Problems from 2024 (Draft)

April 25, 2024

Please enjoy this collection of math problems that I encountered and found interesting throughout the year.

4 April

4.1 Problem Statement

This problem comes from the 2023 GRE Mathematics Test Practice Book available from ETS.

In the complex plane, let C be the circle $\{z = 1 + e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, oriented counterclockwise. What is the value of $\frac{1}{2\pi i}\int_C \left(\frac{\sin z}{z-1}\right)^2 dz$?

Solution

Let $f(z) = \left(\frac{\sin z}{z-1}\right)^2$ which is a holomorphic function with a *pole* (or *non-essential singularity*) at $z = 1$ with order 2. The contour integral over the circle can be calculated by leveraging the *residue theorem* which in our case for one pole states

$$\int_C f(z)\, dz = 2\pi i\, I(C, 1)\ \mathrm{Res}(f, 1) = 2\pi i\ \mathrm{Res}(f, 1). \tag{1}$$

$I(C, 1)$ is the winding number at $z = 1$, which is equal to 1 since the curve C winds around the point counterclockwise a single time. $\mathrm{Res}(f, 1)$ is the residue of f at $z = 1$ which can be found via the *limit formula for higher-order poles* with $n = 2$ to match the degree of the pole

$$\operatorname{Res}(f, z_0 = 1) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n \cdot f(z) \right] \tag{2}$$
$$= \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 \cdot \frac{\sin^2 z}{(z-1)^2} \right] \tag{3}$$
$$= \lim_{z \to 1} 2 \sin z \cos z \tag{4}$$
$$= 2 \sin 1 \cos 1 \tag{5}$$
$$= \sin 2. \tag{6}$$

Now the integral from the problem can be evaluated

$$\frac{1}{2\pi i} \int_C f(z)\, dz = \frac{2\pi i}{2\pi i} \operatorname{Res}(f, 1) = \operatorname{Res}(f, 1) = \sin(2) \approx 0.708. \tag{7}$$

4.2 Problem Statement

This problem comes from chapter 3, exercise 45 in Contemporary Abstract Algebra, 7th Edition by Joseph A. Gallian.

Let H be a subgroup of a finite group G. Suppose that g belongs to G and n is the smallest positive integer such that $g^n \in H$. Prove that n divides $|g|$.

Proof

Let $m = |g|$. For the sake of contradiction, suppose $n \nmid m$ so that $m = kn + r$ for some $k \in \mathbb{Z}^{\geq 0}$ and $r \in \mathbb{Z}$ such that $0 < r < m$. Then, for any $j \in \mathbb{Z}$

$$m = kn + r \tag{8}$$
$$= kn + r + jn - jn \tag{9}$$
$$= (k + j)n + r - jn \tag{10}$$

so

$$e = g^m = g^{(k+j)n + r - jn} \tag{11}$$
$$= g^{(k+j)n} g^{r - jn} \tag{12}$$

so

$$g^{-(k+j)n} = g^{r-jn} \in H \tag{13}$$

Choose j to be the greatest integer such that $0 \leq r - jn < n$. If $r - jn = 0$ then $m = (k+j)n$, but this can't be the case since $n \nmid m$, so $0 < r - jn < n$. However, n is the least positive integer such that $g^n \in H$ so we have a contradiction. Therefore $n|m$, so n divides $|g|$.