Lattices and Topological Systems

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Abstract

In our Bachelor thesis we will examine the concept of lattices and topological systems on an advanced undergraduate level. We will give an introduction to algebraic theories, locales, sum and product systems, Boolean algebras, ideals, filters, Scott opens, compactness and free lattices.

Contents

1 Introduction

In this text, we pursue the development of an algebraic structure called a *topological system* (sometimes abbreviated to simply "system"). This is a structure which generalizes the concept of a set endowed with a topology. Although not strictly necessary, the reader will be more easily familiarized with this topic if acquainted with topologies. We will also develop the theory of lattices, which is needed to define topological systems and which will lead to some interesting results for them.

In Chapter 2, we introduce the basic principles that we will use throughout our thesis, like lattice, frame, join and meet.
In Chapter 3, we explain the universal algebra which will be needed for the rest of our thesis. We will show what laws are needed for (semi)lattices and we will introduce the concept of algebraic system, algebra and presentation.
In Chapter 4, we define the concept of topological system. We will show that topological spaces are a special case of topological systems. Also, we will introduce the concept of locales in this language.
In Chapter 5, we construct new systems from a set of given systems by defining their topological sum and topological product. We will also prove their universal properties of these constructions.
In Chapter 6, we look to equivalent definitions of lattices to be distributive, we will define Boolean algebras and show that they are the same as Boolean rings. Also, we will define ideals and filters.
In Chapter 7, we define the specialization order and define the Scott topology on an arbitrary poset. These in turn allow us to generalize the notion of compactness to systems.
In Chapter 8, we show that finite locales are spatial and find pointless locales.
In Chapter 9, we look at the shape of certain free lattices.

Most material comes from the books [Vickers 1989, *Topology via Logic*] Chapters 1-8 and [Johnstone 1982, *Stone spaces*] Chapter 1. However, some results are new, or combinations of different results. For example, our development of arbitrary topological products in Section 5.3, as well as Theorem 7.11 and Chapter 8.

2 Preliminaries

2.1 Ordered sets

The most basic structure we consider throughout is called a *partially ordered set* or *poset*, for short.

Definition 2.1. A *poset* consists of a set P and an order: a subset $\leq$ (*less than*) of $P \times P$, which satisfies the following three properties:

(i) $\forall a \in P : a \leq a$ (*reflexivity*)

(ii) $\forall a, b, c \in P$: if $a \leq b$ and $b \leq c$ then $a \leq c$ (*transitivity*)

(iii) $\forall a, b \in P$: if $a \leq b$ and $b \leq a$ then $a = b$ (*antisymmetry*).

When no confusion can arise, we refer to the poset $(P, \leq)$ as P.

Simple examples of posets are the natural numbers, and the power set $\mathcal{P}(X)$ of a set X with $\subseteq$ as the order.
Next, we will introduce the two important notions *join* and *meet*, which will be central in our discussion.

Definition 2.2. Let P be a poset and let $X \subset P, y \in P$. Then y is a *meet* (or *greatest lower bound*) for X (denoted $y = \bigwedge X$) iff the following conditions hold:

(i) $\forall x \in X : y \leq x$ (we say y is a *lower bound* for X)

(ii) If $z \in P$ is also a lower bound for X, then $z \leq y$.

Also, we say that y is a *join* (or *smallest upper bound*) for X (denoted $y = \bigvee X$) iff the following hold:

(i) $\forall x \in X : y \geq x$ (we say y is an *upper bound* for X)

(ii) If $z \in P$ is also an upper bound for X, then $z \geq y$.

From these definitions, there follows a *duality principle*: If $y = \bigwedge X$ in $(P, \leq)$, then $y = \bigvee X$ in $(P, \geq)$ (the poset with all inequalities reversed, called the *opposite poset*), and conversely. This means that if we prove something about meets, the dual statement is also true, interchanging all meets and joins, and interchanging $\leq$ and $\geq$.

If meets and joins of some set exist, they are unique. Because if y and y' are both meets of X, then $y \leq y'$ and $y' \leq y$, so $y = y'$ by antisymmetry, and the same holds for joins. Also, let us introduce the notations $\mathbf{0}$ for $\bigvee \varnothing$ and $\mathbf{1}$ for $\bigwedge \varnothing$, if they exist. This implies that $\mathbf{0}$ is the least element of P ($\forall a \in P$: $a \geq \mathbf{0}$) and $\mathbf{1}$ is the greatest element of P ($\forall a \in P$: $a \leq \mathbf{1}$). The notation $\mathbf{0}$ and $\mathbf{1}$ is justified in Section 6.2, where we meet special lattices which can be considered as rings, with $\mathbf{0}$ and $\mathbf{1}$ as units of addition and multiplication respectively.

The following proposition is very useful, since it allows one to retrieve the partial ordering from the meets or joins.

Proposition 2.3. *Let P be a poset. Then $x = \bigwedge \{x, y\} \iff x \leq y \iff y = \bigvee \{x, y\}$.*

Proof When $x = \bigwedge \{x, y\}$, $x \leq y$ by the lower bound property. If $x \leq y$, then x is a lower bound for $\{x, y\}$, and any other lower bound z must have $z \leq x$ so it is indeed the meet. The second equivalence follows by duality. □

Finally, let $f : P \to Q$ be a function between posets. Then f is said to *preserve meets* iff $y = \bigwedge X \Rightarrow f(y) = \bigwedge \{f(x) \mid x \in X\}$. Similarly, f can preserve joins, finite meets, and such.

2.2 Lattices and frames

Definition 2.4. A poset P is a *lattice* iff every finite subset has both a meet and a join. A *lattice homomorphism* is a function between lattices that preserves all these meets and joins.

It is worthwhile to note that a poset is a lattice iff it has all nullary and binary meets and joins. This can easily be proved by induction. We will proceed to denote binary meets and joins with infix notation (and without bracketing). So, for example, we will write $a \wedge b$ instead of $\bigwedge\{a, b\}$. We also wish to note some algebraic properties of $\wedge$ and $\vee$ (the proofs are omitted as they are trivial):

Proposition 2.5. *For all x, y, z, the binary operators $\wedge$ and $\vee$ satisfy the following properties:*

(i)	*commutativity:*	$x \wedge y = y \wedge x$	$x \vee y = y \vee x$
(ii)	*associativity:*	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	$x \vee (y \vee z) = (x \vee y) \vee z$
(iii)	*unit laws:*	$x \wedge \mathbf{1} = x$	$x \vee \mathbf{0} = x$
(iv)	*idempotence:*	$x \wedge x = x$	$x \vee x = x$
(v)	*absorption:*	$x \wedge (x \vee y) = x$	$x \vee (x \wedge y) = x.$

In particular, we are now allowed to write any finite meet or join in infix notation, as associativity holds. Also, in Proposition 3.5 we will show that these ten laws identify a lattice. The attentive reader may note that there is no distributive law above. This is because not every lattice has distributivity, motivating the following definition:

Definition 2.6. A lattice L is said to be *distributive* iff it satisfies (for all $x, y, z \in L$) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Interestingly enough, in a distributive lattice, we also have the other distributive law $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. This can be seen as follows:

$$\begin{aligned}(x \vee y) \wedge (x \vee z) &= (x \vee z) \wedge (x \vee y) = (x \wedge (x \vee y)) \vee (z \wedge (x \vee y)) \\ &= x \vee ((z \wedge x) \vee (z \wedge y)) = (x \vee (z \wedge x)) \vee (y \wedge z) = x \vee (y \wedge z).\end{aligned}$$

Although not every lattice is distributive, a partial result does hold. We will now show that in any lattice $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$. Since $x \geq x \wedge y$ and $y \vee z \geq y \geq x \wedge y$ we see that $x \wedge y$ is a lower bound of $\{x, y \vee z\}$, so less than the meet: $x \wedge (y \vee z) \geq x \wedge y$. Also $x \geq x \wedge z$ and $y \vee z \geq z \geq x \wedge z$, so $x \wedge (y \vee z) \geq x \wedge z$. Now $x \wedge (y \vee z)$ is an upper bound of $\{x \wedge y, x \wedge z\}$, so larger than the join, and $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ follows. By duality, also the inequality $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ holds.

We now give some examples of lattices, first some distributive lattices, and then some non-distributive lattices.

Example 2.7. Examples of distributive lattices are linear orders with two end-points (a maximal element and a minimal element). The join and meet are max and min, respectively. The distributivity is easily checked.
Another example is the power set of any set. Here the ordering is inclusion. The join of two sets is their union, and the meet is their intersection, $\mathbf{0}$ is the empty set and $\mathbf{1}$ is the whole set.
One more example of a distributive lattice is classical logic. There the ordering is implication $\Rightarrow$, meets are conjunctions, joins are disjunctions, $\mathbf{0}$ is the "false" statement and $\mathbf{1}$ is the "true" statement. We easily check that this is indeed an example. In particular, a contradiction (false) implies anything, and anything implies a tautology (true).

Example 2.8. In Figure 2.1 you see the Hasse diagram of two posets. This displays the elements of a poset as points, and a line going upwards from x to y indicates that $x < y$ ($x \leq y$, but $x \neq y$) and there's no z such that $x < z < y$. This is an useful way to show finite posets, including, of course, finite lattices. One can retrieve the ordering from a Hasse diagram in the following way: $x \leq y$ if and only if there's a path from x to y over the lines going only upwards. In these specific examples, we can see that in Figure 2.1a we have $a \wedge (b \vee c) = a \wedge e = a$, but $(a \wedge b) \vee (a \wedge c) = d \wedge d = d$. So this lattice is not distributive. In the same way we can see that these posets are lattices, but not distributive, as we will show. In Figure 2.1b the distributive law also doesn't hold. These are the only essential examples of non-distibutive lattices in the sense that every non-distributive lattice contains five points having meets and joins according to one of the two figures below. We will go further with this in Theorem 6.1.

Note that a lattice homomorphism f is always monotone since

$$a \leq b \Rightarrow a \wedge b = a \Rightarrow f(a) \wedge f(b) = f(a \wedge b) = f(a) \Rightarrow f(a) \leq f(b).$$

Also, for every monotone function g we can note that $g(a) \wedge g(b) \geq g(a \wedge b)$. This follows from the fact that $a \wedge b \leq a$ and $a \wedge b \leq b$, so $g(a \wedge b) \leq g(a)$ and $g(a \wedge b) \leq g(b)$. So $g(a \wedge b)$ is a lower bound of $\{g(a), g(b)\}$, whence smaller than $g(a) \wedge g(b)$. By duality, $g(a \vee b) \geq g(a) \vee g(b)$. Equality does not

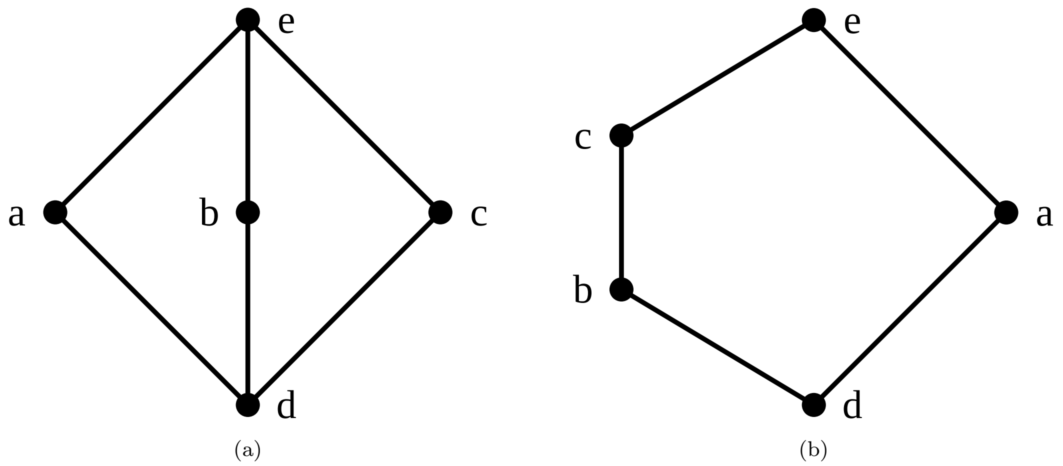


Figure 2.1: Two non-distributive lattices

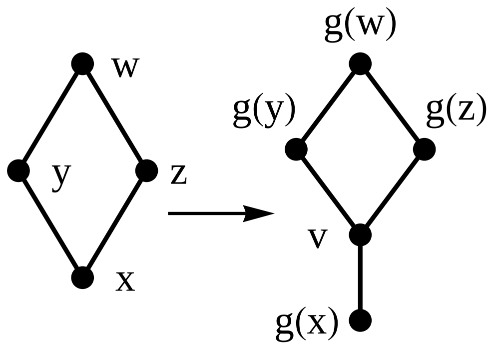


Figure 2.2: A monotone function which is not a lattice homomorphism

always hold, as we can see in Figure 2.2. One easily checks that g in the figure is monotone, but clearly $g(y \wedge z) = g(x) \neq v = g(y) \wedge g(z)$.

We are now prepared to define frames, which will be the platform for most of our coming discussion.

Definition 2.9. A lattice L is a *frame* iff it has the following properties:

(i) *Every* subset has a join.

(ii) Frame distributivity: $x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}$.

A *frame homomorphism* is a function between frames that preserves all joins and finite meets.

So in a frame all joins and all finite meets exist. Interestingly, we can see that these conditions also imply that even the infinite meets exist, as the following proposition shows.

Proposition 2.10. *If in a poset A every set has a join, then every set has a meet.*

Proof Let B be a subset of A. We are going to find a meet for B. Let C be the set of all lower bounds of B. That is, $C = \{a \in A \mid \forall b \in B : a \leq b\}$. C is non-empty, since $\mathbf{0} \in C$. Let c be the join of C, so $c = \bigvee C$. Since every element $b \in B$ is an upper bound for the set C, and c is the least upper bound, $c \leq b$, so c is a lower bound for B. Let d be any other lower bound of B. Then $d \in C$, so $d \leq c$. So c is the greatest lower bound of B. □

So we could have defined a frame to be a set where all joins and all meets exist. But one very important remark is that frame homomorphisms need not to preserve infinite meets! This is a very important remark. And since frame homomorphisms need not to preserve the infinite meets, we choose to let them be no part of the structure. An immediate example of a frame is a finite distributive lattice, as for those, there is no extra requirement. Two frequently-used finite frames are $\mathbb{1}$, the inconsistent frame ($\mathbf{1} = \mathbf{0}$) and $\mathbb{2}$ ($\mathbf{0} \leq \mathbf{1}$). These are the only frames with one, resp. two, elements. An infinite example of a frame is the set of ordinal numbers which are at most ω, so the set $\{1, 2, 3, 4, \ldots, \omega\}$. We now give an example which illustrates the frame distributivity.

Example 2.11. Define the poset A the following way. Let the set $A = \mathbb{Z}^2 \cup \{\mathbf{0}, \mathbf{1}\}$, and we define the order as $x \leq x'$ iff one of the following conditions hold:

(i) $x = \mathbf{0}$,

(ii) $x' = \mathbf{1}$,

(iii) $x = (y, z)$, $x' = (y', z')$ with both $y \leq y'$ and $z \leq z'$.

One can easily check that this is indeed an order which makes A into a lattice with $(x, y) \wedge (x', y') = (\min(x, x'), \min(y, y'))$ and $(x, y) \vee (x', y') = (\max(x, x'), \max(y, y'))$. We see indeed that $\mathbf{0}$ and $\mathbf{1}$ are the empty join and the empty meet respectively. This is even a distributive lattice, it is not hard to check that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. If one of x, y, z is either $\mathbf{0}$ or $\mathbf{1}$, then it is trivial. In all other cases, meet and join are computed componentwise, and on each component the order is linear. Since we know that lattices with linear orders are distributive (cf. Example 2.7), we conclude that A is a distributive lattice.
But we can even say more. All joins, even the infinite ones, in this lattice do exist. It is not hard to see that $\bigvee\{(x_i, y_i) \mid i \in I\} = (\max\{x_i \mid i \in I\}, \max\{y_i \mid i \in I\})$ if both maxima exist. If one of the maxima does not exist, or we take a join where one of the terms is $\mathbf{1}$ then the join is $\mathbf{1}$. If we take the empty join, or the of $\{\mathbf{0}\}$, then the join is $\mathbf{0}$. To illustrate Proposition 2.10, we can now conclude that all infinite meets exist too, and one can easily compute them by replacing max by min and interchanging $\mathbf{0}$ and $\mathbf{1}$. However, A is not a frame. We will now show that the frame distributivity does not hold. Let $y = (3, 2)$ and for all $i \in \mathbb{N}$, let $x_i = (2, i)$. Then $y \wedge \bigvee\{x_i \mid i \in \mathbb{N}\} = y \wedge \mathbf{1} = y = (3, 2)$ but $\bigvee\{y \wedge x_i \mid i \in \mathbb{N}\} = \bigvee\{(2, \min\{2, i\}) \mid i \in \mathbb{N}\} = (2, 2)$. So the frame distributivity does not hold.
So from this example we conclude that a distributive lattice where all joins exist need not be a frame.

Another important example of a frame is a *topology.*

2.3 Topologies

Definition 2.12. Let X be a set. Then a *topology* on X is a *subframe* A of the power set of X, $\mathcal{P}(X)$ - i.e., a subset of $\mathcal{P}(X)$ that itself is a frame. This comes down to A satisfying the following three properties:

(i) $\varnothing, X \in A$

(ii) If $S \subseteq A$ then $\bigcup S \in A$

(iii) If $S \subseteq A$ is finite then $\bigcap S \in A$.

The ordered pair (X, A) is then called a *topological space.*

Usually, one proceeds by calling the elements of A the *open* subsets of X. Again, it is common to refer to (X, A) just with X. The topology A coming with X is then often denoted as ΩX. Note that the opens of a topology are in general not closed under infinite intersections. However, one can easily check that for every set S op opens its meet is the *interior* of $\bigcap S$.

It will be our goal to generalize this concept in order to discuss a broader class of objects. The topologies then will be a specific case of what will be called *topological systems.* In order to do so, it will be convenient to adopt a more algebraic setting, which will be the goal of the next chapter.

3 Algebras

3.1 Algebraic theories

In the previous section we've defined lattices. They consist of a partial order $\leq$ where the finite meets and joins exist. We would like to use some results from algebra on lattices, and also on frames. One thing we would like to use is that in an algebraic theory an algebra can be written as a presentation. A presentation specifies some generators of the algebra, and some relations for these generators, and then this should define the algebra up to isomorphism. This method is well known in group theory, where it is common to write, for example, $\langle a, b \mid a^n = b^2 = 1,\ bab^{-1} = a^{-1}\rangle$ for a dihedral group. The same thing can be done for more general algebraic theories, so we would like to describe lattices and frames as an algebraic theory.

A general *algebraic theory* consists of some operators with a specified *arity* (the number of arguments the operator has). We treat constants as operators with arity zero. Also, there are some *laws*, which are equalities involving variables and operators (however, no quantifiers are allowed).

Example 3.1. In group theory there is a binary operator multiplication, denoted $\cdot$, and a nullary operator, or *constant*, 1. Usual axioms for group theory are

$$\forall xyz((x \cdot y) \cdot z = x \cdot (y \cdot z)), \quad \forall x(x \cdot 1 = 1 \cdot x = x), \quad \forall x \exists y(x \cdot y = y \cdot x = 1).$$

These axioms have to be modified a bit to fit them into the language of algebraic theories, since it these axioms contain quantifiers, which are not allowed in the laws of an algebraic theory. One way to solve this is to add a unary operator $\cdot^{-1}$, the inverse operator. We can now take as group laws:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad x \cdot 1 = 1 \cdot x = x, \quad x \cdot x^{-1} = x^{-1} \cdot x = 1.$$

Definition 3.2. For an algebraic theory $\mathbb{T}$, we define A to be a $\mathbb{T}$-*algebra* iff

(i) A is a set (known as the carrier)

(ii) For each operator ω in $\mathbb{T}$ with arity n there is a function $\omega : A^n \to A$

(iii) Now suppose e is an expression in $\mathbb{T}$ with m (distinct) free variables. If we substitute an m-tuple of elements of A for the variables, and replace the operators by the functions on A as in (ii) then we'll get an element of A again. So e defines a function $A^m \to A$. The last condition is that if $e_1 = e_2$ is a law in $\mathbb{T}$ with m free variables, then the two corresponding functions $e_1, e_2 : A^m \to A$ must be equal.

If A and B are two $\mathbb{T}$-algebras, then a $\mathbb{T}$-*homomorphism*, or if no confusion can arise a homomorphism, from A to B is a function $f : A \to B$ such that for every n-ary $\mathbb{T}$-operator ω the equality $f \circ \omega = \omega \circ f^n$ holds. A $\mathbb{T}$-*isomorphism* is a $\mathbb{T}$-*homomorphism* which is bijective and whose inverse is also a $\mathbb{T}$-*homomorphism.* If there exists a $\mathbb{T}$-isomorphism between two $\mathbb{T}$-algebras A and B, we call A and B ($\mathbb{T}$-)*isomorphic.*
A *sub-$\mathbb{T}$-algebra* B of an $\mathbb{T}$-algebra A is a subset of A such that for each n-ary operator ω the image of $\omega|_{B^n}$ is contained in B. In particular, B has to contain all constants.

If we want to see lattices and frames as an algebraic theory, then there is a small problem. They consist of a partial order $\leq$ which is a relation. But relations do not fit into an algebraic theory. Fortunately, we already saw in Proposition 2.3 that the ordering is connected to meets and joins in the following way: $a \vee b = b$ iff $a \leq b$ iff $a \wedge b = a$. So if given binary operators $\vee$ and $\wedge$, we should be able to define the partial order, and an algebraic theory for meets and joins, as we do below. Let's start easy, with a structure which only contains meets:

Definition 3.3. A poset is a *semilattice* iff every finite subset has a meet.

The following proposition specifies exactly what laws we need for some algebra $(S, \wedge, \mathbf{1})$ to be a semilattice.

Proposition 3.4. *Let SL be the algebraic theory consisting of a binary operator $\wedge$ and a constant $\mathbf{1}$ subject to the laws (i)-(iv) of Definition 2.5. If $(S, \wedge, \mathbf{1})$ is an SL-algebra, and if we define $x \leq y$ iff $x \wedge y = x$, then $(S, \leq)$ is a semilattice.*

Proof First note that the converse, the fact that the laws hold in a semilattice, is trivial.
Now we'll prove that $\le$ defines an order. $\le$ is reflexive by idempotence, it is antisymmetric since if $x \le y \le x$ then $x = x \wedge y = y \wedge x = y$. Finally it is transitive: if $x \le y$ and $y \le z$, then $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$, so $x \le z$.
Now we have to check if $\mathbf{1}$ and $\wedge$, obtained by the algebra, actually correspond to the top element and meets in the poset. By the unit law, $x \le \mathbf{1}$, so $\mathbf{1}$ is indeed the top element. Given x and y, we'll now show that $x \wedge y$ is the meet of x and y. Note that $(x \wedge y) \wedge y = x \wedge (y \wedge y) = x \wedge y$ so that $x \wedge y \le y$ and $x \wedge y = y \wedge x \le x$. Hence $x \wedge y$ is a lower bound. Now suppose that $z \le x$ and $z \le y$. Then $z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z$, so $z \le x \wedge y$. So indeed, $x \wedge y$ is the greatest lower bound of x and y. □

The laws as in Proposition 2.5 are sufficient to describe lattices.

Proposition 3.5. *A lattice P can be described as an algebra with four operators: $\wedge$, $\vee$ binary and $\mathbf{1}$, $\mathbf{0}$ nullary, with the laws as in Proposition 2.5.*

Proof From Proposition 2.5 a lattice satisfies all these conditions. Now suppose we have an algebra P with the operators $\wedge$, $\vee$, $\mathbf{1}$, $\mathbf{0}$, satisfying all the conditions above. From Proposition 3.4 we know that if we define $\le$ by $x \le y$ iff $x \wedge y = x$, then $(P, \le)$ is a semilattice with meets $\wedge$ and top element $\mathbf{1}$. Similarly, we can define $\le'$ by $x \le' y$ iff $y \vee x = y$. By duality, $(P, \le')$ is a poset with joins $\vee$ and bottom element $\mathbf{0}$. Now the only thing left to show is that the relations $\le$ and $\le'$ coincide. Suppose $x \le y$. Then $y = y \vee (y \wedge x) = y \vee (x \wedge y) = y \vee x$, so $x \le' y$. The reverse direction follows by duality. □

In fact, the idempotence laws are not even necessary to describe a lattice. They are a consequence of the absorption and unit laws, since $x = x \wedge (x \vee \mathbf{0}) = x \wedge x$ and $x = x \vee (x \wedge \mathbf{1}) = x \vee x$.

Frames can also be described as an algebraic theory, but there are some complications, since we allow infinite joins. We can describe it with $\mathbf{1}$, $\wedge$ and a proper class of operators: for every cardinal number α an α-ary join $\bigvee_\alpha$, which we will all denote as $\bigvee$ for brevity. Formally, a frame A ought to have a function $\bigvee = \bigvee_\alpha : A^\alpha \to A$ for every cardinal number α. However, this involves an infinite cartesian product, which we don't want to define just now (we will actually define this later, in Definition 5.1). For this reason, we will be a little sloppy with the notation, and view $\bigvee = \bigvee_\alpha$ as a function on subsets of A, with cardinality α, instead. This also makes it unnecessary to state the fact that the join is commutative and idempotent, since this is already implicitly assumed by defining the join as a function on sets rather than ordered tuples.
The theory of frames also has a proper class of laws, namely the four laws of a semilattice, see Proposition 3.4, and for each set I and J and for each $j \in I$ the laws

$$
\begin{aligned}
x_j &= x_j \wedge \bigvee\{x_i \mid i \in I\} && \text{(infinite absorptive law)}\\
y \wedge \bigvee\{x_i \mid i \in I\} &= \bigvee\{y \wedge x_i \mid i \in I\} && \text{(infinite distributive law)}\\
\bigvee\left\{\bigvee\{x_{ij} \mid i \in I\} \middle| j \in J\right\} &= \bigvee\{x_{ij} \mid i \in I, j \in J\}. && \text{(infinite associative law)}
\end{aligned}
$$

Note that we already said that idempotence and commutativity are implicitly assumed for the joins, so these are not laws that we need. According to [Vickers 1989, *Topology via Logic*] page 43 the infinite associative law we stated here is not necessary. We did include him here, because we were unable to prove this law from the others.
We will not prove that these laws suffice to describe frames.

3.2 Presentations

Now we will go deeper into presentations. A presentation of an algebraic theory $\mathbb{T}$ is notated as $\mathbb{T}\langle G \mid R\rangle$, where G is some set of generators, and R is a set of relations. These relations must be equalities between terms involving only generators and operators of the $\mathbb{T}$-algebra (including constants).

Definition 3.6. We say that A is a *model* for a presentation $\mathbb{T}\langle G \mid R\rangle$ iff

(i) A is a $\mathbb{T}$-algebra.

(ii) There is a function $[.] : G \to A$, $g \mapsto [g] = [g]_A$ which assigns an interpretation to every generator (although we'll often leave out the square brackets).

(iii) This function can be extended for every expression e involving generators and operators from $\mathbb{T}$ by replacing the generators g by $[g]$, such that we get an element $[e] \in A$. The last condition is that if $e_1 = e_2$ is a relation in R, then in A the equality $[e_1] = [e_2]$ must hold.

Example 3.7. Consider the presentation $\mathrm{Fr}\langle a, b \mid \rangle$ in the algebraic theory of frames. It has two generators and no relations. If A is a model for this presentation, then A also has to contain the elements $\mathbf{1}$, $\mathbf{0}$, $a \wedge b$ and $a \vee b$. So the frame in Figure 3.1a is a model for this presentation (it's easily verified that it is actually a frame). But there are many more models of this presentation. Take for example the frame in Figure 3.1b. Here $[a] = x = [b]$. Or take the frame in Figure 3.1. However, in some way the first frame was what we meant with our presentation. There we didn't impose extra conditions, like $a = b$ as in the second frame, since it's pointless to use two generators, but then equate them in our frame. The last frame is bigger than we intended to. There's no way we could construct c out of a and b. So we want to have a better definition of presenting a presentation, such that the only frame which presents it is the one which identifies elements only when forced to by the relations, and which is not bigger than we can conclude just by using our generators. This motivates the following definition.

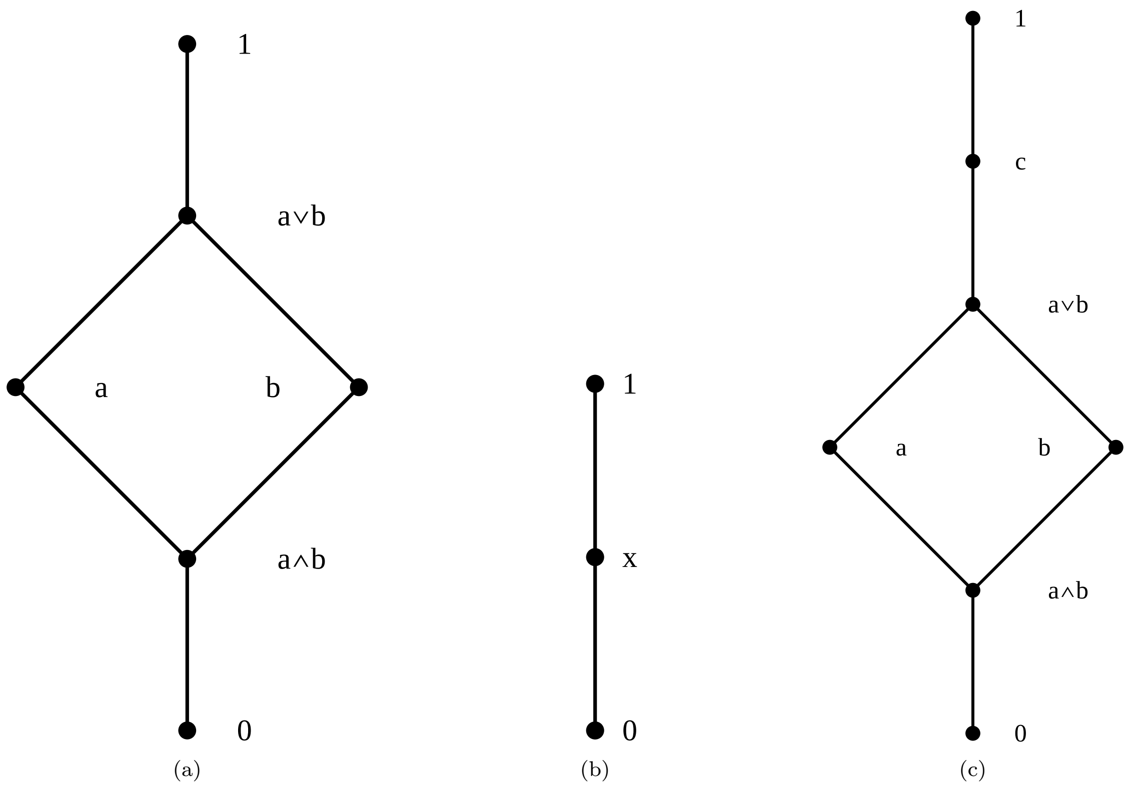


Figure 3.1: Three models for the presentation $\mathrm{Fr}\langle a, b \mid \rangle$.

Definition 3.8. A $\mathbb{T}$-algebra A is *presented* by the presentation $\mathbb{T}\langle G \mid R\rangle$ iff

(i) A is a model for $\mathbb{T}\langle G \mid R\rangle$.

(ii) For every other model B, there is a *unique* $\mathbb{T}$-homomorphism $\theta : A \to B$ such that $\theta([g]_A) = [g]_B$ for every $g \in G$ (the *universal property* of presentations).

The following two propositions ensure that under certain conditions, the presented algebra exists and is unique (up to isomorphism).

Proposition 3.9. *The algebra presented by a presentation is - if it exists - unique up to isomorphism.*

Proof Suppose that both A and B present a presentation, then we have to show that A and B are isomorphic. By the universal property we find a homomorphism $\theta : A \to B$ with $\theta(g_A) = g_B$ for all generators g. Interchanging the roles of A and B we find that there exists a homomorphism $\theta' : B \to A$ with $\theta(g_B) = g_A$. So the composition $\theta' \circ \theta$ maps A to A and leaves all generators g_A fixed. Now using the universal property (letting A play the roles of both A and B), we obtain that $\theta' \circ \theta$ is the unique homomorphism fixing the generators, but clearly the identity maps also does this. So $\theta' \circ \theta = \mathrm{Id}_A$. Analogously, we find that $\theta \circ \theta' = \mathrm{Id}_B$, so θ is an isomorphism, and hence A and B are isomorphic. □

We call an algebraic theory *finitary* if the operations form a set (not a proper class) and each operator has finite arity. A $\mathbb{T}$-*word* constructed from a set X is an expression involving elements of X and operators in $\mathbb{T}$ (including constants, of course). Denote $\mathbb{T}WX$ for all $\mathbb{T}$-words. If $\mathbb{T}$ is finitary, this is a set. The following theorem states the existence of presenting algebras, but we will only give a sketch of the proof.

Proposition 3.10. *If $\mathbb{T}\langle G \mid R\rangle$ is a presentation of a finitary algebraic theory, then there exists an $\mathbb{T}$-algebra which presents $\mathbb{T}\langle G \mid R\rangle$.*

Proof As above, $\mathbb{T}WX$ is the set of all ($\mathbb{T}$-)words, since $\mathbb{T}$ is finitary. Now define an equivalence relation $\sim$ on $\mathbb{T}WX$ stating that two words are equivalent iff we can deduce from the laws of $\mathbb{T}$ and the relations R that these words must be equal. Clearly, this is an equivalence relation, by the basic properties of equality. On the set of equivalence classes $\mathbb{T}WX/\sim$ we can define the operations of $\mathbb{T}$, by stating that for every n-ary operator ω and all words $w_1, \dots, w_n$ we have $\omega([w_1], \dots, [w_n]) = [\omega(w_1, \dots, w_n)]$. This is well-defined, since if $w_i' \sim w_i$ for all i, then $\omega(w_1, \dots, w_n) \sim \omega(w_1', \dots, w_n')$. Now we claim that $\mathbb{T}WX/\sim$ is indeed the $\mathbb{T}$-algebra we are looking for. □

Unfortunately, the theory of frames is not finitary; the join has infinite arity. Or better said: there is an α-ary join for every cardinal number α. However, we can still show that every presentation in this theory has a presenting algebra. We will not do this here, but the proof can be found in [Vickers 1989, *Topology via logic*] pp. 46-50.

Proposition 3.11. *Let $Fr\langle G \mid R\rangle$ be a presentation in the theory of frames. There exists a frame which presents $Fr\langle G \mid R\rangle$.*

The converse, that every algebra in a finitary theory has a presentation, also holds. This is also true for the theory of frames.

Proposition 3.12. *Let $\mathbb{T}$ be a finitary algebraic theory, or the theory of frames, and let A be a $\mathbb{T}$-algebra. Then there exists a presentation $\mathbb{T}\langle G \mid R\rangle$ such that A presents it.*

4 Topological systems

In the previous sections, we have introduced the structures and methods required to conveniently study our main object, *topological systems*. We are now prepared to do so.

4.1 Definition

Definition 4.1. Let A be a frame, and X be a set. We will refer to the elements of A as *opens*, and the elements of X will be called *points*. Also, let $\vDash$ be a subset of $X \times A$. We will denote $(x, a) \in \vDash$ as $x \vDash a$ (x *satisfies* a). Then the triple $(X, A, \vDash)$ is a *topological system* iff $\vDash$ has the following properties (here $S \subset A$):

(i) $\forall x \in X : x \vDash \bigwedge S \iff \forall a \in S : x \vDash a$

(ii) $\forall x \in X : x \vDash \bigvee S \iff \exists a \in S : x \vDash a$

Of course, the first property need only be satisfied for finite subsets S, and the second for all subsets S. If $T = (X, A, \vDash)$ is a topological system, we write $\mathrm{pt}\, T$ for X, ΩT for A and $\vDash_T$ for $\vDash$.

From the properties of $\vDash$ stated above, one can verify immediately that

(i) $x \vDash \mathbf{1}\ \forall x \in X$

(ii) $x \vDash \mathbf{0}$ never holds

(iii) $x \vDash a, a \leq b \Rightarrow x \vDash b$.

We have already seen an example of a topological system, namely, for any topology A on a set X, the triple $(X, A, \in)$ forms a topological system. In fact, it is exactly what we earlier called a topological space. More examples of systems will be encountered later in this chapter. From the theory of topological spaces, there is a notion of continuity for functions; this is exactly the standard continuity property. This notion can be generalized as follows:

Definition 4.2. Let S and T be topological systems. Then a *continuous map* $f : S \to T$ is a pair $(\mathrm{pt}\, f, \Omega f)$ with the following properties:

(i) $\mathrm{pt}\, f : \mathrm{pt}\, S \to \mathrm{pt}\, T$ is just a function

(ii) $\Omega f : \Omega T \to \Omega S$ is a frame homomorphism (note the reversal of direction!)

(iii) $x \vDash_S \Omega f(a) \iff \mathrm{pt}\, f(x) \vDash_T a$.

A continuous function f thus is taken to go in the same direction as $\mathrm{pt}\, f$. In particular, we have a continuous *identity map* for every system T, defined as $Id_T = (Id_{\mathrm{pt}\, T}, Id_{\Omega T})$. In the literature, sometimes the prefixes pt and Ω are omitted when deemed clear from the context. However, we will not adopt this convention as we consider it too confusing.

In the setting of a topological space, these three conditions can be expressed in a simpler form; this will be discussed in more detail in Section 4.2 below. Let us first define composition of continuous functions:

Definition 4.3. Let S, T and U be topological systems, and let $f : S \to T$ and $g : T \to U$ be continuous maps. Then the *composition* of f and g, $g \circ f : S \to U$, is defined by $g \circ f = (\mathrm{pt}\, g \circ \mathrm{pt}\, f, \Omega f \circ \Omega g)$.

It follows immediately that composition of continuous functions yields again a continuous mao. A special case arises when such a composition yields the identity map. In this situation, there is (from the perspective of topological systems) no real need to distinguish between two systems. Let us give such pairs of systems a name:

Definition 4.4. Let S and T be topological systems, and let $f : S \to T$ be a continuous map. Then f is said to be a *homeomorphism* if there exists a continuous map $g : T \to S$ such that $g \circ f = Id_S$ and $f \circ g = Id_T$. Also, if such an f exists between S and T, they are called *homeomorphic* and we write $S \cong T$.

4.2 Topological spaces

Within the general notion of a topological system, there are two important other notions called *(topological) spaces* and *locales*. We will now discuss spaces in some more detail, and in the next section explore some more about locales.

Recall that a topological space T is a pair (X, A) with X a set and A a subframe of $\mathcal{P}(X)$, cf. Section 2.3. In the notation of topological systems, this means $\vDash_T$ is equal to $\in$ (recall, this is set equality). Therefore, we will continue abbreviating a triple $(X, A, \in)$ as (X, A). For spaces, the notion of continuous function takes a simpler form:

Proposition 4.5. *Let (X, A) and (Y, B) be spaces. Then a function $f : X \to Y$ extends to a continuous function $\tilde{f} = (f, f^{-1})$ precisely when $f^{-1}(b) \in A$ holds for all $b \in B$.*

Proof If $\tilde{f}$ is continuous, the latter must hold by definition. For the other direction, it is obvious that $\mathrm{pt}\,\tilde{f} = f$, and from the continuity condition $x \in \Omega\tilde{f}(b) \iff f(x) \in b$ we see that, necessarily, $\Omega\tilde{f}(b) = f^{-1}(b)$, the pre-image of b under f. As f^{-1} preserves intersection and union for any f, we only have to impose that the image of $\Omega\tilde{f}$ is contained in A. This yields the equivalence. □

When considering spaces (and thus, in mainstream topology), there is no need to distinguish between opens that are satisfied by exactly the same points. However, in a general system we have to be more cautious. In fact, if we impose such a restriction on a system T, we will end up with (a system homeomorphic to) a space, called the *spatialization* of T. This space has some interesting properties. Before discussing those, we will introduce some terminology:

Definition 4.6. In a system $(X, A, \vDash)$, for any $a \in A$, the *extent* of a, or $\mathrm{ext}(a)$, is the set $\{x \in X \mid x \vDash a\}$. The extent $\mathrm{ext}(A)$ of the frame A is the set $\{\mathrm{ext}(a) \mid a \in A\}$, viewed as a subframe of $\mathcal{P}(X)$.

It is immediate that ext defines a frame homomorphism from A to $\mathrm{ext}(A)$. In particular, for any space (X, A), $\mathrm{ext} = Id_A$. We will sometimes write ext_T if we want to stress the underlying system T. We are now ready to state what the spatialization of a system is, and prove some properties of it.

Theorem 4.7. *Let $T = (X, A, \vDash)$ be a system. Then the following hold:*

(i) $(X, \mathrm{ext}(A), \in) = (X, \mathrm{ext}(A))$ *is a topological space, denoted* $\mathrm{Spat}\,T$ *and called the spatialization of* T.

(ii) $e = (Id_X, \mathrm{ext})$ *is a continuous function from* $\mathrm{Spat}\,T$ *to* T*, called the spatialization map.*

(iii) *If S is any other topological space, and $f : S \to T$ is continuous, then there is a unique continuous map $g : S \to \mathrm{Spat}\,T$ such that $f = e \circ g$.*

Proof (i): This is clear, as the very definition of a topological system requires $\mathrm{ext}(A)$ to be a frame.
(ii): Again, this is obvious.
(iii): If such a g exists, it will be a map between spaces. Then the definition of composition gives us that $\mathrm{pt}\,f = \mathrm{pt}\,e \circ \mathrm{pt}\,f = \mathrm{pt}\,g$, yielding uniqueness of the points part. Now Proposition 4.5 assures us that Ωg is also uniquely defined. Existence follows from defining g in this manner: for continuity, remark that for any $a \in A$ we have $\Omega g(\mathrm{ext}(a)) = \Omega e \circ \Omega g(a) = \Omega f(a)$, and the latter is open. □

In particular, we see that for a space T, $T \cong \mathrm{Spat}\,T$. A general system satisfying this equation is essentially no different from a space (i.e., it is a space up to homeomorphism). Such systems will be called *spatial*. In particular, Theorem 4.7(iii) holds also if we only require S to be spatial. The following proposition gives two other equivalent conditions for spatiality.

Proposition 4.8. *Let T be a topological system. Then the following three conditions are equivalent:*

(i) *T is spatial*

(ii) $\forall a, b \in \Omega T : (\forall x \in \mathrm{pt}\,T : \ x \vDash a \iff x \vDash b) \Rightarrow a = b$

(iii) $\forall a, b \in \Omega T : (\forall x \in \mathrm{pt}\,T : \ x \vDash a \Rightarrow x \vDash b) \Rightarrow a \leq b$

Proof (iii)⇒(ii): This follows from antisymmetry of $\leq$.
(ii)⇒(iii): Suppose $x \vDash a \Rightarrow x \vDash b$. Then $x \vDash a \iff x \vDash a \wedge b$ and by (ii), $a = a \wedge b$ so $a \leq b$.
(ii)⇒(i): Condition (ii) precisely says that ext is an injective map $\Omega T \to \text{ext}(\Omega T) = \Omega\,\text{Spat}\,T$. It is surjective by definition, whence invertible. Therefore, the spatialization map e is invertible, and thus a homeomorphism.
(i)⇒(ii): If T is spatial, $T \cong \text{Spat}\,T$ holds, and thus $\text{ext}(a) = \text{ext}(b) \Rightarrow a = b$. This is precisely (ii). □

We will not give examples of topological spaces, since we assume the reader already has some experience with topology.

4.3 Locales

In addition to spaces, topologists also study a concept similar to our concept of frame, to which they refer as *locale*. The theory of locales in a sense looks at $\vDash$ as coding a set of index functions. In the language of systems, this makes a point x a function on a frame A, namely by defining

$$x(a) = \begin{cases} \mathbf{1} & x \vDash a \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{4.1}$$

One can check that this defines a frame homomorphism $A \to \mathbb{2}$ (recall that $\mathbb{2}$ is the frame containing only $\mathbf{1}$ and $\mathbf{0}$) for every x.

Definition 4.9. Let A be a frame. Then the *locale* A' corresponding to A has $\text{pt}\,A' = Hom(A, \mathbb{2})$, the set of all frame homomorphisms from A to $\mathbb{2}$. Further, it has $\Omega A' = A$ and defines $\vDash_{A'}$ by requiring $x(a) = \mathbf{1} \iff x \vDash_{A'} a$. Since both $\text{pt}\,A'$ and $\vDash_{A'}$ are defined in terms of A only, we will write A instead of A' for simplicity. A system will be called *localic* if it is homeomorphic to a locale.

As a locale is defined only in terms of its frame, the following proposition will not be a surprise.

Proposition 4.10. *Let A, B be frames (and locales). Continuous functions $A \to B$ are equivalent to frame homomorphisms $B \to A$.*

Proof First, let $f : A \to B$, $f = (\text{pt}\,f, \Omega f)$ be a continuous function. Then $\Omega f : B \to A$ is the corresponding frame homomorphism.
Second, let $g : B \to A$ be a frame homomorphism. Then let $\tilde{g} : A \to B$ be defined by $\Omega\tilde{g} = g$, and for $x \in \text{pt}\,A, b \in B : (\text{pt}\,\tilde{g}(x))(b) = x(\Omega\tilde{g}(b))$. Continuity of $\tilde{g}$ is immediate. □

It is worthwhile to note that the points of a locale A are *all* frame homomorphisms $A \to \mathbb{2}$. This means that every locale corresponds to a frame and vice versa. However, the continuous functions for locales differ from their corresponding frame homomorphism; it is important to keep this in mind. Interestingly enough, there is also a locale corresponding to every system, satisfying similar properties as its spatialization, called the *localification* of a system.

Theorem 4.11. *Let T be a system. The localification of T, $\text{Loc}\,T$, is the locale (corresponding to) ΩT defined above. It has the following properties:*

(i) *There is a continuous map $p : T \to \text{Loc}\,T$ such that $\Omega p = Id_{\Omega T}$, and $\text{pt}\,p$ is defined by sending x to the homomorphism of equation (4.1). This map p is called the localification map.*
(ii) *Let A be a locale, and $f : T \to A$ a continuous map. Then there is a unique continuous map $g : \text{Loc}\,T \to A$ such that $f = g \circ p$.*

Proof The proof of (i) is trivial. For (ii), observe that any g, if it exists, must satisfy $\Omega g = \Omega f$. Since this is a frame homomorphism and $\text{Loc}\,T$, A are locales, we immediately obtain the corresponding (unique) continuous map on virtue of Proposition 4.10. □

Recall that we discussed some properties that were equivalent to spatiality, which made it simpler to identify some system as being spatial. For identifying systems as localic, one has to check that, for every possible frame homomorphism $x : \Omega T \to \mathbb{2}$, there is an unique point associated with it. To make this easier, we now develop some more ways to characterize all such frame homomorphisms. To this end we remark that a frame homomorphism $x : \Omega T \to \mathbb{2}$ is uniquely determined from specifying one of the sets $\{a \in \Omega T \mid x(a) = \mathbf{1}\}$ and $\{a \in \Omega T \mid x(a) = \mathbf{0}\}$ (the so-called $\mathbf{1}$- and $\mathbf{0}$-kernels of x).

Proposition 4.12. *Let ΩT be a frame, and let $I, F \subset \Omega T$. Then F is the $\mathbf{1}$-kernel of a frame homomorphism $x : \Omega T \to \mathbb{2}$ iff*

(i) *for all $a, b \in \Omega T$: if $a \leq b$ and $a \in F$, then $b \in F$ (F is upper closed)*

(ii) *for all finite $S \subset F$: $\bigwedge S \in F$ (F is closed under finite meets)*

(iii) *for all $S \subset \Omega T$: if $\bigvee S \in F$, then $\exists s \in S$: $s \in F$ (F is inaccessible by joins)*

Further, I is the $\mathbf{0}$-kernel of a frame homomorphism x iff

(i) *for all $a, b \in \Omega T$: if $a \leq b$ and $b \in I$, then $a \in I$ (I is lower closed)*

(ii) *for all finite $S \subset \Omega T$: if $\exists a \in I : \bigwedge S \leq a$ then $\exists s \in S : s \leq a$ (I is inaccessible by finite meets)*

(iii) *for all $S \subset I$: $\bigvee S \in I$ (I is closed under joins)*

Proof That a $\mathbf{1}$-kernel satisfies the stated properties is trivial. Suppose now we have a K satisfying the properties for a $\mathbf{1}$-kernel. Let $x_K : \Omega T \to \mathbb{2}$ be defined by $x_K(a) = \mathbf{1} \iff a \in K$. Then we see $x_K(\bigvee_i a_i) = \mathbf{1} \iff \exists i : x_K(a_i) = \mathbf{1}$ by inaccessibility by joins of K. Thus, $\bigvee_i x_K(a_i) = x_K(\bigvee_i a_i)$. Next, we see that for a finite set I, $x_K\left(\bigwedge_{i \in I} a_i\right) = \mathbf{1} \Rightarrow \forall i : x_K(a_i) = \mathbf{1}$ by upper closedness of K. Finally, $\forall i : x_K(a_i) = \mathbf{1} \Rightarrow x_K\left(\bigwedge_{i \in I} a_i\right) = \mathbf{1}$ by closedness under finite meets of K. Therefore, x_K also preserves finite meets, and thus is a frame homomorphism.
For the characterization of $\mathbf{0}$-kernels, observe that for a set K, the three requirements are precisely saying that $\Omega T \backslash K$ is a $\mathbf{1}$-kernel. □

We will now give an examples of a locales.

Example 4.13. One can show that in topology the well-known Hausdorff spaces are all localic. We will not proof this here, but instead refer to [Vickers 1989, *Topology via Logic*] page 66. We can find other locales by taking an arbitrary frame, and construct the points, the frame homomorphisms to $\mathbb{2}$. We will now do this for the frame $\mathrm{Fr}\langle a, b \mid \ \rangle$, the free frame in two generators, displayed in Figure 3.1a. We will look what the possible $\mathbf{1}$-kernels are. The upper closed sets are $\varnothing, \{\mathbf{1}\}, \{\mathbf{1}, a \vee b\}, \{\mathbf{1}, a \vee b, a\}, \{\mathbf{1}, a \vee b, b\}, \{\mathbf{1}, a \vee b, a, b\}, \{\mathbf{1}, a \vee b, a, b, a \wedge b\}, \{\mathbf{1}, a \vee b, a, b, a \wedge b, \mathbf{0}\}$. The only ones which are both inaccessible by joins and closed under meets are $\{\mathbf{1}\}, \{\mathbf{1}, a \vee b, a\}, \{\mathbf{1}, a \vee b, b\}, \{\mathbf{1}, a \vee b, a, b, a \wedge b\}$. These four sets will be the $\mathbf{1}$-kernels of the four points of this locale.

4.4 Spatial locales

As we have already claimed above, Hausdorff spaces are both spatial and localic. However, also from a general system, one can obtain a system which is both spatial and localic. To this end we prove the following proposition:

Proposition 4.14. *Spatialization and localification have the following properties:*

(i) *If (X, A) is a space, then $\mathrm{Loc}(X, A)$ is spatial.*

(ii) *If A is a locale, then $\mathrm{Spat}\, A$ is localic.*

As a consequence, for any system T, $\mathrm{Spat}\,\mathrm{Loc}\, T$ and $\mathrm{Loc}\,\mathrm{Spat}\, T$ are both spatial and localic.

Proof (i): In view of Proposition 4.8, we conclude that X differentiates all opens in A. Since all the points in X give rise to a frame homomorphism by equation (4.1), we conclude that the opens must still be distinguishable, and hence that $\mathrm{Loc}(X, A)$ is spatial.
(ii): Remark A is localic and $\mathrm{Loc}\,\mathrm{Spat}\, A$ is spatial by (i). Furthermore, we have the maps $p : \mathrm{Spat}\, A \to \mathrm{Loc}\,\mathrm{Spat}\, A$ and $e : \mathrm{Spat}\, A \to A$. We can now use Theorems 4.7 and 4.11 to obtain maps $f : \mathrm{Loc}\,\mathrm{Spat}\, A \to A$ and $g : \mathrm{Loc}\,\mathrm{Spat}\, A \to \mathrm{Spat}\, A$ such that $e = f \circ p$ and $f = e \circ g$, and thus $e = e \circ g \circ p$. From this we can conclude (using the uniqueness part of the Theorems) that, $g \circ p = Id_{\mathrm{Spat}\, A}$. Also, $f = e \circ g = f \circ p \circ g$ and therefore $p \circ g = Id_{\mathrm{Loc}\,\mathrm{Spat}\, A}$. Therefore, $\mathrm{Spat}\, A \cong \mathrm{Loc}\,\mathrm{Spat}\, A$ and $\mathrm{Spat}\, A$ is localic. □

5 Sum and product systems

Suppose we are given a set of systems $\{T_i \mid i \in I\}$. Is there a system such that it precisely contains a copy of every point which is in some T_i? And is there a system precisely containing the points of the form $(x_i)_{i \in I}$ with $x_i \in \operatorname{pt} T_i$ (i.e., collections of points indexed by the set I)?

Why would one be interested in specifically these systems? The first case is more natural if studied conversely: Given a system, can we split it up into several systems which are, in some sense, disjoint? To imagine this, one can think of two disjoint, closed intervals in $\mathbb{R}$. In mainstream topology, there is not really a difference between studying them as one system, or as two separate systems. It is desirable that such a difference does not exist if we study general systems either. The motivation for the second system would be, that it allows us to discuss multiple systems at once, much like studying $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$ in mainstream topology. For both of these cases, it seems quite natural that we can extend the approaches of mainstream topology (i.e., spaces) to general systems. This will give rise to the topological sum and product of a collection of spaces.

5.1 Topological sums

In order to be able to construct our desired systems, we need some machinery to express what we need.

Definition 5.1. Suppose we have, for all $i \in I$, a set X_i (an *I-indexed collection* of sets). Then the *disjoint sum* of the X_i, denoted $\coprod_{i \in I} X_i$, is the set $\{(i, x) \mid i \in I, x \in X_i\} \subseteq I \times \bigcup_{i \in I} X_i$. The *Cartesian product* of the X_i, denoted $\bigtimes_{i \in I} X_i$ is the set $\left\{\phi : I \to \bigcup_{i \in I} X_i \;\middle|\; \phi(i) \in X_i\right\}$. In some sense, this can be regarded as selecting a value from each X_i, thus making intuitively clear that this extends the finite Cartesian product.

Definition 5.2. The Cartesian product of an I-indexed collection of frames A_i, denoted $\bigtimes_{i \in I} A_i$, is defined as follows:

(i) The underlying set is $\bigtimes_{i \in I} A_i$ with the A_i considered as their underlying sets.

(ii) Meets and joins are computed component-wise, i.e. $\forall i \in I : \left(\bigvee_{j \in J} \phi_j\right)(i) = \bigvee_{j \in J} \phi_j(i)$ and similarly for meets.

Proposition 5.3. *If all A_i are frames, then $\bigtimes_{i \in I} A_i$ is also a frame.*

Proof All axioms are easily checked by using the fact that meets and joins are computed component-wise, and then using that the axioms hold in all frames A_i. To illustrate this, we will show the frame distributivity. Let $i \in I$ be arbitrary, but fixed:

$$\begin{aligned}\left(\phi \wedge \bigvee_{j \in J} \psi_j\right)(i) &= \phi(i) \wedge \left(\bigvee_{j \in J} \psi_j\right)(i) = \phi(i) \wedge \bigvee_{j \in J} \psi_j(i) \\ &= \bigvee_{j \in j} (\phi(i) \wedge \psi_j(i)) = \bigvee_{j \in J} (\phi \wedge \psi_j)(i) = \left(\bigvee_{j \in J} (\phi \wedge \psi_j)\right)(i)\end{aligned}$$

□

Now that we have these technical details covered, we can proceed with defining the sum of topological systems.

Definition 5.4. Given an I-indexed collection of topological systems T_i, define the system $\sum_{i \in I} T_i = (X, A, \vDash)$, called the *topological sum* of the T_i, as follows:

(i) $X = \coprod_{i \in I} \operatorname{pt} T_i$

(ii) $A = \bigtimes_{i\in I} \Omega T_i$

(iii) $(i,x) \vDash \phi \iff x \vDash_{T_i} \phi(i)$

(iv) $(i,x) \vDash \bigvee_{j\in J} \phi_j \iff (i,x) \vDash \phi_j$ for some $j \in J$.

(v) $(i,x) \vDash \phi \wedge \psi \iff (i,x) \vDash \phi$ and $(i,x) \vDash \psi$.

When considering finite I, we will sometimes use infix notation. E.g., we will sometimes write $T_0 \oplus T_1$ instead of $\sum_{i\in\{0,1\}} T_i$. The fact that this sum is actually a topological system is trivial from the fact that the joins and meets are computed component-wise.

Recalling what we discussed in the motivation for this topological sum, it is desirable that the definition used gives us some structure that is compatible with our intuition. In this light, we prove the following propositions.

Proposition 5.5. *Given an I-ary topological sum system $\sum_{i\in I} T_i$, there is, for every $i \in I$, a unique continuous inclusion function $\iota_i : T_i \to \sum_{i\in I} T_i$.*

Proof The points part is defined by the inclusion, that is, by $\operatorname{pt}\iota_i(x) = (i,x)$. If we want this to be part of a continuous function, we need a frame homomorphism $\Omega\iota_i : \bigtimes_{i\in I} \Omega T_i \to \Omega T_i$ such that $\operatorname{pt}\iota_i(x) \vDash a \iff x \vDash \Omega\iota_i(a)$. From Definition 5.4 we then see that the only choice for $\Omega\iota_i$ would be $\Omega\iota_i(\phi) = \phi(i)$, as one might have guessed. The fact that meets and joins are computed component-wise is precisely saying that $\Omega\iota_i$ is indeed a frame homomorphism. ◻

The following theorem says that our definition of the topological sum is, up to homeomorphism, the best possible.

Theorem 5.6. *If we have an I-indexed collection of continuous functions $f_i : T_i \to S$, there is a unique continuous function $g : \sum_{i\in I} T_i \to S$ such that, for every $i \in I$, $f_i = g \circ \iota_i$.*

Proof If such a continuous function g exists, since it has to satisfy $\operatorname{pt} g((i,x)) = \operatorname{pt} f_i(x)$ and $\Omega g(a) = (i \mapsto \Omega f_i(a))$ for the conditions on the ι_i to hold, we see that g is necessarily unique. On the other hand, since meets and joins are computed component-wise, it is easy to see that g, defined as above, is continuous. ◻

Returning to the example of the two disjoint closed intervals mentioned above, the preceding proposition tells us that, just as one would expect, we can define a continuous function on the union of the intervals by defining one on both of the intervals separately. We conclude with an example of a sum of infinitely many systems.

Example 5.7. Let, for every $n \in \mathbb{N}$, T_n be the locale $\mathbb{2}$. T_n then has only one point, which satisfies $\mathbf{1}$; denote this point by $[2^{-n}]$. Let us now consider the sum system $T = \sum_{n\in\mathbb{N}} T_n$. In T, every open can be understood as a sequence of bits, which in turn can be seen as the binary expansion of a real number in the interval $[0,1]$, by the (clearly bijective) identification

$$\phi \sim \sum_{n\in\mathbb{N}} \phi(n) 2^{-n}.$$

Denote the n-th bit of a real number $x \in [0,1]$ by x_n. Interestingly, the meets and joins now transform to bitwise operators on the real numbers in $[0,1]$. That is, we have (binary) meets corresponding to the bitwise operator AND:

$$x \; AND \; y = \left(\sum_{n\in\mathbb{N}} x_n 2^{-n}\right) AND \left(\sum_{n\in\mathbb{N}} y_n 2^{-n}\right) = \sum_{n\in\mathbb{N}} x_n y_n 2^{-n}$$

which is the real number z having $z_n = 1$ precisely when both $x_n = 1$ and $y_n = 1$. The joins correspond to the operator OR. It is no surprise that this operator takes a set of reals x^i and maps them to the (unique) y satisfying $y_n = 1 \Longleftrightarrow \exists i \in I : x_n^i = 1$. Let us now turn attention to the points, which we called $[2^{-n}]$. As no ambiguity occurs, we retain their names instead of writing the formally correct $(n, [2^{-n}])$ for the corresponding point in $\mathrm{pt}\, T$. We observe that the points are allowing us to approximate real numbers in the following manner: Given a real x, note that, for any $n \in \mathbb{N}$, the equation $x_n = [2^{-n}](x)$ holds. Therefore, we can approximate x by picking a finite $k \in \mathbb{N}$ and then calculating $\sum_{n=1}^{k} [2^{-k}](x) 2^{-n}$. This will yield a rational approximation for x, accurate to the k-th bit. It is easy to see that the points of T allow us to define convergence of sequences of elements of $[0,1]$, and also notions as continuity of functions. On a deeper level, the system T could even be understood as the *definition* of the real interval $[0,1]$.

5.2 Binary topological products

As we have seen, the development of the topological sum flowed rather nicely. Unfortunately, some harder work is required to accomplish the topological product of systems, which will yield tuples of system points to be points of another system. As this matter is relatively complicated, we deem it more insightful to treat the binary case first, before heading to arbitrary products.

To do this, we have to introduce a new kind of frame, which will allow us to give the system the desired properties.

Definition 5.8. Given two frames A and B, the binary *tensor product frame* $A \otimes B$ is the frame defined by (here $a \in A$, $b \in B$)

$$Fr\left\langle a \otimes b \;\middle|\; \textstyle\bigwedge_{i \in I} (a_i \otimes b_i) = (\bigwedge_{i \in I} a_i) \otimes (\bigwedge_{i \in I} b_i)\,, \quad \begin{array}{c} \bigvee_{i \in I} (a_i \otimes b) = (\bigvee_{i \in I} a_i) \otimes b, \\ \bigvee_{i \in I} (a \otimes b_i) = a \otimes (\bigvee_{i \in I} b_i) \end{array} \right\rangle.$$

In the above definition, the first set of relations says precisely that meets are computed component-wise. The last two relations say that the tensor product $\otimes$ distributes over joins. The latter assures us that we in general have a different frame than $A \times B$ defined above and it also means that we cannot in general simplify a join. The former means that we could just as well have chosen $a \otimes \mathbf{1}$ and $\mathbf{1} \otimes b$ as generators, since $a \otimes b$ is the meet of these two. This will be important when we want to extend the product later on. We are now prepared for the introduction of the binary product system.

Definition 5.9. Given two systems S and T, the *binary product system* $S \times T = (X, A, \vDash)$ has its components defined by

(i) $X = \mathrm{pt}\, S \times \mathrm{pt}\, T$

(ii) $A = \Omega S \otimes \Omega T$

(iii) $(s,t) \vDash a \otimes b \Longleftrightarrow s \vDash a$ and $t \vDash b$

(iv) $(s,t) \vDash \bigvee_{i \in I} a_i \otimes b_i \Longleftrightarrow (s,t) \vDash a_i \otimes b_i$ for some $i \in I$.

There is no need to define $\vDash$ for meets separately as they are computed component-wise. Some technical difficulties occur in proving that $S \times T$ is actually a system, but these will be dealt with when we define the arbitrary product (cf. Proposition 5.15 below). For now, let us discuss some properties of this system.

Proposition 5.10. *There exist natural continuous maps $\pi_1 : S \times T \to S$ and $\pi_2 : S \times T \to T$, called* projection maps.

Proof We prove only the existence of π_1, as the proof for π_2 is similar. We see that the only natural choice for $\mathrm{pt}\, \pi_1 : \mathrm{pt}\, S \times \mathrm{pt}\, T \to \mathrm{pt}\, S$ has $\pi_1((s,t)) = s$. The choice for $\Omega\pi_1 : \Omega S \to \Omega S \otimes \Omega T$ has to interact correctly with this definition in order for π_1 to be continuous, and this leads to picking $\Omega\pi_1(a) = a \otimes \mathbf{1}$, which is a frame homomorphism as one readily verifies. □

The word 'natural' in the above proposition refers to the fact that the map does not depend on any extra knowledge of S or T. Of course, more continuous maps could exist. Next, we use these projection maps to establish a factoring property similar to those discussed for spatialization and localification.

Theorem 5.11. *Given continuous maps $f_1 : U \to S$ and $f_2 : U \to T$, there exists a unique map $g : U \to S \times T$ such that $f_1 = \pi_1 \circ g$ and $f_2 = \pi_2 \circ g$.*

Proof Let us first consider the points part. If g exists, it necessarily has $\operatorname{pt} g(u) = (\operatorname{pt} f_1(u), \operatorname{pt} f_2(u))$ by the nature of the projection maps. Now for $\Omega g : \Omega S \otimes \Omega T \to \Omega U$. Because we know that we only have to define Ωg on the generators in such a way that the relations aren't violated, the definition is simplified. Suppose Ωg is known (i.e., g exists). We observe $\Omega g(a \otimes b) = \Omega g(a \otimes \mathbf{1} \wedge \mathbf{1} \otimes b) = \Omega g(\Omega \pi_1(a) \wedge \Omega \pi_2(b)) = \Omega f_1(a) \wedge \Omega f_2(b)$. According to Proposition 3.10 this Ωg will be a frame homomorphism if its definition does not violate the imposed relations. We skip verifying this as it is trivial. We conclude that we can uniquely define a continuous g, as asserted. □

This theorem is the main handle for defining arbitrary products, for which we are now prepared. Several other interesting properties of products will be stated in the arbitrary setting, since these are still proved fairly easily in the arbitrary case. Let us now work out an explicit example to get familiar with this new object.

Example 5.12. Let S be the locale defined by $Fr\langle a \mid \rangle$ (depicted in Figure 3.1b), and let its points x, y be defined by $x \vDash \mathbf{1}, x \vDash a$ and $y \vDash \mathbf{1}$. Similarly, let T be $Fr\langle b \mid \rangle$ with points u and v, corresponding to x and y, respectively. Let us describe $S \times T$. The points will be the set $\{(x,u),(x,v),(y,u),(y,v)\}$. Now to describe the frame. A priori we have nine generators of this frame. However, the existence of the continuous functions of Proposition 5.10 assures us that any generator containing a $\mathbf{0}$ is forced to be $\mathbf{0}$ in the product frame. This leaves the generators $\mathbf{1} \otimes \mathbf{1}$ $(= \mathbf{1}), a \otimes \mathbf{1}, \mathbf{1} \otimes b, a \otimes b$ and $\mathbf{0} \otimes \mathbf{0}$ $(= \mathbf{0})$. Most of the possible expressions with meets and joins can be simplified, as one can verify. Only the expression $a \otimes \mathbf{1} \vee \mathbf{1} \otimes b$ does not correspond to a generator. Further, we have the enforced equality $a \otimes \mathbf{1} \wedge \mathbf{1} \otimes b = a \otimes b$. Therefore, the frame we have described here is nothing but $Fr\langle a \otimes \mathbf{1}, \mathbf{1} \otimes b \mid \rangle$, a structure depicted in Figure 3.1a. Finally, $\vDash$ is easily constructed from $\vDash_S$ and $\vDash_T$ using Definition 5.9(iii)-(iv). So, interestingly, $S \times T \cong Fr\langle a, b \mid \rangle$.

5.3 Arbitrary topological products

Having treated the finite case, we can now proceed with arbitrary products. Our main concern will be the preservation of the unique factoring property of Theorem 5.11. In order to be able to do this, we need the following definition, which is a generalization of Definition 5.8.

Definition 5.13. Given an I-indexed collection of frames ΩT_i, the I-ary *tensor product frame* $\bigotimes_{i \in I} \Omega T_i$ is defined in the following manner: Let $A = \{\otimes_{i \in I} a_i \mid a_i \in \Omega T_i, \#\{a_i \neq \mathbf{1}_{T_i}\} < \infty\}$, i.e. the subset of $\bigtimes_{i \in I} \Omega T_i$ containing precisely the elements with only finitely many components unequal to $\mathbf{1}$. The frame then is

$$Fr\left\langle A \mid \otimes_{i \in I} a_i \wedge \otimes_{i \in I} b_i = \otimes_{i \in I}(a_i \wedge b_i), R\right\rangle$$
$$\text{where } R = \left\{ \bigvee_{j \in J} \otimes_{i \in I} a_{ij} = \otimes_{i \in I} b_i \,\middle|\, \exists i' \in I, b_{i'} = \bigvee_{j \in J} a_{i'j},\ \forall (i \neq i') \forall j \in J : a_{ij} = b_i \right\}.$$

In the above definition, R expresses that we can simplify joins if the elements of the tensor product are equal at all but one position. Also, note that the tensor product generators are just formal symbols we introduce to denote certain elements in $\bigtimes_{i \in I} \Omega T_i$. Interestingly, we need the "finitely many non-$\mathbf{1}$ components" property expressed by A to retain the unique factoring properry we had for binary products (see Theorem 5.17 and Example 5.18 for a more detailed explanation). We are now prepared for the introduction of the arbitrary topological product.

Definition 5.14. Given an I-indexed collection of systems T_i, the product system $\prod_{i \in I} T_i = (X, A, \vDash)$ has its components defined by

(i) $X = \bigtimes_{i \in I} \operatorname{pt} T_i$

(ii) $A = \bigotimes_{i \in I} \Omega T_i$

(iii) $\phi \vDash \otimes_{i\in I} a_i \iff \forall i \in I : \phi(i) \vDash_{T_i} a_i$

(iv) $\phi \vDash \bigvee_{j\in J} \otimes_{i\in I} a_{ij} \iff \exists j \in J : \forall i \in I : \phi(i) \vDash_{T_i} a_{ij}$.

Note that part (iii) of the definition also encapsulates $\vDash$ for meets, as they are computed componentwise. However, for the joins part, we actually have to verify that this $\vDash$ is well-defined. It could be that two joins describing the same open would yield a different $\vDash$. Fortunately, we have the following.

Proposition 5.15. *The $\vDash$ for joins in Definition 5.14 does not depend on the explicit join, but only on the resulting open.*

Proof We will prove this by showing that all the frame homomorphisms of A have this property. To this end, we remark that any frame homomorphism is fixed by its value on the generators of A, and these values are uniquely determined from the values on the $a \in A$ such that at most one component is not $\mathbf{1}$ (as the other generators are formed by finite meets of these). However, for a homomorphism ϕ to be defined correctly on any one component ΩT_i, its values on that component need to be exactly those of an element of $Hom(\Omega T_i, \mathbb{2})$. As this needs to hold for all components, we conclude that any ϕ is an element of $\bigtimes_{i\in I} Hom(\Omega T_i, \mathbb{2})$. Here we view this as a function mapping to $\mathbf{0}$ precisely when a component does (and to $\mathbf{1}$ otherwise) - this means that ϕ satisfies (iii) above. Now it remains to check that ϕ satisfies condition (iv).

Imposing condition (iv) on the frame homomorphism ϕ translates to $\phi\left(\bigvee_{j\in J} \otimes_{i\in I} a_{ij}\right) = \mathbf{1}$ being equivalent to $\bigvee_{j\in J} \phi(\otimes_{i\in I} a_{ij}) = \mathbf{1}$ - that the latter indeed encodes the right hand side above can readily be verified by the reader. However, as we know ϕ to be a frame homomorphism, this equivalence is trivial. □

Having established the product system, it is natural to ask whether or not it interacts in a natural manner with its components. The next proposition is a result in this direction.

Proposition 5.16. *Suppose we have an I-indexed collection of systems T_i. Then, for every $i \in I$, we have a natural continuous map $\pi_i : \prod_{j\in I} T_j \to T_i$.*

Proof For the points part of π_i, we need a function $\mathrm{pt}\,\pi_i : \bigtimes_{j\in I} \mathrm{pt}\,T_j \to \mathrm{pt}\,T_i$, and it is natural to take the evaluation map $\phi \mapsto \phi(i)$. For $\Omega\pi_i$, we need a frame homomorphism $\Omega\pi_i : \Omega T_i \to \bigotimes_{j\in I} \Omega T_j$ such that $\phi \vDash \Omega\pi_i(a) \iff \phi(i) \vDash a$. In other words, the left hand side should only depend on $\phi(i)$. This can be achieved by defining $\Omega\pi_i$ as the frame homomorphism $a \mapsto \otimes_{j\in I} \begin{cases} \mathbf{1} & i \neq j \\ a & i = j \end{cases}$. This clearly is a (natural) frame homomorphism, and so π_i is continuous. □

Having constructed the projection maps, we can now state the analog of Theorem 5.11 for arbitrary products.

Theorem 5.17. *Let S be a system, and suppose we also have an I-indexed collection of systems T_i. Then if we have continuous maps $f_i : S \to T_i$, there is a unique continuous map $g : S \to \prod_{j\in I} T_j$ such that $f_i = \pi_i \circ g$.*

Proof As was the case with the binary product, the points part can only be defined as $\mathrm{pt}\,g : \mathrm{pt}\,S \to \bigtimes_{i\in I}, \mathrm{pt}\,g(x) = (i \mapsto f_i(x))$ for any g that is to satisfy the conditions. The Ωg corresponding to such a hypothetical g would then have to satisfy $\Omega g(\Omega\pi_i(a)) = \Omega f_i(a)$. However, this fixes g on the generators as they can be obtained by finite meets on images of some $\Omega\pi_i$'s. If we verify that defining Ωg on the generators this way does not violate the relations of the frame, Propostion 3.10 tells us that it defines a unique frame homomorphism. We conclude that in this case, g is necessarily unique, and defining it as above yields a continuous map. So let us explicitly verify that no relations are offended. To this end, let $I' \subseteq I$ be finite. Observe a generator of the frame can be written as $\bigwedge_{i'\in I'} f_{i'}(a_{i'})$ for suitable choice of I'. As the first stated condition above says something about the form of the generators, it does not need verification. The second relation (about meets) is satisfied if we can show $\bigwedge_{j\in J}\left(\bigwedge_{i'\in I'} f_{i'}(a_{i'j})\right) = \bigwedge_{i'\in I'} f_{i'}\left(\bigwedge_{j\in J} a_{i'j}\right)$. This is easy, as both expressions are equal to $\bigwedge_{i'\in I'}\left(\bigwedge_{j\in J} f_{i'}(a_{i'j})\right)$ (we use that the $f_{i'}$ are frame homomorphisms and the associativity of meets). For the joins (the set of relations R above) we need to verify $\bigvee_{j\in J}\left(\bigwedge_{\substack{i'\in I'\\ i'\neq i''}} f_{i'}(a_{i'}) \wedge f_{i''}(a_{i''j})\right) = \left(\bigwedge_{\substack{i'\in I'\\ i'\neq i''}} f_{i'}(a_{i'})\right) \wedge f_{i''}\left(\bigvee_{j\in J} a_{i''j}\right)$, which follows from applying frame distributivity, and using that $f_{i''}$ is a frame homomorphism. For generators where the i''-th component is $\mathbf{1}$, there is nothing to prove, and therefore it does not matter that such generators are not covered by the above formula. This concludes the proof. □

Remark that, for the above proof to work, we critically used that we could decompose generators into finite meets of images of some $\Omega\pi_i$ (or, equivalently, elements $\otimes_{i\in I} a_i$ with $a_i \neq \mathbf{1}$ holding for at most one i). If we would have taken all of $\bigtimes_{i\in I}\Omega T_i$ as generators, this would not generally be possible; see Example 5.18. This justifies our choice for the form of the generators of the arbitrary product frame. As a last remark, we note that picking $S = \mathbb{2}$ ($\mathbb{2}$ viewed as locale) gives us a precise characterization of the locale points of the product system (much in the same way as Proposition 5.15 did), and hence of the possible points (cf. Proposition 5.20). In [Vickers 1989, *Topology via Logic*] pp. 82-83, this characterization is used to define $\models$ on the (finite) product, and subsequently to show this definition yields a system. We proceed with an example showing that the definition of the product can't be loosened.

Example 5.18. Suppose we remove the constraint on the form of the generators in Definition 5.13, i.e. we use all of $\bigtimes_{i\in I}\Omega T_i$ as generators. Our goal is to show that we can't retain the factoring property of the preceding theorem if we use this alternate definition. Let $I = \mathbb{N}$ and T_i the locale defined by $Fr\langle i \mid \rangle$, that is, the frame with three points $\mathbf{0} < i < \mathbf{1}$. Furthermore, let S be the locale defined from the subset $\{\frac{1}{4} + \frac{1}{2n} \mid n \in \mathbb{N}\} \cup \{0, \frac{1}{4}, 1\}$ of $\mathbb{Q}$, that is, $\{0, \frac{1}{4}, \ldots, \frac{1}{2}, \frac{3}{4}, 1\}$. Let us now define continuous functions $f_i : S \to T_i$ by the frame homomorphisms $\Omega f_i(i) = \frac{1}{4} + \frac{1}{2i}$. If now the factoring property would be preserved, we would be able to find a unique $f : S \to \prod_{i\in\mathbb{N}} T_i$, or equivalently, a frame homomorphism in the reverse direction. Recall this has only to be defined on the generators. Clearly, $\Omega f(\phi) = 0$ if for some i, $\phi(i) = \mathbf{0}$. Also let us extend this by defining $\Omega f(\phi) = \min_{i\in\mathbb{N}} \frac{1}{4} + \frac{1}{2\phi(i)}$ whenever this minimum exists, but no component is $\mathbf{0}$. The remaining cases are those generators which have infinitely many i's and no $\mathbf{0}$. Obviously, we get two different continuous functions if we define either $f(\phi) = \frac{1}{4}$ for all these cases or $f(\phi) = \mathbf{0}$ in all these cases. Both functions do not violate any conditions. Therefore, the factoring property can no longer hold (uniqueness is violated).

5.4 Relations to spatial and localic systems

It is natural to ask how the systems constructed above look like in specific cases, like when all the composing systems are spatial or localic. For the topological sum, we have some nice results.

Proposition 5.19. *A topological sum of spatial systems is also spatial, and a sum of localic systems is localic.*

Proof For the first claim, note that we have, for two points ϕ, ψ, $\mathrm{ext}(\phi) = \mathrm{ext}(\psi) \iff \forall i \in I : \mathrm{ext}(\phi(i)) = \mathrm{ext}(\psi(i)) \iff \phi = \psi$, where the latter equivalence follows from the spatiality of the summands.
The second claim is a bit more involved. Denote the sum as T for brevity. Consider the localification map $p : T \to \mathrm{Loc}\,T$. Our goal is to show that it gives a bijection on the points. To this end, define functions $f_i : \Omega T_i \to \Omega T$ by (note that the image of $f_i(a)$ is again a function)

$$f_i(a) = j \mapsto \begin{cases} a & \text{if } i = j \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Let us now prove that $\mathrm{pt}\, p$ is injective. Suppose $\mathrm{pt}\, p((i,x)) = \mathrm{pt}\, p((i',x'))$ with $x \in \mathrm{pt}\, T_i$ and $x' \in \mathrm{pt}\, T_{i'}$. Then $(i,x) \vDash f_i(\mathbf{1})$, and thus $(i',x') \vDash f_i(\mathbf{1})$, since (i,x) and (i',x') satisfy the same opens. This means that necessarily $i = i'$. Furthermore, since $\Omega(p \circ \iota_i)$ is surjective, and the domain of $p \circ \iota_i$ is a locale, we can conclude that $\mathrm{pt}(p \circ \iota_i)$ is injective (this is an easy exercise), which shows that $x = x'$.
This leaves us with the surjectivity to prove. To this end, let $u \in \mathrm{pt}\,\mathrm{Loc}\, T$. Then $u \vDash \mathbf{1} = \bigvee_{i \in I} f_i(\mathbf{1})$, so there is an i such that $u \vDash f_i(\mathbf{1})$, and this i is unique since $f_i(\mathbf{1}) \wedge f_{i'}(\mathbf{1}) = \mathbf{0}$ for $i \neq i'$. We conclude that $x = u \circ f_i$ is also a (locale) point, but of T_i (which is localic). It is straightforward to verify that $z = \mathrm{pt}\, p(x)$. □

So we see that the sum behaves very well with respect to the mainstream topological objects. Let us now discuss the product.

Proposition 5.20. *A topological product of localic systems $T = \prod_{i \in I} T_i$ is also localic.*

Proof Suppose we have a locale point x, i.e. a frame homomorphism to $\mathbb{2}$. Then Proposition 3.10 tells us that its definition is fixed by the values on the generators. It is enough to specify x on the images of the $\Omega\pi_i$'s of Proposition 5.16, as the generators are finite meets of these. However, the only possible valid definitions on these are actually determined by the locale points of T_i. Therefore, the locale points of ΩT are precisely the points of T described in Definition 5.14. □

For spaces, however, there is no such general result. **Insert nonspatial Ex** The way mainstream topology deals with this is by introducing the product of spaces as the spatialization of the product introduced here. In the binary case, this corresponds to taking the topology generated by all "rectangles" (cartesian product of opens from the two spaces).

6 More on lattices

6.1 Distributivity

In Example 2.8 we have seen two examples of non-distributive lattices. We also claimed that in every non-distributive lattice, there were five points arranged in this way. We'll state this here again, and prove parts of it.
The algebraic theory of *pseudolattices* is the algebraic theory of lattices without $\mathbf{0}$, $\mathbf{1}$ and the unit laws. So a pseudolattice is a set with two commutative associative idempotent binary operators $\wedge$ and $\vee$, satisfying the absorptive laws. Hence a subpseudolattice is a subset having the same binary meets and joins.

Theorem 6.1. *For a lattice A the following are equivalent:*

(i) *A is a distributive lattice*

(ii) *For all elements $a, x, y \in A$: $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ implies $x = y$*

(iii) *A has none of the two lattices in Figure 2.1 as sub**pseudo**lattice.*

Proof We will only prove (i)⇒(ii)⇒(iii) here. (iii)⇒(i) can be found in [Birkhoff 1967, *Lattice Theory*] page 134.
(i)⇒(ii) Since A is distributive and since $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$, we see that

$$x = x \vee (x \wedge a) = x \vee (y \wedge a) = (x \wedge y) \vee (x \wedge a) = (y \wedge x) \vee (y \wedge a) = y \wedge (x \vee a) = y \wedge (y \vee a) = y.$$

(ii)⇒(iii) Suppose A does contain one of the two lattices in Figure 2.1 as subpseudolattice. Then $a \wedge b = d = a \wedge c$ and $a \vee b = e = a \vee c$, but $b \neq c$. This is a contradiction. □

At first sight, you might think that (ii) follows easily from (iii), but this is rather tricky. If we want to prove it by contradiction, then we might reason the following way:
Suppose $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$, but $x \neq y$. Then there are three cases: $x < y$, $y < x$ or x and y are incomparable. If $x < y$, then the elements $a = a, b = x, c = y, d = a \wedge x = a \wedge y, e = a \vee x = a \vee y$ exactly form the subpseudoframe in Figure 2.1b. This is true because all meets and joins other than $a \wedge b$, $a \wedge c$, $a \vee b$ and $a \vee c$ are forced by the partial ordering, and we know that these two meets are equal, as are the two joins. The second case follows the same way by duality. If we're in the third case, then one might think that the elements $a, b = x, c = y, d = a \wedge x = a \wedge y, e = a \vee x = a \vee y$ form the subpseudoframe in Figure 2.1a. But the catch is, that although this is a subposet, this might not be a subpseudolattice. Nothing guarantees that $x \wedge y = d$ or $x \vee y = e$. A counterexample can be found in Figure 6.1. Note that the lattice in this example is indeed not distributive, and contains one of the five element non-distributive lattices as subpseudolattice, for example $a, y, x \wedge y, a \wedge x, a \vee x$ (but there are four more ways of choosing five such points). But in general, if we would only know that $a \wedge x = a \wedge y$, $a \vee x = a \vee y$ and that x and y are incomparable, then these five points are not a subpseudolattice.

6.2 Boolean algebras

Given a lattice A and an element $a \in A$. An element $x \in A$ is called a *complement* of a iff $a \wedge x = \mathbf{0}$ and $a \vee x = \mathbf{1}$. In a distributive lattice, Theorem 6.1 tells us that every element has at most one complement. Then we write $\neg a$ for the complement of a, if it exists. A *Boolean algebra* is a distributive lattice in which every element has a complement. So a Boolean algebra A has an operator $\neg : A \to A$. One can easily check that $\neg\neg x = x$, showing that $\neq$ is actually a bijection.

Proposition 6.2. *If x and y each have a complement in a distributive lattice, then also $x \vee y$ and $x \wedge y$ have complements, equal to $\neg x \wedge \neg y$ and $\neg x \vee \neg y$, respectively (note that $\neg x \wedge \neg y$ should be interpreted as $(\neg x) \wedge (\neg y)$). These are called the* De Morgan laws.

Proof We calculate

$$(x \vee y) \wedge (\neg x \wedge \neg y) = (x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$$

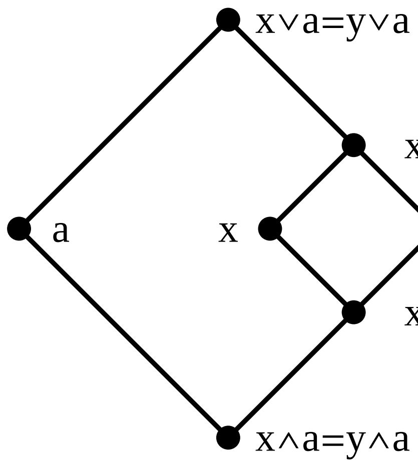


Figure 6.1

and

$$(x \vee y) \vee (\neg x \wedge \neg y) = (x \vee y \vee \neg x) \wedge (x \vee y \vee \neg y) = \mathbf{1} \wedge \mathbf{1} = \mathbf{1},$$

which shows that $\neg x \wedge \neg y$ is the complement of $x \vee y$ (and is unique by distributivity). Since the operation $\neg$ is self-dual, that is, for distributive lattices $\neg x$ is also the complement of x in the *opposite* lattice (the lattice with reversed ordering), the other half of the proposition follows from duality. □

Corollary 6.3. *In a distributive lattice, the elements which have a complement form a sublattice.*

Proof Note that $\neg\mathbf{0} = \mathbf{1}$ and $\neg\mathbf{1} = \mathbf{0}$, since $\mathbf{0} \wedge \mathbf{1} = \mathbf{0}$ and $\mathbf{0} \vee \mathbf{1} = \mathbf{1}$. By Proposition 6.2 the meets and joins of two elements with a complement also have complements. □

We can obtain a ring structure on every Boolean algebra A, by defining the addition as the *symmetric difference*, $a + b = (a \wedge \neg b) \vee (b \wedge \neg a)$ and multiplication by $ab = a \wedge b$ for all $a, b \in A$. We will show below that this indeed is a ring, but first we see that this ring has a special property, namely that $a^2 = a \wedge a = a$ for all $a \in A$ (*idempotence*). Call a ring with 1 a *Boolean ring* if $x^2 = x$ for all elements x in the ring.

Proposition 6.4. *In a Boolean algebra A, defining $a + b = (a \wedge \neg b) \vee (b \wedge \neg a)$ and $ab = a \cdot b = a \wedge b$ makes $(A, +, \cdot, \mathbf{0}, \mathbf{1})$ into a Boolean ring.*

Proof First note that the addition is commutative. We show its associativity using the distributivity of $\wedge$ over $\vee$ multiple times:

$$\begin{aligned}
(a+b)+c &= (((a \wedge \neg b) \vee (b \wedge \neg a)) \wedge \neg c) \vee (c \wedge \neg((a \wedge \neg b) \vee (b \wedge \neg a))) \\
&= (\neg c \wedge ((a \wedge \neg b) \vee (b \wedge \neg a))) \vee (c \wedge ((\neg a \vee b) \wedge (\neg b \vee a))) && \text{(by Proposition 6.2)} \\
&= (\neg c \wedge ((a \wedge \neg b) \vee (b \wedge \neg a))) \vee (c \wedge ((a \wedge b) \vee (\neg b \wedge \neg a))) && \text{(using } x \wedge \neg x = \mathbf{0} \text{ and } x \vee \mathbf{0} = x) \\
&= (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c). && (6.1)
\end{aligned}$$

This last expression is independent of the order of (a, b, c), so $a + (b + c)$ is independent as well. So $a + (b + c) = c + (a + b) = (a + b) + c$, by commutativity.
Now note that $a + \mathbf{0} = (a \wedge \neg\mathbf{0}) \vee (\mathbf{0} \wedge \neg a) = (a \wedge \mathbf{1}) \vee \mathbf{0} = a$ and $a + a = (a \wedge \neg a) \vee (a \wedge \neg a) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$, so $\mathbf{0}$ is a unit under addition, and every element is its own inverse. Thus $(A, +, \mathbf{0})$ forms a commutative group. Note that the multiplication is associative and idempotent. Therefore, only distributivity still needs to be checked:

$$\begin{aligned}
ab + ac &= (a \wedge b \wedge \neg(a \wedge c)) \vee (a \wedge c \wedge \neg(a \wedge b)) \\
&= (a \wedge b \wedge (\neg a \vee \neg c)) \vee (a \wedge c \wedge (\neg a \vee \neg b)) && \text{(by Proposition 6.2)} \\
&= (a \wedge b \wedge \neg c) \vee (a \wedge c \wedge \neg b) && \text{(using } x \wedge \neg x = \mathbf{0} \text{ and } x \vee \mathbf{0} = x) \\
&= a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= a(b + c).
\end{aligned}$$

Thus $(A, +, \cdot, \mathbf{0}, \mathbf{1})$ is a Boolean ring. $\square$

Interestingly, this correspondence works in two ways. In all Boolean rings one can define the structure of a Boolean algebra, as we will see. First we prove the following lemma, showing how strong the idempotence condition for a Boolean ring is.

Lemma 6.5. *A Boolean ring A is commutative and for all $a \in A$, $a = -a$.*

Proof Note that $a + b = (a+b)^2 = a^2 + ab + ba + b^2 = a + b + ab + ba$, so $ab = -ba$. If we now use this for $a = b$, then we get $a = a^2 = -a^2 = -a$, and using this we obtain $ab = -ba = ba$. $\square$

Now we see that $(A, \cdot, 1)$ is a semilattice, since idempotence, associativity and the unit law hold by definition and we just proved the commutativity. But we can get more, as the following proposition states.

Proposition 6.6. *If we define $a \vee b = a + b + ab$, $a \wedge b = ab$, $\neg a = a + 1$, $\mathbf{0} = 0$ and $\mathbf{1} = 1$ in a Boolean ring A, then $(A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \neg)$ is a Boolean algebra.*

Proof We already checked the axioms for $\wedge$. We easily see that $\vee$ is commutative. Also, since

$$a \vee (b \vee c) = a + (b + c + bc) + a(b + c + bc) = a + b + c + ab + bc + ca + abc,$$

which is independent of the order of (a, b, c), we see $\vee$ is associative. Note that $a + 1 + a \cdot 1 = a + 1 + a = 1$ and $a + a + a \cdot a = a$, verifying the axioms for $\vee$. Now we check the absorptive laws:

$$\begin{aligned} a \vee (a \wedge b) &= a + ab + a \cdot ab = a + ab + ab = a \\ a \wedge (a \vee b) &= a(a + b + ab) = a^2 + ab + a^2 b = a + ab + ab = a. \end{aligned}$$

The distributivity is also easily checked:

$$a \wedge (b \vee c) = a(b + c + bc) = ab + ac + abc = ab + ac + (ab)(ac) = (a \wedge b) \vee (a \wedge c).$$

So A is a distributive lattice. Lastly we verify that $a \wedge \neg a = a(1 + a) = a + a^2 = a + a = 0$ and $a \vee \neg a = a + (1 + a) + a(1 + a) = 1 + 3a + a^2 = 1$. So we indeed found that A is a Boolean algebra. $\square$

With what we've done until now, the following result is not surprising anymore.

Theorem 6.7. *Boolean algebras and Boolean rings are equivalent. Also, the Boolean algebra homomorphisms are the same as the Boolean ring homomorphisms.*

Proof For the first part, the only thing left to check is that if we have a Boolean algebra, turn it into a Boolean ring as in Proposition 6.4, and turn that back to a Boolean algebra as in Proposition 6.6, we get the same algebra, and vice versa.
If we start with a Boolean algebra, then using equation (6.1) we get

$$\begin{aligned} a + b + ab &= (a \wedge b) \vee \mathbf{0} \vee (\neg a \wedge b \wedge (\neg a \vee \neg b)) \vee (a \wedge \neg b \wedge (\neg a \vee \neg b)) \\ &= (a \wedge b) \vee (\neg a \wedge b) \vee (a \wedge \neg b) = a \vee b \end{aligned}$$

and $1 + a = (a \wedge \mathbf{0}) \vee (\mathbf{1} \wedge \neg a) = \neg a$. So indeed we obtain the Boolean algebra we started with.
Conversely, if we start with a Boolean ring, then

$$(a \wedge \neg b) \vee (b \wedge \neg a) = (a + ab) \vee (b + ab) = a + ab + b + ab + (a + ab)(b + ab) = a + b + 6ab = a + b.$$

Again we obtain the starting structure.
If a function commutes with $\wedge$, $\vee$ and $\neg$, then it clearly also commutes with $+$ and $\cdot$, and vice versa. So the homomorphisms of these two structures are also the same. $\square$

We can say more about finite Boolean algebras, which we will discuss now.

Definition 6.8. In a Boolean algebra, an element a is called an *atom* iff $a \neq \mathbf{0}$ and there are no elements x with $\mathbf{0} < x < a$. So a is a minimal non-zero element. A system is called *atomic* if every element can be written as a (possibly empty) join of atoms.

Now we are ready for the following proposition.

Proposition 6.9. *Every finite Boolean algebra A is atomic and is isomorphic to the power set of some set, seen as a Boolean algebra with operators union, intersection and complement. Hence the number of elements is a power of two.*

Proof Suppose that there is an element which is not the join of atoms. Since A is finite, we can take a minimal such element, call it x. Clearly, $x \neq 0$ and x is not an atom. So there exists an element $\mathbf{0} < y < x$ which means that y can be written as a join of atoms. Now consider $z = x \wedge \neg y$. Then $z \leq x$. Also $z \neq \mathbf{0}$, because if $z = \mathbf{0}$, then $x \wedge \neg y = \mathbf{0}$ and $x \vee \neg y \geq y \vee \neg y = \mathbf{1}$, so $\neg y$ is the complement of x, which contradicts the fact that the operator $\neg$ is a bijection for Boolean algebras. And $z \neq x$, since else $\neg y \geq x \geq y$, meaning that $\mathbf{0} = y \wedge \neg y = y$. So $\mathbf{0} < z < x$, which means that z can also be written as a join of atoms. Now $z \vee y = (x \vee y) \wedge (\neg y \vee y) = x \wedge \mathbf{1} = x$, so x is the join of two elements which can be written as join of atoms, which is a contradiction with the fact that x cannot be written as a join of atoms.
Now let $S \subseteq A$ be the set of atoms. We define the following map: $f : A \to \mathcal{P}(S)$ with $x \mapsto \{a \in S \mid a \leq x\}$. Every x can be written as the join of some atoms, and clearly these atoms have to be smaller than x. Also, if an atom a does not appear in this join, then $x \wedge a = \mathbf{0}$, since distributivity and the fact that the meet of two different atoms is $\mathbf{0}$. So $x = \bigvee f(x)$, and for $B \subset S$ we have $B = f(\bigvee B)$. So $\bigvee : \mathcal{P}(S) \to A$ is the inverse map of f, which means that f is bijective. Also note that f is an isomorphism between Boolean algebras. So A is isomorphic to $\mathcal{P}(S)$. In turn, this implies $|A| = |\mathcal{P}(S)| = 2^{|S|}$. $\square$

The statement that every finite Boolean algebra is isomorphic to the power set of some set can be generalized in the following way. Define a *complete Boolean algebra* to be a Boolean algebra that is also a frame. Then every complete Boolean algebra which is atomic is isomorphic to the power set of some set.

6.3 Ideals and Filters

In Proposition 6.7 we have seen that Boolean rings and Boolean algebras are essentially the same thing, even the Boolean ring homomorphisms coincide with Boolean algebra homomorphisms. We can exploit this identification, since we can borrow certain ring-theoretical concepts, which can also be described in Boolean algebras. We will do this with the ring-theoretical concept of ideals. We can even generalize this concept to all lattices, in such a way that this coincides with the already known definition of ideal when the lattice is actually a Boolean algebra. The study of these ideals will lead to two interesting results in Chapter 8.

Definition 6.10. Let A be a lattice. A subset I of A is called an *ideal* if the following conditions hold:

(i) I is lower closed. That is, if $a \in I$ and $b \leq a$ then $b \in K$.

(ii) I is closed under joins. So if $a, b \in I$, then also $a \vee b \in I$ and the empty join is in I: $\mathbf{0} \in I$.

An ideal is called a *prime ideal* if it is inaccessible by meets. That is, $\mathbf{1} \notin I$ and $\forall a, b \in A$: if $a \wedge b \in I$, then $a \in I$ or $b \in I$.
A subset satisfying the dual axioms is called a *filter* (or *prime filter*). So a filter is an upper closed subset that is closed under (possibly empty) meets. A prime filter is thus a filter which is inaccessible by (possibly empty) joins.

This definition is justified by the following lemma.

Lemma 6.11. *In a Boolean algebra A, a subset I is an ideal precisely when I is an ideal in the corresponding Boolean ring. The same holds for prime ideals.*

Proof Suppose I is an ideal in the Boolean algebra A. Then $0 \in I$. Now let $a, b \in I$ and $r \in A$. Since $a \wedge \neg b \leq a$ and $b \wedge \neg a \leq b$, we know that $a \wedge \neg b$ and $b \wedge \neg a$ are elements of I, since I is lower closed. Now $a + b = (b \wedge \neg a) \vee (a \wedge \neg b)$ is the join of two elements in I so $a + b = a - b \in I$, where we used that $b = -b$ in a Boolean ring. Since $r \wedge a \leq a$, $r \wedge a = r \cdot a \in I$. So I is an ideal in the Boolean ring A.
Now suppose I is an ideal in the Boolean ring. Of course $\mathbf{0} \in I$. Then, if $a, b \in I$ and $r \leq a$, we know that $r = r \wedge a = r \cdot a \in I$ and we know that $a \vee b = a + b + ab \in I$, so I is an ideal in the Boolean algebra.
The fact that the same holds for prime ideals is trivial. The extra conditions are equivalent for rings and lattices. $\square$

Now we give a characterization of what extra information is necessary for an ideal to be a prime ideal.

Proposition 6.12. *Let I be an ideal of a lattice A. Then the following are equivalent:*

(i) *I is a prime ideal*

(ii) *$A\backslash I$ is a filter*

(iii) *$A\backslash I$ is a prime filter*

(iv) *There is a lattice homomorphism $f : A \to \mathbb{2}$ such that $I = \{a \in A \mid f(a) = \mathbf{0}\}$, the $\mathbf{0}$-kernel of f.*

Proof (i)⇒(ii) Since I is lower closed, $A\backslash I$ is upper closed and the fact that I is inaccessible by meets immediately translates to $A\backslash I$ being closed under meets. So $A\backslash I$ is a filter.
(ii)⇒(iii) Since I is an ideal, it is closed under joins, so $A\backslash I$ is inaccessible by joins.
(iii)⇒(iv) Define $f(a) = \mathbf{0}$ if $a \in I$ and $f(a) = \mathbf{1}$ otherwise. By the fact that $A\backslash I$ is a prime filter, f preserves meets ($A\backslash I$ is closed under meets and upper closed), joins ($A\backslash I$ is inaccessible by joins and upper closed), $\mathbf{0}$ and $\mathbf{1}$.
(iv)⇒(i) Note that $f(\mathbf{1}_A) = \mathbf{1}_{\mathbb{2}}$, so $\mathbf{1} \notin I$. Suppose $a \wedge b \in I$, then $f(a) \wedge f(b) = f(a \wedge b) = \mathbf{0}$, so $f(a) = \mathbf{0}$ or $f(b) = \mathbf{0}$, so $a \in I$ or $b \in I$. □

For the rest of this section, we will work with distributive lattices to prove some stronger results. We will need the following lemma in Theorem 6.14.

Lemma 6.13. *Given an ideal I and an element a in a distributive lattice A, the smallest ideal (by inclusion ordering) containing both I and a is the set $J = \{i \vee (a \wedge x) \mid i \in I, x \in A\}$, which we will call the ideal* generated by I and a.

Proof For $x = \mathbf{0}$ we find that $i = i \vee (a \wedge \mathbf{0}) \in J$, so $\mathbf{0} \in I \subseteq J$. Also for $i = \mathbf{0}$ and $x = \mathbf{1}$ we find $a \in J$. To show that J is lower closed, consider an element b smaller than some element $i \vee (a \wedge x)$. Then by distributivity $b = b \wedge (i \vee (a \wedge x)) = (b \wedge i) \vee (a \wedge (b \wedge x)) \in J$, since $b \wedge i \in I$ by the lower closure of I. Also, I is closed under joins, so $(i_1 \vee (a \wedge x_1)) \vee (i_2 \vee (a \wedge x_2)) = (i_1 \vee i_2) \vee (a \wedge (x_1 \vee x_2)) \in J$, showing that J is also closed under joins. Therefore J is an ideal, containing both a and I.
Now suppose K is also an ideal containing I and a. Then for every $i \in I$, K contains i and a, whence also $i \vee a$. Since taking the join with i is order-preserving, and $a \wedge x \leq a$ for every $x \in A$, $i \vee (a \wedge x) \leq i \vee a$. Because K is lower closed, K contains $i \vee (a \wedge x)$, proving $K \supseteq J$. □

The next theorem states that in every lattice there are enough prime ideals, in some sense. For this result we need the axiom of choice.

Theorem 6.14. *Let I be an ideal and F a filter in a lattice A such that $I \cap F = \varnothing$.*

(i) *There exists an ideal M which is maximal amongst the ideals containing I and disjoint from F. That is, if $J \supseteq I$ is an ideal such that $J \cap F = \varnothing$, then $J \subseteq M$.*

(ii) *If A is a distributive lattice, every M as in* (i) *is a prime ideal.*

Proof (i) The set of ideals in A disjoint from F and containing I is a set partially ordered by inclusion. We will use Zorn's lemma on this poset. It is non-empty, since it contains I. Given a chain of ideals $(J_\kappa)_{\kappa \in K}$, then we claim that $J := \bigcup_{\kappa \in K} I_\kappa$ is again an ideal. Clearly, $\mathbf{0} \in J$. If $a, b \in J$, then $a \in J_\alpha$ and $b \in J_\beta$ for some $\alpha, \beta \in K$. Since $(J_\kappa)_{\kappa \in K}$ is linearly ordered, either $J_\alpha \subseteq J_\beta$ or $J_\beta \subseteq J_\alpha$. In either case, $a, b \in J_\gamma$ for some $\gamma \in K$, so $a \vee b \in J_\gamma$ and thus $a \vee b \in J$. At last, J is lower closed, since $a \in J$ implies $a \in J_\alpha$ for some α and thus $\{x \in A \mid x \leq a\} \subseteq J_\alpha \subseteq J$. So J is an ideal, and thus an upper bound of the chain $(J_\kappa)_{\kappa \in K}$ with respect to the ordering. So by Zorn's lemma, there exists a maximal element M satisfying the conditions.
(ii) Since $\mathbf{1} \in F$ we know that $\mathbf{1} \notin M$. Suppose now we have $a_1 \wedge a_2 \in M$. Consider for $i = 1, 2$ the ideal J_i generated by M and a_i. Suppose that both J_1 and J_2 have a non-empty intersection with F, so there are elements $i_1 \vee (a_1 \wedge x_1)$ and $i_2 \vee (a_2 \wedge x_2)$ in F. Since F is closed under meets, F also contains $(i_1 \vee (a_1 \wedge x_1)) \wedge (i_2 \vee (a_2 \wedge x_2)) = (i_1 \wedge i_2) \vee (i_1 \wedge a_2 \wedge x_2) \vee (i_2 \wedge a_1 \wedge x_1) \vee (a_1 \wedge a_2 \wedge x_1 \wedge x_2)$. Since M is lower closed, it contains all four terms in the join, so the join is also in M. But this contradicts the fact that M and F are disjoint. So either J_1 or J_2 is disjoint from F. Since M was a maximal ideal disjoint from F, either $J_1 \subseteq M$ or $J_2 \subseteq M$, so either $a_1 \in M$ or $a_2 \in M$. This finishes the proof that M is a prime ideal. □

The following is a very useful proposition which states that there are a quite a lot of lattice homomorphisms to $\mathbb{2}$.

Proposition 6.15. *Let a and b be two elements in a distributive lattice A, such that $a \ngeq b$. Then there exists a lattice homomorphism $f : A \to \mathbb{2}$ satisfying $f(a) = 0$ and $f(b) = 1$.*

Proof Let $I = \{x \in A \mid x \leq a\}$ and $F = \{x \in A \mid x \geq b\}$. It can easily be verified that I is an ideal and F is a filter. $a \ngeq b$ implies that I and F are disjoint. By Theorem 6.14 there exists an prime ideal $M \supseteq I$ disjoint from F and by Proposition 6.12, M is the kernel of a lattice homomorphism $f : A \to \mathbb{2}$. Then $a \in I$ implies $f(a) = 0$, and $b \in F$ implies $f(b) = 1$. □

6.4 Complete ideals and filters

In frames A, we can easily modify the definitions of filters to account for the infinite joins we have. Since we're mostly interested in prime ideals and prime filters, we only modify these definitions, but the not-prime case is straightforward.

Definition 6.16. We call a prime ideal I in a frame A *complete* if it is closed under arbitrary joins (instead of just finite ones). We say that a prime filter is complete if it is inaccessible by arbitrary joins.

Actually, we already encountered these concepts, in Proposition 4.12. Translated into these new definitions, the proposition states that the **1**-kernels of frame homomorphisms to $\mathbb{2}$ are exactly the completely prime filters, and the **0**-kernels are exactly the completely prime ideals. So this proposition was actually the frame equivalent to Proposition 6.12. But we can do more in this case. We know that completely prime ideals are closed under arbitrary joins, so given a prime ideal I, we know that $p := \bigvee I$ is also an element of I, and of course, for all elements $i \in I$, $p \geq i$. Actually, since I is lower closed, $i \in I$ if and only if $i \leq p$. We call the elements p *prime* which are the joins of completely prime ideals, so if $\{i \in A \mid i \leq p\}$ is a completely prime ideal. We then easily verify that the prime elements are the elements which satisfy the following condition:

$$\text{for all finite } S \subset A : \bigwedge S \leq p \Rightarrow \exists s \in S : s \leq p. \tag{6.2}$$

So we get a one-to-one correspondence between prime elements and homomorphisms to the frame $\mathbb{2}$, which are the points in the locale A. Studying these will give us more information about locales.

Actually, in the case that A is a complete Boolean algebra (that is, a Boolean algebra that is also a frame), we can easily find the prime elements by the following lemma.

Lemma 6.17. *In a complete Boolean algebra A the prime elements are exactly the complements of the atoms.*

Proof Let p be the complement of an atom. Suppose that for some $S \subset A$, $\bigwedge S \leq p$. This means that $p = p \vee \bigwedge S = \bigwedge\{p \vee s \mid s \in S\}$. This means that there is some $s \in S$ such that $p \leq p \vee s < \mathbf{1}$. Since p is the complement of an atom, there is no $x \in A$ such that $p < x < \mathbf{1}$, hence $p \vee s = p$, so $s \leq p$.
Now suppose that p is not the complement of an atom. For $p = \mathbf{1}$, the property of (6.2) does not hold for $S = \varnothing$. Otherwise, there is some q such that $p < q < \mathbf{1}$. Now $\bigwedge\{q, p \vee \neg q\} = (q \wedge p) \vee (q \wedge \neg q) = p$, but neither $q \leq p$ nor $p \vee \neg q \leq p$ holds (otherwise $\neg q \leq p \leq q$ which would imply $\mathbf{0} = \neg q \wedge q = \neg q$). So p is not a prime element. □

7 Compactness

In mainstream topology, there is the frequently used notion of compactness. We will pursue the development of a generalization of this concept to the framework of topological systems. We will need quite some machinery to be able to do so. In developing this, we will also briefly discuss some results of independent interest.

7.1 Specialization

Definition 7.1. Let T be a system and x, y be two points of it. We then define $x \sqsubseteq y$ (*y specializes x*) to hold precisely when for all $a \in \Omega T$, $x \vDash a \Rightarrow y \vDash a$ holds.

From the definition it is intuitive to think of y as a special case of x, which gives in some sense more information than x does. It is clear that $\sqsubseteq$ is transitive and reflexive. However, antisymmetry does not generally hold. A system for which antisymmetry holds (i.e., $\sqsubseteq$ makes $\mathrm{pt}\, T$ a poset) is said to be a T_0 system (in such a system, no two distinct points satisfy exactly the same opens). An obvious example of a T_0 system is a locale. We can now ask if (and under which conditions) it is possible to define joins on $\mathrm{pt}\, T$ with respect to $\sqsubseteq$. We want such a definition to satisfy, for $S \subseteq \mathrm{pt}\, T$,

$$\bigvee S \vDash a \iff x \vDash a \text{ for some } x \in S.$$

Pitifully such a definition will, in general, not interact correctly with conjunctions of opens (there need not be a point satisfying all components of the meet for (7.1) to hold). We therefore seek to find subsets S such that this problem does not occur. For convenience, for the remainder of this chapter, all systems are considered to be T_0. To be able to proceed we need the following definition.

Definition 7.2. Let $\sqsubseteq$ be as above. A subset $S \subseteq \mathrm{pt}\, T$ is called *directed* precisely when every finite subset of S has an upper bound in S with respect to $\sqsubseteq$.

Note in particular that $S \neq \varnothing$ as it needs an upper bound for the empty set. An immediate example of a directed subset is a linearly ordered subset. Unfortunately, there will now come two conflicting notions of join into play on the points. We will therefore need a distinction between those two notions. This is the goal of the next definition.

Definition 7.3. Let $S \subseteq \mathrm{pt}\, T$. Then the *disjunction* of S, denoted $\bigvee S$, is defined by (7.1) above (if this point exists). If we want to emphasize that S is directed, we use $\bigvee^{\uparrow} S$ instead. The *join* of S, denoted $\bigsqcup S$, is the least upper bound of S with respect to $\sqsubseteq$ (again, if this point exists). Again, if S is directed, we may stress this by writing $\bigsqcup^{\uparrow} S$ instead.

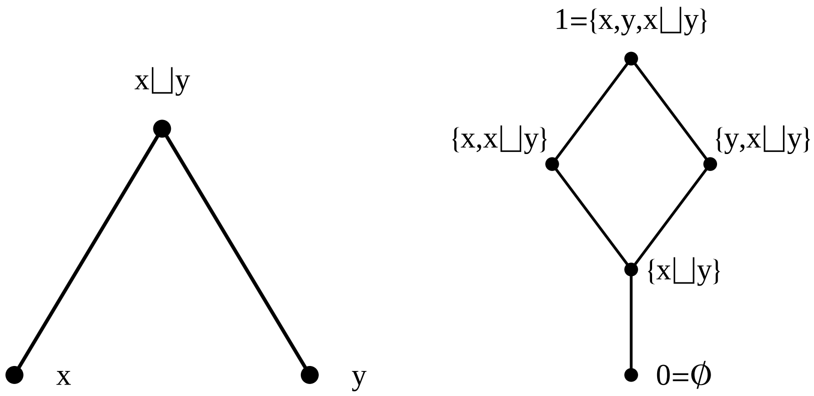
Remark that joins of points are denoted $\bigsqcup$, and for opens they are $\bigvee$. There are some ties between the notions of disjunction and join for points. In fact, the disjunction is stronger than the join, as the following shows.

Proposition 7.4. *If $x = \bigvee S$, then also $x = \bigsqcup S$.*

Proof Suppose $x = \bigvee S$, and let $y \in S$. Then it follows that $y \sqsubseteq x$. Now let x' be another upper bound of S. For an open a, if $x \vDash a$, then $\exists y \in S : y \vDash a$, and therefore $x' \vDash a$. So $x \sqsubseteq x'$ and we have shown $x = \bigsqcup S$. □

Unfortunately, the two definitions are not equivalent. The point being the least upper bound is not necessarily also the join, as the following examples show.

Example 7.5. In Figure 7.1 you will see a system T. One can easily check that it is indeed a topological system. On the points part there is also shown an ordering. It is easy to verify that this is the specialization order. Further note that $x \sqcup y$ is the join of x and y with respect to the specialization order. However, it is not the disjunction of x and y, since the open $x \sqcup y$ satisfies $\{x \sqcup y\}$, but neither x nor y does.
Also note that this system is spatial (no two opens have the same extent) and localic. The latter can be seen in the following way. According to Proposition 4.12, the **1**-kernels of locale points are upper closed, closed under meets and inaccessible by joins. The upper closed sets which are not yet represented by a point are $\{\mathbf{1}\}$ and ΩT, $\varnothing$ and $\{\mathbf{1}, \{x, x \sqcup y\}, \{y, x \sqcup y\}\}$, but the first two of them are accessible by joins and the last two are not closed under meets.



(a) $\mathrm{pt}\,T$ (b) ΩT

Figure 7.1: Representation of the system T using Hasse diagrams.

Example 7.6. Our next example shows that even directed joins need not be disjunctions. We will construct a system T. Let $\mathrm{pt}\,T$ be the ordinal number $\omega + 1$, so the set $\{0, 1, 2, 3, \ldots, \omega\}$. On this set we will have the usual order, which will be our specialization order. For ΩT we will choose the ordinal $\omega + 2$, so the set $\{0, 1, 2, 3, \ldots, \omega, \omega + 1\}$. This time, we take the reversed order. As relation $\vDash$ we will take $x \vDash a$ iff $x \geq a$ (where $\geq$ is the usual order on ordinals). It involves just some easy checking to see that this makes T a system, with the order on the points as specialization order. Now in $\mathrm{pt}\,T$ we have $\omega = \bigsqcup^{\uparrow}\{0, 1, 2, 3, \ldots\}$. But ω is not its disjunction, since $\omega \vDash \omega$, but $n \nvDash \omega$ for every $n < \omega$. So even directed joins need not be directed disjunctions. Note that if we would localify T, we would get a topological system with one more point x, which satisfies $\{0, 1, 2, 3, \ldots\}$ in ΩT. In this system, x would be the directed join and the directed disjunction of $\{0, 1, 2, 3, \ldots\}$. This will hold more general. We will see in the next proposition that in a locale, every directed join is in fact a directed disjunction.

To the reader, it might not be clear what we introduced the notion of directed subset for. We will discuss it more later on, but for now, the following proposition lends some justification.

Proposition 7.7. *In a localic system T, any directed subset S of $\mathrm{pt}\,T$ has a disjunction, and thus by Proposition 7.4, also a join.*

Proof We need $x = \bigvee^{\uparrow} S$ to be defined by (7.1). It needs verification that this does in fact yield a point of the locale T. To this end, let $A \subseteq \Omega T$ and treat x as a function from $\Omega T \to \mathbb{2}$. We need to verify that x defines a frame homomorphism. Then for joins of opens we would have $x(\bigvee A)\mathbf{1} \iff \exists y \in S : y \vDash \bigvee A$. This corresponds to $\exists y \in S, a \in A : y \vDash a$, and therefore $\exists a \in A : x(a) = \mathbf{1}$. What remains to check is the behaviour under meets of this asserted point, and it is enough to check binary meets (since S is non-empty, $x \vDash \mathbf{1}$). Let $a, a' \in \Omega T$. Then $x(a \wedge a') = \mathbf{1}$ precisely when there exist $y, y' \in S : y \vDash a, y' \vDash a'$. Since S is directed, we find a $z \in S : z \vDash a, z \vDash a'$, and therefore conclude that both $x(a) = \mathbf{1}$ and $x(a') = 1$. This means that x is defining a frame homomorphism, and as T is localic, we are done. □

7.2 The Scott topology

Having discussed the specialization ordering, we now try to reason the other way around: Given a set X with an ordering $\leq$, is it possible to choose a system T with points X, such that $\leq$ is precisely the specialization ordering in this system? Clearly, as we are given the points beforehand, the system will be a space. For the characterization of the opens we use the following definition.

Definition 7.8. Let X be a poset with ordering $\leq$. A subset U of X is said to be a *Scott open* iff the following conditions are satisfied:

(i) U is upper closed with respect to $\leq$ (that is, if $x \in U$ and $x \leq y$, then also $y \in U$)

(ii) U is inaccessible by directed joins (if $\bigsqcup^{\uparrow} S \in U$, then $\exists s \in S : s \in U$).

The set of all Scott opens of X is denoted SX.

If we have multiple orderings on X, we stress that they may yield different Scott opens. The next theorem justifies our definition of the Scott opens, showing that they form a topology on X with the desired properties regarding the specialization ordering.

Theorem 7.9. *Let X as above. Then (X, SX) is a topological space, and furthermore*

(i) *The specialization ordering on X coincides with $\leq$.*

(ii) *A function between posets X and Y is continuous with respect to the Scott topologies (referred to as Scott continuous) precisely when it preserves all directed joins.*

Proof Let us verify that SX is a topology. Clearly, $\varnothing \in SX$ and $X \in SX$. Now let $x \in U = \bigcup_{i \in I} U_i$, $U_i \in SX$. Then there exists an i such that $x \in U_i$, and thus if $x \leq y$, $y \in U_i$ and thus $y \in U$. Also, if $\bigsqcup^{\uparrow} S \in U$, it is in some U_i, and therefore $\exists s \in S : s \in U_i$ and thus in U. So $U \in SX$. For finite (thus, binary) intersections, let $x \in U \cap V$ with $U, V \in SX$. Then if $x \leq y$, it follows that $y \in U$ and $y \in V$, so $y \in U \cap V$. Finally, let $\bigsqcup^{\uparrow} S \in U \cap V$. It follows that $\exists u, v \in S : u \in U, v \in V$ and by directedness of S we find a $w \in S : w \in U \cap V$. So $U \cap V \in SX$, and thus SX is a topology.
Now for (i): Let $x, y \in X$ and $x \leq y$. Then by upper closure of the Scott opens it follows that $x \sqsubseteq y$ (recall that$\sqsubseteq$ is the specialization ordering). Suppose now $x \nleq y$. Let $U = \{u \in X \mid u \nleq y\}$. Obviously, $x \in U, y \notin U$. Also, $U \in SX$ as it is upper closed, and if $V \in SX$ is a set such that $\forall v \in V : v \leq y$, we see y is an upper bound for V and thus $\bigsqcup V \leq y$. This means that U is inaccessible by joins, and thus also by directed joins. This means $x \not\sqsubseteq y$, and therefore $\sqsubseteq$ and $\leq$ coincide.
We conclude with the proof of (ii): Suppose $f : X \to Y$ is continuous. Then, as the directed joins in this case all are disjunctions (the least upper bound with respect to $\sqsubseteq$ is necessarily an element of X), we find (letting $S \subseteq SX, b \in SY$) that as $\bigsqcup^{\uparrow} S = \bigvee S$, the following equivalences hold: $\mathrm{pt}\, f(\bigsqcup^{\uparrow} S) \vDash b \iff \bigvee S \vDash \Omega f(b) \iff \exists x \in S : x \vDash \Omega f(b) \iff \bigsqcup^{\uparrow}\{\mathrm{pt}\, f(x) \mid x \in S\} \vDash b$, which proves the statement.
Now let $f : X \to Y$ preserve directed joins. Let $V \in SY$ and $U = f^{-1}(V)$. We have to show $U \in SX$. So suppose $\bigsqcup^{\uparrow} S \in U$. Then $\bigsqcup^{\uparrow}\{f(s) \mid s \in S\} \in V$, so $\exists s \in S : f(s) \in V$ and thus $s \in U$. If $u \sqsubseteq u', u \in U$ we have $f(u') = f(\bigsqcup^{\uparrow}\{u, u'\}) = \bigsqcup^{\uparrow}\{f(u), f(u')\}$ whence $f(u) \sqsubseteq f(u')$. By upper closure of V we conclude $f(u') \in V$ and thus $u' \in U$. So $U \in SX$ and f is continuous. $\square$

As an addition to this interesting theorem, we can establish a surprising link between the Scott topology and spatial, localic systems.

Proposition 7.10. *The opens of a spatial, localic system T are (isomorphic to) a subset of the Scott topology on $\mathrm{pt}\, T$ generated by the specialization ordering.*

Proof It is enough to prove that in a localic system, the extent $\mathrm{ext}(U)$ of any open U is a Scott open. Upper closure of $\mathrm{ext}(U)$ is trivial by the very definition of the specialization ordering, while Proposition 7.7 tells us that any directed join also defines a disjunction, and we immediately conclude that $\mathrm{ext}(U)$ is a Scott open. $\square$

Although we have defined this Scott topology, and proved some interesting results about it, we have not provided a convenient way of generating the Scott opens, given an ordering. The purpose of the following theorem is to provide an aid to make this easier in some specific cases.

Theorem 7.11. *Let $X = \bigcup_{i \in I} X_i$ be a poset such that the X_i are mutually disjoint, and the ordering has the property that for $x \in X_i$ and $y \in X_j$, $x \leq y \Rightarrow i = j$. Then $SX \cong \bigtimes_{i \in I} SX_i$, and thus also $(X, SX) \cong \coprod_{i \in I}(X_i, SX_i)$ (recall the notation from the previous chapter).*

Proof Let $f : SX \to \bigtimes_{i \in I} SX_i$ be defined by $f(U) = (i \mapsto U \cap X_i)$. We check that this is actually well-defined, i.e. that $U \cap X_i \in SX_i$. Upper closure is immediate by the requirement on the ordering. Suppose $x = \bigsqcup^{\uparrow} S \in U \cap X_i$. Then since $U \in SX$, we find an $u \in U \cap S$. However, since we also have $u \leq x$, it follows that $u \in X_i$ and thus $u \in U \cap X_i$. Therefore, f is well-defined. f^{-1} is defined by $f^{-1}(\phi) = \bigcup_{i \in I} \phi(i)$. We will now verify that this actually is a well-defined function. Denote $U = f^{-1}(\phi)$. Let $u \in U, u \leq v$. Then for some $i \in I$ we have $u \in \phi(i)$ and therefore $v \in \phi(i)$ (as necessarily already $v \in X_i$ and $\phi(i)$ is upper closed), so $v \in U$. Lastly let $\bigsqcup^{\uparrow} S \in U$. Then again, there is an $i \in I$ such that $\bigsqcup^{\uparrow} S \in \phi(i)$, and we find $\exists s \in S \cap \phi(i) \subseteq S \cap U$. This makes f a bijection. It remains to check that f and f^{-1} preserve unions and intersections. For f this is trivial as it follows from the laws for unions and intersections themselves. For f^{-1}, it follows trivially from the component-wise operations on $\bigtimes_{i \in I} SX_i$. The statement about (X, SX) then follows from Definition 5.4 if one notes that $X \cong \coprod_{i \in I} X_i$. □

We will now show some examples of Scott topologies.

Example 7.12. The Scott topology of the points part 7.1a in Example 7.5 is in fact the frame in Figure 7.1b. These are exactly the upper closed subsets of $\mathrm{pt}\, T$, and they are all inaccessible by directed joins (although one of them is accessible by an arbitrary join).
The Scott topology of the points part $\mathrm{pt}\, T$ in Example 7.6 is not the frame ΩT in that example. This should not be surprising, since in the Scott topology, directed joins and directed disjunctions should coincide. The frame which is the Scott Topology is (homeomorphic to) $\Omega T \backslash \{\omega\}$, the opens without ω. We can easily check that these are all upper closed sets which are inaccessible by joins. If we would take points including the x in that example, the Scott topology would be ΩT.

7.3 Compactness

Above, we defined the notion of a disjunction of points. Now, using the concept of a conjunction, we are able to define compactness. Throughout, let T be a system.

Definition 7.13. Let $C \subseteq \mathrm{pt}\, T$ be finite. Then the *conjunction* of C, denoted $\bigwedge C$, is defined by (here $a \in \Omega T$)

$$\bigwedge C \vDash a \iff \forall x \in C : x \vDash a \text{ (i.e., iff } C \subseteq \mathrm{ext}(a)). \tag{7.1}$$

What kind of object is this conjunction? We see it is a function from ΩT to $\mathbb{2}$. It is generally not a point, as it will be incompatible with arbitrary joins. However, it can be easily seen that in fact, these conjunctions do interact correctly with directed joins. Also, they preserve finite meets. The attentive reader will note that conjunctions (or, better, their **1**-kernels) therefore are both a Scott open and a filter.

Definition 7.14. Let A be a frame. Then a subset C of A is called a *Scott open filter* iff it is both a Scott open and a filter.

Scott open filters allow us to, in some sense, generalize a point. To make this more intuitive, for Scott open filters F, F', we proceed to write $F \vDash a$ instead of $a \in F$, and $F \sqsubseteq F'$ instead of $F \subseteq F'$. We observe that conjunctions of one point behave the same when considered as real point, or as the corresponding Scott open filter. Remark that for $C \subseteq C' \subseteq \mathrm{pt}\, T$ we have $\bigwedge C' \sqsubseteq \bigwedge C$. It is not the case that all Scott open filters arise from finite conjunctions of points. This roots fundamentally in the fact that finite conjunctions are not enough to describe them all. Therefore, we introduce the following notion.

Definition 7.15. A set $C \subseteq \mathrm{pt}\, T$ (where C is not necessarily finite) is said to be *compact* precisely when $\wedge C$, defined as in (7.1), is a Scott open filter.

So we can immediately conclude that all finite sets are compact. Our question is now: does this notion coincide with the like-named one for topological spaces? This is answered positively. We use the following proposition.

Proposition 7.16. *Let A be a frame, and let $f : A \to \mathbb{2}$ be a monotone function. Then the following are equivalent:*

(i) *f preserves directed joins*

(ii) *if $X \subseteq A$ and $f(\bigvee X) = \mathbf{1}$, then there is a finite $X' \subseteq X$ such that $f(\bigvee X') = \mathbf{1}$.*

Proof (i)⇒(ii): Let $Y = \{\bigvee X' \mid X' \subseteq X \text{ finite}\}$. Then Y is directed, and $\bigvee^{\uparrow} Y = \bigvee X$. Suppose now $f(\bigvee X) = \mathbf{1}$, then since f preserves directed joins, it satisfies an element of Y.
(ii)⇒(i): Let X be directed and suppose $f(\bigvee^{\uparrow} X) = \mathbf{1}$. Then applying our assumption, we find a finite X' such that $f(\bigvee^{\uparrow} X') = \mathbf{1}$. However, as X is directed, we find an upper bound x for X', and subsequently conclude $f(x) = \mathbf{1}$ by monotonicity of f. □

If we now assume that T is spatial, we see that above proposition tells us that Definition 7.15 corresponds to the following, well-known definition for spaces:

C is compact precisely when every open cover has a finite subcover.

We will not go deeper into compactness at this time. For a more comprehensive discussion, we refer to [Vickers 1989, *Topology via Logic*] Chapter 8. Without proof, we lastly state an interesting result from this source:

Proposition 7.17. *Every Scott open filter is a conjunction of some compact set of points.*

8 More on spaces and locales

In the previous chapter, we introduced spaces, locales, as well as spatial and localic systems. In this chapter, we will study these in more detail.

8.1 Finite systems

In this section we will look at finite systems. Interestingly, restricting ourselves to this simpler case gives quite a lot of additional, non-trivial properties. One important such property is the following:

Corollary 8.1. *Every finite locale is spatial. Hence, for finite systems T,* $\mathrm{Spat}\,\mathrm{Loc}\,T \cong \mathrm{Loc}\,T$.

Proof Given a finite locale L. For every $a, b \in \Omega L$ with $a \neq b$ either $a \not\geq b$ or $b \not\geq a$. Now using Proposition 6.15, there is a lattice homomorphism $f : \Omega L \to \mathbb{2}$ with $f(a) \neq f(b)$. Since ΩL is finite, every lattice homomorphism is a frame homomorphism, so f is a frame homomorphism. This means that for every two opens there is a point which distinguishes them, so the system is spatial. We conclude that for every system T, $\mathrm{Loc}\,T$ is spatial, since it is localic. So $\mathrm{Spat}\,\mathrm{Loc}\,T \cong \mathrm{Loc}\,T$. □

This has far-reaching consequences. If we now derive some result for spatial, finite systems, we immediately know that it holds for finite, localic systems as well. In other words, locales reduce to a special kind of spaces, where any two points can be distinguished using opens. For infinite locales, such does not apply. In fact, below we will demonstrate that there are infinite locales without points in the next section!

In mainstream topology, one of the most important problems is to determine whether or not two spaces are homeomorphic. The previous result allows us to prove an interesting result in this direction for finite systems.

$$\begin{tikzcd}
\mathrm{Spat}\,T \arrow[r, "e_T"] \arrow[d, "p_{\mathrm{Spat}\,T}"'] \arrow[drr, "h"] & T \arrow[r, "p_T"] & \mathrm{Loc}\,T \\
\mathrm{Loc}\,\mathrm{Spat}\,T \arrow[rr, "f"] \arrow[urr, "g"] & & \mathrm{Spat}\,\mathrm{Loc}\,T \arrow[u, "e_{\mathrm{Loc}\,T}"']
\end{tikzcd}$$

Figure 8.1: Some natural continuous functions defined from a system T. All the functions stated are unique.

Proposition 8.2. *Let T be a finite system. Then* $\mathrm{Loc}\,\mathrm{Spat}\,T \cong \mathrm{Spat}\,\mathrm{Loc}\,T$ *iff T is spatial (or localic, in view of Proposition 8.1).*

Proof In the diagram (Figure 8.1), p and e are the localification and spatialization maps of the systems in their subscript. From this, we deduce that $\Omega(\mathrm{Loc}\,\mathrm{Spat}\,T) = \mathrm{ext}_T \Omega T$, and $\Omega(\mathrm{Spat}\,\mathrm{Loc}\,T) = \mathrm{ext}_{\mathrm{Loc}\,T}(\Omega T)$ (recall the notation from Definition 4.6). Knowing that finite locales are spatial, we conclude that, necessarily, $|\mathrm{ext}_{\mathrm{Loc}\,T}(\Omega T)| = |\Omega T|$ holds. However, if T is not spatial, we have that $|\mathrm{ext}_T(\Omega T)| < |\Omega T|$. Therefore, in this case a bijection between $\mathrm{ext}_{\mathrm{Loc}\,T}(\Omega T)$ and $\mathrm{ext}_T(\Omega T)$ cannot exist, let alone a frame isomorphism. For the converse statement, assume that T is spatial. Then we have $\mathrm{Loc}\,\mathrm{Spat}\,T \cong \mathrm{Loc}\,T \cong \mathrm{Spat}\,\mathrm{Loc}\,T$, as finite locales are spatial. □

We stress that both directions of this proof use critically the finiteness of the system. Also, if T is spatial, we can describe the explicit homeomorphism between $\mathrm{Loc}\,\mathrm{Spat}\,T$ and $\mathrm{Spat}\,\mathrm{Loc}\,T$. To this end, remark that the functions g, h and f in Figure 8.1 are unique on account of Theorems 4.7 and 4.11. In particular, we have $p_T \circ e_T = e_{\mathrm{Loc}\,T} \circ f \circ p_{\mathrm{Spat}\,T}$. Note also that, by Proposition 4.10, f is uniquely determined from Ωf. If we thus look at the Ω-part of the above, we get that $\Omega e_T \circ \Omega p_T = \Omega p_{\mathrm{Spat}\,T} \circ \Omega f \circ \Omega e_{\mathrm{Loc}\,T}$ or, substituting the known functions, $\mathrm{ext}_T = \Omega f \circ \mathrm{ext}_{\mathrm{Loc}\,T}$. Thus, for $a \in \Omega T$, we know that $\Omega f(\{x \in \mathrm{pt}\,\mathrm{Loc}\,T \mid x \vDash a\}) = \{y \in \mathrm{pt}\,T \mid y \vDash a\}$, and $\Omega f : \mathrm{ext}_{\mathrm{Loc}\,T}(\Omega T) \to \mathrm{ext}_T(\Omega T)$. Since T and $\mathrm{Loc}\,T$ are spatial, we find that the domain and codomain of Ωf both have cardinality $|\Omega T|$. But then, this spatiality also implies the injectivity of Ωf. We conclude that Ωf is a bijection, and trivially it also a frame isomorphism. Therefore, it determines the sought homeomorphism.

8.2 Pointless locales

Given a frame A. Usually, there are many ways to choose a set of points X, and the relation $\models$ such that $(X, A, \models)$ is a topological system. In fact, the locale corresponding to this frame contains all different points, when we view points to be the same when they satisfy the same opens. This is the case, since every point defines a frame homomorphism $A \to \mathbb{2}$. So intuitively, a locale contains a lot of points. Actually, it is the largest T_0 system with this frame. Unfortunately, this intuition does not hold in general. In this section we will look at locales with very few points, and show that there exist locales without a single point. Of course, the locale corresponding to the inconsistent frame (the unique frame with only one point) has no points, since $\mathbf{0} = \mathbf{1}$ in that frame. But there are also non-trivial examples of locales without points. This also means that every topological system with that frame as opens has no points, since every point defines a frame homomorphism to $\mathbb{2}$, but there are none.

Since we know a lot about (complete) Boolean algebras, we will look for examples where the frame is in fact a complete Boolean algebra. We call a complete Boolean algebra *atomic* if every element can be written as a (possibly empty or infinite) join of atoms.

Proposition 8.3. *In a locale with as opens the complete Boolean algebra A, the points correspond one-to-one with the atoms of A. Also, A is atomic iff the locale is spatial.*

Proof The first statement follows directly from Proposition 4.12 and Lemma 6.17.
For the second statement, first suppose that A is atomic, then every element (open) can be written as join of atoms. So every element is uniquely described by the atoms smaller than it, or equivalently, every element is uniquely described by the prime elements larger than it. This means that it is uniquely described by its value of every frame homomorphism, or point. This means that A is spatial. Now suppose that A is not atomic. Then there is some element x which cannot be written as join of atoms. Let $y = \bigvee\{a \text{ atom} \mid a \leq x\}$. Then $\neg x$ and $\neg y$ have the same values for every frame homomorphism to $\mathbb{2}$, since the set of prime elements larger than them are the same. This means that A is not spatial, which completes the proof. $\square$

So we will search for an atomless complete Boolean algebra, to find a pointless locale. We now give an example.

Example 8.4. In a topological space $(X, \mathcal{O})$, we call opens $O \in \mathcal{O}$ a *regular open* iff O is equal to the interior of its closure, that is, $O = \operatorname{int}\overline{O}$. We will show that the regular opens will form a Boolean algebra with the inclusion as ordering.

Lemma 8.5. *In a Topological space $(X, \mathcal{O})$, an open O is regular iff it is the interior of some closed set. Moreover, the regular opens form a Boolean algebra with join $\bigvee_i A_i = \operatorname{int}\overline{\bigcup_i A_i}$, meet $A \wedge B = A \cap B$, complement $\neg A = \operatorname{int}(X \backslash A) = \operatorname{int}(A^c)$ and units $\mathbf{0} = \varnothing$ and $\mathbf{1} = X$.*

Proof If O is a regular open, then it is the interior of $\overline{O}$. If O is the interior of some closed set C, then $O \subseteq C$, and also $\overline{O} \subseteq C$, since $\overline{O}$ is the smallest closed set containing O. Thus $O \subseteq \operatorname{int}\overline{O} \subseteq \operatorname{int} C = O$, so O is regular.
If A and B are regular opens, then also $\bigvee_i A_i$, $A \wedge B$ and $\neg A$ are regular opens (for the meet, use $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$). We will show that these are the Boolean algebra operations corresponding to the inclusion ordering. Clearly $A \wedge B$ is the greatest lower bound of A and B with respect to the inclusion ordering. Note that $A_i \subseteq \bigvee_j A_j$ for all i, and if B is a regular open containing all A_i, then it contains $\bigcup_i A_i$. So it is the interior of some closed set C containing $\bigcup_i A_i$, and thus $\overline{\bigcup_i A_i} \subseteq C$. So $\operatorname{int}\overline{\bigcup_i A_i} \subseteq \operatorname{int} C$, that is $\bigvee_i A_i \subseteq B$. So $\bigvee_i A_i$ is indeed the smallest upper bound. Also note that $\mathbf{0} \leq A \leq \mathbf{1}$ and $A \wedge \neg A = \varnothing = \mathbf{0}$. To show that $A \vee \neg A = \mathbf{1}$, we will show that the complement of $\overline{A \cup \operatorname{int}(A^c)}$ is empty:

$$(\overline{A \cup \operatorname{int}(A^c)})^c = \operatorname{int}(A^c \cap \overline{A}) = \operatorname{int}(A^c) \cap \operatorname{int}(\overline{A}) = \operatorname{int}(A^c) \cap A \subseteq A^c \cap A = \varnothing.$$

The only thing left to show is frame distributivity. So we have to show that

$$LHS := A \cap \left(\operatorname{int}\overline{\bigcup_i B_i}\right) = \operatorname{int}\overline{\bigcup_i (A \cap B_i)} =: RHS.$$

Suppose $x \in LHS$, then $x \in A = \operatorname{int} A$ and $x \in \operatorname{int}\overline{\bigcup_i B_i}$. So there is a open neighbourhood X of x which is on both A and $\overline{\bigcup_i B_i}$. This means that every open subset of X contains some point of $\bigcup_i B_i$, so also of

$A \cap \bigcup_i B_i = \bigcup_i (A \cap B_i)$. This means that $X \subseteq \overline{\bigcup_i (A \cap B_i)}$, which means that $x \in RHS$. So $LHS \subseteq RHS$. The other direction is easy, one can easily verify that $RHS \subseteq A$ and $A \cap B_i \subseteq B_i$ implies $RHS \subseteq \operatorname{int} \overline{\bigcup_i B_i}$. So we conclude that the regular opens in a topological space form a complete Boolean algebra. $\square$

We have now a way to find a lot of complete Boolean algebras. It is not hard to find some which are atomless. For example if we take $\mathbb{R}^n$, with the usual topology, we see that every open ball is a regular open. Since every regular open contains a strictly smaller open ball, no regular open can be an atom. So we have found an atomless complete Boolean algebra. It is quite remarkable that by taking a topology with a lot of points, altering the opens and the meets and joins of opens a bit, we get a frame where we cannot define a single point. This can even be generalized to arbitrary normed vector spaces over $\mathbb{R}$.

9 Freely generated structures

In Section 3.2 we defined presentations and what it means for an algebra to present one. We will now look at a specific class of presentations, namely those with no relations. These are called *free* presentations, and the corresponding algebras are also called *free*. We say that an $\mathbb{T}$-algebra is the free $\mathbb{T}$-algebra generated by X if it presents $\mathbb{T}\langle X \mid \ \rangle$.

9.1 Free semilattices

We first look at an algebra where it's easy to say what the free algebras are, namely the algebra of semilattices.

Proposition 9.1. *The free semilattice generated by the set X is the set FX of finite subsets of X. Meets correspond to taking the union of sets.*

Proof First note that FX is a SL-algebra, that is, a semilattice, with for all $A, B \in FX$, $A \wedge B = A \cup B$ and $\mathbf{1} = \varnothing$. Note that since we defined semilattices with meets and $\mathbf{1}$ we have an order-reversal in this algebra; $\leq$ corresponds to $\supseteq$. The fact that this is indeed a SL-algebra is just easy checking.
Now observe that FX is a model for $SL\langle X \mid \ \rangle$, with function $[x] = \{x\}$ for all $x \in X$. There's nothing else to check, since the presentation does not have any relations.
Now we have to show the universal property. Let S be any other model. We define $\theta : FX \to S$ in the following way. For an arbitrary set $A = \{x_1, x_2, \ldots, x_n\} \in FX$ we define $\theta(A) = [x_1]_S \wedge [x_2]_S \wedge \cdots \wedge [x_n]_S$ which includes the case that $\theta(\varnothing) = \mathbf{1}_S$ (since $\mathbf{1}$ is the empty meet). Note that the bracketing does not matter, since $\wedge$ is associative, and the ordering of x_i also does not matter, since $\wedge$ is commutative. So θ is a function, and we now want to show that it is a semilattice homomorphism. For any two sets A and B, we have $f(A) \wedge f(B) = a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_k = f(A \cup B) = f(A \wedge B)$, since we can delete all double appearances of elements in the expression $a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_k$ by idempotence (and commutativity). Since also $\theta(\varnothing) = \mathbf{1}$, θ is a SL-homomorphism. Note that $\theta(\{x\}) = [x]_S$. We now have to show that θ is unique with these properties. Let ϕ be a semilattice homomorphism $FX \to S$ with $\phi(\{x\}) = [x]_S$. Then $\phi(\{x_1, \ldots, x_n\}) = \phi(\{x_1\} \cup \cdots \cup \{x_n\}) = [x_1]_S \wedge \cdots \wedge [x_n]_S = \theta(\{x_1, \ldots, x_n\})$. So θ is a unique homomorphism with these properties, and hence FX presents $SL\langle X \mid \ \rangle$. □

9.2 Free distributive lattices

We continue not with free lattices, but with free distributive lattices. This is because the distributive law makes the identification of free lattices actually much simpler. We will use the construction of words to determine what the free distributive lattices are. In the last section we proved directly that the set we found was the semilattice presenting the presentation, but that was quite cumbersome, even for such a simply example. This section we will prove it in another way. We will mimic the proof of Proposition 3.10. Suppose we want to find the free distributive lattice generated by some set X. When we are given any word w constructed from X, we will write it in a simpler form to classify all such words. Of course, we will then end up with another word, but if we only equalities which can be derived from the laws of distributive lattices, these words will be the same element. We can first apply the unit laws the equations $a \wedge \mathbf{0} = \mathbf{0}$ and $a \vee \mathbf{1} = \mathbf{1}$. This allows us to remove all occurrences of $\mathbf{0}$ and $\mathbf{1}$ from w, or to reduce w to either $\mathbf{0}$ or $\mathbf{1}$. Then we can apply the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ multiple times to move all joins outside any brackets, and obtain the form $w_1 \vee w_2 \vee \cdots \vee w_n$, where the w_i are meets of elements of X. Furthermore, if $w_i \geq w_j$ for $i \neq j$, that is, w_i is the meet of less elements than w_j, we can remove w_j from the join. Further simplification is not really possible, so this motivates the following attempt of finding the free distributive lattice generated by X. As above, let FX be the set of finite subsets of X, and call a subset A of FX *irredundant* if no two elements of A are comparable with respect to the inclusion ordering.

Proposition 9.2. *The free distributive lattice generated by X is the set GX of all finite irredundant subsets of FX, with suitable ordering. Note that $GX \subseteq \mathcal{P}(\mathcal{P}(X))$.*

Proof We interpret elements of FX as meets of elements of X and elements of GX as joins of elements of FX. For an element T of $F(FX)$ (the set of finite subsets of FX) define $\overline{T} = \{X \in T \mid \forall Y \in T : Y \subseteq$

X implies $Y = X\}$, which is T where the redundant elements are removed. Then it makes sense to define the join of two elements T_1 and T_2 as $T_1 \vee T_2 = \overline{T_1 \cup T_2}$ and the meet as $T_1 \wedge T_2 = \overline{\{S_1 \cup S_2 \mid S_1 \in T_2, S_2 \in T_2\}}$ and to define the constants $\mathbf{0} = \varnothing$ (so as the empty join) and $\mathbf{1} = \{\varnothing\}$ (as the empty meet). It involves a bit of checking to verify this makes GX into a distributive lattice. We will not do this here. The fact that GX is a distributive lattice means that it is actually the distributive lattice we're looking for. According to the proof of Proposition 3.10, the distributive lattice we're looking for can be constructed by the equivalence classes on a equivalence relation on the words. We've now chosen a set of words (the interpretation of GX) in such a way that in every equivalence class there is at least one of our words. If there would be an equivalence class with more than one of our words in it, our set of words cannot possibly be a distributive lattice, since that would imply that the words are the same, since they're equivalent. So we've chosen exactly one word from every equivalence class, and we've chosen the right operations, with means that we've found the distributive lattice we're looking for. □

However, the number of elements of a free distributive lattice in n generators does not appear to be described by a simple formula. At least we find that it is finite, which is not the cases for general lattices, as we will see later.

9.3 Free Boolean algebras

We can say even more about free Boolean algebras. For Boolean algebras we can write all words in almost the same form as for the distributive lattices. We will only handle the case of finite X. First we use the De Morgan laws of Proposition 6.2 and $\neg\neg x = x$ repeatedly, to move all negations in front of generators only. Now proceed with the same steps as above, to write any word as $\mathbf{0}$, $\mathbf{1}$ or $w_1 \vee w_2 \vee \cdots \vee w_n$, where w_i is a meet of elements or complements of elements of X. When $p \in X$, and some w_i contains neither p nor $\neg p$, then we can replace w_i by $(w_i \wedge p) \vee (w_i \wedge \neg p)$. Applying this repeatedly, we can write every word as $a_1 \vee a_2 \vee \cdots \vee a_n$ where a_i is a meet of $|X|$ (remember that X was finite) generators, such that for all $x \in X$ either x or $\neg x$ will appear in a_i. We can even write $\mathbf{0}$ this way, as the empty join, and $\mathbf{1}$ as the join of all possible a_i. So every element is a join of these a_i, which shows that these elements are atoms (recall that atoms are minimal non-zero elements).

Proposition 9.3. *The free Boolean algebra generated by a finite set X is the set $\mathcal{P}(\mathcal{P}(X))$.*

Proof We interpret the elements A of $\mathcal{P}(X)$ as $\bigwedge A \wedge \bigwedge\{\neg x \mid x \in X \backslash A\}$, and we interpret elements of $\mathcal{P}(\mathcal{P}(X))$ as joins of elements of $\mathcal{P}(X)$. Above, we checked that the singleton set (a set with one element) in $\mathcal{P}(\mathcal{P}(X))$ is an atom, since it is the join of one a_i. In this interpretation, for two sets $A, B \in \mathcal{P}(\mathcal{P}(X))$ we see that $A \vee B = A \cup B$, $A \wedge B = A \cap B$ and $\neg A = \mathcal{P}(A) \backslash A$. This clearly is a Boolean algebra. Thus we have found exactly one representative of every equivalence class of $\mathbb{T}WX$ in Proposition 3.10. Therefore, this is the free Boolean algebra we were looking for. □

So we have found that the free Boolean algebra in n generators contains 2^{2^n} elements.

9.4 Free lattices

We will now consider the free lattice in three generators. Until now, all finitely generated structures were finite, but this is not the case for free lattices. We have the following theorem, although we will only give a sketch of the proof here.

Theorem 9.4. *The free lattice in three generators in infinite.*

Proof Let the generators be x, y and z. We will construct a infinite strictly increasing sequence, let $p_0 = x$ and

$$p_{n+1} = x \vee (y \wedge (z \vee (x \wedge (y \vee (z \wedge p_n))))).$$

Since every p_1 is the join of x and something else, $p_1 \geq p_0$. Also, since meets and joins are order preserving - so their composition is, too - we now conclude that $p_{n+1} \geq p_n$. One can also prove that even $p_{i+1} > p_i$. We will not do this here, but instead, we refer to [Johnstone 1982, *Stone spaces*] pages 28-30. When this is proven, there exist an infinite increasing sequence in the free lattice of three generators, so it has to be infinite. □