Formalisation of the fast-marching algorithm, and of the causality property

Jean-Marie Mirebeau*

February 10, 2023

Abstract

Work in progress : plan for the internship

The fast-marching algorithm sethian1996fast,Tsitsiklis:1995EfficientTrajectories [Set96, Tsi95] allows to numerically compute the arrival time of a front in a domain of $\mathbb{R}^d$, in the presence of obstacles, and where the front speed is dictated locally. It has countless applications, in robotic motion planning, image segmentation, seismic traveltime tomography, etc Since its introduction, it has been extended to the computation of shortest paths on manifolds Kimmel:1998Manifold,Bronstein2007Weighted [KS98, BBK07], or in domains equipped with various types of anisotropy Sethian2003OUM,mirebeau2019riemannian [SV03, Mir19].

From the algorithmical standpoint, the fast marching algorithm is a generalization of Dijkstra's shortest path algorithm on graph, and from the analysis standpoint it solves a discretization of the eikonal equation

$$\forall x \in \Omega, \|\nabla u(x)\| = c(x), \qquad\qquad \forall x \in \partial\Omega, u(x) = u_0(x),$$

with $c \in C^0(\Omega,]0, \infty[)$ and $u_0 : \partial\Omega \to]-\infty, \infty]$. The solutions should be understood in the viscosity sense.

The objective of this internship is to formalize the abstract fast marching algorithm, at least one of the two original instantiations sethian1996fast,Tsitsiklis:1995EfficientTrajectories [Set96, Tsi95], and the proof of convergence of the numerical solution. There are a number of optional openings towards related subjects. The report should include a discussion of the various choices and challenges encountered when formalizing the project, as well as a presentation of the main proofs using the lean formatter[1], which allows to write formal proofs and standard Latex proofs along each other[2].

Task : describes an objective that is part of the main goals of the internship

Optional (easy) : optional task, not expected to raise significant challenges. Intended as an immediate illustration of the main objectives.

Optional (medium) : an optional task, with medium expected difficulty. Intended as an application of the main objectives, or a natural connection with another formalism.

Optional (hard) : an optional task, with possibly significant difficulty. Intended as an extension of the main objectives.

*Université Paris-Saclay, ENS Paris-Saclay, CNRS, Centre Borelli, F-91190 Gif-sur-Yvette, France.

[1] Found here : `https://github.com/leanprover-community/format_lean`

[2] Example : `https://www.ma.imperial.ac.uk/~buzzard/docs/lean/sandwich.html`

1 Abstract schemes and their solutions

sec:abstract

Part of this section is copy-pasted from [Mir19, Appendix A]. Concepts and arguments closely related with Definition 1.1 and Proposition 1.4 can be found in [Tsi95, KS98, SV03, Ber95]. See in particular the definition of the causality property in [SV03], and the proof that a stochastic shortest path problem on a graph can be solved in a single pass if there exists an optimal consistently improving policy in [Ber95, chapter 2].

1.1 Fixed point formalism

Throughout this section, we fix a finite set X, and denote by

$$\mathbb{U} :=]-\infty, +\infty]^X$$

the set of functions on X with either real of $+\infty$ values. Here and below, for any $u, v \in \mathbb{U}$, the notation $u \preceq v$ means $u(x) \leq v(x)$ for all $x \in X$.

Task : Define the objects X and $\mathbb{U}$ in Lean. Define the partial order relation on $\mathbb{U}$.

For any $u \in \mathbb{U}$, and any $t \in [-\infty, +\infty]$, we define the function $u^{<t} \in \mathbb{U}$ (resp. $u^{\leq t} \in \mathbb{U}$) by replacing all values above (resp. strictly above) the threshold t with $+\infty$. Explicitly, for all $x \in X$,

$$u^{<t}(x) := \begin{cases} u(x) & \text{if } u(x) < t, \\ +\infty & \text{otherwise.} \end{cases}$$

Task : Choose a syntactically valid Lean notation for $u^{<\lambda}$ and $u^{\leq\lambda}$, and define these objects in lean.

We next introduce the concept of Monotone and Causal operator on the set X.

otoneOperator

Definition 1.1. *An operator on the set X is a map $\Lambda : \mathbb{U} \to \mathbb{U}$. Its is said*

- Monotone, *iff for all $u, v \in \mathbb{U}$ such that $u \preceq v$ one has $\Lambda u \preceq \Lambda v$.*
- Causal, *iff for all $u, v \in \mathbb{U}$ and $t \in [-\infty, +\infty]$ s.t. $u^{<t} = v^{<t}$ one has $(\Lambda u)^{\leq t} = (\Lambda v)^{\leq t}$.*

Formalize the monotony and causality concepts.

graph_operator

Proposition 1.2 (Graph setting)**.** *Let $X_0 \subset X$, let $u_0 : X_0 \to]-\infty, \infty]$, and let $c : (X \setminus X_0) \times X \to]-\infty, \infty]$. For all $u \in \mathbb{U}$ define*

$$\forall x \in X \setminus X_0,\ \Lambda u(x) := \min_{y \in X} \Big(c(x, y) + u(y) \Big), \qquad \forall x \in X_0,\ \Lambda u(x) = u_0(x). \tag{1}$$

{eq:graph_op

Then Λ is monotone. In addition, Λ is causal $\Leftrightarrow$ ($c(x, y) > 0$ for all $x \in X \setminus X_0$, $y \in X$).

Optional (easy) : Prove and formalize Proposition 1.2.

Optional (medium) : Express (1, left) as the matrix product cu in the $(\min, +)$ algebra.

Remark 1.3 (Interpretation of monotony and causality)**.** *A mapping $u \in \mathbb{U}$ should be interpreted as a collection of arrival times at the points of X. The operator value $\Lambda u(x)$ at $x \in X$ should be interpreted as the earliest possible arrival time at $x \in X$, given the rules encoded in the operator Λ, and the initial starting times u. Thus one can rephrase informally the monotony and causality axioms of Definition 1.1 :*

- *(Monotone) By starting earlier, one arrives earlier.*
- *(Causal) The arrival time at a point can be expressed in terms of earlier arrival times.*

1.1.1 The fast marching algorithm

The fast marching method, presented in abstract form in Algorithm 1, computes in a finite number of steps a fixed point of such an operator, see Proposition 1.4.

p:FastMarching

Algorithm 1 Abstract fast marching method

Initialization $n \leftarrow 0$, $t_0 \leftarrow -\infty$, and $u_0 \leftarrow +\infty$ on X.

While $t_n < \infty$ **do**

$$u_{n+1} \leftarrow \Lambda u_n^{\leq t_n}, \qquad t_{n+1} := \min\{u_{n+1}(x);\ x \in X,\, u_{n+1}(x) > t_n\}, \qquad n \leftarrow n+1.$$

Task : Present a formal proof of this result (termination of the fast marching method)

rop:SinglePass

Proposition 1.4. *Let Λ be a monotone and causal operator on a finite set X, to which the fast marching method is applied, see Algorithm 1. Then for some $N < \#(X)$ one has $t_N = +\infty$, so that the algorithm terminates, and one has $\Lambda u_N = u_N$. Furthermore, $u_n \succeq u_{n+1}$ and $u_n^{\leq t_n} = u_{n+1}^{\leq t_n}$ for each $0 \leq n \leq N$.*

Proof. Note that $t_n \leq t_{n+1}$ for all $0 \leq n < N$, by definition of t_{n+1}. We begin with the proof of the last two announced properties, by induction over $n \in \{0, \cdots, N\}$.

Claim: $u_n \succeq u_{n+1}$. Initialization holds since $u_0 \equiv +\infty \succeq u_1$. Then, for $n \geq 1$, we obtain by induction and Monotony of the operator Λ

$$(u_{n-1} \succeq u_n \text{ and } t_{n-1} \leq t_n) \ \Rightarrow\ u_{n-1}^{\leq t_{n-1}} \succeq u_n^{\leq t_n} \ \Rightarrow\ \Lambda u_{n-1}^{\leq t_{n-1}} \succeq \Lambda u_n^{\leq t_n} \ \Leftrightarrow\ u_n \succeq u_{n+1}.$$

Claim: $u_n^{\leq t_n} = u_{n+1}^{\leq t_n}$. Initialization follows from $t_0 = -\infty$. Then, for $n \geq 1$, we obtain

$$u_{n-1}^{\leq t_{n-1}} = (u_{n-1}^{\leq t_{n-1}})^{<t_n}, \qquad\qquad u_n^{\leq t_{n-1}} = u_n^{<t_n} = (u_n^{\leq t_n})^{<t_n},$$

by definition of t_n. Therefore, by induction and Causality of the operator Λ, we obtain

$$(\Lambda u_{n-1}^{\leq t_{n-1}})^{\leq t_n} = (\Lambda u_n^{\leq t_n})^{\leq t_n} \ \Leftrightarrow\ u_n^{\leq t_n} = u_{n+1}^{\leq t_n}.$$

Conclusion of the proof. The sets $X_n := \{x \in X;\, u_n(x) \leq t_n\}$ are strictly increasing for inclusion as n increases, by the above, and are included in the finite domain X. The algorithm therefore terminates, in at most $\#(X)$ iterations. In the last step we have $t_N = +\infty$ and thus $u_N = u_N^{\leq t_N} = u_{N+1}^{\leq t_N} = u_{N+1} = \Lambda u_N$. □

Remark 1.5 (Complexity of the fast marching algorithm)**.** *The fast marching algorithm requires to evaluate $\Lambda u_n^{\leq t_n}$ and $\Lambda u_{n+1}^{\leq t_{n+1}}$ in consecutive iterations of rank n and $n+1$. Observe that the maps $u_{n+1}^{\leq t_{n+1}}$ and $u_n^{\leq t_n} = u_{n+1}^{\leq t_n}$ differ only at the points $A_{n+1} = \{x \in X;\, u_{n+1}(x) = t_{n+1}\}$ which have been Accepted in this iteration (typically a single point). Therefore the recomputation of the operator Λ can be restricted to the points $x \in X$ such that $\Lambda u(x)$ depends on $u(y)$ for some $y \in A_{n+1}$. Since each point is Accepted only once, and assuming that the operator Λ is defined as the Gauss-Siedel update of a numerical scheme with a bounded stencil, the overall number of elementary local evaluations of the operator Λ (regarded as of unit complexity) in the course of the fast marching algorithm is $\mathcal{O}(KM)$ where $K = \#(X)$ and M is the average stencil size.*

The overall optimal complexity of the fast marching algorithm is $\mathcal{O}(KM + K\ln K)$, where the second term comes from the cost of maintaining a Fibonacci heap of all the Considered points, see e.g. Remark 1.7 in Mirebeau:2014EfficientFinsler *[?].*

A linear complexity $\mathcal{O}(KM)$ can be achieved under a strengthened and quantitative variant of the causality property, using a variant of Dial's algorithm, see Tsitsiklis:1995gh,Vladimirsky:2008LabelSetting *[?, ?].*

Tsitsiklis:1995EfficientTraject

A variant of the fast marching algorithm is described in the seminal paper [Tsi95], under a stronger causality assumption. This algorithm does not require a priority queue, and as a result the complexity is $\mathcal{O}(K(\max u_* - \min u_*)/\delta)$, where u_* is the solution, under standard assumptions (instead of $\mathcal{O}(K \ln K)$ for the usual fast marching method).

FastMarching2

Algorithm 2 Abstract fast marching method, with untidy queue and δ-causality

Initialization $u_0 \leftarrow +\infty$ on X, $t_0 \leftarrow \min\{\Lambda u_0(x) \mid x \in X\}$.
For all $n \geq 0$

$$t_n \leftarrow t_0 + n\delta, \qquad u_{n+1} \leftarrow \Lambda u_n^{\leq t_n}, \qquad n \leftarrow n+1.$$

Definition 1.6 (*Optional :* δ-causality)**.** *An operator Λ on X is said δ-causal, where $\delta \in \mathbb{R}$, iff for all $u, v \in \mathbb{U}$ and $t \in [-\infty, +\infty]$ s.t. $u^{<t} = v^{<t}$ one has $(\Lambda u)^{\leq t+\delta} = (\Lambda v)^{\leq t+\delta}$.*

Optional (easy) : Formalize this definition

Proposition 1.7 (*Optional :* Tsitsilikis variant of the fast marching method)**.** *Let Λ be δ_1-causal, for some $\delta_1 > 0$, and let $(u_n)_{n\geq 0}$ be the iterates of Algorithm 2. Then $u_n \geq u_{n+1}$, and $u_n^{\leq t_n} = u_{n+1}^{\leq t_n}$ for all $n \geq 0$.*

In particular, there exists $N > 0$ such that $t_N \geq \max\{u_N(x) \mid x \in X, u_N(x) < \infty\}$, and one has $\Lambda u_N = u_N$ and $u_n = u_N$ for all $n \geq N$.

Proof. TODO. □

Optional (medium) : Formalize the proof

1.1.2 Using monotony alone (without causality)

The causality property, used by the fast marching algorithm, puts a strong constraint on the structure of the operator Λ. If the operator only satisfies the monotony property, then it is nevertheless possible to approximate a fixed point, under mild additional assumptions. The simplest approach is simply to iterate the operator. More efficient variants include fast-sweeping, adaptive Gauss-Siedel iteration, fast iterative method, ...

Monotony, together with mild assumptions, also allows to establish the comparison principle, which is at the foundation of the convergence analysis.

Definition 1.8. *Let Λ be an operator on a finite set X, and let $\overline{u}, \underline{u} \in \mathbb{U}$. Then :*

- *$\overline{u}$ is a sub-solution iff $\Lambda\overline{u} \geq \overline{u}$. (A strict super-solution if $\Lambda\overline{u} > \overline{u}$ on X.)*
- *$\underline{u}$ is a super-solution iff $\Lambda\underline{u} \leq \underline{u}$. (A strict sub-solution if $\Lambda\overline{u} > \overline{u}$ on X.)*

An operator satisfies the comparison principle if any sub-solution $\overline{u}$ and super-solution $\underline{u}$ satisfy $\overline{u} \leq \underline{u}$.

Optional (easy) : Formalize these definitions. Show that an operator which satisfies the comparison principle has at most one fixed point.

Proposition 1.9 (Global iteration). *Let Λ be a monotone and* continuous *operator on X, and let $\overline{u}_0, \underline{u}_0 : X \to \mathbb{R}$ be a finite sub-solution and a super-solution obeying $\overline{u}_0 \leq \underline{u}_0$. Denote $\overline{u}_n := \Lambda^n \overline{u}_0$ and $\underline{u}_n := \Lambda^n \underline{u}_0$. Then $\overline{u}_0 \leq \cdots \leq \overline{u}_n \leq \underline{u}_n \leq \cdots \leq \underline{u}_0$ for all $n \geq 0$, and $\overline{u}_n$ and $\underline{u}_n$ converge to (possibly distinct) fixed points of Λ.*

If Λ admits at most one fixed point (for instance if it satisfies the comparison principle), and if $\overline{u}_0 \leq u_0 \leq \underline{u}_0$, then $u_n := \Lambda^n u_0$ converges to the unique fixed point.

Optional (easy) : Formalize the proof. Note that the assumption $\overline{u}_0 \leq \underline{u}_0$ is automatically satisfied if Λ satisfies the comparison principle.

Definition 1.10. *An operator Λ on X is sub-additive iff $\Lambda[u+t] \leq \lambda u + t$ for any $u \in \mathbb{U}$, $t \geq 0$.*

Optional (easy) : Formalize sub-additivity. Show that the operator (1) associated to a graph satisfies this condition.

rop:comparison **Proposition 1.11** (Weak comparison principle). *Let Λ be a monotone and sub-additive operator, and let $\overline{u}, \underline{u} \in \mathbb{U}$ be a sub-solution and a super-solution. If either one is strict, then $\overline{u} < \underline{u}$ on X.*

Proof. Let $x \in X$ be such that $t := u(x) - v(x)$ is maximal, so that $u \leq v + t$ and $u(x) = v(x) + t$. Assuming that $t \geq 0$ we obtain $u(x) \leq \Lambda u(x) \leq \Lambda[v+t](x) \leq \Lambda v(x) + t \leq v(x) + t = u(x)$, by monotony and subadditivity. If either the first or last inequality is strict, we obtain a contradiction, thus $t < 0$ and therefore $u < v$ as announced. $\square$

Optional (easy) : Formalize the statement and proof

Corollary 1.12 (Comparison principle). *Let Λ be a monotone and sub-additive operator, such that strict sub-solutions are dense in the set of super-solutions (resp. strict super-solutions are dense in the set of super-solutions). The Λ satisfies the comparison principle.*

Optional (easy) : Formalize the statement and proof. (Needs arguments of continuity)

Optional (easy) : Show that the operator (1) associated to the graph setting satisfies this condition.

1.2 Root finding formalism

1.2.1 Discrete degenerate ellipticity

Part of this section is copy-pasted from mirebeau2019riemannian [Mir19, Appendix A].

Task : Formaliser la definition suivante

MonotoneScheme **Definition 1.13.** *A (finite differences) scheme on a finite set X is a continuous map $\mathfrak{F} : \mathbb{R}^X \to \mathbb{R}^X$, written as*

$$(\mathfrak{F}U)(x) := \mathfrak{F}(x, U(x), (U(x) - U(y))_{y \in X \setminus \{x\}}).$$

The scheme is said:

- *Degenerate elliptic iff $\mathfrak{F}$ is non-decreasing w.r.t. the second and (each of) the third variables.*
- *Causal iff $\mathfrak{F}$ only depends on the positive part of (each of) the third variables.*

A discrete map $U \in \mathbb{R}^X$ is called a sub- (resp. strict sub-, resp. super-, resp. strict super-) solution of scheme $\mathfrak{F}$ iff $\mathfrak{F}U \leq 0$ (resp. $\mathfrak{F}U < 0$, resp. $\mathfrak{F}U \geq 0$, resp. $\mathfrak{F}U > 0$) pointwise on X.

f:graph_scheme **Definition 1.14** (Graph setting)**.** *Let $X_0 \subset X$, $c : X \setminus X_0 \to]0, \infty]$, and $u_0 : X_0 \to \mathbb{R}$. Define for all $u \in \mathbb{R}^X$ and all $x \in X$,*

$$\forall x \in X \setminus X_0,\ Fu(x) := \max_{y \in X} \Big(\frac{u(x) - u(y)}{c(x, y)}\Big)_+ - 1, \qquad \forall x \in X_0, Fu(x) := u(x) - u_0(x),$$

where $a_+ := \max\{a, 0\}$.

Optional (easy) : Show that the scheme of Definition 1.14 is DE and causal.

1.2.2 Gauss-Siedel update of a scheme

The next proposition defines an operator Λ as the local Gauss-Siedel update associated with a numerical scheme F. If the scheme is Degenerate elliptic then the resulting operator is Monotone, and Causality also transfers from the scheme to the operator. This property is known regarding Degenerate ellipticity/Monotony Oberman2006ConvergentDifference [Obe06], but appears to be new for Causality.

Task : Formaliser la proposition suivante

ssSiedelCausal **Proposition 1.15.** *Let F be a numerical scheme on X, which is degenerate elliptic in the sense of Definition 1.13. For each $u \in \mathbb{U}$, and each $x \in X$, the mapping*

$$\lambda \in \mathbb{R} \mapsto f(\lambda) := F(x,\, \lambda,\, (\lambda - u(y))_{y \in X \setminus \{x\}}) \tag{2}$$ {eqdef:fLamb

is non-decreasing. We assume that either (i) there exists a unique $\lambda \in \mathbb{R}$ such that $f(\lambda) = 0$, and we set $\Lambda u(x) := \lambda$, or (ii) for all $\lambda \in \mathbb{R}$ one has $f(\lambda) < 0$, and we set $\Lambda u(x) := +\infty$.

Then Λ is a monotone operator on X, in the sense of Definition 1.1. In addition, if the scheme F is causal, then the operator Λ is causal, in the sense of Definitions 1.13 and 1.1 respectively.

Proof. The fact that (2) is non-decreasing directly follows from the degenerate ellipticity of the scheme F.

Monotony of the operator Λ. Let $u, v \in \mathbb{U}$, and let $x \in X$ be arbitrary. If $u \preceq v$, then

$$F(x,\, \lambda,\, (\lambda - u(y))_{y \neq x}) \geq F(x,\, \lambda,\, (\lambda - v(y))_{y \neq x})$$

for all $\lambda \in \mathbb{R}$, by monotony of scheme F. Thus $\Lambda u(x) \leq \Lambda v(x)$, as announced

Causality of the operator Λ. Let $u, v \in \mathbb{U}$, let $t \in [-\infty, \infty]$, and let $x \in X$. If $u^{<t} \equiv v^{<t}$, then

$$F(x, \lambda, \max\{0, \lambda - u(y)\}_{y \neq x}) = F(x, \lambda, \max\{0, \lambda - v(y)\}_{y \neq x})$$

for all $\lambda \leq t$, since indeed $\max\{0, \lambda - u(y)\} = \max\{0, \lambda - v(y)\}$. Assuming the Causality of scheme F, this implies $F(x, \lambda, (\lambda - u(y))_{y \neq x}) = F(x, \lambda, (\lambda - v(y))_{y \neq x})$ for all $\lambda \leq t$. Thus these two functions either (i) have a common root $\lambda \leq t$, $\lambda = \Lambda u(x) = \Lambda v(x)$, or (ii) have (possibly distinct) roots $\Lambda u(x) > t$, $\Lambda v(x) > t$. Finally $(\Lambda u)^{\leq t} = (\Lambda v)^{\leq t}$, as announced. □

Optional (easy) : Show that the Gauss-Siedel update operator of (**??**) is (1).

1.2.3 Using monotony alone

Perron's proof of the existence of a solution. The following proof is adapted from mirebeau2019riemannian [Mir19, Theorem 2.3].

Definition 1.16. *Let F be a scheme on a finite set X, and let $\overline{u}, \underline{u} \in \mathbb{R}^X$. Then :*

- *$\overline{u}$ is a sub-solution iff $F\overline{u} \leq 0$. (A strict super-solution if $F\overline{u} < 0$.)*
- *$\underline{u}$ is a super-solution iff $F\underline{u} \geq 0$. (A strict sub-solution if $F\overline{u} > 0$.)*

A scheme satisfies the comparison principle if any sub-solution $\overline{u}$ and super-solution $\underline{u}$ satisfy $\overline{u} \leq \underline{u}$.

Optional (easy) : Implement this definition, and show that a scheme which satisfies the comparison principle has at most one fixed point.

Proposition 1.17 (Weak comparison principle)**.** *Let F be a DDE scheme, and let $\underline{u}$ be a strict super-solution, and $\overline{u}$ a sub-solution. If either one is strict, then $\overline{u} < \underline{u}$.*

Proof. Let $x \in X$ be such that $\overline{u}(x) - \underline{u}(x)$ is maximal, so that $\overline{u}(x) - \overline{u}(y) \geq \underline{u}(x) - \underline{u}(y)$ for any $y \in X$. Assuming for contradiction that $\overline{u}(x) \geq \underline{u}(x)$ we obtain $0 \geq \mathfrak{F}\overline{u}(x) \geq \mathfrak{F}\underline{u}(x) > 0$ by degenerate ellipticity of the scheme and by definition of a sub- and strict super-solution. This is a contradiction, hence $\overline{u} \leq \underline{u}$. □

Optional (easy) : Formalize the weak comparison principle, and the comparison principle

Corollary 1.18 (Comparison principle)**.** *Let F be a DDE scheme, such that the set of strict sub-solutions is dense in the set of sub-solutions (resp. the set of strict super-solutions is dense in the set of super-solutions). Then F satisfies the comparison principle.*

Proposition 1.19 (Perron-Frobenius solution to a scheme)**.** *Let F be a DDE and continuous, which admits a sub-solution $\overline{u}$ and a super-solution $\underline{u}$ such that $\overline{u} \leq \underline{u}$. Then F admits a solution $u \in \mathbb{R}^X$, satisfying $\overline{u} \leq u \leq \underline{u}$, and defined for all $x \in X$ as*

$$u(x) := \sup\{\tilde{u}(x) \mid \tilde{u} \text{ sub-solution}, \overline{u} \leq \tilde{u} \leq \underline{u}\}. \tag{3}$$

Proof. Note that $\tilde{u} := \overline{u}$ is a sub-solution satisfying $\overline{u} \leq \tilde{u} \leq \underline{u}$, hence the supremum is over a non-empty set. As a result, u is well defined, and $\overline{u} \leq u \leq \underline{u}$.

Consider an arbitrary $x \in X$, and let $\tilde{u}$ be a sub-solution such that $u(x) = \tilde{u}(x)$, which exists by continuity of $\mathfrak{F}$ and a compactness argument. By construction $u \geq \tilde{u}$, hence $\mathfrak{F}u(x) \leq \mathfrak{F}\tilde{u}(x) \leq 0$ by monotony of the scheme, hence u is a sub-solution by arbitraryness of $x \in X$.

Now, assume for contradiction that there exists $x_0 \in X$ such that $\mathfrak{F}u(x_0) < 0$. Then $u(x_0) < \underline{u}(x_0)$, since otherwise we would have $\mathfrak{F}u(x_0) \geq \mathfrak{F}\underline{u}(x_0)$ by monotony, which is a contradiction since $\mathfrak{F}\underline{u}(x_0) \geq 0$. Define $u_\varepsilon(x_0) := u(x_0) + \varepsilon$ and $u_\varepsilon(x) := u(x)$ for all $x \in X \setminus \{x_0\}$. Then u_ε is a sub-solution for any sufficiently small $\varepsilon > 0$, by monotony and continuity of the scheme $\mathfrak{F}$, thus $u(x_0) \geq u_\varepsilon(x_0)$ by construction which is a contradiction. Finally we obtain $\mathfrak{F}u = 0$ identically on X, as announced. □

Optional (medium) : Formalize Perron's existence result discretization

2 Discretizations of the eikonal equation

2.1 Eulerian schemes

2.1.1 General construction

prop:comb_DDE

Proposition 2.1 (Combining DDE schemes)**.** *Let F and G be DDE (resp. causal) schemes over a finite set X. The the following schemes are DDE (resp. causal) as well*

$$\alpha F + \beta G, \qquad \max\{F, G\}, \qquad \min\{F, G\}, \qquad \eta \circ F$$

where $\alpha, \beta \geq 0$, and $\eta : \mathbb{R} \to \mathbb{R}$ is non-decreasing.

Task : Prove this proposition (or a generalization, e.g. to finite sums and maxima).

Proposition 2.2. *Given non-negative coefficients $\alpha : X \times X \times X \to [0, \infty[$, the following scheme is DDE and causal*

$$\forall x \in X \setminus X_0, Fu(x) := \sum_{y,z \in X} \alpha(x, y, z) \max\{0, u(x) - u(y), u(x) - u(z)\}_+^2,$$
$$\forall x \in X_0, Fu(x) := u(x) - u(x_0).$$

Task : Prove this proposition, using the previous proposition and building the scheme from simpler ones.

The classical scheme for the isotropic eikonal equation, on the Cartesian grid $h\mathbb{Z}^d$, reads

$$Fu(x) := \sum_{1 \leq i \leq d} \max\{0, \frac{u(x) - u(x - he_i)}{h}, \frac{u(x) - u(x + he_i)}{h}\}^2 - 1, \tag{4}$$

where $(e_i)_{1 \leq i \leq d}$ denotes the canonical basis.

Task : Prove that it is consistent with the eikonal equation $\|\nabla u(x)\|^2 - 1$, and that it is DDE and causal.

2.1.2 Riemannian metrics, and Selling's algorithm

Consider the scheme on $h\mathbb{Z}^d$

$$Fu(x) := \sum_{1 \leq i \leq I} \max \rho_i(x)\{0, \frac{u(x) - u(x - he_i)}{h}, \frac{u(x) - u(x + he_i)}{h}\}^2 - 1, \tag{5}$$

where I is an integer, $\rho_i(x) \geq 0$ are weights, $e_i \in \mathbb{Z}^d \setminus \{0\}$ are offsets with integer coordinates.

Optional (easy) : Prove that it is consistent with the Riemannian eikonal equation $\|\nabla u(x)\|^2_{D(x)} - 1$, where $D(x) := \sum_{1 \leq i \leq I} \rho_i(x) e_i e_i^\top$.

Optional (medium) : Prove that Selling's decomposition yields a suitable matrix decomposition.

Optional (hard) : Prove that Selling's decomposition is uniquely defined.

2.2 Semi-Lagrangian schemes

2.2.1 Riemannian metrics

Optional (medium) : Construction of an acute stencil.

3 Convergence analysis

ec:convergence

3.1 Viscosity solutions

Task : Formalize the concept of viscosity solution

Task : Prove that the discrete solutions converge to the viscosity solution.

3.2 Quantitative estimates

The main ingredient is the *doubling of variables argument*. See for instance mirebeau2019riemannian [Mir19, Theorem 2.4]

Optional (medium) : Formalize the doubling of variables argument.

Optional (hard) : Formalize the proof of convergence of solutions to the discretized eikonal equation, with rate $\mathcal{O}(\sqrt{h})$.

References

n2007Weighted [BBK07] Alexander M Bronstein, Michael M Bronstein, and Ron Kimmel. Weighted distance maps computation on parametric three-dimensional manifolds. *Journal of Computational Physics*, 225(1):771 – 784, 07 2007.

kas1995dynamic [Ber95] Dimitri P Bertsekas. *Dynamic programming and optimal control*, volume 1 of *Athena scientific Belmont, MA*. Athena scientific Belmont, MA, 00 1995.

:1998Manifold [KS98] R Kimmel and James A. Sethian. Computing geodesic paths on manifolds. *Proceedings of the National Academy of Sciences*, 95(15):8431 – 8435, 07 1998.

2019riemannian [Mir19] Jean-Marie Mirebeau. Riemannian Fast-Marching on Cartesian Grids, Using Voronoi's First Reduction of Quadratic Forms. *SIAM Journal on Numerical Analysis*, 57(6):2608–2655, 00 2019.

gentDifference [Obe06] A M Oberman. Convergent Difference Schemes for Degenerate Elliptic and Parabolic Equations: Hamilton-Jacobi Equations and Free Boundary Problems. *SIAM Journal on Numerical Analysis*, 44(2):879–895, January 2006.

ethian1996fast [Set96] James A. Sethian. A fast marching level set method for monotonically advancing fronts. *Proceedings of the National Academy of Sciences*, 93(4):1591 – 1595, 00 1996.

Sethian2003OUM [SV03] James A. Sethian and Alexander Boris Vladimirsky. Ordered upwind methods for static Hamilton-Jacobi equations: theory and algorithms. *SIAM Journal on Numerical Analysis*, 41(1):325 – 363, 00 2003.

tTrajectories [Tsi95] J.N. Tsitsiklis. Efficient algorithms for globally optimal trajectories. *IEEE transactions on Automatic Control*, 40(9):1528 – 1538, 09 1995.