Problems from 2024

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Abstract

Welcome to my 2024 problem set, a collection of math problems that I encountered and found interesting throughout the year. I hope you find them challenging and enjoy solving these as much as I did!

4 April

4.1 Problem 1

This problem has come from the 2023 GRE Mathematics Test Practice Book available from ETS.

In the complex plane, let C be the circle $\{z = 1 + e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, oriented counterclockwise. What is the value of $\frac{1}{2\pi i} \int_C \left(\frac{\sin z}{z-1}\right)^2 dz$?

Solution

Let $f(z) = \left(\frac{\sin z}{z-1}\right)^2$, which is a holomorphic function with a singularity at $z = 1$. The contour integral over the circle can be calculated by leveraging the *residue theorem* which in our case for one singularity states

$$\int_C f(z)\, dz = 2\pi i\, I(C, 1)\ \mathrm{Res}(f, 1) = 2\pi i\ \mathrm{Res}(f, 1) \tag{1}$$

$I(C, 1)$ is the winding number at $z = 1$, which is equal to 1 since the curve C winds around the point counterclockwise a single time time. $\mathrm{Res}(f, 1)$ is the residue of f at $z = 1$ which is found by examining the coefficients of the laurent series expansion of $f(z)$ at $z = 1$:

$$f(z) = \frac{\sin^2(1)}{(z-1)^2} + \frac{\sin(2)}{z-1} + \cos(2) + \frac{2}{3}(z-1)\sin(2) + \dots \tag{2}$$

We find $\mathrm{Res}(f, 1) = \sin(2)$, the coefficient of the $\frac{1}{z-1}$ term. Then the integral from the problem can be evaluated

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{2\pi i}{2\pi i}\mathrm{Res}(f,1) = \mathrm{Res}(f,1) = \sin(2) \approx 0.708. \tag{3}$$