# Formalization of Walsh-Fourier series


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## Abstract

This master's thesis focuses on the formalization of key objects in harmonic analysis using the Lean Theorem Prover. In particular, dyadic intervals, Haar, Walsh, and Rademacher functions have been formalized, along with theorems describing their fundamental properties. Some basic objects that had previously been missing from Mathlib were identified and subsequently formalized. The main result of this thesis is the introduction, formal proof, and formalization of the integral form of the Walsh–Fourier series.

# Contents

# 1 Introduction

The main goal of formalizing mathematics is to eliminate human error in proofs and guarantee their correctness. Since such mistakes are not uncommon and their existence is well recognized, the formalization of mathematics is becoming more popular. By verifying proofs in a strict logical framework, theorem provers provide a certainty that is difficult to achieve without the help of computers.

One of the notable projects involving theorem prover is the Carleson project, which focuses on the formalization of a generalization of Carleson's theorem, specifically the results proved by Christoph Thiele and his collaborators [Bec+24].

A major difficulty in such projects is that it is necessary to formalize not only the main theorem but also much of the surrounding mathematics used in its proof. These prerequisites often extend far beyond the project itself. As a result, the Carleson project has contributed to the formalization of several important concepts from harmonic analysis, providing a foundation for further work in this area. However, it does not include all the basic definitions: fundamental objects such as dyadic intervals, Haar functions, Walsh functions, and Rademacher functions were not part of the project, since they were not needed for the generalization. These objects are, nevertheless, crucial components of harmonic analysis, and one of this thesis goals was to formalize them.

In chapter 2 we introduce the concepts needed to prove the integral form of the Walsh-Fourier series, alongside some of the proofs of their properties. The provided proofs aim to give a good intuition about the proofs implemented in Lean, which means they are not always the most concise or elegant versions, but rather those that were easier to formalize. Not all formalized properties and proofs are mentioned in the paper; mostly the most important statements are included, while some of the more basic were omitted or only presented as examples. In addition, we often refer to the Lean code, describing some of the steps and choices that were made during the formalization.

In chapter 3 we focused mainly on presenting the mathematical proof of the integral form of the Walsh-Fourier series. This section is presented in less detail than the previous one, as the corresponding Lean code is harder to follow due to its reliance on integrals, which tend to make the code more complex and repetitive.

Finally, chapter 4 presents observations and reflections that arose during the formalization process, including issues that appeared.

All proofs included in this work were carried out by the author, and the definitions are based on [Thi06], [MS13] or [Ste93]. The accompanying code can be found at
`https://github.com/izamandla/carleson/tree/thesis/Carleson/Project`.

## 1.1 Walsh-Fourier series analog of Carleson's theorem

The initial motivation for the thesis was to provide a new proof of the Walsh-Fourier series analog of Carleson's theorem and to formalize this result. Following comes from [Bec+24].

**Theorem 1.1.1** (Classical Carleson)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be a $2\pi$-periodic, continuous function. Then for almost all $x \in [0, 2\pi]$ we have

$$S_N f(x) \to f(x) \quad \text{as } N \to \infty, \tag{1.1}$$

where $S_N f$ is the $N$-th partial Fourier sum defined as

$$S_N f(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}, \tag{1.2}$$

with Fourier coefficients given by

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx. \tag{1.3}$$

Instead of the classical Fourier partial sums, we can consider the Walsh-Fourier partial sums, defined as

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x), \tag{1.4}$$

where $W_i$ is $i$-th Walsh function, for which an analogous theorem holds:

**Theorem 1.1.2.** Let $f : \mathbb{R} \to \mathbb{C}$ be a $2\pi$-periodic, continuous function. Then for almost all $x \in [0, 2\pi]$ we have

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) \to f(x) \quad \text{as } N \to \infty. \tag{1.5}$$

The theorem was first proven by P. Billard [Bil67] and later generalized by R. Hunt [Hun68]. Further results have built on these findings. The classical proof relies on the method using wave packets, whereas the new proof was intended to use results from the Carleson project, in particular the following:

**Theorem 1.1.3** (Linearised Metric Carleson)**.** For all integers $a \geq 4$ and real numbers $1 < q \leq 2$, the following holds. Let $(X, \rho, \mu, a)$ be a doubling metric measure space. Let $\Theta$ be a cancellative compatible collection of functions. Let $Q : X \to \Theta$ be a Borel function with finite range. Let $K$ be a one-sided Calderón-Zygmund kernel on $(X, \rho, \mu, a)$.

Assume that for every $\vartheta \in \Theta$ and every bounded measurable function $g$ on $X$ supported on a set of finite measure, we have

$$\|T_Q^{\vartheta} g\|_2 \leq 2^{a^3} \|g\|_2, \tag{1.6}$$

where

$$T_Q^{\vartheta} f(x) := \sup_{R_1 > 0} \sup_{\rho(x,x') < R_1} \left| \int_{R_1 < \rho(x',y) < R_Q(\vartheta, x')} K(x', y) f(y) \, d\mu(y) \right|. \tag{1.7}$$

Then, for all bounded Borel sets $F$ and $G$ in $X$, and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have

$$\left| \int_G T_Q f \, d\mu \right| \leq \frac{2^{45a^3}}{(q-1)6} \mu(G)^{1-\frac{1}{a}} \mu(F)^{\frac{1}{q}}, \tag{1.8}$$

where

$$T_Q f(x) := \sup_{0<R_1<R_2} \left| \int_{R_1<\rho(x,y)<R_2} K(x,y) f(y) e(Q(x)(y))\, d\mu(y) \right|. \tag{1.9}$$

The idea was to first find an integral form of the Walsh-Fourier series, use it to define the appropriate operator, and then by applying the Linearised Metric Carleson theorem derive the desired convergence result. In the this section, we considered functions defined on $[0, 2\pi)$, but for simplification purposes, in the rest of the paper we will restrict our attention to the interval $[0, 1)$, which allows us to work more conveniently with dyadic intervals and Walsh functions.

## 1.2 Integral form of Walsh-Fourier series

The main result of the thesis is the proof and formalization of the following theorem:

**Theorem 1.2.1.** For every $N \in \mathbb{N}$, let $\mathcal{M}$ be the binary representation set of $N$. Then, for every locally integrable function $f : \mathbb{R} \to \mathbb{R}$ and every $x \in [0, \infty)$, we have

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) = \int_0^1 W_N(x) K_{\mathcal{M}}(x,y) W_N(y) f(y)\, dy, \tag{1.10}$$

where

$$K_{\mathcal{M}}(x,y) = 1 + \sum_{m\in\mathcal{M}} \sum_{I\in\mathcal{I}^m} h_I(x) h_I(y). \tag{1.11}$$

## 1.3 Lean and mathlib

Lean is an interactive theorem prover that enables users to write precise mathematical definitions, statements, and proofs in a formal language. Among other features, it automatically verifies the correctness of proofs and provides tools to reason rigorously about complex mathematical structures. It is open source and has been under active development since 2013.

The most recent version, Lean 4, was released in 2021 and introduces improvements in performance, usability, and language features compared to earlier versions. A central component of Lean is its mathematical library, mathlib, which is largely community-driven and continuously expanding. Mathlib provides foundational theories across a wide range of mathematical fields, facilitating the formalization of advanced results and enabling researchers to build on prior formalizations. It is extensive, comprising approximately 1.8 million lines of code, with thousands of definitions and theorems contributed by over 550 developers. The library is actively maintained, with more than 200 contributions submitted weekly and reviewed by 28 maintainers and 22 reviewers. This thesis utilizes Lean 4 and mathlib to formalize fundamental objects in harmonic analysis.

In recent years, several notable projects have been conducted using Lean. The Carleson project, which this thesis relates to, is one such example; its homepage and GitHub repository can be found respectively at
`https://florisvandoorn.com/carleson/`,

https://github.com/fpvandoorn/carleson.
The project, officially completed in July 2025, was led by Floris van Doorn. Other examples of significant Lean projects include the formalization of Fermat's Last Theorem led by Kevin Buzzard and developments in Equational Theories led by Terence Tao.

# 2 Definitions and properties of formalized objects

## 2.1 Dyadic intervals

First, let us introduce dyadic intervals, which form a natural grid structure that is particularly useful in analysis and probability.

**Definition 2.1.1** (Dyadic interval)**.** $I$ is called dyadic if there exists $k, n \in \mathbb{Z}$ such that $I$ can be written as

$$I_{(k,n)} = [2^k n, 2^k(n+1)).$$

The properties of dyadic intervals are formalized in the Lean file `DyadicStructures`. Aside from basic equivalent definitions and illustrative examples, this includes results relating smaller intervals to larger ones and connecting these relations to the integers $k$ and $n$. There is a special focus on the case $I_{(-k,n)}$, where $k \in \mathbb{N}$ and $0 \leq n < 2^k$, since this corresponds exactly to considering dyadic intervals on the unit interval $[0, 1)$.

Intuitively, dyadic intervals are either nested or disjoint due to their alignment along powers of two. This hierarchical structure is crucial in harmonic analysis and forms the basis for constructions such as Haar bases. The Lean proof of this lemma is somewhat tedious, as it requires careful case distinctions and explicit manipulations of integers and real casts.

**Lemma 2.1.2** (Relationship between dyadic intervals)**.** Let $I_{(k,n)}$ and $I_{(k',n')}$ be dyadic intervals. Then at least one of the following holds:

$$I_{(k,n)} \subseteq I_{(k',n')}, \quad I_{(k',n')} \subseteq I_{(k,n)}, \quad I_{(k,n)} \cap I_{(k',n')} = \emptyset. \tag{2.1}$$

*Proof.* Let us see that if $k = k'$ the first two are true if and only if $n = n'$, and otherwise intervals are disjoint.
Let us choose $k < k'$. We put $p := k' - k$, so $p \geq 1$ and $2^{k'} = 2^k \cdot 2^p$. First let us consider a case where $2^k n < 2^{k'} n'$. Then we get $n < 2^p \cdot n'$, and therefor

$$2^k(n+1) \leq 2^k \cdot 2^p n' = 2^{k'} n',$$

so $I_{(k,n)}$ lies completely to the left of $I_{(k',n')}$ and the intersection is empty.
Now we assume that $2^k n \geq 2^{k'} n'$ and $2^k(n+1) \leq 2^{k'}(n'+1)$. Then for any $x \in I_{(k,n)}$,

$$2^{k'} n' \leq 2^k n \leq x < 2^k(n+1) \leq 2^{k'}(n'+1),$$

so $I_{(k,n)} \subseteq I_{(k',n')}$.
Finally if $2^k(n+1) > 2^{k'}(n'+1)$ then every point of $I_{(k,n)}$ is strictly larger than every point of $I_{(k',n')}$. So $I_{(k,n)} \cap I_{(k',n')} = \varnothing$.

□

## 2.2 Haar function

Now we introduce the Haar function and its scaled versions.

**Definition 2.2.1** (Haar function)**.** The Haar function $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) := \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For a dyadic interval $I_{(k,n)} = [2^k n, 2^k(n+1))$, the scaled Haar function associated with $I_{(k,n)}$ is

$$h_I(x) := 2^{-k/2}\, h\big(2^{-k}x - n\big).$$

For simplification let us write $h_{(k,n)}$. Haar functions play a crucial role in the project, as they allow the construction of Rademacher functions.

In addition to basic properties, in Lean file `Haar` the integrability of Haar functions was shown. This was done by showing that both the Haar function and its scaled versions are bounded, compactly supported, and measurable. For this purpose, the results formalized in the Lean file `BoundedCompactSupport` from the Carleson project were used. In the same file results about the Rademacher function are presented.

## 2.3 Rademacher function

**Definition 2.3.1** (Rademacher function)**.** For fixed $k \in \mathbb{N}$ and $t \in [0, 1)$ define the Rademacher function as

$$r_k(t) = 2^{-k/2} \sum_{0 \le n < 2^k} h_{(-k,n)}(t).$$

**Theorem 2.3.2.** For every $k \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$r_{k+1}(x) = r_k(2x) + r_k(2x - 1). \tag{2.2}$$

*Proof.* We rewrite the Rademacher function as

$$\begin{aligned} r_{k+1}(x) &= 2^{-(k+1)/2} \sum_{n=0}^{2^{k+1}-1} h_{(-(k+1),n)}(x) \\ &= 2^{-(k+1)/2} \sum_{n=0}^{2^k-1} h_{(-(k+1),\,n)}(x) \;+\; 2^{-(k+1)/2} \sum_{n=2^k}^{2^{k+1}-1} h_{(-(k+1),n)}(x). \end{aligned}$$

For $0 \le n < 2^k$

$$\begin{aligned} h_{(-k,n)}(2x) &= 2^{k/2}\, h\Big(2^k \cdot (2x) - n\Big) \\ &= 2^{k/2}\, h\Big(2^{k+1}x - n\Big) = 2^{-1/2}\, h_{(-(k+1),n)}(x), \end{aligned}$$

and for $2^k \leq n < 2^{k+1}$

$$h_{(-k,n)}(2x-1) = 2^{k/2}\, h\Big(2^k \cdot (2x-1) - n\Big)$$
$$= 2^{k/2}\, h\Big(2^{k+1}x - (2^k + n)\Big) = 2^{-1/2}\, h_{(-(k+1),n)}(x).$$

Therefor we get

$$\begin{aligned} r_{k+1}(x) &= 2^{-(k+1)/2} \sum_{n=0}^{2^k-1} 2^{1/2} h_{(-k,n)}(2x) \\ &+ \ 2^{-(k+1)/2} \sum_{n=2^k}^{2^{k+1}-1} 2^{1/2} h_{(-k,n)}(2x-1) = r_k(2x) + r_k(2x-1). \end{aligned} \tag{2.3}$$

□

As a direct corollary of this theorem, we obtain the following localization of $r_{k+1}$ depending on whether $x$ lies in the left or right half of the unit interval.

**Lemma 2.3.3.** Let $k \in \mathbb{N}$ and $x \in [0, \frac{1}{2})$. Then

$$r_{k+1}(x) = r_k(2x). \tag{2.4}$$

*Proof.* We observe that for $x \in [0, \frac{1}{2})$ we have $(2x-1) \notin [0,1)$, so $r_k(2x-1) = 0$.

□

**Lemma 2.3.4.** Let $k \in \mathbb{N}$ and $x \in [\frac{1}{2}, 1)$. Then

$$r_{k+1}(x) = r_k(2x-1). \tag{2.5}$$

*Proof.* We observe that for $x \in [\frac{1}{2}, 1)$ we have $2x \notin [0,1)$, so $r_k(2x) = 0$.

□

## 2.4 Walsh functions and Walsh-Fourier series

**Definition 2.4.1** (Walsh function on $[0,1)$)**.** We define Walsh functions on the unit interval $[0,1)$ by setting $W_0 \equiv 1$ and, recursively, for $n \geq 0$:

$$W_{2n}(x) = \begin{cases} W_n(2x), & x < 0.5, \\ W_n(2x-1), & x \geq 0.5, \end{cases} \tag{2.6}$$

$$W_{2n+1}(x) = \begin{cases} W_n(2x), & x < 0.5, \\ -W_n(2x-1), & x \geq 0.5. \end{cases} \tag{2.7}$$

In lean file `Walsh` the Walsh function was extended to $\mathbb{R}$ by choosing 0 in rest of the places. Due to their recursive definition, proofs of their properties often require induction, therefor the definition was slightly changed in order to make proofs easier to formalize.

```
def walsh (n : ℕ) : ℝ → ℝ
| x =>
if x <0 ∨ 1 ≤ x then 0
else if x < 1/2 then
let m := n / 2
if n = 0 then 1
else walsh m (2 * x)
else
if n = 0 then 1
else
let m := n / 2
if n % 2 = 0 then walsh m (2 * x - 1)
else -walsh m (2 * x - 1)
```

We first note some basic properties of Walsh functions. Their square equals 1 on $[0,1)$ and is zero elsewhere, implying that Walsh functions have norm equal 1. Furthermore, as the intervals on which the function takes the values 1 and $-1$ are of the same length, the integral of a $n$-th Walsh function over $[0,1)$ vanishes, where $n$ has to be greater than 0.

One can observe for every dyadic interval contained in $[0,1)$, its indicator function can be written as a linear combination of Walsh functions.

**Theorem 2.4.2** (Linear combination of Walsh functions)**.** For $M \in \mathbb{N}$ and $0 \leq k < 2^M$, there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that

$$\sum_{j=0}^{2^M-1} f(j)\, W_j(x) = \mathbf{1}_{I_{(-M,k)}}(x). \tag{2.8}$$

*Proof.* We proceed by induction on $M$.

**Base case:** Let $M = 0$. Then $k = 0$, and the dyadic interval is $[0,1)$. Its is enough to take $f(0) = 1$. The claim holds.

**Inductive step:** Let us assume the theorem holds for some $M \geq 0$, with coefficient function $g : \mathbb{N} \to \mathbb{R}$. We consider $M+1$. We split the sum into even and odd indices:

$$\sum_{j=0}^{2^{M+1}-1} f(j)\, W_j(x) = \sum_{j=0}^{2^M-1} f(2j)\, W_{2j}(x) + \sum_{j=0}^{2^M-1} f(2j+1)\, W_{2j+1}(x).$$

First let us consider $k < 2^M$. We choose

$$f(2j) = f(2j+1) = \frac{1}{2}g(j), \quad j = 0, \dots, 2^M - 1.$$

Using the property of Walsh functions which follows from it definition

$$W_j(2x) = \frac{1}{2}\big(W_{2j}(x) + W_{2j+1}(x)\big),$$

we obtain

$$\sum_{j=0}^{2^{M+1}-1} f(j)W_j(x) = \sum_{j=0}^{2^M-1} g(j)W_j(2x) = \mathbf{1}_{I_{(-M,k)}}(2x) = \mathbf{1}_{I_{(-(M+1),k)}}(x).$$

Now let $2^M \le k < 2^{M+1}$. We choose

$$f(2j) = \frac{1}{2}g(j), \quad f(2j+1) = -\frac{1}{2}g(j),$$

and since

$$W_j(2x-1) = \frac{1}{2}\big(W_{2j}(x) - W_{2j+1}(x)\big),$$

we get

$$\sum_{j=0}^{2^{M+1}-1} f(j)W_j(x) = \sum_{j=0}^{2^M-1} g(j)W_j(2x-1) = \mathbf{1}_{I_{(-M,k-2^M)}}(2x-1) = \mathbf{1}_{I_{(-(M+1),k)}}(x).$$

Thus, by induction, the theorem holds for all $M \in \mathbb{N}$. □

By choosing dyadic intervals sufficiently small, we observe that Walsh functions are constant on them. This leads to the following result.

**Theorem 2.4.3** (Walsh functions on dyadic intervals)**.** For any dyadic interval of length $2^{-M}$ in $[0,1)$, the Walsh function $W_n(x)$ such that $n < 2^M$ is constant on that interval, that is there exists $c \in \mathbb{R}$ such that

$$\forall x \in I_{(-M,k)}, \quad W_n(x) = c,$$

where $0 \le k < 2^M$.

*Proof.* We proceed by induction on $M$.

**Base case:** Let $M = 0$. Then $k = 0$, so the dyadic interval is $[0,1)$, and since $n < 1$, we must have $n = 0$. Therefor $W_0(x) = 1$ for all $x \in [0,1)$, so we can take $c = 1$.

**Inductive step:** Let us assume the statement holds for dyadic intervals of length $2^{-M}$. We consider an interval $I_{(-(M+1),k)}$.

If $I_{(-(M+1),k)} \subset [0,1/2)$ then $k < 2^M$ and if $I_{(-(M+1),k)} \subset [1/2,1)$ then $2^M \le k < 2^{M+1}$.

Let us consider first case. We observe that on $[0,1/2)$ we have

$$W_{2n}(x) = W_n(2x), \quad W_{2n+1}(x) = W_n(2x).$$

We can choose $j$ such that $I_{(-(M+1),k)} \subset I_{(-M,j)}$ and then, by the inductive hypothesis, $W_n(2x)$ is constant on $I_{(-M,j)}$. Therefore $W_n(x)$ is constant on $I_{(-(M+1),k)}$.

Now considering $[1/2, 1)$ we use

$$W_{2n}(x) = W_n(2x-1), \quad W_{2n+1}(x) = -W_n(2x-1),$$

and similarly it holds for $I_{(-(M+1),k)}$. Therefor, by induction, the theorem holds for all $M$. $\square$

Walsh functions provide the foundation for constructing Walsh–Fourier series.

**Definition 2.4.4** (Walsh Fourier series)**.** Let $f \in L^2([0,1])$. The Walsh-Fourier series of $f$ is the series

$$\sum_{l=0}^{\infty} \langle f, W_l \rangle W_l,$$

where the Walsh inner product is defined by

$$\langle f, W_l \rangle := \int_0^1 f(x) W_l(x)\, dx.$$

## 2.5 Binary representation set

Before stating the relationship between Walsh and Rademacher functions, we first introduce the notion of a binary representation set, which allows to rewrite a natural number as a sum of powers of two.

**Definition 2.5.1** (Binary representation set)**.** For $N \in \mathbb{N}$, the binary representation set $\mathcal{M}(N)$ is the unique subset of $\mathbb{N}$ such that

$$N = \sum_{m \in \mathcal{M}(N)} 2^m.$$

Although this construction is rather elementary, it was not provided in mathlib. Therefore, it was necessary to define it explicitly together with some basic lemmas about them. The above mathematical definition is not the most convenient one to formalize in Lean. Instead, we rely on a formulation:

```
def binaryRepresentationSet (n : ℕ) : Finset ℕ :=
  Finset.filter (fun m => Nat.testBit n m)
                (Finset.range (Nat.size n + 1))
```

This definition can be interpreted as follows: We look at the binary expansion of natural number $n$ and we take a set of all numbers (smaller than $n$) that refer to positions in which the digit equals 1.

To show the definition used in Lean actually implies **??** 2.5.1, we first look at the behavior of binary representation sets for even and odd numbers.

**Theorem 2.5.2.** For all $M, n, k \in \mathbb{N}$ the following hold:

1. `binaryRepresentationSet (2^M) = {M}`

2. `binaryRepresentationSet (2*n + 1) = insert 0 (binaryRepresentationSet (2*n))`

3. `k ∈ binaryRepresentationSet n ↔ (k+1) ∈ binaryRepresentationSet (2*n)}`

4. `k ∈ binaryRepresentationSet n ↔ (k+1) ∈ binaryRepresentationSet (2*n +1)\ {0}`

*Proof.* 1. The number $2^M$ has binary expansion with a single nonzero digit, namely at position $M$.

2. Multiplication by 2 shifts the binary expansion left, appending a zero in the first position; adding 1 inserts a 1 there.

3. Multiplication by 2 shifts all binary digits one place to the left; hence every $k \in \mathcal{M}(n)$ corresponds to $k+1 \in \mathcal{M}(2n)$.

4. Immediate from (2) and (3).

□

We can now prove that the Lean definition satisfies the intended property.

**Theorem 2.5.3** (Explicit version)**.** For all $n \in \mathbb{N}$ we have

```
∑ k ∈ binaryRepresentationSet n, 2 ^ k = n.
```

*Proof.* We proceed by induction on $n$.

**Base case:** For $n = 0$ the set `binaryRepresentationSet n` is empty, so the sum vanishes and the statement holds.

**Inductive step:** Suppose the claim holds for $n > 0$.

If $n$ is even, say $n = 2m$, then by the lemma above we obtain

```
∑ k ∈ binaryRepresentationSet (2 * m), 2 ^ k =
∑ k ∈ binaryRepresentationSet m, 2 ^ (k + 1) =
2*∑ k ∈ binaryRepresentationSet m, 2 ^ k = 2*m = n
```

If $n$ is odd, say $n = 2m + 1$, then

```
∑ k ∈ binaryRepresentationSet (2 * m + 1), 2 ^ k =∑ k ∈
  binaryRepresentationSet m, 2 ^ (k + 1) + ∑ k ∈ {0}, 2 ^ k = 2*m + 1= n
```

Thus by induction the property holds for all $n \in \mathbb{N}$. □

From now on, we denote by $\mathcal{M}(n)$ the binary representation set of $n$ as defined in the Lean code.

Finally, an interesting result relates XOR of numbers to binary representation sets. The XOR (exclusive or) of two numbers is obtained by comparing their binary digits bitwise and setting each bit to 1 if exactly one of the corresponding bits is 1, and 0 otherwise. We denote it here as $\oplus$.

**Theorem 2.5.4** (XOR characterization)**.** For all $M, N, k \in \mathbb{N}$ we have

$$k = M \oplus N \iff \mathcal{M}(k) = \big(\mathcal{M}(M) \setminus \mathcal{M}(N)\big) \cup \big(\mathcal{M}(N) \setminus \mathcal{M}(M)\big). \tag{2.9}$$

*Proof.* ($\Rightarrow$) Let us assume $k = M \oplus N$. Let $a \in \mathbb{N}$. Then

$$a \in \mathcal{M}(k) \iff k \text{ has a 1 in the } a\text{-th bit} \iff M(a) \oplus N(a) = 1,$$

where $M(a)$ and $N(a)$ denote the $a$-th bits of $M$ and $N$.

Thus exactly one of $M(a), N(a)$ equals so $a \in \mathcal{M}$ or $a \in \mathcal{M}(N) \setminus \mathcal{M}(M)$. Hence

$$\mathcal{M}(k) = \big(\mathcal{M}(M) \setminus \mathcal{M}(N)\big) \cup \big(\mathcal{M}(N) \setminus \mathcal{M}(M)\big).$$

($\Leftarrow$) Conversely, suppose $\mathcal{M}(k) = \big(\mathcal{M}(M) \setminus \mathcal{M}(N)\big) \cup \big(\mathcal{M}(N) \setminus \mathcal{M}(M)\big)$. Then for each $a \in \mathbb{N}$ we have

$$a \in \mathcal{M}(k) \iff \big(M(a) = 1 \wedge N(a) = 0\big) \text{ or } \big(M(a) = 0 \wedge N(a) = 1\big), \tag{2.10}$$

which is equivalent to $M(a) \oplus N(a) = 1$. Therefore the $a$-th bit of $k$ equals the $a$-th bit of $M \oplus N$, for all $a$., and $k = M \oplus N$. □

Worth mentioning is also a fact that if $M$ satisfies $2^M \leq N < 2^{M+1}$, then $M \in \mathcal{M}(N)$.

## 2.6 Relation between Walsh and Rademacher functions.

For easier proofs of some properties of Walsh functions it is crucial to first relate them to the Rademacher functions. This connection allows us to prove their orthogonality and simplifies the treatment of multiplication.

**Theorem 2.6.1.** For $n \in \mathbb{N}$ and $\mathcal{M}$ being binary representation set for $n$ it holds

$$W_n(x) = \prod_{m \in \mathcal{M}} r_m(x). \tag{2.11}$$

.

*Proof.* Let us first consider $x \in [0, 1)$. We conduct the proof by induction on $n$.

**Base case:** If $n = 0$, then $\mathcal{M}(0) = \varnothing$. Since the product over the empty set equals 1 and $W_0(x) = 1$, the statement holds.

**Inductive step:** Suppose the result holds for all natural numbers less than $n$.

For $n$ odd let us write $n = 2k + 1$. If $x < \frac{1}{2}$, then

$$W_{2k+1}(x) = W_k(2x) = \prod_{m \in \mathcal{M}(k)} r_m(2x).$$

From the properties of the binary representation set we have

$$r_0(x) \cdot \prod_{m \in \mathcal{M}(k)} r_{m+1}(x) = \prod_{m \in \mathcal{M}(2k+1)} r_m(x).$$

Since $x \in [0, \frac{1}{2})$ we obtain $r_0(x) = 1$ and $r_{m+1}(x) = r_m(2x)$, which gives the result. If instead $x \geq \frac{1}{2}$, then

$$W_{2k+1}(x) = -W_k(2x - 1) = -\prod_{m \in \mathcal{M}(k)} r_m(2x - 1).$$

Here $r_0(x) = -1$ and $r_{m+1}(x) = r_m(2x - 1)$, so again the result follows.

For $n$ even let us write $n = 2k$. From the binary representation properties we know

$$\prod_{m \in \mathcal{M}(k)} r_{m+1}(x) = \prod_{m \in \mathcal{M}(2k)} r_m(x).$$

If $x < \frac{1}{2}$, then

$$W_{2k}(x) = W_k(2x) = \prod_{m \in \mathcal{M}(k)} r_m(2x),$$

and since $r_{m+1}(x) = r_m(2x)$ the claim holds. If $x \geq \frac{1}{2}$, then

$$W_{2k}(x) = W_k(2x - 1) = \prod_{m \in \mathcal{M}(k)} r_m(2x - 1),$$

and with $r_{m+1}(x) = r_m(2x - 1)$ the claim follows.

Thus the proof is complete for $x \in [0, 1)$. If $x \notin [0, 1)$, both sides of the equality vanish, so the statement also holds in this case. □

As a special case, we obtain $W_{2^n}(x) = r_n(x)$. This shows the importance of the binary representation set in the study of Walsh functions. Moreover, combining this result with the XOR property of binary sets yields the following.

**Theorem 2.6.2.** Let $M, N, k \in \mathbb{N}$ and $x \in \mathbb{R}$ with $0 \leq x < 1$. If $k = M \oplus N$, then

$$W_k(x) \;=\; W_M(x) \cdot W_N(x).$$

Finally, we can prove orthogonality.

**Theorem 2.6.3** (Orthogonality of Walsh functions)**.** Let $m, n \in \mathbb{N}$ with $m \neq n$. Then

$$\langle W_n, W_m \rangle = 0.$$

*Proof.* Let $k = m \oplus n$. By previous theorem we have

$$W_n(x)\, W_m(x) = W_k(x), \qquad x \in [0, 1).$$

Hence

$$\int_0^1 W_n(x)\, W_m(x)\, dx = \int_0^1 W_k(x)\, dx.$$

Since $k \neq 0$ (because $m \neq n$), the integral of $W_k$ over $[0, 1)$ vanishes. Therefore

$$\int_0^1 W_n(x)\, W_m(x)\, dx = \langle W_n, W_m \rangle = 0.$$

□

## 2.7 Properties of the product of Walsh and Haar functions

For simplification purposes, we now consider the product of a $2^M$-th Walsh function and a Haar function scaled to interval of length $2^{-M}$. This object is particularly interesting since it corresponds to the indicator function of a dyadic interval, scaled by a constant.

**Theorem 2.7.1.** Let $M, k \in \mathbb{N}$ and $x \in \mathbb{R}$ with $0 \leq x < 1$. If $k \in \{0, 1, \dots, 2^M - 1\}$, then

$$W_{2^M}(x)\, h_{(-M,k)}(x) = \mathbf{1}_{I_{(-M,k)}}(x) \cdot 2^{M/2}.$$

Moreover, the family of functions defined in this way actually forms an orthogonal basis for the space of functions that are piecewise constant on dyadic intervals of length $2^{-M}$.

**Theorem 2.7.2** (Orthogonality of Walsh–Haar functions)**.** Let $M, k, k' \in \mathbb{N}$ with $k, k' < 2^M$ and $k \neq k'$. Then

$$\int_0^1 W_{2^M}(x)\, h_{(-M,k)}(x)\, W_{2^M}(x)\, h_{(-M,k')}(x)\, dx = 0.$$

**Theorem 2.7.3** (Normalization of Walsh–Haar functions)**.** Let $M, k \in \mathbb{N}$ and $k < 2^M$. Then

$$\int_0^1 \left(W_{2^M}(x)\, h_{(-M,k)}(x)\right)^2 dx = 1.$$

# 3 Integral form of Walsh-Fourier series

The main result provided in this thesis can be stated as following

**Theorem 3.0.1.** For every $N \in \mathbb{N}$ let $\mathcal{M}$ be the binary representation set of $N$. Then, for every locally integrable function $f : \mathbb{R} \to \mathbb{R}$ and every $x \in [0, \infty)$, we have

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) = \int_0^1 W_N(x) K_{\mathcal{M}}(x, y) W_N(y) f(y)\, dy \tag{3.1}$$

with

$$K_{\mathcal{M}}(x, y) = 1 + \sum_{m \in \mathcal{M}} \sum_{n=0}^{2^m - 1} h_{(-m,n)}(x) h_{(-m,n)}(x). \tag{3.2}$$

In order to prove it, let us first state two lemmata

**Lemma 3.0.2.** For all $M > 0$ and $N$ such that $2^M \leq N < 2^{M+1}$

$$\begin{aligned} \sum_{i=0}^{2^M - 1} \langle f, W_i \rangle W_i(x) &= \sum_{n=0}^{2^M - 1} \left\langle f, W_{2^M} h_{(-M,n)} \right\rangle W_{2^M} h_{(-M,n)}(x) \\ &= \sum_{n=0}^{2^M - 1} \left\langle f, W_N h_{(-M,n)} \right\rangle W_N h_{(-M,n)}(x). \end{aligned} \tag{3.3}$$

*Proof.* Lets notice that both $(W_i)_{i=0}^{2^M-1}$ and $(W_{2^M} h_{(-M,i)}(x))_{i=0}^{2^M-1}$ are orthonormal systems and they both span the space of functions that are constant on dyadic intervals of length $2^{-M}$. Therefor, the first equation holds because of the equality of orthogonal projections.

Using (2.11) we have

$$W_N = W_{2^M} \prod_{m \in \mathcal{M} \setminus M} r_m(x).$$

Therefore,

$$\begin{aligned} &\sum_{n=0}^{2^M - 1} \left\langle f, W_N h_{(-M,n)} \right\rangle W_N h_{(-M,n)}(x) \\ = &\int_0^1 W_{2^M}(x) \left( \sum_{n=0}^{2^M - 1} \prod_{m \in \mathcal{M} \setminus M} r_m(x) r_m(y) h_{(-M,n)}(x) h_{(-M,n)}(y) \right) W_{2^M}(y) f(y)\, dy. \end{aligned}$$

Observe that $h_{(-M,n)}(x) h_{(-M,n)}(y) = 1$ whenever both $x, y$ lie in the same half of interval $I = [2^{-M} n, 2^{-M}(n + 1))$ and $h_{(-M,n)}(x) h_{(-M,n)}(y) = -1$ when they are in different halves.

Since the entire interval $I$ is always contained in one half of $[2^{-m}k, 2^{-m}(k+1))$ for some $m \in \mathcal{M} \setminus M$ and $k < 2^m$, it follows that $r_m(x)r_m(y) = 1$. So

$$\sum_{n=0}^{2^M-1} \left\langle f, W_{2^M} h_{(-M,n)} \right\rangle W_{2^M} h_{(-M,n)}(x) = \sum_{n=0}^{2^M-1} \left\langle f, W_N h_{(-M,n)} \right\rangle W_N h_{(-M,n)}(x).$$

□

The part of the proof that uses orthogonal projections need to shown more directly in Lean. This was proven by showing that each one system can be rewritten as linear combination of another one, which was mostly contained in file `coef`.

**Lemma 3.0.3.** For all $M > 0$, $N$ such that $2^M \leq N < 2^{M+1}$ and $N' = N - 2^M$

$$\sum_{i=2^M}^{N} \langle f, W_i \rangle W_i(x) = \sum_{i=0}^{N'} \langle f, r_M W_i \rangle r_M W_i(x). \tag{3.4}$$

*Proof.* Using (2.11) we have

$$\sum_{i=2^M}^{N} \langle f, W_i \rangle W_i(x) = \sum_{i=2^M}^{N} \int_0^1 W_i(x) W_i(y) f(y)\, dy =$$

$$\sum_{i=0}^{N-2^M} \int_0^1 r_M(x) W_i(x) r_M(y) W_i(y) f(y)\, dy = \sum_{i=0}^{N'} \langle f, r_M W_i \rangle r_M W_i(x).$$

□

Now let us begin with the main proof.

*Proof.* We prove the theorem by induction.

**Base case:** Let $N = 0$. Then, by definition, $\mathcal{M}$ is the empty set, so

$$K_{\mathcal{M}}(x, y) = 1.$$

For arbitrary $f$ and $x$, recalling that $W_0 \equiv 1$, we have

$$\sum_{i=0}^{0} \langle f, W_i \rangle W_i(x) = \langle f, W_0 \rangle W_0(x) = \int_0^1 f(y)\, dy$$

$$= \int_0^1 K_{\mathcal{M}}(x, y) f(y)\, dy = \int_0^1 W_0(x) K_{\mathcal{M}}(x, y) W_0(y) f(y)\, dy.$$

**Inductive step:** Now let $N > 0$ be and assume that (3.1) holds for all $n < N$. Let

$$N = \sum_{m \in \mathcal{M}} 2^m.$$

We have $N \neq 0$, so $\mathcal{M}$ is not empty and thus contains a maximal element $M$. Let $\mathcal{M}'$ be $\mathcal{M} \setminus \{M\}$ and

$$N' := \sum_{m \in \mathcal{M}'} 2^m = \sum_{m \in \mathcal{M}} 2^m - 2^M = N - 2^M,$$

so that $N' < N$. Let us observe that

$$\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) = \sum_{i=0}^{2^M-1} \langle f, W_i \rangle W_i(x) + \sum_{i=2^M}^{N} \langle f, W_i \rangle W_i(x). \tag{3.5}$$

Applying (3.5) together with lemmata, we obtain

$$\begin{aligned}
\sum_{i=0}^{N} \langle f, W_i \rangle W_i(x) &= \sum_{n=0}^{2^M-1} \left\langle f, W_N h_{(-M,n)} \right\rangle W_N h_{(-M,n)}(x) + \sum_{i=0}^{N'} \langle f, r_M W_i \rangle r_M W_i(x) \\
&= \int_0^1 W_N(x) \left( \sum_{n=0}^{2^M-1} h_{(-M,n)}(x) h_{(-M,n)}(y) \right) W_N(y) f(y)\, dy \\
&\quad + \int_0^1 r_M(x) W_{N'}(x) K_{\mathcal{M}'}(x,y) r_M(y) W_{N'}(y) f(y)\, dy \\
&= \int_0^1 W_N(x) K_{\mathcal{M}}(x,y) W_N(y) f(y)\, dy.
\end{aligned}$$

□

# 4 Lean

As noted in earlier sections, the files in the Lean project are often named according to the parts of the proof they implement. For instance, `lemmas.lean` contains the lemmas used in the proof of the integral form.

The formalization process was challenging, and several difficulties were encountered. Below, we highlight some of the most important ones.

1. At the beginning of the project, the file `DyadicIntervals` was a bit tedious to work with for a beginner because it required paying close attention to variable types. Conceptually, though, it is a very simple section.

2. Because of the recursive definition of Walsh functions, the file `Walsh` involves repeating similar proofs by induction. This felt quite tiring at first.

3. The file `BinaryRepresentationSet` covers concepts that, intuitively, should already be in Mathlib, but weren't. It was crucial to include this section. Since it mostly deals with structures different from the rest of the project, it was confusing at first, but became manageable over time.

4. Induction is used often in the formalization. While we frequently need the even-odd case induction (e.g., for Walsh functions), the recursion principle `Nat.evenOddRec` is not always suitable because it doesn't always allow to use `generalization`.

5. Several basic lemmas were missing from Mathlib. Some examples:

   a) Integer version `ENNReal.ofReal_pow`, used to compute the length of dyadic intervals.

   b) The theorem $(2n + 1)/2 = n$, coded as `odd_div`.

   c) A theorem stating that subsets of natural numbers can be written as a sum of their odd and even elements, used in code as `sum_of_even_odd_set`.

   d) Division by 2 is injective on subsets of $\mathbb{N}$ containing only even or only odd numbers, coded as `div_of_nat_inj_even` and `div_of_nat_inj_odd`.

6. There was no convenient way to use properties of orthogonal projections. This was needed to rewrite Walsh functions as products of Walsh and Rademacher functions in Lemma 3.2. Although orthogonal projections exist in Lean, using them directly wasn't intuitive. It was simpler to show that both functions can be written as linear combinations of the other, which was a lot of work, and if there was a different structure provided it would simplified a significant part of the project.

7. Handling integrability often requires repeating similar arguments. It would be helpful if a tactic existed to streamline this process. Moreover, at first it was difficult to construct proofs establishing integrability at all. Using the theorems from

`MeasureTheory.BoundedCompactSupport` in the Carleson project significantly simplified this task.

8. The tactic `simp_rw` doesn't always work well with theorems about sums that also involve integrals. For instance, using `Finset.sum_comm` in `simp` would allow to provide cleaner proof, but Lean reports a "possibly looping lemma" error.

There also some other aspects worth mentioning. For example in several parts of the formalization, it was helpful to rewrite lemmas in a more functional form. Doing so often made them easier to apply in later proofs, reduced the need for repeated case analysis, and improved compatibility with Lean's tactic framework. One illustrative example concerns the representation of Walsh functions as linear combinations of indicator functions over dyadic intervals:

```
theorem walshindicator' {M k : ℕ} (hk : k < 2 ^ M) :
∃ (f : ℕ → ℝ),
(fun x ↦ ∑ j ∈ Finset.range (2^M), (walsh j x * f j)) =
(fun x ↦ (Ico (k * 2 ^ (-M : ℤ) : ℝ)
((k+1)* 2 ^ (-M : ℤ) : ℝ)).indicator 1 x)
```

Certain parts of the project could still be improved. In particular, it would be possible to develop code that more closely connects Walsh, Haar, and Rademacher functions with dyadic intervals, resulting in a clearer overall structure.

# 5 Conclusion

The initial goal of this master's thesis was not fully achieved. However, the project produced important results, including a mathematical proof of a crucial theorem necessary for the original goal, as well as a formalization of most of the required structures and their properties. Since the work dealt primarily with foundational objects in harmonic analysis, it could serve as a solid starting point for their future inclusion in MathLib, enabling broader use and accessibility.

The formalization presented here constitutes a relatively complete set of useful theorems, providing a strong foundation for further development. In the future, it would be beneficial to reorganize and refine the code to create an even cleaner structure, improving readability and facilitating extensions. Overall, while the initial goal was only partially reached, this thesis contributes valuable formal results and lays the groundwork for further formalization work.

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