Fluid Dynamics

Chapter 6

General Flow Equations

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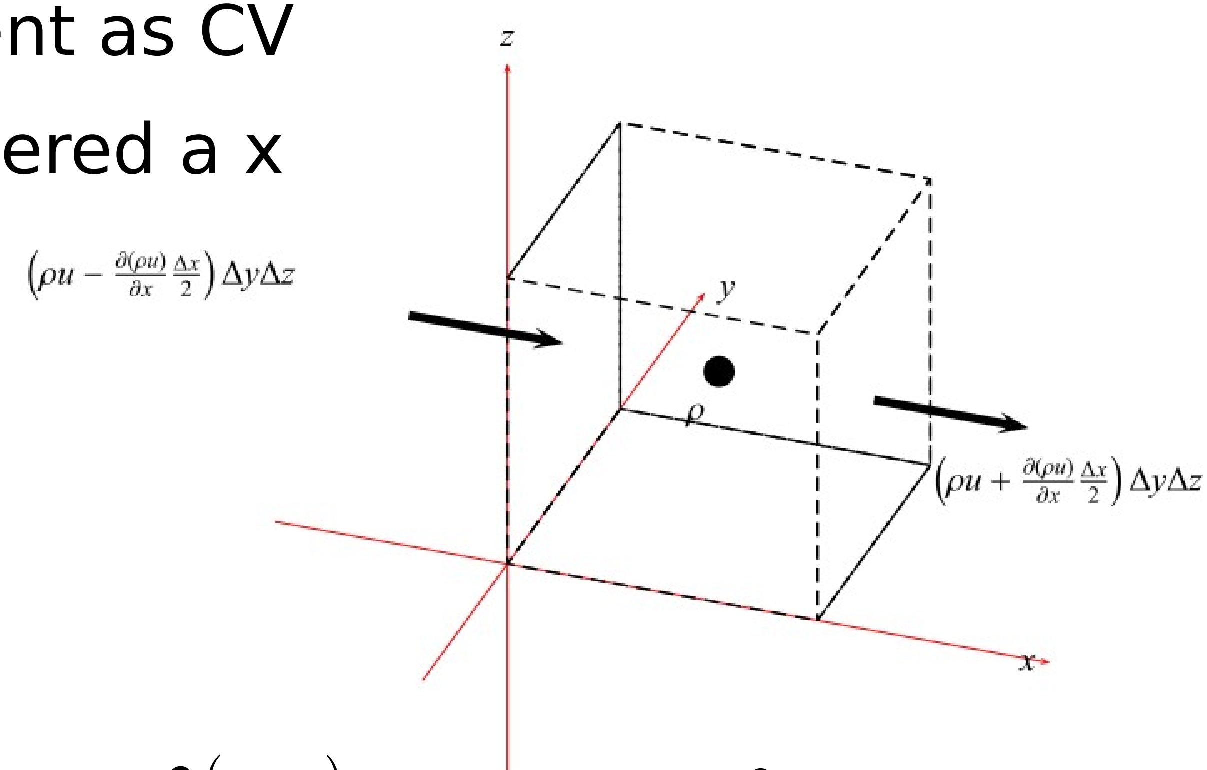
Basics

- Conservation of mass, energy and momentum form the basis to describe fluids
- Previous chapters: only simplified description
- Typical simplifications:
  - incompressible and stationary
  - one-dimensional and without friction
- This even allows for analytical solutions.
- The general form is a system of **coupled, nonlinear, partial differential equations**

Continuity Equation

- Consider a mass flow in the x-direction
- Have a fluid element as CV
- Fluid element centered a x
- Edges Δx, Δy, Δz



$$\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z - \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z + \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z = 0$$

$$\frac{\partial \rho u}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z = 0$$

Continuity Equation

- Other direction similarly. This gives:

$$\Delta x \Delta y \Delta z [\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}] = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- In the last line we introduced the Nabla-Operator. In cartesian coordinates:

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

Momentum Equation – The Navier-Stokes Equation

- The momentum equation is a vector equation.
- Goes back to C.-L. Navier and G.G. Stokes in the middle of the 19th century
- Takes into account mass density ρ, pressure p, the velocity vector **v** = (u,v,w), dynamic viscosity μ, and the gravitational acceleration **g** = (g_x,g_y,g_z).

x-component:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u u)}{\partial x}+\frac{\partial(\rho u v)}{\partial y}+\frac{\partial(\rho u w)}{\partial z}$$

$$=-\frac{\partial p}{\partial x}+\rho g_x+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

Momentum Equation – The Navier-Stokes Equation

y-component:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v u)}{\partial x}+\frac{\partial(\rho v v)}{\partial y}+\frac{\partial(\rho v w)}{\partial z}$$

$$=-\frac{\partial p}{\partial y}+\rho g_y+\frac{\partial}{\partial y}[\mu(2\frac{\partial v}{\partial y}-\frac{2}{3}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}))]$$

$$+\frac{\partial}{\partial x}[\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[\mu(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})]$$

z-component:

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho w u)}{\partial x}+\frac{\partial(\rho w v)}{\partial y}+\frac{\partial(\rho w w)}{\partial z}$$

$$=-\frac{\partial p}{\partial z}+\rho g_z+\frac{\partial}{\partial z}[\mu(2\frac{\partial w}{\partial z}-\frac{2}{3}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}))]$$

$$+\frac{\partial}{\partial x}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]+\frac{\partial}{\partial y}[\mu(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z})]$$

Momentum Equation – The Navier-Stokes Equation

- The above notation in “easy” Cartesian coordinates is a nightmare.
- Use vector analysis notation for a better view:

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\,\mathbf{v}) = -\nabla p + \rho \mathbf{g} + \nabla \cdot \mathbf{T}$$

with stress tensor $\mathbf{T} = \mu\left(\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathbf{T}} - \frac{2}{3}(\nabla \cdot \mathbf{v})\mathbf{I}\right)$

and identity matrix $\mathbf{I}$

Energy Equation

- For flows with temperature differences
- Need energy balance … potentially between many different forms of energy. Potentially lots of physics!
  - e.g. chemistry, radiation, melting, freezing, etc.

$$\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho e u)}{\partial x}+\frac{\partial(\rho e v)}{\partial y}+\frac{\partial(\rho e w)}{\partial z}$$
$$=-\left(\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)\right)$$
$$-p\cdot\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\epsilon+\dot{q}$$

- Includes mass density ρ, internal energy density e, velocity vector **v** = (u,v,w), pressure p, dynamic viscosity μ, thermal conductivity λ, temperature T, a heat source term $\dot{q}$ and heat produced by friction ε