POLYNOMIAL FUNCTIONS (Chapter 5)

LESSON 1 - Graphs of Polynomial Functions

VOCABULARY

x-intercept - where a graph intercepts the x-axis, and the value of y is 0. *Ex.* $(x, 0)$

y-intercept - where the graph intercepts the y-axis, and the value of x is 0. *Ex.* $(0, y)$

Domain - All possible x values on a graph. *Ex.* $\{x | x \in R\}, \{x | -3 \leq x \leq 5, x \in R\}$

Range - All possible y values on a graph. *Ex.* $\{y | x \in R\}, \{y | y \geq 5, y \in R\}$

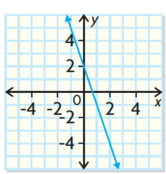
Polynomial function - a function that has exponents that are whole numbers and consists of one or more terms which are separated by $+$ or $-$ sign ; all the coefficients must be real numbers

Degree - the value or the highest exponent in the function.

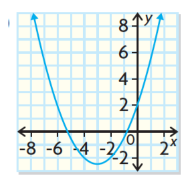
Constant Term - a term in the polynomial with no variable

Leading Coefficient - the number that multiplies the term with the highest power

End Behaviour - describes the shape of the graph, from left to right, on the coordinate plane
Ex.

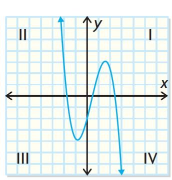


QII to QIV

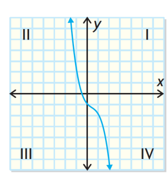


QII to QI

Turning Points - any point where the graph of a function changes from increasing to decreasing or from decreasing to increasing
Ex.



This curve has 2 turning points

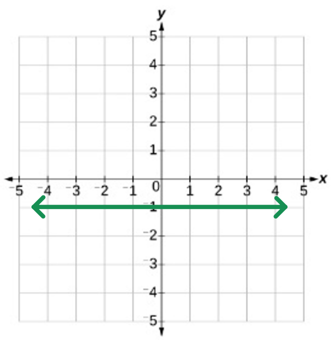
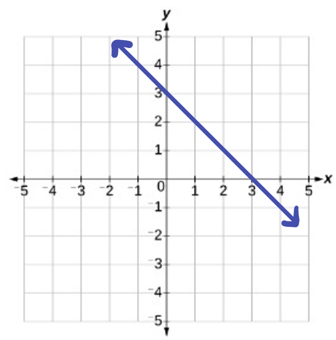
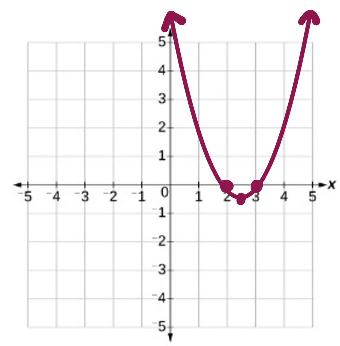


This curve has NO turning points (the y values are always decreasing)

Standard From:
The standard form of a linear function is $f(x) = ax + b$, where $a \neq 0$.
The standard form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$.
The standard form of a cubic function is $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

- We have already studied polynomial functions in one variable in previous math courses:

Examples:

$f(x) = -1$	$f(x) = -x + 3$	$f(x) = x^2 - 5x + 6$
		
Degree : 0	Degree : 1	Degree : 2
Leading Coefficient: 0	Leading Coefficient: -1	Leading Coefficient: 1
Constant Term: -1	Constant Term: 3	Constant Term: 6
x-intercept(s): none	*x*-intercept(s): $x = 3$	*x*-intercept(s): $(2,0)$ & $(3,0)$
y-intercept: $y = -1$	*y*-intercept: $y = 3$	*y*-intercept: $(0,6)$
End Behaviour: QIII to QIV	End Behaviour: QII to QIV	End Behaviour: QII to QI
Number of turning points: none	Number of turning points: none	Number of turning points: one
Domain: $\{x \mid x \in R\}$	Domain: $\{x \mid x \in R\}$	Domain: $\{x \mid x \in R\}$
Range: $\{y \mid y = -1, y \in R\}$	Range: $\{y \mid y \in R\}$	Range: $\{y \mid y \geq -0.25, y \in R\}$

In this investigation, using your graphing calculator, you will examine the effects the degree and the sign of the leading coefficient and the constant term have on the graph of a polynomial function.

Polynomial Function	**Degree**	**Leading Coefficient**	**Constant Term**	**Number of x-intercepts**	**y-intercept**	**Number of Turning Points**	**End Behaviour**	**Domain**	**Range**
$f(x) = 3x - 1$	1	3	$y=-1$	1	-1	0	III to I	$\{x \in R\}$	$\{y \in R\}$
$f(x) =- x^2 + 2x$	2	-1	$y=0$	2	0	1	III to IV	$\{x \in R\}$	$\{y \mid y \leq 1, y \in R\}$
$f(x) = 4$	0	0	$y=4$	0	4	0	II to I	$\{x \in R\}$	$\{y \mid y = 4, y \in R\}$
$f(x) =- 2x^3 - 5x + 3$	3	-2	$y=3$	1	3	0	II to IV	$\{x \in R\}$	$\{y \in R\}$
$f(x) = (x - 2)(x + 4)$	2	1	$y=-8$	2	-8	1	II to I	$\{x \in R\}$	$\{y \mid y \geq -9, y \in R\}$
$f(x) =- 5$	0	0	$y=-5$	0	-5	0	III to IV	$\{x \in R\}$	$\{y \mid y = -5, y \in R\}$
$f(x) = x^3 + 5x^2 + 3x - 1$	3	1	$y=-1$	3	-1	2	III to I	$\{x \in R\}$	$\{y \in R\}$
$f(x) =- 2x$	1	-2	$y=0$	1	0	0	II to I	$\{x \in R\}$	$\{y \in R\}$

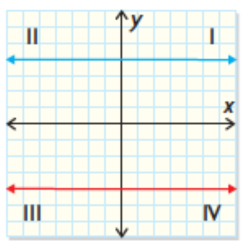
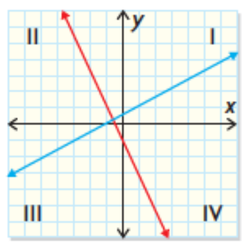
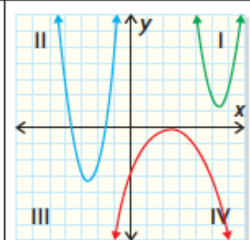
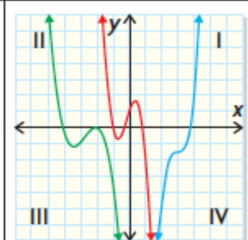
In Summary

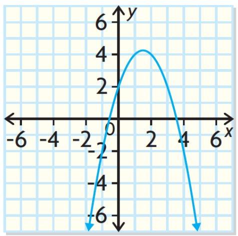
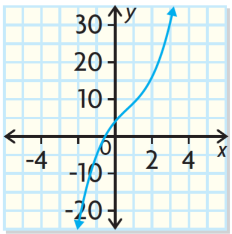
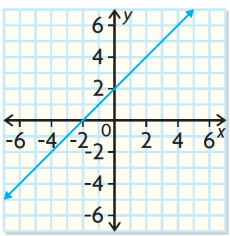
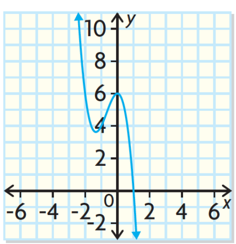
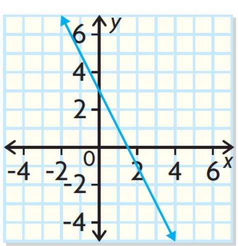
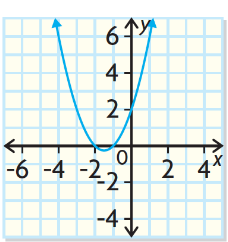
Key Ideas

- A polynomial function in one variable is a function that contains only the operations of multiplication and addition, with real-number coefficients, whole-number exponents, and two variables. The degree of the function is the greatest exponent of the function. For example, $f(x) = 6x^3 + 3x^2 - 4x + 9$ is a cubic polynomial function of degree 3.
- The degree of a polynomial function determines the shape of the function.

Need to Know

- The graphs of polynomial functions of the same degree have common characteristics.
- The chart below shows sample sketches of functions and displays all the possibilities for the x-intercepts, y-intercepts, end behaviour, range, and number of turning points for each type of function.

Type of Function	constant	linear	quadratic	cubic
Degree, *n*	0	1	2	3
Sketch				
Number of *x*-Intercepts	0, except for $y = 0$, for which every point is on the x-axis	1	0, 1, or 2	1, 2, or 3
Number of *y*-Intercepts	1	1	1	1
End Behaviour	Line extends from quadrant II to quadrant I or from quadrant III to quadrant IV.	Line extends from quadrant III to quadrant I or from quadrant II to quadrant IV.	Curve extends from quadrant II to quadrant I or from quadrant III to quadrant IV.	Curve extends from quadrant III to quadrant I or from quadrant II to quadrant IV.
Domain	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$
Range	$\{y \mid y = \text{constant}, y \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}\}$	$\{y \mid y \leq \text{maximum}, y \in \mathbb{R}\}$ or $\{y \mid y \geq \text{minimum}, y \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}\}$
Number of Turning Points	0	0	1	0 or 2

Graph	Degree	Sign of Leading Coefficient	Constant Term	End Behaviour	Domain and Range
	2	— neg	$y=2$	QIII to QIV	$\{x \mid x \in R\}$ $\{y \mid y \leq 4.1, y \in R\}$ (appox)
	3	+ pos	$y=5$	QIII to QI	$\{x \mid x \in R\}$ $\{y \mid y \in R\}$
	1	+ pos	$y=2$	QIII to QI	$\{x \mid x \in R\}$ $\{y \mid y \in R\}$
	3	— neg	$y=6$	QII to QIV	$\{x \mid x \in R\}$ $\{y \mid y \in R\}$
	1	— neg	$y=3$	QII to QIV	$\{x \mid x \in R\}$ $\{y \mid y \in R\}$
	2	+ pos	$y=2$	QII to QI	$\{x \mid x \in R\}$ $\{y \mid y \geq -0.1, y \in R\}$ (appox)

In Summary

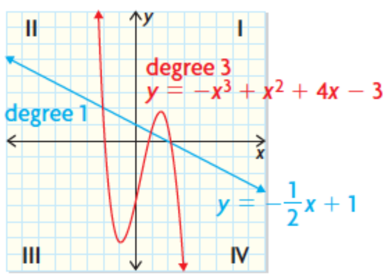
Key Ideas

- When a polynomial function is in standard form:
  - The maximum number of x-intercepts the graph may have is equal to the degree of the function.
  - The maximum number of turning points a graph may have is equal to one less than the degree of the function.
  - The degree and leading coefficient of the equation of a polynomial function indicate the end behaviour of the graph of the function.
  - The constant term in the equation of a polynomial function is the y-intercept of its graph.

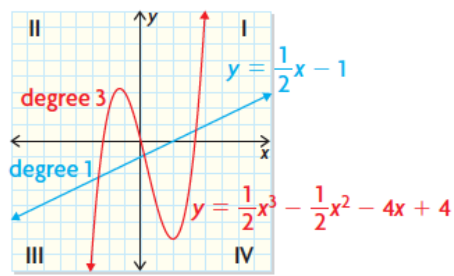
Need to Know

- Linear and cubic polynomial functions with positive leading coefficients have similar end behaviour. Linear and cubic polynomial functions with negative leading coefficients also have similar end behaviour.

If the leading coefficient is negative, then the graph of the function extends from quadrant II to quadrant IV.

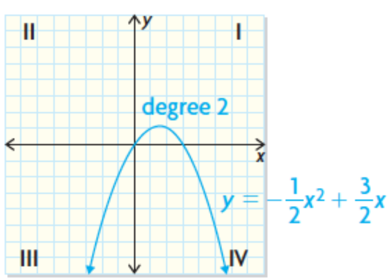


If the leading coefficient is positive, then the graph of the function extends from quadrant III to quadrant I.



- Quadratic polynomial functions have unique end behaviour.

If the leading coefficient is negative, then the graph of the function extends from quadrant III to quadrant IV.



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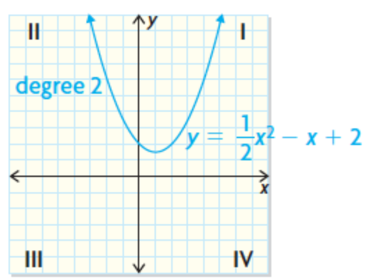
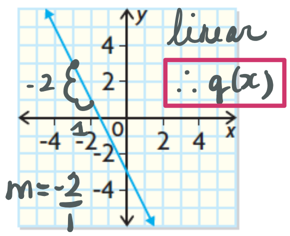


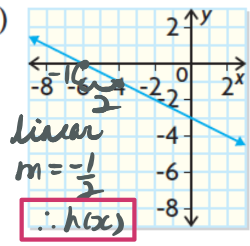
EXAMPLE 1

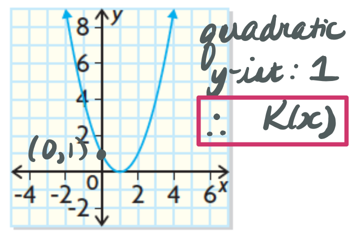
Match each graph with the correct polynomial function.
Justify your reasoning.

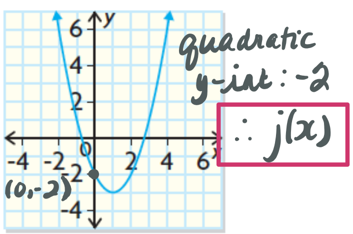
$g(x) = -x^3 + 4x^2 - 2x - 2$ $\quad j(x) = x^2 - 2x - 2$ $\quad p(x) = x^3 - 2x^2 - x - 2$

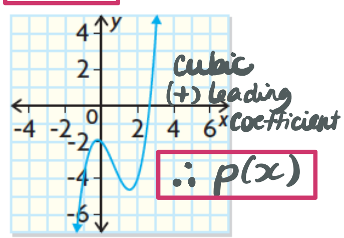
$h(x) = -\frac{1}{2}x - 3$ $\quad k(x) = x^2 - 2x + 1$ $\quad q(x) = -2x - 3$

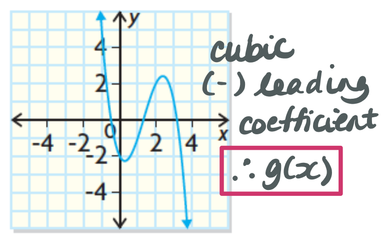
i) 

iii) 

v) 

ii) 

iv) 

vi) 

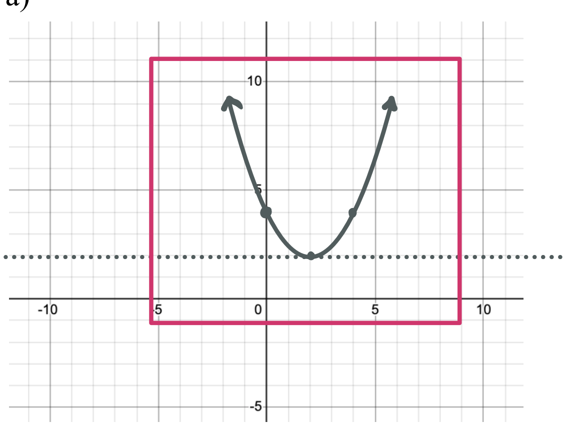
EXAMPLE 2

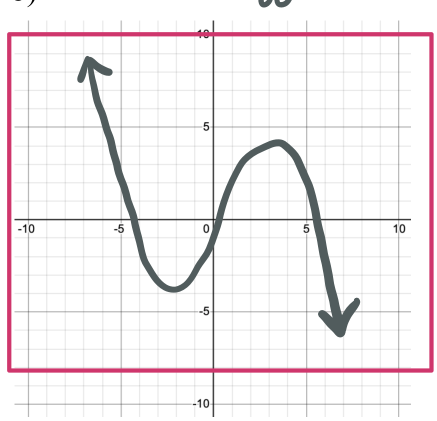
Sketch the graph of a possible polynomial function for each set of characteristics below. What can you conclude about the equation of the function with these characteristics?

a) Range: $\{y \mid y \geq -2, y \in R\}$ QUADRATIC that opens up
y-intercept: 4

b) Range: $\{y \mid y \in R\}$
Turning points: one in quadrant III and another in quadrant I

CUBIC with a negative leading coefficient

a) 

b) 

Practice

1. Write a polynomial function $f(x)$ that satisfies the following conditions:

answers will vary

a) degree 2, x-intercepts of 2 and -1 $f(x) = (x-2)(x+1)$

b) degree 0, y-intercept of 4 $f(x) = 4$

c) degree 1, leading coefficient of 5 $f(x) = x + 5$

d) two turning points, y-intercept of –2 $f(x) = x^3 - 2$

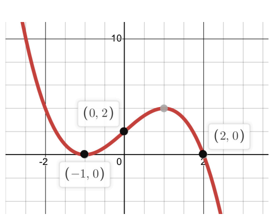
e) extending from quadrant III to quadrant IV, one turning point, y-intercept of 2 $f(x) = -x^2 + x + 2$

2. Match each function with a statement about the graph of the function.

Function	Statement
i. $f(x) = 3x - 7$	iv The graph has exactly one turning point. Quadratic
ii. $g(x) =- 2x + 1$	iii (or ii) The end behaviour is from quadrant 2 to quadrant 4.
iii. $f(x) =- 6x^3 - 5x - 2$	ii The graph has a positive y-intercept.
iv. $f(x) = - x^2 - 3x - 9$	i The end behaviour is from quadrant 3 to quadrant 1.

3. **Describe** the graph of $f(x) =- (x + 1)^2(x - 2)$. Include the intercepts and turning points in your description.

Degree: 3 (cubic)
LC: -1 (QII to QIV)
x-int: $x = -1, 2$
y-int: $(0, 2)$
2 turning points



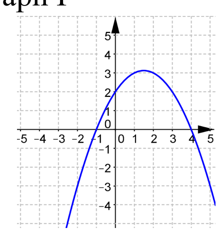
The graph of this cubic function has a negative leading coefficient, therefore extends from QII to QIV. The graph has two turning points. There are two x-intercepts at $x = -1$ and $x = 2$ and the y-intercept is at $y = 2$. Both the domain and range are elements of the reals.

4. Complete the following table.

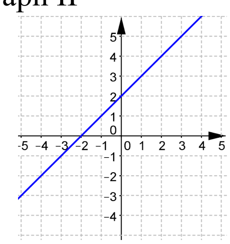
Polynomial Function	Degree	Leading co-efficient	Shape of graph	Constant term	Minimum and maximum # of x-intercepts
$f(x)=6x^2-3x-2$	2	6	quadratic (parabola)	-2	0,1,2 min: 0 max: 2
$g(x)=-\frac{2}{3}x+10$	1	-2/3	linear	10	1 min: 1 max: 1
$h(x)=-x^3+10x+6$	3	-1	cubic	6	1,2,3 min: 1 max: 3
$j(x)=4x^3-2x^2-3x-10$	3	4	cubic	-10	1,2,3 min: 1 max: 3

5. Complete the following table using the graphs below .

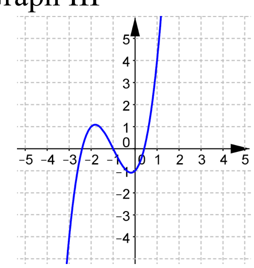
Graph I



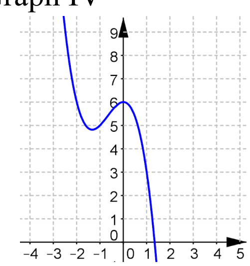
Graph II



Graph III



Graph IV



Polynomial Function	Degree	Leading co-efficient (+ or -)	Constant term (y-int)
I	2	opens down —	2
II	1	rises to the right (QIII to QI) +	2
III	3	QIII to QI +	-1
IV	3	QII to QIV —	6

6. For each type of polynomial function below, write an equation that has a y-intercept of $(0,5)$.

∴ constant term is 5

a) Constant $y=5$

b) Linear $y=x+5$

c) Quadratic $y=x^2+5$

d) Cubic $y=x^3+5$

ZERO function

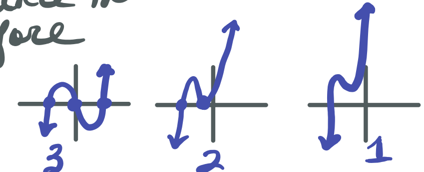
7. **Determine** the x-intercepts *using* your calculator and y-intercept *without* your calculator.

*constant term

Polynomial Function	x-intercept(s)	y-intercept
$f(x) = -2x + 5$	$x = 5/2$	5
$g(x) = x^2 + 2x - 6$	$x \approx -3.646$ $x \approx 1.646$	-6
$h(x) = x^3 - x^2 + 5x - 1$	$x \approx 0.207$	-1
$j(x) = -2x^3 + 4x$	$x \approx -1.414$ $x = 0$ $x \approx 1.414$	0

8. **Explain** why cubic functions may have one, two, or three x-intercepts. You may wish to use sketches to support your answer.

A cubic function starts high and ends low or vice versa and therefore must cross the x-axis at least once. Since the degree is three there can be at most three factors, therefore three x-intercepts.

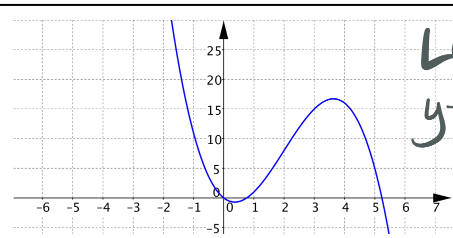


9. **Explain** why quadratic polynomial functions have maximum or minimum values, and cubic polynomial functions have only turning points and "sort of" maximums and minimums.

Quadratic functions start and end in the same direction and therefore have a range that is either greater than or equal to the min/max value. Cubic functions start high and end low (or vice versa) and therefore the vertices of the turning points are not the max/min value of the function.

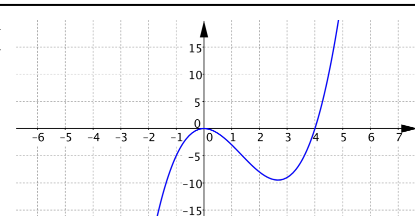
10. Match each graph with the correct polynomial function, *without* using your calculator.

I



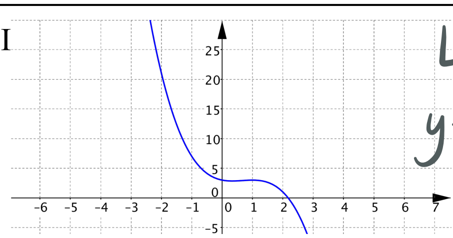
LC −
y-int: 0

II



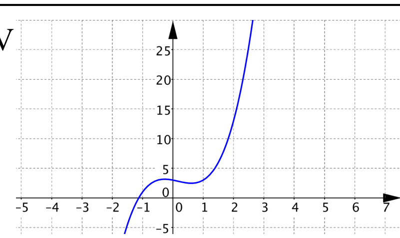
LC +
y-int: 0

III



LC −
y-int ≈ 3

IV



LC +
y-int ≈ 3

I $f(x) = -x^3 + 6x^2 - 4x$

IV $f(x) = 2x^3 - x^2 - x + 3$

III $f(x) = -x^3 + 2x^2 - x + 3$

II $f(x) = x^3 - 4x^2$

Use the following information to answer the next question.

The average retail price of gas in Canada, from 1979 to 2008, can be modeled by the polynomial function

$$P(t) = 0.008t^3 - 0.307t^2 + 4.830t + 25.720$$

where $\boxed{P \text{ is the price of gas}}$ in cents per litre and t is the number of years after 1979.

11. a) **Explain** what the constant term means in the context of this problem.

constant term = 25.720

The initial cost of gas in 1979 was 25.720 ¢/L.

b) **Determine** how many years, to the nearest tenth of a year, it took until the price of gas was 60 cents.

$60 = 0.008t^3 - 0.307t^2 + 4.830t + 25.720$

$\boxed{1}$ $y_1 = 60$
$y_2 = P(t)$

$\boxed{2}$ $x: [-10, 50, 10]$
$y: [-10, 100, 10]$

$\boxed{3}$ calc INTERSECT

$\boxed{4}$ $\boxed{x = 18{,}097}$ $y = 60$

$\boxed{\text{It will take 18.1 years.}}$

Use the following information to answer the next question.

The tide depth of a particular bay can be modeled by the polynomial function

$$d(t) = 0.001t^3 - 0.055t^2 + 0.845t + 0.293$$

where $d(t)$ is the tide depth in metres and t is the number of hours after midnight.

12. a) **Explain** what the y-intercept represents in the context of the question.

y-int: 0.293

The y-intercept represents the initial depth of the tide at midnight.

b) Use the polynomial function to **algrebraically determine** the tide depth at 10:00 a.m.

$d(10) = 0.001(10)^3 - 0.055(10)^2 + 0.845(10) + 0.293$
$= 1 - 5.5 + 8.45 + 0.293$
$= 4.243$

$\boxed{\text{The depth is 4.24m at 10 am}}$

c) How much higher is the depth of the tide at 6:00 am compared to the initial depth of the tide? Round to the nearest tenth of a metre.

$d(6) = 0.001(6)^3 - 0.055(6)^2 + 0.845(6) + 0.293$
$= 3.599$

$d(6) - d(0) = 3.599 - 0.293$
$= 3.306$

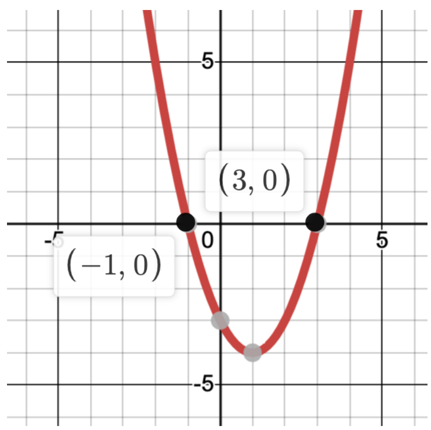
$\boxed{\text{It is 3.3m higher.}}$

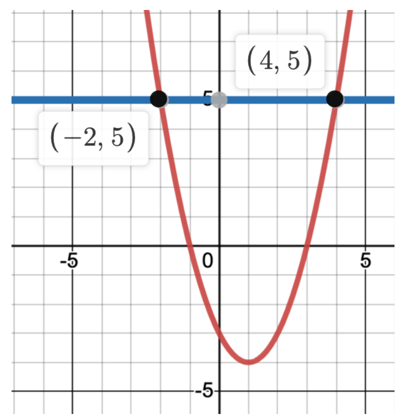
LESSON 2 - Solving Polynomial Equations

Last year we learned some different ways to solve problems involving quadratic relations and functions.
We will extend these to other functions and add a new method.

x – intercepts - Zeros - Roots - Solutions

We know that a *function* has the form $f(x) = x^2 - 2x - 3$ and an *equation* has the form $x^2 - 2x - 3 = \#$

<table>
<tr><th>Function</th><th>Equation</th><th>Equation</th></tr>
<tr><td>$f(x) = x^2 - 2x - 3$</td><td>$x^2 - 2x - 3 = 0$</td><td>$x^2 - 2x - 3 = 5$</td></tr>
<tr><td>Has zeros</td><td colspan="2">Has roots</td></tr>
<tr><td>The zeros are the x- intercepts of the graph.</td><td>Roots = zeros = x- intercepts</td><td>Roots ≠ zeros
Roots = the x-coordinates of the intersection points on the graph of Y1 (left side) & Y2 (right side)</td></tr>
<tr><td rowspan="2">Has no “solution”.</td><td>Solution:
$x = -1$ & $x = 3$


Steps:
1. Y1=$x^2 - 2x - 3$
2. x: $[-5, 5, 1]$ & y: $[-5, 5, 1]$
3. CALC Zero
4. $x = -1$ & $x = 3$</td><td>Solution:
$x = -2$ & $x = 4$


Steps:
1. Y1=$x^2 - 2x - 3$
Y2=5
2. x: $[-7, 5, 1]$ & y: $[-5, 7, 1]$
3. CALC Interesect
4. $x = -2$ & $x = 4$</td></tr>
<tr><td colspan="2">Solutions are the values of the variable which make the equation TRUE.

An equation is said to be VERIFIED if the left-hand side is equal to the right-hand side.</td></tr>
</table>

A few methods to solve equations

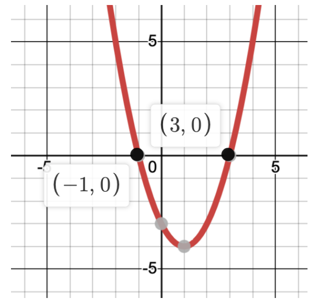
1. **Graphically** (as shown in the previous table.)

- METHOD 1 (find ZEROS)

Solve: $x^2 - 2x - 3 = 0$

Steps:

1. Y1=$x^2 - 2x - 3$
2. x: $[-5, 5, 1]$ & y: $[-5, 5, 1]$
3. CALC Zero
4. $x = -1$ **&** $x = 3$

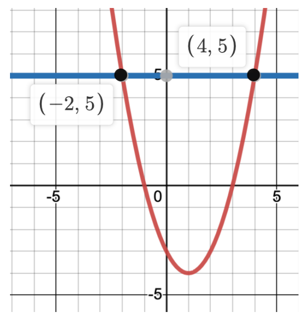


- METHOD 2 (find *x*-coordinates of the intersection points)

Solve: $x^2 - 2x - 3 = 5$

Steps:

1. Y1=$x^2 - 2x - 3$
Y2=5
2. x: $[-7, 5, 1]$ & y: $[-7, 5, 1]$
3. CALC Intersect
4. $x = -2$ **&** $x = 4$



2. **Algebraically**

- Factoring

Linear:

1. Simplify both sides of the equation to obtain the variable term on one side of the equal sign and the constant term on the other.

2. Divide or multiply as needed to isolate the variable.

3. Verify your solution.

Quadratic:

1. Arrange all the terms on one side of the equation so the other side equals 0,

2. Factor the expression.

3. Set each factor equal to 0 and solve each equation.

4. Verify your solution.

- Quadratic Formula (only for quadratics)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Arrange all the terms on one side of the equation so the other side equals 0,

2. Identify the values of a, b and c where $ax^2 + bx + c = 0$ *and then use the quadratic formula to find the solution(s).*

3. Verify your solution.

Example 1

a) Solve the equation $x^2 + x - 4 = 0$ graphically.

$\boxed{1}$ $y_1 = x^2 + x - 4$

$\boxed{2}$ $x: [-10, 10, 1]$
$y: [-10, 10, 1]$

$\boxed{3}$ calc ZERO

$\boxed{4}$ $\boxed{x = -2.562 \quad x = 1.562}$

NOTE:
The graph only shows true solutions. There are no extraneous roots on the graph.

b) Solve the equation $x^2 + x - 4 = -3$ graphically.

$\boxed{1}$ $y_1 = x^2 + x - 4$
$y_2 = -3$

$\boxed{2}$ $x: [-10, 10, 1]$
$y: [-10, 10, 1]$

$\boxed{3}$ calc INTESECT

$\boxed{4}$ $\boxed{x = -1.618 \quad x = 0.618}$

c) **Algebriacally** solve $x^2 - 2x - 3 = 0$ by factoring.

$(x-3)(x+1) = 0$

$x - 3 = 0$
$+3 \quad +3$

$x + 1 = 0$
$-1 \quad -1$

$\boxed{x = 3 \ \& \ x = -1}$

VERIFY:
$x = 3$ ✓

LS
$= (3)^2 - 2(3) - 3$
$= 9 - 6 - 3$
$= 0$

RS
$= 0$

LS=RS

$x = -1$ ✓

LS
$= (-1)^2 - 2(-1) - 3$
$= 1 + 2 - 3$
$= 0$

RS
$= 0$

LS=RS

d) **Algebriacally** solve the equation $x^2 - 2x - 3 = 5$ by factoring.

$x^2 - 2x - 3 = 5$
$-5 \quad -5$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\boxed{x = 4 \ \& \ x = -2}$

VERIFY:
$x = 4$ ✓

LS
$= (4)^2 - 2(4) - 3$
$= 16 - 8 - 3$
$= 5$

RS
$= 5$

LS=RS

$x = -2$ ✓

LS
$= (-2)^2 - 2(-2) - 3$
$= 4 + 4 - 3$
$= 5$

RS
$= 5$

LS=RS

e) **Algebriacally** solve the equation $-(x+3)(x+1)(x-2) = 0$.

$x + 3 = 0$
$-3 \quad -3$

$\boxed{x = -3}$

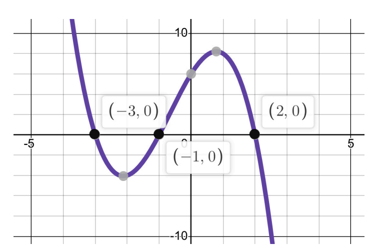
$x + 1 = 0$
$-1 \quad -1$

$\boxed{x = -1}$

$x - 2 = 0$
$+2 \quad +2$

$\boxed{x = 2}$

VERIFY:



f) **Algebriacally** solve the equation $2x^2 - 5x - 3 = 0$ using the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 2$
$b = -5$
$c = -3$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{49}}{4}$$

$$x = \frac{5 \pm 7}{4}$$

$$x = \frac{5+7}{4} = \frac{12}{4} = 3$$

$$x = \frac{5-7}{4} = \frac{-2}{4} = \frac{-1}{2}$$

$$\boxed{x = 3 \quad x = \frac{-1}{2}}$$

g) **Algebriacally** solve the equation $2x^2 - 5x - 3 = \cancel{4}$ using the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$-4 \quad \cancel{-4}$

$$2x^2 - 5x - 7 = 0$$

$a = 2$
$b = -5$
$c = -7$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-7)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{81}}{4}$$

$$x = \frac{5 \pm 9}{4}$$

$$x = \frac{5+9}{4} = \frac{14}{4} = \frac{7}{2}$$

$$x = \frac{5-9}{4} = \frac{-4}{4} = -1$$

$$\boxed{x = 7/2 \quad x = -1}$$

Example 2

Use the following information to answer the next question.

The dimensions of a rectangular prism are 10 cm by 10 cm by 5 cm. When each dimension is increased by the same length, x, the new volume is 1008 cm^3.

a) Write an equation in factored form that represents the volume, V, of the box as a function of x.

$$\boxed{1008 = (10 + x)(10 + x)(5 + x)}$$

b) Calculate the dimensions of the new prism.

$\boxed{1}$ $y_1 = 1008$
$y_2 = (10+x)(10+x)(5+x)$

$\boxed{2}$ $x: [-20, 10, 2]$
$y: [-100, 1500, 100]$

$\boxed{3}$ calc INTESECT

$\boxed{4}$ $\underline{x = 2}$ $y = 1008$

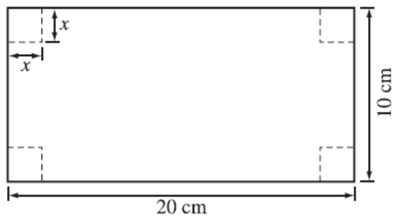
$(10+2) = 12$
$(10+2) = 12$
$(5+2) = 7$

New dimensions are $\boxed{\text{12 cm by 12 cm by 7 cm.}}$

Practice

Use the following information to answer the next question.

A box with no lid is made by cutting four squares of side length x from each corner of a 10 cm by 20 cm rectangular sheet of metal.



An expression that represents the volume of the box is given by $V(x) = x(10 - 2x)(20 - 2x)$, where $0 < x < 5$, where x is the side length of the square being cut out.

1. a) **Explain** the restriction on the domain.

If $x = 0$ or $x = 5$, the volume will be ZERO.
If $x < 0$ or $x > 5$, the volume will be NEGATIVE.
Therefore, $0 < x < 5$, for the volume to exist and be positive.

b) State the value of x, to the nearest hundredth of a centimetre, that gives the maximum volume.

$\boxed{1}$ $y_1 = x(10-2x)(20-2x)$

$\boxed{2}$ $x: [-10, 50, 10]$
$y: [-50, 300, 50]$

$\boxed{3}$ calc MAXIMUM

$\boxed{4}$ $x = 2.1132...$ $y = 192.4500...$

$\boxed{x = 2.11}$

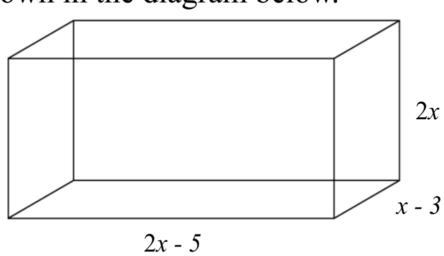
c) State the maximum volume of the box, to the nearest cubic centimetre.

max: $(2.11, 192.45)$

$\boxed{\text{maximum volume is } 192.45 \text{ cm}^3}$

Use the following information to answer the next question.

A gift box has dimensions as shown in the diagram below.

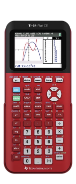


3. a) Write an equation, in factored form, that represents the volume, V, of the box as a function of x.

$\boxed{V(x) = (2x-5)(x-3)(2x)}$

b) **Determine** the value of x that would produce a volume of 24 units^3. Calculate the corresponding dimensions on the box.

$24 = (2x-5)(x-3)(2x)$



$\boxed{1}$ $y_1 = (2x-5)(x-3)(2x)$
$y_2 = 24$

$\boxed{2}$ $x: [-10, 10, 2]$
$y: [-20, 50, 10]$

$\boxed{3}$ calc INTERSECT

$\boxed{4}$ $x = 4$ $y = 24$

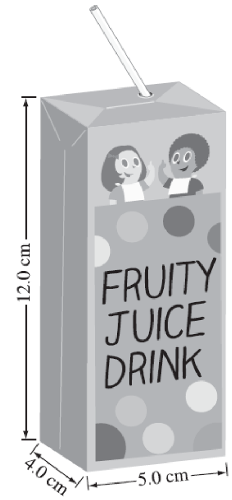
$\boxed{x = 4 \text{ units}}$

Use the following information to answer the next question.

A company wants to change the dimensions of a particular box, which will increase the volume. The box is currently 5.0 cm x 4.0 cm x 12.0 cm, as shown below.

$V = 12 \cdot 4 \cdot 5$

$V = 240 cm^3$



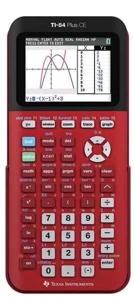
To create the new box, the company will increase each dimension by the same amount. The volume of the box can be calculated by using the formula $V = l \cdot w \cdot h$.

4. The company does not want the volume of the box to exceed $1000 cm^3$. **Determine** the largest amount by which each dimension of the box can be increased, to the nearest tenth of a centimetre and state any restrictions on the variable.

Let x = the amount of increase of each dimension.

$V(x) = (12+x)(4+x)(5+x)$

$1000 = (12+x)(4+x)(5+x)$



$\boxed{1}$ $y_1 = 1000$

$y_2 = (12+x)(4+x)(5+x)$

$\boxed{2}$ $x: [0, 10, 1]$

$y: [0, 1500, 100]$

$\boxed{3}$ calc INTERSECT

$\boxed{4}$ $x = 3.54$ $y = 1000$

The largest amount that each dimension can be increased by is 3.5 cm to stay within the volume restriction.

LESSON 3 - Using Regressions to Model Data - Linear, Quadratic, Cubic

VOCABULARY

Line of Best Fit: A straight line that best approximates the trend in a scatter plot.

Curve of Best Fit: A curve that best approximates the trend of a scatter plot.

Regression Equation: Equation of the line or curve of best fit that results from a statistical analysis of data.

Interpolate: The process used to estimate a value within the domain of a set of data, based on a trend.

Extrapolate: The process used to estimate a value outside the domain of a set of data.

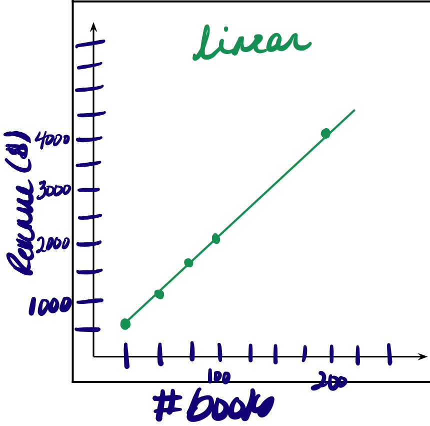
When given an equation it is easy to graph and determine x-intercepts or intersections of graphs. When given a series of data values we can use our calculator to create a line of best fit and a function that models the data. We create a line of best fit by performing a **REGRESSION**. 

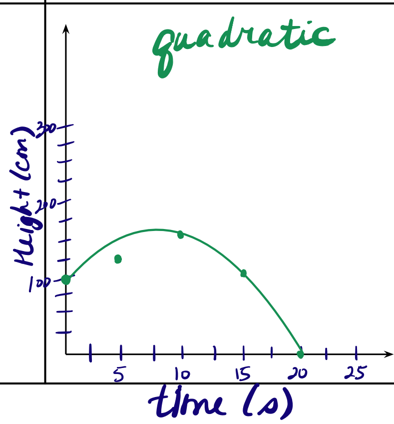
For each of the following scenarios, draw a sketch that approximates the data given.

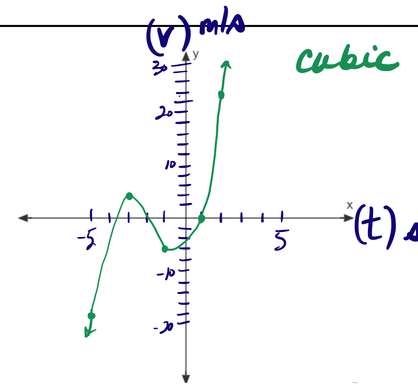
# Books Sold	Revenue ($)
25	500
50	1 000
75	1 500
100	2 000
200	4 000

Time (sec)	Height of a Ball (cm)
0	100
5	120
10	150
15	100
20	0

Time (sec)	Velocity (m/sec)
-5	-18
-3	4
-1	-6
1	0
2	24





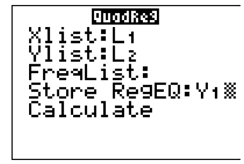


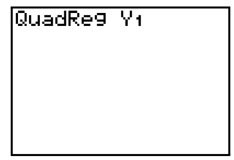
Now that we have a rough idea of what each graph should look like, let's perform a regression to determine the equation of each graph. You should learn following steps.

In your **TI graphing calculator**, use the following steps:

- STAT → EDIT → 1:Edit
  - Enter x-values into L_1, and
  - Enter corresponding y-values into L_2.
- STAT → CALC → 4:LinReg/5:QuadReg/6:CubicReg/

  (choose the function type that matches your sketch/graph)
- Either:
  - "Store RegEQ:" ALPHA → F4 → 1: Y_1, or
  - "Store RegEQ:" VARS → Y-VARS → FUNCTION→ Y_1, or
  - LinReg/QuadReg/CubicReg VARS → Y-VARS → FUNCTION → Y_1





Perform the appropriate Regression and record the function for each of the following..

# Books Sold	Revenue ($)	Time (sec)	Height of a Ball (cm)	Time (sec)	Velocity (m/sec)
25	500	0	100	-5	-18
50	1 000	5	120	-3	4
75	1 500	10	150	-1	-6
100	2 000	15	100	1	0
200	4 000	20	0	2	24
$y = 20x$		$y = -0.91x^2 + 13.89x + 92.29$		$y = x^3 + 5x^2 + 2x - 8$	

For the **first situation**, using your calculator, determine the following:

- The revenue earned if 124 were sold, would be $\boxed{\$2\,480}$.

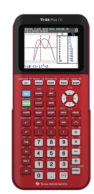
  $x=124$

  $\boxed{3}$ calc VALUE $\boxed{4}$ $x=124$ $y=2480$
- The revenue earned if 550 were sold, would be $\boxed{\$11\,000}$.

  adjust window

  $\boxed{3}$ calc VALUE $\boxed{4}$ $x=550$ $y=11\,000$
- How many books would need to be sold to earn $3200? $Y_2=3200$

  $\boxed{3}$ calc INTERSECT $\boxed{4}$ $x=160$ $y=3200$ $\boxed{\text{160 books}}$



$\boxed{1}$ $y_1 = 20x$

NEW

$\boxed{2}$ $x: [0, 250, 25] \to [0, 600, 25]$

$y: [0, 5000, 500]$

Helpful Hint **for window settings on TI!**

- After entering the data go to the "**Y=**" menu and activate "**Plot1**" at the top of the screen.
- Now press the **ZOOM** button under the screen and select "**9:ZoomStat**"
- The window will automatically adjust to the data entered in the Lists.
- After the window is set appropriately deactivate "**Plot1**".

In Summary

Key Ideas

- A scatter plot is useful when looking for trends in a given set of data.
- If the points on a scatter plot seem to follow a linear trend, then there may be a linear relationship between the independent variable and the dependent variable.

Need to Know

- If the points on a scatter plot follow a linear trend, technology can be used to determine and graph the equation of the line of best fit.
- Technology uses linear regression to determine the line of best fit. Linear regression results in an equation that balances the points in the scatter plot on both sides of the line.
- A line of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the line of best fit on a scatter plot or by using the equation of the line of best fit.

Need to Know

- If the points on a scatter plot follow a quadratic or cubic trend, then graphing technology can be used to determine and graph the equation of the curve of best fit.
- To solve an equation, you can graph the corresponding function of each side of the equation. The x-coordinate of the point of intersection is the solution to the equation.
- Technology uses polynomial regression to determine the curve of best fit. Polynomial regression results in an equation of a curve that balances the points on both sides of the curve.
- A curve of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the curve of best fit on a scatter plot or by using the equation of the curve of best fit.

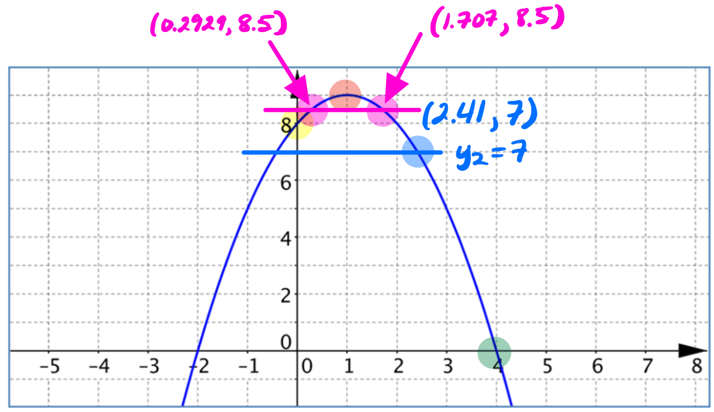
Applications of Polynomial Functions

As we have seen, a great number of questions can be asked for any particular equation or function. We have also seen that particular types of questions correspond to particular characteristics of a function.

Sample Scenario: **A rock is thrown off a bridge into a river below. The path of the rock can be modeled by the equation**

$$h(t) = -t^2 + 2t + 8,$$

where t is the time in seconds and h, is the height in metres above the river.



Sample Question	Characteristic	How to find	Answer
a) Height of a bridge from which a rock is thrown	y-intercept	calc VALUE $x=0$ or constant term	8 m
b) Maximum height of the rock	y-value at the vertex	calc MAXIMUM (1, 9)	9 m
c) Time at which the rock hits the water	positive x-intercept	calc ZERO (-2, 0) & (4, 0)	4 s
d) Time at which the rock is 7 m above the river	value of x when $y = 7$	$y_2 = 7$ calc INTERSECT	2.41 s
e) What are the restrictions on the domain	$t \geq 0$ (time) $t \leq 4$ (hits the water)	N/A	$\{t \mid 0 \leq t \leq 4, t \in \mathbb{R}\}$ or $[0, 4]$
f) What are the restrictions on the range	$h \geq 0$ (min height) $h \leq 9$ (max height)	N/A	$\{h \mid 0 \leq h \leq 9, h \in \mathbb{R}\}$ or $[0, 9]$
g) How long is the rock above 8.5 m	difference between x-values when $y = 8.5$	$y_2 = 8.5$ calc INTERSECT $x_1 = 1.707\ldots$ $x_2 = 0.292\ldots$	$1.707 - 0.292$ 1.41 s

Example 1

Matt buys t-shirts for a company that prints art on t-shirts and resells them. When buying the t-shirts, the price Matt must pay is related to the size of the order. Five of Matt's past orders are listed in the table below.

Number of Shirts	Cost per shirt ($)
500	3.25
700	1.95
200	5.20
460	3.51
740	1.69

a) State the linear regression function that models this data.

$\boxed{y = -0.0065x + 6.5}$

b) Explain what the slope and y-intercept of this linear function represent.

$m = -0.0065$

$y\text{-int} = 6.5$

The slope represents a drop in price of $0.0065 per additional shirts ordered.
The y-intercept is the price when the number of shirts ordered is zero.

c) **Determine** the size of the order needed for the cost of a t-shirt to be $1.50. Show your work.

1 $y_1 = -0.0065x + 6.5$
$y_2 = 1.50$

2 $x: [0, 1000, 100]$
$y: [0, 10, 1]$

3 calc INTERSECT

4 $x = 769.23$

$\therefore$ $\boxed{770 \text{ t-shirts}}$

or

ALGEBRAICALLY

$1.50 = -0.0065x + 6.5$
$-6.5 \qquad\qquad -6.5$

$\dfrac{-5}{-0.0065} = \dfrac{-0.0065x}{-0.0065}$

$x = 769.23\ldots$

$\therefore$ $\boxed{770 \text{ t-shirts}}$

Example 2

Josie hit the ball from the top of a hill. The height of the ball above the green is given in the table below.

Time (s)	1	2	3	4	5
Height (m)	52.5	73.2	74.6	55.8	16.1

a) State the quadratic regression function that models this data. Round to the nearest hundredth if necessary.

$\boxed{y = -10.07x^2 + 51.41x + 11}$

b) **Explain** the meaning of the constant term of the function in this context.

Constant term = 11

The initial height, at zero seconds, is 11m.

c) **Determine** the height of the ball at 2.5 *s*.

1 $y_1 = -10.07x^2 + 51.41x + 11$

2 $x: [0, 10, 1]$
$y: [0, 100, 10]$

3 calc VALUE
$x = 2.5$

4 $y = 76.575$

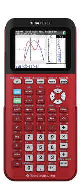
$\therefore$ $\boxed{76.58\ m}$

OR ALGEBRAICALLY

$y = -10.07(2.5)^2 + 51.41(2.5) + 11$
$= \boxed{72.65\ m}$

(not as accurate)

d) **Determine** when the ball hit the ground.



1 $y_1 = -10.07x^2 + 51.41x + 11$

2 $x: [-5, 10, 1]$
$y: [0, 100, 10]$

3 calc ZERO

4 ~~$x = -0.205\ldots$~~ $t > 0$
$x = 5.31\ldots$ $\therefore$ $\boxed{5.31\ s}$

Example 3

A spherical balloon is being inflated. The volume, V, in cubic centimetres is related to the time, t, in seconds.

Time (s)	0	1	2	3	4
Volume (cm³)	33.51	113.10	268.08	523.60	904.78

a) State the cubic regression function that models this data. Round to the nearest hundredth if necessary.

$\boxed{y = 4.19x^3 + 25.13x^2 + 50.27x + 33.51}$

b) **Determine** the volume of the balloon at 10.5 s.

1 $y_1 = 4.19x^3 + 25.13x^2 + 50.27x + 33.51$

2 $x: [0, 15, 1]$
$y: [0, 1000, 100]$

3 calc VALUE
$x = 10.5$

4 $y = 8181.47$
$\therefore \boxed{8181.47 \text{ cm}^3}$

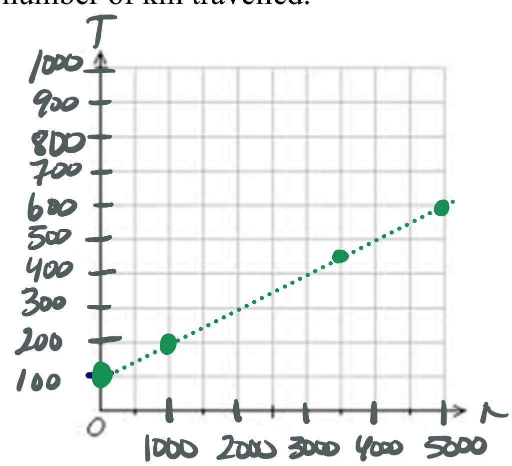
OR ALGEBRAICALLY

$y = 4.19(10.5)^3 + 25.13(10.5)^2 + 50.27(10.5) + 33.51$
$= \boxed{8182.38 \text{ cm}^3}$
(not as accurate)

Practice

1. Triple A Car Rental charges $100 per rental plus 10 cents per km. The total cost, T, in dollars of renting the car can be represented be the formula $T = 100 + 0.10n$, where n is the number of km travelled.

Number of km	**Total Rental Cost ($)**
0	100
1000	200
3500	450
5000	600



a) Completed the table of values and plot the ordered pairs on the grid provided.

b) **Describe** what the ordered pair (0, 100) represents in context of the question.

(0, 100) this represents the initial cost of the rental car with no Kilometers driven.

c) **Determine** the n-intercept of the graph. **Explain** why it is not applicable to this problem.

let $T = 0$ $\quad 0 = 100 + 0.10n$
$-100 \quad -100$
$\frac{-100}{0.10} = \frac{0.10n}{0.10} \rightarrow n = -1000$

The n-intercept is not applicable to this context because the number of Kilometers must be greater than or equal to zero.

d) **Determine** the cost of travelling 2000 km.

n

$C = 100 + 0.10(2000)$
$= 100 + 200$
$= 300$

$\boxed{\$300}$

c

e) If the total cost of the rental is $650, use the graph to find the number of km travelled.

$650 = 100 + 0.10n$
$-100 \quad -100$
$\frac{550}{0.10} = \frac{0.10n}{0.10} \rightarrow n = 5500$

$\boxed{5500 \text{ Km}}$

2. Timothy is interested in how speed plays a role in car accidents. He knows that there is a relationship between the speed of a car and the distance needed to stop.

Speed (km/h)	Distance (m)
22	5.6
42	20.6
49	35
81	76.5
95	105.2

a) State the quadratic regression function that models this data. Round to the nearest hundredth.

$\boxed{y = 0.01x^2 + 0.35x - 6.40}$

b) **Determine** the stopping distances, to the nearest tenth of a metre, at 30 km/h, 60 km/h and 100 km/h.

$\boxed{1}$ $y_1 = 0.01x^2 + 0.35x - 6.40$

$\boxed{2}$ $x: [0, 100, 10]$
$y: [0, 150, 10]$

$\boxed{3}$ calc VALUE (x3)

$\boxed{4}$ $x = 30$ $\boxed{y = 11.84 \text{ m}}$
$x = 60$ $\boxed{y = 45.52 \text{ m}}$
$x = 100$ $\boxed{y = 114.43 \text{ m}}$

c) **Determine** how fast the car can be moving in order to stop within 4 m.

$\boxed{1}$ $y_1 = 0.01x^2 + 0.35x - 6.40$
$y_2 = 4$

$\boxed{2}$ $x: [0, 100, 10]$
$y: [0, 150, 10]$

$\boxed{3}$ calc INTESECT

$\boxed{4}$ $x = 19.94$

$\boxed{19.9 \text{ Km/h}}$

Use the following information to answer the next question.

A design for a new sports arena is tested to measure the pressure placed on the building's exterior, in pounds of force per square foor (lb/ft^2), for various wind speeds, in miles per hour (mph). The data are listed in the table below.

Wind Speed (mph)	Pressure (lb/ft^2)
0	0
40	6.7
50	10.4
60	14.9
70	18.6
80	24.2

These data can be modelled by a quadratic regression function of the form

$$y = ax^2 + bx + c$$

where x is the wind speed, in miles per hour, and y is the pressure, in pounds per square foot.

3. Based on the quadratic regression function, the pressure created by a wind speed of 20 mph, to the nearest tenth of a pound per square foot, is $\boxed{2.2}$ lb/ft^2.

$x = 20$
use calc VALUE

Use the following information to answer the next two questions.

A hockey arena seats 1600 people. The cost of a ticket it $10. At this price, every ticket is sold. To obtain more revenue, the arena management plans to increase the ticket price. A survey was conducted to estimate the potential revenue for different ticket prices, as shown in the table below.

Ticket Price ($)	**Potential Revenue ($)**
10	16 000
15	19 500
20	20 300
25	14 750
30	5 500

The above date can be modelled by a quadratic regression function of the form

$$y = ax^2 + bx + c$$

where x is the ticket price, in dollars, and y is the potential revenue, in dollars.

4. a) **Determine** the ticket price, to the nearest cent, that would maximize the revenue.

$\boxed{1}$ $y_1 = -91x^2 + 3125x - 6340$

$\boxed{2}$ $x: [0, 50, 1]$
$y: [0, 25000, 100]$

$\boxed{3}$ calc MAXIMUM

$\boxed{4}$ $x = 17.17$ $y = 20488.64$

$x = \$17.17$

$\boxed{\$17.17}$

b) State an appropriate domain and range for the function in this context.

Domain: $\{x \mid x \geq 0, x \in R\}$

Range: $\{y \mid y \geq 0, y \in R\}$

Use the following information for the next question.

The height of a bird diving into the ocean, can be represented by the function

$$h(t) = 0.5t^3 - 7.75t^2 + 23.5t + 13.75$$

where h represents the height in feet and t represent the time in seconds.

5. a) The bird catches its prey at the lowest point of its descent. **Determine** how far under the water the bird dives, to the nearest tenth of a foot.

1 $y_1 = 0.5t^3 - 7.75t^2 + 23.5t + 13.75$

2 $x: [0, 20, 1]$
$y: [-50, 50, 5]$

3 calc MINIMUM

4 $x = 8.487$ $y = -39.37$

$\boxed{\text{39.4 ft under the water.}}$

b) **Determine** the change in height, to the nearest tenth of a foot, from the time the bird begins it dive to time the bird catches its prey.

1 $y_1 = 0.5t^3 - 7.75t^2 + 23.5t + 13.75$

2 $x: [0, 20, 1]$
$y: [-50, 50, 5]$

3 calc MAXIMUM

4 $x = 1.845$ $y = 39.375$

max − min
$= 39.375 - (-39.37)$
$= 73.24$

$\boxed{73.2 \text{ ft}}$

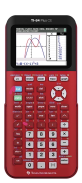
Use the following information for the next question.

The tide depth in Deep Cove, British Columbia, from midnight on January 6, 2011, can be modelled by the polynomial function

$$f(t) = 0.001t^3 - 0.055t^2 + 0.845t + 0.293$$

where $f(t)$ is the tide depth in metres and t is the number of hours after midnight.

6. Use the polynomial function to **determine** the tide depth at 10:00 on January 6.



1 $y_1 = 0.001t^3 - 0.055t^2 + 0.845t + 0.293$

2 $x: [0, 24, 2]$
$y: [-10, 10, 1]$

3 calc VALUE

4 $x = 10$ $y = 4.243$

$\boxed{4.2 \text{ m}}$

Use the following information for the next SIX questions.

A 15-gallon tank is being filled with water and has a pump that will cause it to drain when the amount of water inside the tank hits a certain volume. The volume of water in the tank over a 3-hour period can be modelled by the function

$$y = -2x^2 + 5x + 6$$

where y represents the volume of water in the tank in gallons and x represents the time in hours after noon on a particular day.

7. To determine the volume of water in the tank at noon, the characteristic of the function that should be analyzed is the

A. y-intercept
B. positive x-intercept
C. x-coordinate of the vertex
D. y-coordinate of the vertex

8. To determine the time when the tank is empty, the characteristic of the function that should be analyzed is the

A. y-intercept
B. positive x-intercept
C. x-coordinate of the vertex
D. y-coordinate of the vertex

9. To determine the maximum volume of the tank, the characteristic of the function that should be analyzed is the

A. y-intercept
B. the difference between the x-intercepts
C. x-coordinate of the vertex
D. y-coordinate of the vertex

10. To determine when the maximum volume of the tank is reached, the characteristic of the function that should be analyzed is the

A. y-intercept
B. the difference between the x-intercepts
C. x-coordinate of the vertex
D. y-coordinate of the vertex

11. The maximum volume of water in the tank, to the nearest tenth of a gallon, is 9.1 gallons.

12. To the nearest tenth of an hour, the tank will be full 1.2 hours after noon.

VERTEX:
(1.245, 9.125)

Use the following information for the next two questions.

The rate at which snow fell on a driveway on a particular day can be modelled by

$$y = -3x^2 + 6x$$

where y represents the rate of snowfall in ft^3/h, and x represents the time in hours after midnight.

13. To estimate the length of time that snow fell on this particular day, a student should determine the
    - A. y-intercept
    - B. x-coordinate of the vertex
    - C. y-coordinate of the vertex
    - D. the difference between the x-intercepts

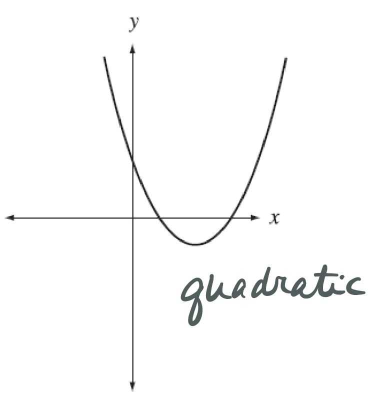
14. a) To the nearest whole number, the greatest rate of snowfall was 3 ft^3/h.

VERTEX: (1, 3)

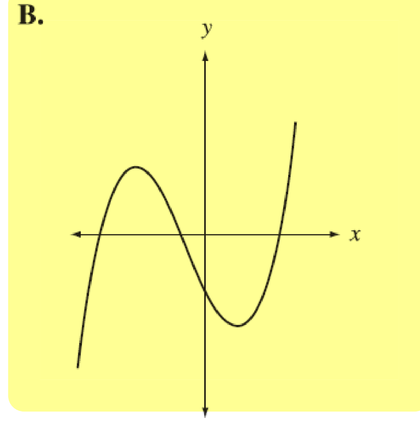
b) To the nearest hour, the rate of snow was the greatest at 1 am.

15. Which of the following graphs would **most likely** represent the graph of a cubic function?

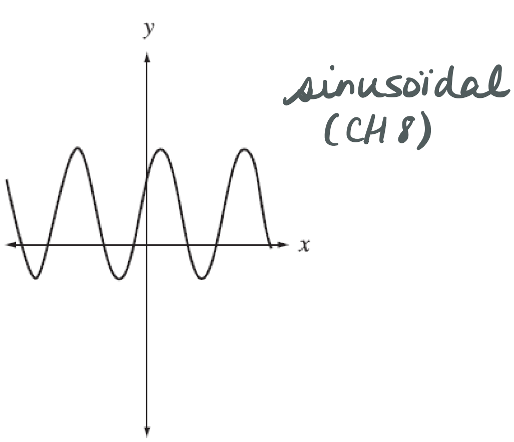
A.



B.



C.



D.