# The Lean-HoTT library

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March 24, 2017

## Lean HoTT library

It is the Lean standard library without Prop, but with univalence and 2 primitive HITs: the n-truncation and quotients (which are interdefinable with pushouts)

Theorems which were already proven last year:

- Loop space of the circle
- Connectedness of suspensions
- The real and complex hopf fibrations
- The Freudenthal suspension theorem
- The long exact sequence of homotopy groups

Formalized parts of chapter 8:

	1	2	3	4	5	6	7	8	9	10
last year	+	+	+	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	-	-	$\frac{1}{2}$	-
now	+	+	+	+	+	$\frac{3}{4}$	+	+	+	$\frac{1}{4}$

#### New Theorems:

- $\pi_n(\mathbb{S}^n) = \mathbb{Z}$  (for  $n \ge 1$ ) and  $\pi_n(\mathbb{S}^3) = \pi_n(\mathbb{S}^2)$  (for  $n \ge 3$ ) (j.w.w. Ulrik Buchholtz)
- The Seifert-van Kampen theorem: (with basepoints)
  - "the fundamental groupoid of a pushout is weakly equivalent (as categories) to the pushout of the fundamental groupoids."
- Whitehead's principle:
  - "A weak equivalence between truncated types is an equivalence."
- Eilenberg-MacLane spaces:
  - We can construct pointed types K(G,n) which are n-truncated and (n-1)-connected and n-th homotopy group  $G,\ldots$

Eilenberg-MacLane spaces:

[Let  $\mathrm{Type}^{=n}_*$  be the universe of *n*-truncated (n-1)-connected types]

- ... then K(-,1) induces an equivalence between the categories  $\operatorname{Grp} \to \operatorname{Type}^{=1}$
- ▶ and for  $n \ge 2$  the functor K(-,n) induces an equivalence  $Ab\mathrm{Grp} \to \mathrm{Type}_*^{=n}$  (j.w.w. Ulrik Buchholtz and Egbert Rijke)

• Properties about the smash product: (main part of my talk)

$$((-) \land B) \dashv (B \rightarrow^* (-))$$
 (natural in  $B$ )

$$A \wedge (B \wedge C) \simeq^* (A \wedge B) \wedge C$$
 (natural in  $A$ ,  $B$  and  $C$ )

(j.w.w. Robin Adams, Mark Bezem, Ulrik Buchholtz, Stefano Piceghello, Egbert Rijke)

- [In Progress] Formalization of spectral sequences, in particular the Serre Spectral Sequence.
  - (j.w.w. Jeremy Avigad, Steve Awodey, Ulrik Buchholtz, Egbert Rijke, Mike Schulman)

# Spectral Sequences (as described by Mike Shulman)

#### Definition A (homologically indexed) spectral sequence consists of

- A family  $(E^r_{p,q})$  of R-modules (or objects in an abelian category) for  $p,q:\mathbb{Z}$  and  $r\geq 2$ . For a fixed r this gives the r-page of the spectral sequence.
- (homo)morphisms  $d^r_{p,q}:E^r_{p,q}\to E^r_{p-r,q+r-1}$  which are called differentials.
- isomorphisms  $\alpha_{p,q}^r: H_{p,q}(E^r) \simeq E_{p,q}^{r+1}$  where  $H_{p,q}(E^r) = \ker(d_{p,q}^r)/\operatorname{im}(d_{p+r,q-r+1}^r).$

#### We build these in the following way:

- Given an iterated fibration sequence;
- We construct an exact couple;
- We iteratively build a derived exact couple;
- These give a spectral sequence, which under certain conditions converges.

#### Iterated fibration sequence

Given a sequence of maps

$$Y_T \xrightarrow{f_T} Y_{T-1} \xrightarrow{f_{T-1}} Y_{T-2} \xrightarrow{f_{T-2}} \cdots$$

Let  $X_s :\equiv \operatorname{fib}_{f_s}$  We build the iterated fibration sequence:

$$X_T \to Y_T \to Y_{T-1}$$

$$X_{T-1} \to Y_{T-1} \to Y_{T-2}$$

$$X_{T-2} \to Y_{T-2} \to Y_{T-3}$$

$$\vdots$$

We want to compute  $\pi_n(Y_T)$  from the homotopy groups of  $X_s$ .

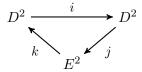
We assume that for every n there is an R that such that  $\pi_n(X_s) = \pi_n(Y_s) = 0$  for  $s \leq R$ .

#### Exact couple

Define  $E_{p,q}^2:\equiv \pi_{p+q}(X_q)$  and  $D_{p,q}^2:\equiv \pi_{p+q}(Y_q)$ . The long exact sequence of homotopy groups gives

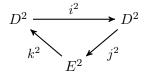
$$\cdots \to \pi_n(X_s) \to \pi_n(Y_s) \to \pi_n(Y_{s-1}) \to \pi_{n-1}(X_s) \to \cdots$$

which gives the exact couple

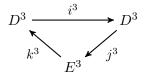


#### Derived Exact couple

From an exact couple



we build a derived exact couple



with  $E^3_{p,q}=H_{p,q}(E^2)$  with differential  $d^2:\equiv j^2k^2:E^2\to E^2.$ 

## Spectral Sequence

We iterate this process, and make construct the exact couple  $(E^{r+1},D^{r+1},i^{r+1},j^{r+1},k^{r+1})$  as the derived couple of  $(E^r,D^r,i^r,j^r,k^r)$ .

Then  $(E^r, d^r)_r$  forms an spectral sequence. For given p, q the sequence  $(E^r_{p,q}$  is eventually constant, and we define the eventual value as  $E^{\infty}_{p,q}$ .

#### Convergence Theorem

Recall: We assume that for every n there is an R that such that  $\pi_n(X_s) = \pi_n(Y_s) = 0$  for  $s \leq R$ .

Theorem There are abelian groups  $B_{n,s}$  with  $B_{n,T}=\pi_n(Y)$  and finite iterated extensions (short exact sequences)

$$E_{n-T,T}^{\infty} \to B_{n,T} \to B_{n,T-1}$$

$$\vdots$$

$$E_{n-s,s}^{\infty} \to B_{n,s} \to B_{n,s-1}$$

$$E_{n-s+1,s-1}^{\infty} \to B_{n,s-1} \to B_{n,s-2}$$

$$\vdots$$

$$E_{n-R,R}^{\infty} \to B_{n,R} \to 0$$

This is denoted  $\pi_{n+q}(X_q) \Rightarrow \pi_{n+q}(Y_T)$ .

# Serre Spectral Sequence

#### **Theorem**

Given a pointed map  $f:X\to B$  with fiber F where B is simply connected. For a spectrum Y we get

$$H^p(B; H^q(F; Y)) \Rightarrow H^{p+q}(X; Y).$$

Here  $H^n(X;Y):\equiv \|X\to Y_n\|_0$  and  $H^n(X;G):\equiv H^n(X;K(G,-))$  for an abelian group G.

# Progress (globally)

#### We have:

- Eilenberg-MacLane spaces
- basic theory of spectra (LES of homotopy groups)
- cohomology theory satisfies Eilenberg-Steenrod axioms
- Basic algebraic constructions
- Long exact sequence of homotopy groups

#### To do:

- Derive an exact couple (in progress)
- The Convergence Theorem
- Spectrification and other constructions on spectra
- Cohomology with local coefficients