Fluid Dynamics

Chapter 3

Conservation Laws for Mass and Energy

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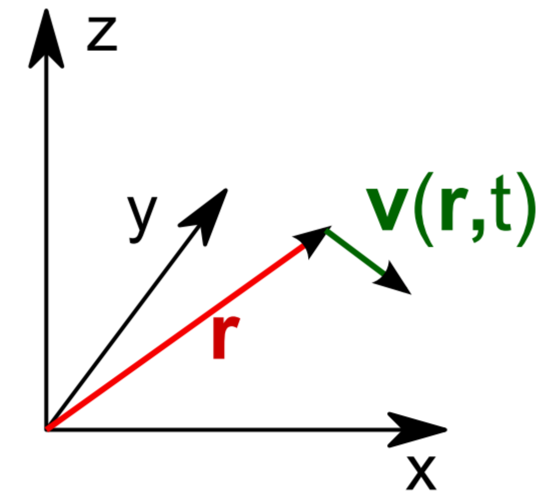
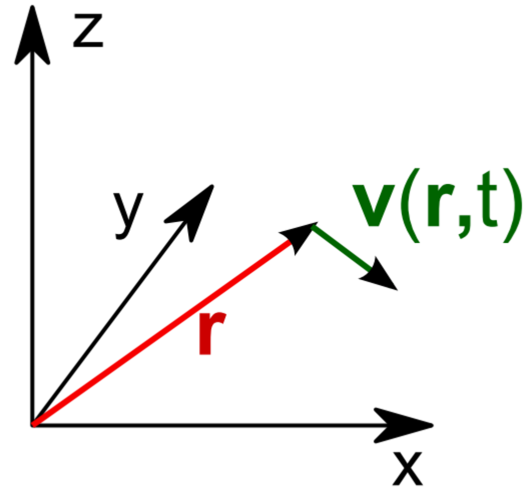
Summer Term 2024

Mathematical Description of a Flow

- Consideration of fluid flows
- Movement of a fluid is in general dependent on
  - the place (x,y,z-coordinates or radius r)
  - the time t
- Approaches for description of fluid flow
  - Euler‘s definition
  - Lagrange definition



Mathematical Description of a Flow



- Euler‘s definition
  - In a given coordinate system with space coordinates **r** the velocity **v**(**r**,t) is given at every point
- Lagrange definition
  - At a starting time t_0 ,the starting position $\mathbf{r}_0$ and velocity $\mathbf{v}_0$ are noted
  - At a later time, the current place $\mathbf{r}(\mathbf{r}_0,\mathbf{v}_0,t)$ and time $\mathbf{v}(\mathbf{r}_0,\mathbf{v}_0,t)$ can be pictured as a function of the starting values for every fluid element

Mathematical Description of a Flow

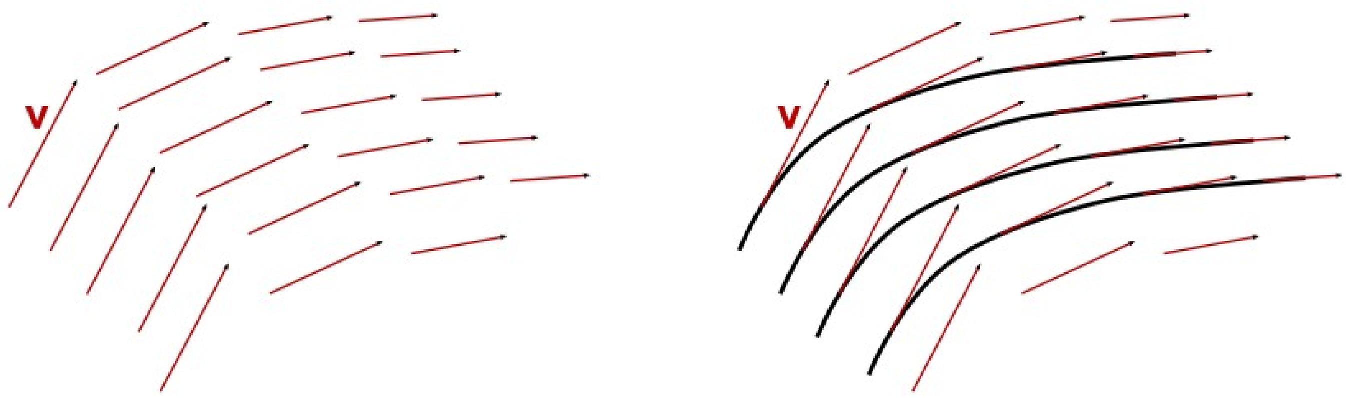
- Euler‘s definition is from a mathematical point of view easier to implement than the Lagrange definition. That is why the Euler definition is commonly used.
- However, thinking about how we are observing flows, we notice that our eyes are following a moving reference point of a flow. This method complies with the Langrange definition.

Mathematical Description of a Flow

- Steady flows
  - The fluid elements change their position with time but the velocity field (and other physical properties) itself stays constant at every place and experiences no temporal change
- Unsteady flows
  - The velocity changes at one or more places within the flow field with time

Streamlines and Pathlines

- Streamlines
  - For a given point in time, curves can be constructed within the flow so that the tangent to these curves are the velocity vectors of the flow



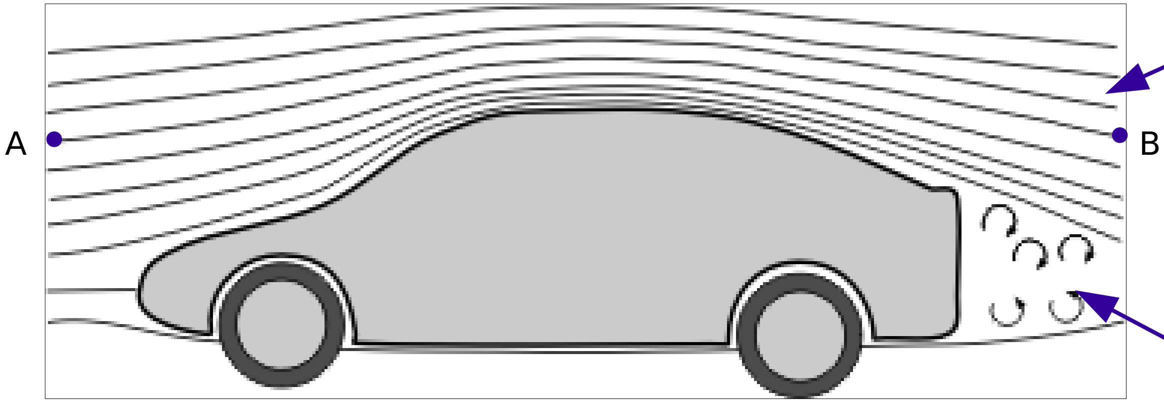
  - This is not only done for one streamline but for the complete section of interest
  - Streamlines indicate where a fluid element is going to move in the next moment

Streamlines and Pathlines

- Pathlines
  - A pathline is the curve in space that is defined by the movement of a specific fluid element within a specified time interval
  - Pathlines are what we see while we are watching a moving reference point in a flow
    - e.g. a leaf on a river
- For steady flows, streamlines and pathlines are identical
- For unsteady flows streamlines and pathlines are different

Streamlines and Pathlines: Examples

- Streamlines around a car:



André Huppertz, https://upload.wikimedia.org/wikipedia/commons/a/ab/Auto_stromlinien.gif
Lizenz: https://creativecommons.org/licenses/by-sa/3.0/deed.de
Point A and B added

In this area the flow is steady, which means it is constant over time

In this area of the vortexes, the flow is unsteady. It changes constantly over time

- Per definition a streamline is only defined for one point in time. But if the flow is steady, which means constant over time, the streamline applies to every point in time.
- In case of steady flow, a fluid element starting in point A moves step by step on a particular streamline. The streamline is also the pathline for this fluid element.
- Result: the fluid element does not leave the streamline and arrives in point B

Streamlines and Pathlines: Examples

- ... and in the experiment:

Deutsches Zentrum für Luft und Raumfahrt, https://upload.wikimedia.org/wikipedia/commons/1/10/Schl%C3%B6rwagen_Windkanal_Modell.jpg

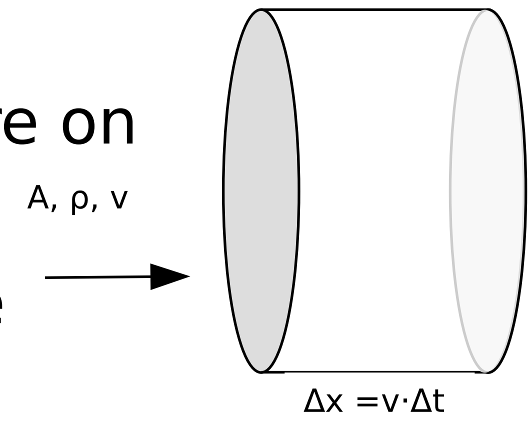
- ...and a simulation:
  https://www.youtube.com/watch?v=09tVUvtyOmk

Conservation of Mass - Continuity Equation

- The mass of a fluid is not altered by a flow
- This is described by the continuity equation
- We are considering the continuity equation just for steady flows
- We are using the mass flow rate to balance the mass in a flow:
  - Mass flow $\dot{m}$ is the mass per time interval that flows through an cross sectional area
  - The unit is kg/s

Conservation of Mass - Mass Flow

- How can the mass flow be calculated?
- Given is a fluid with density ρ and velocity v vertical on the area A
- Let‘s consider the fluid elements that are on the area A at starting time
- After a time Δt they moved the distance Δx = v·Δt



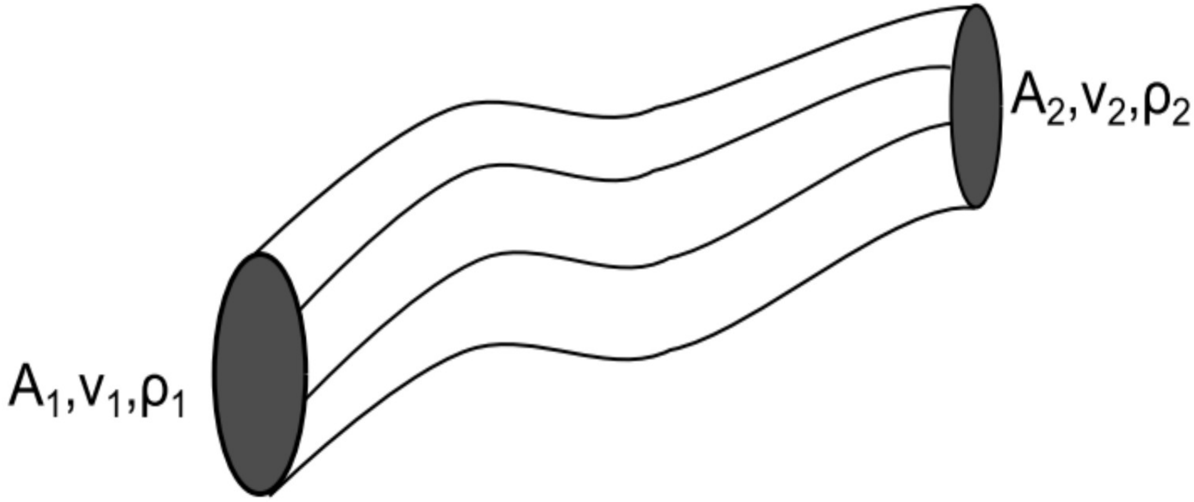
- In the same time interval further fluid elements moved to the area A. The so defined volume Δx·A is filled with the mass of the fluid that moved through the area A. The fluid has the density ρ. The mass Δm in this volume is:

$$\Delta m = \rho \cdot \Delta x \cdot A = \rho \cdot v \cdot \Delta t \cdot A \quad \Rightarrow \quad \boxed{\dot{m} = \frac{\Delta m}{\Delta t} = \rho \cdot v \cdot A}$$

Mass flow through area A

Conservation of Mass - Continuity Equation

- Motivation of the continuity equation for steady flows
- Consideration of a stream pipe:



- Areas A_1 and A_2 vertical to the streamlines
- Streamlines crossing A1 and A2 form the stream pipe
- Shell surface: streamlines that cross the border of the areas A_1 and A_2

Conservation of Mass- Continuity Equation

- Consideration of the mass flow trough the flow pipe
  - The mass flowing into the stream pipe through area A_1 cannot escape trough the shell surface because fluid elements always move with the streamlines in case of steady flow
  - The mass leaves the pipe trough area A_2. The mass flowing through the two areas is the same:

$$\dot{m}_1 = \dot{m}_2$$

  - If density and velocity are constant over the area, the continuity equation is formed

$$\boxed{\rho_1 v_1 A_1 = \rho_2 v_2 A_2}$$

Conservation of Mass - Continuity Equation

- For incompressible flows with constant density the continuity equation can be simplified even more:

$$\rho\, v_1 A_1 = \rho\, v_2 A_2$$
$$v_1 A_1 = v_2 A_2$$
$$\dot{V}_1 = \dot{V}_2$$

- We introduce the volume flow rate $\dot{V}$:

$$\dot{V} = v \cdot A \quad \Rightarrow \quad \dot{m} = \rho \cdot \dot{V}$$

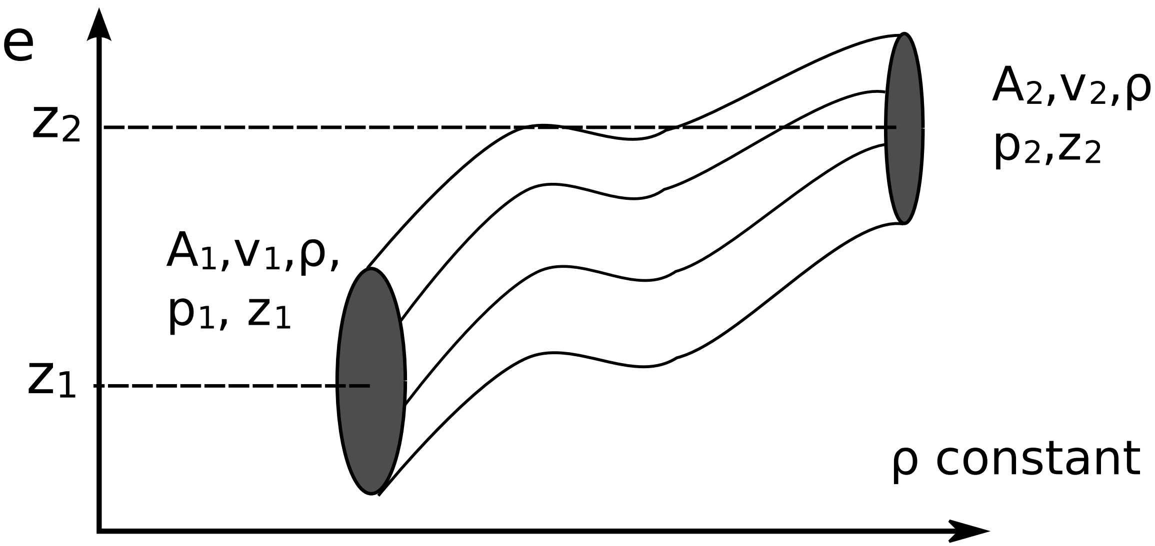
- The volume flow has the unit [volume per time] and is a useful quantity for incompressible fluids
  – e.g. liter/hour
- The general continuity equation for compressible unsteady flows can be found in chapter 6

Conservation of Energy - Bernoulli Equation

- Energy is maintained - no creation or destruction
- But: energy can be transformed from one form to another. For example kinetic energy to potential energy
- And: energy can be transported over system boundaries in both directions
- Relevant forms of energy in this lecture:
  - Kinetic and potential energy
  - Mechanical energy (e.g. pressure)
- We do not consider: thermal and chemical energy
- **Restriction** to
  - isothermal,
  - frictionless („ideal“),
  - steady and
  - incompressible flows (ρ constant)
- Energy balance of these flows: Bernoulli equation

Conservation of Energy- Bernoulli Equation Derivation 1

- restrictions: see slide before!
- Balance area is a flow pipe



- Inlet with area A_1 at height z_1

  with constant quantities:
  - Density ρ
  - Velocity v_1
  - Pressure p_1
- Similarly outlet with area A_2

Important:
The z-axis is pointing **upwards!**

Conservation of Energy- Bernoulli Equation Derivation 1 (continued)

- Energy content within stream pipe depends on inflow through A_1 and outflow through A_2
- Flow is given by volume flow $\dot{V}$ and mass flow $\dot{m}$
- For the balance, we need the energy rate of change (energy change per unit time interval)
- kinetic energy .... $\frac{1}{2} m v^2$
  - Rate of change due to mass flow .... $\frac{1}{2} \dot{m} v^2$
- potential energy .... $m g z$
  - Rate of change due to mass flow .... $\dot{m} g z$
- Work against pressure in the pipe $F \Delta s = p A \Delta s$
  - Rate of change due to volume flow $p A \frac{\Delta s}{\Delta t} = p A v = p \dot{V}$

Conservation of Energy - Bernoulli Equation Derivation 1 (continued)

- „Steady“ means the flow does not change in time in the stream pipe. That means the energy outflow is the equal to the inflow.

$$\dot{m}\frac{v_1^2}{2}+\dot{m}\,g\,z_1+p_1\dot{V}_1=\dot{m}\frac{v_2^2}{2}+\dot{m}\,g\,z_2+p_2\dot{V}_2$$

- Using $\dot{m}=\rho\dot{V}$ gives:

$$\rho\dot{V}_1\frac{v_1^2}{2}+\rho\dot{V}_1 g\,z_1+p_1\dot{V}_1=\rho\dot{V}_2\frac{v_2^2}{2}+\rho\dot{V}_2 g\,z_2+p_2\dot{V}_2$$

- Since the density is constant (‘‘incompressible‘‘) the continuity equation applies in the following form:

$$\dot{V}_1=\dot{V}_2$$

- We divide both sides by volume flow rate $\dot{V}$:

$$\rho\frac{v_1^2}{2}+\rho\,g\,z_1+p_1=\rho\frac{v_2^2}{2}+\rho\,g\,z_2+p_2$$

Conservation of Energy - Bernoulli Equation Derivation 1 (continued)

- There is no special requirement for the position of the areas $A_{1,}$ A_2. We can place them anywhere in the stream pipe. But the equality always applies:

$$\rho\frac{v_1^2}{2}+\rho g z_1+p_1=\rho\frac{v_2^2}{2}+\rho g z_2+p_2$$

- In other words, the total energy of the fluid is constant (friction is ignored!):

$$\boxed{p+\rho\frac{v^2}{2}+\rho g z=\text{const}.}$$

- This is the <u>Bernoulli Equation</u>
  - In general, the upper form of the eq. is used, in order to calculate the energy at one point of the flow as function of another point.

Conservation of Energy - Bernoulli Equation Derivation 1 (continued)

- In this form, the Bernoulli Equation has the unit of pressure:

$$p+\rho\frac{v^2}{2}+\rho g z=\text{const}.$$

- If the eq. Is divided by ρg, the Bernoulli equation has the form

$$\frac{p}{\rho g}+\frac{1}{2g}v^2+z=\text{const}.$$

  - In this form, all terms have the dimension of a height z

- <u>Important for both forms</u>: The z-axis points upwards, because the potential energy E_{pot}=mgz is only correct for this case!

Conservation of Energy - Bernoulli Equation Derivation 1 (continued)

- Example of transformation of energy
  - Kinetic energy is larger at z_2:

$$v_1 A_1 = v_2 A_2$$

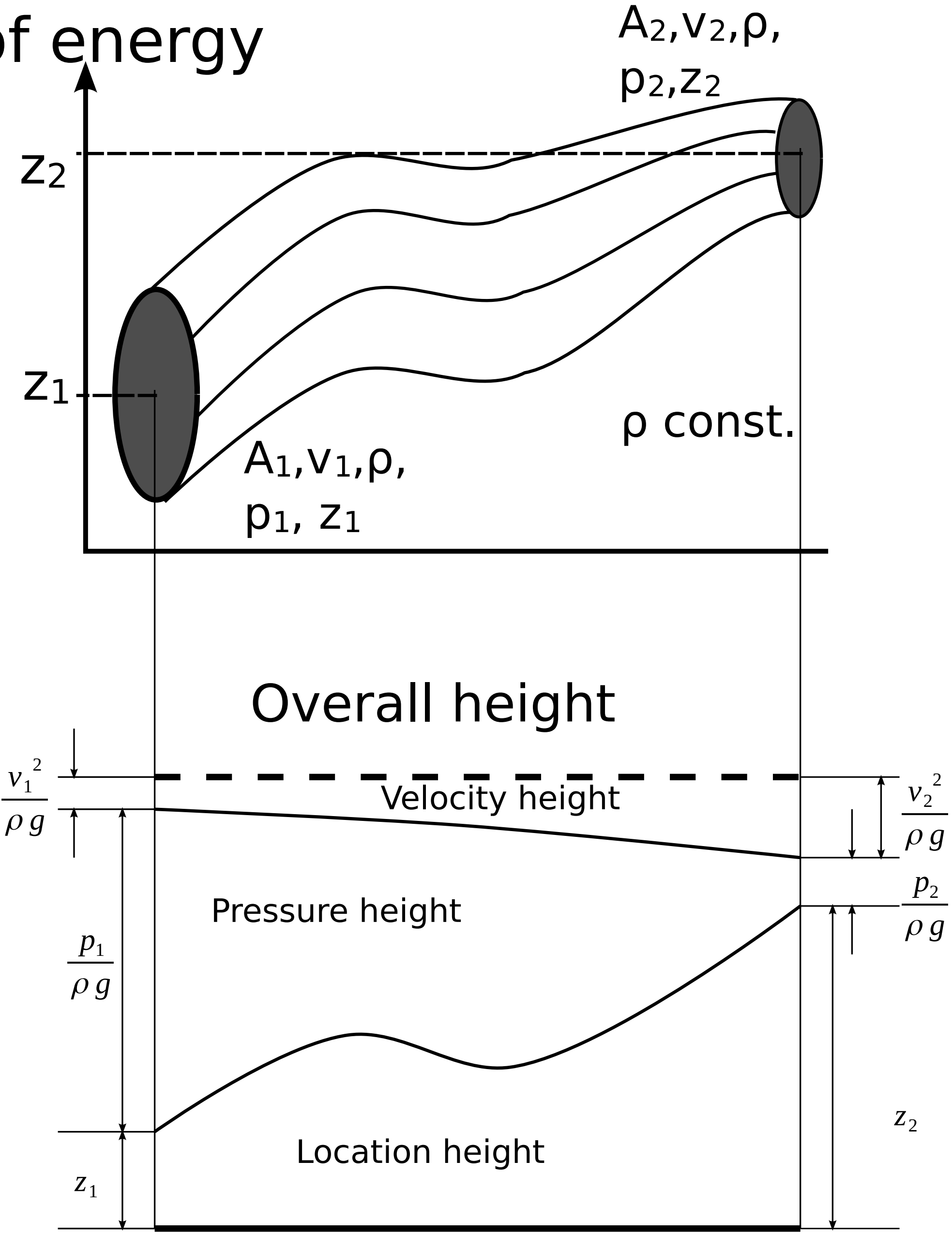
$$A_1 > A_2 \;\Rightarrow\; v_1 < v_2$$

  - Potential energy is larger at z_2 because:

$$z_1 < z_2$$

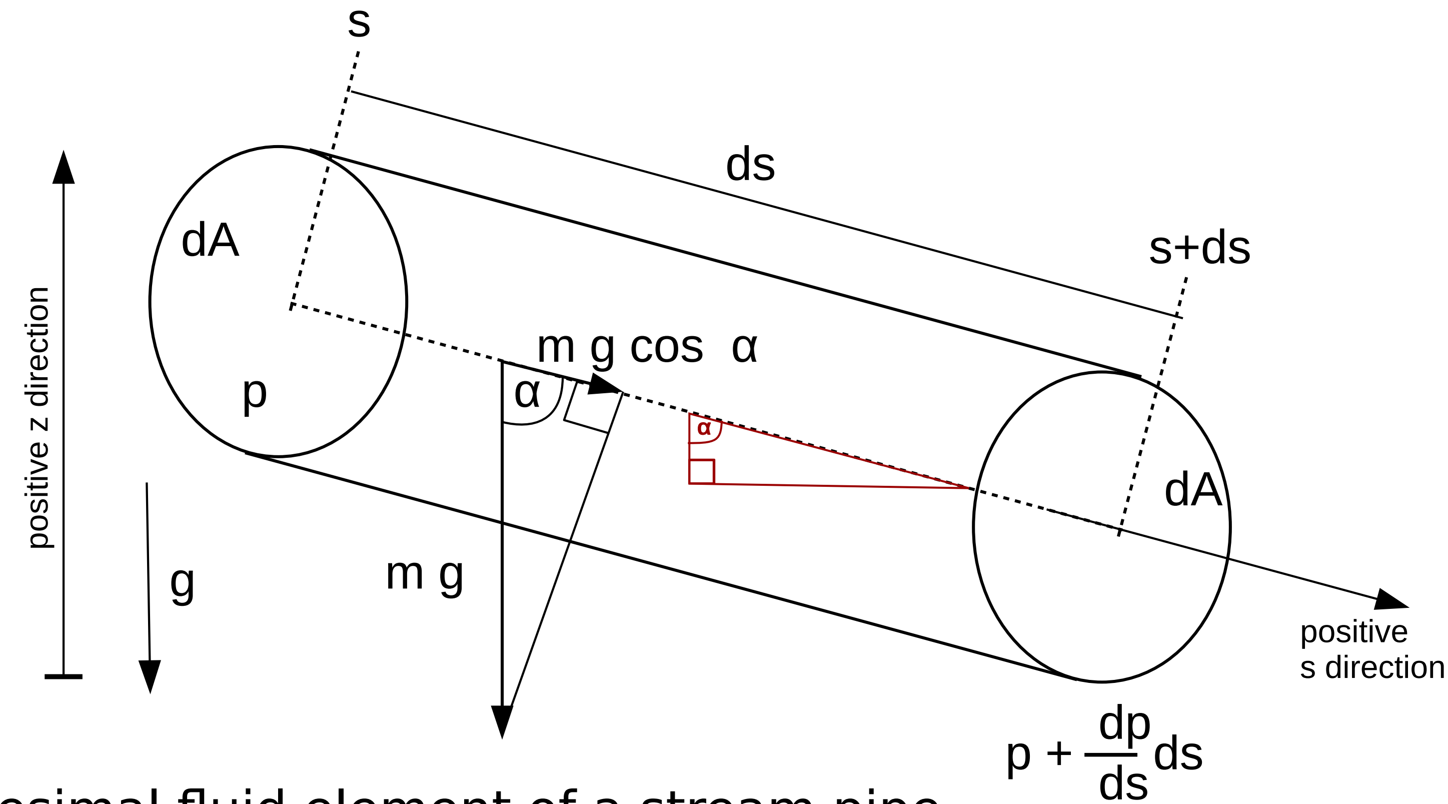
  - Higher kinetic and potential energy at z_2 implies lower pressure energy content. Thus:

$$p_1 > p_2$$



Conservation of Energy - Bernoulli Equation Derivation 2

- Restrictions as before: isothermal, steady, incompressible, ideal flow



- Infinitesimal fluid element of a stream pipe
  - Position given by the length s
  - Constant cross section dA
  - Inclination α against the vertical line
  - Mass of a fluid element: $m = \rho \, \mathrm{d}A \, \mathrm{d}s$

Conservation of Energy - Bernoulli Equation Derivation 2 (continued)

- We want to determine the forces in s direction and set up the equation of motion (Newton's 2nd law) for a fluid element

- Forces:
  - Pressure force at point s: $p(s)\,dA = p\,dA$
  - Pressure force at point s+ds from Taylor series for pressure

$$p(s+ds) \approx p(s) + \frac{dp(s)}{ds}\,ds = p + \frac{dp}{ds}\,ds$$

$$\Rightarrow \quad p(s+ds)\,dA \approx p\,dA + \frac{dp}{ds}\,ds\,dA$$

  - Gravity force: $m\,g\cos(\alpha)$

- All together ... next page

Conservation of Energy - Bernoulli Equation Derivation 2 (continued)

- Before we derive the equation of motion (Newton's 2nd law) for a fluid element, we have to clarify the signs:
  - The pressure force at point s acts in the direction of s ⇒ positive
  - The pressure force at point s+ds on the fluid element acts in the opposite direction of s ⇒ negative
  - Gravity acts in the direction of s ⇒ positive
- Together

$$m\frac{dv}{dt}=\rho\,dA\,ds\frac{dv}{dt}=p\,dA-\left(p\,dA+\frac{dp}{ds}\,ds\,dA\right)+\rho\,ds\,dA\,g\cos\left(\alpha\right)$$

$$\rho\frac{dv}{dt}=-\frac{dp}{ds}+\rho\,g\cos\left(\alpha\right) \qquad (*)$$

  - In the last step, we canceled dA.

Conservation of Energy - Bernoulli Equation Derivation 2 (continued)

- A positive change in s causes a negative change in z. From the red triangle in the illustration of the fluid element on page 22 before we get:

$$\cos(\alpha)\,ds = -dz \quad \Rightarrow \quad \cos(\alpha) = -\frac{dz}{ds}$$

- The flow is steady. The speed does not depend on the time, it depends on the position s: v(s). But the position s changes for our flowing fluid element: s=s(t). For the speed we get v(s(t)). Therefore:

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} \qquad \text{(Derivative Chain Rule)}$$

- The change of the position s(t) by time is the speed itself:

$$\frac{ds}{dt} = v \quad \Rightarrow \quad \frac{dv}{dt} = \frac{dv}{ds}v = v\frac{dv}{ds}$$

Conservation of Energy - Bernoulli Equation Derivation 2 (continued)

- We put that into (*):

$$\rho \frac{dv}{dt} = -\frac{dp}{ds} + \rho\, g \cos(\alpha) \qquad \Big| \quad \cos(\alpha) = -\frac{dz}{ds}$$

$$\Rightarrow \quad \rho \frac{dv}{dt} = -\frac{dp}{ds} - \rho\, g \frac{dz}{ds} \qquad \Big| \quad \frac{dv}{dt} = v \frac{dv}{ds}$$

$$\Rightarrow \quad \rho\, v \frac{dv}{ds} = -\frac{dp}{ds} - \rho\, g \frac{dz}{ds}$$

- Two more tricks:
  - It is $\frac{d}{ds}\left(\frac{1}{2} v^2\right) = v \frac{dv}{ds}$
  - Because g and ρ are constant: $\rho\, g \frac{dz}{ds} = \frac{d}{ds}(\rho\, g\, z)$
- This gives:

$$\rho \frac{d}{ds}\left(\frac{1}{2} v^2\right) = -\frac{dp}{ds} - \frac{d}{ds}(\rho\, g\, z)$$

Conservation of energy - Bernoulli equation Derivation 2 (continued)

- With that we get the following:

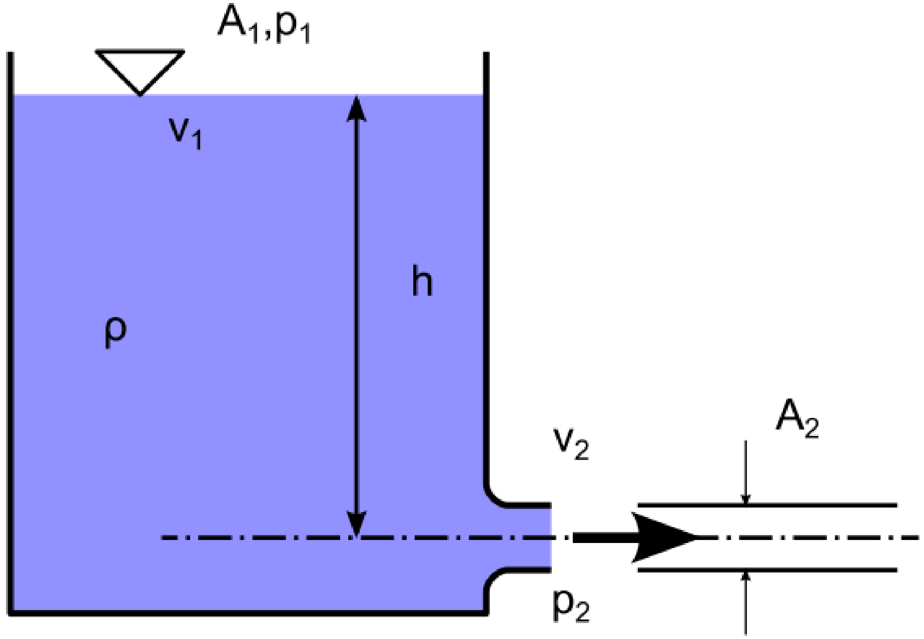
$\rho \frac{d}{ds}\left(\frac{1}{2} v^2\right) = -\frac{dp}{ds} - \frac{d}{ds}(\rho g z)$	ρ=const.
$\Rightarrow \frac{d}{ds}\left(\rho \frac{v^2}{2}\right) = -\frac{dp}{ds} - \frac{d}{ds}(\rho g z)$	All terms on one side
$\Rightarrow \frac{d}{ds}\left(\rho \frac{v^2}{2}\right) + \frac{dp}{ds} + \frac{d}{ds}(\rho g z) = 0$	All terms derivatives by s. Sum up.
$\Rightarrow \frac{d}{ds}\left(\rho \frac{v^2}{2} + p + \rho g z\right) = 0$	

- If the derivative is zero, the term is constant. From this we get again the Bernoulli-Equation:

$$\boxed{\rho \frac{v^2}{2} + p + \rho g z = \text{const.}}$$

Usage of the Bernoulli Equation
Flow from a Water Tank

- Water flows out of a tank through a small outlet.
- Assumption: frictionless, stationary, incompressible
- Looking for: outlet speed v_2
- Approach: We are looking at a streamline between the water surface (z_1) and the outlet (z_2).



Usage of the Bernoulli equation
Flow from a Water Tank

- Calculation of v_2 with Bernoulli and Conti.-Equ.

$$\rho\frac{v_1^2}{2}+\rho g z_1+\cancel{p_1}=\rho\frac{v_2^2}{2}+\rho g z_2+\cancel{p_2}$$

$p_1=p_2=$ ambient pressure, counteract.

$$\Rightarrow \quad \rho\frac{v_1^2}{2}+\rho g z_1=\rho\frac{v_2^2}{2}+\rho g z_2$$

Conti.-Equ.: $\rho v_1 A_1=\rho v_2 A_2$

$$\Rightarrow v_1=\frac{v_2 A_2}{A_1}$$

$$\Rightarrow \quad \rho\frac{\left(\frac{v_2 A_2}{A_1}\right)^2}{2}+\rho g z_1=\rho\frac{v_2^2}{2}+\rho g z_2$$

times 2, divide by ρ, then sort

$$\Rightarrow 2 g z_1-2 g z_2=v_2^2-\left(\frac{A_2}{A_1}\right)^2 {v_2}^2$$

$h=z_1-z_2$ and dissolve by v_2

$$\Rightarrow \quad v_2=\sqrt{\frac{2 g h}{1-\left(\frac{A_2}{A_1}\right)^2}} \quad \Rightarrow \boxed{v_2=\sqrt{2 g h}}$$

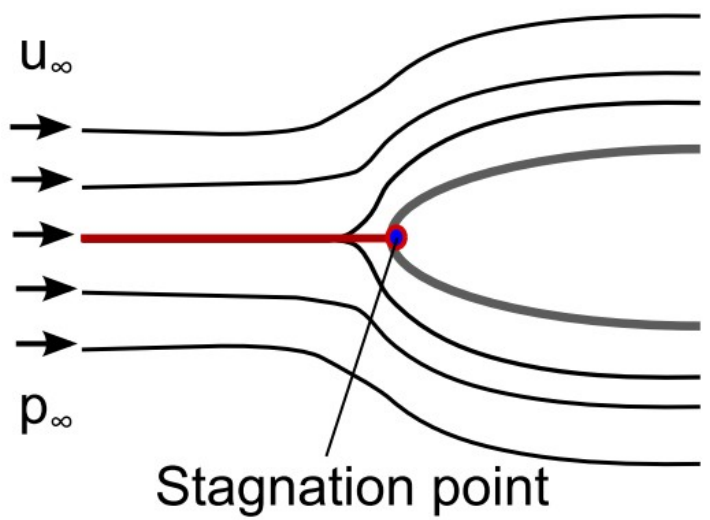
(Torricelli)

The last simplification uses $A_2 / A_1 \ll 1$, if the outflow is <u>much</u> smaller than that water surface in the tank

Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

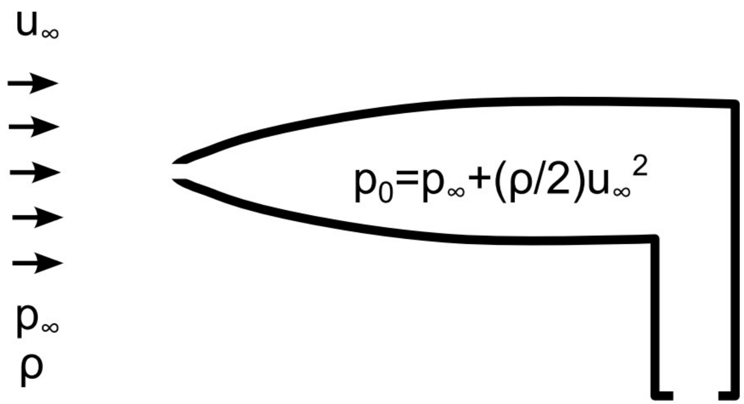
- Separate parts of pressure in a fluid
  - Static pressure p_s
  - Dynamic pressure $p_d = \rho \frac{v^2}{2}$
  - Total pressure $p_t = p_s + p_d + \rho g z$
- Bernoulli:
  The total pressure is constant on a streamline.
- Stagnation point
  - Static pressure: maximum
  - Kinetic energy: minimum
- Stagnation pressure:

$$p_{stagnation} = p_s + p_d = p_\infty + \rho \frac{u_\infty^2}{2}$$

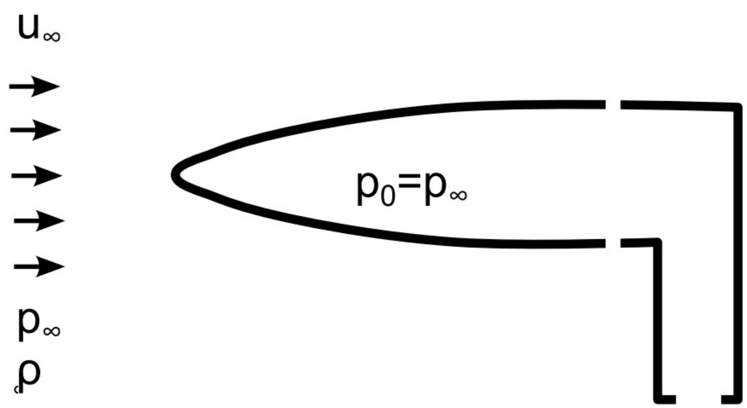


Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

- Pitot tube
  - Measurement of total pressure
  - Static pressure unknown

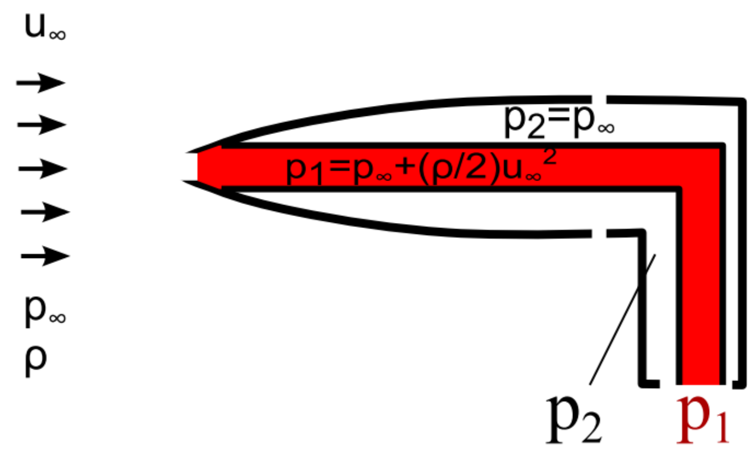


- Pressure probe
  - Measurement of static pressure



Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

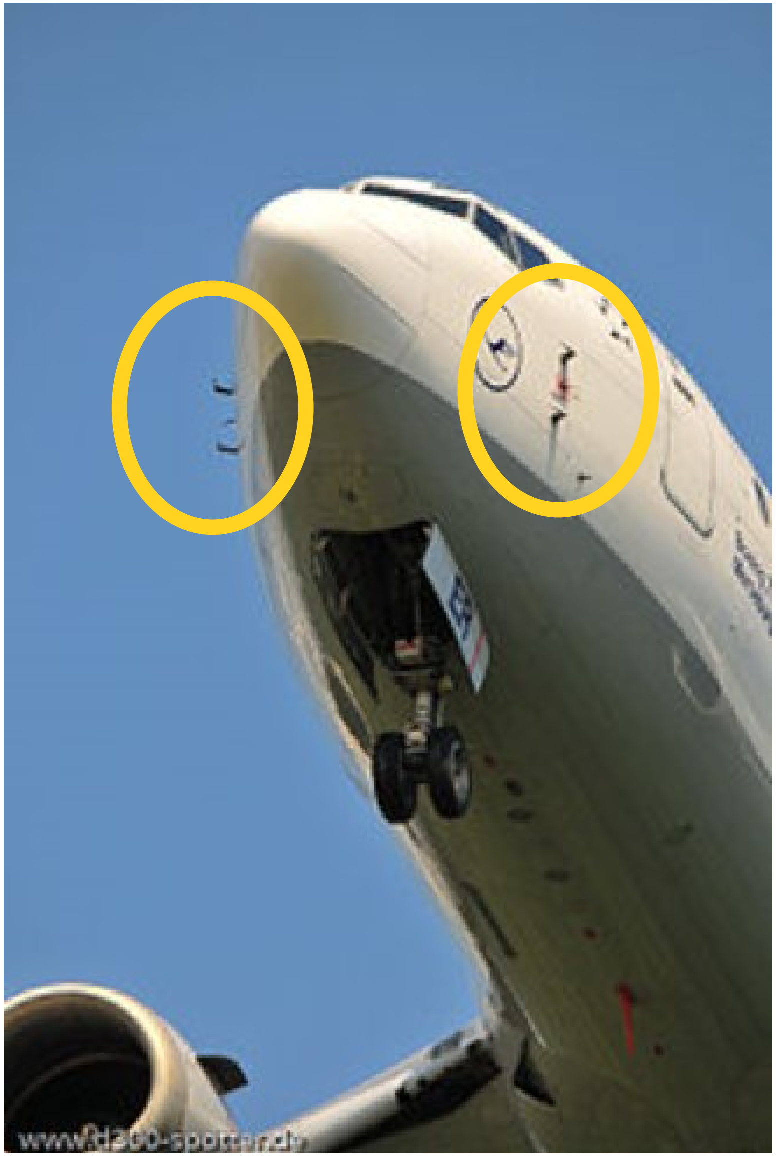
- Prandtl probe
  - Assumption: close streamlines have the same total pressure
  - That's why: Difference in height is negligible
  - With that the flow rate can be determined:



$$p_1 - p_2 = p_\infty + \rho \frac{u_\infty^2}{2} - p_\infty = \rho \frac{u_\infty^2}{2} \quad \Rightarrow \quad u_\infty = \sqrt{\frac{2(p_1-p_2)}{\rho}}$$

  - Use for this device: Measurement of the speed of airflow around aircraft
    - Attention:
      That is not the speed above ground!

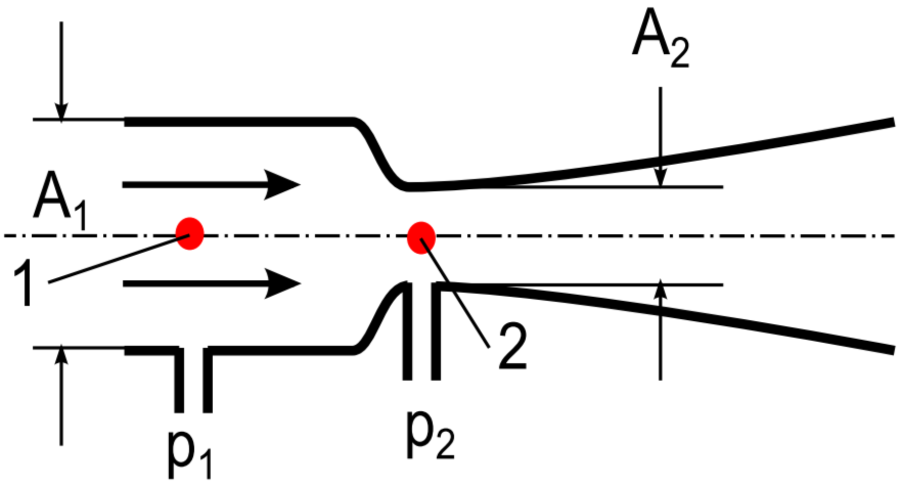
Airplane Pitot Tubes



https://en.wikipedia.org/wiki/Pitot_tube

Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

- Venturi nozzle
  - Measurement of volume flow in a pipe



- Determination of volume flow with Bernoulli and continuity equation

$$p_1+\rho\frac{v_1^2}{2}+\cancel{\rho g z_1}=p_2+\rho\frac{v_2^2}{2}+\cancel{\rho g z_2} \qquad \Big| z_1=z_2$$

Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

- Determination of volume flow (continued)

$$p_1+\rho\frac{v_1^2}{2}=p_2+\rho\frac{v_2^2}{2}$$

Conti.-Equ.: $\rho\, v_1\, A_1=\rho\, v_2\, A_2$

$\Rightarrow v_1=\dfrac{v_2\, A_2}{A_1}$

$$\Rightarrow\quad p_1+\rho\frac{\left(\frac{v_2\, A_2}{A_1}\right)^2}{2}=p_2+\rho\frac{v_2^2}{2}$$

- times 2, divide by ρ, then sort
- use $\Delta p = p_1 - p_2$

$$\Rightarrow\quad 2\frac{\Delta p}{\rho}=v_2^2-\left(\frac{A_2}{A_1}\right)^2 v_2^2$$

Separate for v_2

$$\Rightarrow\quad v_2=\sqrt{\frac{2\,\Delta p}{\rho\left(1-\left(\frac{A_2}{A_1}\right)^2\right)}}$$

$$\Rightarrow\quad \boxed{\dot{V}=v_2\cdot A_2}$$

Volume flow at point 2.
Because it is constant, it is the volume flow through the pipe.

Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

- Important for usage in reality:
  - Consideration of the friction of real fluids
- Coefficient C_d comes as manufacturer information for any particular Venturi nozzle
- From consideration:

$$\dot{V} = v_2 A_2 \quad \Rightarrow \quad \dot{V} \propto A_2 \sqrt{\left(\frac{2 \Delta p}{\rho}\right)}$$

  we get

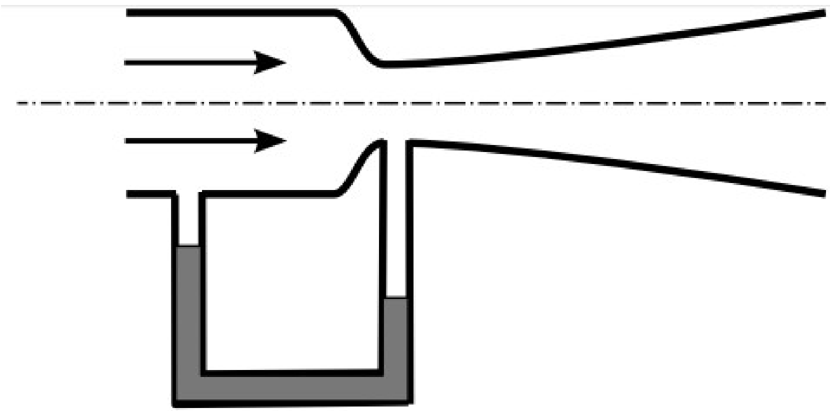
$$\dot{V} = C_d A_2 \sqrt{\left(\frac{2 \Delta p}{\rho}\right)}$$

- The factor $1/\sqrt{\left(1 - \left(A_2 / A_1\right)^2\right)}$ containing the geometry information is included in the coefficient C_d

Usage of the Bernoulli Equation
Measuring Devices for Pressure & Speed

- A quick question of comprehension: Which of the following illustrations is right and which is wrong?

(A)



(B)