Problems from 2024 (Draft)

April 6, 2024

Please enjoy this collection of math problems that I encountered and found interesting throughout the year.

4 April

4.1 Problem 1

This problem comes from the 2023 GRE Mathematics Test Practice Book available from ETS.

In the complex plane, let C be the circle $\{z = 1 + e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, oriented counterclockwise. What is the value of $\frac{1}{2\pi i} \int_C \left(\frac{\sin z}{z-1}\right)^2 \, dz$?

Solution

Let $f(z) = \left(\frac{\sin z}{z-1}\right)^2$ which is a holomorphic function with a *pole* (or *non-essential singularity*) at $z = 1$ with order 2. The contour integral over the circle can be calculated by leveraging the *residue theorem* which in our case for one pole states

$$\int_C f(z)\, dz = 2\pi i \, I(C, 1) \; \mathrm{Res}(f, 1) = 2\pi i \; \mathrm{Res}(f, 1). \tag{1}$$

$I(C, 1)$ is the winding number at $z = 1$, which is equal to 1 since the curve C winds around the point counterclockwise a single time. $\mathrm{Res}(f, 1)$ is the residue of f at $z = 1$ which can be found via the *limit formula for higher-order poles* with $n = 2$ to match the degree of the pole

$$
\begin{aligned}
\operatorname{Res}(f, z_0 = 1) &= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n \cdot f(z) \right] && (2) \\
&= \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 \cdot \frac{\sin^2 z}{(z-1)^2} \right] && (3) \\
&= \lim_{z \to 1} 2 \sin z \cos z && (4) \\
&= 2 \sin 1 \cos 1 && (5) \\
&= \sin 2. && (6)
\end{aligned}
$$

Now the integral from the problem can be evaluated

$$
\frac{1}{2\pi i} \int_C f(z)\, dz = \frac{2\pi i}{2\pi i} \operatorname{Res}(f, 1) = \operatorname{Res}(f, 1) = \sin(2) \approx 0.708. \qquad (7)
$$