Lessons learned from formalizing local convex integration

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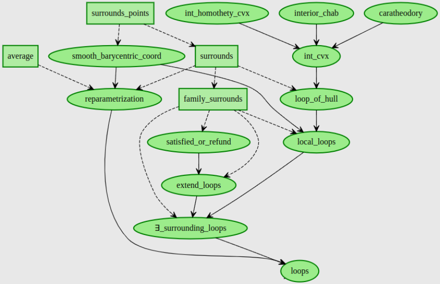
Sphere Eversion Project

This talk is a continuation of Patrick's talk from the morning.

The goal of the Sphere Eversion project is to formalize convex integration, with sphere eversion as immediate application.

Collaboration of Patrick Massot, Oliver Nash and me.

Dependency Graph



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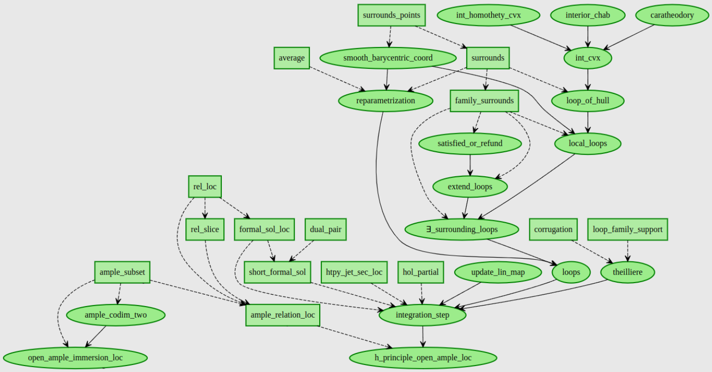

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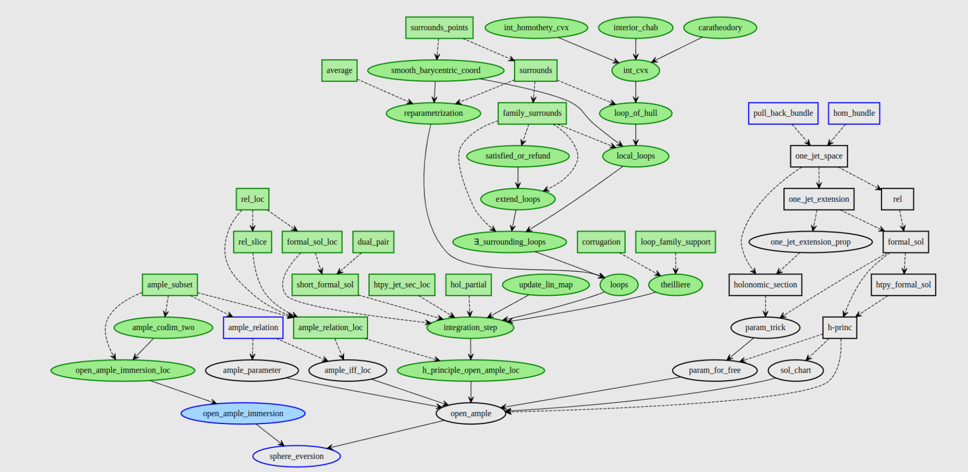
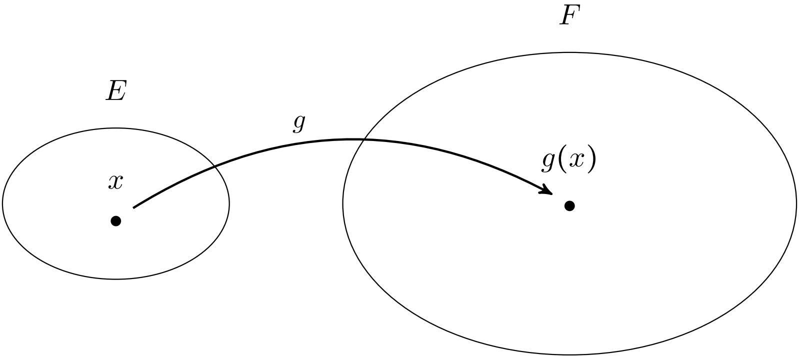


Picture



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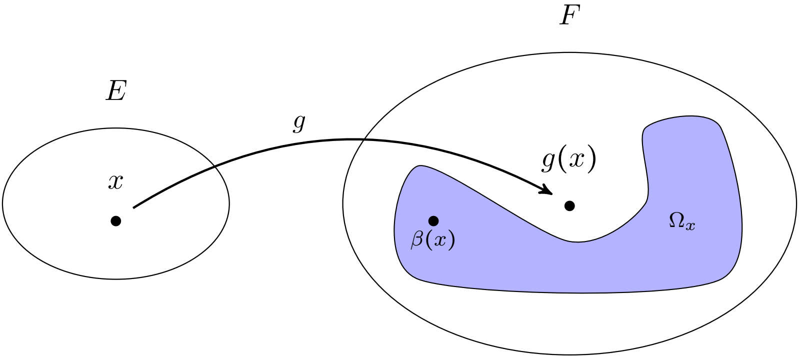

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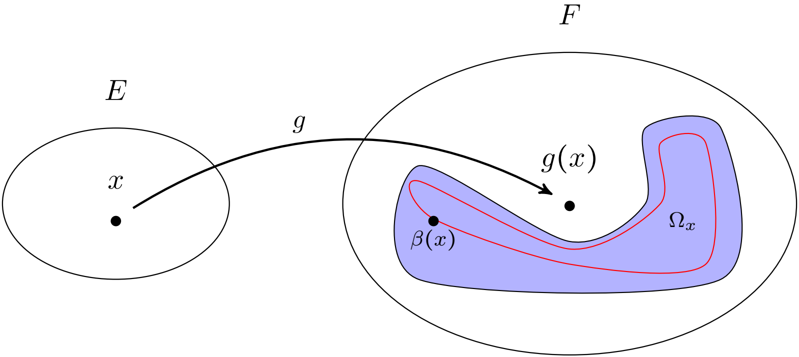

Main Proposition

Proposition

Let Ω be an open set in $E \times F$ such that, for each x the set $\Omega_x := \{y \in F \mid (x, y) \in \Omega\}$ is connected in F.
Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x.
Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.

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Assume that $\beta(x) \in \Omega_x$ for all x.
Suppose that for every x the value $g(x)$ is in the convex hull of Ω_x.
Then there exists a smooth family of loops

$$\gamma : E \times [0,1] \times \mathbb{S}^1 \to F, \quad (x, t, s) \mapsto \gamma_x^t(s)$$

such that, for all $x \in E$, all $t \in \mathbb{R}$ and all $s \in \mathbb{S}^1$,

- $\gamma_x^t(s) \in \Omega_x$
- $\gamma_x^0(s) = \gamma_x^t(1) = \beta(x)$
- *the average of γ_x^1 is $\overline{\gamma_x^1} = g(x)$*

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Let Ω be an open set in $E \times F$ such that, for each x the set $\Omega_x := \{y \in F \mid (x,y) \in \Omega\}$ is connected in F. Let K a compact set in E. Let β and g be smooth maps from E to F.
Assume that $\beta(x) \in \Omega_x$ for all x, and $g(x) = \beta(x)$ near K.
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- *the average of γ_x^1 is $\overline{\gamma_x^1} = g(x)$*
- $\gamma_x^t(s) = \beta(x)$ *if x is near K.*

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Pointwise version

Proposition

Let Ω be an open connected set in F.

Let $\beta, g \in F$.
Assume that $\beta \in \Omega$.
Suppose that g is in the convex hull of Ω.
Then there exists a smooth family of loops

$$\gamma : [0,1] \times \mathbb{S}^1 \to F, \quad (t,s) \mapsto \gamma^t(s)$$

such that for all $t \in \mathbb{R}$ and all $s \in \mathbb{S}^1$,

- $\gamma^t(s) \in \Omega$
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- $\gamma^t(s) \in \Omega$
- $\gamma^0(s) = \gamma^t(1) = \beta$
- γ^1 *surrounds g.*

Pointwise solution

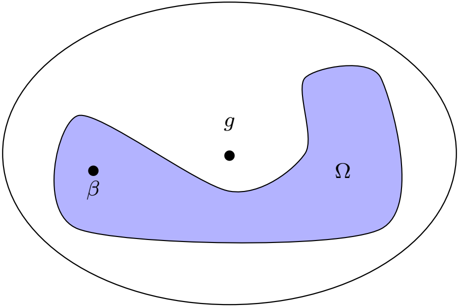
Definition

- A point $x \in F$ is surrounded by points a finite set of points $\{p_i\}$ if those points form an affine basis and there exist weights $w_i \in (0,1)$ with sum 1 such that $x = \sum_i w_i p_i$.
- A set A surrounds x if there is a collection of points in A surrounding x.

Lemma

If a point $x \in F$ lies in the convex hull of an open set P then P surrounds x.

Pointwise solution



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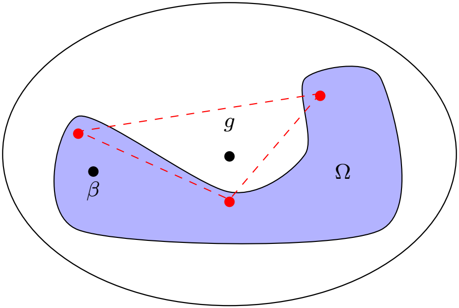
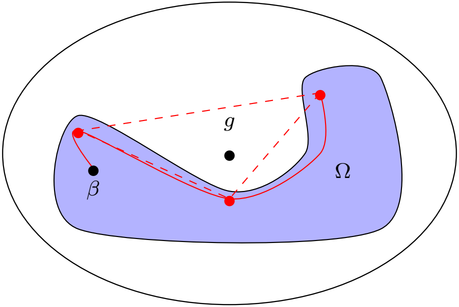


Proof.

By the lemma, pick a collection of points surrounding g.

□

Pointwise solution

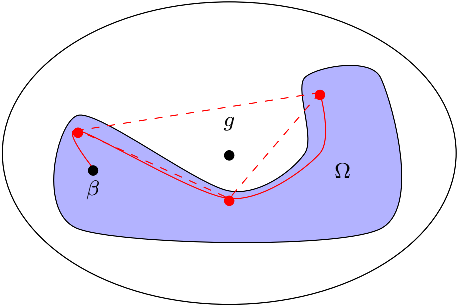


Proof.

By the lemma, pick a collection of points surrounding g. Since Ω is open and connected, we can find a path starting at β through these points.

$\square$

Pointwise solution



Proof.

By the lemma, pick a collection of points surrounding g. Since Ω is open and connected, we can find a path starting at β through these points. Retracing the same path back to β, we obtain a path that is homotopic to the constant path. □

Local solution

Suppose we have such a γ_{x_0} based at $\beta(x_0)$ surrounding $g(x_0)$.

We can extend this to a local solution around x_0:

$$\gamma_x^t(s) = \gamma_{x_0}^t(s) + \beta(x) - \beta(x_0)$$

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Then:

- $\gamma_x^0(s) = \gamma_x^t(1) = \beta(x)$
- For x sufficiently close to x_0 we have $\gamma_x^t(s) \in \Omega_x$
- For x sufficiently close to x_0 the family γ_x^1 surrounds $g(x)$.

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```
∀ᶠ x in 𝓝 x₀, ∀ (t ∈ I) (s ∈ I), (x, γ x t s) ∈ Ω
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```

Local solution

We write $\gamma \in \mathcal{L}(U)$ for such a γ defined on $U \subseteq E$.

So $\gamma \in \mathcal{L}(U)$ means that γ is a continuous family of loops

$$\gamma : E \times [0,1] \times \mathbb{S}^1 \to F, \quad (x,t,s) \mapsto \gamma_x^t(s)$$

such that for every $x \in U$, every $t \in [0,1]$ and every $s \in \mathbb{S}^1$,

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- γ_x^1 surrounds $g(x)$
- $(x, \gamma_x^t(s)) \in \Omega$.

Gluing solutions

We want to glue these local solutions together, to obtain a global solution defined on all of E.

Let's start with two local solutions $\gamma_0 \in \mathcal{L}(U_0)$ and $\gamma_1 \in \mathcal{L}(U_1)$.

We want to get a solution on $U_0 \cup U_1$.

One thing we need is to transition between γ_0 to γ_1 on $U_0 \cap U_1$.

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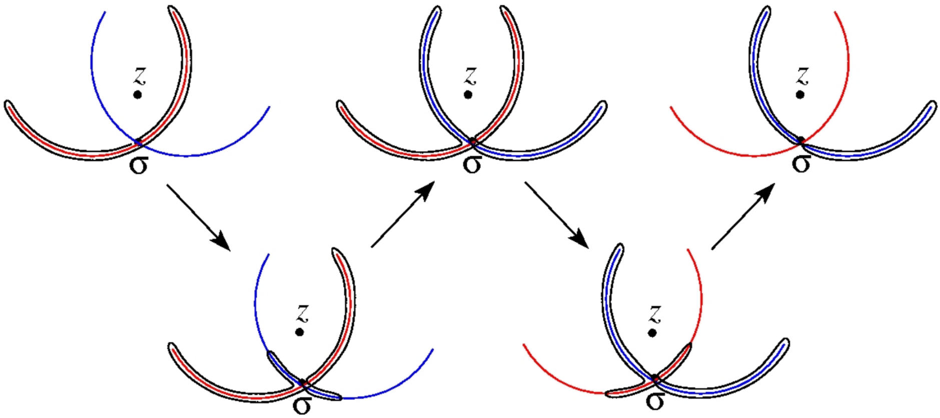
Lemma

If $U \subseteq E$ then $\mathcal{L}(U)$ is contractible: if $\gamma_0, \gamma_1 \in \mathcal{L}(U)$ then there is a continuous homotopy

$$\delta : [0,1] \times E \times [0,1] \times \mathbb{S}^1 \to F, \quad (\tau, x, t, s) \mapsto \delta^t_{\tau,x}(s)$$

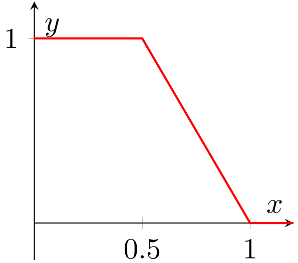
such that $\delta_\tau \in \mathcal{L}(U)$ and $\delta_i = \gamma_i$ for $i \in \{0,1\}$.

Transitioning between solutions



Transitioning between solutions

Let $\rho : \mathbb{R} \to \mathbb{R}$ be the following piecewise affine function:



We can define the homotopy δ as follows:

- $\delta^t_{\tau,x}$ moves along the loop $\gamma^{\rho(\tau)t}_{0,x}$ once on $[0, 1 - \tau]$ (if $\tau < 1$)
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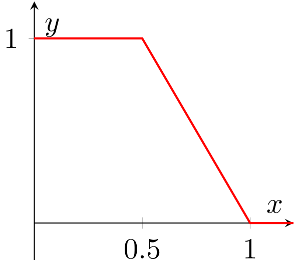

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Note that the image of $\delta^1_{\tau,x}$ contains the image of $\gamma^1_{0,x}$ for $\tau \leq \frac{1}{2}$, and the image of $\gamma^1_{1,x}$ for $\tau \geq \frac{1}{2}$. Hence it will always surround $\beta(x)$.

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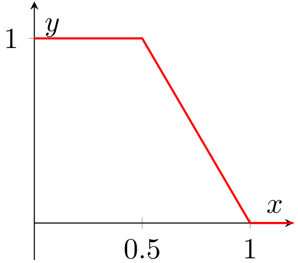

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Is δ continuous?

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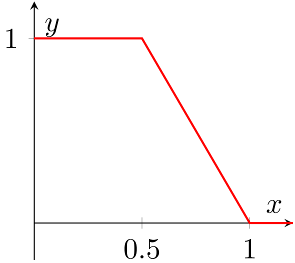

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Is δ continuous? Note that if $\tau \to 1$ then $\delta^1_{\tau,x}$ will move along loop $\gamma^{\rho(\tau)t}_{0,x}$ at a speed that tends to $+\infty$, so we need to show that $\gamma^{\rho(\tau)t'}_{0,x'}$ tends uniformly to the constant loop as $(x', \tau, t') \to (x, 1, t)$.

Transitioning between solutions

```
lemma satisfied_or_refund (γ₀ γ₁ : E → ℝ → loop F)
  (h₀ : γ₀ ∈ ℒ U) (h₁ : γ₁ ∈ ℒ U) :
  ∃ γ : ℝ → E → ℝ → loop F,
    (∀ τ, γ τ ∈ ℒ U) ∧
    γ 0 = γ₀ ∧
    γ 1 = γ₁ ∧
    continuous ↿γ
```

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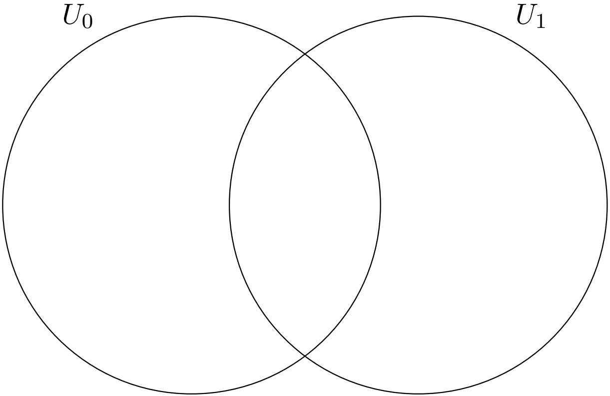
Let $\gamma_0 \in \mathcal{L}(U_0)$ and $\gamma_1 \in \mathcal{L}(U_1)$.
It is hard to get a solution on all of $U_0 \cup U_1$.



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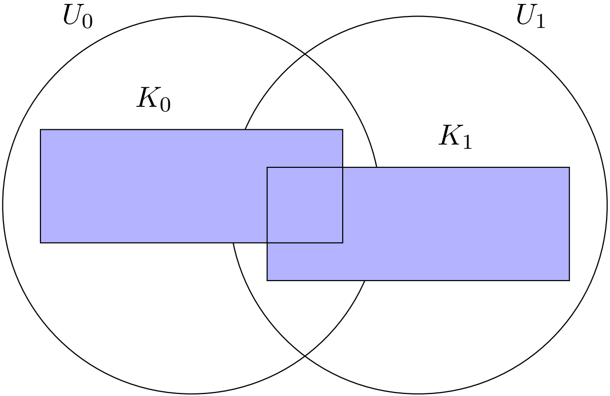

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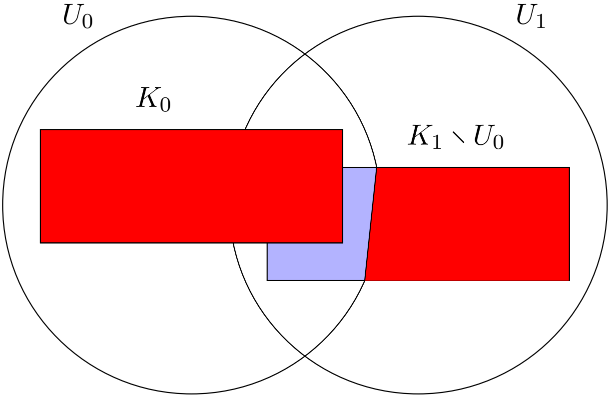

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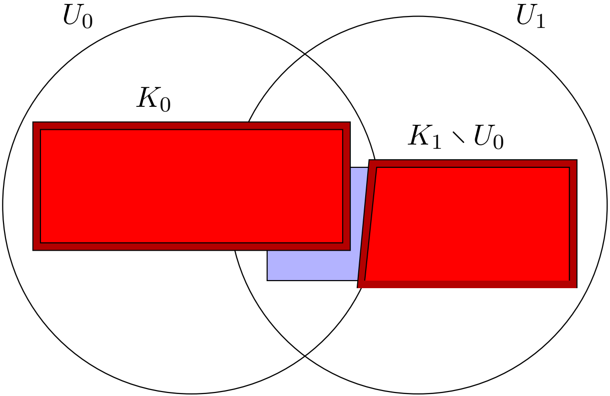

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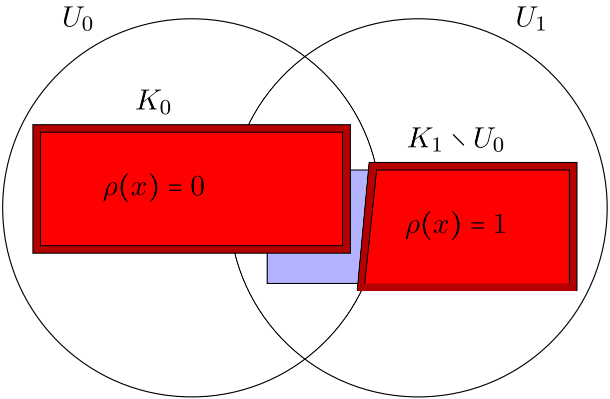

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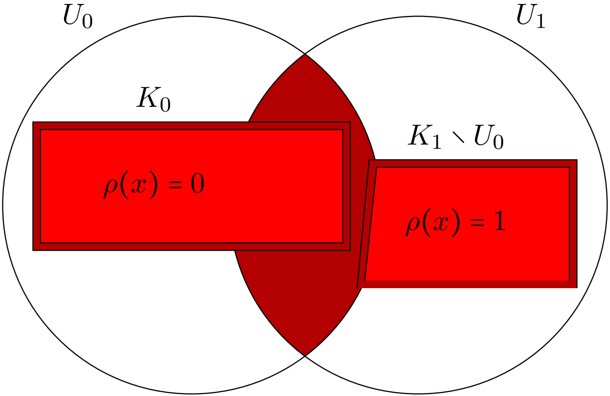

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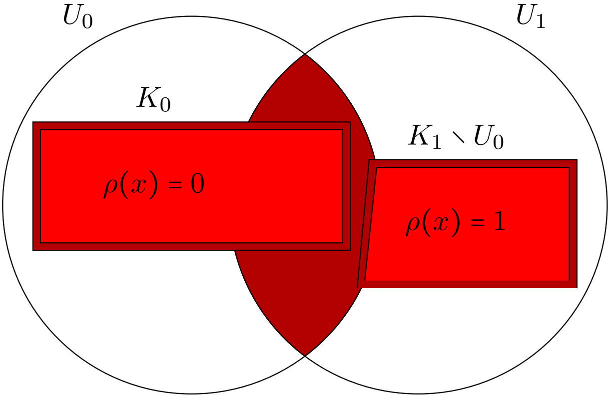

Let δ be a homotopy between γ_0 and γ_1 on $U_0 \cap U_1$. Then $\gamma_x = \delta_{\rho(x),x}$ is a solution in the shaded region.

Gluing solutions

We have shown the following result.

Proposition

If we have solutions $\gamma_i \in \mathcal{L}(U_i)$ with U_i an open neighborhood of the compact set K_i for $i \in \{0,1\}$ then we can find a solution $\gamma \in \mathcal{L}(U)$ for some $U \in \mathcal{N}(K_0 \cup K_1)$ which coincides with γ near K_0.

Induction

Lemma

If $\gamma^0 \in \mathcal{L}(U_0)$ with U_0 a neighborhood of K, then we can find $\gamma \in \mathcal{L}(E)$ which agrees with γ^0 near K.

Proof sketch.

We can find U_i, $i \geq 1$ be a locally finite family of open sets with local surrounding families of loops γ^i and compact subsets $K_i \subseteq U_i$ covering E. By repeatedly gluing these loops to γ^0, we obtain a sequence of families that is eventually constant in each K_i, so we obtain γ in the limit. □

Smoothing

Now that we have a global continuous solution, we need to find a smooth approximation that preserves the following properties:

- γ_x^t is a loop based at $\beta(x)$
- $\gamma_x^0(s) = \beta(x)$
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We could take the convolution with a smooth bump function.

But then we need an argument using partition of unity to ensure that the family is equal to β on part of its domain.

Smoothing

In fact, we can do it without convolution, and using partition of unity in the following form:

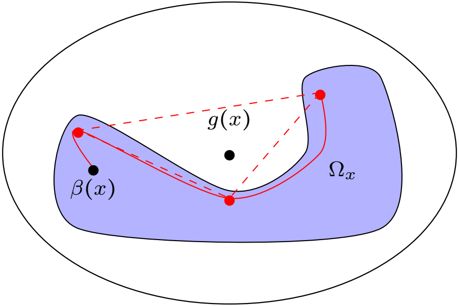
Lemma

If $f : E \to F$ and $\varepsilon : E \to \mathbb{R}_{>0}$ are continuous then there is a C^∞ function $g : E \to F$ with $d(f(x), g(x)) < \varepsilon(x)$ for all x. Moreover, if f is smooth near a closed set C, then we can choose g such that $g(x) = f(x)$ for $x \in C$.

Using this lemma we can get all desired properties for the smooth approximation.

Reparametrization

Finally, we need to reparametrize the loop so that its average is exactly $g(x)$.



Convolution

I formalized the following version of the convolution of two functions.

Given $f : E \to \mathbb{R}$ and $g : E \to F$ the convolution $f \star g : E \to F$ is given by

$$(f \star g)(x) = \int_E f(t)g(x - t)d\mu(t).$$

Here E is an topological abelian group equipped with a Haar measure μ and F is a Banach space.

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This version is useful to smooth a vector-valued function with a bump function (bump functions were defined by Yuri Kudryashov in Lean).

- If f_n is a sequence of bump functions that tends to δ_{x_0} then $(f_n \star g)(x) \to g(x_0)$;
- If f is C^n and has compact support and g is locally integrable then $f \star g$ is C^n.

Convolution

Interestingly, all books that I could find that proves the second property, uses it by computing the partial derivatives of $f \star g$:

$$\frac{\partial}{\partial x_i}(f \star g) = \frac{\partial f}{\partial x_i} \star g.$$

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How to state this? $D(f \star g)_x$ is a linear map $E \to F$ and it should be a convolution of Df with g.

However, the convolution of $Df : E \to E^*$ and $g : E \to F$ is not in the form we defined before.

Convolution

Given $f : E \to F_1$ and $g : E \to F_2$ and a continuous bilinear map $L : \mathrm{Hom}(F_1 \times F_2, F)$ we define the convolution $f \star_L g : E \to F$ is given by

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Theorem

If f is C^1 with compact support and g is locally integrable, then

$$D(f \star_L g) = Df \star_{L'} g.$$

Convolution

Proof.

We compute

$$\begin{aligned} D_{x_0}(f \star_L g) &= D_{x_0} \int_E L(f(x-t), g(t)) d\mu(t) \\ &= \int_E D_{x_0} L(f(x-t), g(t)) d\mu(t) \\ &= \int_E L((-), g(t)) \circ D_{x_0} f(x-t) d\mu(t) \\ &= (Df \star_{L'} g)(x_0). \end{aligned}$$

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Since f has compact support, the integral is dominated by an integrable function, so we are allowed to pull the derivative inside the integral. □

Conclusion

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Lean can help motivate interesting generalizations.