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BACK-OF-THE-ENVELOPE SWAPTIONS IN A VERY PARSIMONIOUS MULTI-CURVE INTEREST RATE MODEL

ROBERTO BAVIERA

Department of Mathematics, Politecnico di Milano
32 p.zza L. da Vinci, I-20133 Milano, Italy
roberto.baviera@polimi.it

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We propose an elementary model in multi-curve setting that allows to price with simple exact closed formulas European swaptions. Swaptions can be both physical delivery and cash-settled ones. The proposed model is very parsimonious: it is a three-parameter multi-curve extension of the two-parameter J. Hull & A. White (1990) [Pricing interest-rate-derivative securities. *Review of Financial Studies* **3**(4), 573–592] model. The model allows also to obtain simple formulas for all other plain vanilla Interest Rate derivatives and convexity adjustments. Calibration issues are discussed in detail.

Keywords: Multi-curve interest rates; parsimonious modeling; cash-settled swaptions; physical delivery swaptions; calibration cascade.

1. Introduction

The financial crisis of 2007 has had a significant impact also on Interest Rate (IR) modeling perspective. On the one hand, multi-curve dynamics have been observed in main inter-bank markets (e.g. EUR and USD). On the other hand, volumes on exotic derivatives have considerably decreased and liquidity has significantly declined even on plain vanilla instruments.

Since the seminal papers on derivative pricing in IR multi-curve dynamics (see, e.g. Henrard 2010, Kenyon 2010, Mercurio & Xie 2012), studies on this topic have grown considerably and nowadays detailed textbooks are available (see, e.g. Henrard 2014, Grbac & Runggaldier 2015). This literature has proposed several possible approaches to the first issue of modeling multi-curve dynamics. However, the main consequence of the second issue, i.e. the need of very parsimonious models, has been largely neglected in current financial literature where the additional complexity of today financial markets is often faced with parameter-rich models. In this paper, the focus is on the two relevant issues of parsimony and calibration.

First, the parsimony feature is crucial: in today (less liquid) markets one often needs to handle models with very few parameters both from a calibration and from a risk management perspective. In several situations it can be useful to have a simple coherent multi-curve model with few parameters: IR models are often the first ingredient in products that involve also other markets (e.g. FX or commodities), and in applications with either counterparty risk (e.g. XVA) or liquidity issues.

In this paper, we focus on a three-parameter multi-curve extension of the well-known two-parameters Hull & White (1990) model. This choice is very parsimonious: one of the most parsimonious multi-curve HJM model in the existing literature is the one introduced by Moreni & Pallavicini (2014) that, in the simplest WG2++ case, requires ten free parameters. Another one has been recently proposed by Grbac *et al.* (2016), that in the simplest model parametrization involves at least seven parameters.

Second, the model should allow for a calibration cascade, the methodology followed by practitioners, that consists in calibrating first IR curves via bootstrap techniques and then volatility parameters on IR options. This cascade is fundamental: the reason is related to the different liquidity of IR derivatives. Instruments used in bootstrap, as FRAs, Short-Term-Interest-Rate (STIR) futures and swaps, are several order of magnitude more liquid than the corresponding options on these instruments.

The proposed model, besides the calibration of the initial *discount* and *pseudo-discount* curves, allows to price with exact and simple closed formulas all plain vanilla IR options: caps/floors, STIR options and European swaptions. While caps/floors and STIR options can be priced with simple modifications of solutions already present in the literature (see, e.g. Henrard 2013, Baviera & Cassaro 2015), in this paper, we focus on pricing European swaption derivatives (hereinafter swaptions) with both settlement styles: physical delivery and cash settlement. We also show the calibration of all model parameters in a market situation.

The main contribution of this paper is twofold. First, we propose a parsimonious multi-curve HJM model, with only three parameters, that allows simple closed formulas for physical delivery (PD) and cash-settled (CS) swaptions. Second, we show in a detailed example the calibration cascade, where the volatility parameters are calibrated via swaptions.

The remainder of the paper is organized as follows. In Sec. 2, we recall the characteristics of a swaption derivative contract in a general multi-curve HJM framework. In Sec. 3, we introduce the parsimonious model within this framework; we also prove model swaption closed formulas. In Sec. 4, we show in detail model calibration. Section 5 concludes.

2. IR Swaptions in a Multi-curve Setting

Multi-curve settings for IRs can be found in Henrard (2014) and Grbac & Runggaldier (2015); the interested reader is referred to these books for a description of multi-curve phenomenology, multi-curve models, and a complete set of references. In this section, we briefly recall IR notation, main definitions and some key relations, with a focus on swaption pricing in a multi-curve setting. In particular, we consider a dual curve setting with the *discount* curve and one *pseudo-discount* curve related to Libor rates of tenor Δ, e.g. 6-months in the Euribor 6 m case.

2.1. *Basic IR definitions and properties*

Let $(\Omega, \mathcal{F}, \mathbb{P})$, with $\{\mathcal{F}_t : t_0 \leq t \leq T^*\}$, be a complete filtered probability space satisfying the usual hypothesis, where t_0 is the value date and T^* a finite time horizon for all market activities. Let us define $B(t,T)$ the *discount* curve at time $t \in [t_0, T^*]$

$$B(t,T) = \mathbb{E}[D(t,T) \,|\, \mathcal{F}_t] \quad \forall T \in [t, T^*], \tag{2.1}$$

where $D(t,T)$ is the stochastic discount.[a] For example, the market standard in the Euro interbank market is to consider as *discount* curve the EONIA curve (also called OIS curve).

The modeling setting for the *discount* curve is the standard single curve one, and the usual relations hold. For example the forward discount $B(t;T,T+\Delta)$ is equal to the ratio $B(t,T+\Delta)/B(t,T)$ and it is martingale in the T-forward measure as a consequence of (2.1).[b]

As in Henrard (2014), also a *pseudo-discount* curve is considered. The following relation holds for Libor rates $L(T,T+\Delta)$, with fixing in T and payment in $T+\Delta$, and the corresponding forward rates $L(t;T,T+\Delta)$ in t

$$B(t,T+\Delta)\, L(t;T,T+\Delta) := \mathbb{E}[D(t,T+\Delta)L(T,T+\Delta) \,|\, \mathcal{F}_t], \tag{2.2}$$

where the tenor Δ is the one that characterizes the *pseudo-discount* curve.

The (forward) *pseudo-discounts* are defined as

$$\tilde{B}(t;T,T+\Delta) := \frac{1}{1+\delta(T,T+\Delta)\, L(t;T,T+\Delta)} \tag{2.3}$$

with $\delta(T,T+\Delta)$ the year-fraction between the two calculation dates for a Libor rate. The multiplicative spread (hereinafter *spread*) is defined as

$$\beta(t;T,T+\Delta) := \frac{B(t;T,T+\Delta)}{\tilde{B}(t;T,T+\Delta)}. \tag{2.4}$$

From Eq. (2.2) one gets

$$B(t,T)\, \beta(t;T,T+\Delta) = \mathbb{E}[D(t,T)\beta(T;T,T+\Delta) \,|\, \mathcal{F}_t], \tag{2.5}$$

[a]The stochastic discount $D(t,T)$ is the quantity used to discount in t any cash-flow paid in $T \geq t$. In the literature (see, e.g. Henrard 2014, Grbac & Runggaldier 2015) $D(t,T)$ is indicated as the ratio between $\mathcal{N}_t/\mathcal{N}_T$, where $\mathcal{N}_t$ is the cash-account numeraire $\mathcal{N}_t = \exp \int_{t_0}^{t} r_s ds$, with r_s the instantaneous rate. In the following, we use only relation (2.1) without introducing the instantaneous rate.

[b]The T-forward measure is defined as the probability measure such that $B(t,T)\,\mathbb{E}^{(T)}[\bullet\,|\mathcal{F}_t] := \mathbb{E}[D(t,T)\,\bullet\,|\mathcal{F}_t]$ (see, e.g. Musiela & Rutkowski 2006).

i.e. $\beta(t; T, T + \Delta)$ is a martingale in the T-forward measure. This is the unique property that process $\beta(t; T, T + \Delta)$ has to satisfy.

Hereinafter, as market standard, all discounts and OIS derivatives refer to the *discount* curve, while forward Libor rates are always related to the corresponding *pseudo-discount* curve via (2.3).

2.2. *Multi-curve HJM framework*

A multi-curve HJM framework (hereinafter MHJM) is specified providing initial conditions for the *discount* curve $B(t_0, T)$ and the *spread* curve $\beta(t_0; T, T+\Delta)$, and indicating their dynamics. Let us consider two generic times $t_\alpha \le t_i$ both in (t_0, T^*). *Discount* and *spread* curves' dynamics, in the MHJM framework we consider in this paper, are

$$\begin{aligned} dB(t; t_\alpha, t_i) &= -B(t; t_\alpha, t_i)[\sigma(t, t_i) - \sigma(t, t_\alpha)] \cdot [d\underline{W}_t + \rho\,\sigma(t, t_\alpha)\,dt] \quad t \in [t_0, t_\alpha], \\ d\beta(t; t_i, t_{i+1}) &= \beta(t; t_i, t_{i+1})[\eta(t, t_{i+1}) - \eta(t, t_i)] \cdot [d\underline{W}_t + \rho\,\sigma(t, t_i)\,dt] \quad t \in [t_0, t_i] \end{aligned} \tag{2.6}$$

where $\sigma(t, T)$ and $\eta(t, T)$ are d-dimensional vectors of adapted processes (in particular in the Gaussian case $\sigma(t, T)$ and $\eta(t, T)$ are deterministic functions of time) with $\sigma(t, t) = \eta(t, t) = 0$, $z \cdot y$ is the canonical scalar product between $z, y \in \Re^d$, and $\underline{W}$ is a d-dimensional Brownian motion with instantaneous covariance $\rho = (\rho_{i\ j=1,\ldots,d})$

$$dW_{i,t} dW_{j,t} = \rho_{ij} dt. \tag{2.7}$$

The modeling framework (2.6) is the most natural extension of the single-curve Heath *et al.* (1992) model. The first equation in (2.6), together with condition (2.1), corresponds to the usual HJM model for the *discount* curve (see, e.g. Musiela & Rutkowski 2006). The second equation in (2.6) is a very general continuous process satisfying condition (2.5) for the *spread*.

We recall that the change of measure is similar to the single curve modeling approach (see, e.g. Musiela & Rutkowski 2006). The process

$$d\underline{W}_t^{(i)} := d\underline{W}_t + \rho\,\sigma(t, t_i)\,dt \tag{2.8}$$

is a d-dimensional Brownian motion in the t_i-forward measure. It is immediate to prove that, given dynamics (2.6), $B(t; t_\alpha, t_i)$ is martingale in the t_α-forward measure and $\beta(t; t_i, t_{i+1})$ is martingale in the t_i-forward measure (thus, satisfying condition (2.5)).

Remark 1. Given Eqs. (2.6), the dynamics for the *pseudo-discounts* (2.3) in the t_i-forward measure is

$$d\tilde{B}(t; t_i, t_{i+1}) = -\tilde{B}(t; t_i, t_{i+1})\tilde{\sigma}_i(t) \cdot [d\underline{W}_t^{(i)} - \rho\,\eta_i(t)\,dt] \quad t \in [t_0, t_i], \tag{2.9}$$

where $\tilde{\sigma}_i(t) := \sigma_i(t) + \eta_i(t)$, $\sigma_i(t) := \sigma(t, t_{i+1}) - \sigma(t, t_i)$ and $\eta_i(t) := \eta(t, t_{i+1}) - \eta(t, t_i)$. Hence, the *pseudo-discount* has a volatility $\tilde{\sigma}_i(t)$ which is the sum of *discount* volatility $\sigma_i(t)$ and of *spread* volatility $\eta_i(t)$.

Let us stress that framework (2.6) is quite general and it allows to reproduce, in a multi-curve setting, all market models and formulas (see, e.g. Mercurio 2009) and their Bachelier extensions (see, e.g. Schachermayer & Teichmann 2008). A model equivalent to the Gaussian case of MHJM (2.6) can be found in Henrard (2013). In Appendix C, we recall, for a Gaussian MHJM, convexity adjustments for FRAs and STIR futures and closed formulas for the other liquid IR options besides swaptions: caps/floors and STIR options. Deductions are similar to the ones in Henrard (2010), Baviera & Cassaro (2015), Henrard (2013).

2.3. *Swaption*

2.3.1. *PD swaption*

A PD swaption is a contract that gives the right to enter, at option's expiry date t_α, in a payer/receiver swap with a strike rate K established when the contract is written.

In Fig. 1, we show the flows related to the swap underlying a PD swaption and the related notation. The underlying swap at expiry date t_α is composed by a floating and a fixed leg. Typically payments do not occur with the same frequency in the two legs (and they can have also different daycount); unfortunately, this feature complicates the notation. Flows end at swap maturity date t_ω. We indicate floating leg payment dates as $\mathbf{t}' := \{t'_\iota\}_{\iota=\alpha'+1\ldots\omega'}$ (in the Euro market, typically versus Euribor-6m with semiannual frequency and Act/360 daycount), and fixed leg payment dates $\mathbf{t} := \{t_j\}_{j=\alpha+1\ldots\omega}$ (in the Euro market, with annual frequency and 30/360 daycount); we define also $t'_{\alpha'} := t_\alpha$, $t'_{\omega'} := t_\omega$.

Let us introduce the following shorthands:

$$
\begin{aligned}
B_{\alpha\, j}(t) &:= B(t; t_\alpha, t_j), \quad j = \alpha+1, \ldots, \omega; && \text{fixed-leg forward } \textit{discounts}\\
B_{\alpha'\, \iota}(t) &:= B(t; t'_{\alpha'}, t'_\iota), \quad \iota = \alpha', \ldots, \omega-1; && \text{floating-leg forward } \textit{discounts}\\
\beta_\iota(t) &:= \beta(t; t'_\iota, t'_{\iota+1}), \quad \iota = \alpha', \ldots, \omega-1; && \text{floating-leg } \textit{spreads}
\end{aligned}
$$

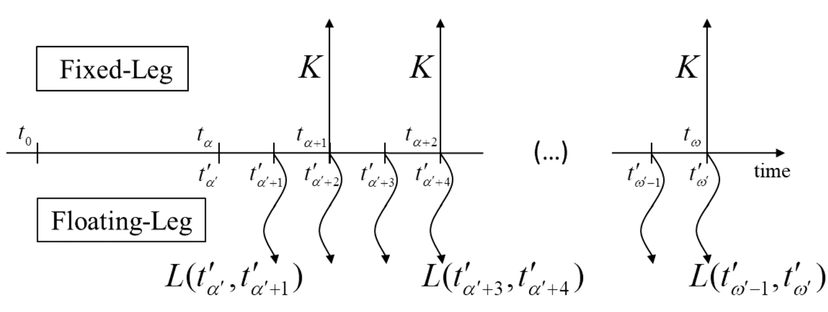

Fig. 1. Flows of a receiver swap forward start in t_α with maturity in t_ω, with fixed rate K (indicated by the straight arrows) and floating rates equal to the corresponding Libor rates (indicated by the curved arrows). In the figure, fixed rate frequency is annual while the floating leg one is semiannual.

$$\delta'_\iota := \delta(t'_\iota, t'_{\iota+1}), \quad \iota = \alpha', \ldots, \omega - 1; \qquad \text{floating-leg daycounts}$$
$$\delta_j := \delta(t_j, t_{j+1}), \quad j = \alpha, \ldots, \omega - 1; \qquad \text{fixed-leg daycounts}$$
$$c_j := \delta_{j-1} K, \quad j < \omega - 1; 1 + \delta_{\omega-1}\, K, j = \omega \qquad \text{fixed-leg flows.}$$

A swap rate forward start in t_α and valued in $t \in [t_0, t_\alpha]$, $S_{\alpha\omega}(t)$, is obtained equating in t the Net-Present-Value of the floating leg and of the fixed leg

$$S_{\alpha\omega}(t) = \frac{\mathcal{N}_{\alpha\omega}(t)}{\mathrm{BPV}_{\alpha\omega}(t)} \tag{2.10}$$

with the forward basis point value

$$\mathrm{BPV}_{\alpha\omega}(t) := \sum_{j=\alpha+1}^{\omega} \delta_{j-1}\, B_{\alpha\, j}(t) \tag{2.11}$$

and the numerator, equal to the expected value in t of swap's floating leg flows divided by $B(t, t_\alpha)$,

$$\begin{aligned} \mathcal{N}_{\alpha\omega}(t) &:= \mathbb{E}\left[\sum_{\iota=\alpha'}^{\omega'-1} D(t, t'_{\iota+1})\, \delta'_\iota\, L(t'_\iota, t'_{\iota+1}) \middle| \mathcal{F}_t\right] / B(t, t_\alpha) \\ &= 1 - B_{\alpha\,\omega}(t) + \sum_{\iota=\alpha'}^{\omega'-1} B_{\alpha'\,\iota}(t)[\beta_\iota(t) - 1], \end{aligned} \tag{2.12}$$

where the last equality is obtained using relations (2.1) and (2.5). Let us observe that $\mathcal{N}_{\alpha\omega}(t)$ is composed by two parts: the term $[1 - B_{\alpha\,\omega}(t)]$, equal to the single curve case, and the remaining sum of $B_{\alpha'\,\iota}(t)[\beta_\iota(t) - 1]$ that corresponds to the *spread* correction present in the multi-curve setting.

PD receiver swaption payoff at expiry date t_α is

$$\mathcal{R}_{\alpha\omega}(t_\alpha) := \mathrm{BPV}_{\alpha\omega}(t_\alpha)\, [K - S_{\alpha\omega}(t_\alpha)]^+ . \tag{2.13}$$

A PD receiver swaption is the expected value at value date of the discounted payoff

$$\mathfrak{R}_{\alpha\omega}(t_0) := \mathbb{E}\{D(t_0, t_\alpha)\, \mathcal{R}_{\alpha\omega}(t_\alpha) \,|\, \mathcal{F}_{t_0}\} = B(t_0, t_\alpha)\, \mathbb{E}^{(\alpha)}\, \{\mathcal{R}_{\alpha\omega}(t_\alpha) \,|\, \mathcal{F}_{t_0}\}, \tag{2.14}$$

where we have also rewritten the expectation in the t_α-forward measure $\mathbb{E}^{(\alpha)}[\bullet]$.

Lemma 2.1. *Under the MHJM framework* (2.6), $\mathcal{N}_{\alpha\omega}(t)$ *and* $\mathrm{BPV}_{\alpha\omega}(t)$ *are martingale processes in the t_α-forward measure for $t \in [t_0, t_\alpha]$, and the PD receiver swaption payoff* (2.13) *reads*

$$\mathcal{R}_{\alpha\omega}(t_\alpha) = \left[\sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_\alpha) + \sum_{\iota=\alpha'+1}^{\omega'-1} B_{\alpha'\iota}(t_\alpha) - \sum_{\iota=\alpha'}^{\omega'-1} \beta_\iota(t_\alpha)\, B_{\alpha'\iota}(t_\alpha)\right]^+ . \tag{2.15}$$

Proof. Point (i) can be deduced writing the dynamics of *discount* and *spread* curves (2.6) in the t_α-forward measure, or equivalently using the relations (2.1) and (2.5). Point (ii) is a direct consequence of PD receiver swaption payoff definition (2.13)

$$\begin{aligned}\mathcal{R}_{\alpha\omega}(t_\alpha) &= [K\,\mathrm{BPV}_{\alpha\omega}(t_\alpha) - \mathcal{N}_{\alpha\omega}(t_\alpha)]^+ \\ &= \left[B(t_\alpha, t_\omega) + K\,BPV_{\alpha\omega}(t_\alpha) + \sum_{\iota=\alpha'}^{\omega'-1} B(t_\alpha, t'_\iota)[1-\beta_\iota(t_\alpha)] - 1\right]^+ .\end{aligned} \tag{2.16}$$

□

This lemma has some relevant consequences. On the one hand, property (i) allows generalizing the Swap Market Model approach in Jamshidian (1997) to swaptions in the multi-curve case, hence it allows obtaining market swaption formulas choosing properly the volatility structure in (2.6). One can get the Black, Bachelier or Shifted-Black market formula (see, e.g. Musiela & Rutkowski 2006, Brigo & Mercurio 2007) where flows are discounted with the *discount* curve and forward Libor rates are related to *pseudo-discounts* via (2.3), as considered in market formulas. Moreover, property (i) implies that put-call parity holds for PD swaptions also in a multi-curve setting.

On the other hand, property (ii) clarifies that a complete specification of the model for swaption pricing requires only the dynamics for the forward *discount* and *spread* curves. This form of PD receiver swaption payoff is used in Sec. 3 in order to deduce a simple closed formula.

2.3.2. *CS swaption*

Par-yield cash settlement is another settlement type that is commonplace in some swaption markets, e.g. the Euro market. In this case, option payoff is a cash amount paid at time t_α. A par-yield CS swaption provides a simplified representation of the net present value of the underlying swap; it has been designed to be a simple approximation of the underlying swap cash value without requiring any knowledge of discount factors. CS swaption cash payoff is obtained by replacing the classical basis point value (2.11) at expiry date t_α with a cash annuity defined as follows:

$$C_{\alpha\,\omega}(S) := \sum_{i=1}^{\omega-\alpha} \frac{\dfrac{1}{m}}{\left(1+\dfrac{S}{m}\right)^i} = \frac{1}{S}\left(1 - \frac{1}{\left(1+\dfrac{S}{m}\right)^{\omega-\alpha}}\right), \quad S > -m, \tag{2.17}$$

where S is the underlying swap rate set at option's expiry and m is the number payments each year in the fixed leg of the underlying swap (e.g. $m = 1$ in the EUR market).

In practice the cash annuity consists in considering the swap rate in the calculation as a par-yield in the discounting formula. Let us note that the cash annuity is defined also for negative swap rates (for $S > -m$) and $C(S = 0)$ is defined as $(\omega - \alpha)/m$.

A CS receiver swaption is the expected value at value date of the discounted cash payoff

$$\mathfrak{R}^{\mathrm{C}}_{\alpha\omega}(t_0) := \mathbb{E}\{D(t_0, t_\alpha)\, C_{\alpha\,\omega}(S_{\alpha\,\omega}(t_\alpha))[K - S_{\alpha\,\omega}(t_\alpha)]^+ \,|\, \mathcal{F}_{t_0}\}. \tag{2.18}$$

The market uses a Black or a Bachelier formula for pricing CS swaptions; the interested reader can find in (Mercurio 2007) the Black-like market formula, we report in Appendix B the Bachelier (also called Normal-Black) market formula.

3. A Multi-curve Gaussian HJM with Closed form Swaptions

3.1. *The model*

In this paper, we consider an elementary 1-dimensional Gaussian model within MHJM framework (2.6). Volatilities for the *discount* curve $\sigma(t, T)$ and for the *spread* curve $\eta(t, T)$ are modeled as

$$\begin{cases} \sigma(t,T) = (1-\gamma) & \tilde{\sigma}(t,T) \\ \eta(t,T) = \gamma & \tilde{\sigma}(t,T) \end{cases} \quad \text{with } \tilde{\sigma}(t,T) := \begin{cases} \sigma \dfrac{1 - e^{-a(T-t)}}{a} & a \in \Re^+ \backslash \{0\} \\ \sigma(T-t) & a = 0 \end{cases} \tag{3.1}$$

with $a, \sigma \in \Re^+$ and $\gamma \in [0, 1]$, the three model parameters.

This model is the most parsimonious (nontrivial) extension of Hull & White (1990) to multi-curve dynamics, for this reason we call it multi-curve Hull White (MHW) model. The limiting cases correspond to some models already known in the literature: the case with $\gamma = 0$ corresponds to the **S0** hypothesis in Henrard (2010), where the *spread* curve is constant over time, while $\gamma = 1$ corresponds to the **S1** assumption in Baviera & Cassaro (2015).

Why have we selected such a parsimonious model?

This selection originates from a calibration issue. As already mentioned in the introduction, in the calibration cascade, "linear" IR products (i.e. depos, FRAs, STIR futures and swaps) are used for the bootstrap of *discount* and *pseudo-discount* initial curves, while the other parameters are calibrated via IR options.

Let us recall two facts related to the IR options available in the market for calibration. First, in the market, liquid IR options are STIR options, caps/floors and swaptions; unfortunately options on OIS are not liquid in the market place (see, e.g. Kenyon 2010, Moreni & Pallavicini 2014, and references therein). Second, in liquid IR options, the main driver in curves' dynamics is the *pseudo-discount* curve $\tilde{B}(t, T)$ via a Libor rate or a swap rate, where the latter can be seen as combinations of Libor rates (see, e.g. Eq. (1.28) in Grbac & Runggaldier 2015); swaption sensitivities with respect to the *discount* curve are lower than the corresponding sensitivities with respect to the *pseudo-discount* curve. Furthermore, $\tilde{B}(t,T)$ dynamics, as shown in Remark 1 for MHJM (2.6), has a volatility related to $\tilde{\sigma}(t,T)$, the sum of $\sigma(t,T)$ and $\eta(t,T)$.

Hence, these two facts lead to the conclusion that a calibration of parameters related to the *pseudo-discount* curve can be achieved quite simply, while it is much more difficult to calibrate volatility parameters specific to the *discount* curve or the *spread* separately. A parsimonious choice should associate a fraction $1-\gamma$ of volatility $\tilde{\sigma}(t,T)$ to the *discount* curve and the remaining fraction γ to the *spread* dynamics; in fact, as previously discussed, options on OIS are not liquid enough and then a separate calibration of $\sigma(t,T)$ and $\eta(t,T)$ in a generic MHJM is not feasible in practice. Moreover, we have selected for $\tilde{\sigma}(t,T)$ the most parsimonious form, i.e. the one of a 1-factor HW model.

Furthermore, MHW model (3.1) presents an additional advantage: it allows pricing all plain vanilla IR options in an elementary way. STIR options and caps/floors Black-like formulas can be obtained as a generalization of the solutions in Henrard (2013) and Baviera & Cassaro (2015). In this paper, we show that it is possible to price also swaptions via a simple closed formula. To the best of our knowledge, MHW model (3.1) is the first multi-curve HJM where all plain vanilla derivatives can be written with simple exact closed formulas that are extensions of Black (1976) formulas (without assuming constant *spreads*).

The remaining part of this subsection is related to two (technical) lemmas. We first show in Lemma 3.1 how to write, within MHW model (3.1), each element in receiver swaption payoff (2.15) as a simple function of one standard normal r.v. x. Then, Lemma 3.2 shows that there exists one unique value x^* such that $S_{\alpha\omega}(t_\alpha)$ is equal to the strike rate K. This result is crucial in order to extend the approach of Jamshidian (1989) to the multi-curve case and grants the possibility to obtain simple closed formulas for PD and CS swaptions, as shown in Sec. 3.2.

Let us define the standard normal r.v.

$$x := \frac{1}{\zeta_\alpha}\int_{t_0}^{t_\alpha} dW_u^{(\alpha)}\,\sigma e^{-a(t_\alpha-u)}, \tag{3.2}$$

where

$$\zeta_\alpha^2 := \begin{cases} \sigma^2\dfrac{1-e^{-2\,a(t_\alpha-t_0)}}{2\,a} & a\in\Re^+\backslash\{0\},\\ \sigma^2(t_\alpha-t_0) & a=0. \end{cases} \tag{3.3}$$

The following lemma holds.

Lemma 3.1. *Discount and spread curves in t_α can be written, according to the MHW model* (3.1) *in the t_α-forward measure, as*

$$\begin{aligned} B_{\alpha'\,\iota}(t_\alpha) &= B_{\alpha'\,\iota}(t_0)\exp\left\{-\varsigma_{\alpha'\,\iota}\,x-\frac{1}{2}\,\varsigma^2_{\alpha'\,\iota}\right\} \qquad \iota=\alpha'+1,\ldots,\omega',\\ \beta_\iota(t_\alpha)\,B_{\alpha'\,\iota}(t_\alpha) &= \beta_\iota(t_0)\,B_{\alpha'\,\iota}(t_0)\exp\left\{-\nu_{\alpha'\iota}x-\frac{1}{2}\nu^2_{\alpha'\iota}\right\} \qquad \iota=\alpha',\ldots,\omega'-1, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\varsigma_{\alpha'\,\iota} &:= (1-\gamma) v_{\alpha'\,\iota},\\ \nu_{\alpha'\,\iota} &:= v_{\alpha'\,\iota} - \gamma v_{\alpha'\,\iota+1}\end{aligned} \tag{3.5}$$

with

$$v_{\alpha'\,\iota} := \zeta_\alpha \, \frac{1-e^{-a(t'_\iota - t'_{\alpha'})}}{a}, \quad \iota = \alpha', \ldots, \omega'. \tag{3.6}$$

Proof. An application of Itô calculus □

Remark 2. Volatilities $\{\varsigma_{\alpha'\,\iota}\}_{\iota=\alpha'+1\ldots\omega'}$ are always positive and they are strictly increasing with ι for $\gamma < 1$ (and identically equal to zero for $\gamma = 1$). The quantities $\{\nu_{\alpha'\,\iota}\}_{\iota=\alpha'\ldots\omega'-1}$ can change sign depending on the value of γ. We observe that

$$\nu_{\alpha'\,\iota} = v_{\alpha'\,\iota+1}(\hat{\gamma}_\iota - \gamma) \tag{3.7}$$

with $\hat{\gamma}_\iota := v_{\alpha'\,\iota}/v_{\alpha'\,\iota+1} \in (0,1)$. Then, when $\gamma = 0$ all $\{\nu_{\alpha'\,\iota}\}_{\iota=\alpha'+1\ldots\omega'-1}$ are positive and $\nu_{\alpha'\,\alpha'} = v_{\alpha'\,\alpha'}$ is zero, while for larger values of γ some $\nu_{\alpha'\,\iota}$ become negative. For γ equal or close to 1 all $\{\nu_{\alpha'\,\iota}\}_{\iota=\alpha'\ldots\omega'-1}$ are negative. Due to these possible negative values, $\{\nu_{\alpha'\,\iota}\}_\iota$ are not volatilities; we call them *extended* volatilities.

A consequence of Lemma 3.1 is that receiver swaption payoff (2.15) in the t_α-forward measure can be written as a function of a unique r.v. x as

$$\mathcal{R}_{\alpha\omega}(t_\alpha) = [f(x)]^+, \tag{3.8}$$

where

$$\begin{aligned} f(x) := &\sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_0)\, e^{-\varsigma_{\alpha j}\, x - \varsigma^2_{\alpha j}/2} + \sum_{\iota=\alpha'+1}^{\omega'-1} B_{\alpha'\iota}(t_0) e^{-\varsigma_{\alpha'\iota}\, x - \varsigma^2_{\alpha'\iota}/2}\\ &- \sum_{\iota=\alpha'}^{\omega'-1} \beta_\iota(t_0) B_{\alpha'\iota}(t_0) e^{-\nu_{\alpha'\iota}\, x - \nu^2_{\alpha'\iota}/2} \end{aligned} \tag{3.9}$$

is a finite sum of exponential functions of x, i.e.

$$f(x) = \sum_i d_i e^{\lambda_i x} \quad \text{with } d_i, \lambda_i \in \Re, \tag{3.10}$$

where some $d_i < 0$ and some $\lambda_i \geq 0$; the number of addends is equal to $\omega - \alpha - 1 + 2(\omega' - \alpha')$. Hence, the swaption looks like a nontrivial spread option, because some (nonconstant) addends in $f(x)$ are negative. In general f is not a monotone function of x, as shown in Fig. 2. For this reason it is not obvious that the equation $f(x) = 0$ admits a unique solution, a property proven in the next lemma.

Lemma 3.2. *According to MHW model* (3.1), $\exists!\, x^*$ *such that* $f(x^*) = 0$, *with* $f(x)$ *defined in* (3.9). *Moreover, the function* f *is greater than zero for* $x < x^*$.

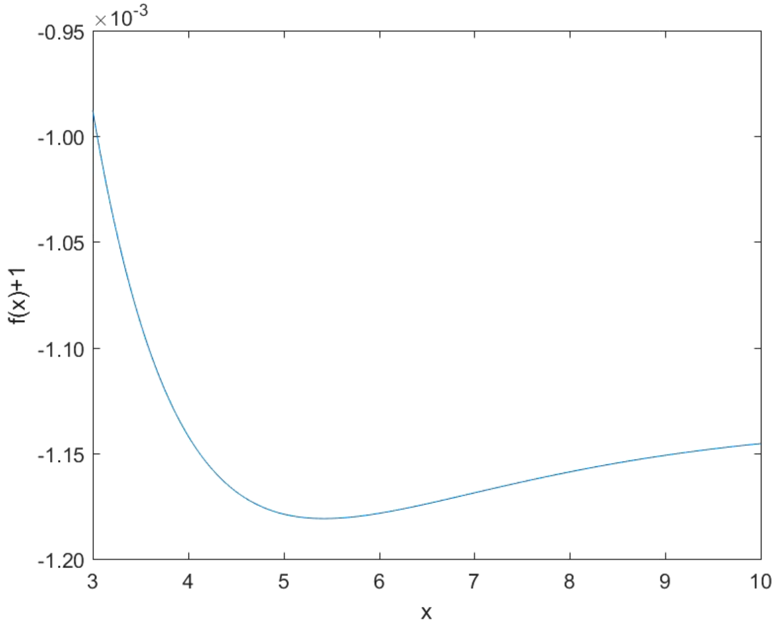

Fig. 2. Plot of $f(x)+1$ as a function of x. We have considered a $1y9y$ swaption. Parameters are $a = 0.1, \sigma = 10, \gamma = 0$; *discount* and *spread* curves are the ones of Sec. 4. We observe that $f(x)$ is not a monotone function.

Proof. See Appendix A. □

In Lemma 3.2, we have proven that, even if the function f, for some parameters choices, is not a decreasing function of x, however there exists a unique value x^* such that $f(x^*) = 0$, i.e. the equality $S_{\alpha\omega}(t_\alpha) = K$ is satisfied for this unique value. This fact allows to extend to MHW the idea of Jamshidian (1989), as shown in Sec. 3.2.

3.2. *Swaption closed formulas*

In the following propositions, we prove that, in the MHW (3.1), closed form solutions hold for both PD and CS receiver swaptions.

Proposition 3.1. *A PD receiver swaption, according to MHW model* (3.1), *can be computed with the closed formula*

$$\mathfrak{R}^{\mathrm{MHW}}_{\alpha\omega}(t_0) = B(t_0, t_\alpha)\Bigg\{ \sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_0) N(x^* + \varsigma_{\alpha j}) + \sum_{\iota=\alpha'+1}^{\omega'-1} B_{\alpha'\iota}(t_0) N(x^* + \varsigma_{\alpha'\iota}) - \sum_{\iota=\alpha'}^{\omega'-1} \beta_\iota(t_0) B_{\alpha'\iota}(t_0) N(x^* + \nu_{\alpha'\iota}) \Bigg\}, \tag{3.11}$$

where $N(\bullet)$ is the standard normal CDF and x^* is the unique solution of $f(x) = 0$, with $f(x)$ defined in (3.9).

Proof. Due to Lemma 3.2, swaption receiver is equivalent to

$$\begin{aligned}
\frac{\mathfrak{R}^{\mathrm{MHW}}_{\alpha\omega}(t_0)}{B(t_0,t_\alpha)} = \mathbb{E}\{f(x)\}^+ = \mathbb{E}\{f(x)\mathbb{1}_{x\le x^*}\} \\
&= \sum_{j=\alpha+1}^{\omega} c_j \mathbb{E}\{B_{\alpha j}(t_0) e^{-\varsigma_{\alpha j}x - \varsigma^2_{\alpha j}\zeta^2_\alpha/2}\mathbb{1}_{x\le x^*}\} \\
&\quad + \sum_{\iota=\alpha'+1}^{\omega'-1} \mathbb{E}\{B_{\alpha\iota}(t_0) e^{-\varsigma_{\alpha\iota}x - \varsigma^2_{\alpha\iota}\zeta^2_\alpha/2}\mathbb{1}_{x\le x^*}\} \\
&\quad - \sum_{\iota=\alpha'}^{\omega'-1} \mathbb{E}\{\beta_\iota(t_0) B_{\alpha\iota}(t_0) e^{-\nu_{\alpha\iota}x - \nu^2_{\alpha\iota}\zeta^2_\alpha/2}\mathbb{1}_{x\le x^*}\} \qquad (3.12)
\end{aligned}$$

and then, after straightforward computations, one proves the proposition. □

The above proposition generalizes the celebrated result of Jamshidian (1989) to this multi-curve HJM model. The main difference is that also negative addends appear in the receiver swaption $\mathfrak{R}^{\mathrm{MHW}}_{\alpha\omega}(t_0)$ and there are *extended* volatilities instead of volatilities.[c]

Proposition 3.2. *A CS receiver swaption, according to MHW model* (3.1), *can be computed with the closed formula*

$$\mathfrak{R}^{\mathrm{C,MHW}}_{\alpha\omega}(t_0) = B(t_0,t_\alpha)\int_{-\infty}^{x^*} dx\, \frac{e^{-x^2/2}}{\sqrt{2\pi}} C_{\alpha\omega}(S(x))[K - S(x)]^+ \qquad (3.13)$$

with

$$S(x) = \frac{\sum_{\iota=\alpha'}^{\omega'-1} B_{\alpha'\iota}(t_0)\beta_\iota(t_0)\exp\{-\nu_{\alpha'\iota}x - \nu^2_{\alpha'\iota}/2\}}{\sum_{j=\alpha+1}^{\omega}\delta_{j-1}B_{\alpha j}(t_0)\exp\{-\varsigma_{\alpha j}x - \varsigma^2_{\alpha j}/2\}} - \frac{\sum_{\iota=\alpha'+1}^{\omega'} B_{\alpha'\iota}(t_0)\exp\{-\varsigma_{\alpha'\iota}x - \varsigma^2_{\alpha'\iota}/2\}}{\sum_{j=\alpha+1}^{\omega}\delta_{j-1}B_{\alpha j}(t_0)\exp\{-\varsigma_{\alpha j}x - \varsigma^2_{\alpha j}/2\}}, \qquad (3.14)$$

[c]In the case when $\gamma = 0$, a similar formula has been proposed in the literature (cf. Theorem 7.1, p. 111, Henrard 2014). Theorem's proof (cf. p. 112, Henrard 2014) considers the zeros of the function $f(x)$ defined in (3.9), or equivalently the values of x such that the underlying swap rate at expiry is equal to K; unfortunately, the proof holds only if $f(x)$ (cf. Eq. (3.10)) has all weights d_i (relative to nonconstant addends) positive and it is a sum of negative exponentials and constants ($\lambda_i \le 0$).

where x^* is the unique solution of $f(x) = 0$, with $f(x)$ defined in (3.9), *and $C_{\alpha\,\omega}(S)$ is defined in Eq.* (2.17).

Proof. A CS swaption is ITM when the corresponding PD one is ITM. As shown in Lemma 3.2, this condition corresponds, for the MHW model in the t_α-forward measure, to impose the condition $x \leq x*$. The result is proven after a direct substitution in CS payoff of the swap rate at the expiry t_α as function of the standard normal x □

Let us recall that the put-call parity holds only in the PD case but not in the CS case (see, e.g. Mercurio 2007). It can be proven that, *mutatis mutandis*, a similar solution holds for a payer swaption in both PD and CS cases.[d]

4. Model Calibration

In this section, we show in detail model calibration of market parameters in the Euro market considering European ATM CS swaptions versus Euribor 6m with the end-of-day market conditions of 10 September 2015 (value date).

The calibration cascade is divided in three steps. First, we bootstrap the *discount* and the *pseudo-discount* curves from 6m-Depo, three FRAs ($1{\times}7, 2{\times}8$ and $3{\times}9$) and swaps (both OIS and versus Euribor 6m); no convexity adjustment is considered. Second, we calibrate the three MHW parameters $\mathbf{p} := (a, \sigma, \gamma)$ with European ATM swaptions versus Euribor 6m on the 10y-diagonal (i.e. considering the $M = 9$ ATM swaptions 1y9y, 2y8y, ..., 9y1y). Third, we recompute the curves in the buckets impacted by convexity adjustments (in the example, the buckets 1 m, 2 m and 3 m obtained via the FRAs).

The *discount* curve is bootstrapped from OIS quoted rates following the standard methodology (see, e.g. Baviera & Cassaro 2015). Their quotes at value date are reported in Table 1 (with market conventions, i.e. annual payments and Act/360 day-count); in the same table we report also the swap rates (annual fixed leg with 30/360 day-count). Then the *pseudo-discount* curve is bootstrapped from 6 m depo, FRAs and swap quoted rates with the same methodology described in Baviera & Cassaro (2015) without considering any convexity adjustment. In Table 2, we show the relevant FRA rates and the Euribor 6 m fixing on the same value date (both with Act/360 day-count). All market data are provided by Bloomberg. Convexity adjustments for FRAs, present in the MHW model, can be neglected because they

[d]All results in this section can be generalized to $\tilde{\sigma}(t,T) := \sigma(t)[1 - e^{-a(T-t)}]/a$ with $\sigma(t)$ a generic positive function of time: in this case the separability condition for Gaussian HJM volatility is satisfied (see, e.g. p. 194 in Henrard 2014). Swaption formulas (3.11) and (3.13) hold. It is enough to redefine x in (3.2) as $x := \int_{t_0}^{t_\alpha} dW_u^{(\alpha)}\,\sigma(u)\,e^{-a(t_\alpha-u)}/\zeta_\alpha$ and ζ_α^2 in (3.3) as $\zeta_\alpha^2 := \int_{t_0}^{t_\alpha} du\,\sigma(u)^2\,e^{-2\,a\,(t_\alpha-u)}$, leaving all other quantities equal. For example, in the piecewise-constant case one uses the same swaption formulas (3.11) and (3.13) with $\zeta_\alpha^2 = \sum_{l=1}^{\alpha}\sigma_l^2\,(e^{-2\,a\,(t_\alpha-t_l)} - e^{-2\,a\,(t_\alpha-t_{l-1})})/2a$.

Table 1. OIS rates and swap rates versus Euribor 6 m in percentages: end-of-day mid quotes (annual 30/360 day-count convention for swaps versus 6 m, Act/360 day-count for OIS) on 10 September 2015.

	OIS rate (%)	Swap rate versus 6m (%)
1w	−0.132	—
2w	−0.132	—
1m	−0.132	—
2m	−0.133	—
3m	−0.136	—
6m	−0.139	—
1y	−0.147	0.044
2y	−0.135	0.080
3y	−0.083	0.154
4y	0.008	0.259
5y	0.122	0.377
6y	0.254	0.512
7y	0.392	0.652
8y	0.529	0.786
9y	0.655	0.909
10y	0.766	1.016
11y	0.866	1.109
12y	0.957	1.195
15y	1.160	1.383

Table 2. Euribor 6 m fixing rate and FRA in percentages (day-count Act/360). FRA rates are end-of-day mid quotes at value date.

	Rate (%)
Euribor 6 m	0.038
FRA 1×7	0.038
FRA 2×8	0.041
FRA 3×9	0.043

do not impact the buckets relevant for the diagonal CS swaptions co-terminal 10y considered in this calibration. In the general case, as discussed in Baviera & Cassaro (2015) one considers IR option straddles that, having a negligible delta, are less influenced by the exact values obtained in curves' bootstrap. In Fig. 3, we show the *discount* and *pseudo-discount* curves obtained via the bootstrapping technique.

We show the CS swaption ATM normal volatilities in basis points (bps)[e] in Table 3. The CS swaption market prices are obtained according to the Normal-Black market formula; market CS swaption pricing formula in the multi-curve setting is reported in Appendix B.

[e] 1bp = 0.01%.

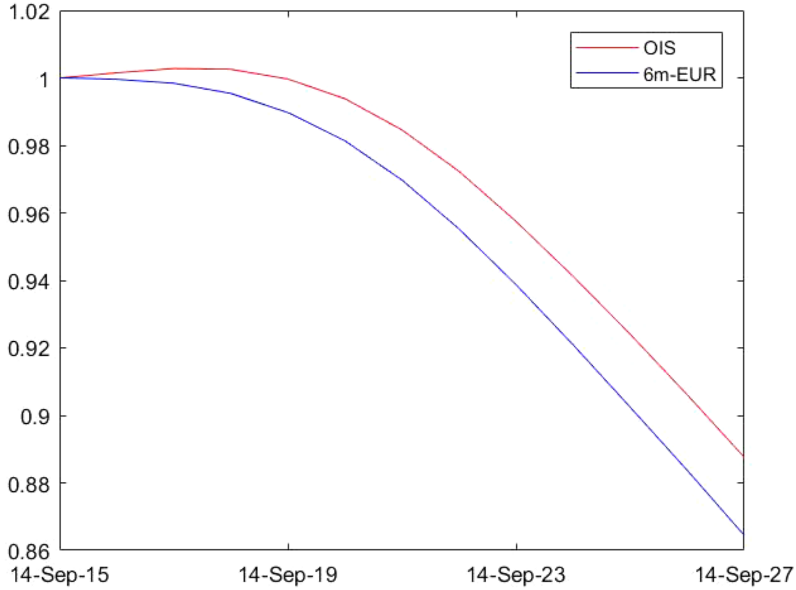

Fig. 3. *Discount* OIS curve (in red) and *pseudo-discount* Euribor-6 m curve (in blue) on 10 September 2015, starting from the settlement date and up to a 12y time horizon.

Table 3. Normal volatilities for ATM diagonal CS swaptions co-terminal 10y in basis points on 10 September 2015, in the Euro market. Swaptions are versus Euribor 6 m.

Expiry	Tenor	Volatility (bps)
1y	9y	64.70
2y	8y	66.78
3y	7y	68.53
4y	6y	70.91
5y	5y	72.36
6y	4y	73.07
7y	3y	73.21
8y	2y	73.51
9y	1y	73.45

We minimize the squared distance between CS swaption model and market prices

$$\texttt{Err}^2(\mathbf{p}) = \sum_{i=1}^{M} [\mathfrak{R}_i^{c,\mathrm{MHW}}(\mathbf{p}; t_0) - \mathfrak{R}_i^{c,\mathrm{MKT}}(t_0)]^2. \tag{4.1}$$

We obtain the parameter estimations minimizing the Err function with respect to model parameters; the solution is stable for a large class of starting points. As estimations we obtain $a = 12.94\%$, $\sigma = 1.26\%$ and $\gamma = 0.07\%$. The difference

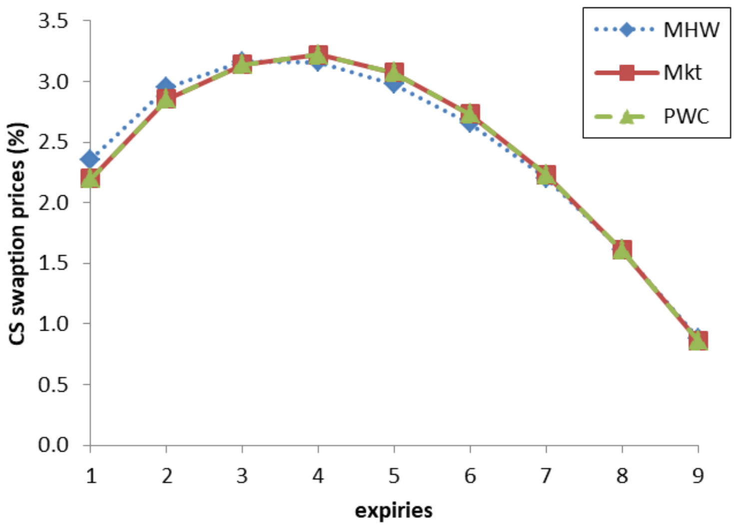

Fig. 4. Market prices for ATM diagonal CS swaptions co-terminal 10y in percentages (squares with continuous red line), the corresponding ones obtained via the MHW calibration (diamonds with dotted blue line) and the ones obtained with the piecewise-constant (PWC) version of the model (triangles with green dashed line) for the 9 expiries considered. The PWC values equal market prices for all practical purposes. Swaptions are versus Euribor 6 m.

between model and market CS swaption prices are shown in Fig. 4: calibration results look good despite the parsimony of the proposed model with differences between market and model prices, in most cases, lower than 10 basis points.[f]

It is interesting to observe that the dependence of the `Err` function with respect to γ is less pronounced compared to the one with respect to a and σ; even if the minimum values for the `Err` function are achieved for very low values of γ, however, differences in terms of squared distance are very small when increasing, even significantly, γ: another evidence that the most relevant dynamics for swaption valuation is the one related to the *pseudo-discount* curve, while it is more difficult to calibrate separately *discount* and *spread* dynamics, where the corresponding volatility depends on γ parameter.

Finally, in the third step one should recompute the buckets influenced by convexity adjustments with the parameters calibrated on the swaption market (the buckets 1 m, 2 m and 3 m obtained via the FRAs): in this case the correction appears negligible for all practical purposes. This step completes the calibration cascade.

[f] Let us note that model generalization discussed in note 4 allows to reproduce all ATM diagonal CS swaption prices. In the piecewise-constant case, a parameter choice with a and γ equal to the three-parameters case and a vector $\sigma = (1.18\%, 1.62\%, 1.92\%, 2.16\%, 2.32\%, 2.41\%, 2.45\%, 2.46\%, 2.45\%)$ replicates exactly market prices. As already discussed, for several applications the parsimony properties of MHW model determine model selection.

As already underlined by Baviera & Cassaro (2015), let us stress that the described three-step calibration cascade is very general and it can be extended to the 3 m curve. In that case STIR options are very liquid. After having bootstrapped the two curves, one considers IR option straddles that, having a negligible delta, depend slightly on the exact values of *discounts* and *pseudo-discounts* obtained in the bootstrap. Often the γ parameter is rather difficult to be calibrated; in those cases its value can be chosen in order to obtain an adequate smoothness of calibrated *pseudo-discount* curve. Finally, one computes convexity adjustments (in general for both FRAs and STIR futures) and recomputes the two curves accordingly. All formulas for the MHW model are reported in Appendix C.

5. Conclusions

Multi-curve IR setting is now the standard. In this paper, we have focused on two issues: parsimony and calibration. In particular, we have replied to two main questions:

(1) Is it possible to consider a parsimonious multi-curve IR model (without assuming constant *spreads*) that allows to obtain simple swaption formulas?
(2) How to implement a calibration cascade, i.e. a procedure that calibrates first the most liquid "linear" IR products (depos, FRAs, STIR futures and swaps) and then IR options?

In this paper, we have introduced the MHW model: a three parameter generalization of the two parameters Hull & White (1990) model, where the additional parameter γ lies in the interval $[0, 1]$. We have proven that the model allows very simple closed formulas for both PD (3.11) and CS swaptions (3.13). The proposed model is simple but not elementary: the application of the Jamshidian trick is not straightforward as shown in Fig. 2. Moreover, MHW model allows Black-like formulas for the other liquid IR options (caps/floors and STIR options) and simple analytical convexity adjustments for FRAs and STIR futures; furthermore, numerical techniques similar to the HW model can be applied.

Model calibration is immediate: we have shown in detail how to implement the calibration cascade on 10 September 2015, end-of-day Euro market conditions. The proposed procedure involves three steps. First, bootstrap *discount* and *pseudo-discount* curves without any convexity adjustment. Second, calibrate IR options. Third, re-evaluate curves considering convexity adjustments.

Finally, let us mention the wide range of possible applications. A very parsimonious model, as the proposed MHW model (3.1), can be the choice of election in challenging tasks where the multi-curve IR dynamics is just one of the modeling elements. Two significant examples are the pricing and the risk management of illiquid corporate bonds (see, e.g. Baviera *et al.* 2019), and the XVA valuation of derivative contracts (see, e.g. Crépey *et al.* 2016).

Notation and shorthands

Symbol	Description
a, σ, γ	MHW (3.1) parameters; $a, \sigma \in \Re^+$ and $\gamma \in [0,1]$
$B(t,T)$	*discount* curve, zero-coupon bond in t with maturity T
$B(t;T,T+\Delta)$	forward *discount* in t between T and $T+\Delta$, $t \le T$
$\tilde{B}(t;T,T+\Delta)$	forward *pseudo-discount* in t between T and $T+\Delta$, $t \le T$
$\beta(t;T,T+\Delta)$	forward *spread* in t between T and $T+\Delta$, $t \le T$
$D(t,T)$	stochastic *discount* between t and T
$\delta(t_j, t_{j+1})$	year-fraction between two payment dates in swap's fixed leg
$\delta(t'_\iota, t'_{\iota+1})$	year-fraction between two payment dates in swap's floating leg
Δ	the lag of *pseudo-discounts*, e.g. 6-months for Euribor-6 m
$\mathbb{E}\left[\bullet\right]$	expectation in the spot measure and expectation with respect to x
$\mathbb{E}^{(\alpha)}\left[\bullet\right]$	expectation in the t_α-forward measure
K	strike rate
$\phi(\bullet)$	the standard normal pdf
$N(\bullet)$	the standard normal CDF
ρ	correlation matrix in $\Re^{d\times d}$ such that $dW_{i,t} dW_{j,t} = \rho_{ij} dt$
$\sigma(t,T)$	MHJM *discount* volatility in $\Re^d$ between t and T
$\tilde{\sigma}(t,T)$	MHW *pseudo-discount* volatility between t and T
$\eta(t,T)$	MHJM *spread* volatility in $\Re^d$ between t and T
$\mathcal{R}_{\alpha\omega}(t_\alpha)$	receiver swaption payoff at expiry
$\mathfrak{R}_{\alpha\omega}(t_0)$	receiver swaption price at value date
t_0	value date
t_α	swaption expiry date
t_ω	underlying swap maturity date
$\mathbf{t} := \{t_j\}_j$	underlying swap fixed leg payment dates, $j = \alpha+1, \dots, \omega$
$\mathbf{t}' := \{t'_\iota\}_\iota$	underlying swap floating leg payment dates, $\iota = \alpha'+1, \dots, \omega'$
$\underline{W}_t$	vector of correlated BMs in $\Re^d$ s.t. $dW_{i,t} dW_{j,t} = \rho_{ij} dt$
$\underline{W}_t^{(i)}$	vector of correlated BMs in the t_i-forward measure
x	standard normal defined in (3.2)
x^*	the unique solution of $f(x) = 0$; $f(x)$ defined in (3.9)
$z \cdot y$	canonical scalar product between z and y in $\Re^d$
z^2	scalar product $z \cdot \rho z$ with $z \in \Re^d$ and ρ correlation matrix

Shorthands

$B_{\alpha\, j}(t)$: $B(t; t_\alpha, t_j)$
$B_{\alpha'\, \iota}(t)$: $B(t; t'_{\alpha'}, t'_\iota)$
$\beta_\iota(t)$: $\beta(t; t'_\iota, t'_{\iota+1})$
δ'_ι : $\delta(t'_\iota, t'_{\iota+1})$
δ_j : $\delta(t_j, t_{j+1})$
c_j : $\delta_{j-1}\, K$ for $j = \alpha+1, \ldots, \omega-1$ and $1 + \delta_{\omega-1} K$ for $j = \omega$
$\upsilon_{\alpha'\, \iota}$: $(1 - e^{-a(t'_\iota - t'_{\alpha'})})\, \zeta_\alpha / a$
$\varsigma_{\alpha'\, \iota}$: $(1-\gamma)\, \upsilon_{\alpha'\, \iota}$
$\nu_{\alpha'\, \iota}$: $\upsilon_{\alpha'\, \iota} - \gamma\, \upsilon_{\alpha'\, \iota+1}$
ζ^2_α : defined in (3.3)
BMs : Brownian motions
CDF : Cumulative Distribution Function
CS : cash-settled (swaption)
IR : Interest Rate
ITM : In The Money
MHJM : Multi-curve HJM framework (2.6)
MHW : Multi-curve Hull White model (3.1)
PD : physical delivery (swaption)
pdf : probability density function
PWC : piecewise-constant (model)
r.v. : random variable
s.t. : such that
w.r.t. : with respect to

Acknowledgments

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Appendix A

Proof of Lemma 3. Let us study $f(x)$ as a function of $x \in \Re$. It is a very regular function ($\mathcal{C}^\infty$), a finite sum of exponentials and constants.
$f(x)$ is composed by two different parts. First, positive addends with negative exponentials (addends in the first two sums in (3.9)); then, the remaining coefficients are always negative. We divide the addends of function f into the two parts. In the first one, $f_+(x)$, we consider the sum of all positive addends, and in the second one, $f_-(x)$, the sum in absolute value of all negative addends, i.e.

$$f(x) =: f_+(x) - f_-(x), \tag{A.1}$$

where both $f_+(x)$ and $f_-(x)$ are positive functions of their argument: $f_+(x)$ is the sum of negative exponentials (or a constant for $\gamma = 1$), while $f_-(x)$ can be the sum of both positive exponentials ($\nu_{\alpha\iota} < 0$), negative exponentials ($\nu_{\alpha\iota} > 0$) and a constant, when at least one $\nu_{\alpha\iota}$ is equal to 0 (only for a finite set of values for γ, for the values of γ equal to one of the $\{\hat{\gamma}_\iota\}_{\iota=\alpha'+1,\ldots,\omega'}$).

Clearly, the solutions of the equation $f(x) = 0$ are the intersections of $f_+(x)$ and $f_-(x)$.

First, let us observe that a positive addend is leading for small x. This fact is a consequence of the following inequalities that hold $\forall \iota = \alpha', \ldots, \omega' - 1$:

$$v_{\alpha'\iota} < v_{\alpha'\iota+1}, \quad \nu_{\alpha'\iota} \leq (1-\gamma) v_{\alpha'\iota} \text{ with } \nu_{\alpha'\iota} = (1-\gamma) v_{\alpha'\iota} \text{ only for } \gamma = 0, \tag{A.2}$$

immediate consequences of volatility definitions (3.1). For all values of $\gamma < 1$ the leading term of $f(x)$ for small x is

$$c_\omega B_{\alpha\omega}(t_0) e^{-(1-\gamma)v_{\alpha\omega}x+\cdots} \tag{A.3}$$

because, due to inequalities (A.2), $-(1-\gamma)\, v_{\alpha\omega}$ is the lowest exponent coefficient that multiplies x among the exponentials in $f(x)$; i.e. there exists always a $\hat{x}$ such that $\forall\, x < \hat{x}\;\; f_+(x) > f_-(x)$.

Then, let us define $\hat{\gamma} := \max_\iota \hat{\gamma}_\iota$ and let us distinguish three cases depending on γ value:

(1) When $\hat{\gamma} \leq \gamma \leq 1$, $f_-(x)$, due to *Remark* 2, is a positive linear combination of positive exponentials (and a positive constant when $\gamma = \hat{\gamma}$). $f_-(x)$ admits one unique intersection with $f_+(x)$, which is a sum of negative exponentials for $\gamma < 1$, while it is a constant for $\gamma = 1$.

(2) When $0 < \gamma < \hat{\gamma}$, $f_-(x)$ is a u-shaped positive function since it is a positive linear combination of positive and negative exponentials (and a constant for some values of γ). Moreover $f_+(x)$ and $f_-(x)$ present one unique intersection, because $f_+(x)$ goes to $+\infty$ for $x \to -\infty$ faster than $f_-(x)$ and to 0 for $x \to +\infty$.

(3) When $\gamma = 0$

$$\begin{aligned} f_+(x) &= \sum_{j=\alpha+1}^{\omega} c_j B_{\alpha j}(t_0) e^{-v_{\alpha j}x - v_{\alpha j}^2/2} + \sum_{\iota=\alpha'+1}^{\omega'-1} B_{\alpha'\iota}(t_0) e^{-v_{\alpha'\iota}x - v_{\alpha'\iota}^2/2} \\ f_-(x) &= \beta_{\alpha'}(t_0) + \sum_{\iota=\alpha'+1}^{\omega'-1} \beta_\iota(t_0) B_{\alpha'\iota}(t_0) e^{-v_{\alpha'\iota}x - v_{\alpha'\iota}^2/2} \end{aligned} \tag{A.4}$$

all addends are negative exponentials and constants. $f_+(x)$ goes to zero for $x \to +\infty$ and, as mentioned above, $f_+(x)$ goes to $+\infty$ for $x \to -\infty$ faster than $f_-(x)$. The two functions admit only one intersection because $\lim_{x\to+\infty} f_-(x) = \beta_{\alpha'}(t_0) > 0$.

We have then proven that, for all parameters choices, there exists a unique value x^* s.t $f(x^*) = 0$. Finally, we note that, since $f_+(x) > f_-(x)$ for small values of x, then the function $f(x)$ is larger than zero for $x < x^*$ in all three cases described above: this observation completes lemma's proof. $\square$

Appendix B

In this appendix, we report the Bachelier (also called Normal-Black) formula for a receiver swaption:

$$\mathfrak{R}^{\text{MKT}}_{\alpha\omega}(t_0) = B(t_0, t_\alpha)\text{BPV}_{\alpha\omega}(t_0)\{[K - S_{\alpha\omega}(t_0)]N(-d) + \sigma_{\alpha\omega}\ \sqrt{t_\alpha - t_0}\phi(d)\}, \tag{B.1}$$

where $N(\bullet)$ is the standard normal CDF, $\phi(\bullet)$ the standard normal density function and $\sigma_{\alpha\omega}$ the corresponding implied normal volatility

$$d := \frac{S_{\alpha\omega}(t_0) - K}{\sigma_{\alpha\omega}\sqrt{t_\alpha - t_0}}. \tag{B.2}$$

This market formula allows for negative IRS. The ATM formula simplifies to

$$\mathfrak{R}^{\text{MKT}}_{\alpha\omega}(t_0) = B(t_0, t_\alpha)\text{BPV}_{\alpha\omega}(t_0)\sigma_{\alpha\omega}\sqrt{\frac{t_\alpha - t_0}{2\pi}}. \tag{B.3}$$

The Bachelier market formula for a cash settlement receiver swaption is

$$\mathfrak{R}^{\text{C,MKT}}_{\alpha\omega}(t_0) = B(t_0, t_\alpha)C_{\alpha\omega}(S(t_0))\{[K - S_{\alpha\omega}(t_0)]N(-d) + \sigma_{\alpha\omega}\sqrt{t_\alpha - t_0}\phi(d)\}. \tag{B.4}$$

Appendix C

In this appendix, we recall the convexity adjustments and IR options' closed formulas for a generic Gaussian MHJM model (see, e.g. Baviera & Cassaro 2015, Henrard 2013). We report only derivative payoff, its price and the convexity adjustments involved.

In particular, MHW model (3.1) is a 1-factor Gaussian model where all volatilities depend on the three parameters $a, \sigma \in \Re^+$ and $\gamma \in [0, 1]$

$$\begin{aligned}
\text{HJM}\,\textit{discount}\text{ vol } \sigma(t,T) &= \gamma\sigma\,\frac{1 - e^{-a(T-t)}}{a},\\
\text{forward}\,\textit{discount}\text{ vol } \sigma_i(t) &= \gamma\sigma\, e^{a\,t}\,\frac{e^{-a\,t_i} - e^{-a\,t_{i+1}}}{a},\\
\text{forward}\,\textit{pseudo-discount}\text{ vol } \tilde{\sigma}_i(t) &= \sigma e^{a\,t}\,\frac{e^{-a\,t_i} - e^{-a\,t_{i+1}}}{a},\\
\textit{spread}\text{ vol } \eta_i(t) &= (1-\gamma)\,\sigma\, e^{a\,t}\,\frac{e^{-a\,t_i} - e^{-a\,t_{i+1}}}{a}.
\end{aligned} \tag{C.1}$$

FRA

FRA payoff in t_i

$$\delta_i \frac{F_i^{\mathrm{FRA}}(t_i) - L(t_i, t_{i+1})}{1 + \delta_i L(t_i, t_{i+1})}. \tag{C.2}$$

Relation between FRA rate and forward at value date t_0

$$F_i^{\mathrm{FRA}}(t_0) = \gamma^{\mathrm{FRA}} L(t_0; t_i, t_{i+1}) + \frac{\gamma^{\mathrm{FRA}} - 1}{\delta_i}, \tag{C.3}$$

with FRA's convexity adjustment

$$\gamma^{\mathrm{FRA}} := \exp - \int_{t_0}^{t_i} \tilde{\sigma}_i(t) \cdot \rho \eta_i(t) dt. \tag{C.4}$$

STIR future

Relation at future expiry t_i between future price, Libor and future rate

$$F_i^{\mathrm{fut}}(t_i) = L(t_i, t_{i+1}) =: 1 - \Phi_i(t_i), \tag{C.5}$$

with $\Phi_i(t)$ future price at time t. At value date

$$F_i^{\mathrm{fut}}(t_0) := 1 - \Phi_i(t_0) = \mathbb{E}[L(t_i, t_{i+1}) \,|\, \mathcal{F}_{t_0}] = \gamma^{\mathrm{fut}} L(t_0; t_i, t_{i+1}) + \frac{\gamma^{\mathrm{fut}} - 1}{\delta_i}, \tag{C.6}$$

with STIR future's convexity adjustment

$$\gamma^{\mathrm{fut}} := \exp - \int_{t_0}^{t_i} \tilde{\sigma}_i(t) \cdot \rho \sigma(t, t_{i+1}) dt. \tag{C.7}$$

STIR option

Payoff of a call on STIR future payoff in t_i with strike K is

$$\mathcal{C}_i(t_i) = [\Phi_i(t_i) - K]^+. \tag{C.8}$$

Price at value date

$$\mathcal{C}_i(t_0) = \frac{B(t_0, t_i)}{\delta_i} [\hat{X} N(-d_2) - \gamma^{\mathrm{opt}} \tilde{B}_i^{-1}(t_0) N(-d_1)], \tag{C.9}$$

where

$$d_{1,2} := \frac{1}{\tilde{\sigma}\sqrt{t_i - t_0}} \ln \frac{\gamma^{\mathrm{opt}} \tilde{B}_i^{-1}(t_0)}{\hat{X}} \pm \frac{1}{2} \tilde{\sigma} \sqrt{t_i - t_0} \tag{C.10}$$

with

$$\hat{X} := 1 + \delta_i (1 - K) \tag{C.11}$$

and

$$\tilde{\sigma}^2 := \frac{1}{t_i - t_0} \int_{t_0}^{t_i} \tilde{\sigma}_i(t)^2 dt. \tag{C.12}$$

Option's convexity adjustment is

$$\gamma^{\text{opt}} := \exp \int_{t_0}^{t_i} \tilde{\sigma}_i(t) \cdot \rho \sigma_i(t) dt. \tag{C.13}$$

Cap

Caplet payoff in t_{i+1} with strike K (and fixing in t_i) is

$$\delta_i [L(t_i, t_{i+1}) - K]^+. \tag{C.14}$$

Caplet price at value date

$$\text{caplet}_i(t_0) = B(t_0, t_{i+1})[(1 + \delta_i L(t_0; t_i, t_{i+1}))N(d_1^{(c)}) - (1 + \delta_i K)\, N(d_2^{(c)})] \tag{C.15}$$

with

$$d_{1,2}^{(c)} := \frac{1}{\tilde{\sigma}\sqrt{t_i - t_0}} \ln \frac{1 + \delta_i L(t_0; t_i, t_{i+1})}{1 + \delta_i K} \pm \frac{1}{2} \tilde{\sigma} \sqrt{t_i - t_0}, \tag{C.16}$$

where $\tilde{\sigma}$ has been defined in (C.12).

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