Categories in HoTT in Lean

Floris van Doorn

Carnegie Mellon University

March 27, 2015

HoTT in Lean

Already formalized

- Most of chapters 1-4 of the book (identity types, sets, equivalences)
- Parts of chapters 7 (n-types) and chapter 9 (Category Theory)

To do: Higher inductive types

HoTT in Lean

Already formalized

- Most of chapters 1-4 of the book (identity types, sets, equivalences)
- Parts of chapters 7 (n-types) and chapter 9 (Category Theory)

To do: Higher inductive types

Precategories

```
structure precategory (ob : Type) : Type :=
  (hom : ob → ob → Type)
  (homH : ∀(a b : ob), is_hset (hom a b))
  (comp : ∀⦃a b c : ob⦄, hom b c → hom a b → hom a c)
  (ID : ∀(a : ob), hom a a)
  (assoc : ∀⦃a b c d : ob⦄ (h : hom c d) (g : hom b c)
    (f : hom a b), comp h (comp g f) = comp (comp h g) f)
  (id_left : ∀⦃a b : ob⦄ (f : hom a b), comp !ID f = f)
  (id_right : ∀⦃a b : ob⦄ (f : hom a b), comp f !ID = f)
```

Natural Transformations

```
variables {C D : Precategory} {F G H : C ⇒ D}
definition nat_trans.compose
  (η : G ⟹ H) (θ : F ⟹ G) : F ⟹ H :=
nat_trans.mk
  (λa, η a ∘ θ a)
  (λa b f, calc
    H f ∘ (η a ∘ θ a) = (H f ∘ η a) ∘ θ a : assoc
                  ... = (η b ∘ G f) ∘ θ a : naturality
                  ... = η b ∘ (G f ∘ θ a) : assoc
                  ... = η b ∘ (θ b ∘ F f) : naturality
                  ... = (η b ∘ θ b) ∘ F f : assoc)
```

Categories

```
definition iso_of_eq {a b : ob} [C : precategory ob]
  (p : a = b) : a ≅ b :=
eq.rec_on p (iso.refl a)

definition is_univalent {ob : Type} (C : precategory ob)
:= ∀(a b : ob), is_equiv (@iso_of_eq ob C a b)

structure category (ob : Type) extends C : precategory ob
:= (iso_of_path_equiv : is_univalent C)
```

Examples

```
definition is_univalent_hset
  : is_univalent Precategory_hset

definition is_univalent_functor
  (D : Category) (C : Precategory)
    : is_univalent (D ^c C)
```

Short-term goals

Short-term goals. Define and prove basic properties about

- Adjunctions of functors
- Equivalences of categories
- Yoneda embedding
- (co)limits

Long-term goals

Long-term goals.

- Define cubical sets
- Prove that cubical sets form a category with families with pi's, sigma's, and a universe.
- Prove that **Kan** cubical sets also have identity types.
- Prove the univalence axiom inside this model
- This gives a model of HoTT inside HoTT