Formalising the h-Principle and Sphere Eversion

Floris van Doorn   Patrick Massot   Oliver Nash

16 January 2023

Formalizing geometry

Can we formalize deep geometric arguments from modern mathematics?

Many formalizations focus on algebra or discrete mathematics.

Potential challenge: Proofs that are given using pictures or geometric intuition.

Sphere eversion

Can we turn a sphere inside out?

Rules:

- No tears or sharp creases;
- It is allowed to self-intersect.

Sphere eversion

Can we turn a sphere inside out?

Rules:

- No tears or sharp creases;
- It is allowed to self-intersect.

Mathematically, we want to transform the inclusion map $\mathbb{S}^2 \to \mathbb{R}^3$.

At each stage of the transformation the map must be an immersion.
f is an immersion $\iff$
f is locally an embedding $\iff$
the image of a small disk under f is 2-dimensional $\iff$
the total derivative of f is injective at each point.

Sphere eversion

What do we mean by “inside out”?

Sphere eversion

What do we mean by "inside out"?

The antipodal map $\mathbb{S}^2 \to \mathbb{R}^3$ that sends (x, y, z) to $(-x, -y, -z)$ is orientation-reversing: it is a composition of three reflections.

Sphere eversion

What do we mean by "inside out"?

The antipodal map $\mathbb{S}^2 \to \mathbb{R}^3$ that sends (x, y, z) to $(-x, -y, -z)$ is orientation-reversing: it is a composition of three reflections.

Any smooth transformation between the inclusion map $\mathbb{S}^2 \to \mathbb{R}^3$ and the antipodal map must interchange the inside and the outside of the sphere.

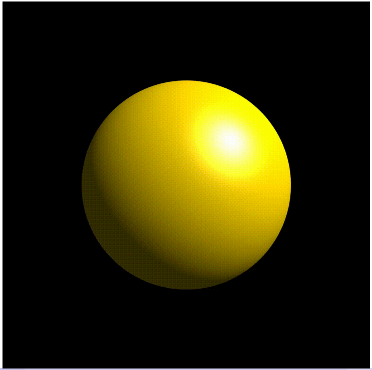
Sphere eversion

Theorem (Smale, 1957)

There is a smooth transformation of immersions

$$\mathbb{S}^2 \to \mathbb{R}^3$$

from the inclusion map to the antipodal map.



The sphere eversion project

Patrick Massot, Oliver Nash and I formalized sphere eversion in Lean.

We don't give an explicit construction, instead we use a general technique called convex integration.

The sphere eversion project

Patrick Massot, Oliver Nash and I formalized sphere eversion in Lean.

We don't give an explicit construction, instead we use a general technique called convex integration.

We use this to prove a deep result in differential topology in Lean, namely Gromov's original homotopy principle (h-principle).

The homotopy principle provides a very general technique to construct solutions to partial differential relations. Sphere eversion follows as a corollary.

Originally proven by Mikhael Gromov in 1973, but we followed a proof by Mélanie Theillière from 2018.

Convex integration (1)

Suppose we want to turn $f : \mathbb{R}^2 \to \mathbb{R}^3$ into an immersion, without changing the function "too much".

Convex integration (1)

Suppose we want to turn $f : \mathbb{R}^2 \to \mathbb{R}^3$ into an immersion, without changing the function “too much”.

We can do this by first ensuring that $\partial_1 f(x) := \frac{\partial f(x)}{\partial x_1} \neq 0$ and next that $\partial_2 f(x)$ is not collinear with $\partial_1 f(x)$.

In both steps we want that $\partial_j f(x)$ lives in some open subset $\Omega_x \subseteq \mathbb{R}^3$.

Convex integration (1)

Suppose we want to turn $f : \mathbb{R}^2 \to \mathbb{R}^3$ into an immersion, without changing the function "too much".

We can do this by first ensuring that $\partial_1 f(x) := \frac{\partial f(x)}{\partial x_1} \neq 0$ and next that $\partial_2 f(x)$ is not collinear with $\partial_1 f(x)$.

In both steps we want that $\partial_j f(x)$ lives in some open subset $\Omega_x \subseteq \mathbb{R}^3$.

Suppose there exists a family of loops $\gamma : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that γ_x takes values in Ω_x and has average $\partial_j f(x)$.

Note: Such loops only exist if $\partial_j f(x)$ is in the convex hull of Ω_x.

Convex integration (2)

We want that $\partial_j f(x)$ lives in some open subset $\Omega_x \subseteq \mathbb{R}^3$.
We have a family of loops $\gamma : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that γ_x takes values in Ω_x and has average $\partial_j f(x)$.

Convex integration (2)

We want that $\partial_j f(x)$ lives in some open subset $\Omega_x \subseteq \mathbb{R}^3$.
We have a family of loops $\gamma : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that γ_x takes values in Ω_x and has average $\partial_j f(x)$.

Now let $N \gg 0$ and replace f by

$$g : x \mapsto f(x) + \frac{1}{N} \int_0^{N x_j} [\gamma_x(s) - \partial_j f(x)] ds.$$

Convex integration (2)

We want that $\partial_j f(x)$ lives in some open subset $\Omega_x \subseteq \mathbb{R}^3$.
We have a family of loops $\gamma : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that γ_x takes values in Ω_x and has average $\partial_j f(x)$.

Now let $N \gg 0$ and replace f by

$$g : x \mapsto f(x) + \frac{1}{N} \int_0^{Nx_j} [\gamma_x(s) - \partial_j f(x)] ds.$$

By a simple computation of partial derivatives, we see that

- $\partial_j g(x) \approx \gamma_x(N x_j) \in \Omega_x$;
- $\partial_i g(x) \approx \partial_i f(x)$ for $i \neq j$;
- $g(x) \approx f(x)$.

Formalization of the homotopy principle

For the paper we formalized the homotopy principle for maps between vector spaces.
In the meantime we also finished the proof for maps between manifolds.

Formalization of the homotopy principle

For the paper we formalized the homotopy principle for maps between vector spaces.
In the meantime we also finished the proof for maps between manifolds.

The project is 15k lines of code and we made 140 pull requests to mathlib in the process, about convexity of sets, parametric integrals, differential geometry and various other topics.

This project took us about a year (part-time).

Formalized version

```
theorem sphere_eversion :
∃ f : ℝ → 𝕊² → ℝ³,
smooth (↿f : ℝ × 𝕊² → ℝ³) ∧
f 0 = (coe : 𝕊² → ℝ³) ∧
f 1 = (-coe : 𝕊² → ℝ³) ∧
∀ t, immersion (f t)
```

The blueprint

We wrote a blueprint with a detailed LaTeX proof.

The sphere eversion project

1 Loops

1.1 Introduction

In this chapter, we explain how to construct families of loops to feed into the corrugation process explained at the end of the introduction. A loop is a map defined on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with values in a finite-dimensional vector space. It can also freely be seen as 1-periodic maps defined on $\mathbb{R}$.

Definition 1.1 ✓

The average of a loop γ is $\bar{\gamma} := \int_{\mathbb{S}^1} \gamma(s)\, ds$.

Throughout this document, E and F will denote finite-dimensional real vector spaces.

Definition 1.2 ✓

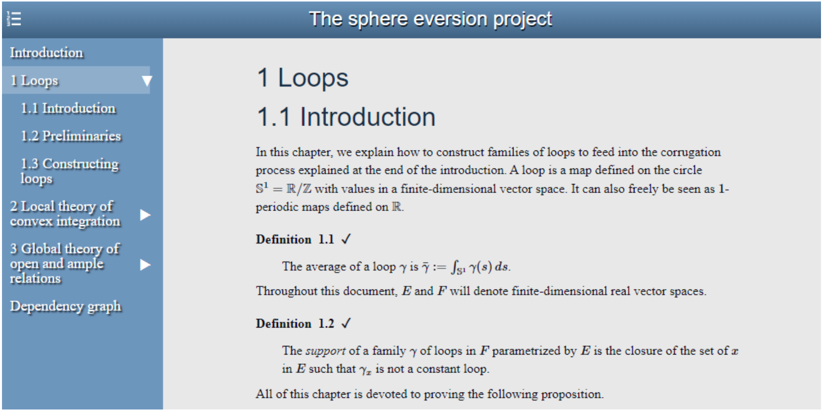
The *support* of a family γ of loops in F parametrized by E is the closure of the set of x in E such that γ_x is not a constant loop.

All of this chapter is devoted to proving the following proposition.

The blueprint

We wrote a blueprint with a detailed LaTeX proof.

The blueprint provides aligned LaTeX and formalized proof

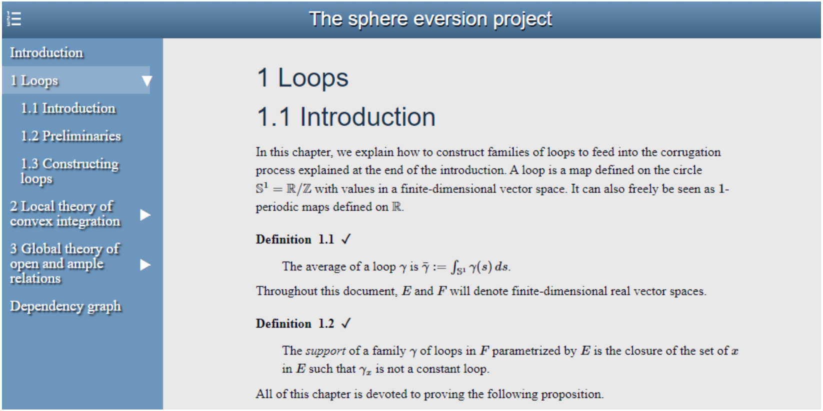


The blueprint

We wrote a blueprint with a detailed LaTeX proof.

The blueprint provides aligned LaTeX and formalized proof

Main benefit: precisely written intermediate lemmas.


The sphere eversion project

Introduction
1 Loops
1.1 Introduction
1.2 Preliminaries
1.3 Constructing loops
2 Local theory of convex integration
3 Global theory of open and ample relations
Dependency graph

1 Loops

1.1 Introduction

In this chapter, we explain how to construct families of loops to feed into the corrugation process explained at the end of the introduction. A loop is a map defined on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with values in a finite-dimensional vector space. It can also freely be seen as 1-periodic maps defined on $\mathbb{R}$.

Definition 1.1 ✓

The average of a loop γ is $\bar{\gamma} := \int_{\mathbb{S}^1} \gamma(s)\, ds$.

Throughout this document, E and F will denote finite-dimensional real vector spaces.

Definition 1.2 ✓

The *support* of a family γ of loops in F parametrized by E is the closure of the set of x in E such that γ_x is not a constant loop.

All of this chapter is devoted to proving the following proposition.

The homotopy principle

Theorem (Gromov, 1973)

If $\mathcal{R}$ is an open and ample[1] partial differential relation for functions between manifolds M and N then $\mathcal{R}$ satisfies the homotopy principle, i.e. any formal solution can be smoothly deformed into a holonomic one inside $\mathcal{R}$.

[1] Ampleness is a geometric condition that ensures that certain convex hulls are large enough for the convex integration argument to work.