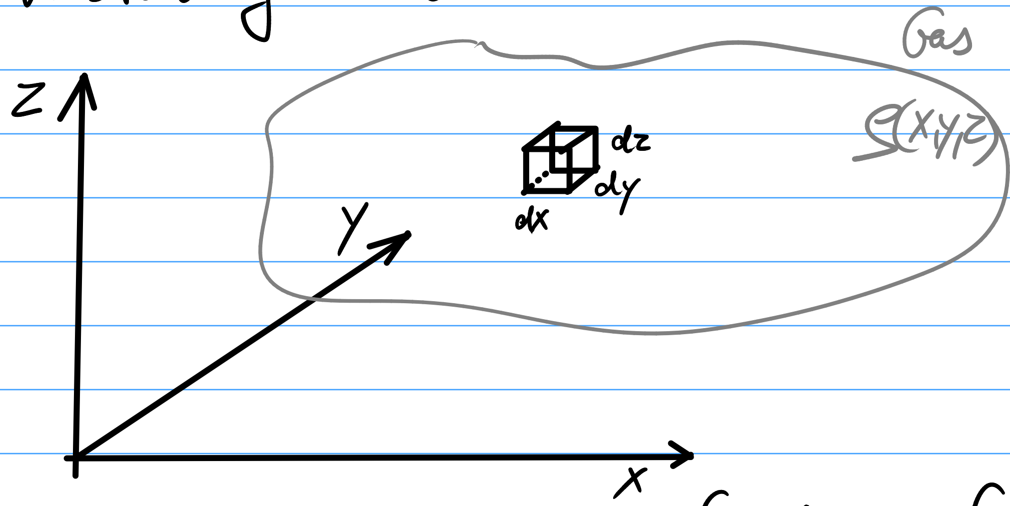
Kap 4, p. 11

Dreidimensional integrieren $dV = dx\,dy\,dz$

Vorstellung dazu:



$dm = \rho(x,y,z)\,dV \quad \Rightarrow \quad \int_V \rho\,dV = \int dm = M$

↑ Masse im Volumenelement dV

Masse des Gases i.d. Wolke

$$\int_V 1\,dV = \text{Volumen in den betreffenden Grenzen}$$

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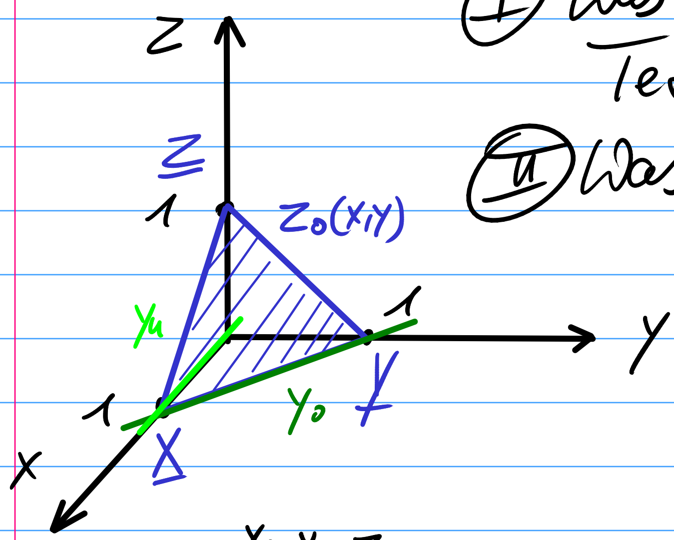


(I) Was ist das Volumen des Tetraeders?

(II) Was ist sein Schwerpunkt?

z_u = in der xy-Ebene

(I) $$V = \int_{x_u}^{x_0}\int_{y_u}^{y_0}\int_{z_u}^{z_0} 1 \; dz\,dy\,dx$$

$$\boxed{z_u = 0} \quad \checkmark$$

$z_0(x,y)$: Ebenengl.

$$\underline{p} = \underline{Y} - \underline{X} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\underline{q} = \underline{Z} - \underline{X} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\Rightarrow \underline{n} = \underline{p} \times \underline{q} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\cdot 1 - 0\cdot 0\\ 0\cdot(-1) - (-1)\cdot 1\\ -1\cdot 0 - 1\cdot(-1)\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Normalenvektor auf der Ebene f. z_0

$$\underline{n}\cdot(\underline{r} - \underline{X}) = 0 \qquad \underline{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Rightarrow \underline{n}\cdot\underline{r} - \underline{n}\cdot\underline{X} = 0 \qquad \underline{n}\cdot\underline{X} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}+1\\0\\0\end{pmatrix} = +1$$

$$x + y + z = +1 \qquad \underline{n}\cdot\underline{r} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = x + y + z$$

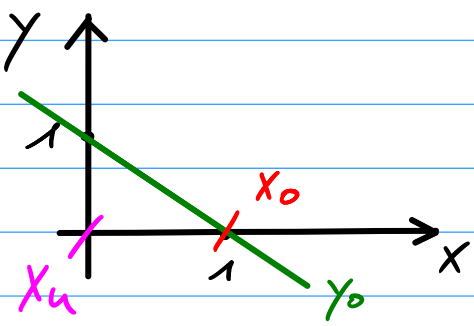
Kap 4, p. 11 (Fortsetzg.)

$$\Rightarrow \boxed{z_o(x,y) = 1 - x - y}$$

$$\boxed{y_u = 0}$$

$$y_o = ?$$

$$\Rightarrow \boxed{y_o(x) = -x + 1}$$



$$\boxed{x_u = 0}$$

$$\boxed{x_o = 1}$$

Etwas umständlich! Wenn $z_o, z_u, y_o, y_u, x_o, x_u$ definiert sind, diese Namen verwenden

$$\Rightarrow V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \, dz \, dy \, dx$$

$$= \int_{x_u}^{x_o} \int_{y_u}^{y_o} \int_{z_u}^{z_o} 1 \, dz \, dy \, dx$$ ist lesbarer!

$$= \int_{x_u}^{x_o} \int_{y_u}^{y_o} [z]_{z_u}^{z_o} \, dy \, dx$$

$$= \int_{x_u}^{x_o} \int_{y_u}^{y_o} z_o - z_u \, dy \, dx$$

$$= \int_{x_u}^{x_o} \int_{y_u}^{y_o} (1-x-y) - 0 \, dy \, dx$$

Kap 4, p. 11 (Fortsetzg.)

$$= \int_{x_u}^{x_o} \left[y - xy - \frac{1}{2}y^2 \right]_{y_u}^{y_o} dx$$

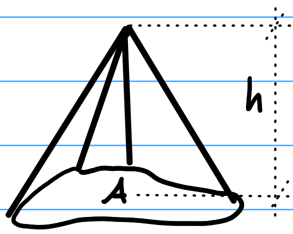
$$= \int_{x_u}^{x_o} \left((1-x) - x(1-x) - \frac{1}{2}\underbrace{(1-x)^2}_{(1-2x+x^2)} \right) - 0 \; dx$$

$$= \int_{x_u}^{x_o} 1 - x - x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2 \; dx$$

$$= \int_{x_u}^{x_o} \frac{1}{2} - x + \frac{1}{2}x^2 \; dx = \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{2}\frac{1}{3}x^3 \right]_0^1$$

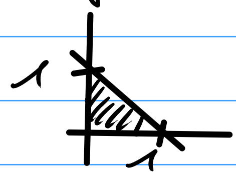
$$= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) - 0 = \frac{1}{6} = V$$

Probe:



$$\boxed{V = \frac{1}{3} A \cdot h}$$

für pyramidenartige Körper



$A = \frac{1}{2}$, $h = 1$

$$\Rightarrow V = \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6} \;\checkmark$$

II Schwerpunkt

$$x_S = \frac{1}{V} \int_V x \, dV$$

y_S, z_S analog

Kap 4, p. 11 (Fortsetzg.)

$$X_S = \frac{1}{V} \underbrace{\int_V x\,dV}_{J_x}$$

$$J_x = \int_{x_u}^{x_o}\int_{y_u}^{y_o}\int_{z_u}^{z_o} x \; dz\,dy\,dx$$

$$= \int_{x_u}^{x_o}\int_{y_u}^{y_o} \left[xz\right]_{z_u}^{z_o} dy\,dx$$

$$= \int_{x_u}^{x_o}\int_{y_u}^{y_o} \underbrace{x(1-x-y)}_{x-x^2-xy} - 0 \quad dy\,dx$$

$$= \int_{x_u}^{x_o} \left[xy - x^2y - \tfrac{1}{2}xy^2\right]_0^{1-x} dx$$

$$= \int_{x_u}^{x_o} \left(x(1-x) - x^2(1-x) - \tfrac{1}{2}x\underbrace{(1-x)^2}_{1-2x+x^2}\right) - 0 \quad dx$$

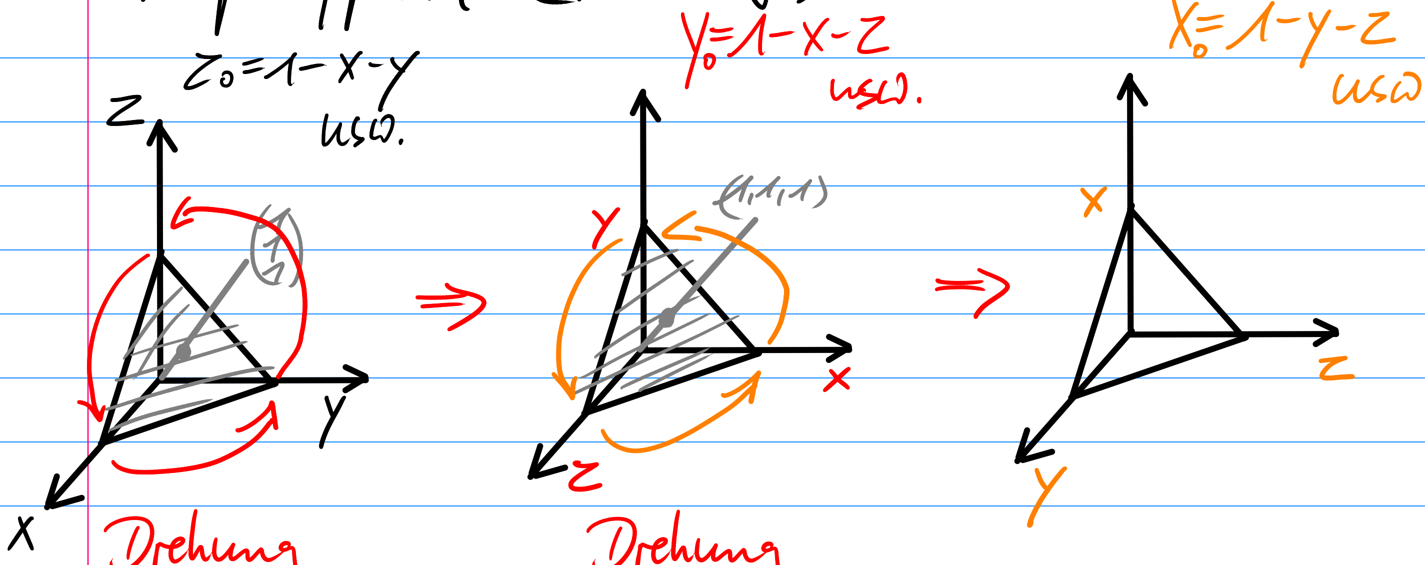
$$= \int_{x_u}^{x_o} x - x^2 - x^2 + x^3 - \tfrac{1}{2}x + x^2 - \tfrac{1}{2}x^3 \; dx$$

$$= \int_{x_u}^{x_o} \tfrac{1}{2}x - x^2 + \tfrac{1}{2}x^3 \; dx = \left[\tfrac{1}{2}\cdot\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{2}\tfrac{1}{4}x^4\right]_0^1$$

$$= \frac{1}{4}\frac{2\cdot 3}{2\cdot 3} - \frac{1}{3}\frac{8}{8} + \frac{1}{8}\frac{3}{3} \qquad = \frac{6-8+3}{24} = \frac{1}{24} = J_x$$

$$\Rightarrow \quad X_S = \frac{1}{V} J_x = \frac{1}{\frac{1}{6}} \cdot \frac{1}{24} = \frac{6}{24} = \boxed{\frac{1}{4} = X_S}$$

Kap. 4, p. 11 (Fortsetzg)



Drehung um Winkelhalbierende   Drehung um Winkelhalbierende

$x_s = \frac{1}{V}\int x \, dV$ letzte Integration in x-Richtung

$z_s = \frac{1}{V}\int z \, dV$ letzte Integration in z-Richtg.

$y_s = \frac{1}{V}\int_V y \, dV$ letzte Integration in y-Richtg.

$\Rightarrow$ Symmetrieüberlegungen

$$x_s = z_s = y_s = \frac{1}{4}$$

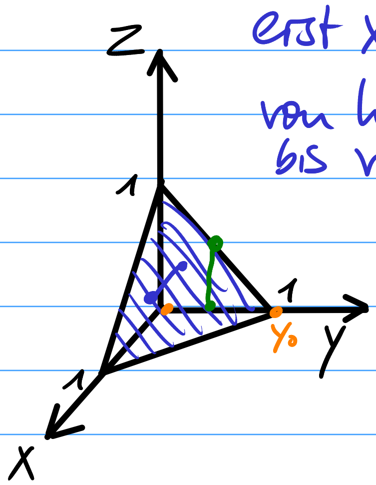
Bisher Integrationsreihenfolge

erst z
dann y
dann x

jetzt anders: x, z, y

für den Schwerpunkt

Kap 4, p. 11 (Fortsetzg.)



erst x-Richtg. von hinten bis vorne: $x_u = 0$, $x_o = 1-y-z$

dann z-Richtg.: unten bis oben
$z_u = 0$
$z_o = 1-y$

dann y-Richtg.: $y_u = 0$
$y_o = 1$

$$x_s = \frac{1}{V}\int_V x\,dV = \frac{1}{V}\underbrace{\int_{y_u}^{y_o}\int_{z_u}^{z_o}\int_{x_u}^{x_o} x\,dx\,dz\,dy}_{J_x}$$

$$J_x = \int_{y_u}^{y_o}\int_{z_u}^{z_o}\int_{x_u}^{x_o} x\,dx\,dz\,dy$$

$$= \int_{y_u}^{y_o}\int_{z_u}^{z_o}\left[\frac{1}{2}x^2\right]_{x_u}^{x_o} dz\,dy$$

$$= \int_{y_u}^{y_o}\int_{z_u}^{z_o}\frac{1}{2}(1-y-z)^2 - 0\;dz\,dy$$

$$= \int_{y_u}^{y_o}\int_{z_u}^{z_o}\frac{1}{2}\left(1+y^2+z^2-2y-2z+2yz\right)dz\,dy$$

Kap 4, p. 20 (Fortsetzg.)

$$= \frac{1}{2}\int_{y_u}^{y_0} \left[z + y^2 z + \frac{1}{3}z^3 - 2yz - 2\cdot\frac{1}{2}z^2 + 2y\frac{1}{2}z^2 \right]_{z_u}^{z_0} dy$$

Fehler ⇒ neue Seite

$$= \frac{1}{2}\int_{y_u}^{y_0} (1-y) + y^2(1-y) + \frac{1}{3}(1-y)^3 - 2y(1-y) - (1-y)^2 + y(1-y)^2 dy$$

$$= \frac{1}{2}\int_{y_u}^{y_0} 1 \cancel{-y} \cancel{+y} - y^2 + \frac{1}{3}(1-3y+3y^2-y^3) - 2y + 2y^2$$
$$- (1-2y+y^2) + y(1-2y+y^2) \quad dy$$

$$= \frac{1}{2}\int_{y_u}^{y_0} \cancel{1} \cancel{-y^2} + \frac{1}{3} \cancel{-y} \cancel{+y^2} - \frac{1}{3}y^3 \cancel{-2y} \cancel{+2y^2}$$
$$\cancel{-1} \cancel{+2y} - y^2 \cancel{+y} \cancel{-2y^2} + y^3 \quad dy$$

$$= \frac{1}{2}\int_{y_u}^{y_0} \frac{1}{3} - y^2 + \frac{2}{3}y^3 \, dy$$

$$= \frac{1}{2}\left[\frac{1}{3}y - \frac{1}{3}y^3 + \frac{2}{3}\frac{1}{4}y^4 \right]_0^1$$

$$= \frac{1}{2}\left(\cancel{\frac{1}{3} - \frac{1}{3}} + \frac{2}{12} \right) = \frac{1}{12} = J_x \quad \text{falsch}$$

⇒ sollte $\frac{1}{24}$ sein ...
Fehler suchen ... s.o.

$$x_S = \frac{1}{V} J_x = \frac{1}{\frac{1}{6}} \cdot \frac{1}{24} = \frac{1}{4} \quad \text{wie zuvor}$$

ABER MIT VIEL MEHR AUFWAND!

Kap 4, p. 20 (Fortsetzg.)

$$= \frac{1}{2}\int_{y_u}^{y_o} \left[z + y^2 z + \frac{1}{3}z^3 - 2yz - 2\cdot\frac{1}{2}z^2 + 2y\frac{1}{2}z^2 \right]_{z_u}^{z_o} dy$$

Fehler korrigiert

$$= \frac{1}{2}\int_{y_u}^{y_o} (1-y) + y^2(1-y) + \frac{1}{3}(1-y)^3 - 2y(1-y) - (1-y)^2 + y(1-y)^2 \, dy$$

$$= \frac{1}{2}\int_{y_u}^{y_o} 1 - y + y^2 - y^3 + \frac{1}{3}(1 - 3y + 3y^2 - y^3) - 2y + 2y^2 - (1 - 2y + y^2) + y(1 - 2y + y^2) \, dy$$

$$= \frac{1}{2}\int_{y_u}^{y_o} \cancel{1} - y + y^2 - \cancel{y^3} + \frac{1}{3} - \cancel{y} + \cancel{y^2} - \frac{1}{3}y^3 - \cancel{2y} + \cancel{2y^2} - \cancel{1} + \cancel{2y} - \cancel{y^2} + \cancel{y} - \cancel{2y^2} + \cancel{y^3} \, dy$$

$$= \frac{1}{2}\int_{y_u}^{y_o} \frac{1}{3} - y + y^2 - \frac{1}{3}y^3 \, dy$$

$$= \frac{1}{2}\left[\frac{1}{3}y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{3}\frac{1}{4}y^4 \right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12} \right) = \frac{1}{2}\,\frac{4 - 6 + 4 - 1}{12}$$

$$= \frac{1}{2}\cdot\frac{1}{12} = \boxed{\frac{1}{24} = J_x}$$ wie zuvor

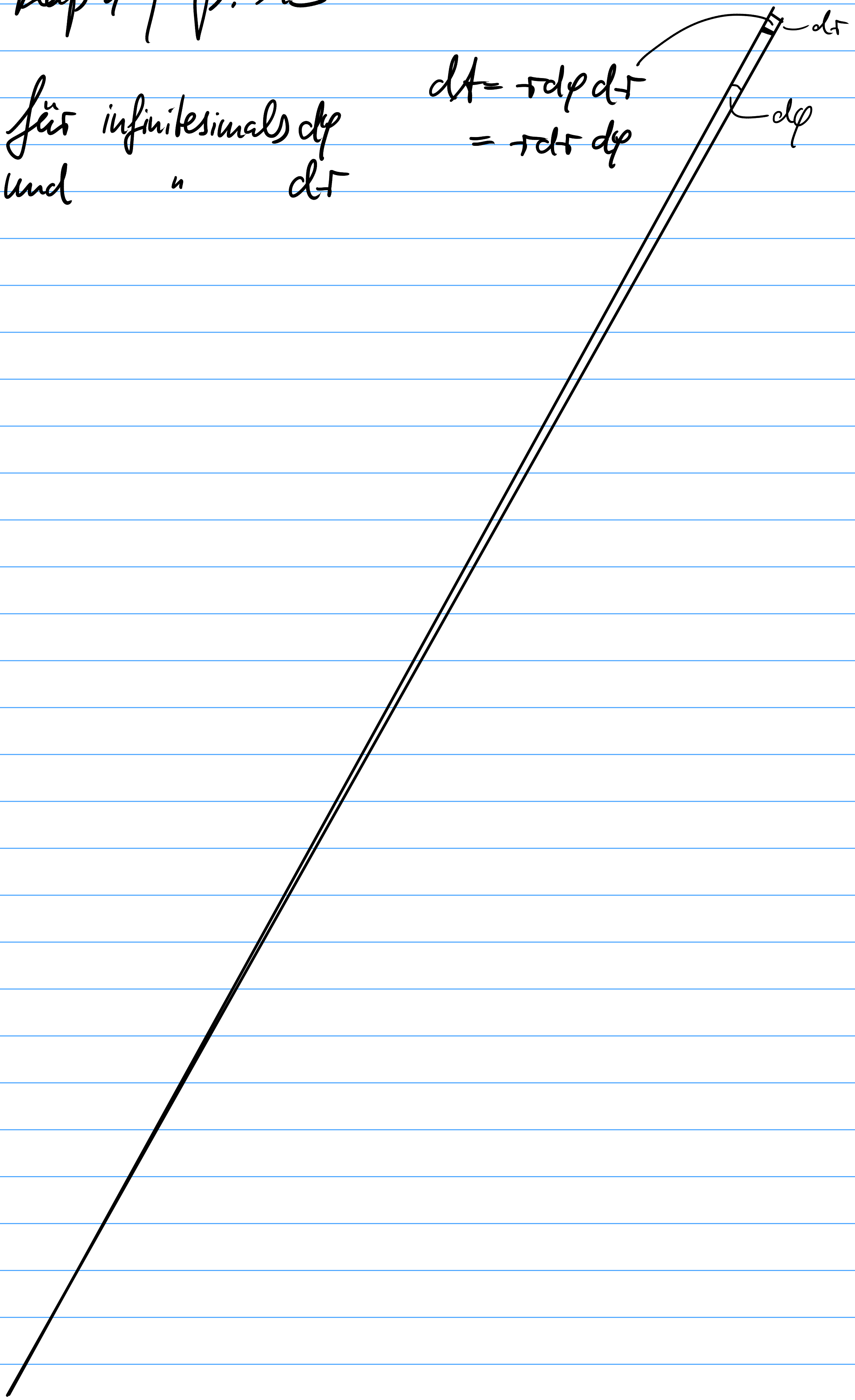
$$x_s = \frac{1}{V} J_x = \frac{1}{\frac{1}{6}}\cdot\frac{1}{24} = \frac{1}{4}$$ wie zuvor

ABER MIT VIEL MEHR AUFWAND!

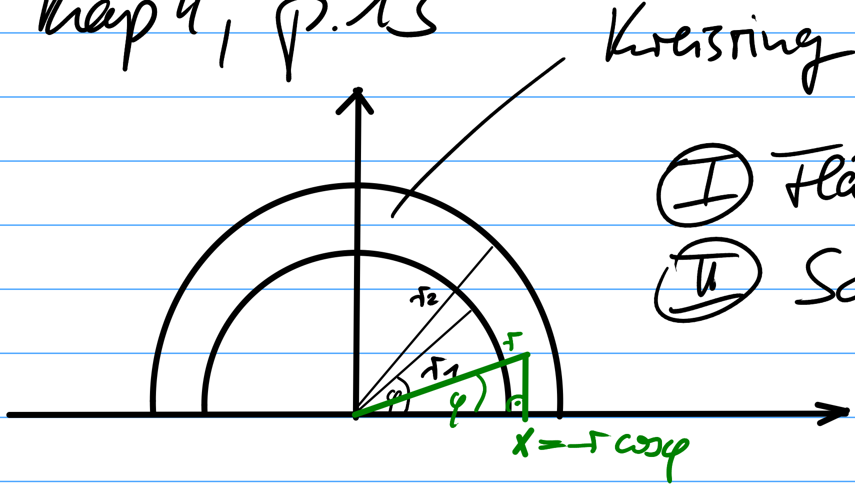
Kap 4 | p. 13

für infinitesimals $d\varphi$
und " dr

$$dA = r\,d\varphi\,dr = r\,dr\,d\varphi$$



Kap 4, p. 13



I Fläche?

II Schwerpunkt?

I über Integral

$$A = \int_0^\pi \int_{r_1}^{r_2} 1 \underbrace{r\,dr\,d\varphi}_{dA} = \int_0^\pi \int_{r_1}^{r_2} r\,dr\,d\varphi$$

$$= \int_0^\pi \left[\frac{1}{2} r^2\right]_{r_1}^{r_2} d\varphi = \int_0^\pi \frac{1}{2} r_2^2 - \frac{1}{2} r_1^2 \, d\varphi$$

$$= \left[\left(\frac{1}{2} r_2^2 - \frac{1}{2} r_1^2\right)\varphi\right]_0^\pi = \underbrace{\frac{1}{2} r_2^2 \pi}_{\text{halbe Kreisfläche zu } r_2} - \underbrace{\frac{1}{2} r_1^2 \pi}_{\text{halbe Kreisfläche zu } r_1}$$

II

$$x_S = \frac{1}{A} \int_A x\,dA$$

$$= \frac{1}{A} \int_0^\pi \int_{r_1}^{r_2} x\, r\,dr\,d\varphi$$

Wir brauchen $x(r,\varphi)$ $= r\cos\varphi$

$$= \frac{1}{A} \int_0^\pi \int_{r_1}^{r_2} r^2 \cos\varphi \, dr\,d\varphi$$

Kap 4, p. 13 (Fortsetzg.)

$$= \frac{1}{A} \int_0^\pi \left[\frac{1}{3} r^3 \cos\varphi \right]_{r_1}^{r_2} d\varphi$$

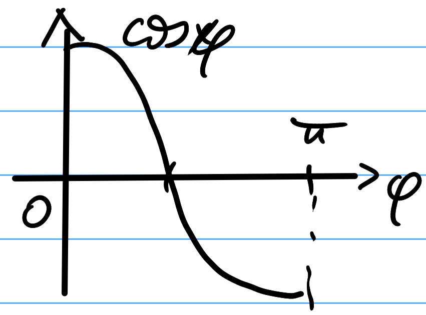
$$= \frac{1}{A} \int_0^\pi \left(\frac{1}{3} r_2^3 - \frac{1}{3} r_1^3 \right) \cos\varphi \, d\varphi$$

$$= \frac{1}{A} \left(\frac{1}{3} r_2^3 - \frac{1}{3} r_1^3 \right) \int_0^\pi \cos\varphi \, d\varphi$$

$$= \frac{1}{A} (\dots) \left[-\sin\varphi \right]_0^\pi = \frac{1}{A} (\dots) (0 - 0)$$

$$= \boxed{0 = x_S}$$

Symmetrie ... hätte man sich denken können.



$$y_S = \frac{1}{A} \int_0^\pi \int_{r_1}^{r_2} y \, r \, dr \, d\varphi \qquad y = r \sin\varphi$$

$$= \frac{1}{A} \int_0^\pi \int_{r_1}^{r_2} r^2 \sin\varphi \, dr \, d\varphi$$

$$= \frac{1}{A} \int_0^\pi \left[\frac{1}{3} r^3 \right]_{r_1}^{r_2} \sin\varphi \, d\varphi = \frac{1}{A} \left[\frac{1}{3} r^3 \right]_{r_1}^{r_2} \int_0^\pi \sin\varphi \, d\varphi$$

$$= \frac{1}{A} \left[\frac{1}{3} r^3 \right]_{r_1}^{r_2} \left[\cos\varphi \right]_0^\pi = \frac{1}{A} \left[\frac{1}{3} r^3 \right]_{r_1}^{r_2} \underbrace{\left(-(-1) - (1) \right)}_{2}$$

$$= \frac{1}{A}\left[\frac{1}{3}r_2^3 - \frac{1}{3}r_1^3\right]\cdot 2$$

$$\boxed{= \frac{\frac{1}{3}r_2^3 - \frac{1}{3}r_1^3}{\frac{1}{2}\pi r_2^2 - \frac{1}{2}\pi r_1^2}\, 2 \quad = y_S}$$