Corporate Bond Liquidity

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Abstract

In the fixed income market, it is quite common to find illiquid corporate bonds which are not traded for several days. It is also quite common to find comparable bonds, from the same issuer, that trade much more frequently. We implement a model proposed by [Baviera & al. (2021)] to estimate illiquid bond prices from available liquid bonds.

The model requires market calibration of three elements:

- discount curves, obtained following the procedure described in [Baviera & Cassaro (2014)],
- corporate credit risk, obtained via the bootstrap of Z-Spreads from available corporate bonds (as is market practice),
- volatility parameters for corporate bonds.

The latter element isn't directly available, as bond options are not always frequently traded. However, since the issuers analysed in this project are major European financial institutions, it is possible to use the dynamics obtained through a calibration on at-the-money diagonal swaptions (as in [Baviera (2019)]) as a proxy for corporate bond dynamics.

1 Multicurve-bootstrap of interest rates' curves for the EUR markets

To obtain discount curves for September 10th, 2015, we are implementing a multicurve bootstrap, according to the model proposed in [Baviera & Cassaro (2014)]. In the following lines we underline the key points in the quoted paper.

Multicurve bootstrap after the Great Financial Crisis has became a standard technique in order to build a discounting curve. The main reasons that led to the adoption of these techniques over the common bootstrap are the following:

- after the crisis there is no longer a unique interbank curve,
- large spreads emerged between different Euribor tenors,
- different interpolation rules were used in the bootstrap by major investment houses provoking some divergences in the discounting curve.

We now introduce the two curves we considered, following the framework developed in [Henrard (2010)].

1.1 EONIA curve

The Euro Overnight Index Average (Eonia) is the average overnight reference rate for which European banks lend to one another in euros. It is an interbank rate calculated by the European Central Bank (ECB) based on the loans made by 28 panel banks.
In particular we consider OIS (Overnight Index Swap) derivatives to bootstrap the *main* discounting curve. The bootstrapping technique is straightforward and, in our case, doesn't require any interpolation, since all the information we are interested in is available in the data provided.

1.2 Euribor curve (6 months)

We want to build a *pseudo-discounting curve* and we do so by using different interest rate derivatives all having the same tenor (6m). More specifically we use FRAs for the shorter maturities and swaps for the longer ones. Since we want to use instruments with the same tenors, in order to compute the discounts of the first maturities, we adopt a *backward* technique: we use the interpolated discounts on the maturities of the FRA to invert the forward discounts and therefore go backward to find the discounts factors on the start date.

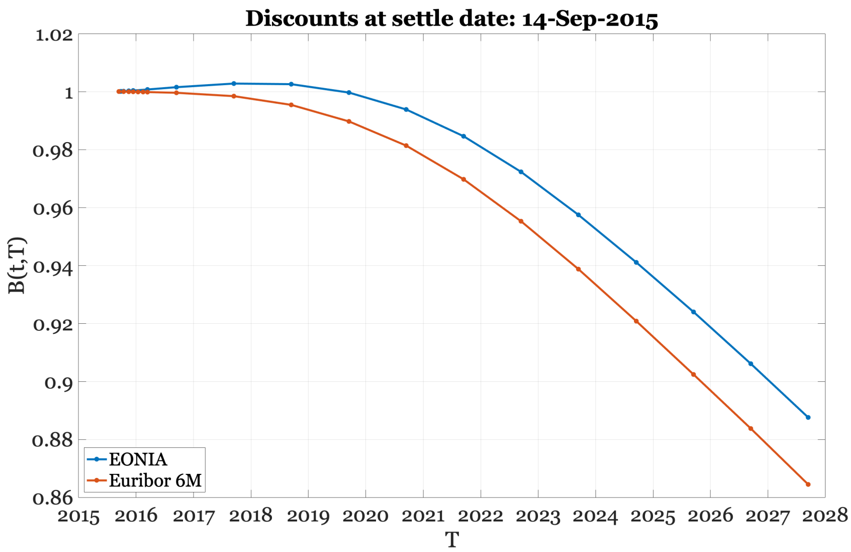
The formulas used in the bootstrap are derived in a framework proposed in [Henrard (2010)] where two main hypothesis are assumed:

1. Independence between the multiplicative *spread* (between the two discounting curves) and the forward discount of the EONIA curve.

$$\text{Spread: } \beta(t, t_s, t_e) \doteq \frac{P^D(t, t_s, t_e)}{P(t, t_s, t_e)} \qquad \text{Forward discount: } P^D(t, t_s, t_e)$$

2. The dynamics of the discounting curve and the spread process are modelled by a Gaussian Heath-Jarrow-Morton model.

By applying this model to our market data we obtain the following discounting curves:



Advantages

The main advantage of the chosen bootstrapping technique is that, in the construction of the Euribor 6m curve, we only use financial instruments with a 6 months tenor. The *backward* step allows us to bootstrap values for maturities lower than 6 months, without having to rely on instruments with different tenors.
Moreover the proposed bootstrapping technique is independent (for all practical purposes) from the chosen interpolation rule. We tried bootstrapping the pseudo-discounting curve using different interpolation rules (*spline on rates*, *spline on discounts*, *log-linear on discounts*) and we obtained results varying only fractions of basis points from the curve computed using linear interpolation on zero rates.

	Spline on rates (10^{-5})	Spline on discounts (10^{-5})	Log-linear on discounts (10^{-5})
Max Error	5.53	4.42	4.23

Criticalities

The independence hypothesis (1.) used by the model isn't by any means the only available model for multicurve bootstrap. [Mercurio & Xie (2012)], and the very Henrard in [Henrard (2013)] propose alternative stochastic descriptions of the multiplicative spread where the independence hypothesis does not hold.

2 Calibration of volatility parameters on diagonal swaptions

Once the discount and pseudo-discount curves have been bootstrapped, the next step is to select and calibrate a market model. The goal is to find the best model in order to price commonly traded interest rate derivatives (European swaptions, in particular) with simple closed formulas. In doing so, the two main aspects to be taken into consideration are the parsimony of the model and the possibility to have a calibration cascade.

Given that, after the Great Financial Crisis, the market is markedly less liquid, choosing a parsimonious model makes it easier to manage both the calibration and the risk management aspects because of a lesser number of parameters to consider.
On the other hand, a model that allows a calibration cascade makes it possible to calibrate first IR curves by bootstrapping and then, using IR options like caps and floors and European swaptions, the volatility parameters.

2.1 Theoretical framework

The chosen model is a 1-dimensional Gaussian model within a multi-curve HJM framework and it is the most parsimonious extension of the Hull & White model. It depends on three parameters: $a, \sigma \in \mathbb{R}^+$ and $\gamma \in [0, 1]$.

The added parameter γ varies in $[0, 1]$ where the extreme cases $\gamma = 0$ and $\gamma = 1$ represent a spread curve constant over time in the first case and pseudo-discounting volatility equal to the volatility of the spread in the second.

This model not only allows to price with simple exact closed formula European swaptions, but also plain vanilla derivatives with formulas that are an extension of Black. From now on it will be referred to as MHW, multi-curve Hull & White.

In order to calibrate the three parameters of the model, a cascade approach is followed.

- Bootstrap the discount and pseudo-discount curves respectively from OIS quoted rates and 6m depo, FRAs and swap quoted rates (described in detail in the previous section).
- Calibrate volatility parameters in order to reproduce market prices for European at-the-money cash settled swaptions vs Euribor 6m on the 10y diagonal, namely with tenors 1y9y, 2y8y, ..., 9y1y.
- Consider the convexity adjustment of FRAs and recompute the curves.

As perfect reproduction of market prices is impossible for a parsimonious model, we minimise the squared distance between model prices and market prices for diagonal swaptions with respect to the three parameters.

$$Err^2(\mathbf{p}) = \sum_{i=1}^{M} [\Re_i^{C,MHW}(\mathbf{p}; t_0) - \Re_i^{C,MKT}(t_0)]^2$$

The market price for a cash-settled receiver swaption is given by the Bachelier formula (also called Normal-Black) and it can be written as:

$$\Re_{\alpha\omega}^{C;MKT}(t_0) = B(t_0, t_\alpha) \cdot C_{\alpha\omega}(S_{\alpha\omega}(t_0)) \cdot [K - (S_{\alpha\omega}(t_0)] \cdot N(-d) + \sigma_{\alpha\omega} \cdot \sqrt{(t_\alpha - t_0)}\Phi(d)$$

For what concerns a cash settlement receiver swaption considering the MHW model, the closed pricing formula is the following:

$$\Re_{\alpha\omega}^{C;MHW}(t_0) = B(t_0, t_\alpha) \int_{-\infty}^{x^*} dx \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot C_{\alpha\omega}(S(x)) \cdot [K - S(x)]^+$$

This expression can be written thanks to the fact that in a MHW framework the payoff of a receiver swaption can be expressed as a simple function of one standard normal random variable x. In particular there exists one unique x^* such that the payoff is equal to 0, or equivalently $S_{\alpha\omega} = K$.

2.2 Practical implementation

MATLAB already offers the opportunity to minimise arbitrary functions, via `fmincon`.

However, trying to minimise the squared error by letting `fmincon` optimise on the three parameters simultaneously is both computationally expensive and highly unstable.

The reason for this phenomenon can be seen in the surface plot of the error function $Err^2(\mathbf{p})$. The variation of γ only slightly modifies the error surface on the first two parameters, a and σ, slightly shifting their optimal values along a valley of the error function.

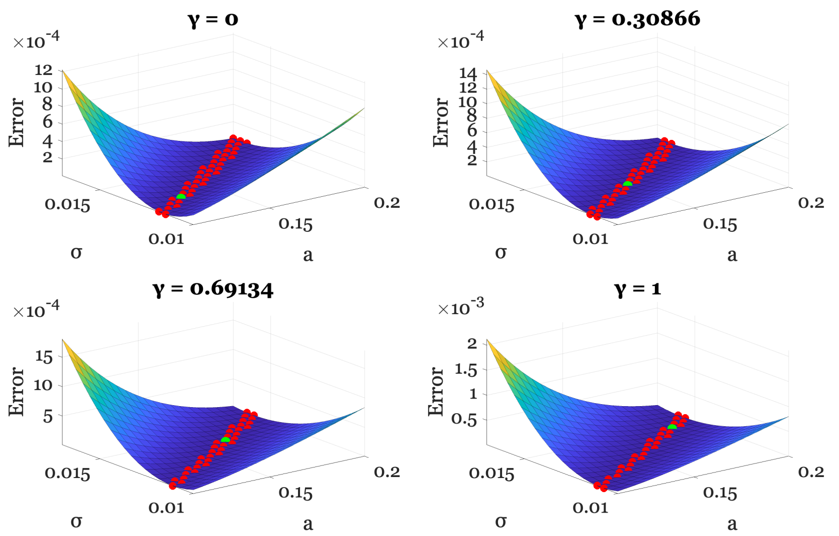


Figure 1: Error surface on a and σ at fixed values for γ. All red points across all four plots have the same order of magnitude, green points denote the optimal parameters given the specific value of γ for the plot. The optimisation problem is badly conditioned (due to what looks like a linear relationship between a and σ) but this problem isn't critical in practice, as parameter estimation is decently robust anyway.

For this reason, we chose to reduce the degrees of freedom of `fmincon`, by running the optimisation at fixed levels of γ. The choice of reducing the degrees of freedom by removing γ, specifically, from `fmincon` was motivated by two main reasons:

1. γ is the only bounded parameter, we can trivially construct a grid spanning all the values of γ and tune the granularity by setting a suitable number of grid points.
2. As we have observed during testing, γ is the least robust parameter, and the error doesn't depend strongly on its value.

In order to properly distinguish between scenarios where γ is exactly 0 (or 1) and scenarios where γ is just close to 0 (or 1), we use a Chebyshev grid to explore possible values, as this kind of grid has higher precision towards the extrema. For each grid value γ_i, we let `fmincon` find the optimal a_i and σ_i under the positivity constraints, thus generating a triplet for each grid point, and we finally select the combination of parameters which produces the minimum error.

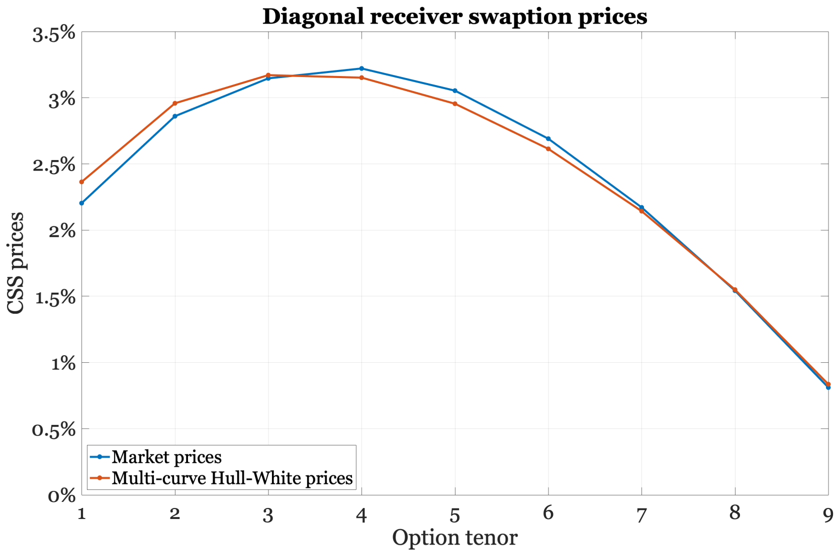
Despite the valley displayed in the previous figure, values of a and σ are decently robust, and hover around the same values (plus or minus a few percentage points) for a large class of starting points. On the other hand, γ is highly unstable, with values as low as 0% and as high as 30% depending on the starting points for a and σ used in `fmincon`.

The chosen implementation of `fmincon`, yields the following parameters:

$$a = 15.59\% \quad \sigma = 1.38\% \quad \gamma = 29.66\%$$

While different implementations converge to different sets of parameters, we found no discernible difference in swaption prices.

The following plot shows the prices of receiver swaption when considering the calibrated MHW model price and their market price.



2.3 Convexity adjustment

The last step of the calibration cascade regards convexity adjustments that is related to the fact that two discount curves are used. This change is necessary in order to consider the difference between the forward and the future market. What it is done is to consider instead of the given FRA mid rates, those obtained by the following expression:

$$F_i^{FRA}(t_0) = \gamma^{FRA} \cdot L(t_0; t_i; t_{i+1}) + \frac{\gamma^{FRA} - 1}{\delta_i}$$

where γ^{FRA} is the convexity adjustment.

Once the new FRA rates are obtained, a new pseudo discount bootstrap is computed. Since the new rates are smaller than those previously used, the new curve is different from the previous one but the difference is negligible and it is only in the first considered months. Given that the convexity adjustments do not affect the buckets of the 10y diagonal swaptions, their impact can be neglected.

3 Pricing of illiquid corporate bonds

3.1 Theoretical framework

Liquidity is a fundamental aspect of financial markets that lends itself to multiple different interpretations, ranging from the tightness of bid-ask spreads to market impact.

The problem tackled in this project is the time-to-liquidate (i.e. the time lag between an investor's decision to sell a security and the effective realisation of the sale). The foundation of the implemented theoretical framework (presented in [Baviera & al. (2021)]) is the formalisation of liquidity as a right that allows the holder to liquidate their position at will.

Sheer liquidity premium

The sheer liquidity premium is defined as the price difference Δ_τ between two defaultable bonds, $\bar{P}$ that can be sold at will, and $\bar{\bar{P}}$ that can only be sold after a time lag τ, that are otherwise identical.

The model assigns a price to this difference, assuming that the bonds in question have zero recovery, and the owner of these bonds is an experienced trader who can:

1. know whether or not the bond issuer defaults before τ,
2. (in case of no default before τ) sell the liquid position through a forward contract at the best possible time $t^* \in [t_0, \tau]$.

If the bond issuer defaults before τ, the experienced trader would immediately sell the liquid bond for $\bar{P}(t_0, t_0, T)$, but would still be forced to hold the illiquid bond through default.

If the bond issuer doesn't default before τ, the experienced trader would sell the liquid position through the optimal forward bond $M_\tau = \bar{P}(t^*, \tau, T)$, while they would have to wait until τ to sell the illiquid bond for $\bar{P}(\tau, \tau, T)$.

Cashflows are summarised in the following table:

Table 1: Comparison of cashflows for liquid and illiquid positions.

Default time	Liquid bond	Illiquid bond	Payment date
$t_D \leq \tau$	$\bar{P}(t_0, t_0, T)$	0	t_0
$t_D > \tau$	M_τ	$\bar{P}(\tau, \tau, T)$	τ

The sheer liquidity premium is obtained by pricing the derivative reproducing the portfolio composed of a long liquid bond and a short illiquid bond, in the hands of an experienced trader.

$$\Delta_\tau = \mathbb{E}[\mathbb{I}_{t_D \leq \tau} \bar{P}(t_0, t_0, T) + D(t_0, \tau)\mathbb{I}_{t_D > \tau}(M_\tau - \bar{P}(\tau, \tau, T))|\mathcal{F}_0]$$

Of course, the formula above represents the sheer liquidity premium for an ideal experienced trader with almost perfect information. In the real world, the sheer liquidity premium would be lower, as a less experienced trader wouldn't be able to extract the optimal value from the liquid position.

Above definitions and derivations rest on the assumption that no coupon payment takes place in $[t_0, \tau]$. This assumption is relatively harmless, as we suppose that the first coupon can be stripped and sold in a liquid market and τ is equal to a couple of months in extremely illiquid markets, meaning that there will be at most one coupon in $[t_0.\tau]$.

3.2 Practical implementation

We developed a MATLAB library which computes the sheer liquidity premium for corporate bonds in a Defaultable HJM framework, following the results published in [Baviera & al. (2021)].

Given the lack of an exact closed formula, it's possible to compute either the lower or the upper bound for the sheer liquidity premium, but the library computes the upper bound by default.

Choosing to use the upper bound presents two major benefits. In the first place, there is a computational advantage, as the lower bound requires numerical integration leading to slower times (as shown in the table below). In addition, the upper bound leads to a more elegant interpretation, by allowing us to create an issuer-specific discount curve which incorporates both default risk and liquidity risk (as specified by the chosen τ). The creation of such a discount curve is not possible for the lower bound, because bonds with different maturities generally produce different lower bounds for the same coupon payment date.

As we report in the following case studies, choosing one bound over the other does not impair accuracy, and their difference is negligible.

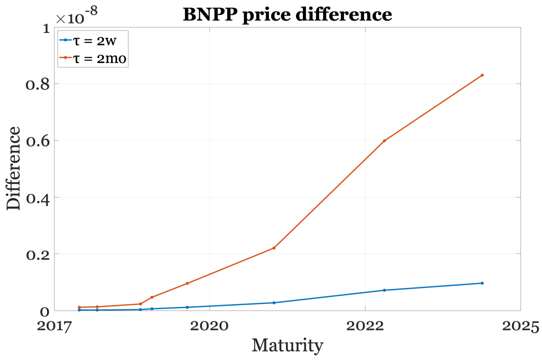
3.3 Application to the European bond market

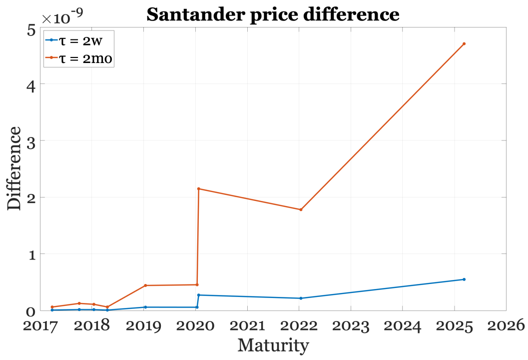
We compute liquidity premiums for two major European financial institutions, BNP Paribas S.A. and Banco Santander S.A. on September 10th, 2015.

Issuer-specific discount curves (and as a consequence Z-Spreads) are bootstrapped from Sr. Unsecured benchmark issues with maturity up to 10 years.

Having chosen two Systemically Important Financial Institutions (SIFIs) allows us to calibrate the only missing component (bond volatilities) even in absence of bond options. The dynamics of the spread between the Euribor and the OIS curve can be used as a proxy for the dynamics of the average credit spread of a SIFI. Parameter estimation for these dynamics has been described in detail in the previous section.

We report the difference between bond prices computed using the upper and the lower bounds for the sheer liquidity premium, to show that the difference is negligible.





Liquidity spread

As is customary among practitioners, we report the liquidity spread $\mathcal{L}(T)$ for the bonds in question, where:

$$\bar{\bar{P}}_\tau(t_0, T) = \sum_{i=1}^{N} c_i e^{-(\mathcal{Y}(T)+\mathcal{L}(T))(t_i - t_0)}$$

Table 2: Invoice prices and liquidity spreads for BNP Paribas S.A. bonds, $\tau = 2\text{w}$.

Maturity	Liquid price	Illiquid price	Liquidity spread (bps)
27-Nov-2017	108.918	107.437 (-1.36%)	19
12-Mar-2018	104.786	103.066 (-1.64%)	18
21-Nov-2018	105.723	103.114 (-2.47%)	17
28-Jan-2019	108.182	105.199 (-2.76%)	17
23-Aug-2019	109.998	106.400 (-3.27%)	17
13-Jan-2021	112.594	106.777 (-5.17%)	15
24-Oct-2022	119.944	111.868 (-6.73%)	13
20-May-2024	115.741	105.733 (-8.65%)	12

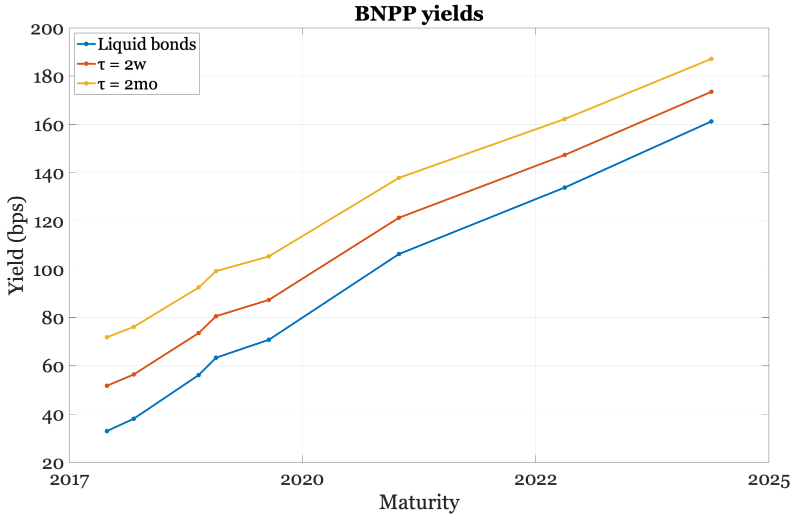
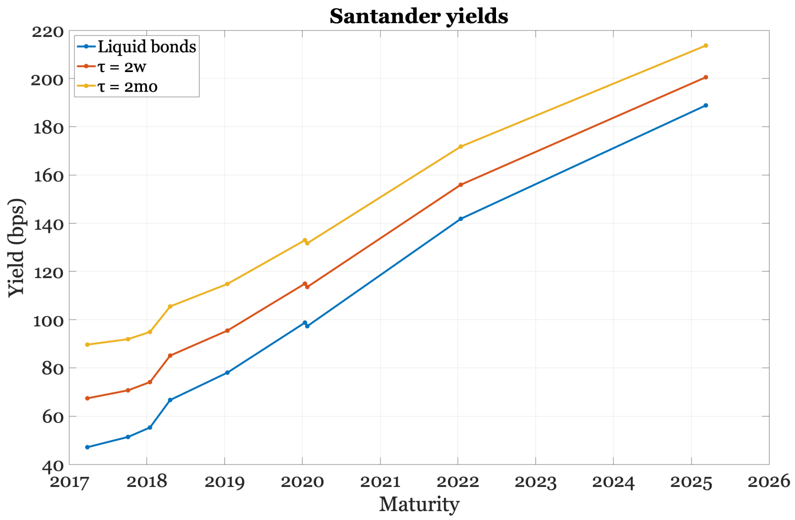


Table 3: Invoice prices and liquidity spreads for Banco Santander S.A. bonds, $\tau = 2\text{w}$.

Maturity	Liquid price	Illiquid price	Liquidity spread (bps)
27-Mar-2017	108.231	106.916 (-1.22%)	20
04-Oct-2017	112.647	110.817 (-1.62%)	19
15-Jan-2018	105.538	103.479 (-1.95%)	19
20-Apr-2018	102.153	99.661 (-2.44%)	18
14-Jan-2019	108.192	104.726 (-3.20%)	17
13-Jan-2020	104.178	99.400 (-4.59%)	16
24-Jan-2020	119.779	114.616 (-4.31%)	16
14-Jan-2022	105.925	98.065 (-7.42%)	14
10-Mar-2025	104.356	92.860 (-11.02%)	12



We can observe that $\mathcal{L}(T)$ decreases, although only slightly, as bond maturity increases.

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