# The Lean-HoTT library and HITs in Lean

Floris van Doorn

Carnegie Mellon University

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# Differences between standard Lean and HoTT Lean

- The lowest universe is neither impredicative, nor proof irrelevant.
- The univalence axiom is assumed.
- There are two HITs primitive, the *n*-truncation and *quotients*.

Given $A : \mathcal{U}$ and $R : A \to A \to \mathcal{U}$ the quotient is:

```
HIT quotient_A(R) :=
```
- $q : A \to quotient_A(R)$
- $\Pi(x, y : A),\ R(x, y) \to q(x) = q(y)$

## Quotients

Using quotients, we can easily define pushouts and colimits, hence we also get suspensions, spheres, wedge products, smash products, . . . .

With some more work we can define the propositional truncation and HITs with 2-constructors.

Work in progress: we can also define $n$-truncations and certain ($\omega$-compact) localizations (Egbert Rijke).

# Lean-HoTT library

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| **1** |   |   |   |   | + | + | + | + | + |    | + | + |   |   |   |
| **2** | + | + | + | + |   | + | + | + | + | + | + | + | + | + | + |
| **3** | + | + | + | + | $\frac{1}{2}$ | + | + | + | + |    | + |   |   |   |   |
| **4** | - | + | + | + |   | + | + | + | + |    |   |   |   |   |   |
| **5** | - |   | $\frac{1}{2}$ | - | - |   |   | $\frac{1}{2}$ |   |    |   |   |   |   |   |
| **6** |   | + | + | + | + | + | + | + | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | + |   |   |   |
| **7** | + | + | + | - | $\frac{3}{4}$ | - | - |   |   |    |   |   |   |   |   |
| **8** | + | + | + | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | - | - | $\frac{1}{2}$ | - |   |   |   |   |   |
| **9** | $\frac{3}{4}$ | + | + | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | - | - | - |    |   |   |   |   |   |

HoTT-Lean: ~35k lines
HoTT-Coq: ~40k lines (in theories/ folder)
HoTT-Agda: ~23k lines (excluding old/ folder)

# Lean-HoTT library

Furthermore, the library includes the following which are not in the book:

- A library of squares, cubes, pathovers, squareovers, cubeovers

  ```
  definition circle.rec {P : S¹ → Type} (Pbase : P base)
    (Ploop : Pbase =[loop] Pbase) (x : circle) : P x
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- A library of pointed types, pointed maps, pointed homotopies, pointed equivalences

  ```
  definition iterated_loop_ptrunc_pequiv
    (n : ℕ₋₂) (k : ℕ) (A : Type*) :
    Ω[k] (ptrunc (n+k) A) ≃* ptrunc n (Ω[k] A)
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- Some more category theory than the book, e.g. limits, colimits and exponential laws:

  ```
  definition functor_functor_iso (C D E : Precategory) :
    (C ^c D) ^c E ≅c C ^c (E ×c D)
  ```

# Homotopy Theory in Lean

Loop space of the circle:

```
definition base_eq_base_equiv : base = base ≃ ℤ
definition fundamental_group_of_circle : π₁(S¹.) = gℤ
```

# Homotopy Theory in Lean

Connectedness of suspensions:

```
definition is_conn_susp (n : ℕ₋₂) (A : Type)
  [H : is_conn n A] : is_conn (n .+1) (susp A)
```

# Homotopy Theory in Lean

The hopf fibration:

```
definition hopf : susp A → Type :=
susp.elim_type A A
  (λa, equiv.mk (λx, a * x) !is_equiv_mul_left)

definition hopf.total (A : Type) [H : h_space A]
  [K : is_conn 0 A] : sigma (hopf A) ≃ join A A

definition circle_h_space : h_space S¹
definition sphere_three_h_space : h_space (S 3)
```

# Homotopy Theory in Lean

The Freudenthal equivalence (ported from Agda):

```
definition freudenthal_pequiv (A : Type*) {n k : ℕ}
  [is_conn n A] (H : k ≤ 2 * n) :
  ptrunc k A ≃* ptrunc k (Ω (psusp A))
```

# LES of homotopy groups

The long exact sequence of homotopy groups.

Given a pointed map $f : X \to Y$.

$$X \xrightarrow{\;f\;} Y$$

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Given a pointed map $f : X \to Y$.

$$F^{(6)} \xrightarrow{\pi_1^{(6)}} F^{(5)} \xrightarrow{\pi_1^{(5)}} F^{(4)} \xrightarrow{\pi_1^{(4)}} F^{(3)} \xrightarrow{\pi_1^{(3)}} F^{(2)} \xrightarrow{\pi_1^{(2)}} F \xrightarrow{\pi_1} X \xrightarrow{f} Y$$

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$$\wr| \qquad \wr| \qquad \wr| \qquad \wr| \qquad \wr|$$

$$\Omega F^{(3)} \qquad \Omega F^{(2)} \qquad \Omega F \qquad \Omega X \qquad \Omega Y$$

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$$\wr\vert \qquad\qquad \wr\vert \qquad\qquad \wr\vert \qquad\qquad\quad \wr\vert \qquad\qquad \wr\vert$$
$$\Omega F^{(3)} \xrightarrow{-\Omega\pi_1^{(3)}} \Omega F^{(2)} \xrightarrow{-\Omega\pi_1^{(2)}} \Omega F \xrightarrow{-\Omega\pi_1} \Omega X \xrightarrow{-\Omega f} \Omega Y$$

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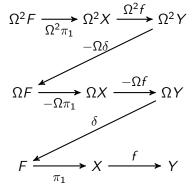
$$\Omega F^{(3)} \xrightarrow{-\Omega\pi_1^{(3)}} \Omega F^{(2)} \xrightarrow{-\Omega\pi_1^{(2)}} \Omega F \xrightarrow{-\Omega\pi_1} \Omega X \xrightarrow{-\Omega f} \Omega Y$$

$$\Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y$$

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$$\Omega F^{(3)} \xrightarrow{-\Omega\pi_1^{(3)}} \Omega F^{(2)} \xrightarrow{-\Omega\pi_1^{(2)}} \Omega F \xrightarrow{-\Omega\pi_1} \Omega X \xrightarrow{-\Omega f} \Omega Y \quad \delta$$

$$\Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \quad -\Omega\delta$$

# LES of homotopy groups

$$\Omega^2 F \xrightarrow[\Omega^2 \pi_1]{} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y$$

$$-\Omega\delta$$

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$$\delta$$

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# LES of homotopy groups

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$$-\Omega\delta$$

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$$\delta$$

$$F \xrightarrow[\pi_1]{} X \xrightarrow{f} Y$$

How do we formalize this?

The obvious thing is to have a sequence $Z : \mathbb{N} \to \mathcal{U}$ and maps $f_n : Z(n+1) \to Z(n)$.

**Problem**: $Z_{3n} = \pi_n(Y)$ doesn't hold definitionally, hence $f_{3n} = \pi_n(f)$ isn't even well-typed.

## LES of homotopy groups

**Better**: Take $Z : \mathbb{N} \times 3 \to \mathcal{U}$, we can define $Z$ by

$$Z_{(n,0)} = \pi_n(Y) \qquad Z_{(n,1)} = \pi_n(X) \qquad Z_{(n,2)} = \pi_n(F).$$

Then we need maps $f_x : Z(succ(x)) \to Z(x)$, where *succ* is the successor function for $\mathbb{N} \times 3$.

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We define chain complexes over an arbitrary type with a successor operation.

# LES of homotopy groups

```
definition homotopy_groups : +6ℕ → Set*
| (n, fin.mk 0 H) := π*[2*n] Y
| (n, fin.mk 1 H) := π*[2*n] X
| (n, fin.mk 2 H) := π*[2*n] (pfiber f)
| (n, fin.mk 3 H) := π*[2*n + 1] Y
| (n, fin.mk 4 H) := π*[2*n + 1] X
| (n, fin.mk k H) := π*[2*n + 1] (pfiber f)
```

## LES of homotopy groups

```
definition homotopy_groups_fun : Π(n : +6ℕ),
  homotopy_groups (S n) →* homotopy_groups n
| (n, fin.mk 0 H) := π→*[2*n] f
| (n, fin.mk 1 H) := π→*[2*n] (ppoint f)
| (n, fin.mk 2 H) := π→*[2*n] (boundary_map f) ∘*
  pcast (ap (ptrunc 0) (loop_space_succ_eq_in Y (2*n)))
| (n, fin.mk 3 H) := π→*[2*n + 1] f ∘* tinverse
| (n, fin.mk 4 H) := π→*[2*n + 1] (ppoint f) ∘* tinverse
| (n, fin.mk 5 H) := (π→*[2*n + 1] (boundary_map f) ∘*
    tinverse) ∘* pcast (ap (ptrunc 0) (loop_space_succ_eq_in
    Y (2*n+1)))
| (n, fin.mk (k+6) H) := begin exfalso, apply lt_le_antisymm
    H, apply le_add_left end
```

# LES of homotopy groups

Then we prove:

- These maps form a chain complex
- This chain complex is exact
- `homotopy_groups (n + 1, k)` are commutative groups.
- `homotopy_groups (0, k + 3)` are groups.
- `homotopy_groups_fun (n + 1, k)` are group homomorphisms.
- `homotopy_groups_fun (0, k + 3)` are group antihomomorphisms (TODO).

# LES of homotopy groups

Corollary:

```
theorem is_equiv_π_of_is_connected {A B : Type*}
  (n k : ℕ) (f : A →* B) [H : is_conn_fun n f]
  (H2 : k ≤ n) : is_equiv (π→[k] f)
```

(caveat: proof currently incomplete for $k = 1$)