Numerical linear algebra

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1 Some key concepts

- Norms
- Singular values
- triangular matrix
- SVD and QR Decomposition
- stability

2 Expanding the definition of Norm from vectors to matrix

Three properties of vector norm:

- $||x|| \geq 0$, and zero value achieved iff x = 0
- $||x+y|| \leq ||x|| + ||y||$
- $||\alpha x|| = |\alpha|||x||$, α is constant

Besides the Euclidean norm we are familiar with, we may expand the definition of norm suppose p is a real number, for $\vec{x} \in \mathbf{C}^m$:

$$||x||_p = (\sum_i^m |x_i|^p)^{1/p}$$

Based on the general definition of vector norm, we can define the induced matrix norm as:

$$||A||_{m,n} = \sup_{x \in \mathbf{C}^m, x \neq 0} \frac{||Ax||_m}{||x||_n}$$

The induced matrix norm would also have three properties of vector norm.

2.1 Frobenius norm

Frobenius norm is a commonly used matrix norm, but it is not a induced norm. we define the a_j to be the jth col of matrix A, we have

$$||A||_F = (\sum_{j=1}^{n} ||a_j||^2)^{.5}$$

or the Frobenius norm can also be found with trace

$$||A||_F = \sqrt{tr(A^\dagger A)} = \sqrt{tr(AA^\dagger)}$$

2.2 usage of matrix norms

We may use the matrix norm to bound $||AB||$ with either induced norm or Frobenius norm. Recall the Cauchy-Schwarz inequality

$$|x^\dagger y| \leq ||x||_p ||y||_q$$

where $1/p + 1/q = 1$.
For induced norm we have:

$$||AB||_{l,n} \leq ||A||_{l,m} ||B||_{m,n}$$

and for Frobenius norm

$$||AB||_F^2 = ||A||_F^2 ||B||_F^2$$

3 singular values

We define the eigen values of a square matrix, and we are trying to expand this concept into a general $m \times n$ matrix. We define $\{\sigma_i\}$ to the the singular values of matrix A, and σ_i would be the ith eigenvalue of $A^\dagger A$, which is guaranteed to be a square matrix. (Assuming $m \geq n$) Denote Σ to be a $n \times n$ diagonal matrix, and U to be a $m \times n$ matrix with orthonormal columns, and V to be a $n \times n$ matrix with orthonormal columns. we have a SVD decomposition of matrix A such that:

$$A = U\Sigma V$$

, the SVD would be unique up to any unitary operations applied to col space of U and V. One may also expand the reduced SVD into full SVD by adding more zeros to make U to be $m \times m$, Σ to be $m \times n$, and V to be $n \times n$.

Numerical linear algebra, LLoyd N. Trefethen; David Dau III

QR factorization:
pg 51 CGS;

pg 58 MGS, more stable;
pg 73 Householder, need to construct Q separately;
pg 98 Householder reduction to Hessenberg matrix;