1. (Ex 12.11) 证明求解热方程 $u_t=\nu u_{xx}$, $(x,t)\in(0,1)\times(0,T)$ 的 Crank-Nicolson 方[illegible]

$(I-\frac{k}{2}A)U^{n+1}=(I+\frac{k}{2}A)U^n+b^n$. 其中 $b^n=\frac{r}{2}[g_0(t_n)+g_0(t_{n+1}), 0, \dots, 0, g_1(t_n)+g_1(t_{n+1})]^T$.

PF: $\frac{U_i^{n+1}-U_i^n}{k}=\frac{1}{2}(f(U_i^n,t_n)+f(U_i^{n+1},t_{n+1}))$

$\Rightarrow U_i^{n+1}-\frac{k}{2}f(U_i^{n+1},t_{n+1})=U_i^n+\frac{k}{2}f(U_i^n,t_n)$

$\Rightarrow U^{n+1}-\frac{k}{2}(AU^{n+1}+g^{n+1})=U^n+\frac{k}{2}(AU^n+g^n)$

$\Rightarrow \quad (I-\frac{k}{2}A)U^{n+1}=(I+\frac{k}{2}A)U^n+\frac{k}{2}(g^n+g^{n+1})$.

$\frac{k}{2}(g^n+g^{n+1})=\frac{k\nu}{2h^2}[g_0(t_n)+g_0(t_{n+1}), 0, \dots, 0, g_1(t_n)+g_1(t_{n+1})]^T=b^n$.

2. (Ex 12.26) 利用单步法的稳定性函数证明 Lem 12.25.

PF: θ 方法的半离散格式单步法迭代公式为 $U^{n+1}=U^n+k(\theta f(U^{n+1})+(1-\theta)f(U^n))$.

将其应用于 $u'=\lambda u$ 得 $U^{n+1}=U^n+k(\theta\lambda U^{n+1}+(1-\theta)\lambda U^n)$.

$\therefore U^{n+1}=\frac{1+k(1-\theta)\lambda}{1-k\theta\lambda}U^n$, $R(z)=\frac{1+(1-\theta)z}{1-\theta z}$. $z=k\lambda$.

θ 方法~~无条件~~稳定 $\Leftrightarrow$ 单步法对半离散格式的全体特征值都绝对稳定.

注意到特征值均在负实轴上, 故只需考虑 $z<0$.

$\begin{cases}|R(z)|\le 1\\ z\le 0\end{cases} \Rightarrow z\le 0$ $(\frac{1}{2}\le\theta\le 1$ 时$)$; $\frac{-2}{1-2\theta}\le z\le 0$ $(0\le\theta<\frac{1}{2}$ 时$)$

故当 $\frac{1}{2}\le\theta\le 1$ 时, θ 方法无条件稳定;

当 $0\le\theta<\frac{1}{2}$ 时, θ 方法稳定 $\Leftrightarrow \frac{-2}{1-2\theta}\le-\frac{4\nu}{h^2}k\le 0 \Leftrightarrow 0\le k\le\frac{h^2}{2(1-2\theta)\nu}$.

3. (Ex 12.41) 证明网格函数 $u\in L^2(h\mathbb{Z})$ 会在一次 Fourier 变换和一次 Fourier 逆变换后恢复原状.

PF: $(\mathcal{F}u)(\xi)=\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}e^{-imh\xi}u_m h$.

$$\begin{aligned}(\mathcal{F}^{-1}\circ\mathcal{F}(u))_m&=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{imh\xi}\sum_{n\in\mathbb{Z}}e^{-inh\xi}u_n h\,d\xi\\&=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\sum_{n\in\mathbb{Z}}e^{i(m-n)h\xi}u_n h\,d\xi\\&=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{i(m-n)h\xi}u_n h\,d\xi\\&=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}u_m h\,d\xi\\&=u_m.\end{aligned}$$

$\therefore (\mathcal{F}^{-1}\circ\mathcal{F})(u)=u$.

4. (Ex 12.48) 利用 Von Neumann 分析证明 Lem 12.25.

PF: ~~$U_j^{n+1}=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\chi_j(h\xi)\hat{U}^n(\xi)d\xi=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ijh\xi}\hat{U}^{n+1}(\xi)d\xi$.~~

~~$\chi_j(h\xi)=$~~ 在迭代式 $-\theta rU_{j-1}^{n+1}+(1+2\theta r)U_j^{n+1}-\theta rU_{j+1}^{n+1}=(1-\theta)rU_{j-1}^n+[1-2(1-\theta)r]U_j^n+(1-\theta)rU_{j+1}^n$ 的两端同时 Fourier 变换得

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}(-\theta re^{i(j-1)h\xi}+(1+2\theta r)e^{ijh\xi}-\theta re^{i(j+1)h\xi})\hat{U}^{n+1}(\xi)d\xi$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}((1-\theta)re^{i(j-1)h\xi}+[1-2(1-\theta)r]e^{ijh\xi}+(1-\theta)re^{i(j+1)h\xi})\hat{U}^n(\xi)d\xi.$$

因此有 $\hat{U}^{n+1}=\hat{U}^n(\xi)g(h\xi)$. $g(h\xi)=\frac{(1-\theta)re^{i(j-1)h\xi}+[1-2(1-\theta)r]e^{ijh\xi}+(1-\theta)re^{i(j+1)h\xi}}{-\theta re^{i(j-1)h\xi}+(1+2\theta r)e^{ijh\xi}-\theta re^{i(j+1)h\xi}}=\frac{2(1-\theta)r\cos h\xi+1-2(1-\theta)r}{-2\theta r\cos h\xi+1+2\theta r}$.

$|g(h\xi)|\le 1+Ck$, $\forall\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]$?

由 $-1 \le \frac{\cdots}{-2\theta r\cos h\xi + 1 + 2\theta r} \le 1$ 得 $\begin{cases} \cdots \\ \cancel{\quad}(\cos h\xi - 2)r \le 0 \end{cases}$

$\because -1 \le \cos h\xi \le 1$，$\therefore$ 当 $\theta \in [\frac{1}{2}, 1]$ 时，不等式恒成立；$\theta \in [0, \frac{1}{2})$ 时，$r \le \frac{1}{2(1-2\theta)}$。即 $k \le \frac{h^2}{2(1-2\theta)\nu}$。

因此 Lem 12.25 得证.

使用 Von Neumann 分析证明稳定性避开了 Lem 12.21 的矩阵特征值，只需作 Fourier 变换，更具有普适应用价值.

5. (Ex 12.81) 证明 Beam-Warming 方法具有二阶时间、空间精度.

Pf：当 $a>0$ 时，方法的 LTE 为

$$T(x,t) = u(x_j, t_n+k) - u(x_j, t_n) - \frac{\mu}{2}\left(3u(x_j,t_n) - 4u(x_{j-1},t_n) + u(x_{j-2},t_n)\right) + \frac{\mu^2}{2}\left(u(x_j,t_n) - 2u(x_{j-1},t_n) + u(x_{j-2},t_n)\right)$$

$$= u + ku_t + \frac{k^2}{2}u_{tt} - u - \frac{\mu}{2}\left(3u - 4(u - hu_x + \frac{h^2}{2}u_{xx}) + (u - 2hu_x + 2h^2u_{xx})\right) + \frac{\mu^2}{2}\left(u - 2(u - hu_x + \frac{h^2}{2}u_{xx}) + (u - 2hu_x + 2h^2u_{xx})\right)\Big|_{(x_j,t_n)} + O(k^3+h^3)$$

$$= ku_t + \frac{k^2}{2}u_{tt} + \mu h u_x + \frac{\mu^2}{2}h^2u_{xx} + O(k^3+h^3)$$

$$= ku_t + \frac{k^2}{2}u_{tt} + aku_x + \frac{a^2k^2}{2}u_{xx} + O(k^3+h^3).$$

$\because u_t + au_x = 0$，$\therefore u_t = -au_x$. $u_{tt} = -au_{xt} = -a(u_t)_x = a^2u_{xx}$.

$\therefore T(x,t) = k^2u_{tt} + O(k^3+h^3)$. $\therefore$ Beam-Warming 方法具有二阶时空精度.

6. (Ex 12.82) 证明 Beam-Warming 方法在 $\mu \in [0,2]$ 和 $\mu \in [-2,0]$ 上收敛. 重现 $\mu = 0.8, 1.6, 2, 2.4$ 的收敛点图像.

Pf：~~当 a>0 时，方法可重写为~~ 当 $a>0$ 时，

$$B = -\frac{a}{2h}\begin{bmatrix} 3 & & \cdots & 1 & -4 \\ -4 & 3 & & \cdots & 1 \\ 1 & -4 & 3 & \cdots & \\ & 1 & -4 & 3 & \cdots \\ & & \ddots & \ddots & \ddots \\ & & 1 & -4 & 3 \\ & & & 1 & -4 & 3 \end{bmatrix} + \frac{a^2k}{2h^2}\begin{bmatrix} 1 & & \cdots & 1 & -2 \\ -2 & 1 & & \cdots & 1 \\ 1 & -2 & 1 & \cdots & \\ & 1 & -2 & 1 & \cdots \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 & 1 \end{bmatrix}$$

$$Bw^p = -\frac{a}{2h}\begin{bmatrix} 3 & & \cdots & 1 & -4 \\ -4 & 3 & & \cdots & 1 \\ 1 & -4 & 3 & \cdots & \\ & 1 & -4 & 3 & \cdots \\ & & \ddots & \ddots & \ddots \\ & & 1 & -4 & 3 \\ & & & 1 & -4 & 3 \end{bmatrix}\begin{bmatrix} e^{2\pi iph} \\ e^{2\pi i\cdot 2ph} \\ e^{2\pi i\cdot 3ph} \\ e^{2\pi i\cdot 4ph} \\ \vdots \\ e^{2\pi i\cdot mph} \\ e^{2\pi i\cdot(m+1)ph} \end{bmatrix} + \frac{a^2k}{2h^2}\begin{bmatrix} 1 & & \cdots & 1 & -2 \\ -2 & 1 & & \cdots & 1 \\ 1 & -2 & 1 & \cdots & \\ & 1 & -2 & 1 & \cdots \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} e^{2\pi iph} \\ e^{2\pi i\cdot 2ph} \\ e^{2\pi i\cdot 3ph} \\ e^{2\pi i\cdot 4ph} \\ \vdots \\ e^{2\pi i\cdot mph} \\ e^{2\pi i\cdot(m+1)ph} \end{bmatrix}$$

$$= \left(-\frac{a}{2h}(3 - 4e^{-2\pi iph} + e^{-4\pi iph}) + \frac{a^2k}{2h^2}(1 - 2e^{-2\pi iph} + e^{-4\pi iph})\right)w^p$$

$$\cancel{= (1-e^{-2\pi iph})\left(-\frac{a}{2h}(3-e^{-2\pi iph}) + \frac{a^2k}{2h^2}(1-e^{-2\pi iph})\right)w^p}.$$

$$\cancel{\lambda_p = (1-e^{-2\pi iph})}$$

$$z_p = k\lambda_p = -\frac{\mu}{2}(3 - 4e^{-2\pi iph} + e^{-4\pi iph}) + \frac{\mu^2}{2}(1 - 2e^{-2\pi iph} + e^{-4\pi iph}).$$

$$= e^{-2\pi iph}\left[(\mu^2 - 2\mu)(\cos(2\pi ph) - 1) - i\mu\sin(2\pi ph)\right].$$

当 $\mu \in [0,2]$ 时，有

$$1 + z_p = e^{-2\pi iph}\left[(\mu^2-2\mu)(\cos(2\pi ph)-1) - i\mu\sin(2\pi ph) + e^{2\pi iph}\right]$$

$$= e^{-2\pi iph}\left[(\mu-1)^2\cos(2\pi ph) + (2-\mu)\mu + i(1-\mu)\sin(2\pi ph)\right]$$

$$\overset{\eta := \mu-1}{=} e^{-2\pi iph}\left[\eta^2\cos(2\pi ph) + 1 - \eta^2 - i\eta\sin(2\pi ph)\right], \quad \eta \in [-1,1].$$

记 $c = \cos(2\pi ph) \in [-1,1]$，则 $\sin^2(2\pi ph) = 1 - c^2$.

$$|1+z_p|^2 = \left(\eta^2 c + 1 - \eta^2\right)^2 + \left(\eta\sin(2\pi ph)\right)^2$$

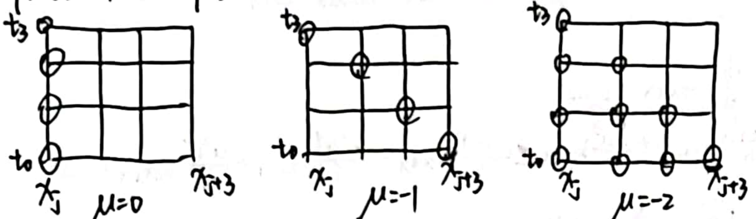
$= \eta^4 c^2 + 2\eta^2(1-\eta^2)c + (1-\eta^2)^2 + \eta^2(1-c^2)$

$= \eta^2(\eta^2-1)(c-1)^2 + 1 \le 1$. $\therefore$ Warm-Beamer 方法在 $\mu\in[0,2]$ 时绝对稳定.

$a<0$, $\mu\in[-2,0]$ 情形同理.

作图和代码见附录.

7. (Ex 12.86) 作出网格点 (x_j, t_3) 在上风法 $(a<0, \mu=0,-1,-2)$ 下的数值依赖域.

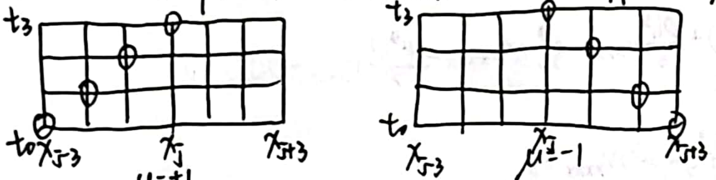


$\mu=0$: $U_j^{n+1} = U_j^n$

$\mu=-1$: $U_j^{n+1} = U_{j+1}^n$

$\mu=-2$: $U_j^{n+1} = 2U_{j+1}^n - U_j^n$

8. (Ex 12.88) 作出网格点 (x_j, t_3) 在 Lax-Wandroff 方法 $(\mu=+1,-1)$ 下的数值~~稳定~~依赖域.



$\mu=+1$: $U_j^{n+1} = U_{j-1}^n$

$\mu=-1$: $U_j^{n+1} = U_{j+1}^n$

9. (Ex 12.97) 证明 Leapfrog 方法的 modified equation 也是 $v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = 0$.

但若加入更高次项, Leapfrog 方法将加入 $\epsilon_f v_{xxxx}$, 而 Lax-Wendroff 方法将加入 $\epsilon_w v_{xxxx}$.

PF: 对于 Leapfrog 方法:

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} = -\frac{a}{2h}(U_{j+1}^n - U_{j-1}^n).$$

$$\frac{v(x_j, t_n+k) - v(x_j, t_n-k)}{2k} = -\frac{a}{2h}(v(x_j+h, t_n) - v(x_j-h, t_n)).$$

$$v_t + \frac{k^2}{6}v_{ttt} + \frac{k^4}{120}v_{ttttt} + O(k^6) = -a\left(v_x + \frac{h^2}{6}v_{xxx} + \frac{h^4}{120}v_{xxxxx}\right) + O(h^6).$$

$$v_t + av_x = -\frac{1}{6}(k^2 v_{ttt} + ah^2 v_{xxx}) - \frac{1}{120}(k^4 v_{ttttt} + ah^4 v_{xxxxx}) + O(h^6).$$

$$v_{ttt} = -av_{xtt} - \frac{1}{6}(k^2 v_{ttttt} + ah^2 v_{xxxtt}) + O(h^4)$$

$$= a^2 v_{xxt} + \frac{a}{6}(k^2 v_{xtttt} + ah^2 v_{xxxxt}) - \frac{1}{6}(k^2 a^5 v_{xxxxx} + h^2 a^3 v_{xxxxx}) + O(h^4)$$

$$= -a^3 v_{xxx} - \frac{a^2}{6}(k^2 v_{xxttt} + ah^2 v_{xxxxx}) + \frac{a}{6}(k^2 a^4 v_{xxxxx} + h^2 a^2 v_{xxxxx}) - \frac{1}{6}(k^2 a^5 v_{xxxxx} + h^2 a^3 v_{xxxxx}) + O(h^4)$$

$$= -a^3 v_{xxx} + \frac{1}{2}(k^2 a^5 - h^2 a^3)v_{xxxxx} + O(h^4).$$

$$v_{ttttt} = -a^5 v_{xxxxx} + O(h^2).$$

$$\therefore v_t + av_x + \frac{k^2}{6}\left(-a^3 v_{xxx} + \frac{1}{2}(k^2 a^5 - h^2 a^3)v_{xxxxx}\right) - \frac{k^4}{120}a^5 v_{xxxxx} + \frac{ah^2}{6}v_{xxx} + \frac{ah^4}{120}v_{xxxxx} + O(h^6) = 0.$$

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} + \frac{ah^4}{120}(1-10\mu^2+9\mu^4) + O(h^6) = 0.$$

故保留到三阶的方程为 $v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} \cancel{+\frac{ah^4}{120}(1-10\mu^2+9\mu^4)} = 0$

保留到五阶的方程为 $v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} + \frac{ah^4}{120}(1-10\mu^2+9\mu^4) = 0$.

对于 Lax-Wendroff 方法:

$$\cancel{\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}(U_{j+1}^n - U_{j-1}^n) + \frac{ak}{2h^2}(U_{j+1}^n - 2U_j^n)}$$

$$U_j^{n+1} - U_j^n = -\frac{\mu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{\mu^2}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

$$v(x_j, t_n+k) - v(x_j, t_n) = -\frac{\mu}{2}(v(x_j+h, t_n) - v(x_j-h, t_n)) + \frac{\mu^2}{2}(v(x_j+h, t_n) - 2v(x_j, t_n) + v(x_j-h, t_n))$$

$$kv_t + \frac{k^2}{2}v_{tt} + \frac{k^3}{6}v_{ttt} + \frac{k^4}{24}v_{tttt} + O(k^5) = -\mu\left(hv_x + \frac{h^3}{6}v_{xxx} + O(h^5)\right) + \mu^2\left(\frac{h^2}{2}v_{xx} + \frac{h^4}{24}v_{xxxx} + O(h^6)\right).$$

$v_{tttt} = a^4 v_{xxxx} + O(h)$.

$v_{ttt} = -a v_{xtt} - \frac{k}{2} v_{tttt} + \frac{a\mu h}{2} v_{ttxx} + O(h^2)$

$= -a\left(-a v_{xxt} - \frac{k}{2} v_{xttt} + \frac{a\mu h}{2} v_{xxxt}\right) - \cancel{\frac{k}{2}a^4 v_{xxxx}} + \cancel{\frac{a^3\mu h}{2} v_{xxxx}} + O(h^2)$

$= a^2\left(-a v_{xxx} - \frac{k}{2} v_{xxtt} + \frac{a\mu h}{2} v_{xxxx}\right) + \cancel{\frac{k}{2}a^4 v_{xxxx}} + \cancel{\frac{a^3\mu h}{2} v_{xxxx}} - \cancel{\frac{k}{2}a^4 v_{xxxx} + \frac{a^3\mu h}{2} v_{xxxx}} + O(h^2)$

$= -a^3 v_{xxx} + O(h^2)$.

$v_{tt} = -a v_{xt} - \frac{k}{2} v_{ttt} + \frac{a\mu h}{2} v_{xxt} - \frac{k^2}{6} v_{tttt} - \frac{ah^2}{6} v_{txxx} + O(h^3)$

$= -a\left(-a v_{xx} - \frac{k}{2} v_{xtt} + \frac{a\mu h}{2} v_{xxx} - \frac{k^2}{6} v_{xttt} - \frac{ah^2}{6} v_{xxxx}\right) + \frac{ka^3}{2} v_{xxx} - \frac{\mu h a^2}{2} v_{xxx} - \frac{k^2 a^4}{6} v_{xxxx} + \frac{a^2h^2}{6} v_{xxxx} + O(h^3)$

$= a^2 v_{xx} + \cancel{\frac{ka^3}{2} v_{xxx}} - \cancel{\frac{\mu h a^2}{2} v_{xxx}} - \frac{k^2 a^4}{6} v_{xxxx} + \frac{a^2h^2}{6} v_{xxxx}) + O(h^3)$

$= a^2 v_{xx} + \frac{a^2h^2}{3}(1-\mu^2) v_{xxxx} + O(h^3)$.

∴ $v_t + a v_x + \frac{k}{2}\left(\cancel{a^2 v_{xx}} + \frac{1}{3}(1-\mu^2) v_{xxxx}\right) - \cancel{\frac{\mu h a}{2} v_{xx}} + \frac{k^2}{6}(-a^3 v_{xxx}) + \frac{ah^2}{6} v_{xxx} + \frac{k^3}{24} a^4 v_{xxxx} - \cancel{\frac{a\mu^3 h^3}{24}} v_{xxxx} + O(h^5) = 0$.

$v_t + a v_x + \frac{ah^2}{6}(1-\mu^2) v_{xxx} + \frac{ah^3}{8}(\mu-\mu^3) v_{xxxx} + O(h^5) = 0$.

故保留到四阶的方程为 $v_t + a v_x + \frac{ah^2}{6}(1-\mu^2) v_{xxx} + \frac{ah^2}{8}(\mu-\mu^3) v_{xxxx} = 0$.

10. (Ex 12.98) 证明 Beam-Warming 方法的 modified equation 是 $v_t + a v_x + \frac{ah^2}{6}(-2+3\mu-\mu^2) v_{xxx} = 0$.

因此有 $C_p(\xi) = a + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^2$, $C_g(\xi) = a + \frac{ah^2}{2}(\mu-1)(\mu-2)\xi^2$.

PF: $U_j^{n+1} = U_j^n - \frac{\mu}{2}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{\mu^2}{2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$

$v(x_j, t_n+k) - v(x_j, t_n) = -\frac{\mu}{2}\left(3v(x_j,t_n) - 4v(x_j-h,t_n) + v(x_j-2h,t_n)\right) + \frac{\mu^2}{2}\left(v(x_j,t_n) - 2v(x_j-h,t_n) + v(x_j-2h,t_n)\right)$

$k v_t + \frac{k^2}{2} v_{tt} + \frac{k^3}{6} v_{ttt} + O(k^4) = -\frac{\mu}{2}\left(2h v_x - \frac{2}{3}h^3 v_{xxx}\right) + \frac{\mu^2}{2}\left(h^2 v_{xx} - h^3 v_{xxx}\right)$

$v_{ttt} = -a^3 v_{xxx} + O(h)$.

$v_{tt} = -a v_{xt} - \frac{k}{2} v_{ttt} + \frac{a\mu h}{2} v_{xxt} + O(h^2)$

$= -a\left(-a v_{xx} - \frac{k}{2} v_{xtt} + \frac{a\mu h}{2} v_{xxx}\right) + \cancel{\frac{k}{2}a^3 v_{xxx}} - \cancel{\frac{a^2\mu h}{2} v_{xxx}} + O(h^2)$

$= a^2 v_{xx} + O(h^2)$.

$v_t + a v_x + \cancel{\frac{k}{2}a^2 v_{xx}} + \frac{1}{3}ah^2 v_{xxx} - \frac{k^2}{6}a^3 v_{xxx} - \cancel{\frac{1}{2}\mu a h v_{xx}} + \frac{1}{2}ah^2\mu v_{xxx} + O(h^3) = 0$.

$v_t + a v_x + \frac{1}{6}ah^2(-2+3\mu-\mu^2) v_{xxx} + O(h^3) = 0$.

故保留到四阶的方程为 $v_t + a v_x + \frac{1}{6}ah^2(-2+3\mu-\mu^2) v_{xxx} = 0$.

$\omega(\xi) = a\xi + \frac{ah^2}{6}(2-3\mu+\mu^2) = a\xi + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^3$. $C_p(\xi) = \frac{\omega(\xi)}{\xi} = a + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^2$.

$C_g(\xi) = \frac{d\omega(\xi)}{d\xi} = a + \frac{ah^2}{2}(\mu-1)(\mu-2)$.

当 $1<\mu<2$ 时, $|C_p(\xi)| < |a|$, 而当 $0<\mu<1$ 时, $|C_p(\xi)| > |a|$. 因此 Beam-Warming 方法的振动比实际早.

11. (Ex 12.99) 当 $\mu=1$ 时, Lax-Wendroff 方法和 leapfrog 方法的数值结果如何?

PF: $\mu=1$ 时, Lax-Wendroff 方法退化为 $U_j^{n+1} = U_{j-1}^n$. ~~leap~~ 此时格点处的解均为精确解.

leapfrog 方法退化为 $U_j^{n+1} = U_j^{n-1} - U_{j+1}^n + U_{j-1}^n$. 归纳可知, 假设 n 步前解均为精确解即 $U_j^n = U_{j-n}^0$ 时,

n+1 步也是精确解, $U_j^{n+1} = U_{j-n+1}^0 - U_{j+1-n}^0 + U_{j+n}^0 = U_{j-1-n}^0$.

12.(Ex 12.101) 利用 Von Neumann 分析 Lax-Friedrichs 方法导出 $g(\xi)=\cos(\xi h)-\mu i\sin(\xi h)$.
对于 μ 取何值时方法稳定?

PF: $U_j^{n+1}=\frac{1}{2}(U_{j+1}^n+U_{j-1}^n)-\frac{\mu}{2}(U_{j+1}^n-U_{j-1}^n)$.

两端同时作 Fourier 变换得

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ijh\xi}\hat{u}^{n+1}(\xi)d\xi=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\left[\frac{1}{2}(e^{i(j+1)h\xi}+e^{i(j-1)h\xi})-\frac{\mu}{2}(e^{i(j+1)h\xi}-e^{i(j-1)h\xi})\right]\hat{u}^n(\xi)d\xi$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ijh\xi}(\cos h\xi-\mu i\sin h\xi)\hat{u}^n(\xi)d\xi.$$

$g(\xi)=\cos h\xi-\mu i\sin h\xi$.

$|g(\xi)|\le 1\Rightarrow\cos^2 h\xi+\mu^2\sin^2 h\xi\le 1\Rightarrow 1+(\mu^2-1)\sin^2 h\xi\le 1\Rightarrow|\mu|\le 1$.

13.(Ex 12.102) 利用 Von Neumann 分析 Lax-Wendroff 方法导出 $g(\xi)=1-2\mu^2\sin^2\frac{\xi h}{2}-i\mu\sin\xi h$.
μ 取何值时方法稳定?

PF: $U_j^{n+1}=U_j^n-\frac{\mu}{2}(U_{j+1}^n-U_{j-1}^n)+\frac{\mu^2}{2}(U_{j+1}^n-2U_j^n+U_{j-1}^n)$.

两端同时作 Fourier 变换得

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ijh\xi}\hat{u}^{n+1}(\xi)d\xi=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ijh\xi}(1-\mu i\sin h\xi-2\mu^2\sin^2\frac{h\xi}{2})\hat{u}^n(\xi)d\xi.$$

$g(\xi)=1-\mu i\sin h\xi-2\mu^2\sin^2\frac{h\xi}{2}$.

$|g(\xi)|\le 1\Rightarrow(1-2\mu^2\sin^2\frac{h\xi}{2})^2+\mu^2\sin^2 h\xi\le 1$.

$\Rightarrow(1-\mu^2(1-\cos h\xi))^2+\mu^2\sin^2 h\xi\le 1$

$\Rightarrow\mu^2+\mu^4+2(\mu^2-\mu^4)\cos h\xi+(\mu^4-\mu^2)\cos^2 h\xi\le 0$

$\Rightarrow(\mu^4-\mu^2)(1-\cos h\xi)^2\le 0$.

$\Rightarrow\mu^4-\mu^2\le 0\Rightarrow|\mu|\le 1$.