1. 给出 Thm 5.7 的详细证明.

PF: ~~只需证 <·,·> 在 C[a,b] 上的对称性、正定性~~

(1) 正定性.

$\langle u,u\rangle = \int_a^b \rho(t)u(t)\overline{u(t)}dt = \int_a^b \rho(t)|u(t)|^2 dt \geq 0.$

(2) 绝对性.

若 $\langle u,u\rangle = \int_a^b \rho(t)|u(t)|^2 dt = 0$, 则因为 $\rho(t) \neq 0$, 所以 $u(t) \equiv 0$.

(3) 线性性.

$\langle au+v,w\rangle = \int_a^b \rho(t)(au(t)+v(t))\overline{w(t)}dt = a\int_a^b \rho(t)u(t)\overline{w(t)}dt + \int_a^b \rho(t)v(t)\overline{w(t)}dt = a\langle u,w\rangle + \langle v,w\rangle.$

(4) 对称性.

$\langle v,u\rangle = \int_a^b \rho(t)v(t)\overline{u(t)}dt = \int_a^b \overline{\rho(t)u(t)\overline{v(t)}}dt = \overline{\langle u,v\rangle}.$

所以 $\langle\cdot,\cdot\rangle$ 是内积运算, $C[a,b]$ 是内积空间.

2. 考虑第一类切比雪夫多项式.

(1) 证明它们在 $C[-1,1]$ 上正交. 内积如 Thm 5.7 定义, $\rho(x)=\frac{1}{\sqrt{1-x^2}}$.

PF: $\langle T_n,T_m\rangle = \int_{-1}^{1}\rho(t)T_n(t)T_m(t)dt = \int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\cos(n\arccos t)\cos(m\arccos t)dt$

$= -\int_{-1}^{1}\cos(n\arccos t)\cos(m\arccos t)\, d\arccos t$

$= \int_0^{\pi}\cos(n\theta)\cos(m\theta)d\theta = \int_0^{\pi}\frac{\cos[(n-m)\theta]+\cos[(n+m)\theta]}{2}d\theta$

$= \begin{cases} \left(\frac{\sin[(n-m)\theta]}{2(n-m)} + \frac{\sin[(n+m)\theta]}{2(n+m)}\right)\Big|_0^{\pi} \\ \left(\frac{\theta}{2}+\frac{\sin 2n\theta}{4n}\right)\Big|_0^{\pi} \end{cases} = \begin{cases} 0, & n\neq m \\ \frac{\pi}{2}, & n=m \end{cases}$

$\therefore \{T_n\}$ 正交.

(2) 将前三个第一类切比雪夫多项式正则化, 使其成为标准正交组.

Sol: $\langle T_n,T_n\rangle = \frac{\pi}{2}$. 所以 $\|T_n\|_2 = \sqrt{\frac{\pi}{2}}$.

而 $T_1(x)=x$, $T_2(x)=2x^2-1$, $T_3(x)=4x^3-3x$,

所以 $T_1^*(x)=\sqrt{\frac{2}{\pi}}x$, $T_2^*(x)=\sqrt{\frac{2}{\pi}}(2x^2-1)$, $T_3^*(x)=\sqrt{\frac{2}{\pi}}(4x^3-3x)$.

3. 连续函数的最小平方估计.

在内积如 Thm 5.7 定义下, 用二次多项式估计函数 $y(x)=\sqrt{1-x^2}$, $x\in[-1,1]$.

(1) $\rho(x)=\frac{1}{\sqrt{1-x^2}}$, 用傅里叶展开.

Sol: $\langle y,T_0^*\rangle = \langle y,\sqrt{\frac{1}{\pi}}\rangle = \sqrt{\frac{1}{\pi}}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}\,dt = 2\sqrt{\frac{1}{\pi}}$

$\langle y,T_1^*\rangle = \langle y,\sqrt{\frac{2}{\pi}}x\rangle = \sqrt{\frac{2}{\pi}}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}\,t\,dt = 0.$

$\langle y,T_2^*\rangle = \langle y,\sqrt{\frac{2}{\pi}}(2x^2-1)\rangle = \sqrt{\frac{2}{\pi}}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}(2t^2-1)dt = \sqrt{\frac{2}{\pi}}\left(\frac{2}{3}t^3-t\right)\Big|_{-1}^{1} = -\frac{2}{3}\sqrt{\frac{2}{\pi}}.$

~~$p(x)=\sqrt{\frac{2}{\pi}}\left[2-\frac{2}{3}(2x^2-1)\right]=\sqrt{\frac{2}{\pi}}\left(-\frac{4}{3}x^2+\frac{8}{3}\right)$.~~

~~$p(x)=2\sqrt{\frac{1}{\pi}}-\frac{2}{3}\sqrt{\frac{2}{\pi}}(2x^2-1)$.~~

$p(x) = 2\sqrt{\frac{1}{\pi}}\cdot\sqrt{\frac{1}{\pi}} - \frac{2}{3}\sqrt{\frac{2}{\pi}}\cdot\sqrt{\frac{2}{\pi}}(2x^2-1) = \frac{10-8x^2}{3\pi}.$

$\frac{}{\sqrt{1-x^2}}$ 的正则方程组.

Sol: $G(1,x,x^2) = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{bmatrix}$

~~$\langle x,x\rangle = \langle 1,x^2\rangle = \int_{-1}^{1}$~~

$\langle 1,1\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \Big|_{-1}^{1} = \pi.$

$\langle 1,x\rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = 0.$

$\langle x,x\rangle = \langle 1,x^2\rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\,dx \xlongequal{x=\cos\theta} -\int_{0}^{\pi} \frac{\cos^2\theta}{\sin\theta}\,d\cos\theta = \int_0^{\pi} \cos^2\theta\,d\theta = \frac{\pi}{2}.$

$\langle x,x^2\rangle = \int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}}\,dx = 0.$

$\langle x^2,x^2\rangle = \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\,dx \xlongequal{x=\cos\theta} -\int_{0}^{\pi} \frac{\cos^4\theta}{\sin\theta}\,d\cos\theta = \int_0^{\pi} \cos^4\theta\,d\theta = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{3\pi}{8}.$

$\therefore G(1,x,x^2) = \begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{bmatrix}.$

$c = [\langle y,1\rangle, \langle y,x\rangle, \langle y,x^2\rangle]^T.$

$\langle y,1\rangle = \int_{-1}^{1} dx = 2.$

$\langle y,x\rangle = \int_{-1}^{1} x\,dx = 0.$

$\langle y,x^2\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}.$

$\therefore \begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{10}{3\pi} \\ 0 \\ -\frac{8}{3\pi} \end{bmatrix}$

$p(x) = \frac{10-8x^2}{3\pi}.$

14. 用正交多项式进行离散最小二乘估计

考虑 Ex 5.48 的销售记录表格.

(1) 从线性无关集 $\{1,x,x^2\}$ 构造正交多项式. 内积定义为 $\langle u(t),v(t)\rangle = \sum_{i=1}^{N} \rho(t_i)u(t_i)v(t_i)$.

其中 $N=12$, $\rho(x)=1$.

Sol: $\langle u_0,u_0\rangle = \langle 1,1\rangle = \sum_{i=1}^{12} 1 = 12.\quad u_0^* = \frac{u_0}{\|u_0\|_2} = \frac{1}{2\sqrt{3}}.$

$\langle u_0^*,u_1\rangle = \langle \frac{1}{2\sqrt{3}}, x\rangle = \frac{1}{2\sqrt{3}}\sum_{i=1}^{12} x_i = \frac{78}{2\sqrt{3}}.$

$v_1 = u_1 - \langle u_0^*,u_1\rangle u_0^* = x - \frac{13}{2}$

$\langle v_1,v_1\rangle = \langle x-\frac{13}{2}, x-\frac{13}{2}\rangle = \sum_{i=1}^{12}(x_i-\frac{13}{2})^2 = 143.\quad u_1^* = \frac{v_1}{\|v_1\|_2} = \frac{x-\frac{13}{2}}{\sqrt{143}}.$

$\langle u_0^*,u_2\rangle = \langle \frac{1}{2\sqrt{3}}, x^2\rangle = \frac{1}{2\sqrt{3}}\sum_{i=1}^{12} x_i^2 = \frac{650}{2\sqrt{3}}.$

$\langle u_1^*,u_2\rangle = \langle \frac{x-\frac{13}{2}}{\sqrt{143}}, x^2\rangle = \frac{1}{\sqrt{143}}\sum_{i=1}^{12}(x_i-\frac{13}{2})x_i^2 = \frac{1859}{\sqrt{143}}.$

$v_2 = u_2 - \langle u_0^*,u_2\rangle u_0^* - \langle u_1^*,u_2\rangle u_1^* = x^2 - 13x + \frac{91}{3}.$

$\langle v_2,v_2\rangle = \langle x^2-13x+\frac{91}{3}, x^2-13x+\frac{91}{3}\rangle = \sum_{i=1}^{12}(x_i^2-13x_i+\frac{91}{3})^2 = \frac{4004}{3}.$

$u_2 = \frac{v_2}{\|v_2\|_2} = \sqrt{\frac{3}{4004}}\,(x^2-13x+\frac{91}{3}).$

(2) 求最佳线性估计 $\hat{\varphi}=\sum_{i=0}^{2}a_i x^i$，使得 $\|y-\hat{\varphi}\| \le \|y-\sum_{i=0}^{2}b_i x^i\|$，$\forall b_i \in \mathbb{R}$. 并验证 $\hat{\varphi}$ 和讲义中的 $\hat{\varphi}$ 相同.

Sol：$\langle u_0^*, y\rangle = \langle \frac{1}{2\sqrt{3}}, y\rangle = \frac{1}{2\sqrt{3}}\sum_{i=1}^{12} y_i = \frac{1662}{2\sqrt{3}}$.

$\langle u_1^*, y\rangle = \langle \frac{x-\frac{13}{2}}{\sqrt{143}}, y\rangle = \frac{1}{\sqrt{143}}\sum_{i=1}^{12}(x_i-\frac{13}{2})y_i = \frac{589}{\sqrt{143}}$.

$\langle u_2^*, y\rangle = \langle \sqrt{\frac{3}{4004}}(x^2-13x+\frac{91}{3}), y\rangle = \sqrt{\frac{3}{4004}}\sum_{i=1}^{12}(x_i^2-13x_i+\frac{91}{3})y_i = 12068\sqrt{\frac{3}{4004}}$.

$$\hat{\varphi} = \frac{1662}{2\sqrt{3}}\cdot\frac{1}{2\sqrt{3}} + \frac{589}{\sqrt{143}}\cdot\frac{x-\frac{13}{2}}{\sqrt{143}} + 12068\sqrt{\frac{3}{4004}}\cdot\sqrt{\frac{3}{4004}}(x^2-13x+\frac{91}{3})$$

$$= 9.04196x^2 - 113.427x + 386.$$

(3) 假设有另一张销售表格的格式与本题相同，N 和 x_i 相同，但 y_i 不同，则哪些计算结果可重复使用，哪些不能？这种可重用性体现了正交多项式相对于正则方程组的哪些优点？

Sol：正交多项式结果 $u_0^*, u_1^*, u_2^*, \dots$ 可重用. $\langle u_0^*, y\rangle, \langle u_1^*, y\rangle, \langle u_2^*, y\rangle$ 的值不可重用.

假设要用 p 次多项式拟合，则对于正交多项式，在用 $O(np^2)$ 次运算计算出 $u_0^*, u_1^*, \dots, u_p^*$ 后，对每个不同的 y，只需用 $O(np)$ 次运算计算出 $\langle u_0^*, y\rangle, \langle u_1^*, y\rangle, \dots, \langle u_p^*, y\rangle$ 即可. 而对于正则化方法，在用 $O(np^2)$ 次运算计算出 $G(1, x, \dots, x^p)$ 后，对于每个不同的 y，~~还需~~用 $O(np)$ 次运算计算出 c 后，还需再用 $O(p^3)$ 次运算解方程组才能得到 $\hat{\varphi}$. 在 p 较大时正则方程组方法的运算量远大于正交多项式方法.