Kap 2, p. 9 Bsp. Taylor-Polynom bei $\boxed{x_0 = 0}$

$$f(x) = \frac{1}{1-x} = \frac{Z}{N} \qquad f'(x) = \frac{Z' \cdot N - Z \cdot N'}{N^2}$$

umständlich

$$= (1-x)^{-1}$$

Kettenregel mit

äußere Ableitung

innere Ableitung

$$f'(x) = (-1)(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$$

$$f''(x) = (-1)(-2)(1-x)^{-3}(-1)(-1) = 1 \cdot 2\,(1-x)^{-3}$$

$$f'''(x) = 1 \cdot 2\,(-3)(1-x)^{-4}(-1) = 1 \cdot 2 \cdot 3\,(1-x)^{-4}$$

$$f^{IV}(x) = 1 \cdot 2 \cdot 3\,(-4)(1-x)^{-5}(-1) = 1 \cdot 2 \cdot 3 \cdot 4\,(1-x)^{\underset{-(4+1)}{-5}}$$

$$\Rightarrow \text{allg.:} \quad f^{(k)}(x) = k!\,(1-x)^{-k-1} \Rightarrow \boxed{f^{(k)}(x_0) = k!}$$

$$T_n(x) = \sum_{k=0}^{n} \frac{\cancel{k!}}{\cancel{k!}} (x - \underset{=0}{x_0})^k = \sum_{k=0}^{n} x^k$$

$$= 1 + x + x^2 + x^3 + \ldots + x^n$$

Kap 2, p. 9 $x_0 = 0$

$$f(x) = e^x = f'(x) = f''(x) = f'''(x) = \ldots$$

$$f(x_0) = e^0 = 1 = f'(x_0) = f''(x_0) = \ldots$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \underset{1=0}{(x-x_0)^k} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots$$

$x_0 = 0$

$$f(x) = \sin x = f^{IV}(x) \qquad f(x_0) = 0 = f^{IV}(x_0)$$

$$f'(x) = \cos x = f^{V}(x) \qquad f'(x_0) = 1 = f^{V}(x_0)$$

$$f''(x) = -\sin x = f^{VI}(x) \qquad f''(x_0) = 0 = f^{VI}(x_0)$$

$$f'''(x) = -\cos x = f^{VII}(x) \qquad f'''(x_0) = -1 = f^{VII}(x_0)$$

usw

$\Rightarrow$ alle geraden Ableitungen sind Null!

$\Rightarrow$ es bleiben nur Terme für ungerades k übrig $\Rightarrow$ $k = 2\ell+1$ $\quad \ell = 0, 1, 2, 3, \ldots$

$$f^{(k)}(x_0) = f^{(2\ell+1)}(x_0) = (-1)^\ell$$

$$T_n = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\underbrace{(2\ell+1)!}_{k!\ \text{f.}\ k=2\ell+1}} \underset{1=0}{(x-x_0)^{\overbrace{2\ell+1}^{k}}} = \sum_{\ell=0}^{n} (-1)^\ell \frac{x^{2\ell+1}}{(2\ell+1)!}$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} \mp \ldots$$

Kap. 2, p. 9

$f(x) = \cos(x)$ ist Abl. von $\sin x$

$T_{\cos}(x)$ ist Abl. von $T_{\sin}(x)$

$$\Rightarrow T_{n,\cos}(x) = \left(T_{n,\sin}(x)\right)'$$

$$= \left(\sum_{\ell=0}^{n} (-1)^\ell \frac{x^{2\ell+1}}{(2\ell+1)!}\right)'$$

$$= \sum_{\ell=0}^{n} (-1)^\ell \frac{1}{\underbrace{(2\ell+1)!}_{(2\ell)!\,(2\ell+1)}} x^{2\ell} \cdot (2\ell+1)$$

Erinnerung: $m! = (m-1)! \cdot m$

$$= \underbrace{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot (m-2)(m-1)}_{(m-1)!} \cdot m$$

$$= \sum_{\ell=0}^{n} (-1)^\ell \frac{\cancel{2\ell+1}}{(2\ell)!\,\cancel{(2\ell+1)}} x^{2\ell}$$

$$= \sum_{\ell=0}^{n} (-1)^\ell \frac{x^{2\ell}}{(2\ell)!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \pm \ldots$$

Kap. 2, p.9 $\qquad x_0 = 0$

$$f(x) = (1-x)^n$$
$$f'(x) = n(1-x)^{n-1}(-1)$$
$$f''(x) = n(n-1)(1-x)^{n-2}(-1)^2$$
$$f'''(x) = n(n-1)(n-2)(1-x)^{n-3}(-1)^3$$
$$f^{IV}(x) = n(n-1)(n-2)(n-3)(1-x)^{n-4}(-1)^4$$
$$= \frac{n!}{(n-4)!}(1+x)^{n-4}$$

$$\Rightarrow f^{(k)}(x) = \frac{n!}{(n-k)!}(1+x)^{n-k} \quad (-1)^k$$

$$f^{(k)}(x_0) = \frac{n!}{(n-k)!} \quad (-1)^k$$

$$T_m(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-\underbrace{x_0}_{=0})^k \qquad \text{allg. Formel}$$

$$= \sum_{k=0}^{m} \frac{(-1)^k \frac{n!}{(n-k)!}}{k!} x^k$$

Binomialreihe

$$= \sum_{k=0}^{m} (-1)^k \frac{n!}{k!(n-k)!} x^k = \sum_{k=0}^{m} (-1)^k \binom{n}{k} x^k$$

Binomial-koeffizient

b.w.

$$\binom{n}{k} = \frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot \ldots \quad k}$$

Binomialkoeffizient so ausrechnen.

$$f(x)=\ln(1+x) \qquad f(x_0)=0 \qquad x_0=0$$

Term für $k=0$ verschwindet.

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$$f''(x) = (-1)(1+x)^{-2} = (-1)\cdot 1\,(1+x)^{-2}$$

$$f'''(x) = (-1)(-2)(1+x)^{-3} = (-1)^2\cdot 1\cdot 2\,(1+x)^{-3}$$

$$f^{IV}(x) = (-1)(-2)(-3)(1+x)^{-4} = (-1)^3\, 1\cdot 2\cdot 3\,(1+x)^{-4}$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\,(1+x)^{-k}$$

$$f^{(k)}(x_0) = (-1)^{k-1}(k-1)! \qquad k\geq 1 \quad (!)$$

$$T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}\cancel{(k-1)!}}{\underset{\cancel{(k-1)!}\cdot k}{k!}}(x-\underbrace{x_0}_{=0})^k$$

$$= \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \mp \ldots$$

Kap 2, p. 13

Reihenentwicklg. f. $(1-x)^{-1/2}$

$\Rightarrow$ Binomialkoeffizienten $\binom{n}{k}$ mit $n = -\frac{1}{2}$

$\binom{-\frac{1}{2}}{0} = (-1)^0 1$ („über 0 ist immer 1")

$\binom{-\frac{1}{2}}{1} = (-1)^1 \frac{-\frac{1}{2}}{1} = (-1)^1 - \frac{1}{2}$

$\binom{-\frac{1}{2}}{2} = (-1)^2 \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1 \cdot 2} = (-1)^2 \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2}$

$\binom{-\frac{1}{2}}{3} = (-1)^3 \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1 \cdot 2 \cdot 3} = (-1)^3 \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1 \cdot 2 \cdot 3}$

alle „Minuse" heben sich weg

Kap 2, p. 15

$$\int_0^{\pi/2} \sin^n t \, dt$$

$$= \int_0^{\pi/2} \underbrace{\sin t}_{v'(t)} \underbrace{\sin^{n-1} t}_{u(t)} \, dt$$

$\hookrightarrow v(t) = -\cos t$

$$= \left[\sin^{n-1}(t) \cdot (-\cos t)\right]_0^{\pi/2}$$

$$- \int_0^{\pi/2} \underbrace{(n-1)\sin^{n-2} t \cdot \cos t}_{u'(t)} \, (-\cos t) \, dt$$

$$= \underset{\text{wg Grenzen}}{0} + \int_0^{\pi/2} (n-1) \sin^{n-2} t \underbrace{\cos^2 t}_{1-\sin^2 t} \, dt$$

$$= \int_0^{\pi/2} (n-1) \sin^{n-2} t \, (1 - \sin^2 t) \, dt$$

$$= \int_0^{\pi/2} (n-1) \sin^{n-2} t - (n-1) \sin^n t \, dt$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} t \, dt - (n-1) \int_0^{\pi/2} \sin^n t \, dt$$

Kap 2, p. 15 (Fortsetzg)

$(n-1)\int_0^{\pi/2} \sin^n t \, dt$ auf die andere Seite der Glg.

$$\Rightarrow n \int_0^{\pi/2} \sin^n t \, dt = (n-1)\int_0^{\pi/2} \sin^{n-2} t \, dt$$

$$\int_0^{\pi/2} \sin^n t \, dt = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} t \, dt$$

✓

Kap 2, p. 22

$$\sum_{n=1}^{\infty} \frac{1}{(2n)!} \quad : \qquad a_n = \frac{1}{(2n)!}$$

$$a_{n+1} = \frac{1}{(2(n+1))!} = \frac{1}{(2n+2)!}$$

Quotientenkriterium

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!}}{\cancel{(2n)!}\,(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0 < 1$$

$\Rightarrow$ Die Reihe konvergiert absolut.

Kap 2, p. 22

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$a_n = \frac{1}{n}$$

$$a_{n+1} = \frac{1}{n+1}$$

Quotientenkriterium

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= 1$$

$\Rightarrow$ Kriterium macht kein Aussage!

$\Rightarrow$ Wie dann? Tricks.

$$\sum_{n=1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Ersetze einzelne Zahlen der Summe durch kleinere $\Rightarrow$ neue Summe kleiner

$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{} + \underbrace{\dots}_{}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$= \infty$$

divergiert aber trotzdem!

$\Rightarrow$ Original divergiert garantiert.