Uncertainty in Environments

Partially Observable and Stochastic environments have uncertainty. The agent must act, without perfect certainty.

The rational action depends on maximizing the expected utility.

MEU: Maximum Expected Utility

We use probability to calculate which action produces MEU.

Probability

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \dots\}$ Sample space

ω = possible world, member of Ω

Probability of ω.

$$0 \le P(\omega) \le 1$$

Must have some world

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

Proposition

$$\phi = \{\omega_i, \omega_j, \omega_k, \dots\} \subseteq \Omega$$

Proposition
subset of sample space

$$P(\phi) = \sum_{\omega \in \phi} P(\omega)$$

Probability

unconditional probability
prior probability
prior

degree of belief in a proposition in the abscence of any other information

evidence

information that has been revealed, that is relevant.

Conditional probability
posterior probability
posterior

probability of proposition given evidence.

posterior

$$\underbrace{P(a|b)}_{\text{posterior}} = \frac{P(a \wedge b)}{P(b)} \quad \begin{matrix} \leftarrow \text{prior} \\ \leftarrow \text{prior} \end{matrix}$$

probability of a, "given b and no other evidence"

product rule

$$P(a \wedge b) = P(a|b)\,P(b)$$

Probability Example 2 fair distinct 6-sided dice

$\Omega = \{ (1,1), (1,2), (1,3), \dots, (6,6) \}$ Sample space.

$P((3,2)) = 1/36$ why? prior or posterior?

ϕ = 11 is rolled = $\{(5,6), (6,5)\}$ proposition.

$P(\phi) = \sum_{\omega \in \phi} P(\phi) = 1/36 + 1/36 = 1/18$

$e = (d_1 = 5)$ evidence

$$P(\phi | e) = \frac{P(\phi \wedge e)}{P(e)} = \frac{1/36}{1/6} = 1/6$$ why?

Probability | Example | Your turn | 2 fair distinct 6-sided dice

ϕ = doubles = ?

$P(\phi) = ?$

$e = (d_1 = 2)$

$P(\phi | e) = ?$

Probabity Axioms

$P(\neg a) = 1 - P(a)$

$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$

1-die example

$P(d_1 = 3) = 1/6$

$P(d_1 \neq 3) = P(\neg\, d_1 = 3) = 1 - P(d_1 = 3) = 1 - 1/6 = 5/6$

$P(d_1 = 1 \vee d_1 = 2 \vee d_1 = 4 \vee d_1 = 5 \vee d_1 = 6) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 5/6$ ✓

2-die example

$P(d_1 = 2 \vee d_2 = 2) = P(d_1 = 2) + P(d_2 = 2) - P(d_1 = 2 \wedge d_2 = 2) = 1/6 + 1/6 - \frac{1}{36} = 11/36$

ϕ = at least one 2 was rolled = $\{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (1,2), (3,2), (4,2), (5,2), (6,2)\}$

$P(\phi) = 11/36$ ✓

Probability Example   Your Turn   2-distinct, fair, 6 sided dice

What is the probability of not rolling a sum of 10?

$P(T=10) = ?$

$P(\neg T=10) = ?$

What is the probability of not rolling a sum of 6, 7, or 8?

$P(T=6 \vee T=7 \vee T=8) = ?$

$P(\neg (T=6 \vee T=7 \vee T=8)) = ?$

Probability Example   Your Turn   2-distinct, fair, 6-sided dice

What is the probability of the first die rolling 5, or rolling a total of 9?

$P(d_1 = 5 \vee T = 9) = ?$

$P(d_1 \text{ is odd } \vee T \text{ is even}) = ?$

Probability Distributions

Tables of probabilities for all possible conditions.

Example: Random Variable X has 3 possible values : a, b, c, with probabilities : 0.25, 0.40, 0.35 respectively.

$P(X=a) = 0.25$, $P(X=b) = 0.40$, $P(X=c) = 0.35$

$\vec{P}(X) = \langle 0.25, 0.40, 0.35 \rangle$ probability distribution of X.

or

X_i	$P(X_i)$
a	0.25
b	0.40
c	0.35

Probability Distributions

For distributions with more random variables, the distribution is a large table.

Example: X has values a, b, c

Y has values d, e

$\vec{P}(X, Y)$ is a $|X| \times |Y|$ table $=$ $3 \times 2 = 6$ elements.

Y_j

X_i		d	e
	a	0.12	0.13
X_i	b	0.30	0.10
	c	0.05	0.30

$P(X_i, Y_j)$

$P(X = a \wedge Y = e) = ?$

$P(X = c \wedge Y = d) = ?$

$P(X = b) = ?$

$P(Y = e) = ?$

Probability Distributions   Cavity Example

Random Variables: Cavity, Toothache, Catch each has possible values of true or false.

The full joint probability distribution table:

	toothache		¬toothache	
	catch	¬catch	catch	¬catch
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

$$P(cavity) = \sum_{\omega:\text{ where Cavity is true}} P(\omega) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$$

$$P(toothache \lor cavity) = \sum_{\substack{\omega,\text{ where Toothache is true}\\ \text{or cavity is true}}} P(\omega) =$$

Probability Distributions

$$\vec{P}(Y) = \sum_{z \in Z} P(Y, z)$$

$$\vec{P}(Y) = \sum_{z \in Z} \vec{P}(Y \mid z) P(z)$$

Marginalization

Given joint distribution, $\vec{P}(Y, Z)$, we can find $\vec{P}(Y)$, using marginalization. (also called summing out.)

Conditioning

Variant of marginalization, using conditional probabilities.

Probability Distributions

Conditioning Example

We want the probability of a cavity, given a toothache.

$$P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \wedge \text{toothache})}{P(\text{toothache})}$$

using the product rule!

we can find the values from summing out on the joint distribution

$$P(\text{cavity} \mid \text{toothache}) = \frac{0.108 + 0.012}{0.108 + 0.012 + 0.016 + 0.064} = \frac{0.12}{0.20} = 0.6$$

Let's look at the opposite:

$$P(\neg\text{cavity} \mid \text{toothache}) = \frac{P(\neg\text{cavity} \wedge \text{toothache})}{P(\text{toothache})} = \frac{0.016 + 0.064}{0.2} = 0.4$$

Hey, they add to 1 !

Probability Distribations

Your Turn

$P(cavity \mid catch) = ?$

$P(\neg cavity \mid catch) = ?$

Probability Distributions   Normalization

Notice:

$$P(\text{cavity} \mid \text{toothache}) = \frac{P(\text{cavity} \wedge \text{toothache})}{P(\text{toothache})} = \alpha\, P(\text{cavity} \wedge \text{toothache})$$

$$P(\neg\text{cavity} \mid \text{toothache}) = \frac{P(\neg\text{cavity} \wedge \text{toothache})}{P(\text{toothache})} = \alpha\, P(\neg\text{cavity} \wedge \text{toothache})$$

if $\alpha = \frac{1}{P(\text{toothache})}$. Also, $P(\text{cavity} \mid \text{toothache}) + P(\neg\text{cavity} \mid \text{toothache}) = 1$.

$$\alpha \left[P(\text{cavity} \wedge \text{toothache}) + P(\neg\text{cavity} \wedge \text{toothache}) \right] = 1$$

$$\alpha = \frac{1}{P(\text{cavity} \wedge \text{toothache}) + P(\neg\text{cavity} \wedge \text{toothache})}$$

We can normalize useng α, without knowing $P(\text{toothache})$!

Independence in Probability

If two random variables are independent, then:

$P(a|b) = P(a)$, $P(b|a) = P(b)$, $P(a \wedge b) = P(a)\,P(b)$

Example: We don't think the weather today and the health of the patient's teeth are related.

$$P(cloudy \mid cavity) = P(cloudy)$$

$$P(cavity \mid cloudy) = P(cavity)$$

$$P(cloudy \wedge cavity) = P(cloudy)\,P(cavity)$$

Bayes' Rule

$$P(b|a) = \frac{P(a|b)\,P(b)}{P(a)}$$

easily derived from the product rule.

this is often useful if you think of it like this:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})\,P(\text{cause})}{P(\text{effect})}$$

For example:

$$P(\text{cavity}|\text{toothache}) = \frac{P(\text{toothache}|\text{cavity})\,P(\text{cavity})}{P(\text{toothache})}$$

Bayes' Rule

Example: Stiff necks and meningitis

s = stiff neck, m = meningitis

$P(s|m) = 0.7$ from studies of meningitis patients

$P(m) = \frac{1}{50{,}000}$ from large population studies

$P(s) = 0.01$ from studies of patient complaints.

A patient complains of a stiff neck. What is the probability they have meningitis?

$P(m|s) =$

Bayes' Rule   Multiple Evidences   Conditional Independence

$$P(\text{cause} \mid \text{effect}_1 \wedge \text{effect}_2) = \alpha\, P(\text{effect}_1 \mid \text{cause})\, P(\text{effect}_2 \mid \text{cause}) \cdot P(\text{cause})$$

This uses conditional independence: we claim that the probability of any effect_i is independent of any other effect_j, if we know cause or $\neg$cause.

This is true in many real world cases, and close to true in many more.

Conditional Independence Example

$P(cavity \mid toothache \wedge catch) = ?$

$P(\neg cavity \mid toothache \wedge catch) = ?$