Fluid Dynamics

Chapter 4

The Momentum Equation

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The Momentum Equation

- The momentum of a fluid can change due to external forces
- A typical engineering task is to estimate these forces and to design devices and systems to withstand these forces
- In order to deal with this tasks, we need a mathematical tool:

the „Reynolds Transport Theorem (RTT)“

- The following description of the RTT follows the treatment in the book ”Engineering Fluid Mechanics”, 10th Edition SI Version, by Donald F. Elger et al., Wiley, ISBN: 978-1-118-31875-1

The Reynolds Transport Theorem (RTT) Introduction

- What is a system?
  - a certain volume to be examined
  - separated from the environment by a system boundary
- Closed system
  - no mass transport through boundary
    => mass is constant
- Open system
  - mass transfer through boundary (in and out)
  - definition of a Control Volume CV
  - surface of CV is called Control Surface CS

The Reynolds Transport Theorem (RTT) Introduction

- Extensive property:
  - depends on system mass
  - e.g. mass, momentum, energy
- Intensive property:
  - independent of the system mass
  - e.g. density, pressure, temperature
  - often created by dividing two extensive properties

    Example; Extensive property B of a fluid with mass m gets transferred into an intensive property b according to:

$$b = \frac{B}{m}$$

The Reynolds Transport Theorem (RTT) Introduction

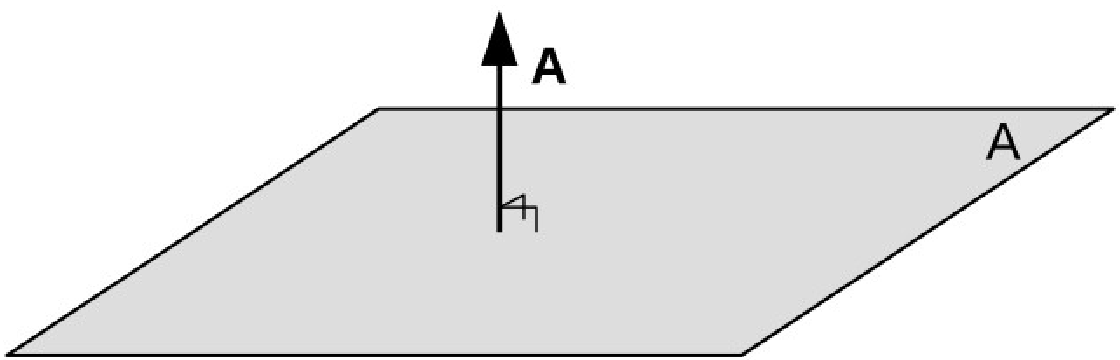
- A fluid mass m is distributed within a large control volume (CV). The fluid has the density ρ.
- The amount of B in the CV is derived by integrating b over the CV:

$$\boxed{B = \int_{CV} b\,\rho\,dV}$$

  - Consider a very small partial volume ΔV. Its mass is Δm = ρΔV. Then ΔB = bΔm = bρΔV. The integral is the sum over all small partial volumes.
  - Note: If the density ρ is not constant in the large CV, areas with a lower density will contribute less to the total value B according to the product b*ρ.

The Reynolds Transport Theorem (RTT) Introduction

- Required: An expression for the transport of a property B of a fluid through the area A.
- We define the surface vector **A** of a plane area A as follows:
  - Perpendicular to the surface A
  - Length of the vector **A** is the size of the area A



The Reynolds Transport Theorem (RTT)
Introduction

- The volume flow rate transported through area A is defined as the dot product (Skalarprodukt) of the surface vector A and the velocity v of the fluid flow:

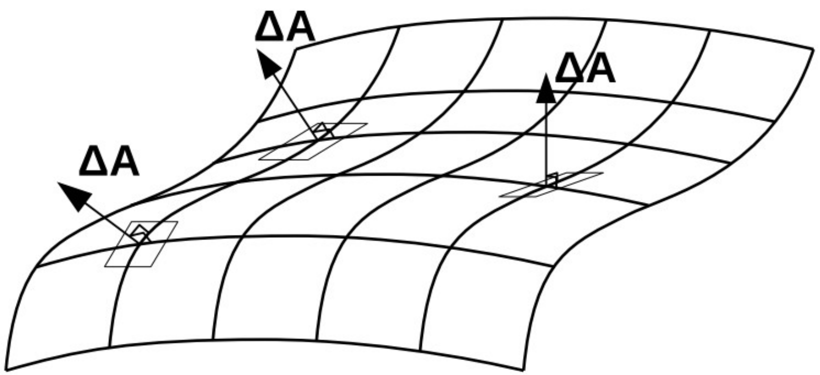
$$\dot{V} = \mathbf{v} \cdot \mathbf{A}$$

  - *Why is the dot product used here?*

Answer: The velocity component perpendicular to the surface is defining the volume flow rate. The tangential velocity component does not contribute to the flow through area A. The scalar product of the velocity **v** with the vertical surface **A** vector gives exactly the volume flow rate transported through the corresponding area

The Reynolds Transport Theorem (RTT) Introduction

- How to apply this to curved surfaces?
  - Split surface A into pieces ΔA
  - Definition of surface vectors that are locally perpendicular to the surface:



  - The following then applies to the volume flow through a very small area ΔA: $\Delta \dot{V} = \mathbf{v} \cdot \mathbf{\Delta A}$
  - For the entire area follows: $\dot{V} \approx \sum \Delta \dot{V} = \sum \mathbf{v} \cdot \mathbf{\Delta A}$
  - Attention: **v** is no longer necessarily constant!

The Reynolds Transport Theorem (RTT) Introduction

- In the limit ΔA → dA the sum becomes an integral

$$\dot{V} = \int_A \mathbf{v} \cdot \mathbf{d\,A}$$

- For the mass flow

$$\dot{m} = \int_A \rho\, \mathbf{v} \cdot \mathbf{d\,A}$$

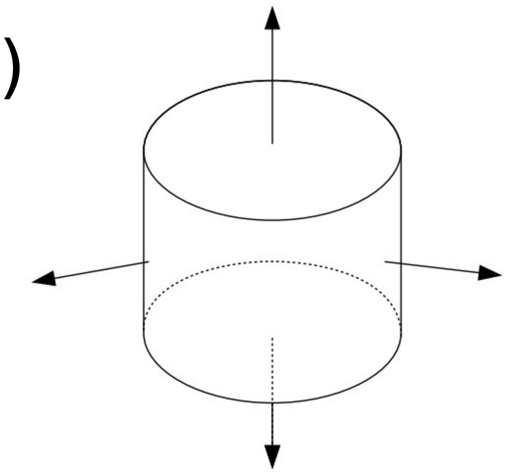
- And generally for an intense property b

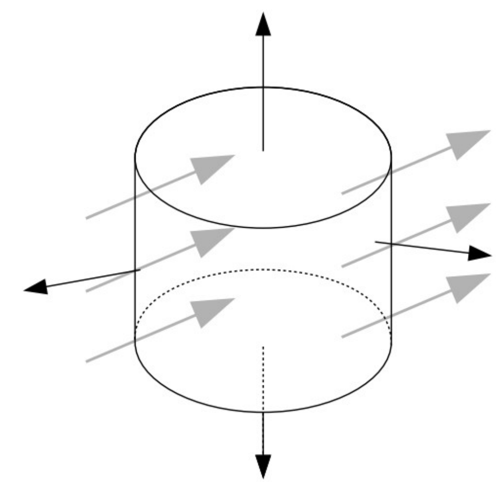
$$\boxed{\dot{B} = \int_A b\, \rho\, \mathbf{v} \cdot \mathbf{d\,A}}$$

- Note: **v** is the velocity <u>relative</u> to the surface A here. If $\mathbf{v}_F$ is the velocity of the fluid and $\mathbf{v}_A$ the velocity of a moving surface A, then $\mathbf{v} = \mathbf{v}_F - \mathbf{v}_A$.

The Reynolds Transport Theorem (RTT) Introduction

- What does that mean for the transport of B through the surface CS of a control volume CV?
- Black vectors: The surface vectors **A** of the CV point to the outside (convention)
- Grey vectors: velocity vectors **v**
  - Inlet: negative dot product **v·A**
  - Outlet: positive dot product **v·A**
- The integral over the entire upper area CS describes a net rate of B:

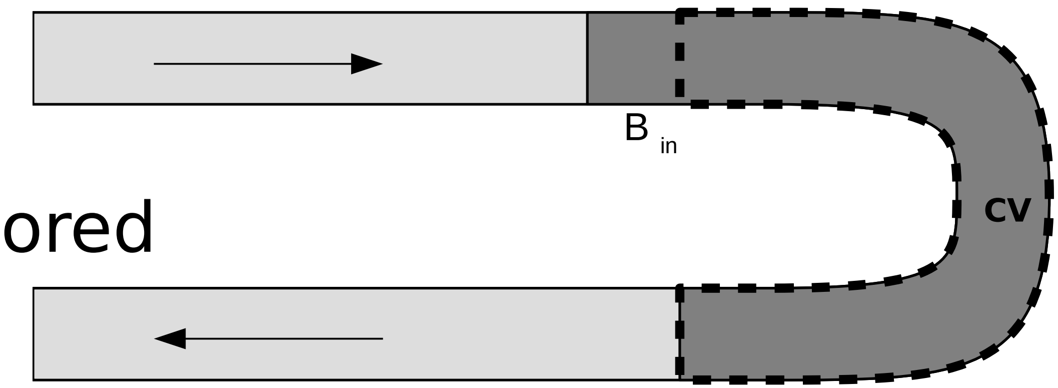




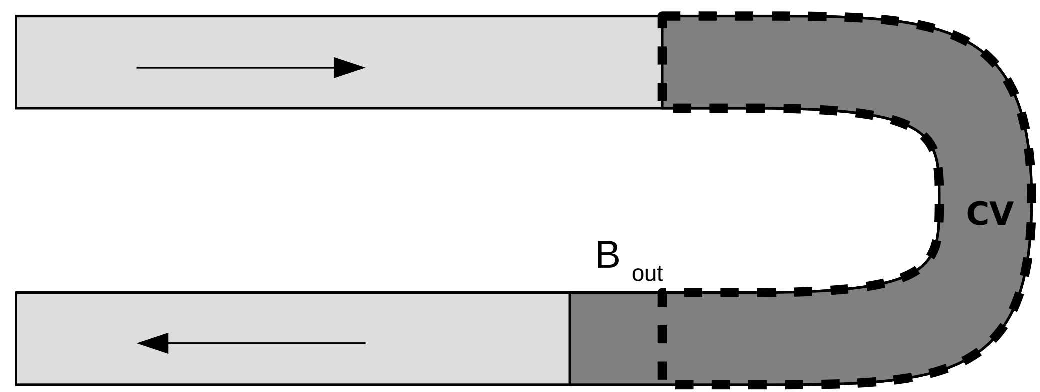
$$\boxed{\dot{B}_{net} = \int_{CS} b\,\rho\,\mathbf{v}\cdot\mathbf{d\,A}}$$

The Reynolds Transport Theorem (RTT)
The RTT – A Motivation

- Tool for formulating balance equations for flows using control volumes CV
- CS: dashed line
- Flow direction: arrows
- Closed system: dark colored part of the fluid
- An extensive property B is transported within the closed sytem



top: time t / below: time $t+\Delta t$



The Reynolds Transport Theorem (RTT)
The RTT – A Motivation

- Time t:
  - Some parts of the fluid within the closed system are not within the CV yet: Amount B_{in}
- Time t+Δt:
  - Some parts of the fluid in the closed system have already left the CV: Amount B_{out}
- The amount of B in the closed system B_{closed} can be described by the amount of B in CV B_{CV}:

$$B_{closed}(t) = B_{CV}(t) + B_{in}$$

$$B_{closed}(t+\Delta t) = B_{CV}(t+\Delta t) + B_{out}$$

The Reynolds Transport Theorem (RTT)
The RTT – A Motivation

- Rate of change of B in the closed system:

$$\frac{d\,B_{closed}}{dt} = \lim_{\Delta t \to 0}\left(\frac{B_{closed}(t+\Delta t) - B_{closed}(t)}{\Delta t}\right)$$

$$= \lim_{\Delta t \to 0}\left(\frac{B_{CV}(t+\Delta t) - B_{CV}(t) + B_{out} - B_{in}}{\Delta t}\right)$$

$$= \frac{d\,B_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in} = \frac{d\,B_{CV}}{dt} + \dot{B}_{net}$$

- With the integrals for B_{CV} and $\dot{B}_{net}$ we get:

$$\boxed{\frac{d\,B_{closed}}{dt} = \frac{d}{dt}\int_{CV} b\,\rho\,dV + \int_{CS} b\,\rho\,\mathbf{v}\cdot\mathbf{dA}}$$

as a general form of the
Reynolds Transport Theorem *(RTT)*

The Reynolds Transport Theorem (RTT)
The RTT – A Motivation

- The general form of the RTT is somewhat bulky
- Therefore simplifications are introduced:
  - Often there are only a few inlet and outlet ports
  - Consequence: The surface integral vanishes except for the ports, because only at the ports $\mathbf{v} \neq 0$

$$\boxed{\frac{d\,B_{closed}}{dt} = \frac{d}{dt}\int_{CV} b\,\rho\,dV + \sum_{inlets,outlets}\int_{port} b\,\rho\,\mathbf{v}\cdot\mathbf{dA}}$$

- If the inlets and outlets are only flat surfaces and ρ, v and b are constant on these surfaces, the integral becomes the product of these constants[1]:

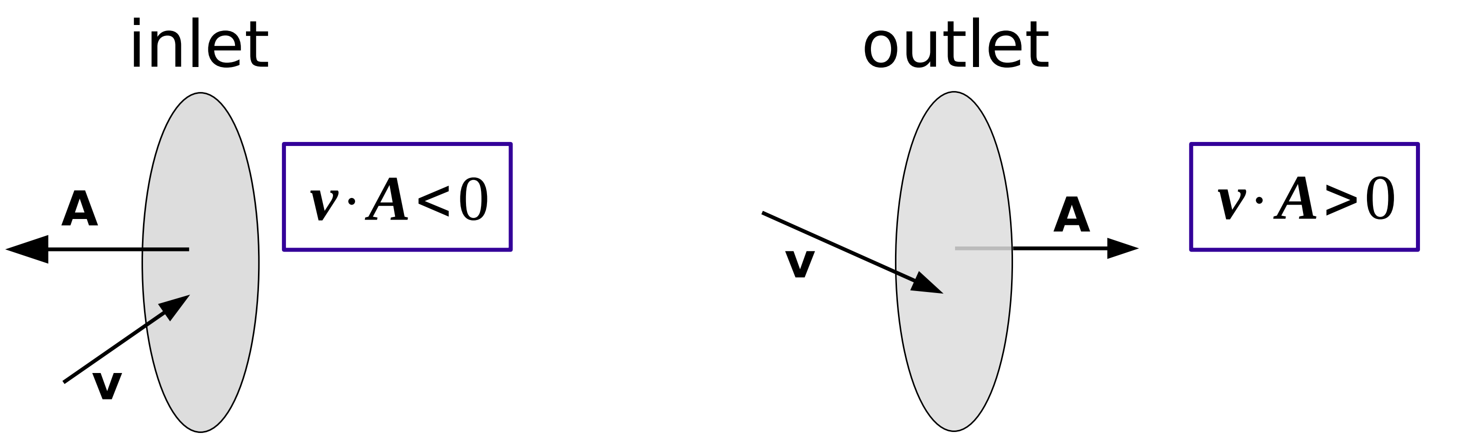
$$\boxed{\frac{d\,B_{closed}}{dt} = \frac{d}{dt}\int_{CV} b\,\rho\,dV + \sum_{inlets,outlets}\left(b\,\rho\,\mathbf{v}\cdot\mathbf{A}\right)_{port}}$$

[1] We remember: The surface vector is constant on flat surfaces

The Reynolds Transport Theorem (RTT)
The RTT – A Motivation

A final simplification. We remember:

- If **v** is perpendicular to surface A, the mass flow can be calculated as: $\dot{m} = \rho\, v\, A$
- If **v** is not perpendicular to surface A, the mass flow can be calculated based on the dot product of velocity and area vector: $\dot{m} = \rho\, \boldsymbol{v} \cdot \boldsymbol{A}$
- The sign of the mass flow depends on the dot product $\boldsymbol{v} \cdot \boldsymbol{A}$



The Reynolds Transport Theorem (RTT)
The RTT – A Motivation

A final simplification. (continued)

- Therefore, inlets with positive numerical values of $\dot{m}_{in}$ get a – sign: $\rho\,\boldsymbol{v}\cdot\boldsymbol{A}=-\dot{m}_{in}$
- ... and outlets with a positive numerical value of $\dot{m}_{out}$ have a + sign: $\rho\,\boldsymbol{v}\cdot\boldsymbol{A}=+\dot{m}_{out}$

- This allows the ports in the RTT to be simplified:

$$\boxed{\frac{d\,B_{closed}}{dt}=\frac{d}{dt}\int_{CV} b\,\rho\,dV+\sum_{outlets} b_{out}\,\dot{m}_{out}-\sum_{inlets} b_{in}\,\dot{m}_{in}}$$

- Important: The properties b_{out} and b_{in} must still have their own sign.

Use of the RTT: The Continuity Equation

- First application of the RTT, the continuity equation (b=1):

$$b=1 \;\Rightarrow\; B=\int_{CV} b\,\rho\,dV=\int_{CV} \rho\,dV=m_{CV} \qquad \text{... the mass in the CV}$$

$$\int_{CS} b\,\rho\,\mathbf{v}\cdot\mathbf{dA}=\int_{CS} \rho\,\mathbf{v}\cdot\mathbf{dA}=\dot{m}_{net}=+\sum_{outlets}\dot{m}_{out}-\sum_{inlets}\dot{m}_{in}$$

- In the last step, we wrote down the signs of the mass flows according to the rules previously explained

- Mass is constant within the closed systems (no nuclear reactions):

$$\frac{dm_{closed}}{dt}=0=\frac{d}{dt}\int_{CV}\rho\,dV+\int_{CS}\rho\,\mathbf{v}\cdot\mathbf{dA}$$

$$\Rightarrow\;\frac{d}{dt}m_{CV}+\sum_{outlets}\dot{m}_{out}-\sum_{inlets}\dot{m}_{in}=0$$

$$\Leftrightarrow\;\frac{d}{dt}m_{CV}=-\sum_{outlets}\dot{m}_{out}+\sum_{inlets}\dot{m}_{in}$$

Use of the RTT: The Continuity Equation

- The result is the continuity equation:

$$\boxed{\frac{d}{dt} m_{CV} = -\sum_{outlets} \dot{m}_{out} + \sum_{inlets} \dot{m}_{in}}$$

- The term $\frac{d}{dt} m_{CV}$ describes the change of mass in the control volume.
  - $\frac{d}{dt} m_{CV} > 0$ means the mass in the CV increases
  - $\frac{d}{dt} m_{CV} < 0$ means the mass in the CV decreases
- A mass flow $\dot{m}_{in}$ increases the mass in the CV (plus: positive input for $\frac{d}{dt} m_{CV}$)
- A mass flow $\dot{m}_{out}$ decreases the mass in the CV (minus: negative input for $\frac{d}{dt} m_{CV}$)

Use of the RTT: The Continuity Equation

- For steady state flow, the mass in the CV is constant ⇒ m_{CV} = const. ⇒ $\frac{d}{dt} m_{CV} = 0$

$$\frac{d}{dt} m_{CV} = 0 = -\sum_{outlets} \dot{m}_{out} + \sum_{inlets} \dot{m}_{in}$$

$$\Rightarrow \qquad \sum_{outlets} \dot{m}_{out} = \sum_{inlets} \dot{m}_{in}$$

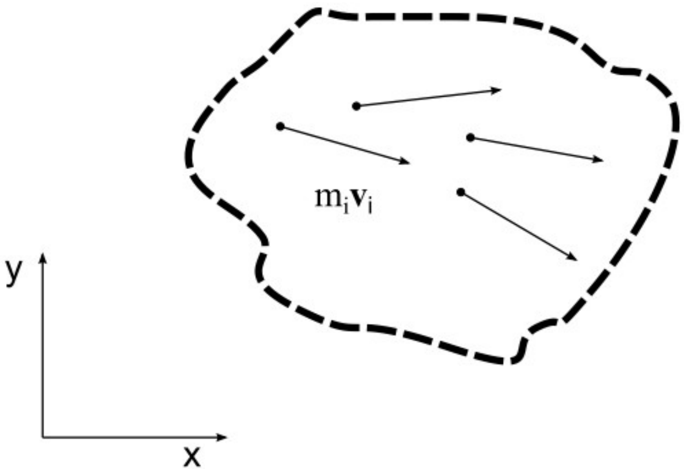
  – “Everything that flows in has to flow out“

Video – Reynolds Transport Theorem (RTT)

https://www.youtube.com/watch?v=3HMq1O0xI_4

Use of the RTT: The Momentum Equation

- Second usage of the RTT: **b=v** … but why?
- We break down a fluid into fluid elements i with mass Δm_i and velocity $\mathbf{v}_i$
- The total momentum of the system is:



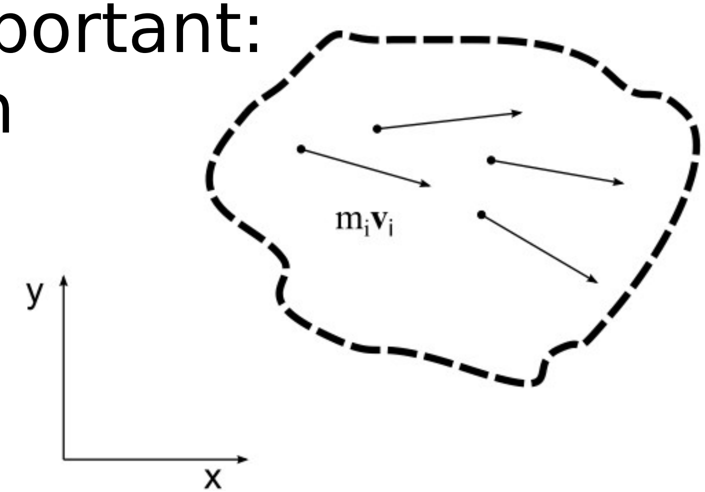
$$\boldsymbol{I} = \sum_i \Delta m_i \boldsymbol{v_i} = \sum_i \rho_i \Delta V_i \boldsymbol{v_i} = \sum_i \boldsymbol{v_i} \rho_i \Delta V_i$$

$$= \int \boldsymbol{v}\, \rho \, \mathrm{d} V \quad \text{in limes } \Delta V \to 0$$

- From **b=v** we get that **B** is the total momentum **I**
- Attention: **B** and **I** are vectors!
  - The velocity can be decomposed in x,y and z components: $\mathbf{v} = (v_x, v_y, v_z)$

Use of the RTT: The Momentum Equation

- The change of momentum is caused by forces.
- The fluid elements exert forces on one another. Newton's laws apply to each element.
- Esp. the 3rd Newtonian Law is important: “When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body”.



- Therefore, the internal forces that the fluid elements exerts on one another cancel out.
- As a consequence, the total momentum can only be changed by external forces!

Use of the RTT: The Momentum Equation

- The change of the momentum of the fluid in the closed system due to external forces is:

$$\sum(\mathbf{F})_{ext}=\frac{d\,\mathbf{I}_{closed}}{dt}$$

- The application of the RTT results in a general balance equation for the total momentum:

$$\boxed{\sum(\boldsymbol{F})_{ext}=\frac{d\,\boldsymbol{I}_{closed}}{dt}=\frac{d}{dt}\int_{CV}\boldsymbol{v}\,\rho\,dV+\int_{CS}(\rho\,\boldsymbol{v})\,\boldsymbol{v}\cdot\boldsymbol{dA}}$$

- Applying the same simplifications as for the continuity equation:

$$\boxed{\sum(\boldsymbol{F})_{ext}=\frac{d\,\boldsymbol{I}_{closed}}{dt}=\frac{d}{dt}\int_{CV}\boldsymbol{v}\,\rho\,dV+\sum_{outlets}\boldsymbol{v}_{out}\,\dot{m}_{out}-\sum_{inlets}\boldsymbol{v}_{in}\,\dot{m}_{in}}$$

Use of the RTT: The Momentum Equation

- For a steady state flow the total momentum in the CV does not change:

$$\frac{d}{dt}\int_{CV} \boldsymbol{v}\,\rho\,dV = 0$$

- Therefore, the simplified momentum equation for steady state flow is:

$$\boxed{\sum(\boldsymbol{F})_{ext} = \frac{d\,\boldsymbol{I}_{closed}}{dt} = \sum_{outlets} \boldsymbol{v}_{out}\,\dot{m}_{out} - \sum_{inlets} \boldsymbol{v}_{in}\,\dot{m}_{in}}$$

- Note: These are **vector** equations!
  That means that we have to deal with separate equations for the x-, y- and z- component in a Cartesian coordinate system

Use of the RTT: The Momentum Equation

What kind of external forces can affect the fluid?

- Gravity

$$\boldsymbol{F}_g = \int_{CV} \boldsymbol{g}\, \rho \, dV = m\, \boldsymbol{g}$$

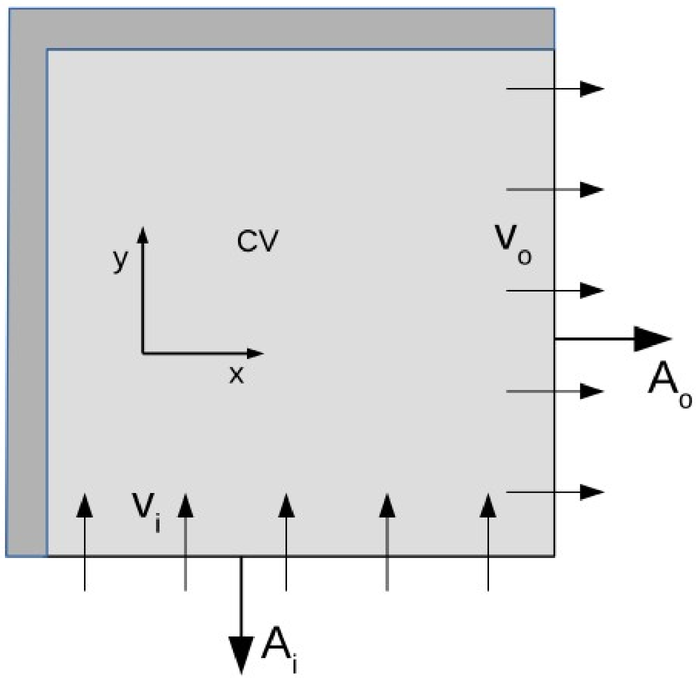
- Pressure forces on open boundaries of the CV
  - They are always directed against the outward surface vector. This is considered by the minus in the equation:

$$\boldsymbol{F}_p = -\int_{A} p\, \boldsymbol{dA} \quad \left(= -p\, \boldsymbol{A} \ \text{ for a flat surface and } \ p = const.\right)$$

- Pressure forces from walls $\mathbf{F}_w$. These forces are often the ones we want to calculate!
- Friction forces along surfaces are difficult to calculate! <u>We are going to ignore them here.</u>

The Momentum Equation: A First Example

- Steady state flow through a square CV with two walls
- Inlet:
  - Area vector $\mathbf{A}_i=(0,-A)$
  - Velocity $\mathbf{v}_i=(0,v)$
- Outlet:
  - Area vector $\mathbf{A}_o=(A,0)$
  - Velocity $\mathbf{v}_o=(v,0)$
- Mass flow $\dot{m}_o=\dot{m}_i=\rho\, v\, A$

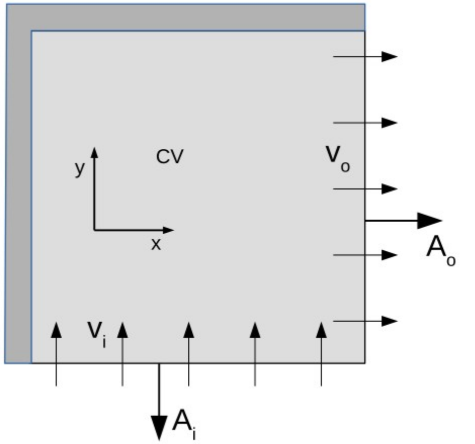


- To be calculated: Change in momentum of the fluid in y-direction:

$$F_{ext,y}=\sum_{outlets} v_{o,y}\dot{m}_0-\sum_{inlets} v_{i,y}\dot{m}_i=(0(\rho\, v\, A))-(v(\rho\, v\, A))=-\rho\, v^2 A$$

The Momentum Equation: A First Example

- The change in momentum of the fluid in y-direction is:
  $F_{ext,y} = -\rho v^2 A$
- What does this mean?
- The fluid flows at the inlet in positive y-direction. The momentum in y-direction entering the CV within the time interval Δt is:

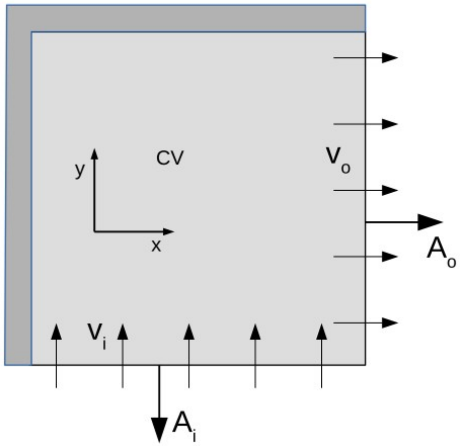


$$\Delta I_y = v\,\Delta m = v\,\rho\,\Delta V = v\,\rho\,\Delta y\,A \qquad \text{with } \Delta V = \Delta y\,A$$
$$= v\,\rho\,v\,\Delta t\,A = \rho v^2 A\,\Delta t \qquad \text{with } \Delta y = v\,\Delta t$$

- At the outlet, the flow is only in x-direction. Therefore, no momentum is leaving in y-direction. Because the flow is steady state, no momentum may accumulate in the CV.
- The momentum ΔI_y that flows into the system per unit of time must therefore be neutralized by opposing external forces. Forces cause changes in momentum:

$$F_{ext,y} = -\frac{\Delta I_y}{\Delta t} = -\rho\,v^2 A$$

The Momentum Equation: A First Example



- Important: The force that is calculated by momentum balance is the force **on the fluid!**
- Often you want to calculate the force **on the wall** (stability).
- According to Newton's 3rd law, the force on the wall is in opposite direction to the force on the fluid!
- Here: The force on the horizontal upper wall is

$$F_{wall,y} = \rho v^2 A$$

- More examples for employing the momentum equation:
  - Force on walls
  - Wind turbine
  - Jet engine

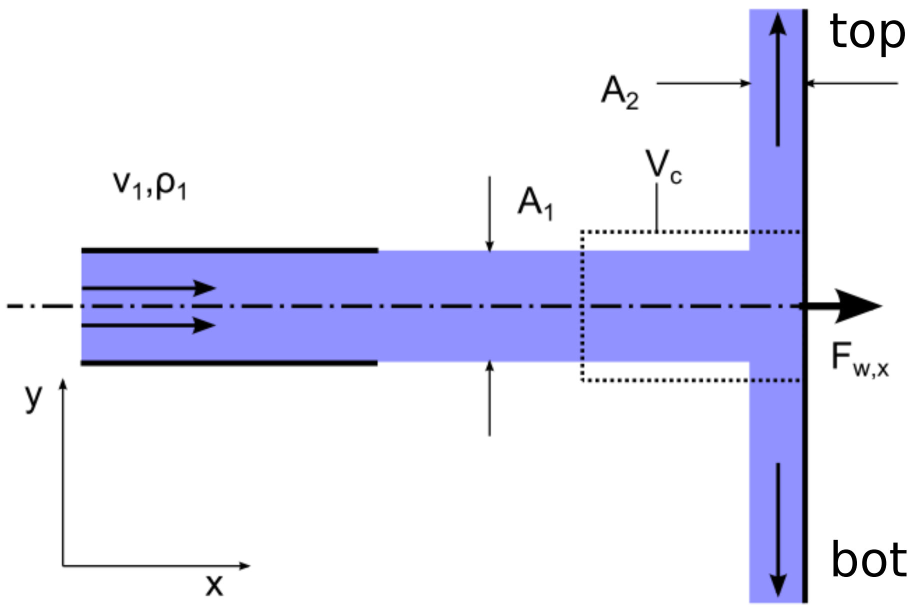
The Momentum Equation: Typical Procedure

The general approach for solving a task with the momentum equation is:

- Choose a control volume. Most important criterium: **You know** the properties of the flow (velocity, pressure,...) on the control surface (CS)!
  … or **you can calculate** these properties.
- Calculate the missing properties employing continuity and Bernoulli equation.
- Solve the momentum equation for the components you are interested in.
  - Use the components of the surface vectors and velocities correctly.

The Momentum Equation - Applications
Force of a Water Jet on a Flat Wall

- Steady state flow:
  - Water jet with velocity v_1 and density ρ_1 hits a wall and is deflected in two directions
  - Water is incompressible
  - Friction and gravity are ignored
  - Water jet in air: The pressure on the jet is ambient pressure everywhere. Therefore, the pressure balances itself out.



- <u>To be determined</u>: The force on the wall
- Continuity equation: $\dot{m} = \rho \dot{V} = \rho v A$

The Momentum Equation - Applications
Force of a Water Jet on a Flat Wall

- The general momentum equation:

$$\sum_{outlets} \rho \dot{V} \mathbf{v} - \sum_{inlets} \rho \dot{V} \mathbf{v} = \sum \mathbf{F}$$

- x-direction:

$$\left(\rho \dot{V} v_x\right)_{out,top} + \left(\rho \dot{V} \left(v_x\right)\right)_{out,bot} - \left(\rho \dot{V} v_x\right)_{\mathrm{in}} = F_{w,x} \qquad \Big|\ v_{x,out,top} = v_{x,out,bot} = 0$$

$$-\left(\rho \dot{V} v_x\right)_{\mathrm{in}} = F_{w,x}$$

$$-\rho A v_1^2 = F_{w,x}$$

Result: external force on the fluid

- y-direction:

$$\left(\rho \dot{V} v_y\right)_{out,top} + \left(\rho \dot{V} \left(-v_y\right)\right)_{out,bot} - \left(\rho \dot{V} v_y\right)_{\mathrm{in}} = F_{w,y}$$

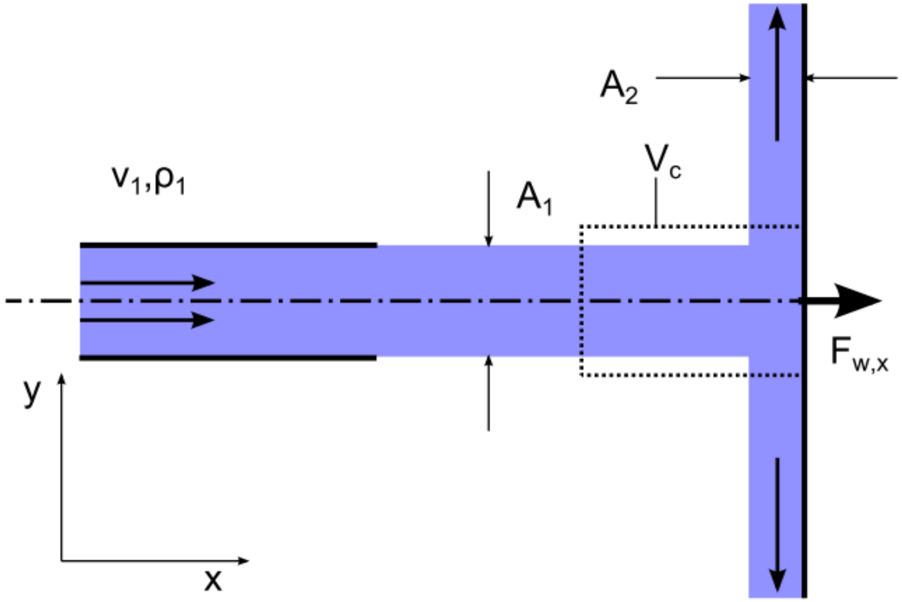
$$0 = F_{w,y}$$

Result: no external force on the fluid

  - At the inlet: $v_{y,in}=0$; at the outlets: identical mass flow in opposite direction. The outlet momentum balances each other out

The Momentum Equation - Applications
Force of a Water Jet on a Flat Wall

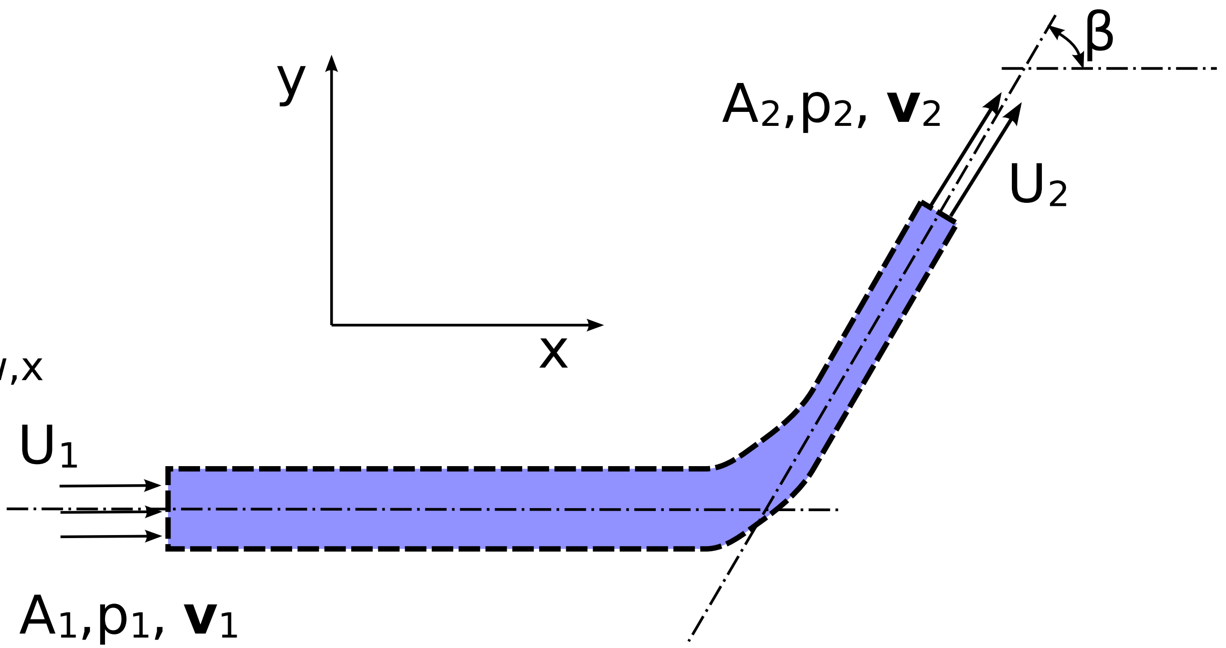
- We found the force in x-direction
  on the fluid: $F_{w,x} = -\rho v_1^2 A$
- To be determined:
  The force on the wall!



- Result: $F_{wall,x} = \rho v_1^2 A$

The Momentum Equation - Applications
Force Acting on the Fluid in a Pipe Elbow

- Steady state water flow in a pipe elbow
- Gravity and friction are ignored, water is assumed to be incompressible
- Given:
  U_1, A_1, p_1, A_2, β
- To be determined:
  Force on the fluid $F_{w,x}$
  (only in x-direction)



- Nomenclature:
  U is the velocity perpendicular to the corresponding area (scalar). $\mathbf{v}=(v_x,v_y)$ is the velocity vector.

The Momentum Equation - Applications
Force Acting on the Fluid in a Pipe Elbow

- Change in cross section results in change of velocity ⇒ Calculate the velocity at the outlet using the continuity equation:

$$\dot{V}_1 = \dot{V}_2 \;\Leftrightarrow\; U_1 \cdot A_1 = U_2 \cdot A_2 \;\Rightarrow\; U_2 = \frac{A_1}{A_2} \cdot U_1$$

- Relationship between outlet and inlet pressure due to energy conversation. Calculation of outlet pressure using Bernoulli equation

$$p_1 + \frac{\rho}{2} U_1^2 = p_2 + \frac{\rho}{2} U_2^2$$

$$p_2 = p_1 + \frac{\rho}{2}\left(U_1^2 - U_2^2\right)$$

Continuity equation for U_2

$$p_2 = \frac{\rho}{2}\left(1 - \frac{A_1^2}{A_2^2}\right) U_1^2 + p_1$$

The Momentum Equation - Applications
Force Acting on the Fluid in a Pipe Elbow

- General momentum equation

$$\sum_{outlets} \rho \dot{V} \mathbf{v} - \sum_{inlets} \rho \dot{V} \mathbf{v} = \sum \mathbf{F}$$

- x-component for force in x-direction

$$\rho \dot{V}_{out} v_{2,x} - \rho \dot{V}_{\mathrm{in}} v_{1,x} = \sum F_x$$

- The pressure is different at inlet and outlet port and needs to be determined

$$\boldsymbol{A}_1 = \begin{pmatrix} -A_1 \\ 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} A_2 \cos\beta \\ A_2 \sin\beta \end{pmatrix}$$

(Area vectors are facing outwards!
At the inlet to the left, therefore: -)

$$F_{p,1,x} = -p_1 A_{1,x} = -p_1 \left(-A_1\right) = p_1 A_1$$
$$F_{p,2,x} = -p_2 A_{2,x} = -p_2 A_2 \cos\beta$$

The Momentum Equation - Applications
Force Acting on the Fluid in a Pipe Elbow

- Momentum equation for mass flow

$$\sum_{outlets} \rho \dot{V} \mathbf{v} - \sum_{inlets} \rho \dot{V} \mathbf{v} = \sum \mathbf{F}$$

- x-component for force in x-direction

$$\rho \dot{V}_{out} v_{2,x} - \rho \dot{V}_{\text{in}} v_{1,x} = \sum F_x \qquad \Big| \quad v_{1,x} = U_1 \;\; , \;\; v_{2,x} = U_2 \cos\beta$$

$$\Rightarrow \;\; \rho \dot{V}_{out} U_2 \cos(\beta) - \rho \dot{V}_{\text{in}} U_1 = \sum F_x \qquad \Big| \quad \dot{V}_{in} = U_1 A_1 \;\; , \;\; \dot{V}_{out} = U_2 A_2$$

$$\Rightarrow \;\; \rho A_2 U_2^2 \cos(\beta) - \rho A_1 U_1^2 = \sum F_x \qquad \Big| \quad \sum F_x = F_{w,x} + F_{p,1,x} + F_{p,2,x}$$

$$\Rightarrow \;\; \rho A_2 U_2^2 \cos(\beta) - \rho A_1 U_1^2 = F_{w,x} + p_1 A_1 - p_2 A_2 \cos(\beta)$$

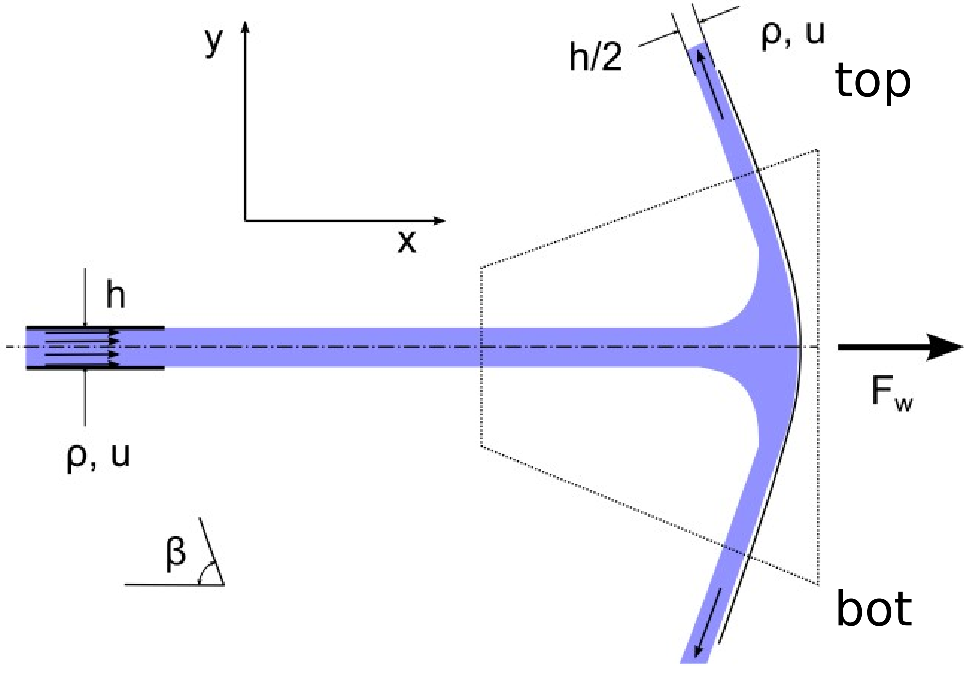
$$\Rightarrow \;\; \underline{F_{w,x} = A_2 \cos(\beta)\left(p_2 + \rho U_2^2\right) - \left(p_1 + \rho U_1^2\right) A_1}$$

(Result)

- The equations derived before for U_2 and p_2 need to be inserted into this final equation

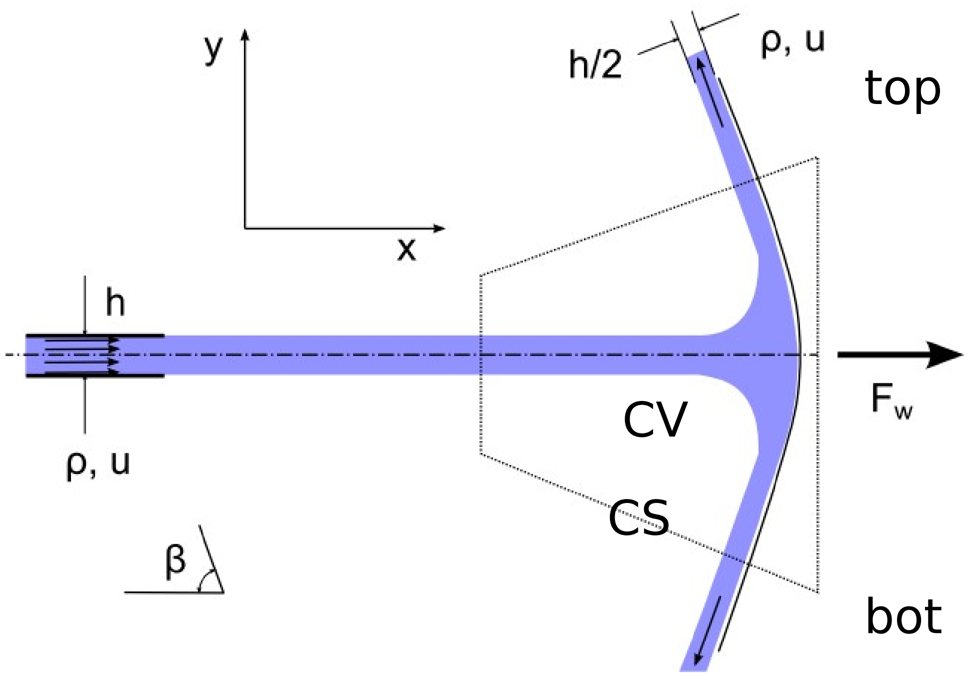
Momentum Equation - Applications
Force on a bent Wall

- Wall is not moving, steady state flow
- Rectangular jet with area A_{in}=b*h with velocity u in x-direction and fluid density ρ.
- Jet deflected in top and bottom direction according to the angle ß
- Outlet area: A_{out}=bh/2
- <u>To be determined:</u> Force F_{wall} acting from the fluid onto the wall.



Momentum Equation - Applications
Force on a bent Wall

- Assumptions:
  - The fluid is incompressible
  - The assembly is horizontally orientated, gravitation can be ignored
  - Friction is neglected
  - The jet is in the air: constant pressure in the system
- Choose CV so that jet is perpendicular to the CS



Momentum Equation - Applications
Force on a bent Wall

- Velocities:

$$A_{out}=A_{out,top}+A_{out,bot}=b\frac{h}{2}+b\frac{h}{2}=hb=A_{in}$$

for that: $u_{in}=u_{out}=u$.

- Velocity vectors:

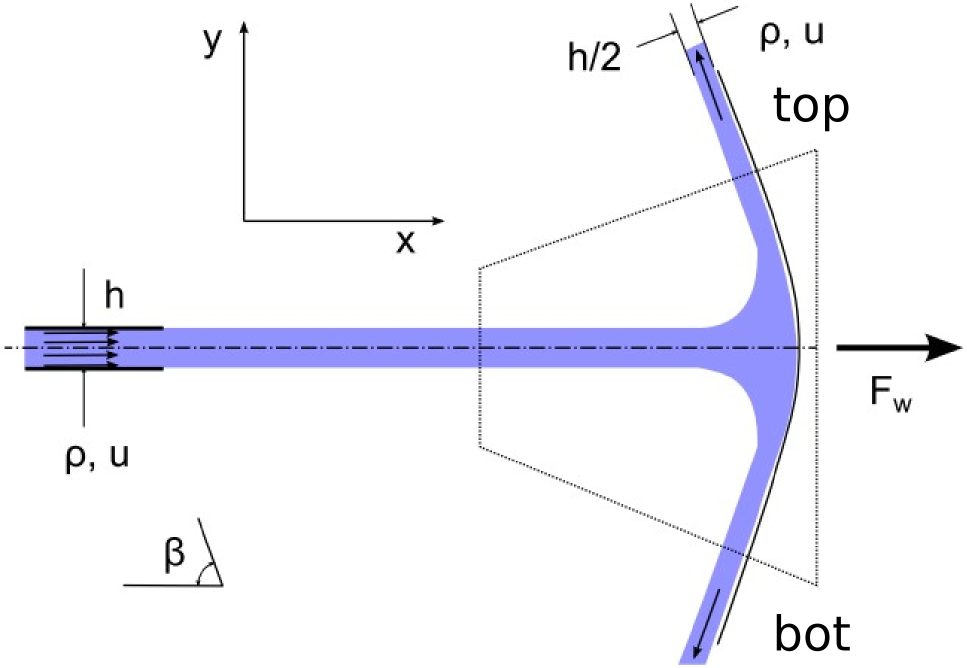
$$\boldsymbol{v}_{in}=\begin{pmatrix}u\\0\end{pmatrix}\quad \boldsymbol{v}_{out}=\begin{pmatrix}-u\cos\beta\\ u\sin\beta,-u\sin\beta\end{pmatrix}$$

- Volume flows:

$$\dot{V}_{in}=u\,A_{in}=u\,h\,b$$

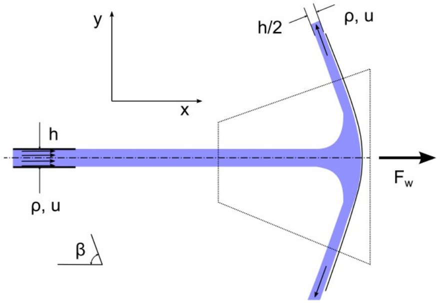
$$\dot{V}_{out,top}=u\,A_{out,top}=u\frac{h}{2}b$$

$$\dot{V}_{out,bot}=u\,A_{out,bot}=u\frac{h}{2}b$$



Momentum Equation - Applications
Force on a bent Wall

- General momentum equation



$$\sum_{outlets} \rho \dot{V} \mathbf{v} - \sum_{inlets} \rho \dot{V} \mathbf{v} = \sum \mathbf{F}$$

- x-direction:

$$\sum_{top,bot} \left(\rho \dot{V} v_x\right)_{out} - \left(\rho \dot{V} v_x\right)_{in} = F_{w,x}$$

$$\left(\rho \dot{V} v_x\right)_{out,top} + \left(\rho \dot{V} v_x\right)_{out,bot} - \left(\rho \dot{V} v_x\right)_{in} = F_{w,x}$$

$$\rho \left(u \frac{h}{2} b\right)\left(-u \cos\beta\right) + \rho \left(u \frac{h}{2} b\right)\left(-u \cos\beta\right) - \rho \left(u h b\right) u = F_{w,x}$$

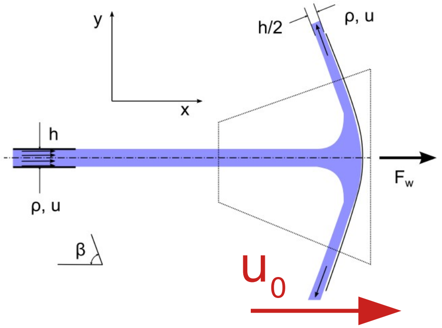
$$\left(-2\rho \frac{h}{2} b u^2 \cos(\beta)\right) - \left(\rho h b u^2\right) = F_{f,x}$$

$$\Rightarrow F_{w,x} = -\rho h b u^2 \left(1 + \cos(\beta)\right)$$ Force on the fluid

$$\Rightarrow \underline{F_{wall,x} = \rho h b u^2 \left(1 + \cos(\beta)\right)}$$ Force on the wall

Momentum Equation - Applications
Force at a bent and moving Wall

- Now:
  Moving wall with velocity u_0 in x-direction



- Choose the control volume to move with with the velocity u_0, too.
- Inlet velocity into CV: $u_{in} = u - u_0$.
- Outlet velocity from continuity equation:

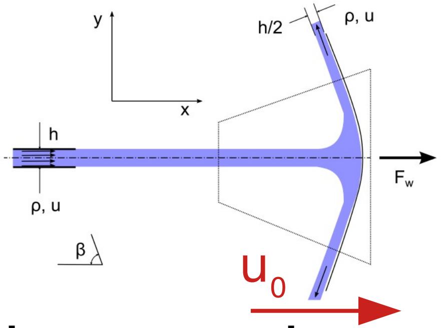
$$u_{in} A_{in} = (u_{out} A_{out})_{bot} + (u_{out} A_{out})_{top}$$
$$\Rightarrow (u - u_0) A_{in} = u_{out} (A_{(out,bot)} + A_{(out,top)}) = u_{out} A_{in}$$
$$\Rightarrow u_{out} = u - u_0 = u_{in}$$

- Result: Compared to the non-moving wall, we just replaced u by $(u\text{-}u_0)$: $F_{wall,x} = \rho h b (u - u_0)^2 (1 + \cos(\beta))$

Momentum Equation - Applications
Force at a bent and Moving Wall

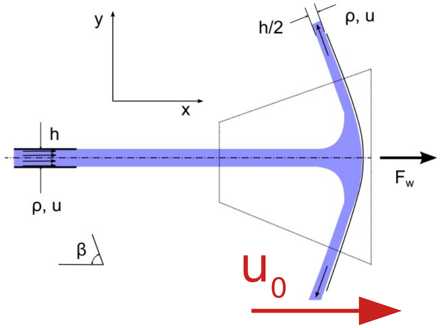
- Mechanical power of the moving wall:

$$P = F_{wall,x} \cdot u_0 = u_0 \rho h b (u - u_0)^2 (1 + \cos\beta)$$
$$= \rho h b (u^2 u_0 - 2 u u_0^2 + u_0^3)(1 + \cos(\beta))$$



- If $u_0 = 0$ the wall is not moving and does not generate mechanical power
- If $u_0 = u$ the power is also 0!
  - The wall moves with the same velocity as the jet. Therefore, the jet cannot push the wall.
- For which velocity u_0 of the wall does the power reach its maximum? In other words: What is the best possible u_0 for power generation?
- Calculating local maximum: differentiation of P wrt. u_0.

Momentum Equation - Applications
Force at a bent and Moving Wall



- Calculation of local maximum:
  - Derivative of P for u_0:

$$\frac{dP}{du_0}=\rho h b\left(1+\cos\left(\beta\right)\right)\left(u^2-4uu_0+3u_0^2\right)$$

  - Derivative = 0:

$$\frac{dP}{du_0}=0$$

$$\rho h b\left(1+\cos\left(\beta\right)\right)\left(u^2-4uu_0+3u_0^2\right)=0$$

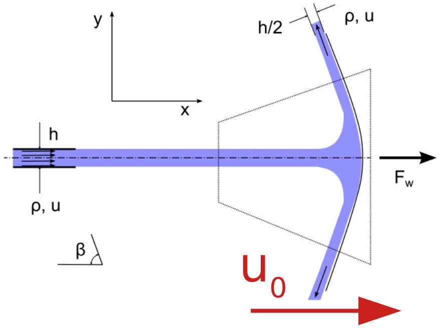
$$u^2-4uu_0+3u_0^2=0$$

$$\Rightarrow u_{0,1/2}=\frac{2}{3}u\pm\frac{1}{3}u$$ Solve for u_0

$$\Rightarrow u_{0,1}=u \;\Rightarrow\; P_1=0$$ 1. Solution not of interest

$$\Rightarrow u_{0,2}=\frac{u}{3} \;\Rightarrow\; P_2=\frac{4}{27}u^3\rho h b\left(1+\cos\left(\beta\right)\right)$$ Is this the max.?

Momentum Equation - Applications
Force at a bent and Moving Wall



- Calculation local maximum (cont.):
  - 2nd derivative of P wrt. u_0:

$$\frac{d^2 P}{du_0^{\;2}} = \rho h b (1+\cos(\beta))(-4u+6u_0)$$

  - Sign of the 2nd derivative at $u_0 = u_{0,2} = u/3$:

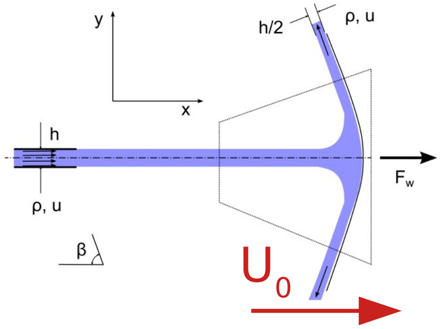
$$\frac{d^2 P}{du_0^{\;2}}(u_{0,2}) = \rho h b (1+\cos(\beta))(-4u+6u_{0,2})$$

$$= \rho h b (1+\cos(\beta))(-4u+6\frac{u}{3})$$

$$= \rho h b (1+\cos(\beta))(-2u) \quad < \; 0 \quad \Rightarrow \textbf{Max.}$$

  - Maximal power: $$P_{max} = \frac{4}{27} u^3 \rho h b (1+\cos(\beta))$$

Momentum Equation - Applications
Force at a bent and Moving Wall



- Finally, the solution is:
  - Power $$P=\rho\, h\, b\left(u-u_0\right)^2 u_0\left(1+\cos\beta\right)$$
  - Maximal power
$$P_{max}=\frac{4}{27}u^3\rho\, h\, b\left(1+\cos\left(\beta\right)\right)$$

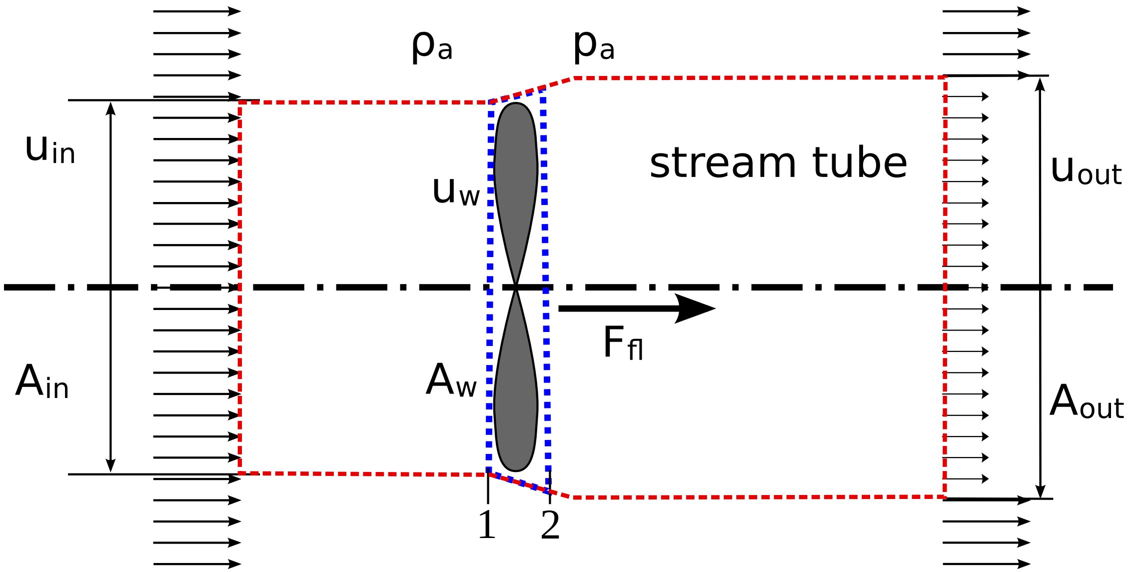
- Application of this idea is a water turbine for power generation.
- We assumed the movement to be linear. If the jet is hitting a wheel tangentially, this is a good assumption.
- For maximal power, the turning resistance of the wheel needs to be so large, that the circumferential velocity is ⅓ of the velocity of the jet.
- If the blades of the wheel are bend even stronger (smaller β), the jet is deflected more and the generated power will increase.

Momentum Equation - Applications
Wind turbine – Theory of Rankine / Froude

- How much power can be generated by a wind turbine?
- For the following consideration of Rankine and Froude, we need to assume some simplifications :
  - incompressible fluid
  - the rotation of the turbine has no influence on the axial air velocity
  - The forces are not influenced by the detailed geometry of the turbine (e.g. number of blades)
  - Changes in velocity occur without dissipation
    - A sudden change in velocity would for example result in a sudden change of kinetic energy (Bernoulli eq.)

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- Two control volumes are assumed
  - In red: far field control volume
  - In blue: near field control volume



- The diameter of the near field is the diameter of the rotor
- The far field is defined by the stream pipe of the air transported through the turbine. On the left side the pipe has the same diameter as the turbine, on the right side the diameter is larger, to give the same volume flow for the slower air decelerated by the turbine.

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- Axial flow in x-direction
  ⇒ Momentum equation in x-direction (far field CV)
- Checking the conditions:
  - The fluid is assumed to be incompressible
  - Friction and gravity are neglected
  - The stream pipe is an open system, the pressure equals ambient pressure

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- We expect the flow to be decelerated. We are interested in the force F_{fl} that is responsible for the deceleration

- Momentum equation in x-direction in the far field CV:

$$\sum \boldsymbol{F} = \sum_{outlets} \dot{m}\boldsymbol{v} - \sum_{inlets} \dot{m}\boldsymbol{v}$$

$$F_{fl} = \sum_{outlets} \dot{m} v_x - \sum_{inlets} \dot{m} v_x$$

$$\Rightarrow \quad F_{fl} = \dot{m}\left(u_{out} - u_{in}\right)$$

- The mass flow is constant. Continuity equation for all three areas:

$$\dot{m} = \rho A_{in} u_{in} = \rho A_w u_w = \rho A_{out} u_{out}$$

$$\Rightarrow \quad F_{fl} = \rho A_w u_w \left(u_{out} - u_{\mathrm{in}}\right) \qquad \text{with } \dot{m} = \rho A_w u_w$$

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- The deceleration of the fluid can be translated into a difference in pressure.
- The resulting force slows down the flow in the near field CV.
- The difference in pressure can be calculate by employing two Bernoulli equations:
  - Inlet → 1 $$p_a + \rho \frac{u_{\mathrm{in}}^2}{2} = p_1 + \rho \frac{u_w^2}{2}$$
  - Outlet → 2 $$p_a + \rho \frac{u_{out}^2}{2} = p_2 + \rho \frac{u_w^2}{2}$$
  - 2 - 1: $$p_2 - p_1 = \frac{\rho}{2}\left(u_{out}^2 - u_{\mathrm{in}}^2\right)$$

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- The force induced by the difference in pressure between 1 and 2 is calculated by multiplying the pressure difference with the area A_w:

$$F_{fl,p} = A_w (p_2 - p_1) = A_w \frac{\rho}{2} (u_{out}^2 - u_{\mathrm{in}}^2)$$

- The pressure force is the force that decelerates the flow in the far field CV. Equalizing F_{fl} and $F_{fl,p}$:

$$\rho\, A_w u_w (u_{out} - u_{\mathrm{in}}) = A_w \frac{\rho}{2} (u_{out}^2 - u_{\mathrm{in}}^2)$$

$$\Rightarrow \quad u_w (u_{out} - u_{\mathrm{in}}) = \frac{1}{2} (u_{out}^2 - u_{\mathrm{in}}^2) = \frac{1}{2} (u_{out} - u_{\mathrm{in}})(u_{out} + u_{\mathrm{in}})$$

- So the velocity at the turbine according to Froude and Rankine is:

$$u_w = \frac{1}{2} (u_{out} + u_{\mathrm{in}})$$

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- We estimated the force on the fluid and the flow velocity. Now we can calculate the turbine power
  - Force on the turbine = minus the force on the fluid!

$$
\begin{aligned}
P &= -F_{fl} u_w \\
&= -A_w \frac{\rho}{2}\left(u_{out}^2 - u_{in}^2\right)\frac{1}{2}\left(u_{in} + u_{out}\right) \\
&= A_w \frac{\rho u_{in}^3}{4}\left(1 + \frac{u_{out}}{u_{in}}\right)\left(1 - \left(\frac{u_{out}}{u_{in}}\right)^2\right)
\end{aligned}
$$

Transfer the - sign into the first bracket, then factor out u_{in}

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- Calculation of the maximal power
- Definition: $x=\frac{u_{out}}{u_{in}}$ and: $C=A_w\frac{\rho u_{in}^3}{4}$
- With that:

$$P=C(1+x)(1-x^2) \qquad P'=C(1-2x-3x^2) \qquad P''=C(-2-6x)$$

$$P'=0 \;\Rightarrow\; x=-1 \;\vee\; x=\frac{1}{3}$$

$$P''\left(x=\frac{1}{3}\right)<0 \;\Rightarrow\; \text{Maximum at} \quad x=\frac{1}{3}=\frac{u_{out}}{u_{in}}$$

- The max. theoretical power of a wind turbine is:

$$P=A_w\frac{\rho u_{\mathrm{in}}^3}{4}\left(1+\frac{u_{out}}{u_{\mathrm{in}}}\right)\left(1-\left(\frac{u_{out}}{u_{\mathrm{in}}}\right)^2\right)$$

$$\underline{P_{max}}=A_w\frac{\rho u_{\mathrm{in}}^3}{4}\left(1+\frac{1}{3}\right)\left(1-\left(\frac{1}{3}\right)^2\right)=A_w\frac{\rho u_{\mathrm{in}}^3}{4}\frac{4}{3}\frac{8}{9}=\underline{A_w\rho u_{\mathrm{in}}^3\frac{8}{27}}$$

Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- The max. power is defined by the air flow:
  - Kinetic energy $\frac{1}{2} m u_{in}^2$
  - Power = Energy / Time
  - We replace mass with mass flow $\dot{m} = \rho A_w u_{in}$
  - With that:

$$P_{wind,\mathrm{in}} = A_w \frac{\rho}{2} u_{\mathrm{in}}^3$$

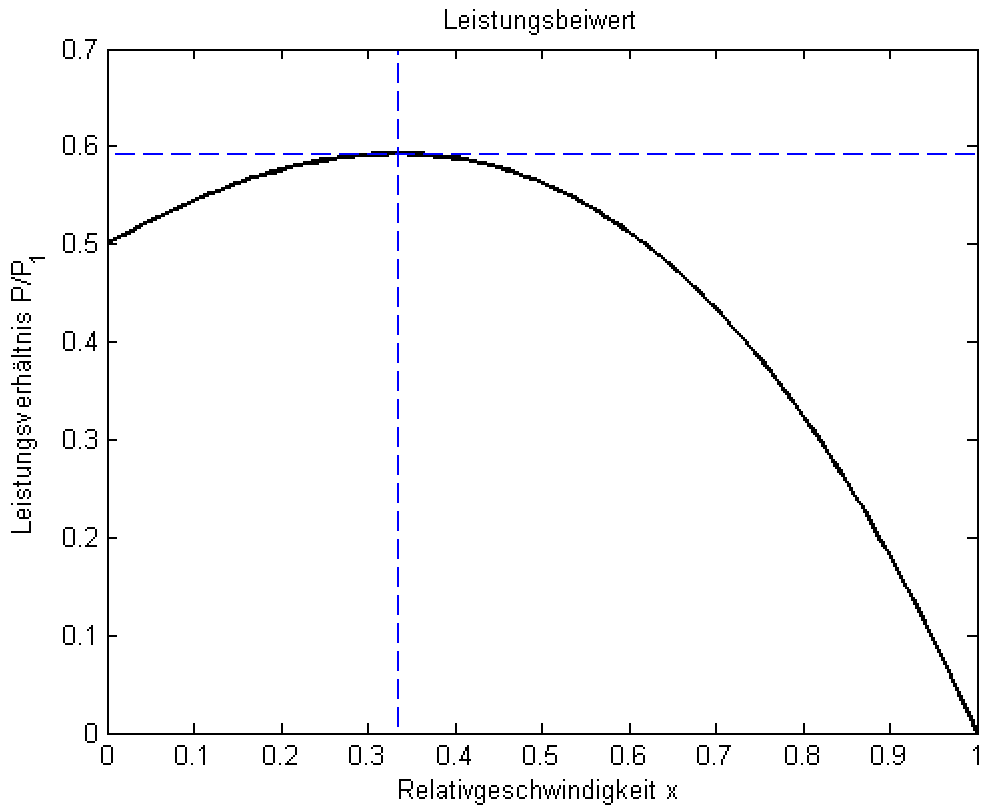
- We introduce the so called coefficient of power C_p:

$$C_P = \frac{P}{P_{wind,\mathrm{in}}}$$

$$C_{P,max} = \frac{P_{max}}{P_{wind,\mathrm{in}}} = \frac{A_w \rho u_{\mathrm{in}}^3 8/27}{A_w \rho / 2 u_{\mathrm{in}}^3} = \frac{16}{27} = 0.5926$$

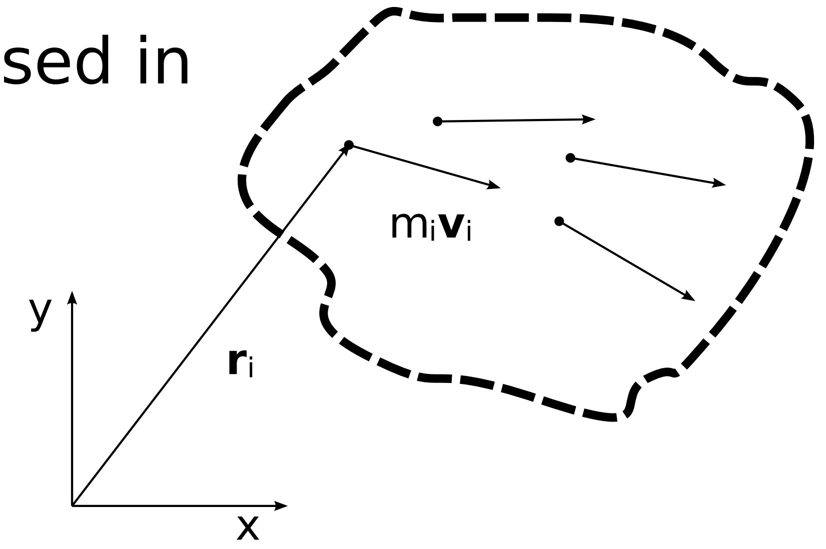
Momentum Equation - Applications
Wind turbines – Theory of Rankine / Froude

- According to the result, a maximum of 16/27 of the air flow energy can be converted into mechanical energy of the rotating turbine.
- C_p as function of relative velocity is shown in the graph on the right
- C_p is also called the Betz coefficient of power after the German engineer Albert Betz



Application of the RTT: Angular Momentum Equation

- Based on the RTT approach, similar to the linear momentum, an equation for the angular momentum can be derived.
- The angular momentum equation is used for the design of rotating machines (e.g. pumps and turbines)
- We assume, the fluid is decomposed in fluid elements i with mass Δm_i and velocity $\mathbf{v}_i$.
- Introducing the radius $\mathbf{r}_i$ in the xy-plane, an angular momentum around the z-axis exist



- The angular momentum **L** can be calculated as:

$$\sum_i \boldsymbol{r}_i \times (\Delta m_i \boldsymbol{v}_i) = \sum_i \boldsymbol{r}_i \times (\rho_i \Delta V_i \mathbf{v}_i) \overset{\lim \Delta V \to 0}{=} \int (\mathbf{r} \times \mathbf{v}) \rho \, dV = \boldsymbol{L}$$

Application of the RTT: Angular Momentum Equation

- What does the change over time of the angular momentum mean? Differentiation:

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}\sum_i \boldsymbol{r}_i \times (m_i \boldsymbol{v}_i) = \sum_i \boldsymbol{r}_i \times \frac{d}{dt}(m_i \boldsymbol{v}_i) = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum \boldsymbol{T}$$

- The change of the angular momentum over time is a torque.

- This corresponds to the 2nd law of Newton for linear momentum. Momentum and force are replaced by angular momentum and torque.

- Application of RTT for **b = r** x **v** and **B = L**

$$\boxed{\sum \boldsymbol{T} = \frac{d\boldsymbol{L}_{closed}}{dt} = \frac{d}{dt}\int_{CV} \rho\left(\boldsymbol{r}\times\boldsymbol{v}\right)dV + \int_{CS} \rho\left(\boldsymbol{r}\times\boldsymbol{v}\right)\left(\boldsymbol{v}\cdot\boldsymbol{dA}\right)}$$

Video on Angular Momentum

https://www.youtube.com/watch?v=2Oc-Ucx_4Ug

Application of the RTT: Angular Momentum Equation

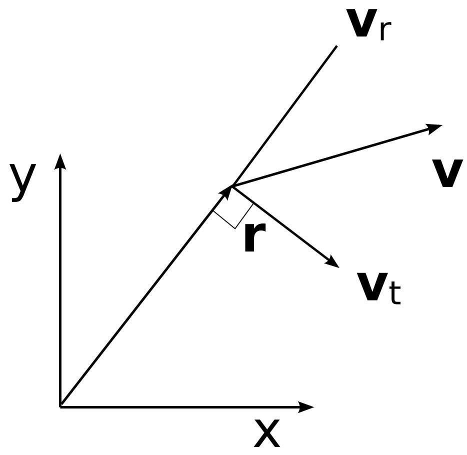
- Simplified assumptions:
  - Steady state flow: $\frac{d}{dt}\int_{CV} \rho\left(\boldsymbol{r}\times\boldsymbol{v}\right) dV = 0$
  - Constant properties at plane in- and outlets
  - Signs of mass flow needs to be considered
- This reduces the general equation to:

$$\sum \boldsymbol{T} = \sum_{outlets} \boldsymbol{r}\times\boldsymbol{v}_{out}\,\dot{m}_{out} - \sum_{inlets} \boldsymbol{r}\times\boldsymbol{v}_{in}\,\dot{m}_{in}$$

- Note: **T** is the torque acting <u>on the Fluid</u>!
- In general, the mechanical usable torque (e.g. torque acting on turbine) is of interest $\mathbf{T}_{\mathrm{mech}} = -\mathbf{T}$
- In most applications, there is a defined axis of rotation. We define that the rotation takes place around the z-axis.

Application of the RTT: Angular Momentum Equation

- Decomposition of the vector **v** in:
  - tangential component $\mathbf{v}_t$
  - radial component $\mathbf{v}_r$



$$\boldsymbol{v}=\boldsymbol{v}_r+\boldsymbol{v}_t \;\Rightarrow\; \boldsymbol{r}\times\boldsymbol{v}=\boldsymbol{r}\times\boldsymbol{v}_r+\boldsymbol{r}\times\boldsymbol{v}_t=\boldsymbol{r}\times\boldsymbol{v}_t$$

the cross product of parallel vectors is zero

- Rotation around z-axis (in xy-plane):

$$\boldsymbol{r}\times\boldsymbol{v}_t=r\,v_t\sin(90°)\,\boldsymbol{e}_z=r\,v_t\,\boldsymbol{e}_z \qquad r=|\boldsymbol{r}|\;,\; v_t=|\boldsymbol{v}_t|$$

$\mathbf{e}_z$ is the unit vector in z-direction

- Therefore, the z-component of the angular momentum equation is:

$$\boxed{\sum_{outlets}\dot{m}\,r\,v_t-\sum_{inlets}\dot{m}\,r\,v_t=\sum_{outlets}\rho\,\dot{V}\,r\,v_t-\sum_{inlets}\rho\,\dot{V}\,r\,v_t=\sum T_z}$$

Angular Momentum Equation – Pumps and Turbines

- For continuous-flow machines a distinction is made between:
  - Pumps
  - Turbines
- Pumps supply energy to the fluid.
- Turbines extract energy from the fluid
- In both cases the energy is converted:

$$E_{pot} + E_{kin} \leftrightarrow E_{mech}$$

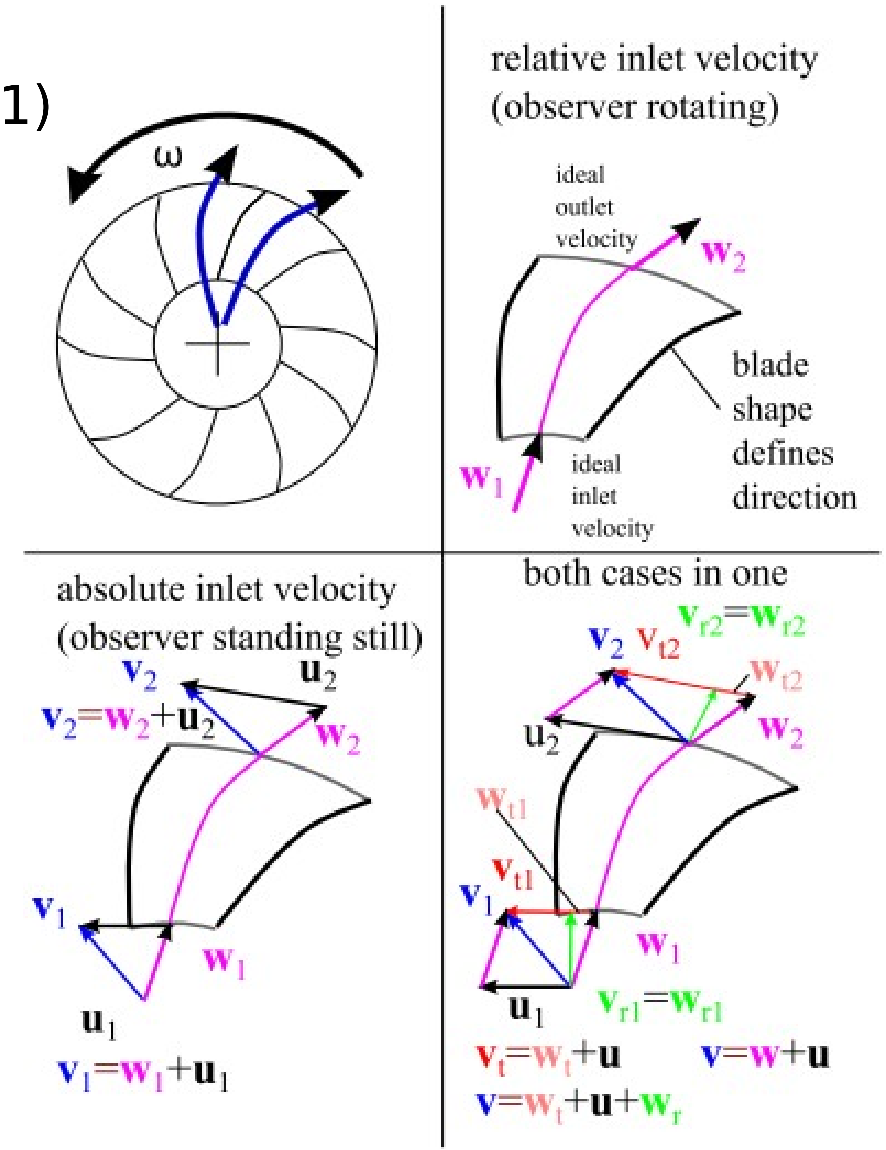
Angular Momentum Equation – Pumps and Turbines

- Rotating parts
  - Angular velocity **ω**
  - Tangential velocity **u** = **ω** x **r**
    - Note: $\mathbf{u} \perp \mathbf{r}$, which means, **u** is a tangential vector!
- We consider **2** reference systems!
  - Fixed position system: absolute velocity $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t$
  - Rotating system: relative velocity $\mathbf{w} = \mathbf{w}_r + \mathbf{w}_t$
  - Effective: $\mathbf{v} = \mathbf{w} + \mathbf{u} \quad \Leftrightarrow \quad \mathbf{w} = \mathbf{v} - \mathbf{u}$
- The <u>inlet- or outlet angle β</u> is defined based on the rotating system:

$$\boxed{\tan(\beta) = \frac{w_r}{w_t}}$$

Angular Momentum Equation – Pumps and Turbines

- Example for a radial pump:
  - Inlet in center of the wheel (index 1)
  - Outlet outside (index 2)
  - Angular velocity $\boldsymbol{\omega}$
  - Tangential velocity $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$
  - Absolute velocity $\mathbf{v}$
  - Relative velocity $\mathbf{w}$

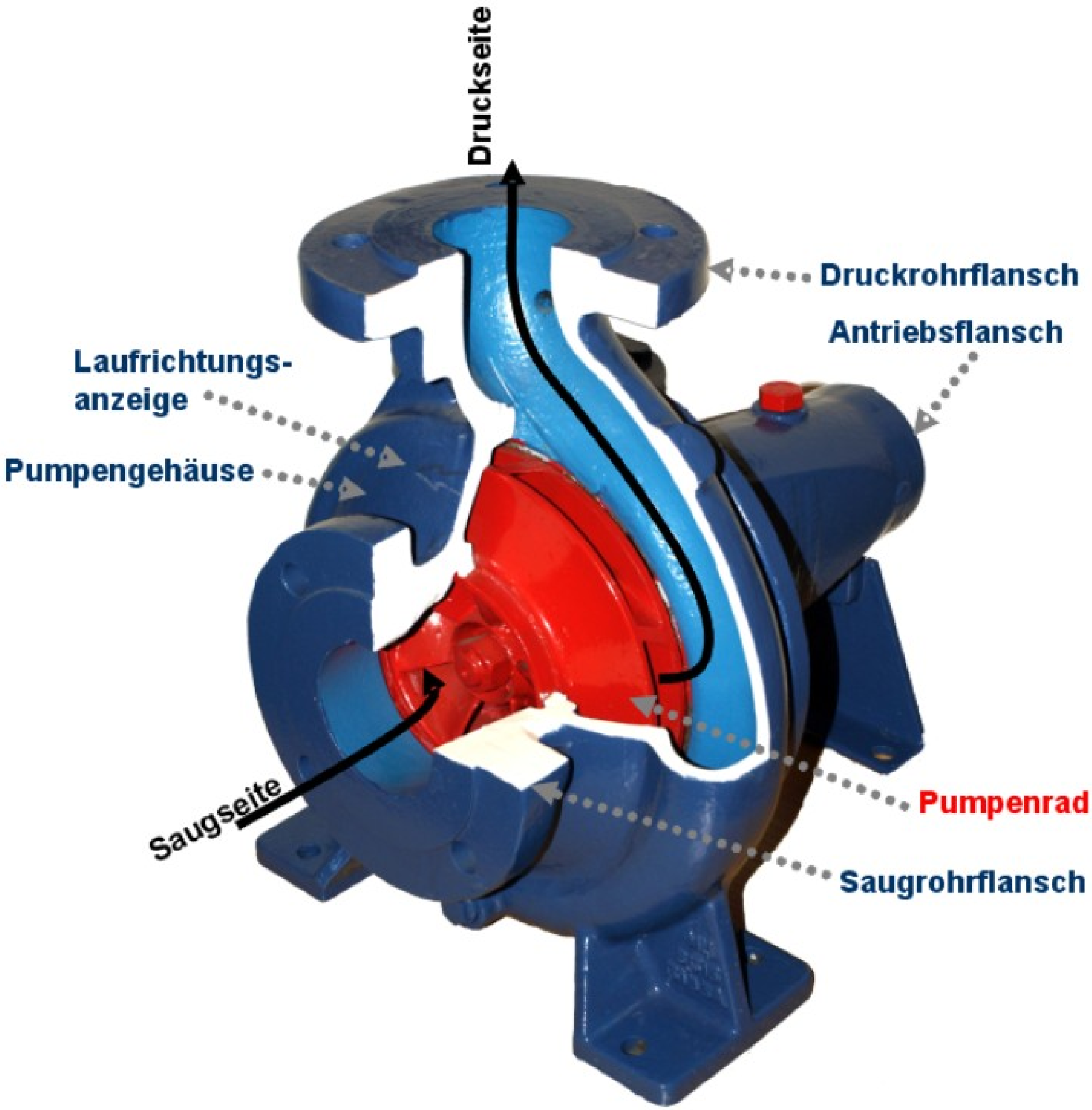


- Since $\mathbf{u}$ is only tangential:

$$\boldsymbol{v}=\boldsymbol{w}+\boldsymbol{u} \;\Rightarrow\; \begin{cases} \boldsymbol{v}_t=\boldsymbol{w}_t+\boldsymbol{u} \\ \boldsymbol{v}_r=\boldsymbol{w}_r \end{cases}$$

- Hint: **v,w,u** are vectors. For vector addition decompose vectors into its tangential and radial components!

Radial Pump



https://de.wikipedia.org/wiki/Radialpumpe#/media/Datei:Kreiselpumpe_Bezeichnungen.png

Radial Pump

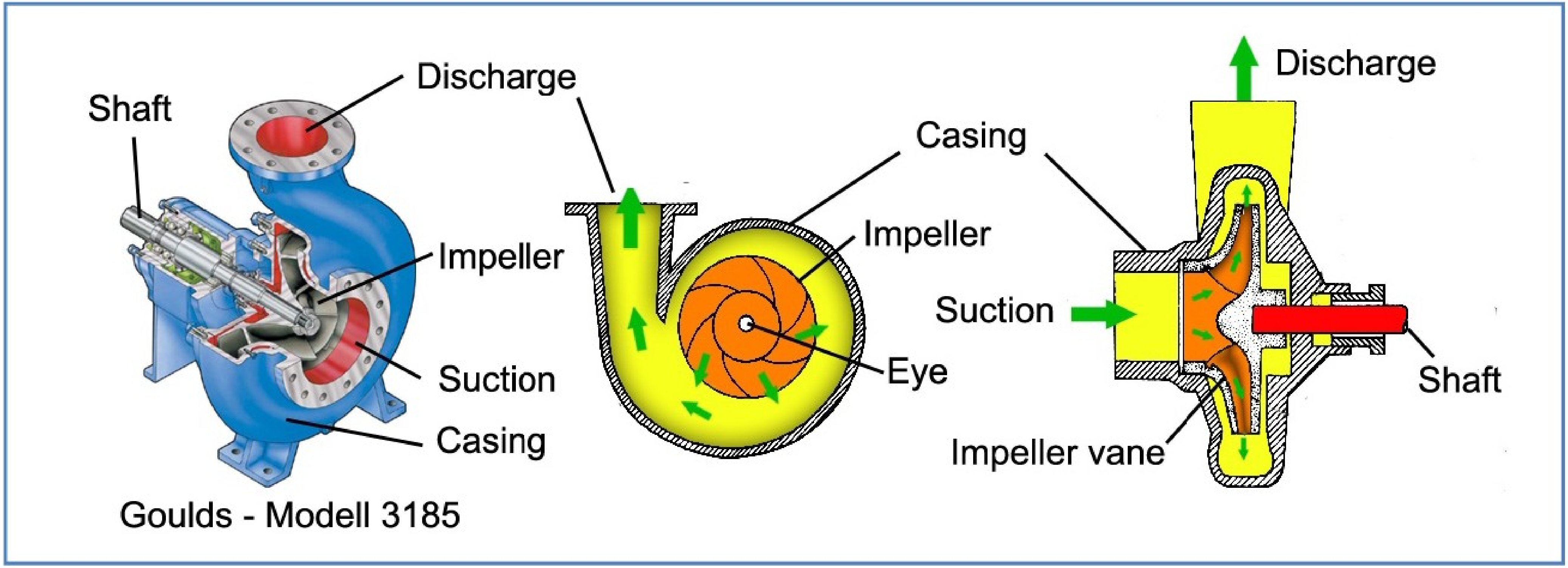
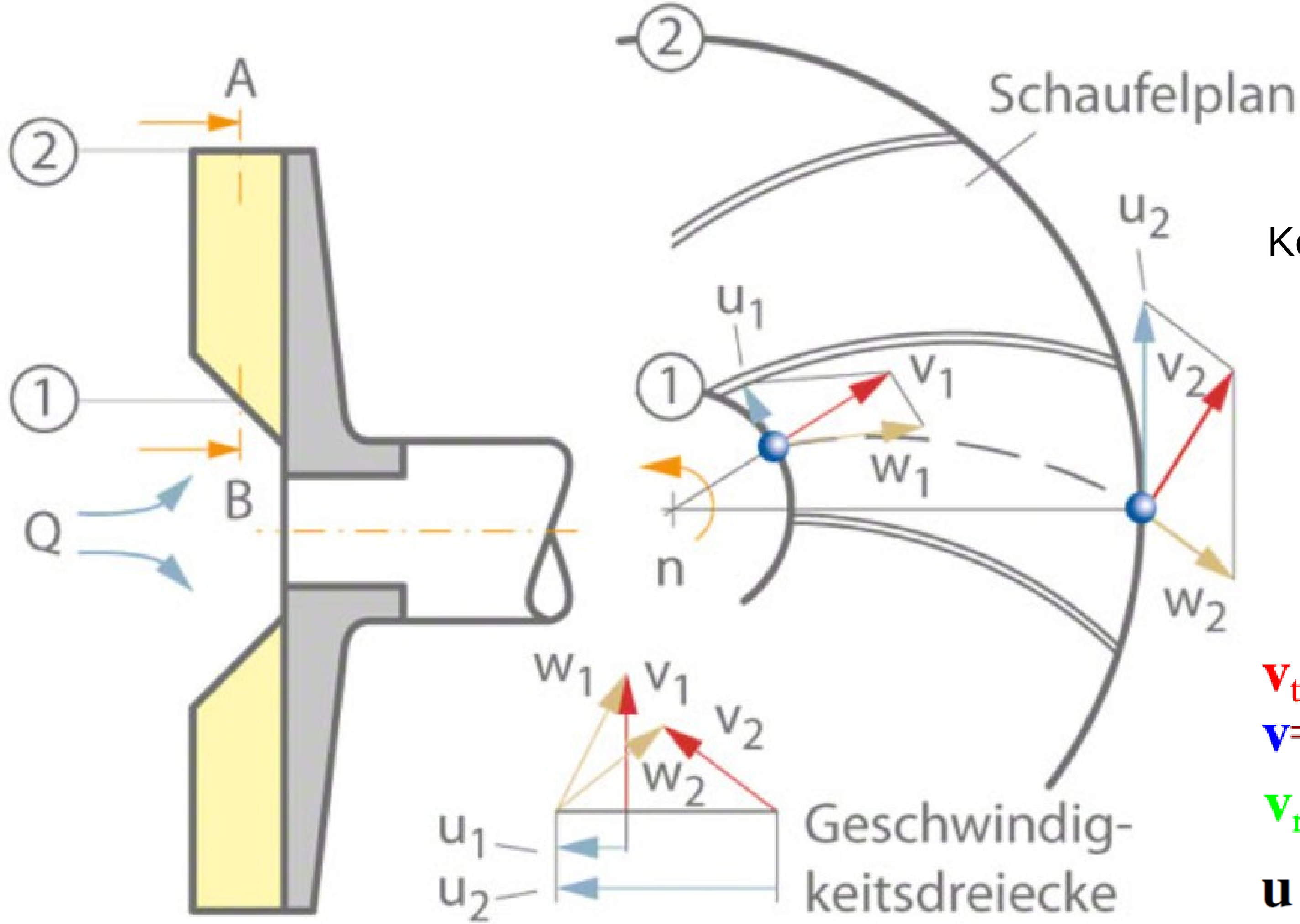


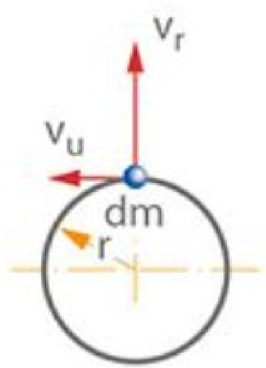
Figure 6 – Principle of a centrifugal pump[1]

Prabhu T, Hydrodynamic design of a centrifugal pump impeller, https://shodhganga.inflibnet.ac.in/handle/10603/14078

Velocity Triangles – Radial Pump



Konti: $w_{r1} * A_1 = w_{r2} * A_2$



$\mathbf{v}_t = \mathbf{w}_t + \mathbf{u}$ $\quad$ $\mathbf{v} = \mathbf{w} + \mathbf{u}$

$\mathbf{v} = \mathbf{w}_t + \mathbf{u} + \mathbf{w}_r$

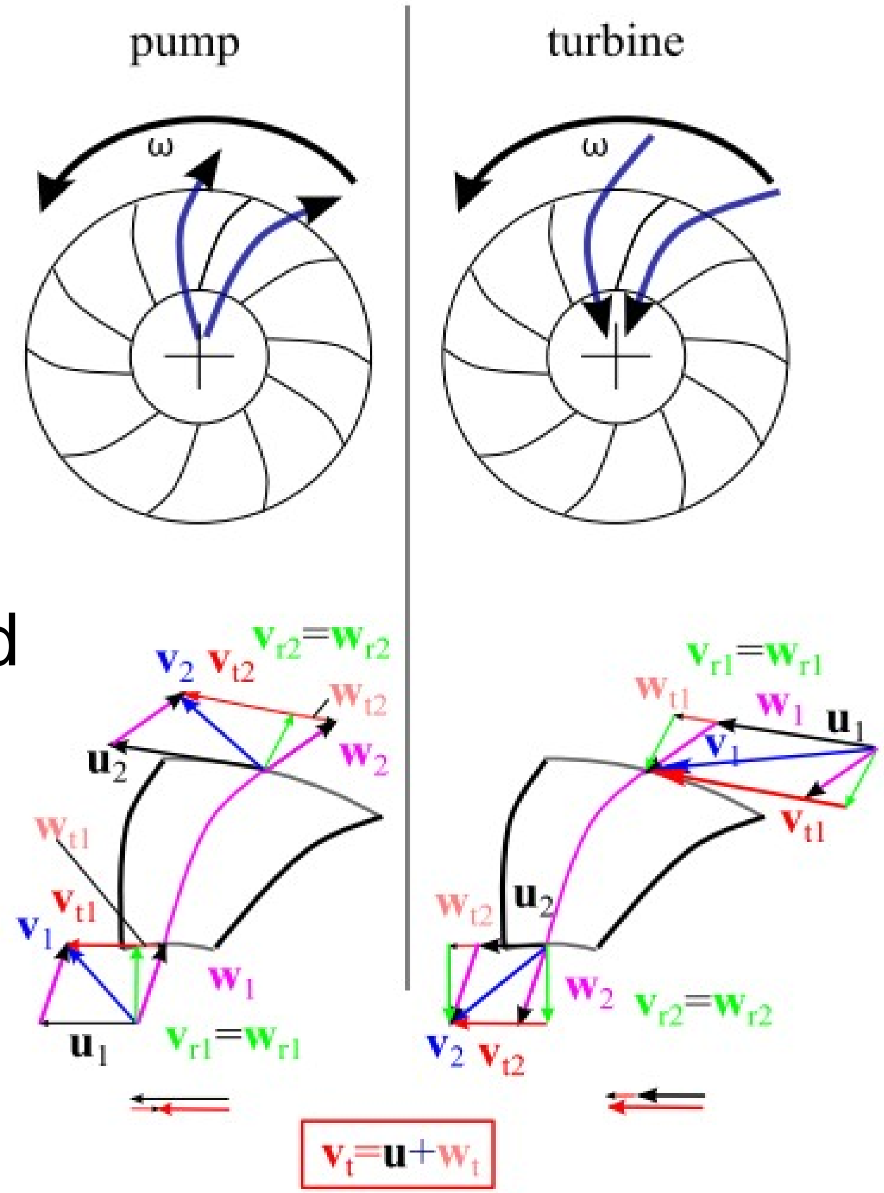
$\mathbf{v}_{r1} = \mathbf{w}_{r1}$

$\mathbf{u} = \omega \times \mathbf{r}$

https://www.ksb.com/kreiselpumpenlexikon/geschwindigkeitsdreieck/186444/

Angular Momentum Equation – Pumps and Turbines

- Radial pump
  - Supplies energy to fluid
  - Tangential inlet velocity of the fluid and blade velocity in the opposite direction
- Radial turbine
  - Extracts energy from fluid
  - Tangential inlet velocity in direction of blade velocity



Angular Momentum Equation – Pumps and Turbines

- Euler turbine equation:
  - From the angular momentum equation follows:

$$\boxed{T_{mech} = \dot{m}\left(r_1 v_{t1} - r_2 v_{t2}\right)} \qquad T_{mech} = -T$$

  - The power P is:

$$\boxed{\begin{aligned} P = \omega\, T_{mech} &= \omega\, \dot{m}\left(r_1 v_{t1} - r_2 v_{t2}\right) \\ &= \dot{m}\left(u_1 v_{t1} - u_2 v_{t2}\right) \end{aligned}} \qquad u = r \cdot \omega$$

  - The specific work y is the mechanical power divided by mass flow:

$$\boxed{y = \frac{P}{\dot{m}} = \left(u_1 v_{t1} - u_2 v_{t2}\right)} \qquad \text{unit} \quad \frac{m^2}{s^2}$$

- Note: Difference in pressure and in radial velocity do not contribute to the angular momentum, because they have no “lever“.