Fluid Dynamics Exercises

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Chapter 1

Introduction

1.1 Simple shear flow

Two plates with the surface A are sheared with velocity u. The plates have the distance h.

Given:

$A = 1\ \mathrm{m}^2$	$u = 1\ \frac{\mathrm{m}}{\mathrm{s}}$	$\mu = 1 \cdot \frac{\mathrm{kg}}{\mathrm{m \cdot s}}$	$h = 1\ \mathrm{m}$

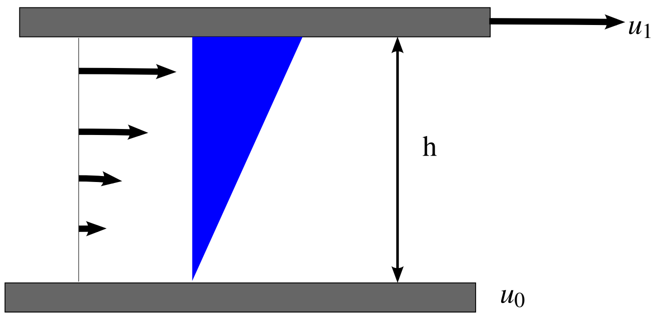

Figure 1.1: Simple shear flow

Task:

a) Which force is needed to move the upper plate?

Solution:

$$
\begin{aligned}
F &= \tau \cdot A \\
&= \frac{\mathrm{d}u}{\mathrm{d}y}\mu \cdot A \\
&= \frac{\Delta u}{\Delta y}\mu \cdot A \\
&= \frac{1\ \frac{\mathrm{m}}{\mathrm{s}}}{1\ \mathrm{m}} 1\ \frac{\mathrm{kg}}{\mathrm{m} \cdot \mathrm{s}} \cdot 1\ \mathrm{m}^2 \\
&= 1\ \mathrm{N}
\end{aligned}
$$

Chapter 2

Hydrostatics

Hydrostatics or fluid statics is an branch of fluid dynamics. It is under the condition that all fluid elements rest under a stable equilibrium. It is important among other thing for the design and calculation of the buoyancy of ships, big reservoirs, and manometers.

Pressure dependent on depth

$$p(z) = p_0 + \rho z g$$

Buoyancy

Buoyancy is an upward force exerted by a fluid that opposes the weight of an immersed object. The principle was first used by Archimedes of Syracuse (287 to 212 BC) to check a king's crown. The crown displaced more water than the gold bar. Therefore the density of the crown was lower and part of the gold was replaced by a less dense material.

$$F_B = \rho_f g V_{Solid}$$

2.1 Pressure measurement

A U-shaped tube filled with air connects a water vessel with a mercury filled manometer as is depicted in figure 2.1. Please be aware that the density of air is very small ($\rho_{air} << \rho_{water} < \rho_{mercury}$).

Given:

$z_1 = 0.5$ m	$h = 0.25$ m	$\rho_{water} = 1000\ \frac{kg}{m^3}$	$\rho_{mercury} = 13600\ \frac{kg}{m^3}$	$g = 10\ \frac{m}{s^2}$	$p_a = 1$ bar

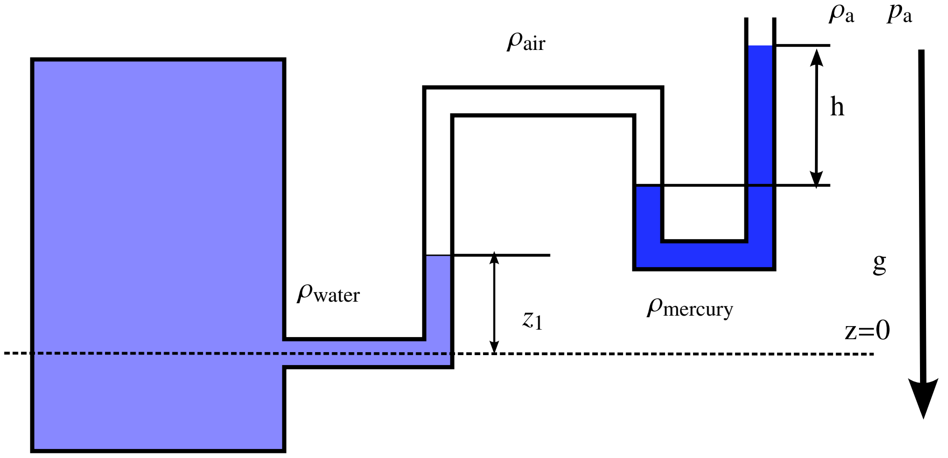

Figure 2.1: Sketch of a pressurized vessel

Task:

Determine

a) The gauge pressure inside the vessel (gauge is the relative pressure) at $z = 0$!

b) The absolute pressure inside the vessel at $z = 0$!

Solution:

a) $p_{\text{gauge}} = 39$ kPa

b) $p_{\text{absolut}} = 139$ kPa

a) The gauge pressure inside the vessel (gauge is the relative pressure) at $z = 0$!

Since the density of air is not provided and it is very small the difference of the pressure caused by the difference in air height can be neglected.
The gauge pressure is defined at the level $z = 0$. Above this level there is water in a height of z_1 and there is additionally a mercury measurement device with a deflection of h.

$$\begin{aligned}
\Delta p = p_g &= \rho_{water} \cdot g z_1 + \rho_{mercury} \cdot g h \\
p_g &= 1000\ \frac{\text{kg}}{\text{m}^3} \cdot 10\ \frac{\text{m}}{\text{s}^2} \cdot 0.5\ \text{m} + 13600\ \frac{\text{kg}}{\text{m}^3} \cdot 10\ \frac{\text{m}}{\text{s}^2} \cdot 0.25\ \text{m} \\
&= 5000\ \text{Pa} + 34,000\ \text{Pa} = 39\ \text{kPa}
\end{aligned}$$

b) The absolute pressure inside the vessel at $z = 0$!

$$p_{abs} = 39\ \text{kPa} + 100\ \text{kPa} = 139\ \text{kPa}$$

2.2 Pistons

In three communicating water chamber the water surface is loaded with pressure by three pistons with different weights.

Given:

$F_1 = 1650$ N	$F_2 = 900$ N	$F_3 = 1500$ N
$A_1 = 0.06$ m^2	$A_2 = 0.03$ m^2	$A_3 = 0.045$ m^2
$\rho_{water} = 1000\ \frac{\text{kg}}{\text{m}^3}$	$g = 10\ \frac{\text{m}}{\text{s}^2}$	

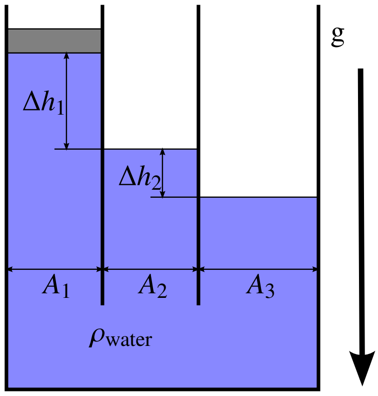

Figure 2.2: Sketch of pistons providing pressure in a water tank

Task:
Determine

a) Δh_1 and Δh_2

Solution:

a) $\Delta h_1 = 0.25$ m and $\Delta h_2 = 0.33$ m

a) Δh_1 and Δh_2

Calculate the pressure induced by the pistons:

$$\begin{aligned}
p_1 &= \frac{F_1}{A_1} = \frac{1650}{0.06} = 27,500 \text{ Pa} \\
p_2 &= \frac{F_2}{A_2} = \frac{900}{0.03} = 30,000 \text{ Pa} \\
p_3 &= \frac{F_3}{A_3} = \frac{1500}{0.045} = 33,333 \text{ Pa}
\end{aligned}$$

Going from piston 1 to piston 2:

$$\begin{aligned}
\Delta p &= \rho g h_1 \\
h_1 &= \frac{p_2 - p_1}{\rho g} = \frac{2500}{10000} \text{ m} = 0.25 \text{ m}
\end{aligned}$$

Going from piston 2 to piston 3:

$$\begin{aligned}
\Delta p &= \rho g h_2 \\
h_2 &= \frac{p_3 - p_2}{\rho g} = \frac{3333}{10000} \text{ m} = 0.333 \text{ m}
\end{aligned}$$

2.3 Vertical force equilibrium

A body with the density $\rho_b = 850\frac{\text{kg}}{\text{m}^3}$ is submerging in three different fluids which have strictly separated phases. The liquids have following densities:

Given:	density liquid 1 ρ_1	$800\frac{\text{kg}}{\text{m}^3}$	gravity acceleration g	$10\frac{\text{m}}{\text{s}^2}$
	density liquid 2 ρ_2	$850\frac{\text{kg}}{\text{m}^3}$	cube size overall a	1m
	density liquid 3 ρ_3	$900\frac{\text{kg}}{\text{m}^3}$	length fluid two l_2	0.2m
	density body ρ_b	$850\frac{\text{kg}}{\text{m}^3}$		

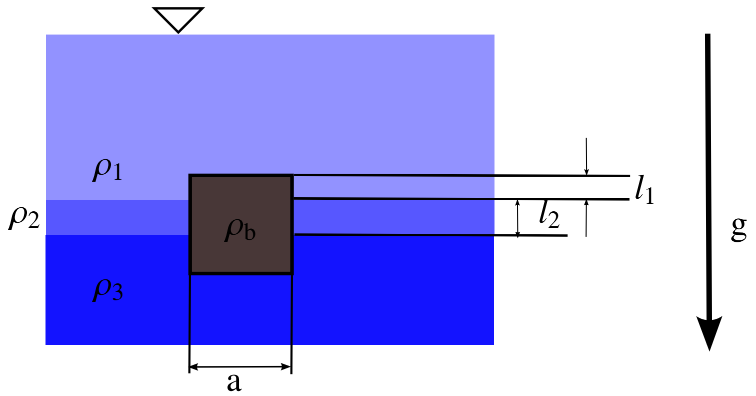

Figure 2.3: Submerging cube floating in three strictly separated fluids.

Determine:

a) the length l_1 of the cube location in liquid 1

Solution:

a) $l_1 = 0.4$ m

a) the length l_1 of the cube location in liquid 1

Equilibrium of Forces:

$$\begin{aligned}
V_b\rho_b g &= V_1\rho_1 g + V_2\rho_2 g + V_3\rho_3 g \qquad ; \frac{1}{g} \\
V_b\rho_b &= V_1\rho_1 + V_2\rho_2 + V_3\rho_3 \qquad \text{; resolve V} \\
a^3\rho_b &= l_1 a^2\rho_1 + l_2 a^2\rho_2 + l_3 a^2\rho_3 \qquad ; \frac{1}{a^2} \\
a\rho_b &= l_1\rho_1 + l_2\rho_2 + l_3\rho_3 \qquad ; l_3 = a - l_2 - l_1 \\
a\rho_b &= l_1\rho_1 + l_2\rho_2 + (a - l_2 - l_1)\rho_3 \\
a\rho_b - l_2\rho 2 - a\rho 3 + l_2\rho 3 &= l_1\rho_1 + -l_1\rho_3 \\
l_1 &= \frac{a\rho_b - l_2\rho 2 - a\rho 3 + l_2\rho 3}{\rho_1 - \rho_3} \\
l_1 &= \frac{1\cdot 850 - 0.2\cdot 850 - 1\cdot 900 + 0.2\cdot 900}{800 - 900}\text{m} \\
l_1 &= \underline{\underline{0.4\text{m}}}
\end{aligned}$$

2.4 Horizontal force and torque

A wall separates two liquids, unknown is the resulting force and its location on the wall:

density liquid 1 ρ	$1000\dfrac{\text{kg}}{\text{m}^3}$
B	10m
h_1	5m
h_2	3m
gravity acceleration g	$g = 10\dfrac{\text{m}}{\text{s}^2}$

Determine:

a) the resultant force on the sluice gate

b) the load application point

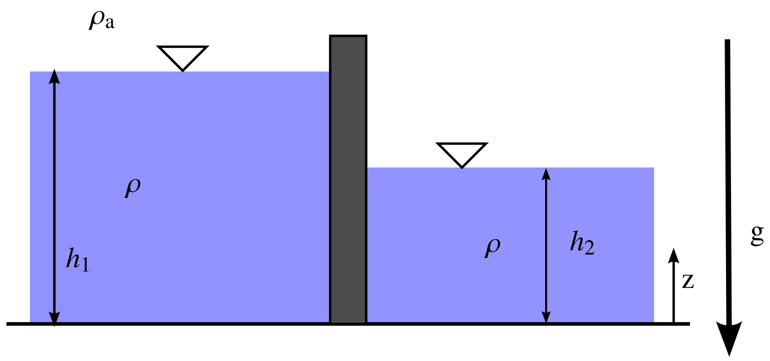

Figure 2.4: Sketch of two basins, unknown resulting force and location on the separation wall

Solution:

a) $F_s = 800$ kN

b) $z_s = 2.04$ m

a) the resultant force on the sluice gate

$$
\begin{aligned}
\mathrm{d}F_s &= B \cdot (p_1(z) - p_2(z))\,\mathrm{d}z \\
p_1(z) &= p_a + \rho g(h_1 - z) \text{ for } z \leq h_1 \\
p_2(z) &= p_a + \rho g(h_2 - z) \text{ for } z \leq h_2 \\
p_2(z) &= p_a \text{ for } h_2 \leq z \leq h_1 \\
F_s &= \int_0^{h_2} B\rho g\,((h_1 - \cancel{z}) - (h_2 - \cancel{z}))\,\mathrm{d}z + \int_{h_2}^{h_1} B\rho g\,(h_1 - z)\,\mathrm{d}z \\
&= \rho g B\left([(h_1 - h_2)\cdot z]_0^{h_2} + \left[h_1 \cdot z - \frac{1}{2}z^2\right]_{h_2}^{h_1}\right) \\
&= \rho g B\left(h_1 h_2 - h_2^2 + h_1^2 - \frac{1}{2}h_1^2 - h_1 h_2 + \frac{1}{2}h_2^2\right) \\
&= \rho g B\left(\frac{1}{2}h_1^2 - \frac{1}{2}h_2^2\right) \\
&= \rho g B\frac{1}{2}\left(h_1^2 - h_2^2\right) = 800 \text{ kN}
\end{aligned}
$$

b) the load application point

$$
\begin{aligned}
M_s &= F_s \cdot z_s = \int_0^{h_1} z \cdot \mathrm{d}F \\
&= \int_0^{h_2} B\rho g\,((h_1 - \cancel{z}) - (h_2 - \cancel{z}))\,z\mathrm{d}z + \int_{h_2}^{h_1} B\rho g\,(h_1 - z)\,z\mathrm{d}z \\
&= \rho g B\left(\left[(h_1 - h_2)\cdot \frac{z^2}{2}\right]_0^{h_2} + \left[h_1 \cdot \frac{z^2}{2} - \frac{z^3}{3}\right]_{h_2}^{h_1}\right) \\
&= \rho g B\left(\left[(h_1 - h_2)\cdot \frac{{h_2}^2}{2}\right] + \left[h_1 \cdot \frac{{h_1}^2}{2} - \frac{{h_1}^3}{3} - h_1 \cdot \frac{{h_2}^2}{2} + \frac{{h_2}^3}{3}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \rho g B\left(\cancel{h_1 \cdot \frac{h_2{}^2}{2}} - \frac{h_2{}^3}{2} + \frac{h_1{}^3}{2} - \frac{h_1{}^3}{3} - \cancel{h_1 \cdot \frac{h_2{}^2}{2}} + \frac{h_2{}^3}{3}\right) \\
&= \rho g B\left(\frac{h_1{}^3}{6} - \frac{h_2{}^3}{6}\right) \\
\Rightarrow z_s &= \frac{M_s}{F_s} \\
z_s &= \frac{\rho g B\left(\frac{h_1{}^3}{6} - \frac{h_2{}^3}{6}\right)}{\rho g B\left(\frac{1}{2}h_1^2 - \frac{1}{2}h_2^2\right)} \\
&= \frac{\left(\frac{h_1{}^3}{6} - \frac{h_2{}^3}{6}\right)}{\left(\frac{1}{2}h_1^2 - \frac{1}{2}h_2^2\right)} = \frac{1}{3} \cdot \frac{h_1^3 - h_2^3}{h_1^2 - h_2^2} \\
&= \frac{1}{3} \cdot \frac{125 - 27}{25 - 9}\ \mathrm{m} = 2.04\ \mathrm{m}
\end{aligned}
$$

2.5 Pressure on a triangular orifice

The triangular orifice of a weir is closed with a plate.
Given:
| $\rho = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ | $B = 1$ m | $g = 10\ \frac{\mathrm{m}}{\mathrm{s}^2}$ |

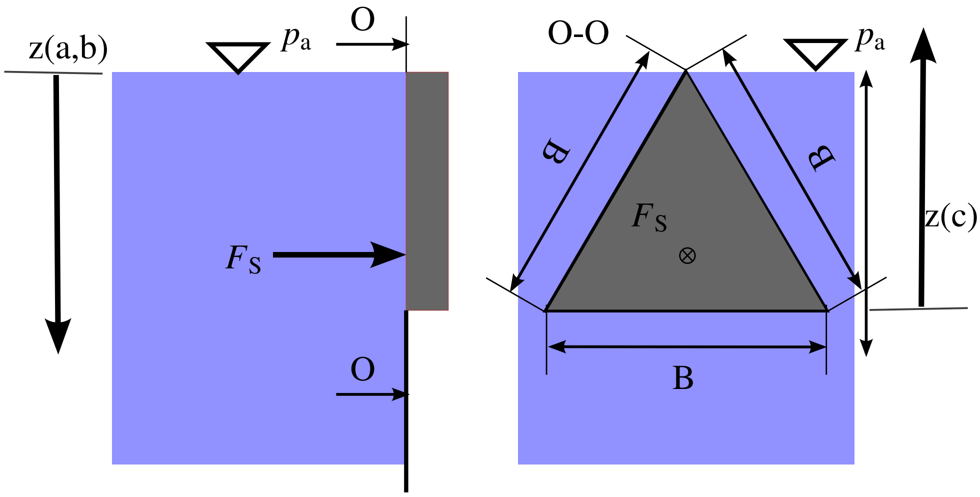

Figure 2.5: Sketch of force on triangle because of water pressure

Determine:

a) the resultant closing force on the plate!

b) the load application point!

c) check both results by staring z from the bottom of the triangle!

Solution:

a) $F_s = 2500$ N

b) $Z_s = 0.65$ m

a) the resultant closing force on the plate!

- Split the triangle in two similar ones. Use this to determine the width to be used along with the integration in the z direction

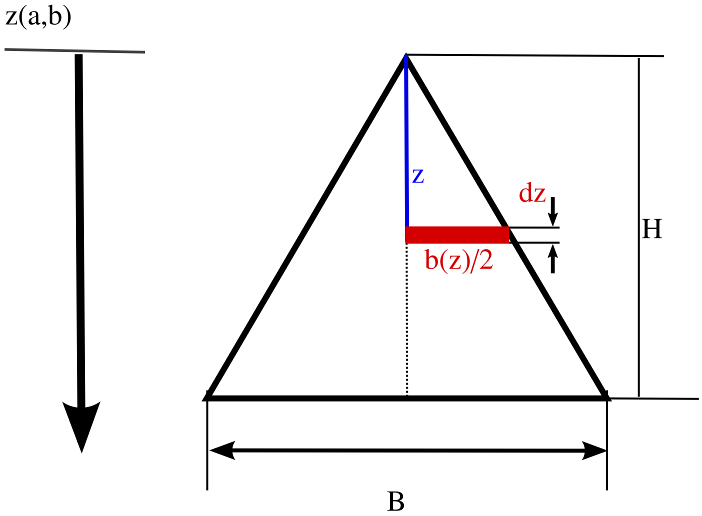

Figure 2.6: Determine the integration width and limits

- trigonometry gives:

$$\frac{1}{2}b(z) = z\tan(\alpha)$$
$$b(z) = 2z\tan(\alpha)$$

- Equilateral triangle $\Rightarrow \alpha = 30^\circ \Rightarrow \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1/2}{\cos\alpha}$

- And this may be used to get

$$b(z) = 2z\tan(\alpha) = 2z\frac{1/2}{\cos\alpha} = \frac{z}{\cos(\alpha)}$$

- integration

$$
\begin{aligned}
\mathrm{d}F_s &= p(z) \cdot b(z)\mathrm{d}z \\
F_s &= \int_0^{B\ \cos(\alpha)} \rho g z \cdot \frac{z}{\cos(\alpha)} \mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)} \int_0^{B\ \cos(\alpha)} z^2 \mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)} \left[\frac{z^3}{3}\right]_0^{B\ \cos(\alpha)} \\
&= \frac{\rho g}{\cos(\alpha)} \frac{B^3 \cos\alpha^3}{3} \\
&= \rho g \frac{B^3 \cos\alpha^2}{3} \\
&= 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} 10\ \frac{\mathrm{m}}{\mathrm{s}^2} \frac{1^3\ \mathrm{m}^3 \left(\frac{\sqrt{3}}{2}\right)^2}{3} = 2500\ \mathrm{N}
\end{aligned}
$$

or:

$$
\begin{aligned}
F_s &= \int_0^{B\ \cos(\alpha)} \rho g z \cdot 2z \tan(\alpha)\, \mathrm{d}z \\
&= \rho g 2 \tan(\alpha) \int_0^{B\ \cos(\alpha)} z^2 \mathrm{d}z \\
&= \rho g 2 \tan(\alpha) \left[\frac{z^3}{3}\right]_0^{B\ \cos(\alpha)} \\
&= \rho g 2 \frac{\sin(\alpha)}{\cos(\alpha)} \frac{B^3 (\cos\alpha)^3}{3} \\
&= \rho g 2 \sin(\alpha) \frac{B^3 (\cos\alpha)^2}{3} \\
&= \rho g 2 \frac{1}{2} \frac{B^3 \left(\frac{\sqrt{3}}{2}\right)^2}{3} \\
&= 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} 10\ \frac{\mathrm{m}}{\mathrm{s}^2} \frac{1^3\ \mathrm{m}^3 \left(\frac{\sqrt{3}}{2}\right)^2}{3} = 2500\ \mathrm{N}
\end{aligned}
$$

b) the load application point!

$$M_s = F_s \cdot Z_s$$

$$\begin{aligned}
\mathrm{d}M_s &= p(z) \cdot b(z) z \mathrm{d}z \\
M_s &= \int\limits_0^{B\,\cos(\alpha)} \rho g z^2 \cdot \frac{z}{\cos(\alpha)} \mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)} \int\limits_0^{B\,\cos(\alpha)} z^3 \mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)} \left[\frac{z^4}{4} \right]_0^{B\,\cos(\alpha)} \\
&= \frac{\rho g}{\cos(\alpha)} \frac{B^4 (\cos\alpha)^4}{4} \\
&= \rho g \frac{B^4 (\cos\alpha)^3}{4} \\
&= 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} 10\ \frac{\mathrm{m}}{\mathrm{s}^2} \frac{1^4\ \mathrm{m}^3 \left(\frac{\sqrt{3}}{2}\right)^3}{4} = 1623.80\ \mathrm{Nm} \\
\Rightarrow Z_s &= \frac{1623.80\ \mathrm{Nm}}{2500\ \mathrm{N}} = 0.65\ \mathrm{m}
\end{aligned}$$

c) check both results by starting z from the bottom of the triangle!

$$\begin{aligned}
\mathrm{d}F_s &= p(z) \cdot b(z) \mathrm{d}z \\
F_s &= \int\limits_0^{B\,\cos(\alpha)} \rho g (B \cos(\alpha) - z) \cdot \frac{(B \cos(\alpha) - z)}{\cos(\alpha)} \mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)} \int\limits_0^{B\,\cos(\alpha)} (B \cos(\alpha) - z)^2 \,\mathrm{d}z \\
&= -\frac{\rho g}{3 \cos(\alpha)} \left[(B \cos(\alpha) - z)^3 \right]_0^{B\,\cos(\alpha)} \\
&= \rho g \frac{B^3 (\cos\alpha)^2}{3} \\
&= 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} 10\ \frac{\mathrm{m}}{\mathrm{s}^2} \frac{1^3\ \mathrm{m}^3 \left(\frac{\sqrt{3}}{2}\right)^2}{3} = 2500\ \mathrm{N}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{d}M_s &= p(z)\cdot b(z)z\mathrm{d}z \\
M_s &= \int\limits_0^{B\,\cos(\alpha)} \rho g(B\cos(\alpha)-z)\cdot\frac{(B\cos(\alpha)-z)}{\cos(\alpha)}z\mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)}\int\limits_0^{B\,\cos(\alpha)} (B\cos(\alpha)-z)^2\,z\mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)}\int\limits_0^{B\,\cos(\alpha)} \left((B\cos(\alpha))^2-2\cdot B\cos(\alpha)\,z+z^2\right)z\mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)}\int\limits_0^{B\,\cos(\alpha)} \left(z(B\cos(\alpha))^2-2z^2\cdot B\cos(\alpha)+z^3\right)\mathrm{d}z \\
&= \frac{\rho g}{\cos(\alpha)}\left[\frac{z^2}{2}(B\cos(\alpha))^2-\frac{2z^3}{3}\cdot B\cos(\alpha)+\frac{z^4}{4}\right]_0^{B\,\cos(\alpha)} \\
&= \frac{\rho g}{\cos(\alpha)}\left[\frac{(B\cos(\alpha))^2}{2}(B\cos(\alpha))^2-\frac{2(B\cos(\alpha))^3}{3}\cdot B\cos(\alpha)+\frac{(B\cos(\alpha))^4}{4}\right] \\
&= \frac{(B\cos(\alpha))^4\rho g}{\cos(\alpha)}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] \\
&= B^4(\cos(\alpha))^3\rho g\frac{1}{12} \\
&= \frac{1}{12}1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}1^4\mathrm{m}^4 10\,\frac{\mathrm{m}}{\mathrm{s}^2}\left(\frac{\sqrt{3}}{2}\right)^3 \\
&= 541.27\text{ Nm}\Rightarrow Z_s=\frac{541.27\text{ Nm}}{2500\text{ N}}=0.217\text{ m}
\end{aligned}
$$

In the other coordinate system this would be the location of $B\cos(\alpha)-0.217$ m $=$ 1 m$\left(\frac{\sqrt{3}}{2}\right)-0.217$ m $=0.649$ m.

2.6 Balloon

A balloon has a total mass of 600 kg and a volume of 700 m^3. The air properties on the ground are given below:

Given:

$p_0 = 101.325$ kPa	$\rho_0 = 1.225\ \frac{\text{kg}}{\text{m}^3}$	$g = 10\ \frac{\text{m}}{\text{s}^2}$

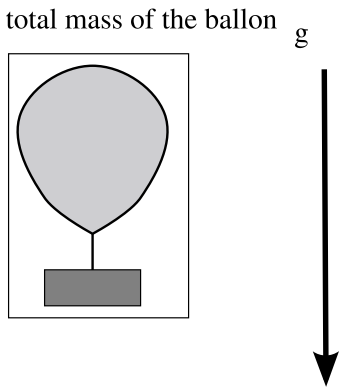

Figure 2.7: Sketch of the balloon with gondola

Task:

Determine

a) the height h_{max} where the balloon is rising up to in an isothermal atmosphere!

Solution:

a) $h_{max} = 2954$ m

a) the height h_{max} where the balloon is rising up to in an isothermal atmosphere!

Calculate the average density of the whole balloon:

$$\rho_{balloon} = \frac{600 \text{ kg}}{700 \text{ m}^3} = \frac{6}{7} \frac{\text{kg}}{\text{m}^3}$$

We include this in the density equation of the script (2.5), since the average density of the balloon has to be equal of the density of the air at the height z:

$$\begin{aligned}
\rho_{balloon} &= \rho_0 \cdot e^{\frac{-z \cdot g}{R_i T_0}} \quad \text{with } \frac{p_0}{\rho_0} = R_i \cdot T \\
\frac{\rho_{balloon}}{\rho_0} &= e^{\frac{-z \cdot g \cdot \rho_0}{p_0}} \\
\ln\left(\frac{\rho_{balloon}}{\rho_0}\right) &= \frac{-z \cdot g \cdot \rho_0}{p_0} \\
-z &= \ln\left(\frac{\rho_{balloon}}{\rho_0}\right) \frac{p_0}{g \cdot \rho_0} \\
-z &= \ln\left(\frac{(\frac{6}{7})}{1.225}\right) \cdot \frac{101325 \frac{\text{N}}{\text{m}^2}}{1.225 \frac{\text{kg}}{\text{m}^3} \cdot 10 \frac{\text{m}}{\text{s}^2}} \\
z &= (-1) \cdot (-0.3571) \cdot 8271.43 \text{ m} \\
z &= 2954 \text{ m}
\end{aligned}$$

Chapter 3

Continuity and Energy Equation

This part of the practical work deals with the continuity and the energy equation. The energy equation is also called Bernoulli equation. If we only apply constant density, the continuity equation gives us the correlation of the speeds in the streamline at the investigated positions.

- Continuity equation

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

- Bernoulli equation

$$p + \rho \frac{v^2}{2} + \rho g z = \text{const.}$$

3.1 Closed container with outflow

In a closed container there are two non-mixed liquids with the densities ρ_1 and ρ_2. Above the two liquids there is an absolute pressure $p_{abs} = p_o + \Delta p$. At the bottom of the container there is a small orifice ($A_1 >> A_2$, see figure 3.1) where the liquid discharges frictionless. The flow-field is in steady state condition and the filling level is changing very slow.

Given:

$h_1 = 6$ m	$h_2 = 4$ m	$\rho_1 = 1000\ \frac{\text{kg}}{\text{m}^3}$	$\rho_2 = 800\ \frac{\text{kg}}{\text{m}^3}$	$g = 10\ \frac{\text{m}}{\text{s}^2}$
$p_0 = 1$ bar	$\Delta p = 0.5$ bar			

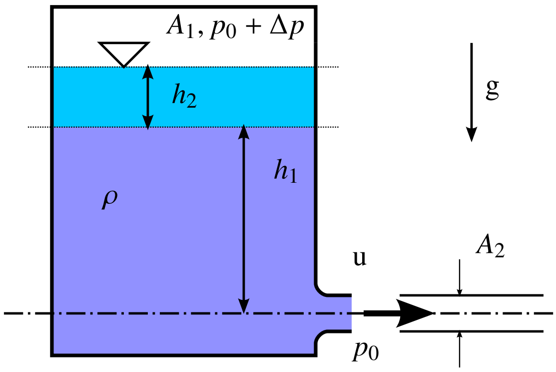

Figure 3.1: Sketch of a water reservoir under pressure with two fluids

Task:
Determine

a) the velocity u coming out of the orifice for the given level of fluids!

Solution:

a) $u = 16.85\ \frac{\text{m}}{\text{s}}$

a) the velocity u coming out of the orifice for the given level of fluids!

Bernoulli equation from point 1 (top of liquid 2; A_1) and point 2 (outlet with A_2). Please Note: The pressure at the interface between liquid 1 and 2 is $p_0 + \Delta p + \rho_2 \cdot h_2 \cdot g$.

$$
\begin{aligned}
p + \rho \frac{u^2}{2} + \rho g h &= \text{const.} \\
\cancel{p_0} + \rho_1 \frac{u^2}{2} &= \cancel{p_0} + \Delta p + \rho_2 \cdot h_2 \cdot g + \rho_1 \frac{\cancelto{0}{u_1^2}}{\cancel{2}} + \rho_1 \cdot h_1 \cdot g \\
\rho_1 \frac{u^2}{2} &= \Delta p + (\rho_1 \cdot h_1 + \rho_2 \cdot h_2) \cdot g \\
u^2 &= \left(\frac{2\Delta p + 2(\rho_1 \cdot h_1 + \rho_2 \cdot h_2) \cdot g}{\rho_1} \right) \\
u &= \sqrt{\frac{2(\Delta p + \rho_1 \cdot h_1 \cdot g + \rho_2 \cdot h_2 \cdot g)}{\rho_1}} \\
u &= \sqrt{\frac{2(50,000\ \frac{\mathrm{N}}{\mathrm{m}^2} + 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 6\ \mathrm{m} \cdot 10\ \frac{\mathrm{m}}{\mathrm{s}^2} + 800\ \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 4\ \mathrm{m} \cdot 10\ \frac{\mathrm{m}}{\mathrm{s}^2})}{1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}}} \\
u &= \sqrt{\frac{284\ \frac{\mathrm{N}}{\mathrm{m}^2}}{1\ \frac{\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}^2}}{\mathrm{m}^3 \cdot \frac{\mathrm{m}}{\mathrm{s}^2}}}} = 16.85\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

3.2 Free fall water stream

Water is frictionless discharging out of a big reservoir driven by the gravity force

Given:

| $h_1 = 0.1$ m | $h_2 = 1.5$ m | $D = 0.1$ m |

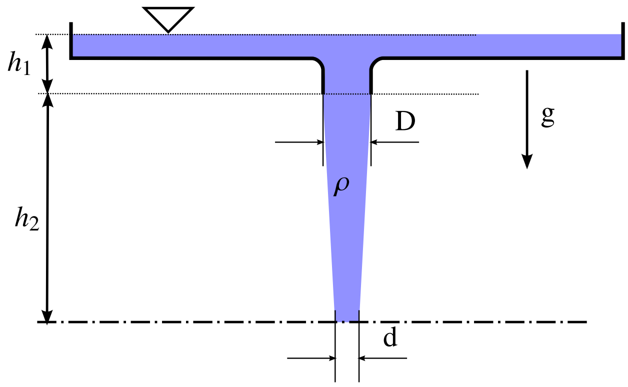

Figure 3.2: Sketch of the water discharging and falling

Task:

Determine

a) the diameter d of the stream at the height h_2 below the orifice!

Solution:

a) $d = 0.05$ m

a) the diameter d of the stream at the height h_2 below the orifice!

The volume flow is constant at the outflow of the orifice and h_2 below the orifice. But the velocity at the two locations are unknown.

- Bernoulli equation from top to the orifice

$$\begin{aligned}
\cancel{p_a} + \rho \cancel{\frac{v_0^2}{2}}^{0} + h_1 \cdot \rho \cdot g &= \cancel{p_a} + \rho \frac{v_D^2}{2} \\
\frac{v_D^2}{2} &= h_1 \cdot g \\
v_D &= \sqrt{2 \cdot h_1 \cdot g} = \sqrt{2 \cdot 0.1 \text{ m} \cdot 10 \ \frac{\text{m}}{\text{s}^2}} = \sqrt{2} \ \frac{\text{m}}{\text{s}}
\end{aligned}$$

- Bernoulli equation from top to the h_2 below the orifice

$$\begin{aligned}
\cancel{p_a} + \rho \cancel{\frac{v_0^2}{2}}^{0} + (h_2 + h_1) \cdot \rho \cdot g &= \cancel{p_a} + \rho \frac{v_d^2}{2} \\
\frac{v_d^2}{2} &= (h_2 + h_1) \cdot g \\
v_d &= \sqrt{2 \cdot (h_2 + h_1) \cdot g} = \sqrt{2 \cdot 1.6 \text{ m} \cdot 10 \ \frac{\text{m}}{\text{s}^2}} = \sqrt{32} \ \frac{\text{m}}{\text{s}}
\end{aligned}$$

- continuity equation between the orifice and h_2 below the orifice

$$\begin{aligned}
\frac{D^2}{4} \pi \cdot v_D &= \frac{d^2}{4} \pi \cdot v_d \\
D^2 \cdot v_D &= d^2 \cdot v_d \\
d^2 &= D^2 \cdot \frac{v_D}{v_d} \\
d &= \sqrt{D^2 \cdot \frac{v_D}{v_d}} = 0.1 \text{ m} \cdot \sqrt{\frac{\sqrt{2}}{\sqrt{32}}} = 0.1 \text{ m} \sqrt{\frac{1}{4}} = 0.1 \text{ m} \frac{1}{2} = 0.05 \text{ m}
\end{aligned}$$

3.3 Flow measurement

A constant frictionless water flow through a pipe is measured by means of a Venturi nozzle. The pressure difference is measured by a mercury filled U-manometer. The water in the U-manometer should NOT be neglected, but the difference in height of the water column that results from the different diameters in the Venturi nozzle shall be neglected.

Given:

$d_1 = 0.08$ m	$d_2 = 0.06$ m	$\Delta h = 0.49$ m	$\rho_W = 1000\ \frac{\text{kg}}{\text{m}^3}$	$\rho_M = 13,600\ \frac{\text{kg}}{\text{m}^3}$
$g = 10\ \frac{\text{m}}{\text{s}^2}$				

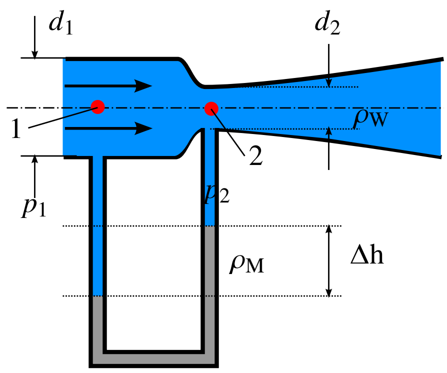

Figure 3.3: Sketch of a pressure measurement using a Venturi nozzle

Task: Determine the water volume flow rate!

Solution:

a) $\dot{V} = 0.0380\ \frac{\text{m}^3}{\text{s}}$

Task: Determine the water volume flow rate!

We get the flow velocity as a result of the pressure difference.

- Calculate the pressure difference from hydrostatics:

$$\begin{aligned}
\Delta p &= \Delta h \cdot \Delta \rho \cdot g \\
\Delta p &= 0.49 \text{ m} \cdot 12600 \, \frac{\text{kg}}{\text{m}^3} \cdot 10 \, \frac{\text{m}}{\text{s}^2} = 61740 \text{ Pa}
\end{aligned}$$

- Use the continuity equation between point 1 and 2 to get a relation between v_1 and v_2.

$$\begin{aligned}
v_1 \cdot \rho_W \cdot \frac{d_1^2}{4}\pi &= v_2 \cdot \rho_W \cdot \frac{d_2^2}{4}\pi \\
v_1 \cdot d_1^2 &= v_2 \cdot d_2^2 \\
v_2 &= v_1 \frac{d_1^2}{d_2^2}
\end{aligned}$$

- Use the Bernoulli equation between point 1 and 2 to compute v_1.

$$\begin{aligned}
\cancel{p_1} + \rho_W \frac{v_1^2}{2} &= \cancel{p_1} - \Delta p + \rho_W \frac{v_2^2}{2} \\
\rho_W \frac{v_1^2}{2} &= -\Delta p + \rho_W \frac{v_2^2}{2} \qquad \text{with conti.} \\
\rho_W \frac{v_1^2}{2} &= -\Delta p + \rho_W \frac{v_1^2 \cdot \left(\frac{d_1^4}{d_2^4}\right)}{2} \\
\rho_W \frac{v_1^2 \left(1 - \left(\frac{d_1^4}{d_2^4}\right)\right)}{2} &= -\Delta p \\
v_1^2 \left(1 - \frac{d_1^4}{d_2^4}\right) &= -\frac{2\Delta p}{\rho_W} \\
v_1^2 &= -\frac{2\Delta p}{\rho_W \cdot \left(1 - \frac{d_1^4}{d_2^4}\right)} \\
v_1 &= \sqrt{\frac{2\Delta p}{\rho_W \cdot \left(\frac{d_1^4}{d_2^4} - 1\right)}}
\end{aligned}$$

- Compute the volume flow rate from $\dot{V} = A_1 \cdot v_1$

$$\dot{V} = \pi \frac{d_1^2}{4} \sqrt{\frac{2\Delta p}{\rho_W \cdot \left(\frac{d_1^4}{d_2^4} - 1\right)}}$$

$$\dot{V} = \pi \frac{(0.08)^2}{4} \text{ m}^2 \sqrt{\frac{123480 \ \frac{\text{N}}{\text{m}^2}}{1000 \ \frac{\text{kg}}{\text{m}^3} \cdot \left(\frac{0.08^4}{0.06^4} - 1\right)}}$$

$$\dot{V} = \pi \frac{0.0064}{4} \text{ m}^2 \sqrt{\frac{123480 \ \frac{\text{N}}{\text{m}^2}}{1000 \ \frac{\text{kg}}{\text{m}^3} \cdot 2.16}}$$

$$\dot{V} = 0.0380 \ \frac{\text{m}^3}{\text{s}} = 38\ell/s$$

3.4 Downward conical nozzle

At the end of a vertical pipe with a downward flow there is a truncated conical nozzle[1]. The diameter decreases form d_1 to d_2 over the Length L. The static pressure is identical at the location of d_1 and d_2 and is the ambient pressure p_a.

Given:

$d_1 = 1$ m	$d_2 = 0.6$ m	$L = 2.0$ m	$g = 10\ \frac{\text{m}}{\text{s}^2}$

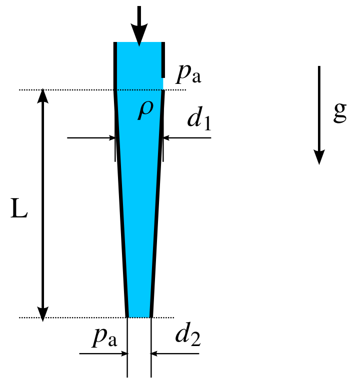

Figure 3.4: Sketch of a conical nozzle with same static pressure p_a at the locations d_1 and d_2

Task:

- Determine the volume flow rate.

Solution:

a) $\dot{V} = 1.917\ \frac{\text{m}^3}{\text{s}}$

[1] Düse in Form eines geraden Kegelstupfs

- Determine the volume flow rate.

The volume flow rate is constant between the positions with diameters d_1 and d_2. But the velocity at the two locations is unknown.

- Bernoulli equation between locations with d_1 to d_2:

$$\begin{aligned}
\cancel{p_a} + \rho\frac{v_1^2}{2} + L \cdot \rho \cdot g &= \cancel{p_a} + \rho\frac{v_2^2}{2} \\
\frac{v_1^2}{2} + L \cdot g &= \frac{v_2^2}{2}
\end{aligned}$$

- Use continuity equation between locations with d_1 to d_2:

$$\begin{aligned}
\frac{d_1^2}{4}\pi \cdot v_1 &= \frac{d_2^2}{4}\pi \cdot v_2 \\
v_2 &= \frac{d_1^2}{d_2^2} v_1
\end{aligned}$$

- Use this in Bernoulli equation

$$\begin{aligned}
\frac{v_1^2}{2} + L \cdot g &= \frac{v_1^2 \cdot \left(\frac{d_1^4}{d_2^4}\right)}{2} \\
v_1^2 \cdot \left(1 - \frac{d_1^4}{d_2^4}\right) &= -2L \cdot g \\
v_1 &= \sqrt{\frac{2L \cdot g}{\left(\frac{d_1^4}{d_2^4} - 1\right)}}
\end{aligned}$$

- Volume flow rate follows from:

$$\begin{aligned}
\dot{V} &= \frac{d_1^2}{4}\pi \cdot \sqrt{\frac{2L \cdot g}{\left(\frac{d_1^4}{d_2^4} - 1\right)}} \\
\dot{V} &= \frac{1}{4}\,\mathrm{m}^2\pi \cdot \sqrt{\frac{2 \cdot 2\,\mathrm{m} \cdot 10\,\frac{\mathrm{m}}{\mathrm{s}^2}}{\left(\frac{1^4}{0.6^4} - 1\right)}} = 1.9167\,\frac{\mathrm{m}^3}{\mathrm{s}}
\end{aligned}$$

3.5 Carburetor

Ambient air flow is accelerated in the Venturi nozzle of a carburetor (Vergaser) as shown in the graph. Thereby the fuel is sucked from the tank through the small pipe (cross section Area A_p) into the carburetor. Friction shall be neglected.

Given:

$A_p = 2\ \mathrm{mm}^2$	$\Delta h = 0.02\ \mathrm{m}$	$\rho_a = 1.2\ \frac{\mathrm{kg}}{\mathrm{m}^3}$	$\rho_f = 840\ \frac{\mathrm{kg}}{\mathrm{m}^3}$	$g = 10\ \frac{\mathrm{m}}{\mathrm{s}^2}$

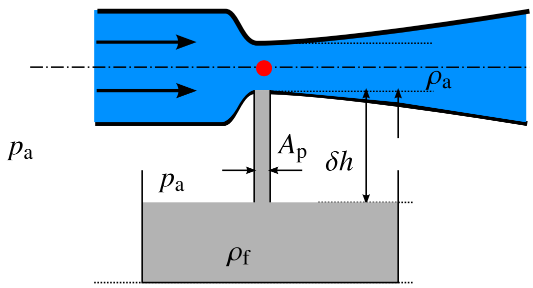

Figure 3.5: Sketch of a carburetor

Task:
Determine the velocity of the air in the nozzle if $7.2\ \frac{\mathrm{l}}{\mathrm{h}}$ of fuel are to be delivered!
Hint: Set up the Bernoulli equation for the air flow and the fuel flow!

Solution:

a) $v_{air} = 31.30\ \frac{\mathrm{m}}{\mathrm{s}}$

Determine the velocity of the air in the nozzle if 7.2 $\frac{\mathrm{l}}{\mathrm{h}}$ of fuel are to be delivered!

- Use the Bernoulli equation for the fuel to determine Δp

First we need to know the speed of the fuel flow.

$$
\begin{aligned}
v_{fuel} \cdot A_p &= 7.2\ \frac{\mathrm{l}}{\mathrm{h}} \\
v_{fuel} &= \frac{\frac{7.2}{1000 \cdot 3600}\ \frac{\mathrm{m}^3}{\mathrm{s}}}{2 \cdot 10^{-6}\ \mathrm{m}^2} = 1\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}
$$

Bernoulli for the fuel:

$$
p_a = p_c + \frac{1}{2} v_f^2 \rho_f + \rho_f \cdot g \cdot \Delta h
$$

Bernoulli for the air:

$$
p_a = p_c + \frac{1}{2} v_a^2 \rho_a
$$

Combine the two equations:

$$
\begin{aligned}
\cancel{p_c} + \frac{1}{2} v_f^2 \rho_f + \rho_f \cdot g \cdot \Delta h &= \cancel{p_c} + \frac{1}{2} v_a^2 \rho_a \\
v_a^2 &= \frac{v_f^2 \rho_f + 2\rho_f \cdot g \cdot \Delta h}{\rho_a} \\
v_a &= \sqrt{\frac{v_f^2 \rho_f + 2\rho_f \cdot g \cdot \Delta h}{\rho_a}}
\end{aligned}
$$

$$
v_a = \sqrt{\frac{1^2\ \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 840\ \frac{\mathrm{kg}}{\mathrm{m}^3} + 2 \cdot 840\ \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 10\ \frac{\mathrm{m}}{\mathrm{s}^2} \cdot 0.02\ \mathrm{m}}{1.2}} = 31.30\ \frac{\mathrm{m}}{\mathrm{s}}
$$

3.6 Two water reservoirs

Two water reservoirs of different height are connected by a pipe. The water flows frictionless from the upper to the lower reservoir. At the end of the pipe is a nozzle with the cross section area A_N to prevent cavitation inside the pipe. The upper and lower reservoirs have a very large surface compared to the pipe cross section area $A_{upper}; A_{lower} >> A > A_N$.

Given:

$A = 1\ \text{m}^2$	$A_N = 0.1\ \text{m}^2$	$h = 5\ \text{m}$	$H = 80\ \text{m}$	$s = 16\ \text{m}$	$p_0 = 10^5\ \text{Pa}$	$\rho = 1000\ \frac{\text{kg}}{\text{m}^3}$	$g = 10\ \frac{\text{m}}{\text{s}^2}$

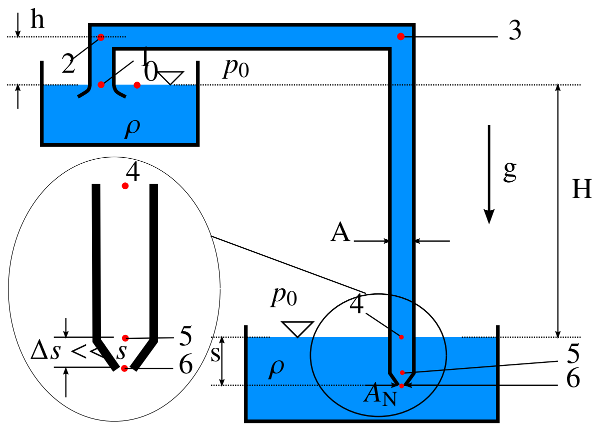

Figure 3.6: Sketch of the two water reservoirs

Task:

a) Determine the water volume flow rate!

b) Sketch the curve of the static pressure along the seven points of the pipe!

c) Determine the cross section A_N^* of the nozzle where the water starts to vaporize somewhere in the pipe (vapor pressure $p_v = 2500$ Pa)!

Solution:

a) $\dot{V} = 4\ \frac{\text{m}^3}{\text{s}}$

b) See full solution.

c) $A_N^* = 0.244\ \text{m}^2$

a) Determine the water volume flow rate!

- We need the velocity at the nozzle outlet:

First we need to know the pressure p_6 at the outlet

$$p_6 = p_0 + \rho \cdot g \cdot s$$

Bernoulli from point 0 to point 6:

$$\begin{aligned}
\cancel{p_0} + \frac{1}{2}\cancelto{0}{v_0^2}\rho + \rho \cdot g \cdot (H + \cancel{s}) &= \cancel{p_0} + \rho \cdot g \cdot \cancel{s} + \rho \frac{1}{2} v_6^2 + \cancelto{0}{\rho \cdot g \cdot 0} \\
\rho \cdot g \cdot H &= \rho \frac{1}{2} v_6^2 \\
v_6 &= \sqrt{2 \cdot g \cdot H} = \sqrt{1600 \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2} = 40\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

Volume flow:

$$\dot{V} = v_6 \cdot A_N = 40\ \frac{\mathrm{m}}{\mathrm{s}} \cdot 0.1\ \mathrm{m}^2 = 4\ \frac{\mathrm{m}^3}{\mathrm{s}}$$

b) Sketch the curve of the static pressure along the seven points of the pipe!

Calculate dynamic pressure for cross section areas A_N and A:

$$p_{d;A_N} = \rho \frac{v_6^2}{2} = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} \frac{40^2}{2} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 800,000\ \mathrm{Pa}$$

Speed at point 4 with cross section area A:

$$v_A = \frac{\dot{V}}{A} = \frac{4\ \frac{\mathrm{m}^3}{\mathrm{s}}}{1\ \mathrm{m}^2} = 4\ \frac{\mathrm{m}}{\mathrm{s}}$$

$$p_{d;A} = \rho \frac{v_A^2}{2} = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} \frac{4^2}{2} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 8,000\ \mathrm{Pa}$$

c) Determine the cross section A_N^* of the nozzle where the water starts to vaporize somewhere in the pipe (vapor pressure $p_v = 2500$ Pa)!

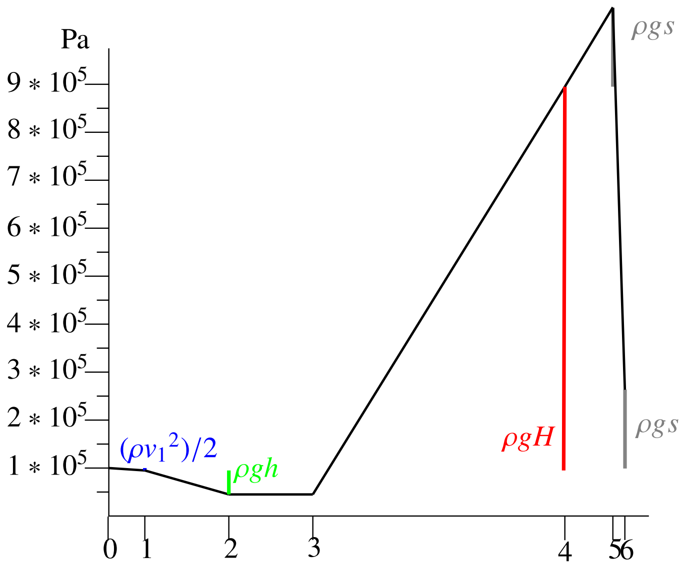

Figure 3.7: Static pressure sketch

We know that the lowest static pressure is at locations between 2 and 3. The static inlet pressure is reduced by the difference in height h and the dynamic pressure of the pipe velocity with the surface A. We also know that the volume flow is constant and the velocity of point 6 is just dependent on the height H.

Bernoulli equation from point 0 to point 3∗:

$$p_0 \quad = \quad 2500 \text{ Pa} + \rho \frac{v_{3*}^2}{2} + \rho g h$$

Continuity equation from point 3∗ to point 6:

$$v_{3*} \cdot A \quad = \quad v_6 \cdot A_N^*$$
$$v_{3*} = \frac{A_N^*}{A} v_6$$

The continuity equation included in the Bernoulli equation leads to:

$$
\begin{aligned}
100,000 \text{ Pa} &= 2500 \text{ Pa} + \rho \frac{v_6^2 \cdot \left(\frac{A_N^*}{A}\right)^2}{2} + \rho g h \\
100,000 \text{ Pa} &= 2500 \text{ Pa} + \rho g H \cdot \left(\frac{A_N^*}{A}\right)^2 + \rho g h \\
97,500 \text{ Pa} - \rho g h &= \rho g H \cdot \left(\frac{A_N^*}{A}\right)^2 \\
\frac{97,500 \text{ Pa} - \rho g h}{\rho g H} &= \left(\frac{A_N^*}{A}\right)^2 \\
\left(\frac{A_N^*}{A}\right) &= \sqrt{\frac{97,500 \text{ Pa}}{\rho g H} - \frac{h}{H}} \\
A_N^* &= 1 \text{ m}^2 \sqrt{\frac{97,500 \text{ Pa}}{1000 \,\frac{\text{kg}}{\text{m}^3} \cdot 10 \,\frac{\text{m}}{\text{s}^2} \cdot 80 \text{ m}} - \frac{5}{80}} = 0.244 \text{ m}^2
\end{aligned}
$$

Chapter 4

Momentum equation

Here we want to briefly remind us about the momentum equation.

Variant for a stationary, incompressible flow,
where $\mathbf{v}$ and $\mathbf{A} = A\mathbf{n}$ need not to be constant (variant 1):

$$\sum \mathbf{F} = \rho \int_{CS} \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) \mathrm{d}A$$

additionally assume constant velocities an inlets and outlets
as well as plane inlets and outlets (variant 2):

$$\sum \mathbf{F} = \sum_{ports} (\rho\, \mathbf{v})\mathbf{v} \cdot \mathbf{A}$$

additionally write the signs of the mass flow rates explicitly (variant 3):

$$\sum \mathbf{F} = + \sum_{outlets} \mathbf{v}_o\, \dot{m}_o - \sum_{inlets} \mathbf{v}_i\, \dot{m}_i$$

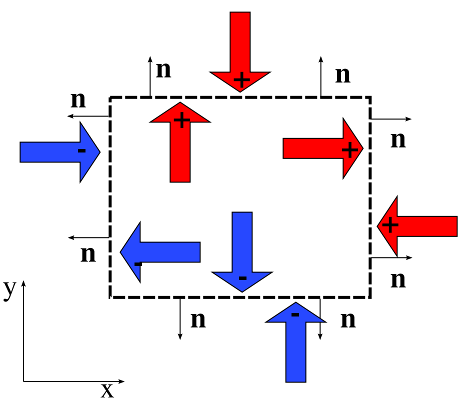

Figure 4.1: Effective signs for in- and outflow in a simple geometry.

As an example, we want to assume a mass-flow ($\mathbf{v} = 1\ \frac{\mathrm{m}}{\mathrm{s}}$; $\rho = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$; $A = 1\ \mathrm{m}^2$) going into a control volume in x-direction and exiting in negative x-direction, cf. figure 4.2. We want to compute the force on the wall necessary to turn the flow. Note that the inlet and outlet area normal vector is $\mathbf{n} = (-1, 0)$ and thus $\mathbf{A} = A\mathbf{n} = (-A, 0)$. Inflow velocity is $\mathbf{v}_{in} = (v, 0)$ and outflow velocity is $\mathbf{v}_{out} = (-v, 0)$.

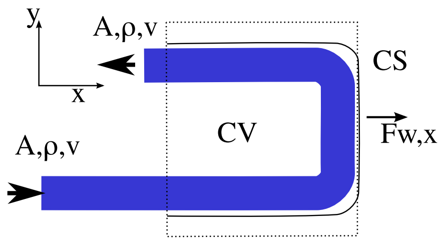

Figure 4.2: Control volume (CV) with control surface (CS) and flow definitions

The momentum equation in x-direction is what we need:

$$
\begin{aligned}
&\text{variant 1:} \quad \sum \mathbf{F} = && \rho \int_{CS} \mathbf{v}(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A = \rho \int_{in} \mathbf{v}(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A + \rho \int_{out} \mathbf{v}(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A \\
&\text{variant 1:} \quad \sum F_x = && \rho \int_{in} v_x(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A + \rho \int_{out} v_x(\mathbf{v}\cdot\mathbf{n})\mathrm{d}A \\
&\text{variant 1:} \quad \sum F_x = && \rho v_x(\mathbf{v}\cdot\mathbf{n}) \int_{in} \mathrm{d}A + \rho v_x(\mathbf{v}\cdot\mathbf{n}) \int_{out} \mathrm{d}A \quad \text{Integrands are constant here.} \\
&\text{variant 1:} \quad F_{w,x} = && \rho v(v\cdot(-1))A + \rho\cdot(-v)\cdot((-v)\cdot(-1))A \\
&\text{variant 1:} \quad F_{w,x} && = -2\rho v^2 A \quad = -2\cdot 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}\cdot 1^2 \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \cdot 1\ \mathrm{m}^2 \\
&\text{variant 1:} \quad F_{w,x} && = -2000\ \mathrm{N}
\end{aligned}
$$

$$
\begin{aligned}
&\text{variant 2:} \quad \sum \mathbf{F} && = \sum_{ports} (\rho\,\mathbf{v})\mathbf{v}\cdot\mathbf{A} \\
&\text{variant 2:} \quad F_{w,x} && = (\rho v)\, v(-A) + (\rho(-v))\,(-v)(-A) \\
&\text{variant 2:} \quad F_{w,x} && = -2\rho v A v \quad = -2\cdot 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}\cdot 1^2 \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \cdot 1\ \mathrm{m}^2 \\
&\text{variant 2:} \quad F_{w,x} && = -2000\ \mathrm{N}
\end{aligned}
$$

Take velocities all positive, but take signs from figure 4.1:

$$
\begin{aligned}
&\text{simple:} \quad \sum F_x && = -\rho A v v - \rho A v v \text{ ; with sign definition from the picture} \\
&\text{simple:} \quad F_{w,x} && = -2\rho v^2 A \quad = -2\cdot 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}\cdot 1^2 \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \cdot 1\ \mathrm{m}^2 \\
&\text{simple:} \quad F_{w,x} && = -2000\ \mathrm{N}
\end{aligned}
$$

4.1 Rectangular water jet hitting a blade

A rectangular water jet (height h, width b) impinges onto a rectangular blade (seen from x-direction) with the velocity $u_x = U$. After hitting the bent plate the jet is divided into two streams of the same volume flow (height $h/2$). Ignore gravity and friction.

Given:

$h = 0.1$ m	$b = 0.2$ m	$u_x = 10\ \frac{\text{m}}{\text{s}}$	$\rho_w = 1000\ \frac{\text{kg}}{\text{m}^3}$

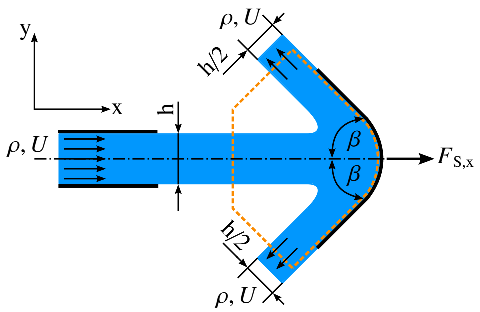

Figure 4.3: Jet hitting a blade

Tasks:

a) Determine the force $F_{S,x}$ (as a function $f(\beta)$) affecting the fluid.

b) For which angle β does the force reach its maximum?

Solution:

a) $F_{S,x} = -2\ kN\ (1 + \cos\beta)$

b) $\beta = 0°$

- Task a
- solution continuity equation:

$$
\begin{aligned}
A_1 \cdot v &= A_2 \cdot v \\
hbU_{in} &= 2\frac{h}{2}bU_{out} \\
U_{in} &= U_{out} = U
\end{aligned}
$$

- solution momentum equation:

$$
\begin{aligned}
\text{simple}: \quad & -\dot{m}(\mathbf{v})_1 + \dot{m}(\mathbf{v})_2 && = \sum F_x \\
\text{simple}: \quad & -\rho AUU - \rho AU\cos(\beta)\,U && = F_{S,x} \text{ ; with sign definition from the picture} \\
\text{simple}: \quad & F_{S,x} && = -(\rho hbU)\,U - (\rho hbU)\,U \cdot \cos(\beta) \\
\text{simple}: \quad & F_{S,x} && = -\rho hbU^2(1 + \cos(\beta)) \\
\text{simple}: \quad & F_{S,x} && = -1000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 10^2 \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \cdot 0.2 \cdot 0.1\ \mathrm{m}^2(1 + \cos(\beta)) \\
\text{simple}: \quad & F_{S,x} && = -2000(1 + \cos(\beta))\ \mathrm{N}
\end{aligned}
$$

The unknown force is $F_{S,x} = -2000(1 + \cos(\beta))$ N.

- Task b

We search for the extreme points of our force function:

$$
\begin{aligned}
\frac{\mathrm{d}F_{S,x}}{\mathrm{d}\beta} &= \frac{\mathrm{d}(-2000\ \mathrm{N}(1 + \cos(\beta))}{\mathrm{d}\beta} = 0 \\
0 &= -2000\ \mathrm{N}(0 - \sin(\beta)) \quad \Rightarrow \underline{\beta_1 = 0^\circ} \ ; \ \beta_2 = 180^\circ
\end{aligned}
$$

The maximum force is at angle $\beta = 0^\circ$.

4.2 Rectangular water jet hitting a wall

An inclined (= geneigt) (angle β) rectangular water jet (height h, width b) strikes onto a vertical wall with the velocity U and breaks into upward and downward jets (velocity U, height εh and $(1-\varepsilon)h$ resp.). Ignore gravity and friction.

Given:

- h, b, U, ρ_W, β

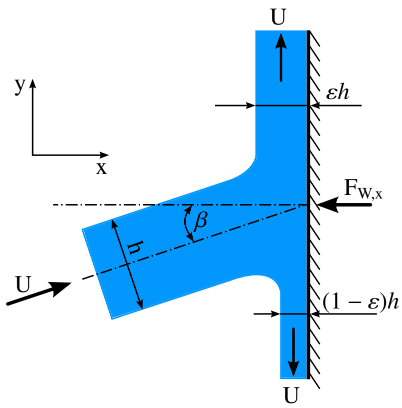

Figure 4.4: Jet hitting a wall

Tasks:

a) Determine the force $F_{W,x}$.

b) Derive a function for ε dependent on β.

Solution:

a) $F_{W,x} = \rho U^2 bh \cos\beta$

b) $\varepsilon = \frac{1}{2}(1 + \sin\beta)$

- Task a
- solution of the momentum equation in x-direction:

$$\begin{aligned}
-\dot{m}(\mathbf{v}) &= -F_{W,x} \\
-\rho_W U h b \cos(\beta)\, U &= -F_{W,x} \\
\underline{F_{W,x}} &= \underline{\rho_W U^2 h b \cos(\beta)}
\end{aligned}$$

The unknown force is $F_{W,x} = \rho_W U^2 h b \cos(\beta)$.

- Task b
- solution momentum equation in y-direction, summ is 0 since just pressure acts on the wall (no friction):

$$\begin{aligned}
-\dot{m}(\mathbf{v})_{in} + \dot{m}(\mathbf{v})_{out1} + \dot{m}(\mathbf{v})_{out2} &= 0 \\
-\rho_W h b U \sin(\beta)\, U + \rho_W \epsilon h b U U - \rho_W (1-\epsilon) h b U U &= 0 \\
-\sin(\beta) + \epsilon - (1-\epsilon) &= 0 \\
-\sin(\beta) - 1 + 2\epsilon &= 0 \\
2\epsilon &= 1 + \sin(\beta) \\
\underline{\epsilon} &= \underline{\frac{1+\sin(\beta)}{2}}
\end{aligned}$$

4.3 Rectangular water jet hitting a moving blade

A rectangular (height h, width b) water jet with the absolute velocity U is impinging onto a moving rectangular blade (seen from x-direction) with the velocity u_0. After hitting the bent plate the jet is divided into two streams of the same volume flow (height $h/2$). Ignore gravity and friction!

Given:

- Power: $P = F_{S_x} \cdot U_0 = \rho bh(U - U_0)^2 U_0(1 + \cos\beta)$

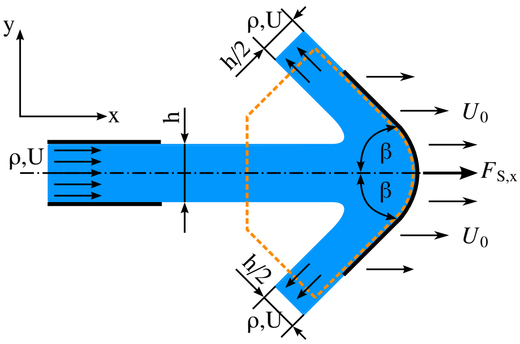

Figure 4.5: Moving blade

Task: For which velocity of the blade U_0 does the power P reach its maximum?

Solution:

a) $U_0 = \frac{1}{3}U$

Task: For which velocity of the blade U_0 does the power P reach its maximum?

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}U_0} &= \rho hb(1+\cos\beta)(U^2 - 4UU_0 + 3U_0^2) = 0 \\
0 &= U^2 - 4UU_0 + 3U_0^2 \\
0 &= \overbrace{\frac{U^2}{3}}^{q} - \overbrace{\frac{4}{3}U}^{p}\; U_0 + U_0^2 \\
U_{0,1/2} &= \frac{2}{3}U + \sqrt{\left(\frac{2}{3}U\right)^2 - \frac{1}{3}U^2} \\
U_{0,1/2} &= \frac{2}{3}U \pm \frac{1}{3}U \\
\Rightarrow U_{0,1} &= U \Rightarrow P_1 = 0 \\
\underline{\Rightarrow U_{0,2}} &= \underline{\frac{1}{3}U} \Rightarrow P_2 = \frac{4}{27}U^3\rho hb(1+\cos\beta)
\end{aligned}
$$

4.4 Rocket

A rocket is moving with velocity v_1. The air in front is displaced radially. The velocity of the exhaust jet is v_E, outside the exhaust jet it is v_1.

Given: | $v_1 = 1000\ \frac{\text{m}}{\text{s}}$ | $v_E = 4500\ \frac{\text{m}}{\text{s}}$ | $\rho_1 = 1.0\ \frac{\text{kg}}{\text{m}^3}$ | $\rho_E = 0.3\ \frac{\text{kg}}{\text{m}^3}$ | $A_R = 1\ \text{m}^2$ |

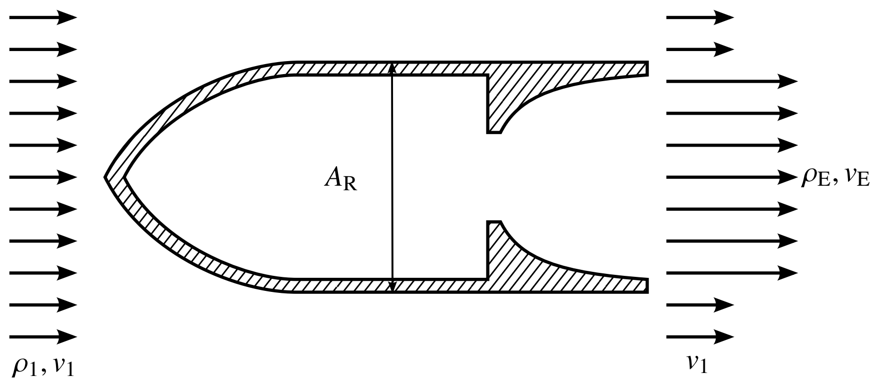

Figure 4.6: Rocket

Tasks:
Determine

a) the displaced air mass flow,

b) the thrust (Schub) in kN,

c) the Power in MW.

Solution:

a) $\dot{m} = 1000\ \frac{\text{kg}}{\text{s}}$

b) $F_t = 6075$ kN

c) $P = 6075$ MW

- Task a, the displaced air mass flow

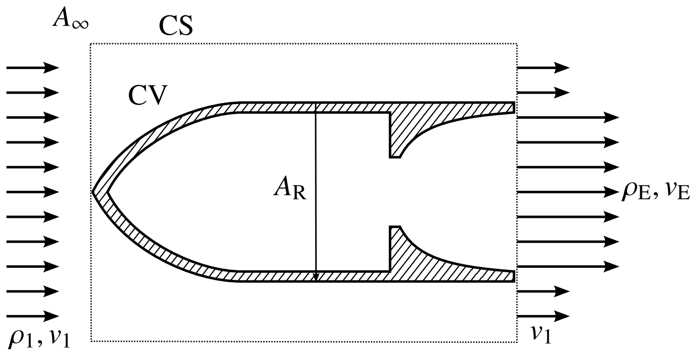

Figure 4.7: Rocket

continuity equation

$$
\begin{aligned}
A_\infty v_1 \rho_1 &= (A_\infty - A_R) v_1 \rho_1 + \Delta \dot{m} \\
A_R v_1 \rho_1 &= \Delta \dot{m} \\
\Delta \dot{m} &= 1 \text{ m}^2 1.0 \,\frac{\text{kg}}{\text{m}^3} 1000 \,\frac{\text{m}}{\text{s}} = 1000 \,\frac{\text{kg}}{\text{s}}
\end{aligned}
$$

- Task b, the thrust in kN

momentum equation

v_x is the x-velocity of the displaced air.

$$
\begin{aligned}
F_t &= -A_\infty (v_1)^2 \rho_1 + \Delta \dot{m} v_x + A_R \rho_E u_E^2 + (A_\infty - A_R) \rho_1 v_1^2 \\
F_t &= \Delta \dot{m} v_x + A_R \rho_E u_E^2 - A_R \rho_1 v_1^2
\end{aligned}
$$

from $A_\infty >> A_R \Rightarrow v_x \approx v_1$

$$
\begin{aligned}
F_t &= (A_R v_1 \rho_1) v_1 + A_R \rho_E u_E^2 - A_R \rho_1 v_1^2 \\
F_t &= A_R \rho_E u_E^2 \\
F_t &= 1 \text{ m}^2 \cdot 0.3 \,\frac{\text{kg}}{\text{m}^3} \cdot (4500)^2 \left(\frac{\text{m}}{\text{s}}\right)^2 = 6075 \text{ kN}
\end{aligned}
$$

- Task c, power in MW

$$
P = F_t \cdot v_1 = 6075 \text{ kN} \cdot 1000 \,\frac{\text{m}}{\text{s}} = 6075 \text{ MW}
$$

4.5 Wind turbine

The wind velocity is u_{in}, the pressure drop over the wind wheel (area A_w) form point (1) to (2) is Δp. Please apply Rankine's simplified theory!

Given: | $u_{in} = 10\ \frac{\mathrm{m}}{\mathrm{s}}$ | $p_1 - p_2 = 50.4$ Pa | $\rho_a = 1.2\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ | $A_w = 10\ \mathrm{m}^2$ |

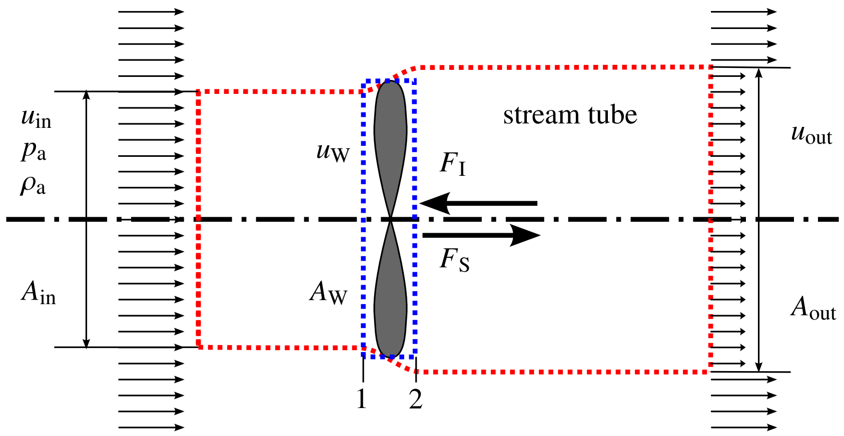

Figure 4.8: Wind turbine

Task:
Determine

a) Determine the generated power of the wind turbine!

Solution:

a) $P = 3.528$ kW

a) Power of the turbine

1. velocity at the outlet, since power is $F_S \cdot u_w$, unknown: $u_w = \frac{u_{in}+u_{out}}{2}$

Bernoulli from (in) to (1):

$$\rho_a \frac{u_{in}^2}{2} + p_a = \rho_a \frac{u_w^2}{2} + p_1$$

Bernoulli from (out) to (2):

$$\rho_a \frac{u_{out}^2}{2} + p_a = \rho_a \frac{u_w^2}{2} + p_2$$

subtract the equations:

$$\rho_a \frac{u_{in}^2}{2} - \rho_a \frac{u_{out}^2}{2} = p_1 - p_2 = \Delta p$$

insert known Δp:

$$\begin{aligned}
\Delta p &= \rho_a \left(\frac{u_{in}^2}{2} - \frac{u_{out}^2}{2} \right) \\
\frac{2\Delta p}{\rho_a} &= u_{in}^2 - u_{out}^2 \\
u_{out}^2 &= u_{in}^2 - \frac{2\Delta p}{\rho_a} \\
u_{out} &= \sqrt{10^2 \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - \frac{2 \cdot 50.4 \ \frac{\mathrm{N}}{\mathrm{m}^2}}{1.2 \ \frac{\mathrm{s}^2\mathrm{N}}{\mathrm{m}^4}}} \\
u_{out} &= 4 \ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

calculate u_w:

$$u_w = \frac{u_{in} + u_{out}}{2} = \frac{14}{2} \ \frac{\mathrm{m}}{\mathrm{s}} = 7 \ \frac{\mathrm{m}}{\mathrm{s}}$$

calculate F_S:

$$F_S = \Delta p \cdot A_w = 50.4 \ \frac{\mathrm{N}}{\mathrm{m}^2} \cdot 10 \ \mathrm{m}^2 = 504 \ \mathrm{N}$$

calculate P:

$$P = F_S \cdot u_w = 504 \ \mathrm{N} \cdot 7 \ \frac{\mathrm{m}}{\mathrm{s}} = 3528 \ \mathrm{W} = \underline{3.528 \ \mathrm{kW}}$$

4.6 Turbine flow

A sketch of a radial turbine flow is given. Flow direction is from outer to inner radius. The absolute velocity $\mathbf{v}_1$ is graphically given. Pleas draw the following velocities in a qualitative way: relative velocity $\mathbf{w}_1$, peripheral velocity $\mathbf{u}_1$, the radial velocity $\mathbf{v}_{r1}$.

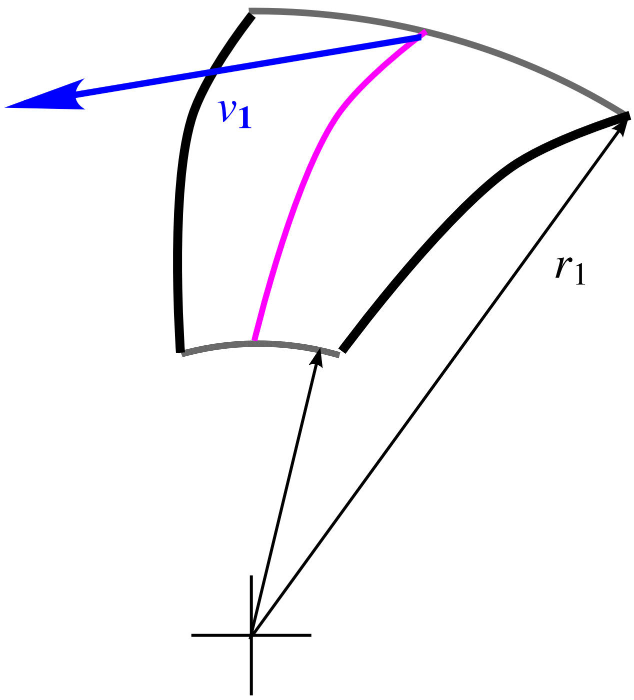

Figure 4.9: Turbine, given $\mathbf{v}_1$

Solution: see drawing:

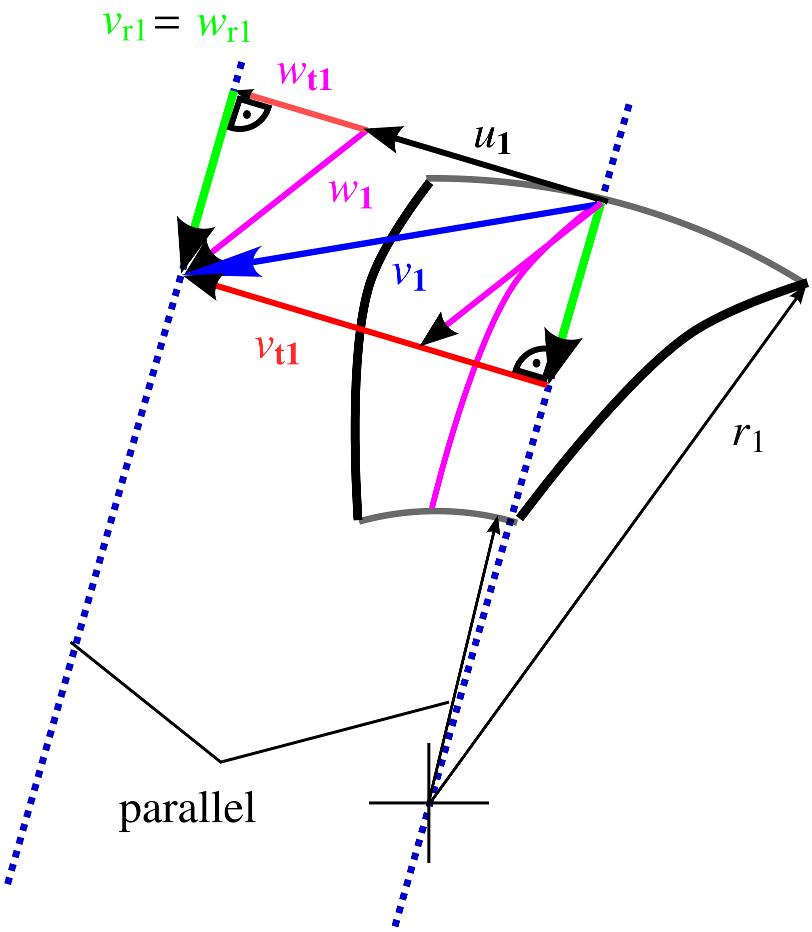

Figure 4.10: Solution turbine

4.7 Pump flow

A radial water pump with an outer blade wheel radius r_2 and an angular speed of $n = 3000$ rpm (round per minute) has an absolute outflow velocity $\mathbf{v}_2$ with an angle of 20° to its tangential direction. The absolute inflow velocity is in radial direction.

Given: | $r_2 = 0.15$ m | $|v_2| = 45\ \frac{\text{m}}{\text{s}}$ | $n = 3000$ rpm | $\rho_w = 1000\ \frac{\text{kg}}{\text{m}^3}$ |

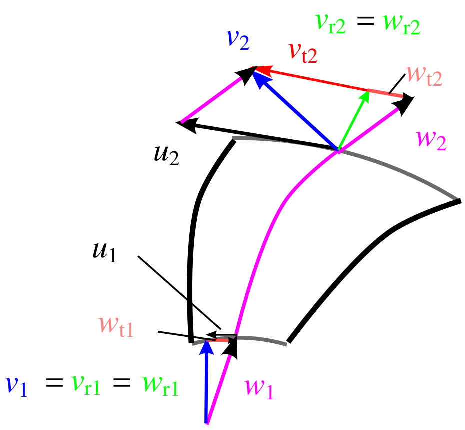

Figure 4.11: Water pump

Task:
Determine

a) the specific work Y in $\frac{\text{m}^2}{\text{s}^2} = \frac{\text{J}}{\text{kg}}$ of the pump!

b) the power for a volume flow of $60\ \frac{\text{l}}{\text{s}}$ (liter/s)!

Solution:

a) $Y = -1.993\ \frac{\text{kJ}}{\text{kg}}$

b) $P = -119.6$ kW

a) specific work Y

$$Y_p = \cancel{u_1 \cdot v_{1t}}^{0} - u_2 \cdot v_{2t}$$

1. determine v_t out of $|\mathbf{v}|$

$$v_t = \cos(20°)\,|\mathbf{v}|$$

2. determine u_2

$$u_2 = \omega \cdot r_2 = n \cdot 2\pi r_2 = \frac{3000}{60} 2\pi r_2$$

$$\begin{aligned}
Y_p &= -\cos(20°)\,|\mathbf{v}|\frac{3000}{60}2\pi r_2 \\
&= -45\ \frac{\text{m}}{\text{s}}\cos(20°)\ \frac{3000}{60\text{s}}2\pi 0.15\ \text{m} \\
&= -1992.6\ \frac{\text{m}^2}{\text{s}^2} \\
&= \underline{-1993\ \frac{\text{J}}{\text{kg}} = -1.993\ \frac{\text{kJ}}{\text{kg}}}
\end{aligned}$$

The Power is:

$$P = \dot{m}Y$$

3. determine $\dot{m}$

$$\dot{m} = \dot{V}\rho$$

$$\begin{aligned}
P &= \dot{m}Y \\
&= \dot{V}\rho Y \\
&= -60\ \frac{1}{\text{s}}1000\ \frac{\text{kg}}{\text{m}^3}1.993\ \frac{\text{kJ}}{\text{kg}} \\
&= -0.06\ \frac{\text{m}^3}{\text{s}}1000\ \frac{\text{kg}}{\text{m}^3}1.993\ \frac{\text{kJ}}{\text{kg}} \\
&= -\underline{119.56\ \text{kW}}
\end{aligned}$$

4.8 Turbine flow, calculation and sketch of vectors

A sketch of a radial turbine flow is given. Flow direction is from outer to inner radius. The absolute velocity v_1 is 20 $\frac{\text{m}}{\text{s}}$ with an angle of $\alpha_1 = 20°$. The specific work of the turbine is $Y = 0.3 \, \frac{\text{kJ}}{\text{kg}}$. The outflow velocity v_2 has no component in tangential direction to optimize the power of the turbine. Please be aware that just the magnitudes of the flow vectors are given. All other information is included in the sketch.

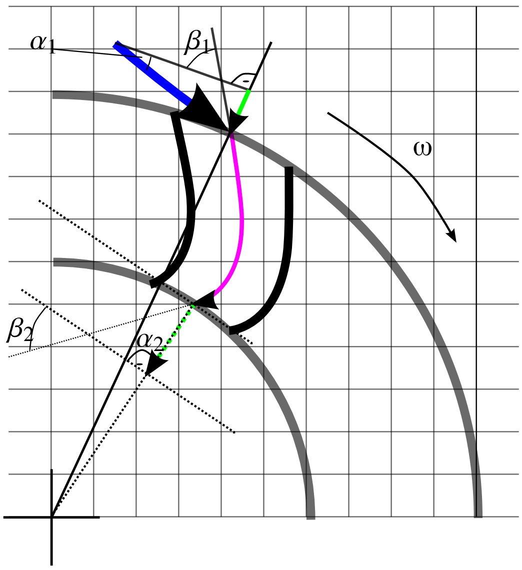

Figure 4.12: Turbine: angle definition

Given: | $r_1 = 10$ m | $r_2 = 6$ m | $v_1 = 20 \, \frac{\text{m}}{\text{s}}$ | $\alpha_1 = 20°$ | $Y = 0.3 \, \frac{\text{kJ}}{\text{kg}}$ |

Task:

Determine

a) The circumferential velocity at r_1: u_1

b) The inflow angle: β_1

c) The radial outflow velocity relative to the rotation: w_{r2}

d) The outflow angle: β_2

Draw the result in the given sketch (5 $\frac{\text{m}}{\text{s}}$ = 1cm).

Solutions: $u_1 = 15.96 \, \frac{\text{m}}{\text{s}}$, $\beta_1 = 67.52°$, $w_{r2} = 11.40 \, \frac{\text{m}}{\text{s}}$, $\beta_2 = 50°$.

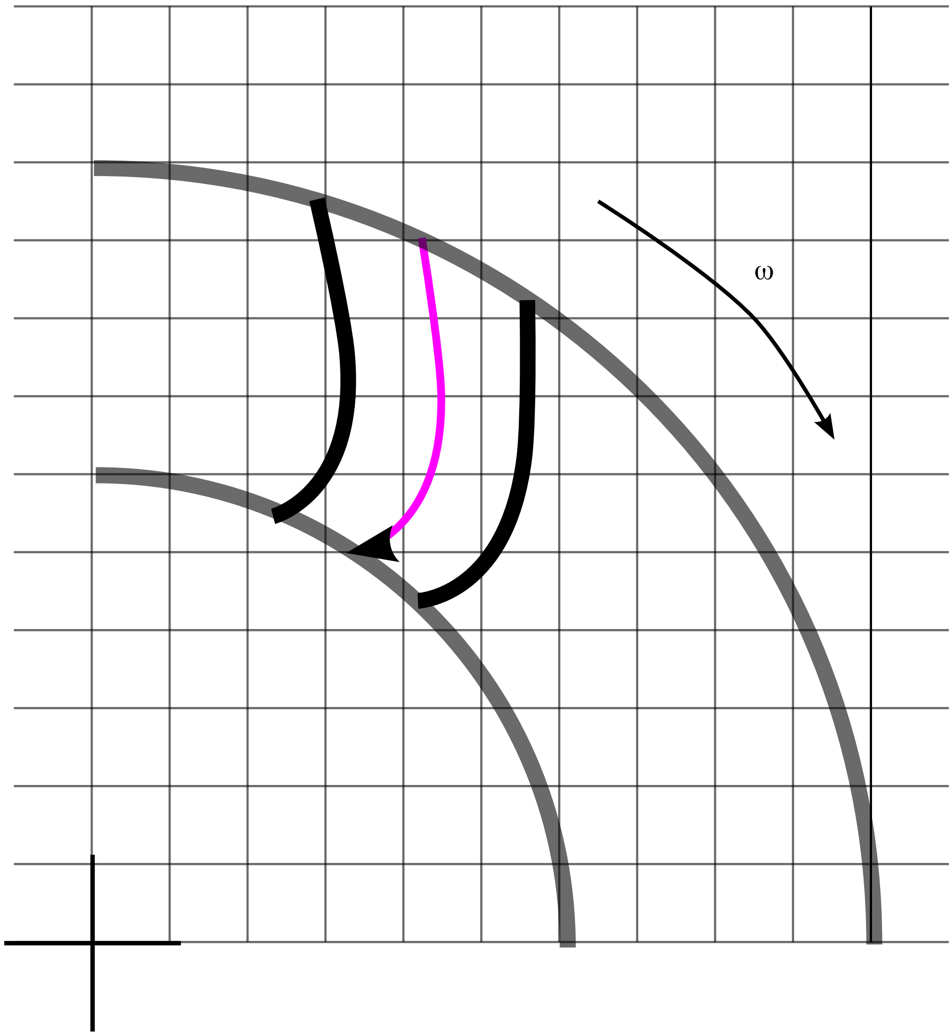

Figure 4.13: Turbine: to be used by the students

Solution:

a) u_1

$$
\begin{aligned}
Y_t &= v_{t1} \cdot u_1 - v_{t2} \cdot u_2 \\
v_{t1} &= \cos(20^\circ)\, v_1 \\
Y_t &= \cos(20^\circ)\, v_1 \cdot u_1 \\
0.3\,\frac{\text{kJ}}{\text{kg}} &= 18.79\,\frac{\text{m}}{\text{s}} \cdot u_1 \\
u_1 &= \tfrac{300}{18.79}\,\tfrac{\text{m}}{\text{s}} = 15.96\,\tfrac{\text{m}}{\text{s}}
\end{aligned}
$$

b) β_1

We need the speed difference between the u_1 and v_{t1}:

$$
w_{t1} = v_{t1} - u_1 = 18.79\,\frac{\text{m}}{\text{s}} - 15.96\,\frac{\text{m}}{\text{s}} = 2.83\,\frac{\text{m}}{\text{s}}
$$

The radial speed v_{r1} is:

$$
w_{r1} = v_{r1} = \sin(20^\circ) \cdot v_1 = 6.84\,\frac{\text{m}}{\text{s}}
$$

The angle β_1 is:

$$
\beta_1 = \arctan\left(\frac{w_{r1}}{w_{t1}}\right) = 67.52^\circ
$$

c) w_{r2}

To make use of the continuity equation we need a full area. For this we need the width of the turbine perpendicular to the drawing plane. Say, this width is t. Then we may write:

$$
\begin{aligned}
\rho \cdot t \cdot r_1 \cdot 2 \cdot \pi \cdot w_{r1} &= \rho \cdot t \cdot r_2 \cdot 2 \cdot \pi \cdot w_{r2} \\
r_1 \cdot w_{r1} &= r_2 \cdot w_{r2} \\
w_{r2} &= \frac{r_1}{r_2} \cdot w_{r1} = \frac{10}{6} \cdot 6.84\,\frac{\text{m}}{\text{s}} = 11.40\,\frac{\text{m}}{\text{s}}
\end{aligned}
$$

d) β_2

The turbine's radial velocity depends linearly on the radius and we know the radius and the radial velocity at the inlet.

$$u_2 = \frac{r_2}{r_1} \cdot u_1 = \frac{3}{5} \cdot 15.96\ \frac{\mathrm{m}}{\mathrm{s}} = 9,58\ \frac{\mathrm{m}}{\mathrm{s}}$$

$w_{t2} = u_2$ (but opposite direction) and $w_{r2} = v_{r2}$

$$\beta_2 = \arctan\left(\frac{w_{r2}}{w_{t2}}\right) = 50^\circ$$

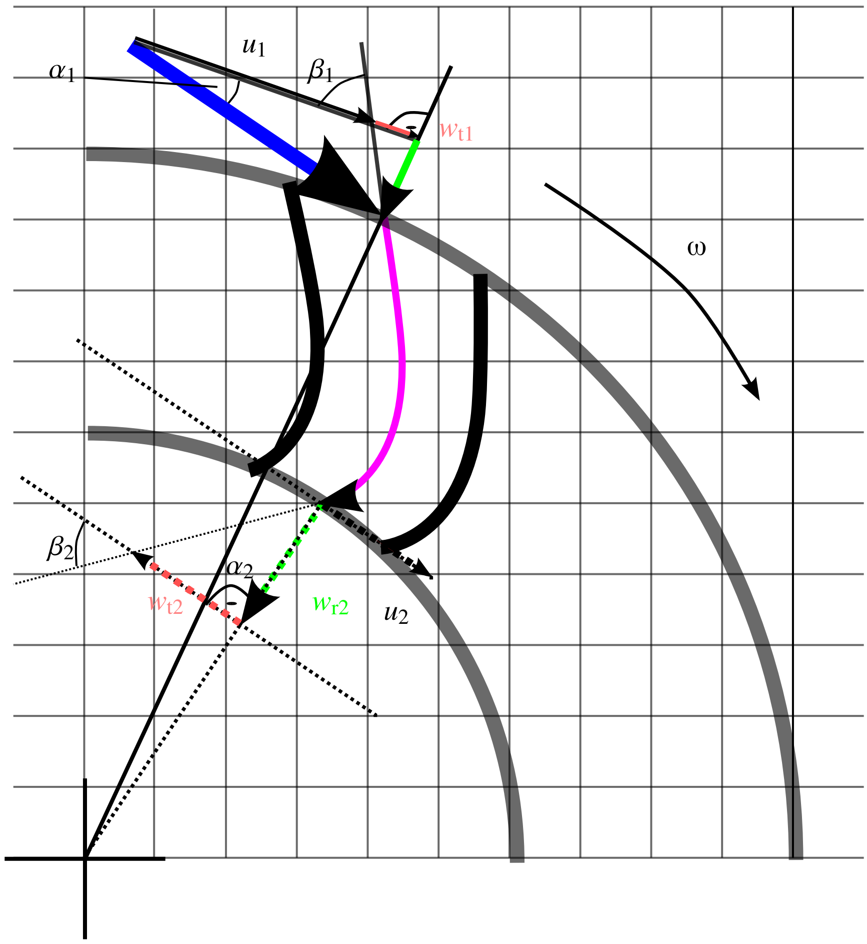

Figure 4.14: Turbine: Solution

Chapter 5

Friction

- Hint 1

The Reynolds number for pipes can be generated with the dynamic viscosity μ in $\left[\frac{\mathrm{kg}}{\mathrm{m \cdot s}}\right]$ and the kinematic viscosity $\nu = \frac{\mu}{\rho}$ in $\left[\frac{\mathrm{m}^2}{\mathrm{s}}\right]$. The diameter d and the average velocity of the pipe $\bar{u}$ are also used.

$$\mathrm{Re} = \frac{\rho \bar{u} d}{\mu} = \frac{\bar{u} d}{\nu}$$

- Hint 2

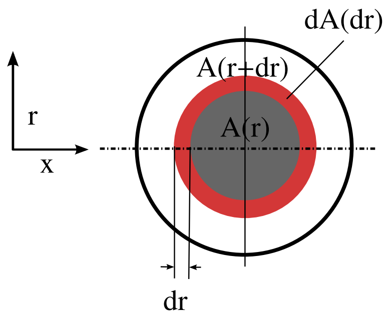

Figure 5.1: Infinitesimal element

$$\begin{aligned}
\mathrm{d}A &= A(r+\mathrm{d}r) - A(r) \\
&= \pi(r+\mathrm{d}r)^2 - \pi r^2 \\
&= \pi(r^2 + 2r\mathrm{d}r + \overset{very small}{\cancel{(\mathrm{d}r)^2}}) - \pi r^2 \\
&= \pi 2r\mathrm{d}r
\end{aligned}$$

5.1 Laminar pipe flow

A laminar pipe flow is given. Using polar coordinates x, r the velocity is $u(r) = \frac{p_1 - p_2}{4\mu l}(r_0^2 - r^2)$ with pressures p_1, p_2, dynamic viscosity μ, pipe length l and pipe radius r_0

Given:

$p_1 - p_2$	μ	l	r_0

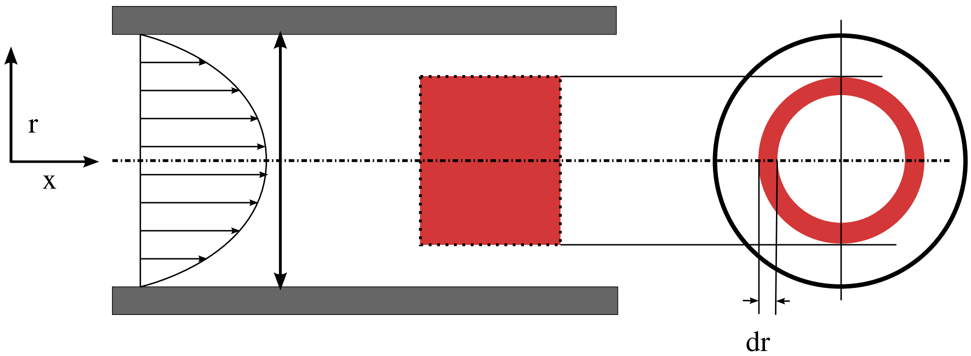

Figure 5.2: Sketch of a laminar pipe

Tasks:

a) Formulate the infinitesimal small area dA of the marked ring!

b) Determine the volume flow $\dot{V}$ (Hagen-Poiseuille equation)!

Solution:

a) $\mathrm{d}A = 2\pi r \mathrm{d}r$

b) $\dot{V} = \frac{\pi(p_1 - p_2)}{8\mu l} r_0^4$

a) infinitesimal small area dA of the marked ring, quick solution

1. we take in the radius in the middle as reference as shown in figure 5.3

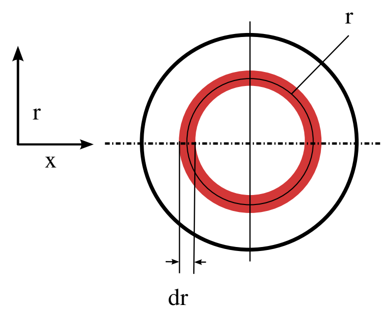

Figure 5.3: Sketch of a laminar pipe

$$\mathrm{d}A = 2\pi r \mathrm{d}r$$

b) the volume flow

1. we formulate the integral of the velocity flowing through the pipe cross surface

$$\begin{aligned}
\dot{V} &= \int_A v(r)\mathrm{d}A \\
&= \int_0^{r_0} v(r) 2\pi r \mathrm{d}r \\
&= \int_0^{r_0} \frac{p_1 - p_2}{4\mu l}(r_0^2 - r^2) 2\pi r \mathrm{d}r \\
&= \frac{p_1 - p_2}{4\mu l} 2\pi \left[\frac{r_0^2 r^2}{2} - \frac{r^4}{4}\right]_0^{r_0} \\
&= \frac{p_1 - p_2}{2\mu l} \pi \left[\frac{r_0^4}{2} - \frac{r_0^4}{4} - 0\right] \\
&= \frac{p_1 - p_2}{2\mu l} \pi \frac{r_0^4}{4} \\
&= \frac{p_1 - p_2}{8\mu l} \pi r_0^4
\end{aligned}$$

5.2 Hagen-Poiseuille equation

The fluid in the inner cylinder volume (l_1, r_1) of a syringe (Injektionsspritze) is pressed out by a piston (just the left large volume of the syringe), pushed by the constant force F. The fluid leaves the syringe through the needle (l_2, r_2). The pressure drop within the cylinder 1 (l_1) can be neglected. The fluid has the dynamic viscosity μ. The inertia energy should be also neglected. The needle's flow complies with the assumptions of the Hagen-Poiseuille equation. The assumption is that steady state was already reached.

Given:

$F = 10$ N	$l_1 = 0.03$ m	$l_2 = 0.08$ m	$r_1 = 0.005$ m	$r_2 = 2 \cdot 10^{-4}$ m	$\mu = 2 \cdot 10^{-3}\ \frac{\text{kg}}{\text{m} \cdot \text{s}}$

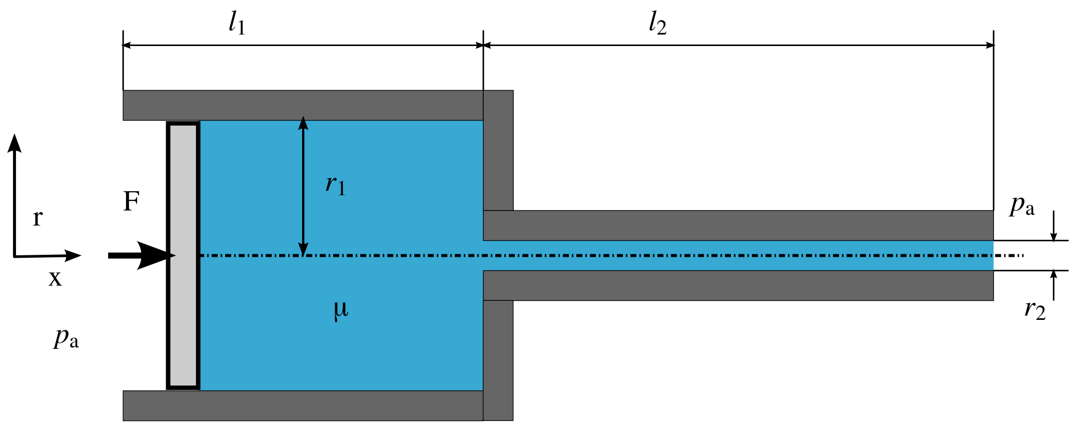

Figure 5.4: Sketch of a syringe

Tasks:

a) Which time Δt is needed to empty the syringe?

Solution:

a) $\Delta t = 4.71$ s

a) time to empty the syringe?

1. Δp?

$$\Delta p = \frac{F}{A} = \frac{F}{\pi r_1^2}$$

2. Volume flow, which has to be emptied

$$\dot{V} = \frac{V}{\Delta t} = \frac{\pi r_1^2 l_1}{\Delta t}$$

3. Volume flow according to Δp (Hagen-Poiseuille)

$$\dot{V} = \frac{p_1 - p_2}{8\mu l_2} \pi r_2^4$$

4. Equalize volume flow

$$\begin{aligned}
\frac{p_1 - p_2}{8\mu l_2} \pi r_2^4 &= \frac{\pi r_1^2 l_1}{\Delta t} \\
\frac{8\mu l_2}{p_1 - p_2} \frac{\pi r_1^2 l_1}{\pi r_2^4} &= \Delta t \\
\Delta t &= \frac{8\mu l_2 \pi r_1^2}{F} \frac{r_1^2 l_1}{r_2^4} \\
\Delta t &= \frac{8\mu l_2 \pi l_1}{F} \frac{r_1^4}{r_2^4} \\
\Delta t &= \pi \frac{8 \cdot 2 \cdot 10^{-3}\ \frac{\mathrm{kg}}{\mathrm{m \cdot s}} \cdot 8 \cdot 10^{-2}\ \mathrm{m}\ 3 \cdot 10^{-2}\ \mathrm{m}}{10\ \mathrm{N}} \cdot \frac{(5 \cdot 10^{-3}\ \mathrm{m})^4}{(2 \cdot 10^{-4}\ \mathrm{m})^4} \\
\Delta t &= = 4.712\ \mathrm{s}
\end{aligned}$$

5.3 Laminar or turbulent pipe

10 m^3 heating oil with dynamic viscosity μ and the density ρ are flowing through a supply pipe with the length l and the diameter d per hour.

Given:

$\rho = 900\ \frac{kg}{m^3}$	$l = 1500$ m	$\mu = 0.036\ \frac{kg}{m \cdot s}$	$d = 0.05$ m

Tasks:

a) Identify whether it is a laminar or a turbulent pipe flow!

b) Determine the necessary pressure difference to transport the oil flow!

Solution:

a) laminar Re= 1769

b) $\Delta p = 9.78$ bar

a) Determine Reynolds number

1. Determine velocity

$$\bar{u} = \frac{\dot{V}}{A} = \frac{\frac{10}{3600}\frac{\mathrm{m}^3}{\mathrm{s}}}{0.025^2\pi\ \mathrm{m}^2} = 1.4147\ \frac{\mathrm{m}}{\mathrm{s}}$$

2. Reynolds number

$$\mathrm{Re} = \frac{\rho\bar{u}d}{\mu} = \frac{1.414\ \frac{\mathrm{m}}{\mathrm{s}}900\ \frac{\mathrm{kg}}{\mathrm{m}^3}\cdot 0.05\ \mathrm{m}}{0.036\ \frac{\mathrm{kg}}{\mathrm{m\cdot s}}} = 1769.4$$

The flow is laminar since the Reynolds Number is below 2300.

b) Pressure difference

1. Pressure drop

$$\begin{aligned}
\frac{\Delta p_l}{\frac{\rho}{2}\bar{u}^2} &= \frac{l}{d}\frac{64}{\mathrm{Re}} \\
\Delta p_l &= \frac{l}{d}\cdot\frac{32}{\mathrm{Re}}\rho\bar{u}^2 \\
\Delta p_l &= \frac{1500}{0.05}\cdot\frac{32}{1769.4}900\ \frac{\mathrm{kg}}{\mathrm{m}^3}(1.414)^2(\frac{\mathrm{m}}{\mathrm{s}})^2 \\
\Delta p_l &= 977847\ \mathrm{Pa} = 9.78\ \mathrm{bar}
\end{aligned}$$

5.4 Moody diagram

1200 m^3 water with the kinematic viscosity ν and the density ρ are flowing through a horizontal steel pipe with the length l, diameter d, and roughness ϵ per hour.

<u>Given:</u>

$\rho = 1000\ \frac{kg}{m^3}$	$l = 2000$ m	$\nu = 1.13 \cdot 10^{-6}\ \frac{m^2}{s}$	$d = 0.5$ m	$\epsilon = 0.1$ mm

<u>Tasks:</u>

a) Identify whether it is a laminar or a turbulent pipe flow!

b) Determine the necessary pressure difference to transport the water flow!

<u>Solution:</u>

a) turbulent Re= $751,000$

b) $\Delta p = 0.865$ bar

a) Determine Reynolds number

1. Determine velocity

$$\bar{u} = \frac{\dot{V}}{A} = \frac{\frac{1200}{3600}\ \frac{\mathrm{m}^3}{\mathrm{s}}}{0.5^2 \frac{\pi}{4}\ \mathrm{m}^2} = 1.698\ \frac{\mathrm{m}}{\mathrm{s}}$$

2. Reynolds number

$$\mathrm{Re} = \frac{\bar{u}d}{\nu} = \frac{1.698\ \frac{\mathrm{m}}{\mathrm{s}} \cdot 0.5\ \mathrm{m}}{1.13 \cdot 10^{-6}\ \frac{\mathrm{m}^2}{\mathrm{s}}} = 751,173.7$$

3. surface roughness k

$$k = \frac{\epsilon}{d} = \frac{0.1}{500} = 0.0002$$

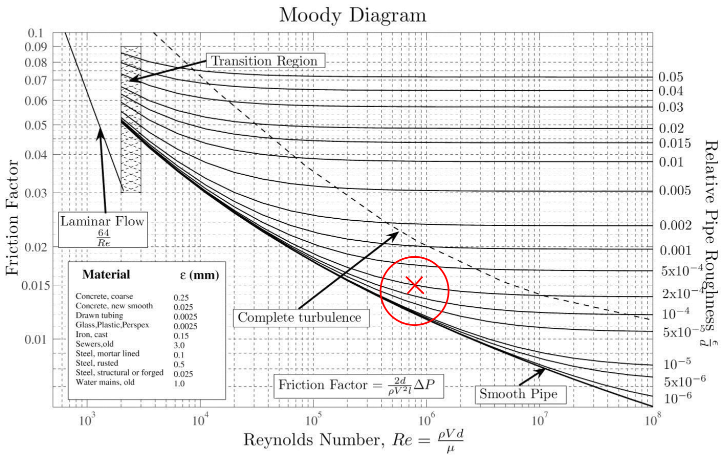

Figure 5.5: Moody Diagram including working point, Source: www.wikimedia.org

From the moody diagram we can determine that this is a turbulent flow.

b) Pressure difference

1. Pressure drop, $\lambda = 0.015$ from Moody diagram

$$\begin{aligned}
\lambda &= \frac{d}{l} \cdot \frac{\Delta p_l}{\frac{\rho}{2}(\bar{u})^2} \\
\Delta p_l &= \lambda \cdot \frac{l}{d} \cdot \frac{\rho}{2}(\bar{u})^2 \\
&= 0.015 \frac{2000}{0.5} \cdot 500 \cdot 1.698^2\ \frac{\mathrm{kg}}{\mathrm{m}^3} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 86460\ \mathrm{Pa} = 0.865\ \mathrm{bar}
\end{aligned}$$

5.5 Laminar and turbulent pipe flow

27 m^3 oil with the kinematic viscosity ν and the density ρ are flowing through a horizontal steel pipe with the length l, diameter d, and roughness k per hour.

Given:

$\rho = 860\ \frac{kg}{m^3}$	$l = 750$ m	$\nu = 8 \cdot 10^{-5}\ \frac{m^2}{s}$	$d = 0.1$ m	$k = 0.002$

Tasks:

a) Determine the necessary pressure difference to transport the oil flow!

b) Calculate the needed power P of the pump (efficiency $\eta_P = 0.7$) (Hint: $P = \frac{\dot{V}\Delta p_l}{\eta_P}$)!

c) Now the customer (Auftraggeber) wants to quadruple the volume flow. How do the required (erforderlich) pressure difference an power change?

d) The provided pump hs only 40 kW as a maximum power. Is it sufficient to enlarge the pipe's diameter to 0.12 m?

Solution:

a) $\Delta p_l = 1.58$ bar

b) $P = 1.69$ kW

c) $\Delta p_l = 19.3$ bar; $P = 82.7$ kW (based on $\lambda = 0.041$)

d) $\Delta p_l = 8.02$ bar; $P = 34.4$ kW (based on $\lambda = 0.0425$) $\Rightarrow$ OK!

a) Determine Reynolds number

1. Determine velocity

$$\bar{u} = \frac{\dot{V}}{A} = \frac{\frac{27}{3600}\,\frac{\text{m}^3}{\text{s}}}{0.1^2\frac{\pi}{4}\ \text{m}^2} = 0.955\ \frac{\text{m}}{\text{s}}$$

2. Reynolds number

$$\text{Re} = \frac{\bar{u}d}{\nu} = \frac{0.955\ \frac{\text{m}}{\text{s}} \cdot 0.1\ \text{m}}{8 \cdot 10^{-5}\ \frac{\text{m}^2}{\text{s}}} = 1194$$

The flow is laminar.

$$\lambda = \frac{64}{\text{Re}} = 0.0536$$

3. Pressure difference

$$\begin{aligned}
\lambda &= \frac{d}{l} \cdot \frac{\Delta p_l}{\frac{\rho}{2}(\bar{u})^2} \\
\Delta p_l &= \lambda \cdot \frac{l}{d} \cdot \frac{\rho}{2}(\bar{u})^2 \\
&= 0.0536\frac{750}{0.1} \cdot 430 \cdot 0.955^2\ \frac{\text{kg}}{\text{m}^3}\left(\frac{\text{m}}{\text{s}}\right)^2 = 157629\ \text{Pa} = 1.58\ \text{bar}
\end{aligned}$$

b) Power

$$P = \frac{\dot{V}\Delta p_l}{\eta_P} = \frac{\frac{27}{3600}\frac{\text{m}^3}{\text{s}}157629\ \text{Pa}}{0.7} = 1688\ \text{W} = 1.69\ \text{kW}$$

c) 4 times the flow ?

1. Determine velocity

$$\bar{u} = \frac{\dot{V}}{A} = \frac{\frac{108}{3600}\,\frac{\text{m}^3}{\text{s}}}{0.1^2\frac{\pi}{4}\ \text{m}^2} = 3.820\ \frac{\text{m}}{\text{s}}$$

2. Reynolds number

$$\text{Re} = \frac{\bar{u}d}{\nu} = \frac{3.820\ \frac{\text{m}}{\text{s}} \cdot 0.1\ \text{m}}{8 \cdot 10^{-5}\ \frac{\text{m}^2}{\text{s}}} = 4775$$

The flow is turbulent.

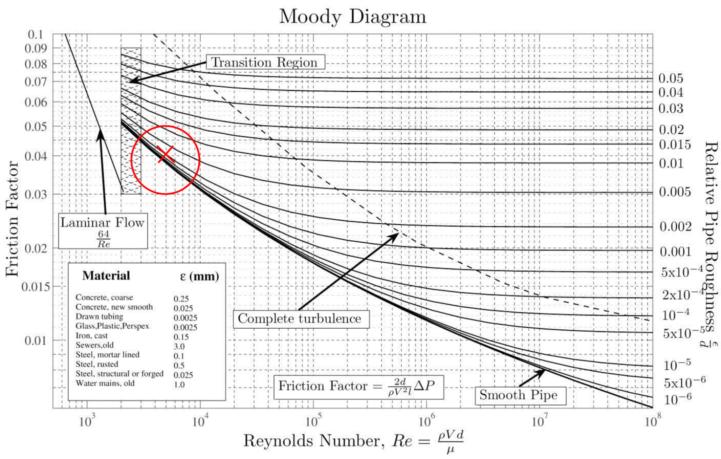

Figure 5.6: Moody Diagram including working point, Source: www.wikimedia.org

1. Pressure drop, $\lambda = 0.041$ from Moody diagram

$$\lambda = \frac{d}{l} \cdot \frac{\Delta p_l}{\frac{\rho}{2}(\bar{u})^2}$$

$$\Delta p_l = \lambda \cdot \frac{l}{d} \cdot \frac{\rho}{2}(\bar{u})^2$$

$$= 0.041 \frac{750}{0.1} \cdot 430 \cdot 3.820^2 \ \frac{\text{kg}}{\text{m}^3} \left(\frac{\text{m}}{\text{s}}\right)^2 = 1,929,195 \text{ Pa} = 19.3 \text{ bar}$$

2. Power

$$P = \frac{\dot{V}\Delta p_l}{\eta_P} = \frac{\frac{108 \text{ m}^3}{3600 \text{ s}} 1,929,195 \text{ Pa}}{0.7} = 82680 \text{ W} = 82.680 \text{ kW}$$

d) 0.12 m diameter pipe

1. Determine velocity

$$\bar{u} = \frac{\dot{V}}{\Delta A} = \frac{\frac{108}{3600} \frac{\text{m}^3}{\text{s}}}{0.12^2 \frac{\pi}{4} \text{ m}^2} = 2.652 \ \frac{\text{m}}{\text{s}}$$

2. Reynolds number

$$\text{Re} = \frac{\bar{u}d}{\nu} = \frac{3.820 \ \frac{\text{m}}{\text{s}} \cdot 0.12 \text{ m}}{8 \cdot 10^{-5} \ \frac{\text{m}^2}{\text{s}}} = 3979$$

The flow is turbulent.

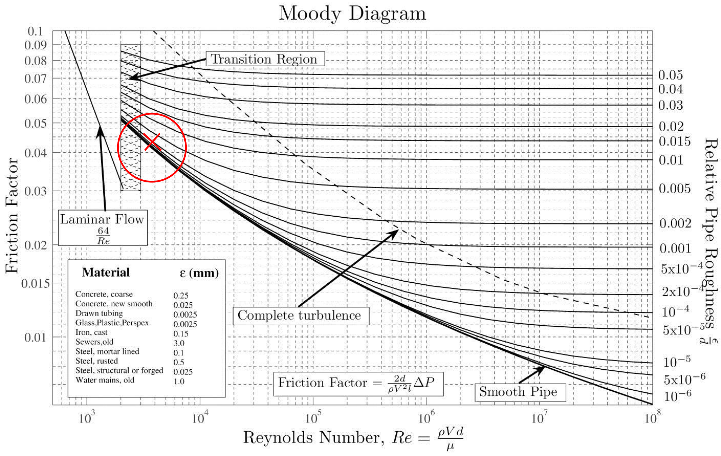

Figure 5.7: Moody Diagram including working point, Source: www.wikimedia.org

1. Pressure drop, $\lambda = 0.0425$ from Moody diagram

$$\begin{aligned}
\lambda &= \frac{d}{l} \cdot \frac{\Delta p_l}{\frac{\rho}{2}(\bar{u})^2} \\
\Delta p_l &= \lambda \cdot \frac{l}{d} \cdot \frac{\rho}{2}(\bar{u})^2 \\
&= 0.0425 \frac{750}{0.12} \cdot 430 \cdot 2.652^2 \ \frac{\text{kg}}{\text{m}^3} \left(\frac{\text{m}}{\text{s}}\right)^2 = 8803,665 \text{ Pa} = 8.8 \text{ bar}
\end{aligned}$$

2. Power

$$P = \frac{\dot{V}\Delta p_l}{\eta_P} = \frac{\frac{108 \text{ m}^3}{3600 \text{ s}} 8803,665 \text{ Pa}}{0.7} = 34442 \text{ W} = 34.4 \text{ kW}$$

5.6 Water reservoir with pressure drop

Water is flowing out of a big water reservoir. The pipe flow is frictionless except the loss due to the sudden change from area A_1 to A_2.

Given:

$\rho = 1000\ \frac{\text{kg}}{\text{m}^3}$	$\frac{A_1}{A_2} = 0.5$	$A_1 = 20\ \text{cm}^2$	$h = 5\ \text{m}$	$g = 10\ \frac{\text{m}}{\text{s}^2}$	$c_p = 4187\ \frac{\text{J}}{\text{kgK}}$

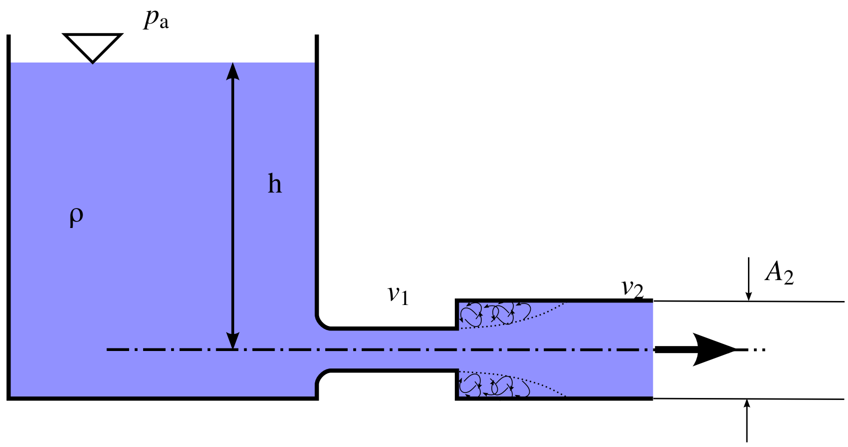

Figure 5.8: Sketch of a water reservoir

Tasks:

a) Calculate the velocities v_1 and v_2!

b) Calculate the pressure loss Δp_l and the loss factor ζ!

c) How much mechanical energy is dissipated into heat per second (P_l)?

d) Calculate the temperature rise due to the dissipation!

Hint: $\delta T = \frac{P}{\dot{m} \cdot c_p} = \frac{\Delta p_l}{\rho \cdot c_p}$

Solution:

a) $v_1 = 14.14\ \frac{\text{m}}{\text{s}}$; $v21 = 7.07\ \frac{\text{m}}{\text{s}}$

b) $\Delta p_l = 0.25$ bar; $\zeta = 0.25$

c) $P = 707$ W

d) $\Delta T = 0.006$ K

a) Calculate the velocities

1. Bernoulli equation

$$
\begin{aligned}
p_a + \rho g h &= p_a + \frac{\rho}{2} v_2^2 + \Delta p_l \\
\rho g h &= \frac{\rho}{2} v_2^2 + \Delta p_l \\
\Delta p_l &= \rho g h - \frac{\rho}{2} v_2^2
\end{aligned}
$$

2. From script; sudden expansion of a pipe

$$
\Delta p_L = \frac{\rho}{2} v_1^2 \left(1 - \frac{A_1}{A_2}\right)^2
$$

3. Continuity equation

$$
A_2 \cdot v_2 = A_1 \cdot v_1
$$

4. equalize Δp_l

$$
\begin{aligned}
\frac{\rho}{2} v_1^2 \left(1 - \frac{A_1}{A_2}\right)^2 &= \rho g h - \frac{\rho}{2} v_2^2 \text{ with continuity} \\
\frac{\rho}{2} v_1^2 \left(1 - \frac{A_1}{A_2}\right)^2 &= \rho g h - \frac{\rho}{2} v_1^2 \left(\frac{A_1}{A_2}\right)^2 \\
g h &= \frac{1}{2} v_1^2 \left(1 - \frac{A_1}{A_2}\right)^2 + \frac{1}{2} v_1^2 \left(\frac{A_1}{A_2}\right)^2 \\
v_1^2 &= \frac{g h}{\left(\frac{1}{2} - \frac{A_1}{A_2} + \left(\frac{A_1}{A_2}\right)^2\right)} \\
v_1 &= \sqrt{\frac{g h}{\left(\frac{1}{2} - \frac{A_1}{A_2} + \left(\frac{A_1}{A_2}\right)^2\right)}} \\
v_1 &= \sqrt{\frac{10 \cdot 5}{\frac{1}{2} - 0.5 + 0.25}} \ \frac{\text{m}}{\text{s}} \\
v_1 &= 14.14 \ \frac{\text{m}}{\text{s}}
\end{aligned}
$$

With continuity:

$$
v_2 = \frac{A_1}{A_2} \cdot v_1 = 0.5 \cdot 14.14 \ \frac{\text{m}}{\text{s}} = 7.07 \ \frac{\text{m}}{\text{s}}
$$

b) Δp_l

$$\zeta = \left(1 - \frac{A_1}{A_2}\right)^2 = (1 - 0.5)^2 = 0.25$$

$$\begin{aligned}
\Delta p_l &= \frac{\rho}{2} v_1^2 \cdot \zeta \\
&= 500 \ \frac{\mathrm{kg}}{\mathrm{m}^3} 14.14^2 \ \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 0.25 \\
&= 25,000 \ \mathrm{Pa} = 0.25 \ \mathrm{bar}
\end{aligned}$$

c) P

$$\begin{aligned}
P &= \dot{V} \Delta p_l \\
&= A_1 \cdot v_1 \cdot 25,000 \ \mathrm{Pa} \\
&= 0.002 \ \mathrm{m}^2 \cdot 14.14 \ \frac{\mathrm{m}}{\mathrm{s}} \cdot 25,000 \ \mathrm{Pa} \\
&= 707 \ \mathrm{W}
\end{aligned}$$

d) ΔT

$$\begin{aligned}
\Delta T &= \frac{P}{\dot{m} c_p} \\
&= \frac{\dot{V} \cdot \Delta p_l}{\dot{V} \rho c_p} \\
&= \frac{\Delta p_l}{\rho c_p} \\
&= \frac{25.000 \ \mathrm{Pa}}{1000 \ \frac{\mathrm{kg}}{\mathrm{m}^3} 4187 \ \frac{\mathrm{J}}{\mathrm{kgK}}} \\
&= 0.006 \ \mathrm{K}
\end{aligned}$$

Chapter 6

Appendix: Exam style Exercises

6.1 Floating Cube

A solid cube with edge length L and density ρ is floating in water with a density $\rho_W = 1000\text{kg/m}^3$. The emerging height is $h_1 = 50\text{mm}$ (left figure). Then the same cube is put in glycerin with $\rho_G = 1350\text{kg/m}^3$ (right figure). The emerging height is now $h_2 = 76\text{mm}$.

Determine the density ρ and the edge length L of the cube.

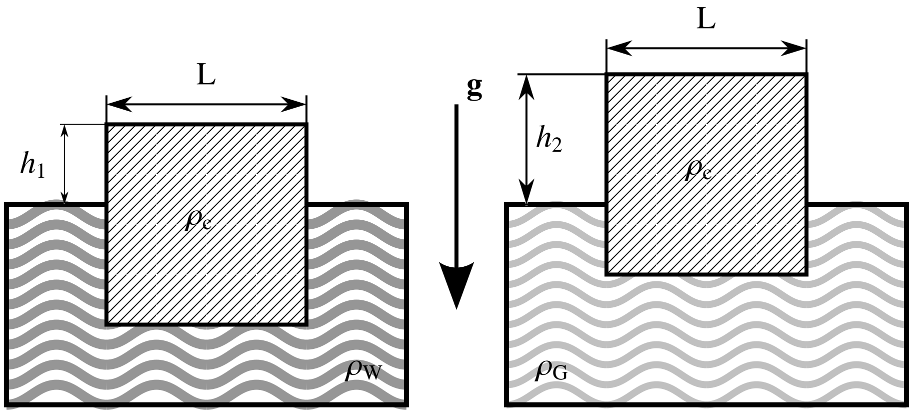

Figure 6.1: Cube floating in water and glycerin.

Solution:

- $\rho = 667,3\ \frac{\text{kg}}{\text{m}^3}$
- $L = 0,15\ \text{m}$

Floating Cube: Solution

A solid cube with edge length L and density ρ is floating in water with a density $\rho_W = 1000\text{kg/m}^3$. The emerging height is $h_1 = 50$mm (left figure). Then the sae cube is put in glycerin with $\rho_G = 1350\text{kg/m}^3$ (right figure). The emerging height is now $h_2 = 76$mm.

Determine the density ρ and the edge length L of the cube.

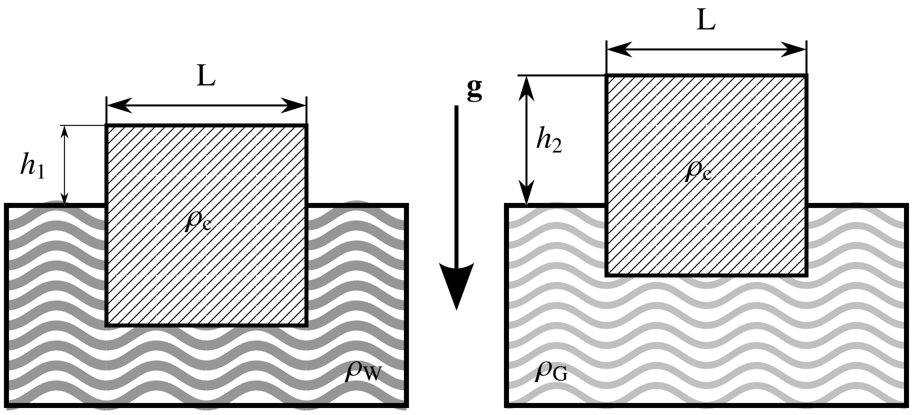

Figure 6.2: Cube floating in water and glycerin.

Solution:

$$
\begin{aligned}
\rho L^3 &= \rho_W L^2(L - h_1) = \rho_G L^2(L - h_2) \\
\rho_W(L - h_1) &= \rho_G(L - h_2) \\
L &= \frac{\rho_W h_1 - \rho_G h_2}{\rho_W - \rho_G} = 0.15 \text{ m}
\end{aligned}
$$

$$
\begin{aligned}
\rho L^3 &= \rho_W L^2(L - h_1) \\
\rho L &= \rho_W(L - h_1) \\
\rho &= \rho_W \frac{L - h_1}{L} = 667,3 \ \frac{\text{kg}}{\text{m}^3}
\end{aligned}
$$

6.2 Reservoir with double ended outflow

Water flows steadily from a big reservoir (assume z_0 = const. through a pipe with different cross section areas A_1, A_2, A_3 to two different outflows, c.f. the figure below. Ambient pressure applies at the reservoir surface and at the outflows. Treat the flow as inviscid.
Given are: $p_a = 1$ bar, $z_0 = 10$ m, $z_1 = 8,20$ m, $z_2 = 6,80$ m, $\rho_W = 1000\ \frac{\text{kg}}{\text{m}^3}$, $g = 10\ \frac{\text{m}}{\text{s}^2}$, $A_1 = 0.2\ \text{m}^2$, $A_3 = 0.4\ \text{m}^2$,

a) Determine the flow velocities v_1 and v_2 at the outflows 1 and 2, respectively.

b) Determine the area A_2 for which the the volume flow rates through A_1 and A_2 become equal.

c) For the case from b), determine the static pressure p_3 at point 3.

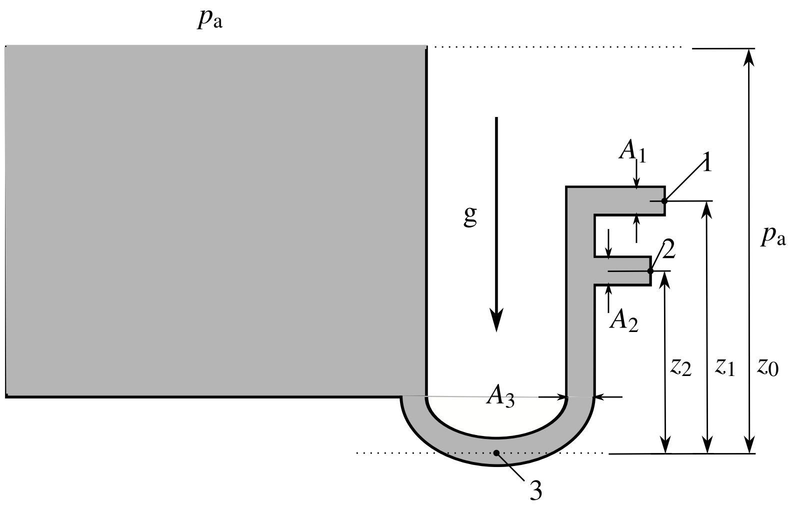

Figure 6.3: Reservoir with an double ended outflow.

<u>Solution:</u>

a) $v_1 = 6\ \frac{\text{m}}{\text{s}}$ and $v_2 = 8\ \frac{\text{m}}{\text{s}}$

b) $A_2 = 0,15\ \text{m}^2$

c) $p_3 = 1,82$ bar

Reservoir with double ended outflow: Solution

Water flows steadily from a big reservoir (assume z_0 = const. through a pipe with different cross section areas A_1, A_2, A_3 to two different outflows, c.f. the figure below. Ambient pressure applies at the reservoir surface and at the outflows. Treat the flow as inviscid.
Given are: $p_a = 1$ bar, $z_0 = 10$ m, $z_1 = 8,20$ m, $z_2 = 6,80$ m, $\rho_W = 1000\ \frac{\text{kg}}{\text{m}^3}$, $g = 10\ \frac{\text{m}}{\text{s}^2}$, $A_1 = 0.2\ \text{m}^2$, $A_3 = 0.4\ \text{m}^2$,

a) Determine the flow velocities v_1 and v_2 at the outflows 1 and 2, respectively.

b) Determine the area A_2 for which the the volume flow rates through A_1 and A_2 become equal.

c) For the case from b), determine the static pressure p_3 at point 3.

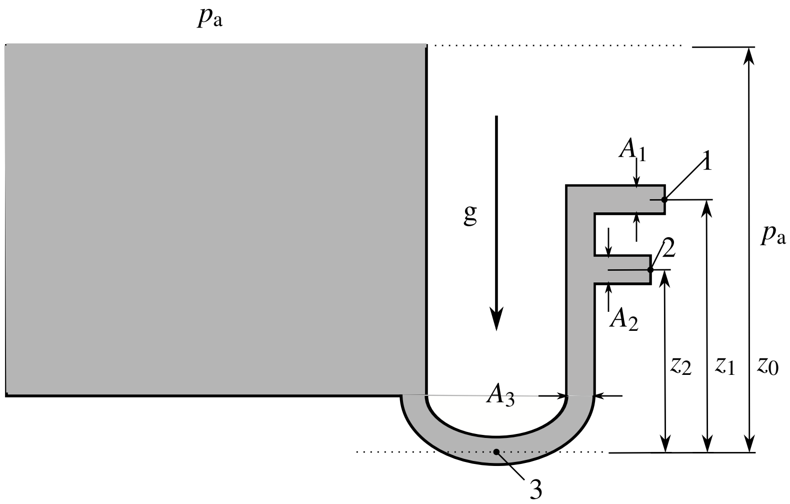

Figure 6.4: Reservoir with an double ended outflow.

Solution:

a) Bernoulli between reservoir surface and outflow 1:
$p_a + \frac{\rho}{2}v_0^2 + \rho_W g z_0 = p_a + \frac{\rho}{2}v_1^2 + \rho_W g z_1$
Constant z_0 implies $v_0 = 0$, solve for v_1: $v_1 = \sqrt{2g(z_0 - z_1)} = 6\ \frac{\text{m}}{\text{s}}$
Similar for v_2: $v_2 = \sqrt{2g(z_0 - z_2)} = 8\ \frac{\text{m}}{\text{s}}$

b) $A_1 v_1 = A_2 v_2 \Rightarrow A_2 = A_1 \frac{v_1}{v_2} = 0,15\ \text{m}^2$

c) $A_3 v_3 = A_1 v_1 + A_2 v_2 = 2A_2 v_2$, because we have A_2 chosen for the outflows to be equal. $\Rightarrow v_3 = 2v_2 \frac{A_2}{A_3} = 2 \cdot 8\ \frac{\text{m}}{\text{s}} \cdot \frac{0.15\ \text{m}^2}{0.4\ \text{m}^2} = 6\ \frac{\text{m}}{\text{s}}$
Bernoulli between reservoir surface and point 3:
$p_a + \frac{\rho}{2}v_0^2 + \rho_W g z_0 = p_3 + \frac{\rho}{2}v_3^2 + \rho_W g z_3$

$v_0 = 0$ as before, $z_3 = 0$:

$$\begin{aligned}
p_a + \rho_W g z_0 &= p_3 + \frac{\rho}{2} v_3^2 \\
\Rightarrow p_3 &= p_a + \rho_W g z_0 - \frac{\rho}{2} v_3^2 \\
&= 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 10\ \frac{\mathrm{m}}{\mathrm{s}^2} \cdot 10\ \mathrm{m} + 100000\ \mathrm{Pa} - 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \frac{(6\ \frac{\mathrm{m}}{\mathrm{s}})^2}{2} \\
&= 1,82\ \mathrm{bar}
\end{aligned}$$

6.3 Two reservoirs separated by a sluice gate

A rectangular sluice gate of width B separates two reservoirs with liquids of different density $\rho_1 < \rho_2$ that are filled to the same height H=3L. In the sluice gate there is a quadratic opening with edge length L that is closed by a plate, c.f. figure below.
Given: $B = 5$ m, $H = 6$ m, $L = 2$ m, $\rho_1 = 800\ \frac{\text{kg}}{\text{m}^3}$, $\rho_2 = 1000\ \frac{\text{kg}}{\text{m}^3}$, $g = 10\ \frac{\text{m}}{\text{s}^2}$.

a) Determine the resultant force F_G on the entire sluice gate including the plate.

b) Determine the resultant force F_P on the plate only.

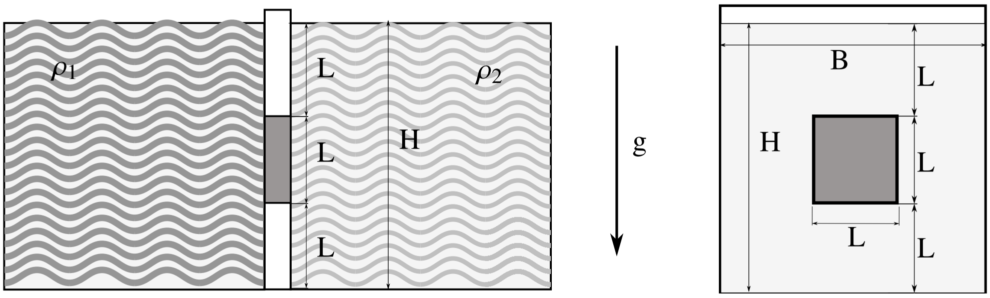

Figure 6.5: Two reservoirs separated by a sluice gate.

<u>Solution:</u>

a) 180 kN

b) 24 kN

Two reservoirs separated by a sluice gate: Solution

A rectangular sluice gate of width B separates two reservoirs with liquids of different density $\rho_1 < \rho_2$ that are filled to the same height H=3L. In the sluice gate there is a quadratic opening with edge length L that is closed by a plate, c.f. figure below.
Given: $B = 5$ m, $H = 6$ m, $L = 2$ m, $\rho_1 = 800\ \frac{\text{kg}}{\text{m}^3}$, $\rho_2 = 1000\ \frac{\text{kg}}{\text{m}^3}$, $g = 10\ \frac{\text{m}}{\text{s}^2}$.

a) Determine the resultant force F_G on the entire sluice gate including the plate.

b) Determine the resultant force F_P on the plate only.

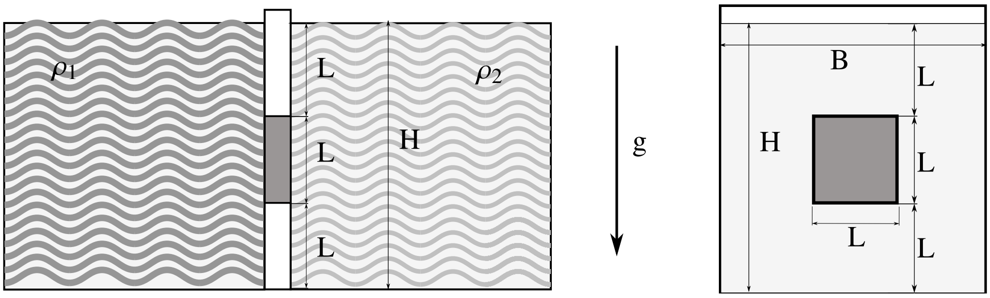

Figure 6.6: Two reservoirs separated by a sluice gate.

<u>Solution:</u>

a) $$F_R = F_1 - F_2 = B\int_0^H (p_a + \rho_1 g z)\mathrm{d}z - B\int_0^H (p_a + \rho_2 g z)\mathrm{d}z$$
$$= B\int_0^H (\rho_1 - \rho_2) g z \mathrm{d}z = B[(\rho_1 - \rho_2) g [\frac{1}{2}z^2]_0^H$$
$$= 5\text{ m} \cdot 200\ \frac{\text{kg}}{\text{m}^3} \cdot 10\ \frac{\text{m}}{\text{s}^2} \cdot \frac{6^2}{2}\ \text{m}^2 = 180\text{ kN}$$

b) $$F_P = F_1 - F_2 = L\int_L^{2L} (p_a + \rho_1 g z)\mathrm{d}z - L\int_L^{2L} (p_a + \rho_2 g z)\mathrm{d}z$$
$$= L\int_L^{2L} (\rho_1 - \rho_2) g z \mathrm{d}z = L[(\rho_1 - \rho_2) g [\frac{1}{2}z^2]_L^{2L}$$
$$= 2\text{ m} \cdot 200\ \frac{\text{kg}}{\text{m}^3} \cdot 10\ \frac{\text{m}}{\text{s}^2} \cdot (\frac{4^2}{2} - \frac{2^2}{2})\ \text{m}^2 = 24\text{ kN}$$

6.4 Pressure in a Flow Heater

To avoid overheating, a flow heater[1] is switched on only, if the volume flow is sufficiently high. Here it needs to be $12 l/\text{min}$ or higher.

a) In a first design, the flow heater is controlled by the pressure difference $\Delta p = p_2 - p_1$ of a frictionless Venturi nozzle, cf. left figure below. What is the pressure trigger for switching on the heater?

b) For cost saving reasons, in a second design, the smooth nozzle is replaced by a sudden change in diameter, cf. right figure below. Except for this step the flow is still frictionless and the flow is fully developed at point 2 again. How does the pressure trigger need to be changed to keep the same safety level of the volume flow rate of $12 l/\text{min}$.

Given: $A_1 = 1\ \text{cm}^2$, $A_2 = 2\ \text{cm}^2$, $\rho = 1000\ \frac{\text{kg}}{\text{m}^3}$.

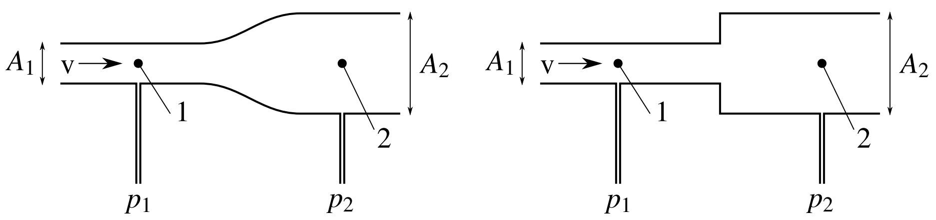

Figure 6.7: Two variants of a part of a flow heater to measure a pressure difference.

Solution:

a) $\Delta p = 1500$ Pa

b) $\Delta p = 1000$ Pa

[1]Durchlauferhitzer

Pressure in a Flow Heater: Solution

To avoid overheating, a flow heater[2] is switched on only, if the volume flow is sufficiently high. Here it needs to be $12 l/\mathrm{min}$ or higher.

a) In a first design, the flow heater is controlled by the pressure difference $\Delta p = p_2 - p_1$ of a frictionless Venturi nozzle, cf. left figure below. What is the pressure trigger for switching on the heater?

b) For cost saving reasons, in a second design, the smooth nozzle is replaced by a sudden change in diameter, cf. right figure below. Except for this step the flow is still frictionless and the flow is fully developed at point 2 again. How does the pressure trigger need to be changed to keep the same safety level of the volume flow rate of $12 l/\mathrm{min}$.

Given: $A_1 = 1\ \mathrm{cm}^2$, $A_2 = 2\ \mathrm{cm}^2$, $\rho = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$.

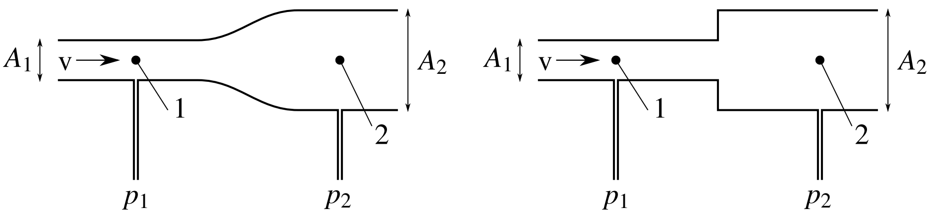

Figure 6.8: Two variants of a part of a flow heater to measure a pressure difference.

- Volume flow rate: $\dot{V}_1 = \dot{V}_1 = 12 l/\mathrm{min} = \frac{0{,}012 m^3}{60 s} = 2 \cdot 10^{-4} \frac{m^3}{s}$, $\dot{V} = vA$ with velocity v and area A
- $\dot{V}_1 = v_1 A_1 \Rightarrow v_1 = \frac{\dot{V}_1}{A_1} = \frac{2 \cdot 10^{-4} \frac{m^3}{s}}{1\ \mathrm{cm}^2} = 2 \frac{m}{s}$
- $\dot{V}_2 = v_2 A_2 \Rightarrow v_2 = \frac{\dot{V}_2}{A_2} = \frac{2 \cdot 10^{-4} \frac{m^3}{s}}{2\ \mathrm{cm}^2} = 1 \frac{m}{s}$

a) $p_1 + \frac{\rho}{2} v_1^2 + \rho g z_1 = p_2 + \frac{\rho}{2} v_2^2 + \rho g z_2 \qquad z_1 = z_2$

$$\Rightarrow \Delta p = p_2 - p_1 = -\frac{\rho}{2}(v_2^2 - v_1^2) = -\frac{1000}{2} \frac{\mathrm{kg}}{\mathrm{m}^3}(1^2 - 2^2)(\frac{\mathrm{m}}{\mathrm{s}})^2 = 1500\ \mathrm{Pa}$$

b) $p_1 + \frac{\rho}{2} v_1^2 + \rho g z_1 = p_2 + \frac{\rho}{2} v_2^2 + \rho g z_2 + \delta p_L \qquad z_1 = z_2$

$$\Rightarrow \Delta p = p_2 - p_1 = -\frac{\rho}{2}(v_2^2 - v_1^2) - \frac{\rho}{2} v_1^2 (1 - \frac{A_1}{A_2})^2 = 1000\ \mathrm{Pa}$$

[2]Durchlauferhitzer

6.5 Jet against an inclined Plate

An inclined plate is moved with a velocity of $U = 5\ \frac{\text{m}}{\text{s}}$ against a horizontal water jet, c.f. the figure below. The water jet has a rectangular cross section of height $h = 0.1$ m and width $b = 0.1$ m and a velocity of $v = 10\ \frac{\text{m}}{\text{s}}$. The jet breaks up into an upward and downward jet with velocity v, width b and heights of ϵh and $(1-\epsilon)h$, respectively. The water density is $\rho = 1000\ \frac{\text{kg}}{\text{m}^3}$ and the inclination angle of the plate is $\alpha = 70°$. Ignore gravity and friction.

a) Derive a function for ϵ depending on α.

b) Determine the propulsion force F_P needed to move the plate.

Hint: Use a coordinate system like depicted in the figure.

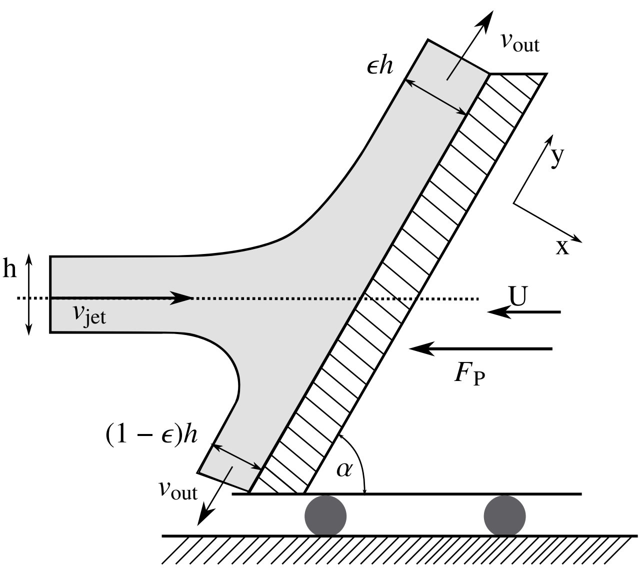

Figure 6.9: A square jet hitting on an inclined plate.

Solution:

a) $\epsilon = \frac{1}{2}(1 + \cos\alpha)$

b) $F_P = -\rho \cdot h \cdot b \cdot (v + U)^2 \sin^2\alpha = 1986{,}8$ N

Jet against an inclined Plate: Solution

An inclined plate is moved with a velocity of $U = 5\ \frac{\mathrm{m}}{\mathrm{s}}$ against a horizontal water jet, c.f. the figure below. The water jet has a rectangular cross section of height $h = 0.1$ m and width $b = 0.1$ m and a velocity of $v_{\mathrm{jet}} = 10\ \frac{\mathrm{m}}{\mathrm{s}}$. The jet breaks up into an upward and downward jet with velocity v_{out}, width b and heights of ϵh and $(1-\epsilon)h$, respectively. The water density is $\rho = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$ and the inclination angle of the plate is $\alpha = 70°$. Ignore gravity and friction.

a) Derive a function for ϵ depending on α.

b) Determine the propulsion force F_P needed to move the plate.

Hint: Use a coordinate system like depicted in the figure.

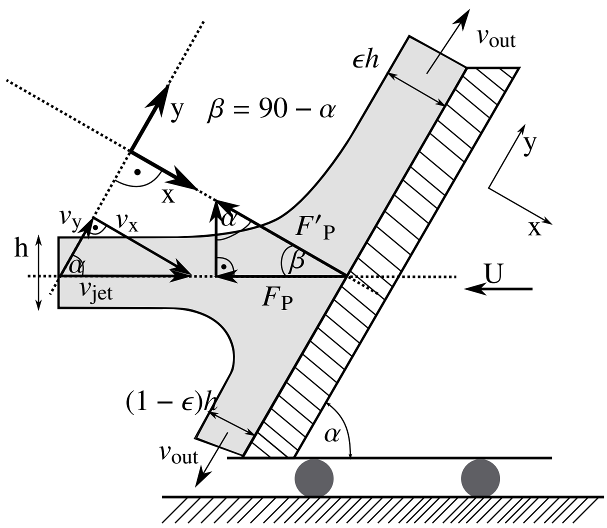

Figure 6.10: A square jet hitting on an inclined plate.

- The flow is incompressible and inviscid.
- Coordinate system 1 (CS1): x-axis parallel to the ground, y-axis vertically upward.
- Coordinate system 2 (CS2): y-axis parallel to the inclined plate and upward, x-axis orthogonal to y-axis and moving with velocity U as shown in figure 6.10.

a) The grey area is the control volume and is moving with the plate. Use CS2! In CS2, the inflow velocity is $v_{\mathrm{in}} = v_{\mathrm{jet}} + U$. The continuity equation for incompressible flow then gives:

$$\dot{V}_{\text{in}} = \dot{V}_{\text{out}}$$

$$\Rightarrow (v_{\text{jet}} + U) \cdot b \cdot h = v_{\text{out}} \cdot b \cdot \epsilon h + v_{\text{out}} \cdot b \cdot (1-\epsilon)h = v_{\text{out}} \cdot b \cdot h$$

$$\Rightarrow v_{\text{out}} = (v_{\text{jet}} + U) = v_{\text{in}}$$

Write up the y-component of the moment equation in CS2.

$$(\rho v_y \dot{V})_{\text{in}} - (\rho v_y \dot{V})_{\text{out}} + \underbrace{\sum F_y}_{\text{no forces in the y-direction in CS2}} = 0$$

$$\Rightarrow \rho \cdot v_{\text{in}} \cdot \cos\alpha \cdot h \cdot b \cdot v_{\text{in}}$$
$$- (\rho \cdot v_{\text{out}} \cdot \epsilon h \cdot b \cdot v_{\text{out}} - \rho \cdot v_{\text{out}} \cdot (1-\epsilon) h \cdot b \cdot v_{\text{out}}) = 0$$

$$\Rightarrow \rho \cdot (v_{\text{jet}} + U) \cdot \cos\alpha \cdot h \cdot b \cdot (v_{\text{jet}} + U)$$
$$- (\rho \cdot (v_{\text{jet}} + U) \cdot \epsilon h \cdot b \cdot (v_{\text{jet}} + U) - \rho \cdot (v_{\text{jet}} + U) \cdot (1-\epsilon) h \cdot b \cdot (v_{\text{jet}} + U)) = 0$$

divide by $(v_{\text{jet}} + U)^2, h, b, \rho$

$$\Rightarrow \cos\alpha - \epsilon + 1 - \epsilon = 0$$

$$\Rightarrow \epsilon = \frac{1}{2}(1 + \cos\alpha)$$

b) We need F_P, the force to move the plate horizontally. This is with respect to CS1. The force F_P is in the same direction as the force on the fluid! So we have the same sign as for the forces on the fluid.

However, using CS2 is more convenient to set up a momentum equation, because we already noticed that there are no forces in the y-direction. So in CS2, the only force available will be along the x-axis in CS2. And this force F'_P in CS2 we need to express in CS1 to get the horizontal force that is needed to move the plate.

Thus, from figure 6.10, we infer $F_P = F'_P \sin\alpha \Leftrightarrow F'_P = \frac{F_P}{\sin\alpha}$. As before, in the control volume that is moving with the plate, the incoming jet velocity is $v_{\text{in}} = v_{\text{jet}} + U$. Now we may write up the x-component of the moment equation in CS2. Note that there is no outflow in the x-direction when using CS2.

$$(\rho \cdot v_{\text{in},x} \cdot \dot{V})_{\text{in}} - (\rho \cdot v_{\text{out},x} \cdot \dot{V})_{\text{out}} + F'_P = 0$$

$$\Rightarrow \rho \cdot (v_{\text{jet}} + U) \cdot \sin\alpha \cdot h \cdot b \cdot (v_{\text{jet}} + U) + \frac{F_P}{\sin\alpha} = 0$$

$$\Rightarrow F_P = -\rho \cdot h \cdot b \cdot \sin^2\alpha \left(v_{\text{jet}} + U\right)^2$$

$$\Rightarrow F_P = -1000\ \frac{\text{kg}}{\text{m}^3} \cdot 0,1\ \text{m} \cdot 0,1\ \text{m} \cdot \sin(70°) \cdot \left(10\ \frac{\text{m}}{\text{s}} + 5\ \frac{\text{m}}{\text{s}}\right)^2 = -1986,8\ \text{N}$$

In a coordinate system with the x-axis parallel to the jet inflow, the force is in the negative x-direction. Given engineering sign conventions, the force F_P may be taken positive in the direction shown in the figure.

6.6 Rough Pipeline

A circular horizontal pipeline is made out of concrete and has a diameter of 20 cm and a surface roughness of 8 mm. It delivers 340 m^3 of water per hour (density 1000 kg/m^3, dynamic viscosity 10^{-3} Pa$\cdot s$). The length of the pipeline is 24 km.The pressure loss is is compensated for by a number of pumps which deliver a power of 1.150 MW each. How many pumps are needed to compensate for the pressure loss?

<u>Solution:</u> Three pumps.

Rough Pipeline: Solution

A circular horizontal pipeline is made out of concrete and has a diameter of 20 cm and a surface roughness of 8 mm. It delivers 340 m^3 of water per hour (density 1000 kg/m^3, dynamic viscosity 10^{-3} Pa$\cdot s$). The length of the pipeline is 24 km.The pressure loss is is compensated for by a number of pumps which deliver a power of 1.150 MW each. How many pumps are needed to compensate for the pressure loss?

- Radius $r = 0.1m$.
- Flow velocity: $v = \dot{V}/A = (340m^3/3600s)/(\pi \cdot 0.1^2 m^2) = 3.01m/s$
- Reynolds-Zahl: Re $= (3.01 * 0,2)/(10^{-3}/1000) = 602000 \gg 2300$. This means, the flow is turbulent.
- Relative roughness: $k = \epsilon/D = 0.008/0.2 = 0.04$
- Need pressure loss factor from Moody diagram: $\lambda \approx 0,066$
- The pressure loss then follows from $\Delta p = \lambda(L/D)(\rho/2)v^2 = 0,066 \cdot 24000/0,2 \cdot (1000/2) \cdot 3,01^2 = 35877996$ Pa
- The power needed to maintain the flow is then $P = F \cdot v = \Delta p \cdot A \cdot v =$ 35877996 Pa $\cdot \pi \cdot 0.1^2 m^2 \cdot 3.01m/s = 3392693W = 3.393MW$
- With each pump delivering 1,150 MW, one needs three pumps to deliver enough power to maintain the flow rate.