Lower Bounds and an Exact Method for the Capacitated Vehicle Routing Problem

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ABSTRACT

In this paper we consider the problem in which a fleet of M vehicles stationed at a central depot is to be optimally routed to supply customers with known demands subject to vehicle capacity constraints. This problem is referred as the Capacitated Vehicle Routing Problem (CVRP). We present an exact algorithm for solving the CVRP based on a Set Partitioning formulation of the problem. We describe a procedure for computing a valid lower bound to the cost of the optimal CVRP solution that finds a feasible solution of the dual of the LP-relaxation of the set partitioning formulation without generating the entire set partitioning matrix. The dual solution obtained is then used to limit the set of the feasible routes containing the optimal CVRP solutions. The resulting Set Partitioning problem is solved by using a branch and bound algorithm. Computational results are presented for a number of problems derived from the literature. The results show the effectiveness of the proposed method in solving problems up to about 100 customers.
Keywords: Vehicle Routing, Set Partitioning, Cutting planes

1. INTRODUCTION

The Capacitated Vehicle Routing Problem (CVRP) is the problem of designing, for a fleet of m identical vehicles of capacity Q located at a central depot, a number of feasible routes in order to supply a set of n customers. Every route performed by a vehicle must start and end at the depot and the load carried must be less than or equal to the vehicle capacity. It is assumed that the "cost matrix" of the least cost paths between each pair of customers is known. The cost of a route is computed as the sum of the costs of the arcs forming the route. The objective is to design vehicle routes (one route for each vehicle), so that all customers are visited exactly once and the sum of the route costs is minimized. In this paper, the cost matrix is assumed to be symmetric. The CVRP is $\mathcal{NP}$-hard as it is a natural generalization of the Travelling Salesman Problem (TSP).

Real-world CVRP's (see Ball et al. [9], Bodin, Mingozzi and Maniezzo [10] and Toth and Vigo [44]) involve, in addition to vehicle capacity restrictions, complicated constraints like time-windows to visit customers, customer-vehicle incompatibilities, mixed deliveries or collections on the same route, multiple interacting depots, etc.

The practical importance of the CVRP provides the motivation for the effort involved in the development of heuristic algorithms (see the surveys of Golden et al. [23], Laporte and Semet [33] and of Gendreau, Laporte and Potvin [22]). Annotated bibliographies on the CVRP were given by Laporte and Osman [32] and by Laporte [29]. The bibliography of Laporte and Osman [32] contains over 500 articles on vehicle routing. Laporte and Nobert [30] presented an extensive survey which was entirely devoted to exact methods for the CVRP and gave a complete and detailed analysis of the state of the art up to the late eighties. Other surveys covering exact algorithms were given by Laporte [28] and Fisher [19]. The chapters of Toth and Vigo [43], Naddef and Rinaldi [39] and Bramel and Simchi-Levi [11] in the book edited by Toth and Vigo [44] survey the most effective exact methods proposed in the literature so far.

The best known exact algorithms for the symmetric CVRP are briefly surveyed below. These methods can be classified into the following categories: branch–and–bound, branch–and–cut, dynamic programming and Set Partitioning (SP) based methods.

The effectiveness of branch–and–bound algorithms largely depends on the quality of the lower bounds used to limit the search tree. Christofides, Mingozzi and Toth [14] describe a Lagrangean bound based on the SP formulation where the columns correspond to the set of q-routes. A q-route is a not necessarily simple cycle covering the depot and a subset of customers whose total demand is equal to q. Some customers may be visited more than once, so the set of valid CVRP routes is strictly contained in the set of q-routes. The resulting branch–and–bound algorithm could solve instances with up to 25 customers. Christofides, Mingozzi and Toth [14] also describe a lower bound based on spanning trees where the depot has degree k, with $m \leq k \leq 2m$. Fisher [20] describe a Lagrangean bound based on m-trees, which are spanning trees having degree $2m$ in the depot, and an exact branch–and–bound algorithm that has optimally solved a well-known problem with 100 customers. Miller [37] dualized vehicle capacity constraints in a Lagrangen fashion and worked on the resulting b-matching relaxation. The resulting branch–and–bound algorithm solved several problems containing up to 50 customers.

Christofides, Mingozzi and Toth [15] present three dynamic programming formulations of the CVRP and introduce the *state space relaxation* method for relaxing the dynamic programming recursions in order to obtain valid lower bounds on the value of the optimal solutions.

Balinski and Quandt [8] introduced the SP formulation of the CVRP, where each column corresponds to a route. This formulation is not practical as it involves an expo-

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nential number of variables. Agarwal, Marthur and Salkin [1] proposed a column generation algorithm on a modified SP problem where column costs are given by a linear function over the customers yielding a lower bound on the actual route cost. Hadjiconstantinou, Christofides and Mingozzi [26] describe a method for computing a feasible dual solution of the LP-relaxation of the SP formulation that is based on the computation of k-shortest paths and q-paths. The resulting CVRP lower bound is superior to the lower bound obtained by Christofides, Mingozzi and Toth [14] and the branch–and–bound algorithm is able to optimally solve problems involving up to 50 customers. Mingozzi, Christofides and Hadjiconstantinou [38] present a new method for solving the SP formulation of the CVRP. They describe a procedure for computing a valid lower bound that combines in an additive way different feasible solutions of the dual of the LP-relaxation of the SP formulation without generating the entire SP matrix. The dual solution obtained is then used to limit the set of the feasible routes containing the optimal CVRP solutions. The resulting SP problem is solved by using a branch–and–bound algorithm. Mingozzi, Christofides and Hadjiconstantinou [38] report optimal solutions of problems up to 50 customers.

Branch–and–cut methods extend to the symmetric CVRP the successful results of polyhedral combinatorics developed for the TSP by Chvátal [16] and by Grötschel and Padberg [24, 25]. These methods are mainly based on cutting-plane techniques used to strengthen the LP–relaxation of the two–index vehicle flow CVRP formulation by adding families of valid inequalities. Using comb inequalities within a branch–and–cut method Cornuéjols and Harche [17] were able to solve the 50 customer test problem described in Christofides and Eilon [13]. Araque et al. [2] describe a branch–and–cut procedure based on multistar inequalities for the case where customers have unit demands, which can consistently solve to optimality instances with up to 60 customers. A more sophisticated branch–and–cut algorithm for the symmetric CVRP has been proposed by Augerat et al. [4]. In addition to capacity constraints, these authors have used new classes of valid inequalities, such as comb and extended comb inequalities, generalized capacity constraints and hypotour inequalities. These new inequalities led to significant improvements in the quality of the bound. Other branch-and-cut methods have been proposed by Ralphs et al. [41], by Baldacci, Hadjiconstantinou and Mingozzi [7] and by Lysgaard, Letchford, and Eglese [36]. Baldacci, Hadjiconstantinou and Mingozzi [7] describe a branch–and–cut algorithm that is based on a two–commodity network flow formulation of the CVRP. Lysgaard, Letchford, and Eglese [36] proposed a branch-and-cut algorithm that uses a variety of valid inequalities, including capacity, framed capacity, comb, partial multistar, hypotour and classical Gomory mixed integer cuts. The algorithms of Augerat et al. [4], of Baldacci, Hadjiconstantinou and Mingozzi [7] and of Lysgaard, Letchford, and Eglese [36] were able to solve a 135-customer problem which is the largest CVRP problem solved to date.

For a complete survey of branch–and–cut methods for the CVRP see Naddef and Rinaldi [39].

The best exact method currently available for the CVRP has been proposed by Fukasawa et al. [21]. This method combines the branch-and-cut of Lysgaard, Letchford, and Eglese [36] with the SP approach. Besides the well-known capacity constraints, they also use framed capacity, strengthened comb, multistar, partial multistar, generalized multistar and hypotour inequalities, all presented in Lysgaard, Letchford, and Eglese [36]. The columns of the SP correspond to the set of q-routes that strictly contains the set of valid CVRP routes. Since the resulting formulation has an exponential number of both columns and rows, this leads to column and cut generation for computing the lower bound and to a branch–and–cut–and–price algorithm for solving the CVRP. The computational results indicates that the new bounding procedure obtains lower bounds that are superior to those given by previous methods. However, this procedure is time consuming as the LP–relaxation of the master problem is usually highly degenerate and degeneracy implies alternative optimal dual solutions. Consequently, the generation of new columns and their associated variables may not change the value of the objective function of the master problem, the master problem may become large, and the overall method may become slow computationally. Moreover, in some CVRP instances, the increase in the lower bound with respect to the one achieved by the pure branch–and–cut is very small and is not worth the computing time required by the additional SP approach. The exact algorithm presented by Fukasawa et al. [21] decides at the root node, according to the best balance between running time and bound quality, either to use the branch–and–cut method of Lysgaard, Letchford, and Eglese [36] or the new proposed branch–and–cut-and-price strategy. The computational results reported by Fukasawa et al. [21] show that this algorithm is very consistent on solving instances from the literature with up to 135 customers.

In this paper we present an exact algorithm for solving the SP formulation of the CVRP that is based on the exact methods for the CVRP proposed by Christofides, Mingozzi and Toth [14], Mingozzi, Christofides and Hadjiconstantinou [38], Baldacci, Bodin and Mingozzi [6].

The method proposed avoids the drawbacks of the column and cut generation of Fukasawa et al. [21] as follows.

- We use an additive bounding procedure that combines three heuristics, called H^1, H^2 and H^3, to compute an effective dual solution of the LP–relaxation of the SP formulation improved by additional cuts. In H^1 and H^2 we add to the SP formulation capacity inequalities while in H^3 we add both capacity and clique inequalities. H^1 and H^2 are based on a relaxation of the SP formulation using variable splitting and relaxing in a Lagrangean fashion both partitioning and capacity constraints.
  H^1 is an extension of the bounding method proposed by Christofides, Mingozzi and Toth [14] and it is based on the q-route relaxation of the CVRP routes.
  H^2 is an iterative procedure that considers valid CVRP routes but uses column and cut generation.
  H^3 is a column and cut generation procedure that attempts to close the duality gap of the near optimal dual solution produced by H^1 and H^2. In order to stabilize the procedure, we impose lower and upper bounds on

the dual variables to restrict alternative optimal dual solutions of the master to those solutions having only few variables equal to zero.

- The final dual solution achieved is used to generate a reduced SP problem containing only the routes whose reduced cost is smaller than the gap between an upper bound and the lower bound achieved. Then, the resulting reduced problem is solved by an integer programming solver.

Our computational results over the main instances from the literature show that this algorithm is competitive with both the exact methods of Lysgaard, Letchford, and Eglese [36] and of Fukasawa et al. [21].

2. COMPUTATIONAL RESULTS

The experiments were performed on a Pentium 4 personal computer at 2.6 GHz equipped with 3 Gb of RAM and running under Windows XP.

We have considered five classes of test problems taken from the literature called A, B, E, M and P, respectively. Classes A, B and P have been produced by Augerat [3]. Problem class M has been produced by Christofides, Mingozzi and Toth [12] while class E has been generated by Christofides and Eilon [13].

In the following we denote by BCP the branch–and–cut–and–prize algorithm of Fukasawa et al. [21] and by ESP the exact method described in this paper.

Table 1, summarizes the results of BCP and ESP over all problem classes. Table 1 shows the following columns:

- $\#i$: number of instances belonging to the problem class;
- $\#o$: the total number of instances solved to optimality;
- $\#BC$: the total number of instances solved by BCP without using the column generation but using the branch–and–cut of Lysgaard, Letchford, and Eglese [36].
- LB_B : lower bound obtained at the root node of BCP;
- $\%LB_B$: percentage ratio of lower bound LB_B computed as 100.0 LB_B/z^*;
- t_{LB_B} : time in seconds of a Pentium 4 running at 2.4 Ghz for computing lower bound LB_B;
- $\%LB'$: percentage ratio of the new lower bound LB' proposed in this paper.
- $t_{LB'}$: time in seconds for computing LB';
- t_{ESP} : total computing time in seconds of the exact method ESP.

A comparison between lower bound LB_B and lower bound LB' shows that our lower bound LB' is superior to LB_B in all problem classes.

Since our Pentium 4 2.6 GHz is about 10% faster than the Pentium 4 2.4 GHz used by Fukasawa et al., tables 1, indicates that our bounding procedure is on average faster than the one of Fukasawa et al. on problem classes A, E, M and P but not for class B instances. Note that class B instances is composed of *clustered* instances which are particular difficult for column generation methods.

Tab. 1: All problem classes: BCP

		BCP				ESP			
$Class$	$\#i$	$\#o$	$\%LB_B$	t_{LB_B}	t_B	$\#o$	$\%LB'$	$t_{LB'}$	t_{ESP}
A	20	20	99.2	199.6	2155.1	20	99.8	43.6	128.3
B	6	6	99.4	163.3	15597.3	6	99.8	243.2	1251.5
E-M	10	7	98.7	1486.2	56788.7	6	99.3	262.2	1593.4
P	16	16	99.0	95.8	3312.8	16	99.7	13.2	116.1
	52	49	99.1	486.2	19463.5	48	97.9	140.6	772.3

Indeed, table 1 shows that 14 out 20 instances of class B were solved to optimality by Fukasawa et al. not using BCP but using the branch–and–cut of Lysgaard, Letchford, and Eglese [36].

The results concerning ESP and BCP can be analyzed as follows. ESP solved to optimality all instances of classes A and B but it cannot solve to optimality one instance of class E and two instances of class P because of memory limits of ESP.

On the instances solved to optimality by both methods, ESP is on average faster than BCP. In particular, both exact methods found the optimal solutions to instances E–n76–k10 and E–n101–k14, two of the most famous CVRP instances so far unsolved. The computing times in seconds for instances E–n76–k10 and E–n101–k14 of ESP were 174.4 and 1230.0, respectively, whereas BCP requires 80722.0 and 116284.0 seconds, respectively.

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