1. 辛普森公式.

(1) 证明 $[-1,1]$ 上的辛普森公式可由 $\int_{-1}^{1} y(t)\,dt = \int_{-1}^{1} P_3(y;-1,0,0,1;t) + E^S(y)$ 导出.

其中 $y\in C^4[-1,1]$, $P_3(y;-1,0,0,1;t)$ 是 y 满足 $P_3(-1)=y(-1)$, $P_3(0)=y(0)$, $P_3'(0)=y'(0)$, $P_3(1)=y(1)$ 的插值多项式.

PF: $\int_{-1}^{1} P_3(y;-1,0,0,1;t)\,dt$

$$= \int_{-1}^{1}\left[y(-1) + y[-1,0](t+1) + y[-1,0,0](t+1)t + y[-1,0,0,1](t+1)t^2\right]dt$$

$$= 2y(-1) + 2(y(0)-y(-1)) + \frac{2}{3}(y'(0)-y(0)+y(-1)) + \frac{2}{3}\cdot\frac{1}{2}(y(1)-2y'(0)-y(-1))$$

$$= \frac{1}{3}(y(-1)+4y(0)+y(1)) = I^S(y).$$

(2) 求 $E^S(y)$.

Sol: $E^S(y) = \int_{-1}^{1} y(t)\,dt - \frac{1}{3}(y(-1)+4y(0)+y(1))$.

(3) 由 (1)(2) 及变量替换, 导出组合辛普森公式并证明其误差估计.

Sol: 组合辛普森公式满足

$$I_n^S(y) = \int_{x_0}^{x_2} P_3(y;x_0,x_1,x_1,x_2;t)\,dt + \int_{x_2}^{x_4} P_3(y;x_2,x_3,x_3,x_4;t)\,dt + \dots + \int_{x_{n-2}}^{x_n} P_3(y;x_{n-2},x_{n-1},x_{n-1},x_n;t)\,dt$$

$$= \frac{h}{3}(y(x_0)+4y(x_1)+y(x_2)) + \frac{h}{3}(y(x_2)+4y(x_3)+y(x_4)) + \dots + \frac{h}{3}(y(x_{n-2})+4y(x_{n-1})+y(x_n))$$

$$= \frac{h}{3}(y(x_0)+4y(x_1)+2y(x_2)+4y(x_3)+2y(x_4)+\dots+4y(x_{n-1})+y(x_n)).$$

这与 Def 6.19 给出的公式相同.

$$E^S(y) = \int_{-1}^{1}\left[y(t)-P_3(y;-1,0,0,1;t)\right]dt \overset{\text{Thm 2.37}}{=\!=\!=} \int_{-1}^{1}\frac{f^{(4)}(\xi(t))}{24}t^2(t-1)(t+1)\,dt$$

$$= -\int_{-1}^{1}\frac{f^{(4)}(\xi(t))}{24}t^2(1-t)(1+t)\,dt = -\frac{f^{(4)}(\xi)}{24}\int_{-1}^{1}t^2(1-t)(1+t)\,dt = -\frac{f^{(4)}(\xi)}{24}\cdot\frac{4}{15} = -\frac{f^{(4)}(\xi)}{90}.$$

$$E_n^S(y) = \sum_{k=0}^{\frac{n}{2}-1} -\frac{f^{(4)}(\xi_k)}{90}\cdot h^5 = -\frac{f^{(4)}(\xi)}{90}\cdot h^5\cdot\frac{n}{2} = -\frac{b-a}{180}h^4 f^{(4)}(\xi).$$

2. 估计 $\int_0^1 e^{-x^2}dx$ 精确到小数点后六位, 即绝对误差小于 0.5×10^{-6}, 需多少个子区间.

(1) 用组合梯形公式

Sol: $I = \int_0^1 e^{-x^2}dx \approx 0.746824133$.

$I_n^T(f) = h\left(\frac{1}{2}f(0) + f(h) + f(2h) + \dots + f((n-1)h) + \frac{1}{2}f(1)\right)$

$I_{350}^T(f) \approx 0.746823632$, $E_{350}^T(f) \approx 5.01\times10^{-7}$.

$I_{351}^T(f) \approx 0.746823635$, $E_{351}^T(f) \approx 4.98\times10^{-7}$.

因此需至少 351 个子区间.

(2) 用组合辛普森公式

Sol: $I_n^S(f) = \frac{h}{3}\left(f(0) + 4f(h) + 2f(2h) + 4f(3h) + \dots + 4f((n-1)h) + f(1)\right)$

$I_{10}^S(f) \approx 0.746824948$, $E_{10}^S(f) \approx 8.15\times10^{-7}$.

$I_{12}^S(f) \approx 0.746824526$, $E_{12}^S(f) \approx 4.13\times10^{-7}$.

因此需到少 12 个子区间.

3. 高斯-拉盖尔积分公式

(1) 求多项式 $\pi_2(t) = t^2+at+b$ 使其在 $\rho(t) = e^{-t}$ 下与 $\mathbb{P}_1$ 正交.

PF: $\int_0^\infty \pi_2(t)\rho(t)\,dt = \int_0^\infty (t^2+at+b)e^{-t}\,dt = 2+a+b = 0$

$\int_0^\infty \pi_2(t)\rho(t)\,t\,dt = \int_0^\infty (t^2+at+b)te^{-t}\,dt = 6+2a+b = 0$

$\Rightarrow \begin{cases} a = -4 \\ b = 2 \end{cases}$.

$\pi_2(t) = t^2-4t+2$.

(2) 导出两点高斯-拉盖尔求积式 $\int_0^\infty f(t)e^{-t}dt = w_1 f(t_1)+w_2 f(t_2)+E_2(f)$. 并将 $E_2(f)$ ~~表示~~ 用 $f^{(4)}(T)$ 表示.

PF: $\pi_2(t) = t^2-4t+2=0 \Rightarrow t_1 = 2-\sqrt{2},\ t_2 = 2+\sqrt{2}$.

$\cancel{I_2^G(1) = w_1+w_2}$

$$I(1)=I_2^G(1) \Rightarrow w_1+w_2=\int_0^\infty \rho(t)dt=\int_0^\infty e^{-t}dt=1.$$

$$I(t)=I_2^G(t) \Rightarrow w_1t_1+w_2t_2=\int_0^\infty t\rho(t)dt=\int_0^\infty te^{-t}dt=1.$$

$$\Rightarrow \begin{cases} w_1 = \frac{2+\sqrt{2}}{4} \\ w_2 = \frac{2-\sqrt{2}}{4} \end{cases}.$$

$$\therefore \int_0^\infty f(t)e^{-t}dt = \frac{2+\sqrt{2}}{4}f(2-\sqrt{2})+\frac{2-\sqrt{2}}{4}f(2+\sqrt{2})+E_2(f).$$

$$E_2(f)=\frac{f^{(4)}(T)}{24}\int_0^\infty e^{-t}(t^2-4t+2)^2dt=\frac{f^{(4)}(T)}{24}(24-8\times6+20\times2-8\times1+4)=\frac{1}{2}f^{(4)}(T).$$

(3) 用(2)中结论估计 $I=\int_0^\infty \frac{1}{1+t}e^{-t}dt$. 比较真实误差和(2)中计算的误差并求出(2)中未知的 T.

Sol: $I_2(f)=\frac{2+\sqrt{2}}{4}\cdot\frac{1}{1+2-\sqrt{2}}+\frac{2-\sqrt{2}}{4}\cdot\frac{1}{1+2+\sqrt{2}}=\frac{4}{7}\approx 0.571428571$.

$E_2(f)=I(f)-I_2(f)\approx 0.024918790$.

$f^{(4)}(t)=\left(\frac{1}{1+t}\right)^{(4)}=\frac{24}{(1+t)^5}$. $\quad \frac{24}{(1+T)^5}=E_2(f) \Rightarrow T\approx 2.951271$.

4. 高斯公式的余项. 考虑如下 Hermite 插值问题: 求 $p\in\mathbb{P}_{2n-1}$ s.t. $\forall m=1,2,\dots,n$. $p(x_m)=f_m$, $p'(x_m)=f'_m$.

则存在基本 Hermite 插值多项式 h_m, g_m. s.t. $p(t)=\sum_{m=1}^n[h_m(t)f_m+g_m(t)f'_m]$. 类似于 Lagrange 插值多项式.

(1) 求如下形式 $h_m(t)=(a_m+b_mt)l_m^2(t)$ 和 $g_m(t)=(c_m+d_mt)l_m^2(t)$ 的 h_m, g_m.

~~Sol: $p(x_m)=\ \forall j\neq m,\ l_m(x_j)=0.\ \therefore p(x_m)=$~~

~~Sol: $p(x_m)=\sum_{k=1}^n[h_k(t)f_k+g_k(t)f'_m]=\sum_{m=1}^n[(a_m+b_mx_m)f_m+(c_m+d_mx_m)f'_m]l_m^2(x_m)$~~

Sol: $p(x_m)=\sum_{k=1}^n[h_k(x_m)f_k+g_k(x_m)f'_k]=\sum_{k=1}^n[(a_k+b_kx_m)f_k+(c_k+d_kx_m)f'_k]l_k^2(x_m)$

$=(a_m+b_mx_m)f_m+(c_m+d_mx_m)f'_m=f_m$.

$p'(x_m)=\sum_{k=1}^n[h'_k(x_m)f_k+g'_k(x_m)f'_k]$ $\cancel{=\sum_{k=1}^n[(b_k+d_k)l_k(x_m)+(a_k+b_kx_m+c_k+d_kx_m)l'_k(x_m)f'_k]l_k(x_m)}$

$\cancel{=(b_m+d_m)f_k+(a_m+b_mx_m+c_m+d_mx_m)f'_m\sum_{i\neq}}$

$=\sum_{k=1}^n[(b_kf_k+d_kf'_k)l_k(x_m)+[(a_k+b_kx_m)f_k+(c_k+d_kx_m)f'_k]\cdot 2l'_k(x_m)]l_k(x_m)$.

$=(b_mf_m+d_mf'_m)+2[(a_m+b_mx_m)f_m+(c_m+d_mx_m)f'_m]\cdot\sum_{i\neq m}\frac{1}{x_m-x_i}=f'_m$.

$$\Rightarrow \begin{cases} a_m+x_mb_m=1 \\ c_m+x_md_m=0 \\ 2(a_m+x_mb_m)l'_m(x_m)+b_m=0 \\ 2(c_m+x_md_m)l'_m(x_m)+d_m=1 \end{cases} \Rightarrow \begin{cases} a_m=1+2x_ml'_m(x_m) \\ b_m=-2l'_m(x_m) \\ c_m=-x_m \\ d_m=1 \end{cases}. \quad \left(l'_m(x_m)=\sum_{i\neq m}\frac{1}{x_m-x_i}\right)$$

(2) 导出求积式 $\cancel{I_n(f)=\sum_{k=1}^n[w_kf_k+\mu_kf']}$ $I_n(f)=\sum_{k=1}^n[w_kf(x_k)+\mu_kf'(x_k)]$, 满足 $E_n(p)=0$, $\forall p\in\mathbb{P}_{2n-1}$

Sol: 令 $I_n(f)=\int_a^b p(t)dt=\sum_{k=1}^n[w_kf(x_k)+\mu_kf'(x_k)]$.

其中, $w_k=\int_a^b h_k(t)dt=\int_a^b[1+2(x_k-t)l'_k(x_k)]l_k^2(t)dt$,

$\mu_k=\int_a^b g_k(t)dt=\int_a^b(t-x_k)l_k^2(t)dt$.

因为 $p(t)$ 是对 f 的 $2n-1$ 次 Hermite 插值, 所以对 $p\in\mathbb{P}_{2n-1}$, $p(t)\equiv f$.

即 $\forall p\in\mathbb{P}_{2n-1}$, $E_n(p)=0$.

(3) 结点多项式或结点需满足什么条件，可使得 $\mu_k=0,\ \forall k=1,2,\dots,n$.

Sol: ~~即~~ $\forall k$, $\mu_k=\int_a^b (t-x_k)\ell_k^2(t)\,dt = \int_a^b \frac{V_n^2(t)}{t-x_k}\,dt = 0$.

6.5. 证明 Lem 6.43. 如何选择 h 可最小化误差上界？设计一个四阶精度的基于对称模板的公式，导出其误差上界并最小化。比较二阶精度的公式和四阶精度的公式，可以观察到什么？

Pf: (1) $D^2u(\bar{x}) = \frac{u(\bar{x}-h)-2u(\bar{x})+u(\bar{x}+h)}{h^2}$

$$= \frac{1}{h^2}\Big[u(\bar{x}) - hu'(\bar{x}) + \frac{h^2}{2}u''(\bar{x}) - \frac{h^3}{6}u'''(\bar{x}) + \frac{h^4}{24}u^{(4)}(\bar{x}) - \frac{h^5}{120}u^{(5)}(\bar{x}) - 2u(\bar{x}) + u(\bar{x}) + hu'(\bar{x}) + \frac{h^2}{2}u''(\bar{x}) + \frac{h^3}{6}u'''(\bar{x}) + \frac{h^4}{24}u^{(4)}(\bar{x}) + \frac{h^5}{120}u^{(5)}(\bar{x}) + O(h^6)\Big]$$

$$= u''(\bar{x}) + \frac{h^2}{12}u^{(4)}(\bar{x}) + O(h^{(4)}).$$

$\therefore |u''(\bar{x}) - D^2u(\bar{x})| = \frac{h^2}{12}|u^{(4)}(\xi)|$ $\xi\in(x-h,x+h)$.

当 u 的计算存在误差 $\varepsilon\in[-E,E]$ 时，

$$|u''(\bar{x}) - D^2\tilde{u}(\bar{x})| \le |u''(\bar{x}) - D^2u(\bar{x})| + |D^2u(\bar{x}) - D^2\tilde{u}(\bar{x})| \le \frac{h^2}{12}|u^{(4)}(\xi)| + \frac{4E}{h^2}.$$

(2) 取 $h=\left(\frac{|u^{(4)}(\bar{x})|}{12}\cdot 4E\right)^{\frac14} = \left(\frac{E|u^{(4)}(\bar{x})|}{3}\right)^{\frac14}$，误差上界为 $2\left(\frac{E|u^{(4)}(\bar{x})|}{3}\right)^{\frac12}$.

(3) 设 $D_2^2u(\bar{x}) = \frac{Au(\bar{x}-2h)+Bu(\bar{x}-h)+Cu(\bar{x})+Bu(\bar{x}+h)+Au(\bar{x}+2h)}{h^2}$.

$\cancel{= (4A+B)u''(\bar{x}) + \frac{1}{12}(16A+B)u^{(4)}(\bar{x}) + \frac{1}{360}(64A+B)u^{(6)}(\bar{x})}$

$$= \frac{1}{h^2}\Big[(2A+2B+C)u(\bar{x}) + (4A+B)h^2u''(\bar{x}) + \frac{1}{12}(16A+B)h^4u^{(4)}(\bar{x}) + \frac{1}{360}(64A+B)h^6u^{(6)}(\bar{x}) + O(h^8)\Big].$$

$$\therefore \begin{cases} 2A+2B+C=0 \\ 4A+B=1 \\ 16A+B=0 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{12} \\ B=\frac43 \\ C=-\frac52 \end{cases}.$$

~~即~~ $\cancel{D_2^2u(\bar{x}) = \frac{1}{12h^2}\left[-u(\bar{x}-2h)+\frac43u(\bar{x}-h)-\frac52u(\bar{x})+\frac43u(\bar{x}+h)\right.}$

即 $D_2^2u(\bar{x}) = \frac{-u(\bar{x}-2h)+16u(\bar{x}-h)-30u(\bar{x})+16u(\bar{x}+h)-u(\bar{x}+2h)}{12h^2}$.

且 $D_2^2u(\bar{x}) = u''(\bar{x}) + \frac{h^4}{360}(64A+B)u^{(6)}(\bar{x}) = u''(\bar{x}) - \frac{h^4}{90}u^{(6)}(\bar{x}) + O(h^{(6)})$.

当 u 的计算存在误差 $\varepsilon\in[-E,E]$ 时，

$$|u''(\bar{x}) - D_2^2\tilde{u}(\bar{x})| \le \frac{h^4}{90}|u^{(6)}(\xi)| + \frac{16E}{3h^2}.\quad \xi\in(x-2h,x+2h).$$

取 $h=\left(\frac{|u^{(6)}(\bar{x})|}{90}\cdot\frac{16E}{3}\cdot\frac12\right)^{\frac16} = \left(\frac{4E|u^{(6)}(\bar{x})|}{135}\right)^{\frac16}$，误差上界为 $3\left(\frac{640E^2}{|u^{(6)}(\bar{x})|}\right)^{\frac13}$.

(4) ①四阶精度的误差上界只在二阶精度的~~基础~~ $O(E^{\frac12})$ 基础上提升到 $O(E^{\frac23})$.

②二阶精度公式的误差上界和 u 的高阶导数正相关，但四阶精度公式的误差上界和 u 的高阶导数负相关。