Lessons learned from formalizing local convex integration

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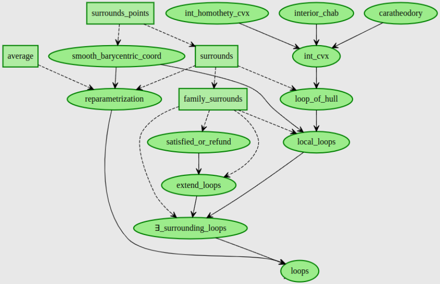
Dependency Graph



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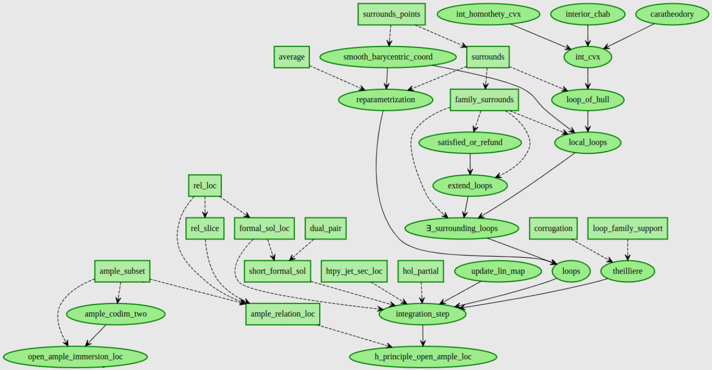

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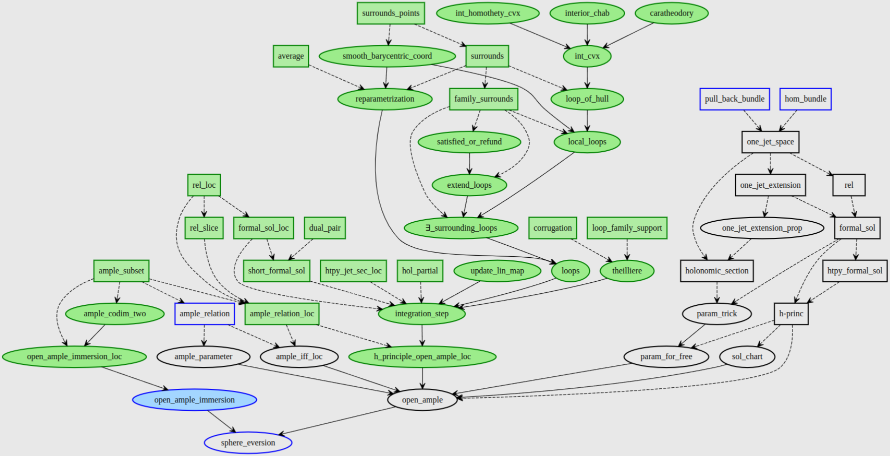
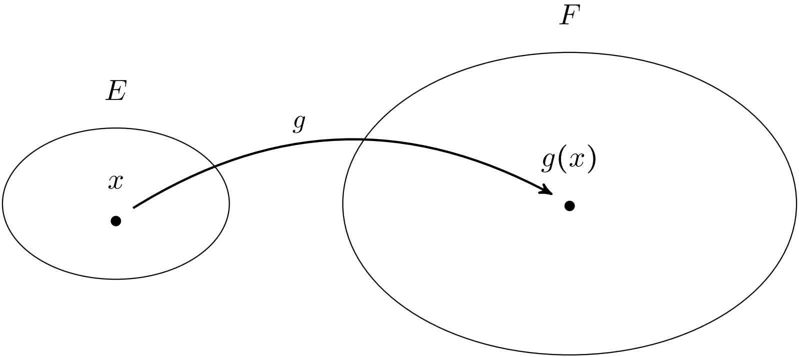


Picture



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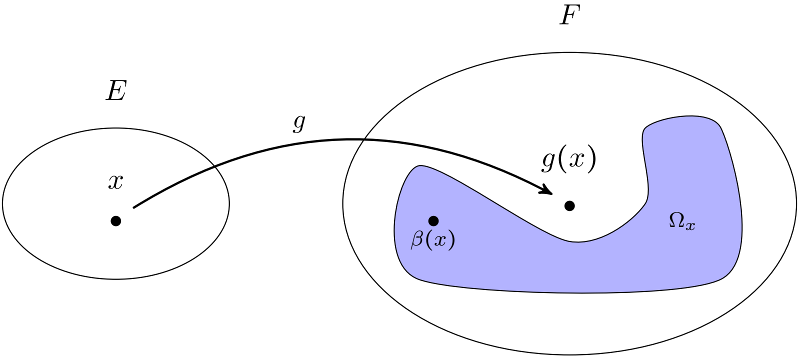

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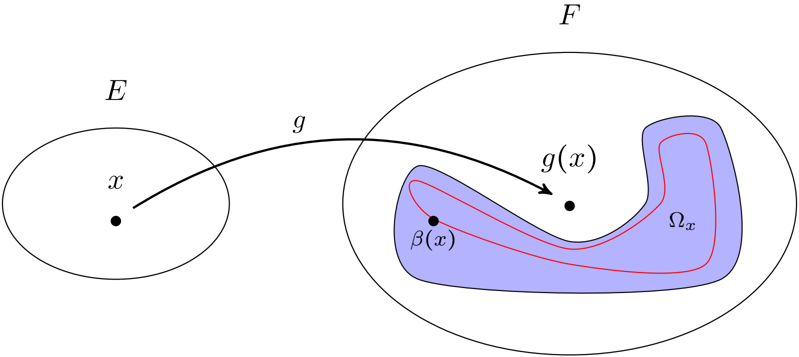

Main Proposition

Proposition

Let Ω be an open set in $E \times F$ such that, for each x the set $\Omega_x := \{y \in F \mid (x, y) \in \Omega\}$ is connected in F.
Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x.
Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.
Then there exists a smooth family of loops

$$\gamma : E \times [0,1] \times \mathbb{S}^1 \to F, \quad (x, t, s) \mapsto \gamma_x^t(s)$$

such that, for all $x \in E$, all $t \in \mathbb{R}$ and all $s \in \mathbb{S}^1$,

- $\gamma_x^t(s) \in \Omega_x$
- $\gamma_x^0(s) = \gamma_x^t(1) = \beta(x)$
- *the average of γ_x^1 is $\overline{\gamma_x^1} = g(x)$*

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Let Ω be an open set in $E \times F$ such that, for each x the set $\Omega_x := \{y \in F \mid (x, y) \in \Omega\}$ is connected in F. Let K a compact set in E. Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x, and $g(x) = \beta(x)$ near K.
Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.
Then there exists a smooth family of loops

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- *the average of γ_x^1 is* $\overline{\gamma_x^1} = g(x)$
- $\gamma_x^t(s) = \beta(x)$ *if x is near K.*

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- *the average of* γ_x^1 *is* $\overline{\gamma_x^1} = g(x)$

Pointwise version

Proposition

Let Ω be an open connected set in F.

Let $\beta, g \in F$.
Assume that $\beta \in \Omega$.
Suppose that g is in the convex hull of Ω.
Then there exists a smooth family of loops

$$\gamma : [0,1] \times \mathbb{S}^1 \to F, \quad (t,s) \mapsto \gamma^t(s)$$

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such that for all $t \in \mathbb{R}$ and all $s \in \mathbb{S}^1$,

- $\gamma^t(s) \in \Omega$
- $\gamma^0(s) = \gamma^t(1) = \beta$
- γ^1 *surrounds* g

Pointwise solution

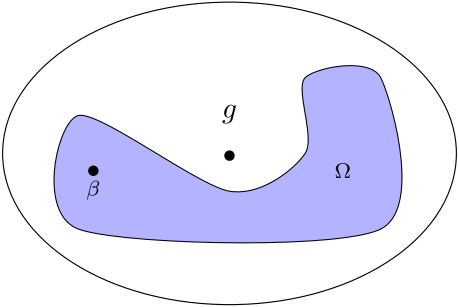
Definition

- A point $x \in F$ is surrounded by points a finite set of points $\{p_i\}$ if those points form an affine basis and there exist weights $w_i \in (0,1)$ with sum 1 such that $x = \sum_i w_i p_i$.
- A set A surrounds x if there is a collection of points in A surrounding x.

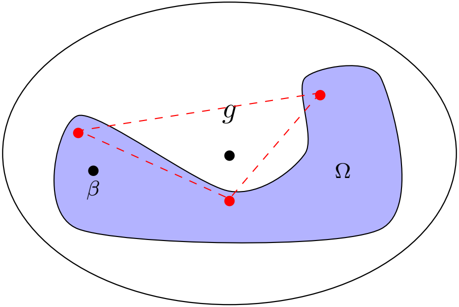
Lemma

If a point $x \in F$ lies in the convex hull of an open set P then P surrounds x

Pointwise solution



Pointwise solution



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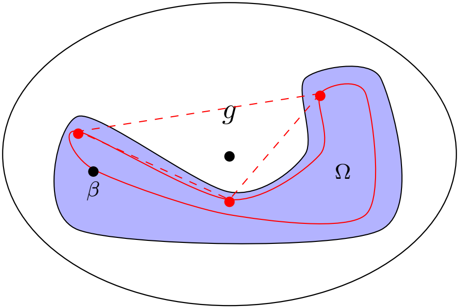

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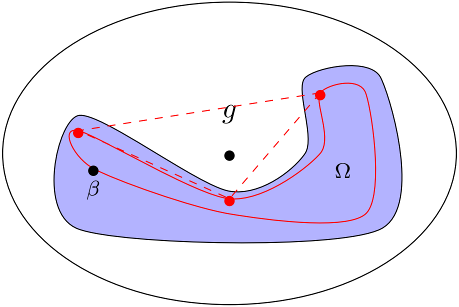

Proof.

By the lemma, pick a collection of points surrounding g. Since Ω is connected, we can find a path starting at β through these points. Retracing the same path back to β, we obtain a path that is homotopic to the constant path. $\square$

Local solution

Suppose we have such a γ_{x_0} based at $\beta(x_0)$ surrounding $g(x_0)$.

We can extend this to a local solution around x_0:

$$\gamma_x^t(s) = \gamma_{x_0}^t(s) + \beta(x) - \beta(x_0)$$

Then:

- For x sufficiently close to x_0 we have $\gamma_x^t(s) \in \Omega_x$
- $\gamma_x^0(s) = \gamma_x^t(1) = \beta(x)$
- For x sufficiently close to x_0 we have γ_x^1 surrounds $g(x)$

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∀ᶠ x in 𝓝 x₀, ∀ (t ∈ I) (s ∈ I), (x, γ x t s) ∈ Ω
∀ᶠ x in 𝓝 x₀, (γ x 1).surrounds (g x)
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Local solution

We write $\gamma \in \mathcal{L}(U)$ for such a γ defined on $U \subseteq E$.

$\gamma \in \mathcal{L}(U)$ means that γ is a continuous family of loops

$$\gamma : E \times [0,1] \times \mathbb{S}^1 \to F, \quad (x,t,s) \mapsto \gamma_x^t(s)$$

such that for every $x \in U$, every $t \in [0,1]$ and every $s \in \mathbb{S}^1$,

- γ_x^t is based at $\beta(x)$
- $\gamma_x^0(s) = \beta(x)$
- γ_x^1 surrounds $g(x)$
- $(x, \gamma_x^t(s)) \in \Omega$.

Gluing solutions

We want to glue these local solutions together, to obtain a global solution defined on all of E.

Let's start with two local solutions $\gamma_0 \in \mathcal{L}(U_0)$ and $\gamma_1 \in \mathcal{L}(U_1)$.

We want to get a solution on $U_0 \cup U_1$ (or similar).

One thing we need is to transition between γ_0 to γ_1 on $U_0 \cap U_1$.

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One thing we need is to transition between γ_0 to γ_1 on $U_0 \cap U_1$.

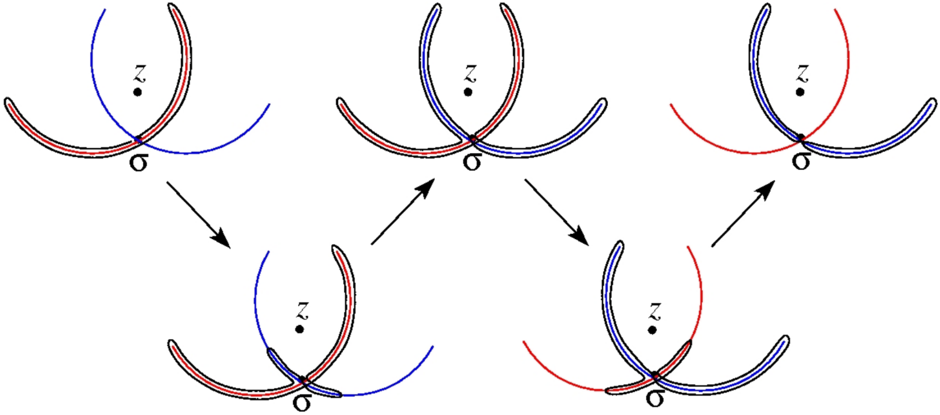
Lemma

If $U \subseteq E$ then $\mathcal{L}(U)$ is path-connected: if $\gamma_0, \gamma_1 \in \mathcal{L}(U)$ then there is a continuous homotopy

$$\delta : [0,1] \times E \times [0,1] \times \mathbb{S}^1 \to F, \quad (\tau, x, t, s) \mapsto \delta^t_{\tau,x}(s)$$

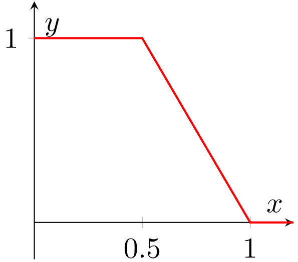
such that $\delta_\tau \in \mathcal{L}(U)$ and $\delta_i = \gamma_i$ for $i \in \{0,1\}$.

Transitioning between solutions



Transitioning between solutions

Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following piecewise affine function:



We can define the homotopy δ as follows:

- $\delta^t_{\tau,x}$ moves along the loop $\gamma^{\rho(\tau)t}_{0,x}$ once on $[0, 1 - \tau]$ (if $\tau < 1$)
- $\delta^t_{\tau,x}$ moves along the loop $\gamma^{\rho(1-\tau)t}_{1,x}$ once on $[1 - \tau, 1]$ (if $\tau > 0$)

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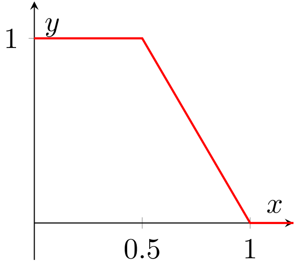

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- $\delta^t_{\tau,x}$ moves along the loop $\gamma^{\rho(1-\tau)t}_{1,x}$ once on $[1-\tau, 1]$ (if $\tau > 0$)

Note that the image of $\delta^1_{\tau,x}$ contains the image of $\gamma^1_{0,x}$ for $\tau \leq \frac{1}{2}$, and the image of $\gamma^1_{1,x}$ for $\tau \geq \frac{1}{2}$. Hence it will always surround $\beta(x)$.

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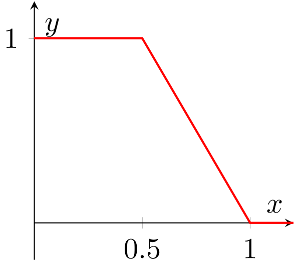

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Is δ continuous?

Transitioning between solutions

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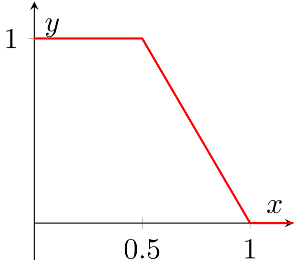

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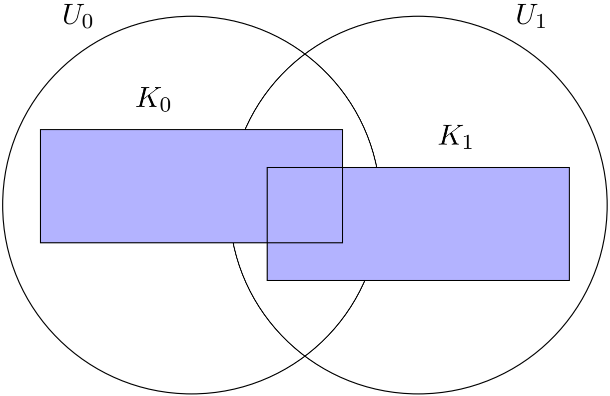
Is δ continuous? Note that if $\tau \to 1$ then $\delta^1_{\tau,x}$ will move along loop $\gamma^{\rho(\tau)t}_{0,x}$ at a speed that tends to $+\infty$, so we need to show that $\gamma^{\rho(\tau)t'}_{0,x'}$ tends uniformly to the constant loop as $(x', \tau, t') \to (x, 1, t)$

Gluing solutions

Let $\gamma_0 \in \mathcal{L}(U_0)$ and $\gamma_1 \in \mathcal{L}(U_1)$.
It is hard to get a solution on all of $U_0 \cup U_1$.
Pick $K_i \subset U_i$ compact.



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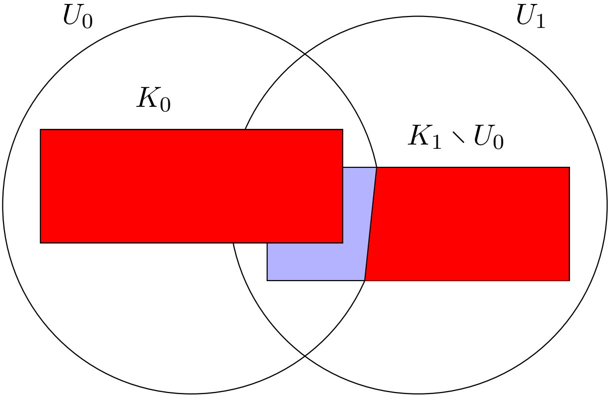

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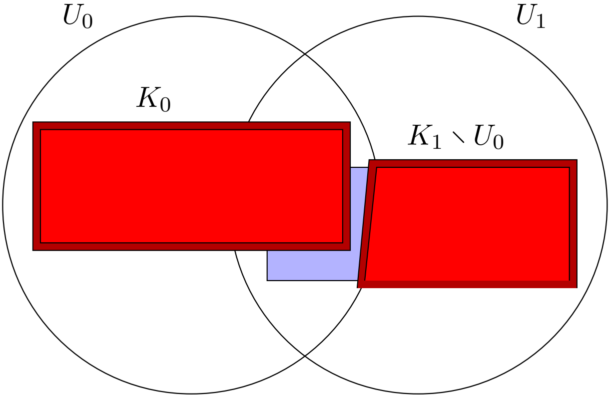

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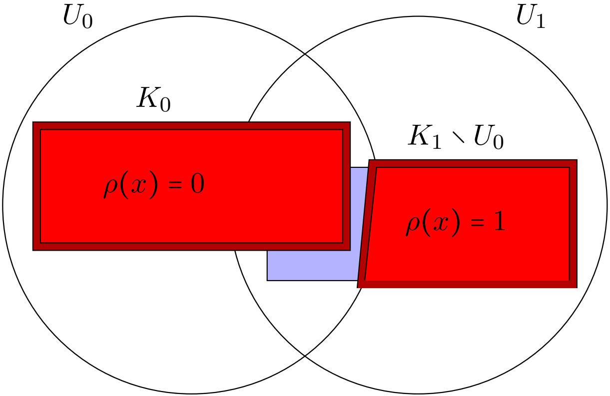

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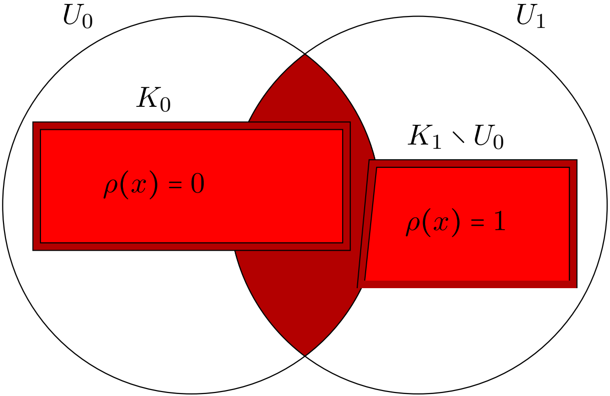

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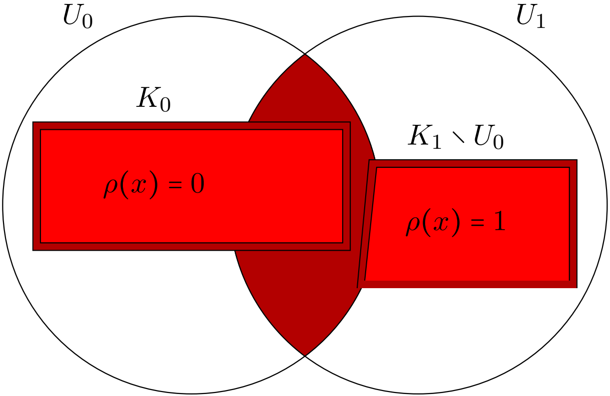

Let δ be a homotopy between γ_0 and γ_1 on $U_0 \cap U_1$. Then $\gamma_x = \delta_{\rho(x),x}$ is a solution in the shaded region.

Gluing solutions

So we have shown the following.

Proposition

If we have solutions $\gamma_i \in \mathcal{L}(U_i)$ with U_i an open neighborhood of the compact set K_i for $i \in \{0,1\}$ then we can find a solution $\gamma \in \mathcal{L}(U)$ for some $U \in \mathcal{N}(K_0 \cup K_1)$ which coincides with γ near K_0.

Induction

Lemma

If $\gamma^0 \in \mathcal{L}(U_0)$ *with* U_0 *a neighborhood of* K*, then we can find* $\gamma \in \mathcal{L}(E)$ *which agrees with* γ^0 *near* K*.*

Proof sketch.

We can find U_i, $i \geq 1$ be a locally finite family of open sets with local surrounding families of loops γ^i and compact subsets $K_i \subseteq U_i$ covering E. By repeatedly gluing these loops to γ^0, we obtain a sequence of families that is eventually constant in each K_i, so we obtain γ in the limit. □

Smoothing

Now that we have a global continuous solution, we need to find a smooth approximation that preserves the following properties:

- γ_x^t is a loop based at $\beta(x)$
- $\gamma_x^0(s) = \beta(x)$
- γ_x^1 surrounds $g(x)$
- $(x, \gamma_x^t(s)) \in \Omega$.

We could take the convolution with a smooth bump function to obtain a smooth function.

But then we need to use a partition of unity to ensure that the family is equal to β on part of its domain.

Smoothing

In fact, we can do it without convolution, and using partition of unity in the following form:

Lemma

if $f : E \to F$ and $\varepsilon : E \to \mathbb{R}_{>0}$ are continuous then there is a C^∞ function $g : E \to F$ with $d(f(x), g(x)) < \varepsilon(x)$ for all x. Moreover, if f is smooth near a closed set C, then we can choose g such that $g(x) = f(x)$ for $x \in C$.

Reparametrization

Finally, we need to reparametrize the loop so that its average is exactly $g(x)$.

I will skip this proof.

Convolution

At first I formalized the properties of convolution, and we needed the following case.

Given $f : E \to \mathbb{R}$ and $g : E \to F$ the convolution $f \star g : E \to F$ is given by

$$(f \star g)(x) = \int f(t)g(x-t)dt.$$

This version is useful to smooth a vector-valued function with a bump function (bump functions were defined by Yuri Kudryashov in Lean).

In particular we have

- If f_n is a sequence of bump functions that tends to δ_x then $f_n \star g \to g(x)$;
- If f is C^n and has compact support and g is locally integrable then $f \star g$ is C^n.

Convolution

Interestingly, all books that I could find that proves the second property, uses it by computing the partial derivatives of $f \star g$:

$$\frac{\partial}{\partial x_i}(f \star g) = \frac{\partial f}{\partial x_i} \star g.$$

None of these books computes the total derivative of the derivative.

How to state this? $D(f \star g)_x$ is a linear map $E \to F$ and it should be a convolution of Df with g.

However, the convolution of $Df : E \to E^*$ and $g : E \to F$ is not in the form we defined before.

Convolution

Given $f : E \to F_1$ and $g : E \to F_2$ and a continuous bilinear map $L : \hom(F_1 \times F_2, F)$ we define the convolution $f \star_L g : E \to F$ is given by

$$(f \star_L g)(x) = \int L(f(t), g(x - t))dt.$$

This has the regular properties of convolution.
To compute the derivative, let

$$L' : \hom(\hom(E, F_1) \times F_2, \hom(E, F))$$

given by $L'(M, y) = L(M(-), y)$.

Theorem

If f is C^1 with compact support and g is locally integrable, then

$$D(f \star_L g) = Df \star_{L'} g$$

Moreover, if f is C^n then $f \star_L g$ is C^n.

Convolution

Proof.

Since f has compact support, the integral is dominated by an integrable function.
Therefore

$$\begin{aligned} D_{x_0}(f \star_L g) &= D_{x_0} \int L(f(x-t), g(t))dt \\ &= \int D_{x_0} L(f(x-t), g(t))dt \\ &= \int L((-), g(t)) \circ D_{x_0} f(x-t)dt \\ &= (Df \star_{L'} g)(x_0). \end{aligned}$$

By induction we get that if f is C^n then $f \star_L g$ is C^n. □

Conclusion

Carefulness needed

Cool generalizations

Thank You