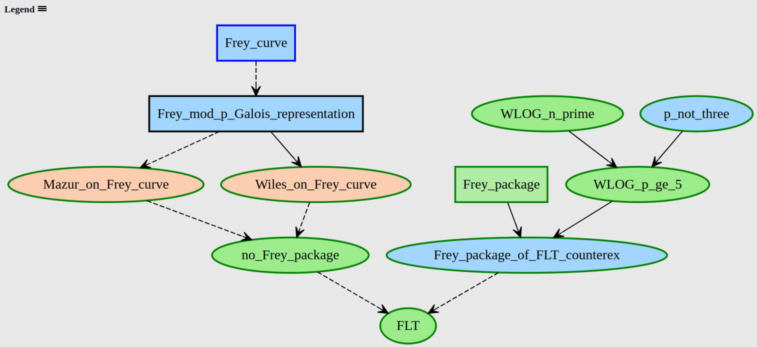

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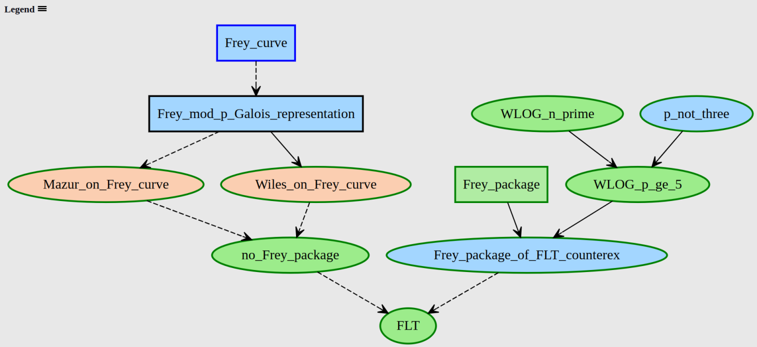

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We’re going live in April.

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Thanks a lot for your time!