Towards a formalized proof of Carleson's theorem

Floris van Doorn

University of Bonn

17 June 2024

j.w.w. Christoph Thiele, Lars Becker, Asgar Jamneshan, Rajula Srivastava and Leo Diederink

Background: Fourier transform

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then its Fourier transform $\mathcal{F}f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\mathcal{F}f(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\, dx.$$

The inverse Fourier transform $\mathcal{F}^{-1}$ is almost the same, up to a sign in the exponential

$$\mathcal{F}^{-1}g(x) := \int_{-\infty}^{\infty} g(\xi) e^{2\pi i x \xi}\, d\xi.$$

Properties of the Fourier transform

- If f is C^1 and both f and $\frac{df}{dx}$ are integrable, then:

$$\mathcal{F}(\tfrac{df}{dx})(\xi) = 2\pi i\xi\mathcal{F}f(\xi).$$

- If f and g are integrable, then:

$$\mathcal{F}(f \star g)(\xi) = \mathcal{F}f(\xi)\mathcal{F}g(\xi).$$

Properties of the Fourier transform

- If f is C^1 and both f and $\frac{df}{dx}$ are integrable, then:

$$\mathcal{F}(\tfrac{df}{dx})(\xi) = 2\pi i\xi\mathcal{F}f(\xi).$$

- If f and g are integrable, then:

$$\mathcal{F}(f \star g)(\xi) = \mathcal{F}f(\xi)\mathcal{F}g(\xi).$$

Theorem (Fourier Inversion)

If $f : \mathbb{R} \to \mathbb{R}$ is integrable, continuous at x and $\mathcal{F}f$ is also integrable, then:

$$\mathcal{F}^{-1}\mathcal{F}f(x) = x.$$

These properties make Fourier transforms very useful for differential equations.

Weaker conditions on the function

Example: Let $f := \chi_{[-\frac{1}{2},\frac{1}{2}]}$ be a box function. It has fourier transform

$$\mathcal{F}f(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

Note: $\mathcal{F}f$ is not integrable on $\mathbb{R}$.

Weaker conditions on the function

Example: Let $f := \chi_{[-\frac{1}{2},\frac{1}{2}]}$ be a box function. It has fourier transform

$$\mathcal{F}f(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

Note: $\mathcal{F}f$ is not integrable on $\mathbb{R}$.

We can define the Fourier transform for a wider class of functions by defining

$$g_R(\xi) := \int_{-R}^{R} f(x)e^{-2\pi i\xi x}\,dx.$$

and considering the limit of g_R as $R \to \infty$.

Weaker conditions on the function

Example: Let $f := \chi_{[-\frac{1}{2},\frac{1}{2}]}$ be a box function. It has fourier transform

$$\mathcal{F}f(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

Note: $\mathcal{F}f$ is not integrable on $\mathbb{R}$.

We can define the Fourier transform for a wider class of functions by defining

$$g_R(\xi) := \int_{-R}^{R} f(x)e^{-2\pi i\xi x}\,dx.$$

and considering the limit of g_R as $R \to \infty$.

Important: Whether this limit converges depends on the topology you use for this limit:

- Pointwise convergence
- L^p-convergence: $\|f\|_{L^p}^p := \int |f(x)|^p dx$.

Fourier inversion with weaker conditions

In what generality does the Fourier inversion theorem hold?

- If $f \in L^2$ then $\mathcal{F}f$ is well-defined using the L^2-norm, and $\mathcal{F}f \in L^2$. In this case, we have $\mathcal{F}^{-1}\mathcal{F}f = f$ w.r.t. to the L^2-norm.

Carleson's theorem

Theorem (Lennart Carleson, 1966)

If $f \in L^2$*. Then for almost every* x *we have* $\mathcal{F}^{-1}\mathcal{F}f(x) = f(x)$.

Carleson's theorem

Theorem (Lennart Carleson, 1966)

If $f \in L^2$. Then for almost every x we have $\mathcal{F}^{-1}\mathcal{F}f(x) = f(x)$.

We also have the following generalization:

Theorem (Richard Hunt, 1968)

If $f \in L^p$ for some $1 < p \leq 2$. Then for almost every x we have $\mathcal{F}^{-1}\mathcal{F}f(x) = f(x)$.

These theorems have very hard proofs.

Counterexamples and remarks

- We cannot remove the "almost every" from the statement: there are continuous L^2 functions where the limit diverges for some x (of measure 0).
- There are L^1 functions where the limit defining $\mathcal{F}^{-1}\mathcal{F}f(x)$ diverges for all points x.
- If we consider continuous periodic functions and replace the Fourier transform by the Fourier series, then the Fourier series can also diverge everywhere.
- However, if we use Cesaro summation, by taking the average of the first n-terms of the Fourier series and then taking the limit, then this will converge to the right value. This is Fejér's theorem.
- If f is a function in multiple variables, versions of Carleson's theorem also hold. One has to be very careful about the shape of the integration domain that tends to infinity. If the shape is spherical, then this is still an open problem.

Formalizing Carleson's theorem

In October last year I started discussing with Christoph Thiele to formalize Carleson's theorem.

He and his group just finished a preprint of a generalization of Carleson's theorem to spaces of homogenous type.

Since then, they have written a detailed self-contained blueprint, that I want to formalize:

`http://florisvandoorn.com/carleson/`

You are very welcome to help!

Blueprint

The original proof was about 30 pages, but that became 100 pages when writing the proofs out in detail, plus 30 pages to prove classical Carleson's theorem as a corollary.

It has 10 sections:

- Section 1: statement of the generalized metric Carleson's theorem;
- Section 2: statement of 6 propositions used in the proof;
- Section 3: proof of metric Carleson from the propositions;
- Sections 4-9: each section proves one of the 6 propositions;
- Section 10: proof of classical Carleson's theorem from the generalization.

Observations

- The work is separated by mathematicians that barely know Lean, and formalizers that barely know analysis (or at least: are not experts). How much friction will this cause?

Observations

- The work is separated by mathematicians that barely know Lean, and formalizers that barely know analysis (or at least: are not experts). How much friction will this cause?
  - The blueprint definitely doesn't Mathlib best practices, and is sometimes tricky to formalize verbatim.

Observations

- The work is separated by mathematicians that barely know Lean, and formalizers that barely know analysis (or at least: are not experts). How much friction will this cause?
  - The blueprint definitely doesn't Mathlib best practices, and is sometimes tricky to formalize verbatim.
  - When reformulating results, it can be tricky as a non-expert to know if the result still holds.

Observations

- The work is separated by mathematicians that barely know Lean, and formalizers that barely know analysis (or at least: are not experts). How much friction will this cause?
  - The blueprint definitely doesn't Mathlib best practices, and is sometimes tricky to formalize verbatim.
  - When reformulating results, it can be tricky as a non-expert to know if the result still holds.
- I hope to show that it is feasible to formalize (some) current research in Fourier analysis.

Observations

- The work is separated by mathematicians that barely know Lean, and formalizers that barely know analysis (or at least: are not experts). How much friction will this cause?
    - The blueprint definitely doesn't Mathlib best practices, and is sometimes tricky to formalize verbatim.
    - When reformulating results, it can be tricky as a non-expert to know if the result still holds.
- I hope to show that it is feasible to formalize (some) current research in Fourier analysis.
- To beware: Mathlib-compatibility of the project.

In conclusion

- This project is a formalization of both current research and a very hard classical theorem.
- Please help if you are interested! I will lead a working group on this formalization this week.
    - I will make a `#Carleson` stream on the Lean Zulip channel for coordination.

Thank you for listening

`http://florisvandoorn.com/carleson/`