

Problems from 2024 (Draft)

May 9, 2024

Please enjoy this collection of math problems that I encountered and found interesting throughout the year.

4 April

4.1 Problem Statement

This problem comes from the 2023 GRE Mathematics Test Practice Book available from ETS.

In the complex plane, let C be the circle $\{z = 1 + e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, oriented counterclockwise. What is the value of $\frac{1}{2\pi i} \int_C \left(\frac{\sin z}{z-1}\right)^2 dz$?

Solution

Let $f(z) = \left(\frac{\sin z}{z-1}\right)^2$ which is a holomorphic function with a *pole* (or *non-essential singularity*) at $z = 1$ with order 2. The contour integral over the circle can be calculated by leveraging the *residue theorem* which in our case for one pole states

$$\int_C f(z) dz = 2\pi i I(C, 1) \operatorname{Res}(f, 1) = 2\pi i \operatorname{Res}(f, 1). \quad (1)$$

$I(C, 1)$ is the winding number at $z = 1$, which is equal to 1 since the curve C winds around the point counterclockwise a single time. $\operatorname{Res}(f, 1)$ is the residue of f at $z = 1$ which can be found via the *limit formula for higher-order poles* with $n = 2$ to match the degree of the pole

$$\operatorname{Res}(f, z_0 = 1) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n \cdot f(z)] \quad (2)$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 \cdot \frac{\sin^2 z}{(z-1)^2} \right] \quad (3)$$

$$= \lim_{z \rightarrow 1} 2 \sin z \cos z \quad (4)$$

$$= 2 \sin 1 \cos 1 \quad (5)$$

$$= \sin 2. \quad (6)$$

Now the integral from the problem can be evaluated

$$\frac{1}{2\pi i} \int_C f(z) dz = \frac{2\pi i}{2\pi i} \operatorname{Res}(f, 1) = \operatorname{Res}(f, 1) = \sin(2) \approx 0.708. \quad (7)$$

4.2 Problem Statement

This problem comes from chapter 3, exercise 45 in Contemporary Abstract Algebra, 7th Edition by Joseph A. Gallian.

Let H be a subgroup of a finite group G . Suppose that g belongs to G and n is the smallest positive integer such that $g^n \in H$. Prove that n divides $|g|$.

Proof

Let $m = |g|$. For the sake of contradiction, suppose $n \nmid m$ so that $m = kn + r$ for some $k \in \mathbb{Z}^{\geq 0}$ and $r \in \mathbb{Z}$ such that $0 < r < m$. Then, for any $j \in \mathbb{Z}$

$$m = kn + r \quad (8)$$

$$= kn + r + jn - jn \quad (9)$$

$$= (k+j)n + r - jn \quad (10)$$

so

$$e = g^m = g^{(k+j)n+r-jn} \quad (11)$$

$$= g^{(k+j)n} g^{r-jn} \quad (12)$$

so

$$g^{-(k+j)n} = g^{r-jn} \in H \quad (13)$$

Choose j to be the greatest integer such that $0 \leq r - jn < n$. If $r - jn = 0$ then $m = (k + j)n$, but this can't be the case since $n \nmid m$, so $0 < r - jn < n$. However, n is the least positive integer such that $g^n \in H$ so we have a contradiction. Therefore $n|m$, so n divides $|g|$.

5 May

5.1 Problem Statement

Find the area enclosed by an astroid, a curve given by the following parametric equations:

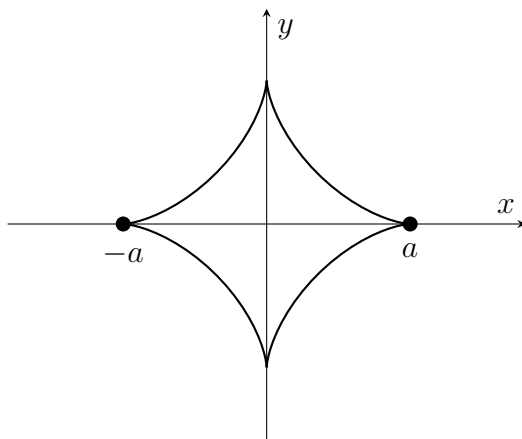
$$x = a \cos^3 \theta \quad (14)$$

$$y = a \sin^3 \theta \quad (15)$$

for some constant a .

Solution

The curve for an astroid looks like this:



Notice that it is symmetric about the x -axis so we can find the area of the top half and then double it. The total area can be found by way of the following integral

$$A = 2 \int_{-a}^a y \, dx = 2 \int_{\pi}^0 y \frac{dx}{d\theta} \, d\theta \quad (16)$$

since $x = a$ when $\theta = 0$ and $x = -a$ when $\theta = \pi$ and where

$$\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta. \quad (17)$$

Then

$$A = 2 \int_{\pi}^0 a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) \, d\theta \quad (18)$$

$$= -6a^2 \int_{\pi}^0 \sin^4 \theta \cos^2 \theta \, d\theta \quad (19)$$

$$= 6a^2 \int_0^{\pi} \sin^2 \theta (\sin \theta \cos \theta)^2 \, d\theta \quad (20)$$

$$= 6a^2 \int_0^{\pi} \frac{1 - \cos 2\theta}{2} \left(\frac{1}{2} \sin 2\theta \right)^2 \, d\theta \quad (21)$$

$$= \frac{6a^2}{8} \int_0^{\pi} (1 - \cos 2\theta) \sin^2 2\theta \, d\theta \quad (22)$$

$$= \frac{6a^2}{8} \int_0^{\pi} \sin^2 2\theta - \sin^2 2\theta \cos 2\theta \, d\theta \quad (23)$$

$$= \frac{6a^2}{8} \int_0^{\pi} \frac{1 - \cos 4\theta}{2} - \sin^2 2\theta \cos 2\theta \, d\theta \quad (24)$$

$$= \frac{6a^2}{8} \left[\int_0^{\pi} \frac{1}{2} - \frac{1}{2} \cos 4\theta - \sin^2 2\theta \cos 2\theta \, d\theta \right] \quad (25)$$

$$= \frac{6a^2}{8} \left[\frac{1}{2} \theta - \frac{1}{8} \sin 4\theta - \frac{1}{6} \sin^3 2\theta \right]_0^{\pi} \quad (26)$$

$$= \frac{6a^2}{8} \left[\frac{1}{2} \pi - 0 \right] \quad (27)$$

$$= \frac{3\pi a^2}{8}. \quad (28)$$