Fluid Dynamics

Chapter 2

Hydrostatics – Aerostatics

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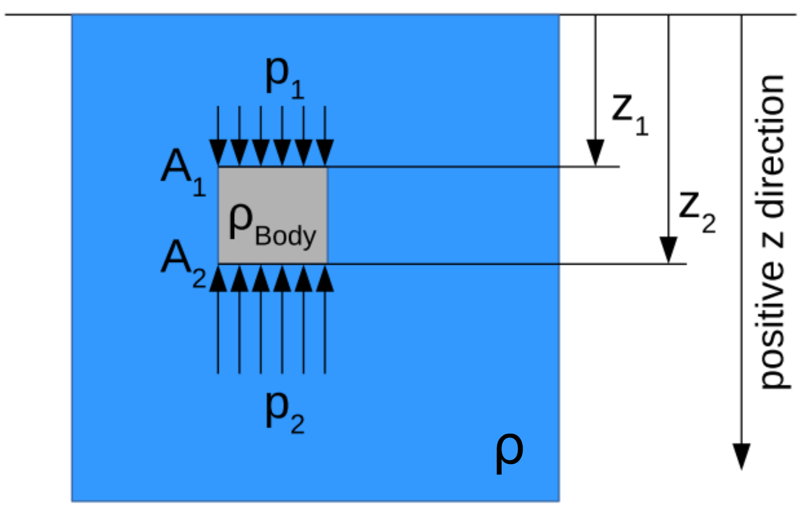
Fundamentals

- Hydro- and aerostatics deal with static fluids
  - As a consequence the fluids are in balance
- Hydrostatics
  - Assumption: the density of liquids is constant!
  - Application examples:
    - Design/calculations of the buoyancy of ships
    - Design of big reservoirs
    - Pressure gauge
- Aerostatics
  - Density is not constant!
  - Gases change their density as function of pressure

Hydrostatic Pressure: Derivation (1)

- 1. Case: Immersed cube
  - Floating
  - At rest (static)
  - Fluid has constant density



- Coordinate system:
  - z-Axis **positive pointing downwards**
  - z=0 is the surface of the fluid
- Force balance of the cube:

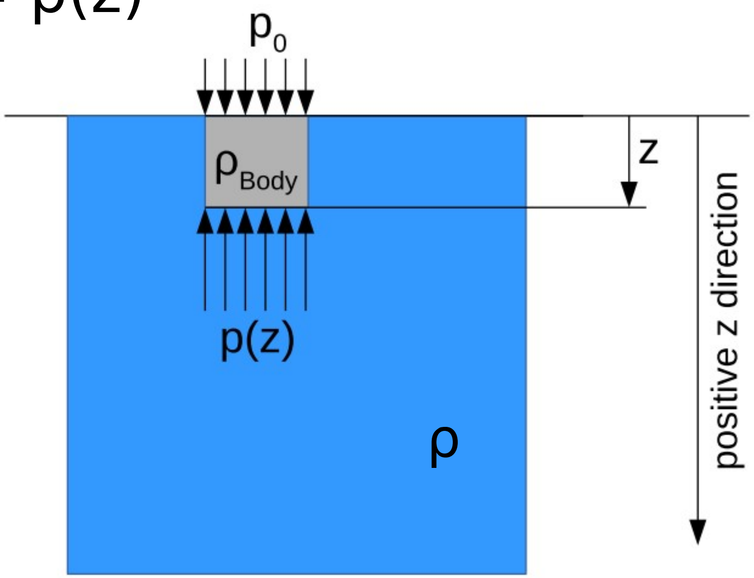
$$mg + p_1 A_1 = p_2 A_2;$$
$$mg + p_1 A = p_2 A;$$
$$\rho_{Body} A l g + p_1 A = p_2 A$$
$$p_2 = p_1 + \rho_{Body} l g$$

$$A_1 = A_2 = A \qquad l = z_2 - z_1$$
$$m = \rho_{Body} A l$$

Edge length

Hydrostatic Pressure: Derivation (2)

- 2. Case: cube on the liquid surface: $l = z$
  - Pressure on bottom of the cube: $p_2 = p(z)$
  - Pressure on the surface is equal to the air pressure: $p_1 = p_0 = \text{const.}$
  - Special case:
    The cube is a fluid instead of a solid. The cube is just delimited in our mind from the rest of the fluid:
    $\rho_{Body} = \rho$



- Insert in the previous equation:

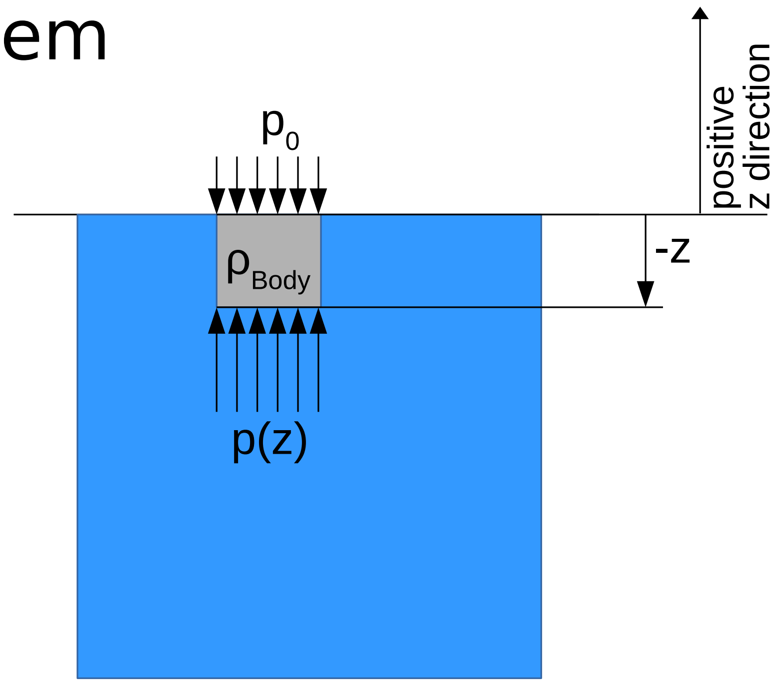
$$p(z) = p_0 + \rho\, g\, z$$ Basic equation of hydrostatics

- Pressure curve as a function of depth (z-coordinate)
- Validity: fluids with constant density!

Hydrostatic Pressure: Derivation (3)

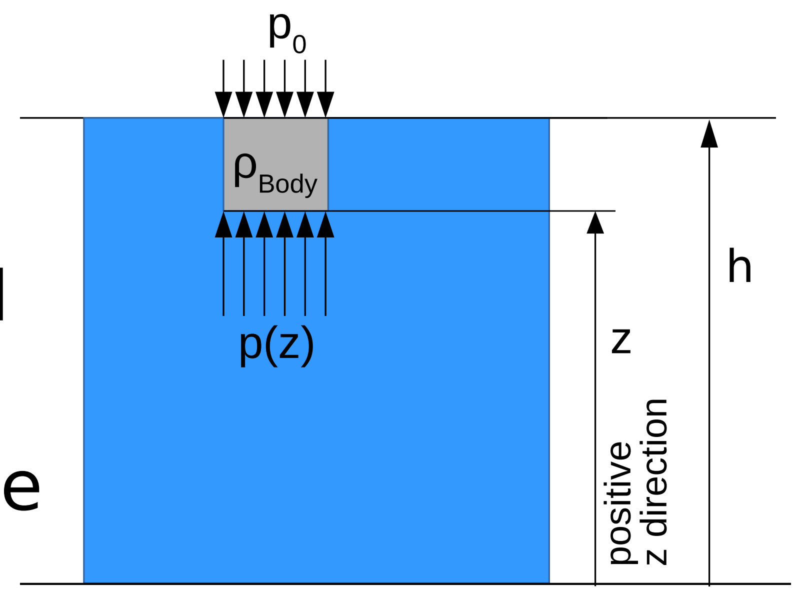
- Now a different coordinate system
  - z-axis **positive pointing upwards**
  - z=0 is the liquid surface
- Result: replace z with -z

$$p(z) = p_0 - \rho\, g\, z$$



Hydrostatic Pressure: Derivation (4)

- And a final coordinate system
  - z-axis **positive pointing upwards**
  - z=0 is the ground of the fluid at depth h
- Now the edge length of the cube becomes $l = h - z$

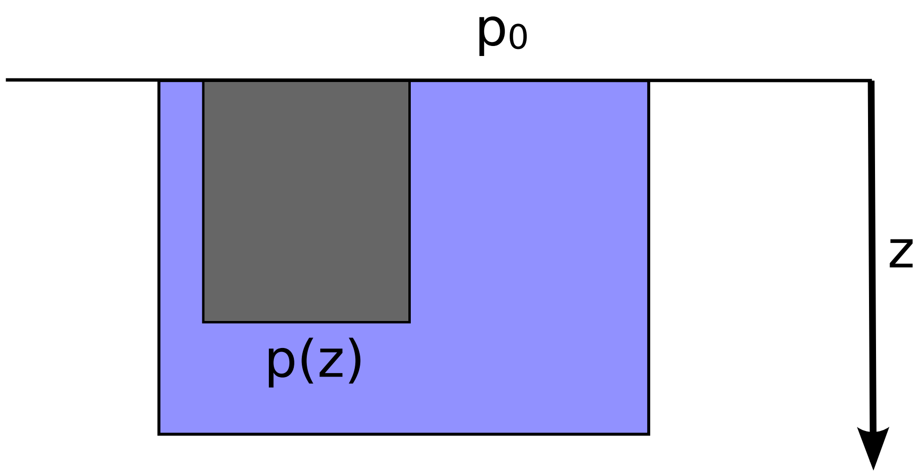


- Result:

$$p(z) = p_0 + \rho g (h - z)$$ Basic equation of hydrostatics at depth h

Hydrostatic Pressure: Thought Experiment Consolidation

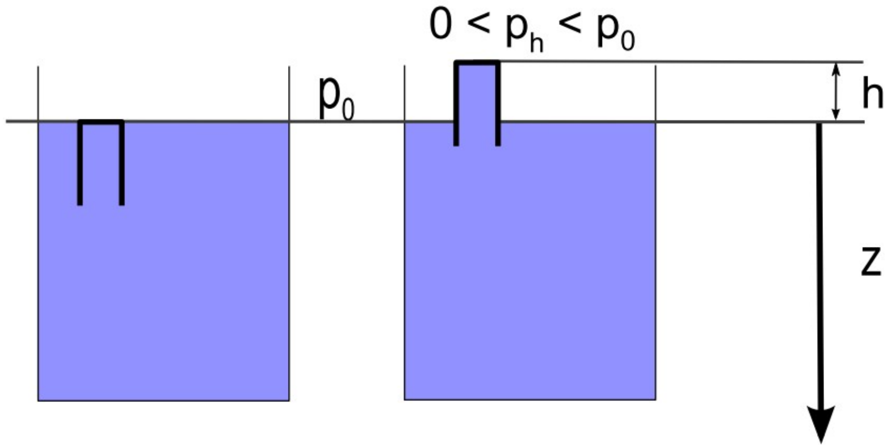
- Reservoir with resting liquid and constant density
- Consolidation of the liquid in the grey area
- All other physical quantities stay unchanged
- Result: The forces in the fluid stay unchanged!



- Pressure only depends on the depth. It does not matter if it is a liquid or a solid.

Hydrostatic Pressure: Liquid Column

- A pipe with a closed end is completely immersed in a reservoir with a liquid as shown in the picture on the left

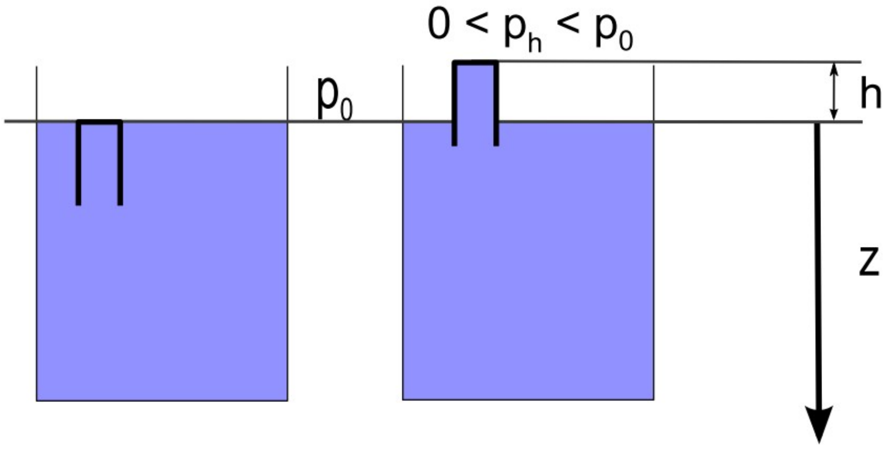


- The pipe gets lifted. The liquid rises with the pipe
- However, fluids cannot transmit tensile forces!

  How can formation of the liquid column be explained?

Hydrostatic Pressure: Liquid Column

- Explanation:



- The ambient air pressure acts on the surface in the reservoir, but not on the surface of the liquid at the closed end of the tube
- The ambient pressure forces the liquid upwards
- The pressure p_h at the upper end of the liquid column equals the ambient pressure reduced by the pressure caused by the weight of the liquid column

Hydrostatic Pressure: Liquid Column

- Exercise:
  Calculate the maximum height of the liquid column for the following fluids:
  - Water
  - Mercury
  - Given:

$$\rho_{H_20} = 1{,}000\,\frac{\mathrm{kg}}{\mathrm{m}^3},$$

$$\rho_{Hg} = 13{,}600\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

$$p_o = p_{atm} = 101{,}300\,\mathrm{Pa}$$

$$g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Hydrostatic Pressure: Liquid Column

- Solution: The maximum height is reached when the ambient pressure p_0 is not able to push the liquid column any higher. That means that the pressure caused by the weight of the liquid neutralizes the ambient pressure. At this moment the pressure at the upper end of the liquid column becomes 0 Pa. So: $p(\mathrm{h}_{\mathrm{max}}) = 0\,\mathrm{Pa}$

  Basic equation of hydrostatics for $z = h_{max}$:

$$p_{h_{max}} = p(z = h_{max}) = p_0 + \rho\, h_{max}\, g = 0 \quad \Rightarrow \quad h_{max} = -\frac{p_0}{\rho\, g}$$

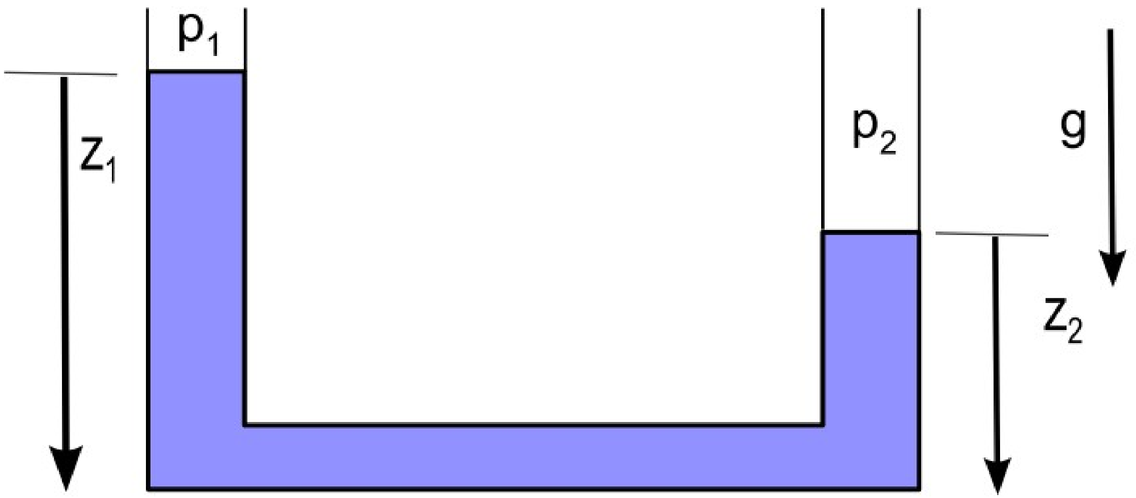
  Insert in equation:

$$h_{H_20} = -10.33\,\mathrm{m}$$
$$h_{Hg} = -0.76\,\mathrm{m}$$

Attention: z positiv points down, z=0 on the surface. That’s why h_{max} is negative!

Hydrostatic Pressure: Pressure Gauge

- Use of the basic equation of hydrostatics



  - Calculation of pressure differences between p_1 and p_2. The pressure of the liquid columns are equal at the bottom:

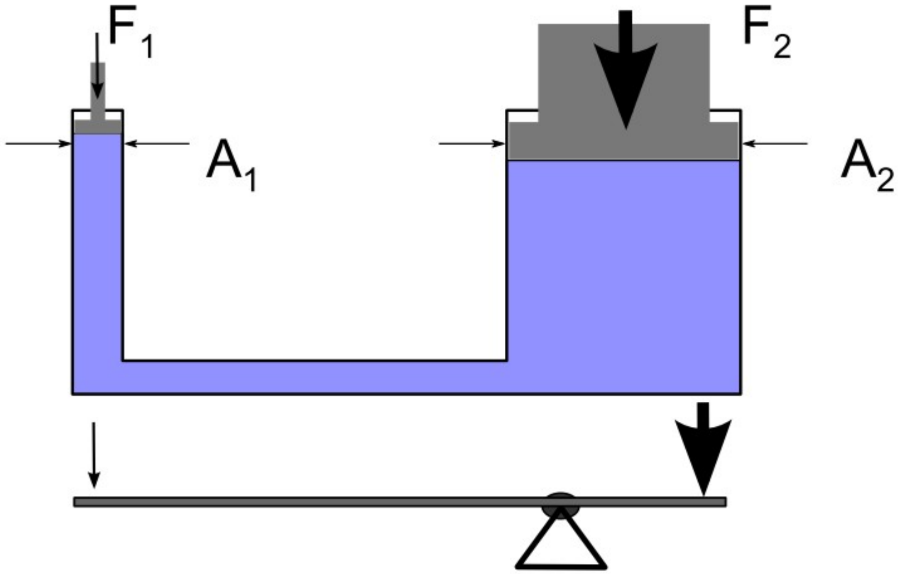
$$p_1 + \rho\, g\, z_1 = p_2 + \rho\, g\, z_2$$

$$\Delta p = p_1 - p_2 = \rho\, g \left(z_2 - z_1\right)$$

One pressure must be known to calculate the other one. In many applications it is sufficient to know Δp.

Hydrostatic Pressure: Hydraulic Press

- A hydraulic press transforms small forces into big ones



  - The pressure must be equal on both sides

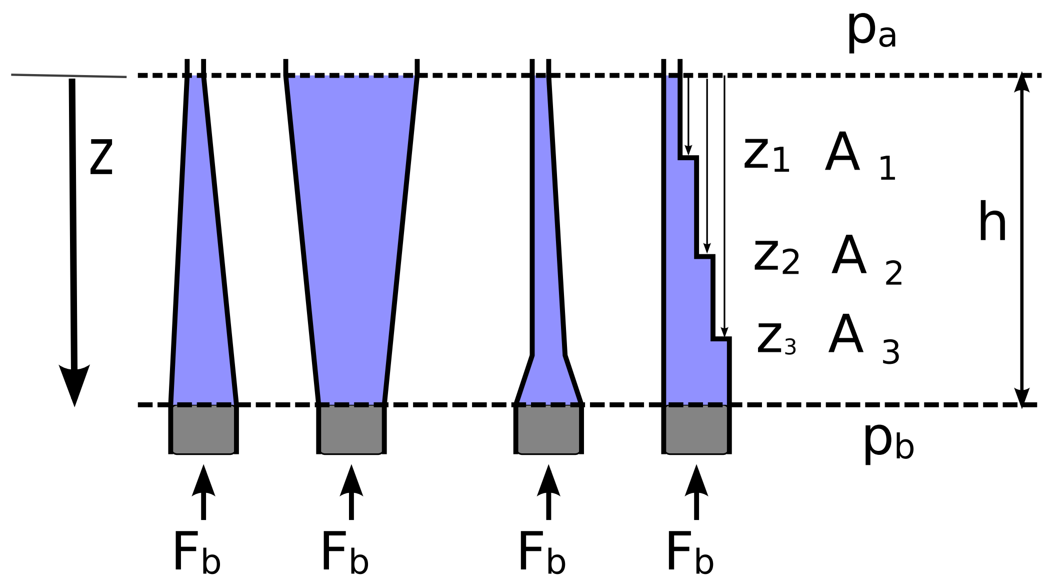
$$p_1 = p_2 \quad \Leftrightarrow \quad \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \Rightarrow \quad F_2 = F_1 \cdot \frac{A_2}{A_1}$$

  - The small force F_1 can be scaled up by the factor A_2/A_1, therefore heavy loads can be lifted

- Comparable to mechanical lever law

Hydrostatic Pressure: Hydrostatic Paradoxon

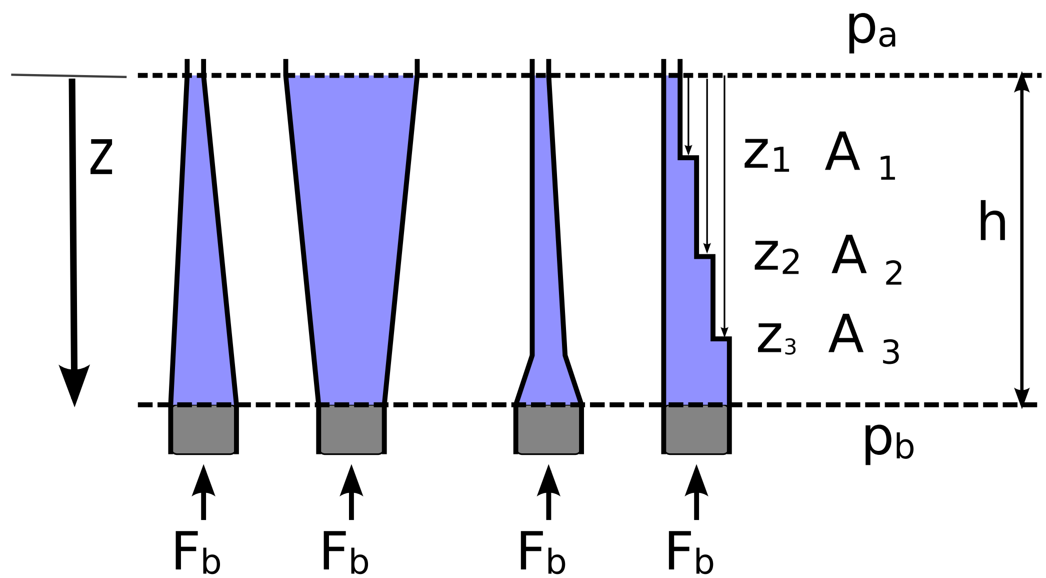
- Different reservoirs with the same ground area A_b, closed with a lid on the bottom
- Identical height of the fluid but different in volume



- How large is the closing force F_b for each reservoir?

Hydrostatic Pressure: Hydrostatic Paradoxon

- Different reservoirs with the same ground area A_b, closed with a lid on the bottom
  - Identical height of the fluid but different in volume



  - Paradoxon:
    The same closing force is needed for every column.

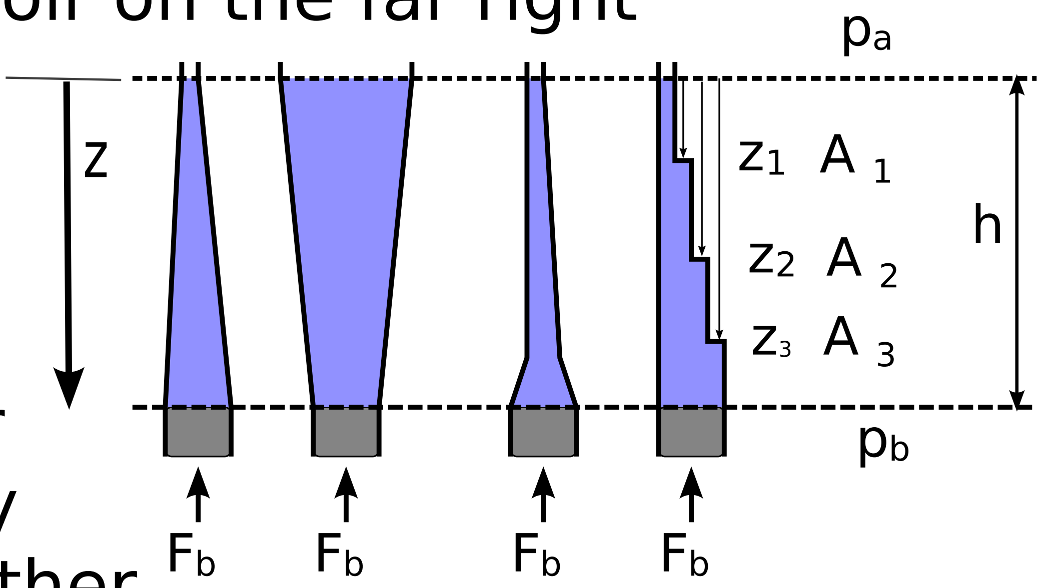
    ⇒ identical pressure p_b!

$$F_b = A_b \left(p_a + \rho g h \right) \quad \Rightarrow \quad p_b = p_a + \rho g h = \frac{F_b}{A_b}$$

  - **Only the height of the liquid column determines the hydrostatic pressure on the ground!**

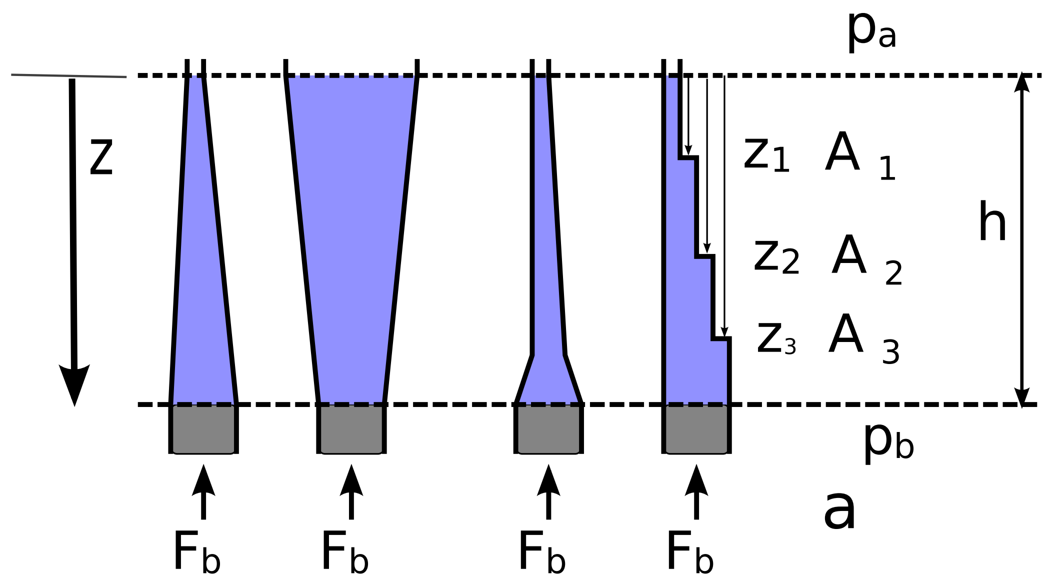
Hydrostatic Pressure: Hydrostatic Paradoxon

- Example for the reservoir on the far right
- Area A_1, A_2, A_3 in depth z_1, z_2, z_3
- The volume gets split into four perpendicular columns. One for every area A_1, A_2, A_3 and another one for the height h
- Every column acts with a certain force on the lid. The sum of these forces equals F_b, the force needed to close the column with the lid



Hydrostatic Pressure: Hydrostatic Paradoxon

(continued)



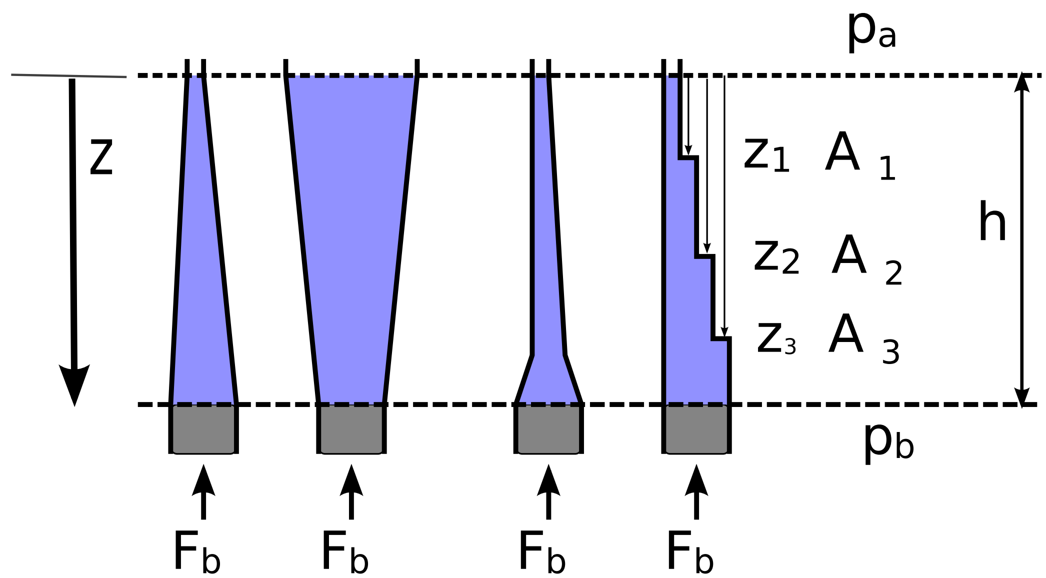
- Pressure at the depth z_1:
  $$p(z_1) = p_a + \rho\, g\, z_1$$
- This pressure acts at the upper end of the water column right under the area A_1. This causes force pointing downwards:
  $$F_1 = p(z_1)\, A_1$$
- On top of that acts the force caused by the weight of the water on the lid. With the volume $A_1(h\text{-}z_1)$. So:
  $$F_{G1} = \rho\, g\, A_1 (h - z_1)$$
- Contribution of the column under A_1:

$$F_1 + F_{G1} = p(z_1)\, A_1 + \rho\, g\, A_1 (h - z_1) = (p_a + \rho\, g\, z_1)\, A_1 + \rho\, g\, A_1 (h - z_1)$$

Hydrostatic Pressure: Hydrostatic Paradoxon

(continued)

- The same for the columns under the areas A_2 and A_3



- Contribution of the column under A_2:

$$F_2 + F_{G2} = p(z_2) A_2 + \rho\, g\, A_2 (h - z_2) = (p_a + \rho\, g\, z_2) A_2 + \rho\, g\, A_2 (h - z_2)$$

- Contribution of the column under A_3:

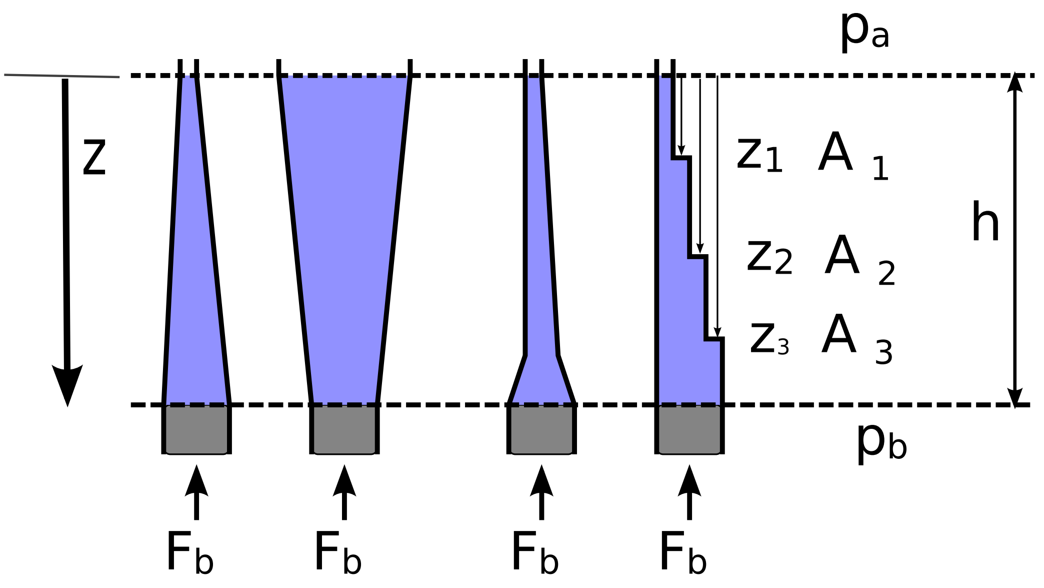
$$F_3 + F_{G3} = p(z_3) A_3 + \rho\, g\, A_3 (h - z_3) = (p_a + \rho\, g\, z_3) A_3 + \rho\, g\, A_3 (h - z_3)$$

Hydrostatic Pressure: Hydrostatic Paradoxon

(continued)

- Area of column 4: A_b - A_1 - A_2 - A_3
- Pressure at the depth h for this column

$$p(h) = p_a + \rho g h$$



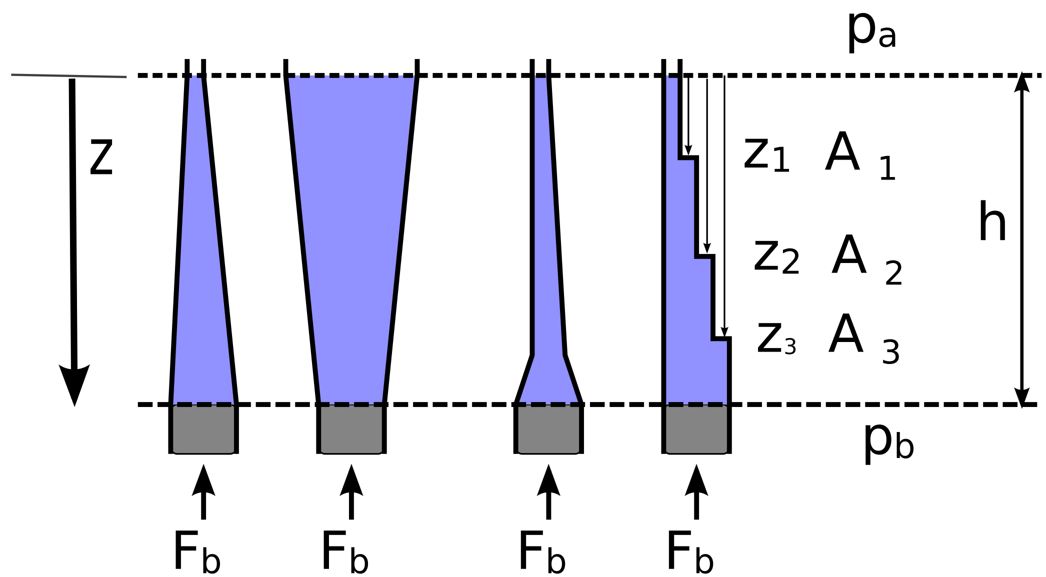
- Force on the lid of this column:

$$F_4 = (A_b - A_1 - A_2 - A_3)\, p(h) = (A_b - A_1 - A_2 - A_3)(p_a + \rho g h)$$

Hydrostatic Pressure: Hydrostatic Paradoxon

(continued)

- All contributions of the four columns summed up:

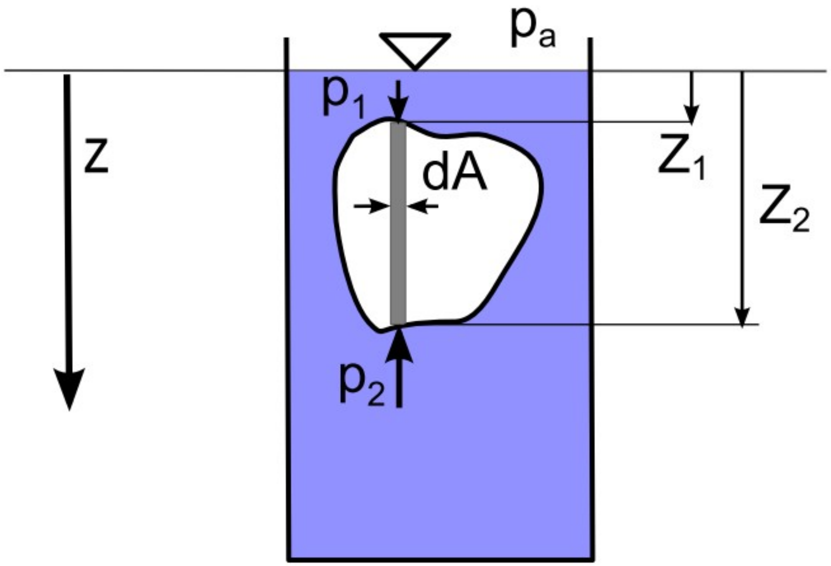


$$
\begin{aligned}
F_b &= F_1 + F_{G1} + F_2 + F_{G2} + F_3 + F_{G3} + F_4 \\
&= \left(p_a + \rho\, g\, z_1\right) A_1 + \rho\, g\, A_1 \left(h - z_1\right) \\
&\quad + \left(p_a + \rho\, g\, z_2\right) A_2 + \rho\, g\, A_2 \left(h - z_2\right) \\
&\quad + \left(p_a + \rho\, g\, z_3\right) A_3 + \rho\, g\, A_3 \left(h - z_3\right) \\
&\quad + \left(A_b - A_1 - A_2 - A_3\right)\left(p_a + \rho\, g\, h\right) \\
&= A_b \left(p_a + \rho\, g\, h\right) = A_b\, p_b
\end{aligned}
$$

Multiply and summarize brackets

Buoyancy (Auftrieb)

- Buoyancy is an upwards pointing force of a fluid that counteracts with the gravity force of the immersed object



- Consider a randomly shaped body that is completely immersed
- We split the body in a group of columns with the base area dA
- Hence, the buoyancy force can be defined by the basic equation of hydrostatics

Buoyancy (continued)

- The buoyancy affecting one column results from the difference in pressure between the upper and the bottom end:

$$\mathrm{d}\,F_B = p_2\,\mathrm{dA} - p_1\,\mathrm{dA} = \rho_f\, g\left(z_2 - z_1\right)\mathrm{dA}$$

- The total buoyancy force results from the integration over the whole area

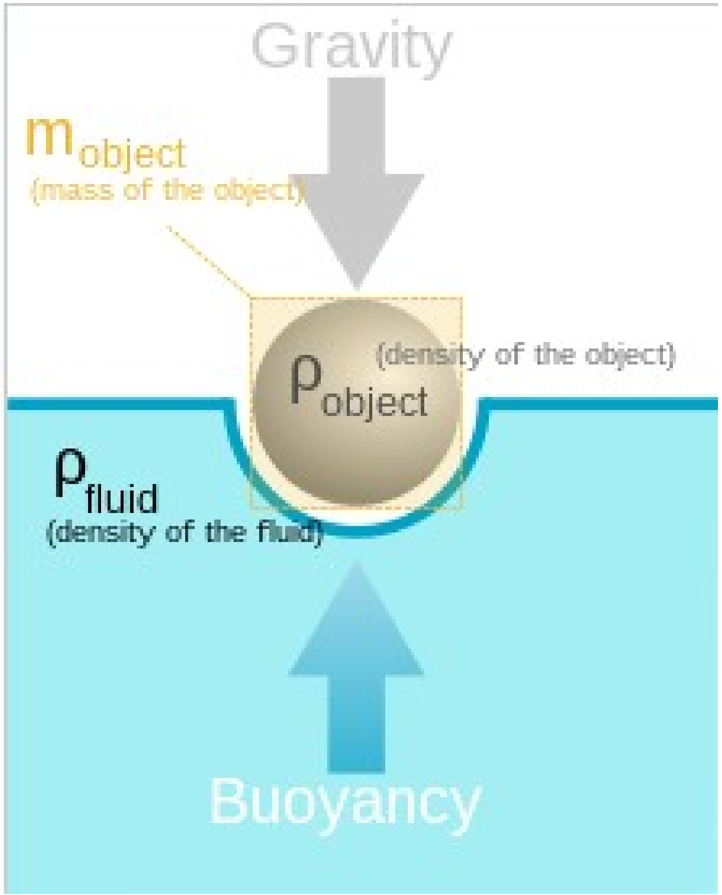
$$F_B = \int dF_B = \int_A \rho_f\, g\left(z_2 - z_1\right) dA = \rho_f\, g \int_A \left(z_2 - z_1\right) dA = \rho_f\, g\, V_{Body}$$

  – The last integral of the height z_2-z_1 over the area is the volume of the immersed part of the body and hence the displaced fluid

- The buoyancy force corresponds to the gravity force of the displaced fluid $$F_B = \rho_f\, g\, V_{Body}$$

Buyoncy example

- Calculate the buoyoncy force acting on a floting ball with a diameter of 10 cm.The density of water is 1000 kg/m³

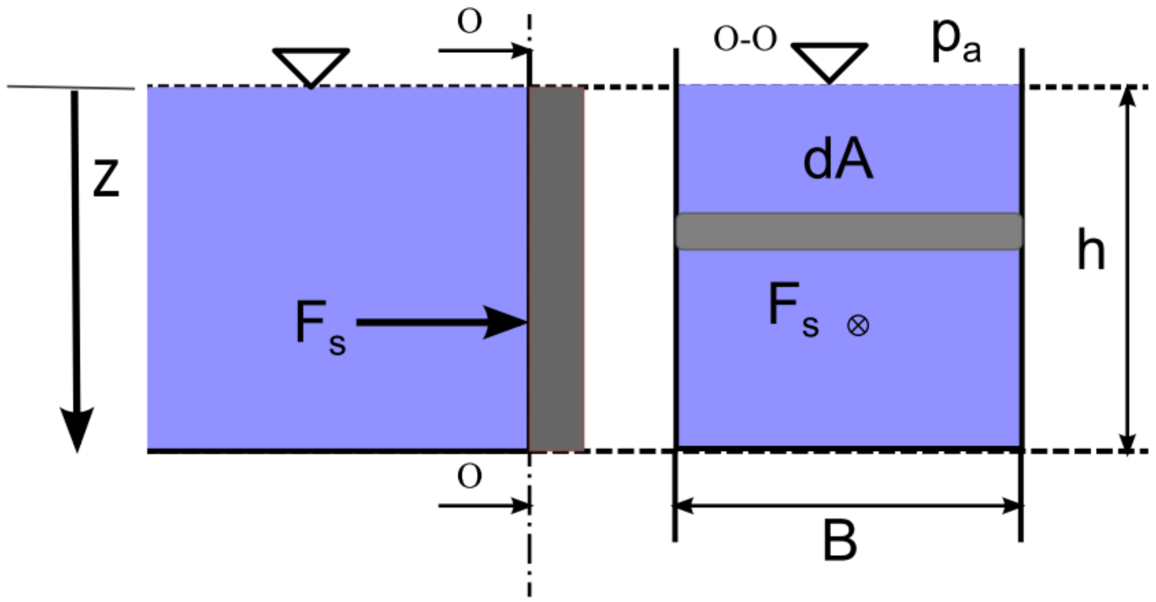


$$F_B = \rho_f \, g \, V_{Body}$$

https://en.wikipedia.org/wiki/Buoyancy

Hydrostatic Pressure on Walls

- Application: construction of water reservoirs
- Goal: calculation of the resulting force on a wall as well as the point of force application



  - Two views of a wall of the reservoir pictured opposite
  - Area A = B*h

.

  - For the calculation of the resulting force on the area with the width B, a small stripe of the area dA is considered

Hydrostatic Pressure on Walls

- The net force onto the area dA is caused by the pressure force of the water along one side of the wall and the air pressure along the other side:

$$\mathrm{d}\,F = p(z)\,\mathrm{d}\,A - p_a\,\mathrm{d}\,A = (p_a + \rho\,g\,z)\,\mathrm{d}\,A - p_a\,\mathrm{d}\,A = \rho\,g\,z\,\mathrm{d}\,A$$

- The area of one segment is dA = B*dz.
- The substitution force F_s is calculated by integration of the fluid covered area:

$$F_S = \int dF = \int_A \rho\,g\,z\,\mathrm{d}\,A = \int_0^h \rho\,g\,z\,B\,\mathrm{d}\,z = \rho\,g\,B \int_0^h z\,\mathrm{d}\,z = \rho\,g\,A\,\frac{h}{2}$$

Hydrostatic Pressure on Walls

- Often the point of force application is also of interest
- Consideration:
  - The point of force application results in a lever regarding a reference point
  - Substitution force times lever is the correct torque
- Torque for segment dA: $dM = z\,dF = z\left(\rho\, g\, z\,\mathrm{dA}\right) = z\left(\rho\, g\, z\right) B\,\mathrm{dz}$
- Resulting torque regarding z=0:

$$M_S = \int dM = \int_A z\,\mathrm{d}F = \int_A z\left(\rho\, g\, z\right) B\,\mathrm{dz} = \rho\, g\, B\,\frac{h^3}{3} = \rho\, g\, A\,\frac{h^2}{3}$$

- Determination of the force application point z_s for F_s

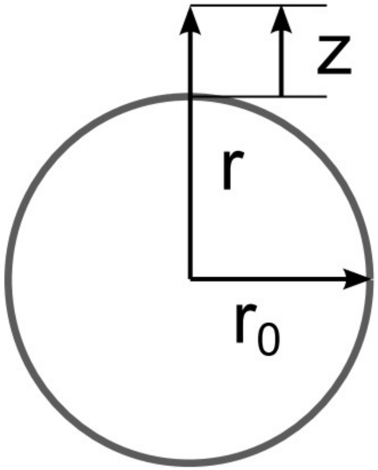
$$M_S = z_S F_S \quad \Rightarrow \quad z_S = \frac{M_S}{F_S} = \frac{\rho\, g\, A\,\frac{h^2}{3}}{\rho\, g\, A\,\frac{h}{2}} = \frac{2}{3} h$$

Aerostatics

- Aerostatics is the study of stationary gases. The state equation of ideal gases is of use here:

$$\frac{p}{\rho}=R_i T$$

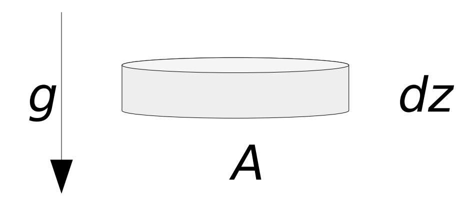
- Demarcation to hydrostatics: the density of gases depends on pressure and temperature
- Aerostatics is important for questions regarding the atmosphere
- We want to deduce a relation between the pressure *p* and a vertical coordinate *z*



Aerostatics

- The vertical coordinate points upwards, against the gravity force. In this direction the pressure will decrease

- Considering a stationary, flat cylinder with area *A* and height *dz*. The difference *dp* of the pressures on the top an the bottom needs to be compensated by the gravity force



$$dF_p = dF_G \quad \Rightarrow \quad A\,dp = -g\,\rho\,A\,dz \quad \Rightarrow \quad dp = -\rho\,g\,dz$$

- The density *ρ* depends on *p* and *T*, for example an ideal gas:

$$\rho = \frac{p}{R_i T}$$

Aerostatics

- With the equation *$\rho=\rho(p,T)$* we can transform it to:

$$dp = -\rho\, g\, dz$$
$$\Leftrightarrow dz = -\frac{1}{g}\frac{dp}{\rho\left(p,T\right)}$$

- Integrating both sides

$$z = \int_0^z dz = -\frac{1}{g}\int_{p_0}^{p\left(z\right)} \frac{dp}{\rho\left(p,T\right)}$$

- To integrate over pressure *p*, the function *$T(p)$* needs to be known, which is in general not the case
- Simplification: $T = T_0$ = constant
  - isothermal atmosphere

Aerostatics

- For an isothermal atmosphere: $$\rho = \frac{p}{R_i T_0}$$
- As a result

$$z = -\frac{1}{g} \int_{p_0}^{p(z)} \frac{dp}{\rho(p, T_0)} = -\frac{R_i T_0}{g} \int_{p_0}^{p(z)} \frac{dp}{p} = -\frac{R_i T_0}{g} \ln\left(\frac{p(z)}{p_0}\right)$$

- Solving for *p(z)*:

$$z = -\frac{R_i T_0}{g} \ln\left(\frac{p(z)}{p_0}\right)$$

$$e^{-\frac{z g}{R_i T_0}} = \frac{p(z)}{p_0}$$

$$\boxed{p(z) = p_0 \cdot e^{-\frac{z g}{R_i T_0}}}$$

Aerostatics

- Solving for p(z) results in:

$$p(z) = p_0 \mathrm{e}^{-\frac{zg}{R_i T_0}}$$

- And for the density:

$$\rho(z) = \rho_0 \mathrm{e}^{-\frac{zg}{R_i T_0}} \quad \text{with} \quad \rho_0 = \frac{p_0}{R_i T_0}$$

Note: The assumption of a constant temperature is reasonable for a height up to approx. 8000 m

Aerostatics

- For an isothermal atmosphere: $$\rho = \frac{p}{R_i T_0}$$
- As a result

$$z = -\frac{1}{g} \int_{p_0}^{p(z)} \frac{dp}{\rho(p, T_0)} = -\frac{R_i T_0}{g} \int_{p_0}^{p(z)} \frac{dp}{p} = -\frac{R_i T_0}{g} \ln\left(\frac{p(z)}{p_0}\right)$$

- Solving for p(z) results in: $$p(z) = p_0 \mathrm{e}^{-\frac{zg}{R_i T_0}}$$
- And for the density: $$\rho(z) = \rho_0 \mathrm{e}^{-\frac{zg}{R_i T_0}} \quad \text{mit} \quad \rho_0 = \frac{p_0}{R_i T_0}$$
- Note: The assumption of a constant temperature is reasonable for a height up to approx. 8000 m

Altitude of a mountain

The barometer on top of a mountain is showing 70 kPa. At sea level a pressure of 101,3 kPa is measured. t= 20°C, Ri=290 J/(kg*K), g=10 m/s^2

Task: Calculate the altitude of the mountain

Vladischern, https://de.dreamstime.com