NumPDE Homework #3 Exercises for 11.1 – 11.3. 樊睿 3200102142

1. (Ex 11.9) 证明 $\|T\| = \inf\{M\geq 0 : \forall x\in\mathbb{F}^n,\ \|Tx\|\leq M\|x\|\}$ 是 $\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$ 上的一个范数.

PF: (1) 正定性. $\forall T\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$, $\|T\|\geq 0$.

(2) 分离性. 若 $\|T\|=0$, 则 $\forall x\in\mathbb{F}^n$, $\|Tx\|\leq 0$. $\therefore \|Tx\|=0$.

$\therefore \forall j=1\dots n$, $\|Te_j\|=0$. $Te_j=0$. $\therefore \forall i=1\dots m, j=1\dots n$, $\mathrm{Mat}(T)_{i,j}=0$.

$\therefore \mathrm{Mat}(T)=O_{n\times m}$. $T=0$.

(3) 齐次性. $\forall T\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$, $\alpha\in\mathbb{F}$,

$$\begin{aligned}\|\alpha T\| &= \inf\{M\geq 0 : \forall x\in\mathbb{F}^n,\ \|\alpha Tx\|\leq M\|x\|\}\\ &= \inf\{M\geq 0 : \forall x\in\mathbb{F}^n,\ |\alpha|\,\|Tx\|\leq M\|x\|\}\\ &= |\alpha|\cdot\inf\{M\geq 0 : \forall x\in\mathbb{F}^n,\ \|Tx\|\leq M\|x\|\}\\ &= |\alpha|\,\|T\|.\end{aligned}$$

(4) 三角不等式. $\forall T_1, T_2\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$,

若 $\forall x\in\mathbb{F}^n$, $\|T_1x\|\leq M_1\|x\|$, ~~且~~ $\forall x\in\mathbb{F}^n$, $\|T_2x\|\leq M_2\|x\|$,

则 $\|(T_1+T_2)(x)\|\leq\|T_1x\|+\|T_2x\|\leq(M_1+M_2)\|x\|$.

$\therefore M_1+M_2\in\{M\geq 0 : \forall x\in\mathbb{F}^n,\ \|(T_1+T_2)x\|\leq M\|x\|\}$.

$\therefore \|T_1+T_2\| = \inf\{M\geq 0 : \forall x\in\mathbb{F}^n,\ \|(T_1+T_2)x\|\leq M\|x\|\}\leq\|T_1\|+\|T_2\|$.

2. (Ex 11.13) 证明 若定义 $d(T,S)=\|T-S\|$, 则 $\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$ 在度量 d 下成为度量空间.

PF: ~~由~~ $\forall T, S, U\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$.

(1) $\|T-S\|\geq 0$.

(2) $\|T-S\|=0 \Leftrightarrow T-S=0 \Leftrightarrow T=S$.

(3) $d(T,S)=\|T-S\|=\|T-U+U-S\|\leq\|T-U\|+\|U-S\|=d(T,U)+d(U,S)$.

$\therefore \mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$ 在 d 下成为度量空间.

3. (Ex 11.16) 证明 Frobenius 范数 $|T|=\left(\sum_{j=1}^n\|Te_j\|^2\right)^{\frac12}$ 是 $\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$ 上的一个范数.

PF: (1) 正定性. $\forall T\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$, $|T|\geq 0$.

(2) 分离性. 若 $|T|=0$, 则 $\forall j=1\dots n$, $Te_j=0$. $\therefore \mathrm{Mat}(T)=O_{m\times n}$. $T=0$.

(3) 齐次性. $\forall T\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$, $\alpha\in\mathbb{F}$,

$$|\alpha T|=\left(\sum_{j=1}^n\|\alpha Te_j\|^2\right)^{\frac12}=\left(\sum_{j=1}^n|\alpha|^2\|Te_j\|^2\right)^{\frac12}=|\alpha|\left(\sum_{j=1}^n\|Te_j\|^2\right)^{\frac12}=|\alpha|\,|T|.$$

(4) 三角不等式. $\forall T_1, T_2\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$,

~~$|T_1+T_2|=\left(\sum_{j=1}^n\|(T_1+T_2)e_j\|^2\right)^{\frac12}$~~ $|T|=\left(\sum_{j=1}^n\|Te_j\|^2\right)^{\frac12}=\left(\sum_{i=1}^m\sum_{j=1}^n|a_{ij}|^2\right)^{\frac12}$. $a_{ij}=\mathrm{Mat}(T)_{i,j}$.

~~令~~ 设 $a_{ij}=\mathrm{Mat}(T_1)_{i,j}$, $b_{ij}=\mathrm{Mat}(T_2)_{i,j}$, 则

$$\begin{aligned}|T_1+T_2| &= \left(\sum_{i=1}^m\sum_{j=1}^n(a_{ij}+b_{ij})^2\right)^{\frac12}=\left(\sum_{i=1}^m\sum_{j=1}^n(a_{ij}^2+b_{ij}^2+2a_{ij}b_{ij})\right)^{\frac12}\leq\left(\sum_{i=1}^m\sum_{j=1}^n(a_{ij}^2+b_{ij}^2)+2\left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2\cdot\sum_{i=1}^m\sum_{j=1}^n b_{ij}^2\right)^{\frac12}\right)^{\frac12}\\ &= \left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2\right)^{\frac12}+\left(\sum_{i=1}^m\sum_{j=1}^n b_{ij}^2\right)^{\frac12}=|T_1|+|T_2|.\end{aligned}$$

4. (Ex 11.20) 证明: 若 $S\in\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$, $T\in\mathcal{L}(\mathbb{F}^m,\mathbb{F}^k)$, 则 $|TS|\leq|T|\,|S|$.

PF: 设 $a_{ij}=\mathrm{Mat}(T)_{ij}$, $b_{ij}=\mathrm{Mat}(S)_{ij}$, 则

$$|TS|=\left(\sum_{i=1}^k\sum_{j=1}^n\left(\sum_{l=1}^m b_{il}a_{lj}\right)^2\right)^{\frac12}\leq\left(\sum_{i=1}^k\sum_{j=1}^n\left(\sum_{l=1}^m b_{il}^2\sum_{l=1}^m a_{lj}^2\right)\right)^{\frac12}=\left(\sum_{i=1}^k\sum_{l=1}^m b_{il}^2\sum_{j=1}^n\sum_{l=1}^m a_{lj}^2\right)^{\frac12}=|T|\,|S|.$$

5. (Ex 11.36) 证明 $\det e^X = e^{\mathrm{Trace}\,X}$.

PF: 设 $X = P^{-1}JP$, P 为~~对称~~正交阵, J 为若当标准型, 则 $e^X = e^{P^{-1}JP} = P^{-1}e^J P$.

$$\det e^X = \det(P^{-1}e^J P) = \det e^J \xlongequal{J\text{为上三角阵}} \cancel{\det e}\prod_{i=1}^{n} e^{\lambda_i} = e^{\sum_{i=1}^{n}\lambda_i} = e^{\mathrm{Trace}\,X}.$$

6. (Ex 11.50) 证明 IVP ~~的解对初值~~满足 Lipschitz 条件时, 解对~~x~~初值不过度敏感, 即

若 v, w 是同一个 IVP 的解, $v(a)=v_0$, $w(a)=w_0$, 则 $\|v(t)-w(t)\| \le \|v_0-w_0\| \exp\{L(t-a)\}$.

PF: 设 $\begin{cases} v' = f(v,t), & v(a)=v_0 \\ w' = f(w,t), & w(a)=w_0 \end{cases}$, 则 $\begin{cases} v(t) = v_0 + \int_0^t f(v(s),s)\,ds \\ w(t) = w_0 + \int_0^t f(w(s),s)\,ds \end{cases}$.

$$\begin{aligned}\|v(t)-w(t)\| &= \|v_0 - w_0 + \int_a^t \big(f(v(s),s) - f(w(s),s)\big)\,ds\| \\ &\le \|v_0-w_0\| + \int_a^t \|f(v(s),s) - f(w(s),s)\|\,ds \\ &\le \|v_0-w_0\| + \int_a^t L\|v(s)-w(s)\|\,ds\end{aligned}$$

由 Gronwall 不等式得 $\|v(t)-w(t)\| \le \|v_0-w_0\| \exp\{L(t-a)\}$.

7. (Ex 11.100) 对梯形法和中点法计算前五个系数 C_j.

Sol: 梯形法: $U^{n+1} = U^n + \frac{k}{2}\big(f(U^n,t_n) + f(U^{n+1},t_{n+1})\big)$.

$s=1$, $\alpha = \{-1, 1\}$, $\beta = \{\frac{1}{2}, \frac{1}{2}\}$.

$C_0 = \alpha_0 + \alpha_1 = 0$,

$C_1 = -\beta_0 + \alpha_1 - \beta_1 = 0$,

~~$C_2 = \frac{1}{2}\alpha_1 - 2\beta_1 =$~~

$C_2 = \frac{1}{2}\alpha_1 - \beta_1 = 0$,

$C_3 = \frac{1}{6}\alpha_1 - \frac{1}{2}\beta_1 = -\frac{1}{12}$,

$C_4 = \frac{1}{24}\alpha_1 - \frac{1}{6}\beta_1 = -\frac{1}{24}$.

中点法: $U^{n+1} = U^{n-1} + 2kf(U^n, t_n)$.

$s=2$, $\alpha = \{-1, 0, 1\}$, $\beta = \{0, 2, 0\}$.

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = 0$.

$C_1 = -\beta_0 + \alpha_1 - \beta_1 + 2\alpha_2 = 0$.

$C_2 = \frac{1}{2}\alpha_1 - \beta_1 + 2\alpha_2 \cancel{-2\beta_2} = 0$.

$C_3 = \frac{1}{6}\alpha_1 - \frac{1}{2}\beta_1 + \frac{4}{3}\alpha_2 = \frac{1}{3}$.

$C_4 = \frac{1}{24}\alpha_1 - \frac{1}{6}\beta_1 + \frac{2}{3}\alpha_2 = \frac{1}{3}$.

8. (Ex 11.102) 利用特征多项式~~解释~~表示 $\|\mathcal{L}u(t_n)\| = O(k^3)$ 的条件.

Sol: $\|\mathcal{L}u(t_n)\| = O(k^3)$ 要求 $C_0 = C_1 = C_2 = 0$.

即 $C_0 = \sum_{j=0}^{s}\alpha_j = \rho(1) = 0$,

$C_1 = \sum_{j=0}^{s} j\alpha_j - \sum_{j=0}^{s}\beta_j = \rho'(1) - \sigma(1) = 0$.

$C_2 = \sum_{j=0}^{s}\frac{j^2}{2}\alpha_j - \sum_{j=0}^{s} j\beta_j \cancel{= \rho''(1)}$

$= \frac{1}{2}\sum_{j=0}^{s} j(j-1)\alpha_j + \frac{1}{2}\sum_{j=0}^{s} j\alpha_j - \sum_{j=0}^{s} j\beta_j = \frac{1}{2}\rho''(1) + \frac{1}{2}\rho'(1) - \sigma'(1) = 0$.

~~9. (Ex 11.103) 导出下表中 LMM 的待定系数.~~

~~Sol: (1) Adams-Bashforth: $\alpha_s = 1, \alpha_{s-1} = -1, \alpha_{s-2} = \dots = \alpha_0 = 0$.~~

~~① $s=1, p=1$. $C_0 = C_1 = 0 \Rightarrow \alpha_1 - \beta_0 - \beta_1 = 0 \Rightarrow \beta_0 = 0$.~~

~~② $s=2, p=2$. $C_0 = C_1 = C_2 = 0 \Rightarrow \begin{cases}\alpha_1 - \beta_0 - \beta_1 = 0 \\ \alpha_1 + 2 \\ \frac{1}{2}\alpha_1 - \beta_1 = 0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_0\end{bmatrix} = \begin{bmatrix}1 \\ \frac{1}{2}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_1 \\ \beta_0\end{bmatrix} =$~~

9. (Ex 11.103) 导出表中 LMM 的待定系数.

Sol: (1) Adams Bashforth: $\alpha_s=1, \alpha_{s-1}=-1, \alpha_{s-2}=\dots=\alpha_0=0$.

① $s=1, p=1$. $C_0=C_1=0 \Rightarrow \alpha_1-\beta_0=0 \Rightarrow \beta_0=1$.

② $s=2, p=2$. $C_0=C_1=C_2=0 \Rightarrow \begin{cases}(\alpha_1+2\alpha_2)-(\beta_0+\beta_1)=0 \\ (\frac{1}{2}\alpha_1+2\alpha_2)-\beta_1=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{3}{2}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{3}{2} \\ -\frac{1}{2}\end{bmatrix}$.

③ $s=3, p=3$. $C_0=C_1=C_2=C_3=0 \Rightarrow \begin{cases}(\cancel{\alpha_1}+2\alpha_2+3\alpha_3)-(\beta_0+\beta_1+\beta_2)=0 \\ (\cancel{\frac{1}{2}\alpha_1}+2\alpha_2+\frac{9}{2}\alpha_3)-(\beta_1+2\beta_2)=0 \\ (\frac{4}{3}\alpha_2+\frac{9}{2}\alpha_3)-(\frac{1}{2}\beta_1+2\beta_2)=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}\beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{5}{2} \\ \frac{19}{6}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{23}{12} \\ -\frac{16}{12} \\ \frac{5}{12}\end{bmatrix}$.

④ $s=4, p=4$. $C_0=C_1=C_2=C_3=C_4=0 \Rightarrow \begin{cases}(3\alpha_3+4\alpha_4)-(\beta_0+\beta_1+\beta_2+\beta_3)=0 \\ (\frac{9}{2}\alpha_3+8\alpha_4)-(\beta_1+2\beta_2+3\beta_3)=0 \\ (\frac{9}{2}\alpha_3+\frac{32}{3}\alpha_4)-(\frac{1}{2}\beta_1+2\beta_2+\frac{9}{2}\beta_3)=0 \\ (\frac{27}{8}\alpha_3+\frac{32}{3}\alpha_4)-(\frac{1}{6}\beta_1+\frac{4}{3}\beta_2+\frac{9}{2}\beta_3)=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ \frac{9}{2} & 2 & \frac{1}{2} & 0 \\ \frac{9}{2} & \frac{4}{3} & \frac{1}{6} & 0\end{bmatrix}\begin{bmatrix}\beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{7}{2} \\ \frac{37}{6} \\ \frac{175}{24}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{55}{24} \\ -\frac{59}{24} \\ \frac{37}{24} \\ -\frac{9}{24}\end{bmatrix}$.

(2) Adams-Moulton: $\alpha_s=1, \alpha_{s-1}=-1, \alpha_{s-2}=\dots=\alpha_0=0$.

① $s=1, p=1$. $C_0=C_1=0 \Rightarrow \alpha_1-\beta_1=0 \Rightarrow \beta_1=1$.

② $s=1, p=2$. $C_0=C_1=C_2=0 \Rightarrow \begin{cases}\alpha_1-(\beta_0+\beta_1)=0. \\ \frac{1}{2}\alpha_1-\beta_1=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{1}{2}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{1}{2} \\ \frac{1}{2}\end{bmatrix}$.

③ $s=2, p=3$. $C_0=C_1=C_2=C_3=0 \Rightarrow \begin{cases}(\alpha_1+2\alpha_2)-(\beta_0+\beta_1+\beta_2)=0 \\ (\frac{1}{2}\alpha_1+2\alpha_2)-(\beta_1+2\beta_2)=0 \\ (\frac{1}{6}\alpha_1+\frac{4}{3}\alpha_2)-(\frac{1}{2}\beta_1+2\beta_2)=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}\beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{3}{2} \\ \frac{7}{6}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{5}{12} \\ \frac{8}{12} \\ -\frac{1}{12}\end{bmatrix}$.

④ $s=3, p=4$. $C_0=C_1=C_2=C_3=C_4=0 \Rightarrow \begin{cases}(2\alpha_2+3\alpha_3)-(\beta_0+\beta_1+\beta_2+\beta_3)=0 \\ (2\alpha_2+\frac{9}{2}\alpha_3)-(\beta_1+2\beta_2+3\beta_3)=0 \\ (\frac{4}{3}\alpha_2+\frac{9}{2}\alpha_3)-(\frac{1}{2}\beta_1+2\beta_2+\frac{9}{2}\beta_3)=0 \\ (\frac{2}{3}\alpha_2+\frac{27}{8}\alpha_3)-(\frac{1}{6}\beta_1+\frac{4}{3}\beta_2+\frac{9}{2}\beta_3)=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ \frac{9}{2} & 2 & \frac{1}{2} & 0 \\ \frac{9}{2} & \frac{4}{3} & \frac{1}{6} & 0\end{bmatrix}\begin{bmatrix}\beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{5}{2} \\ \frac{19}{6} \\ \frac{65}{24}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{9}{24} \\ \frac{19}{24} \\ -\frac{5}{24} \\ \frac{1}{24}\end{bmatrix}$.

⑤ $s=4, p=5$. $C_0=C_1=C_2=C_3=C_4=C_5=0 \Rightarrow \begin{cases}(3\alpha_3+4\alpha_4)-(\beta_0+\beta_1+\beta_2+\beta_3+\beta_4)=0 \\ (\frac{9}{2}\alpha_3+8\alpha_4)-(\beta_1+2\beta_2+3\beta_3+4\beta_4)=0 \\ (\frac{9}{2}\alpha_3+\frac{32}{3}\alpha_4)-(\frac{1}{2}\beta_1+2\beta_2+\frac{9}{2}\beta_3+8\beta_4)=0 \\ (\frac{27}{8}\alpha_3+\frac{32}{3}\alpha_4)-(\frac{1}{6}\beta_1+\frac{4}{3}\beta_2+\frac{9}{2}\beta_3+\frac{32}{3}\beta_4)=0 \\ (\frac{81}{40}\alpha_3+\frac{128}{15}\alpha_4)-(\frac{1}{24}\beta_1+\frac{2}{3}\beta_2+\frac{27}{8}\beta_3+\frac{32}{3}\beta_4)=0\end{cases} \Rightarrow \begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 8 & \frac{9}{2} & 2 & \frac{1}{2} & 0 \\ \frac{32}{3} & \frac{9}{2} & \frac{4}{3} & \frac{1}{6} & 0 \\ \frac{32}{3} & \frac{27}{8} & \frac{2}{3} & \frac{1}{24} & 0\end{bmatrix}\begin{bmatrix}\beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}1 \\ \frac{7}{2} \\ \frac{37}{6} \\ \frac{175}{24} \\ \frac{781}{120}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0\end{bmatrix}=\begin{bmatrix}\frac{251}{720} \\ \frac{646}{720} \\ -\frac{264}{720} \\ \frac{106}{720} \\ -\frac{19}{720}\end{bmatrix}$

(3) Backward differentiation: $\beta_{s-1}=\dots=\beta_0=0$. 方程组欠定, 故指定 $\alpha_s=1$.

① $s=1, p=1$. $C_0=C_1=0 \Rightarrow \begin{cases}\alpha_0+\alpha_1=0 \\ \alpha_1-\beta_1=0\end{cases} \Rightarrow \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1 \\ \beta_1\end{bmatrix}=\begin{bmatrix}1 \\ 1\end{bmatrix}$. $\begin{bmatrix}\alpha_1 \\ \beta_1\end{bmatrix}=\begin{bmatrix}-1 \\ 1\end{bmatrix}$.

② $s=2, p=2$. $C_0=C_1=C_2=0 \Rightarrow \begin{cases}\alpha_0+\alpha_1+\alpha_2=0 \\ (\alpha_1+2\alpha_2)-\beta_2=0 \\ (\frac{1}{2}\alpha_1+2\alpha_2)-2\beta_2=0\end{cases} \Rightarrow \begin{bmatrix}-1 & -1 & 0 \\ -1 & 0 & 1 \\ -\frac{1}{2} & 0 & 2\end{bmatrix}\begin{bmatrix}\alpha_1 \\ \alpha_0 \\ \beta_1\end{bmatrix}=\begin{bmatrix}1 \\ 2 \\ 2\end{bmatrix}$. $\begin{bmatrix}\alpha_1 \\ \alpha_0 \\ \beta_1\end{bmatrix}=\begin{bmatrix}-\frac{4}{3} \\ \frac{1}{3} \\ \frac{2}{3}\end{bmatrix}$.

③ $s=3, p=3$. $C_0=C_1=C_2=C_3=0 \Rightarrow \begin{cases}\alpha_0+\alpha_1+\alpha_2+\alpha_3=0 \\ (\alpha_1+2\alpha_2+3\alpha_3)-\beta_3=0 \\ (\frac{1}{2}\alpha_1+2\alpha_2+\frac{9}{2}\alpha_3)-3\beta_3=0 \\ (\frac{1}{6}\alpha_1+\frac{4}{3}\alpha_2+\frac{9}{2}\alpha_3)-\frac{9}{2}\beta_3=0.\end{cases} \Rightarrow \begin{bmatrix}-1 & -1 & -1 & 0 \\ -2 & -1 & 0 & 1 \\ -2 & -\frac{1}{2} & 0 & 3 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{9}{2}\end{bmatrix}\begin{bmatrix}\alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_3\end{bmatrix}=\begin{bmatrix}1 \\ 3 \\ \frac{9}{2} \\ \frac{9}{2}\end{bmatrix} \Rightarrow \begin{bmatrix}\alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_3\end{bmatrix}=\begin{bmatrix}-\frac{18}{11} \\ \frac{9}{11} \\ -\frac{2}{11} \\ \frac{6}{11}\end{bmatrix}$.

④ $s=4, p=4$. $C_0=C_1=C_2=C_3=C_4=0 \Rightarrow \begin{cases}\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4=0 \\ (\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4)-\beta_4=0 \\ (\frac{1}{2}\alpha_1+2\alpha_2+\frac{9}{2}\alpha_3+8\alpha_4)-4\beta_4=0 \\ (\frac{1}{6}\alpha_1+\frac{4}{3}\alpha_2+\frac{9}{2}\alpha_3+\frac{32}{3}\alpha_4)-8\beta_4=0. \\ (\frac{1}{24}\alpha_1+\frac{2}{3}\alpha_2+\frac{27}{8}\alpha_3+\frac{32}{3}\alpha_4)-\frac{32}{3}\beta_4=0.\end{cases} \Rightarrow \begin{bmatrix}-1 & -1 & -1 & -1 & 0 \\ -3 & -2 & -1 & 0 & 1 \\ -\frac{9}{2} & -2 & -\frac{1}{2} & 0 & 4 \\ -\frac{9}{2} & -\frac{4}{3} & -\frac{1}{6} & 0 & 8 \\ -\frac{27}{8} & -\frac{2}{3} & -\frac{1}{24} & 0 & \frac{32}{3}\end{bmatrix}\begin{bmatrix}\alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_4\end{bmatrix}=\begin{bmatrix}1 \\ 4 \\ 8 \\ \frac{32}{3} \\ \frac{32}{3}\end{bmatrix} \Rightarrow \begin{bmatrix}\alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_4\end{bmatrix}=\begin{bmatrix}-\frac{48}{25} \\ \frac{36}{25} \\ -\frac{16}{25} \\ \frac{3}{25} \\ \frac{12}{25}\end{bmatrix}$

~~10. 对三阶 BDF, 求出其特~~

10. (Ex 11.108) 对三阶 BDF, 求出其特征多项式并利用 Thm 11.106 证明它的精度为3.

PF: 由上一题, 三阶 BDF 公式为 $u^{n+3}-\frac{18}{11}u^{n+2}+\frac{9}{11}u^{n+1}-\frac{2}{11}u^{n}=\frac{6}{11}kf(u^{n+3},t_{n+3})$.

特征多项式为 $\rho(\zeta)=\zeta^3-\frac{18}{11}\zeta^2+\frac{9}{11}\zeta-\frac{2}{11}$, $\sigma(\zeta)=\frac{6}{11}\zeta^3$.

$$\frac{\rho(z)}{\sigma(z)}=\frac{z^3-\frac{18}{11}z^2+\frac{9}{11}z-\frac{2}{11}}{\frac{6}{11}z^3}=\frac{(z-1)^3+\frac{15}{11}(z-1)^2+\frac{6}{11}(z-1)}{\frac{6}{11}(z-1+1)^3}=\frac{11}{6}\left((z-1)^3+\frac{15}{11}(z-1)^2+\frac{6}{11}(z-1)\right)\left(1-3(z-1)+6(z-1)^2-10(z-1)^3+O((z-1)^4)\right)$$

$$=(z-1)-\frac{1}{2}(z-1)^2+\frac{1}{3}(z-1)^3-\frac{1}{2}(z-1)^4+O((z-1)^5)$$

$$=\log(z)-\frac{1}{4}(z-1)^4+O((z-1)^5).$$ ∴ 三阶 BDF 的精度为3.

11. (Ex 11.109). 证明 s步LMM的精度为p, 当且仅当在将该LMM应用于 $u_t=g(t)$ 时, LMM的解对所有次数小于p的多项式精确, 但不对所有次数等于p的多项式精确. 假设初值 u_0 任意, 初始数据 $V^0\dots V^{s-1}$ 精确.

~~PF: "⇒": ∵ LMM的精度为p. ∴ $U^{T/k}=u(T)+\sum_{n=p+1}^{\infty}C_nk^nu^{(n)}(t_N)$.~~

~~∵ deg(g)<p, ∴ ∀n≥p deg(u)~~

PF: "⇒". ∵ LMM的精度为p, ∴ $U^N=u(T)+\sum_{n=p+1}^{\infty}C_nk^nu^{(n)}(t_N)$.

∵ $\deg(g)<p$, $u=\int g(t)dt$, ∴ $\deg(u)\le p$. ∴ $\forall n\ge p+1$, $u^{(n)}\equiv 0$.

∴ $U^N=u(T)$. 即 LMM对次数小于p的多项式g精确.

∵ $C_{p+1}\neq 0$, ∴ 取 $g(t)=t^p$, $u_0=0$, 则 $u(t)=\frac{1}{p+1}t^{p+1}$. $U^N=u(T)+C_{p+1}k^{p+1}\cdot p!\neq u(T)$.

"⇐": 反证. 若 LMM的精度为 $p'<p$, 则同必要性证明可知 LMM的解对p'次多项式 $g(t)=t^{p'}$ 不精确. 矛盾. 所以 LMM的精度不小于p. 若LMM的精度大于p, 则由必要性知对任意p次多项式, LMM的解都精确. 矛盾. 所以 LMM的精度等于p.

12. (Ex 11.113) 证明 $p_M(z)=z^s+\sum_{j=0}^{s-1}\alpha_jz^j$ 是 $M=\begin{bmatrix}0&1&&&\\&0&1&&\\&&\ddots&\ddots&\\&&&0&1\\-\alpha_0&-\alpha_1&\dots&-\alpha_{s-2}&-\alpha_{s-1}\end{bmatrix}\in\mathbb{C}^{s\times s}$ 的特征多项式.

PF: $\mathrm{chf}(M)=\det(zI-M)=\det\begin{bmatrix}z&-1&&&\\&z&-1&&\\&&\ddots&\ddots&\\&&&z&-1\\\alpha_0&\alpha_1&\dots&\alpha_{s-2}&z+\alpha_{s-1}\end{bmatrix}=z^{s-1}(z+\alpha_{s-1})+z^{s-2}\alpha_{s-2}+z^{s-3}\alpha_{s-3}+\dots+\alpha_0=p_M(z)$.

13. (Ex 11.119) ~~证明~~ 设 θ_n 是齐次差分方程 $\theta_{n+s}+\sum_{j=0}^{s-1}\alpha_j\theta_{n+j}=0$, α_j 为常数, 初值为 $\begin{bmatrix}\theta_0\\\theta_{-1}\\\vdots\\\theta_{-s+2}\\\theta_{-s+1}\end{bmatrix}=\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix}$,

则非齐次方程 $y_{n+s}+\sum_{j=0}^{s-1}\alpha_jy_{n+j}=\psi_{n+s}$ (初值为 $y_0,y_1,\dots,y_{s-1}$) 的解为

$y_n=\sum_{j=0}^{s-1}\theta_{n-j}\tilde{y}_j+\sum_{j=s}^{n}\theta_{n-j}\psi_j$. 其中

$$\begin{bmatrix}\tilde{y}_{s-1}\\\tilde{y}_{s-2}\\\tilde{y}_{s-3}\\\vdots\\\tilde{y}_1\\\tilde{y}_0\end{bmatrix}=\begin{bmatrix}1&\theta_1&\theta_2&\dots&\theta_{s-2}&\theta_{s-1}\\&1&\theta_1&\dots&\theta_{s-3}&\theta_{s-2}\\&&1&\dots&\theta_{s-4}&\theta_{s-3}\\&&&\ddots&\vdots&\vdots\\&&&&1&\theta_1\\&&&&&1\end{bmatrix}^{-1}\begin{bmatrix}y_{s-1}\\y_{s-2}\\y_{s-3}\\\vdots\\y_1\\y_0\end{bmatrix}$$

PF: 当 $n\le s-1$ 时, 结论显然成立. 下面设结论对 case $s-1, s-2, \dots, n+s-1$ 均成立, 则

$$y_{n+s}=-\sum_{i=0}^{s-1}\alpha_iy_{n+i}+\psi_{n+s}=-\sum_{i=0}^{s-1}\alpha_i\left(\sum_{j=0}^{s-1}\theta_{n+i-j}\tilde{y}_j+\sum_{j=s}^{n+i}\theta_{n+i-j}\psi_j\right)+\psi_{n+s}$$

$$=\sum_{j=0}^{s-1}\theta_{n-j}\tilde{y}_j+\sum_{j=s}^{n+s-1}\theta_{n-j}\psi_j+\psi_{n+s}=\sum_{j=0}^{s-1}\theta_{n-j}\tilde{y}_j+\sum_{j=s}^{n+s}\theta_{n-j}\psi_j.$$

14. (Ex 11.124) 用和 Lem 11.121 类似的方法证明 Lem 11.123：收敛的LMM是一致的.

~~PF~~: 考虑方程 $u'(t)=f(t)=0,\ u(0)=1$ 和 $u'(t)=f(t)=1,\ u(0)=0$.

PF: 对 IVP $u'(t)=0,\ u(0)=1$. 由收敛性, $\forall T>0$, $\lim\limits_{\substack{k\to 0\\ Nk=T}} U^N = \lim\limits_{\substack{k\to 0\\ Nk=T}} \frac{1}{\alpha_s}\sum_{j=0}^{s-1} -\alpha_j U^{N-s+j} = 1$.

$\therefore \frac{1}{\alpha_s}\sum_{j=0}^{s-1} -\alpha_j = 1$. $\sum_{j=0}^{s}\alpha_j = 1$.

对 IVP $u'(t)=1,\ u(0)=0$. 由收敛性, $\forall T>0$, $\lim\limits_{\substack{k\to 0\\ Nk=T}} U^N = \frac{1}{\alpha_s}\left(-\sum_{j=0}^{s-1}\alpha_j U^{N-s+j} + k\sum_{j=0}^{s}\beta_j\right) = T$.

$\cancel{\lim_{k\to 0}}\frac{1}{\alpha_s}\left(-\sum_{j=0}^{s-1}\alpha_j k(N-s+j) + k\sum_{j=0}^{s}\beta_j\right) = kN$. $\cancel{k\sum_{j=0}^{s}\alpha_j kN}$

$k\sum_{j=0}^{s}\alpha_j(N-s+j) = k\sum_{j=0}^{s}\beta_j$. $\therefore \sum_{j=0}^{s}\alpha_j(N-s) + \sum_{j=0}^{s} j\alpha_j = \cancel{k}\sum_{j=0}^{s}\beta_j$.

$\because \sum_{j=0}^{s}\alpha_j = 0$, $\therefore \sum_{j=0}^{s} j\alpha_j = \sum_{j=0}^{s}\beta_j$.

Ex 11.141见上机作业文档。