Problems from 2024 (Draft)

April 5, 2024

Please enjoy this collection of math problems that I encountered and found interesting throughout the year.

4 April

4.1 Problem 1

This problem comes from the 2023 GRE Mathematics Test Practice Book available from ETS.

In the complex plane, let C be the circle $\{z = 1 + e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, oriented counterclockwise. What is the value of $\frac{1}{2\pi i} \int_C \left(\frac{\sin z}{z-1}\right)^2 \, dz$?

Solution

Let $f(z) = \left(\frac{\sin z}{z-1}\right)^2$ which is a holomorphic function with a non-essential singularity (or pole) at $z = 1$ with order 2. The contour integral over the circle can be calculated by leveraging the *residue theorem* which in our case for one singularity states

$$\int_C f(z)\, dz = 2\pi i\, I(C, 1)\ \mathrm{Res}(f, 1) = 2\pi i\ \mathrm{Res}(f, 1). \tag{1}$$

$I(C, 1)$ is the winding number at $z = 1$, which is equal to 1 since the curve C winds around the point counterclockwise a single time. $\mathrm{Res}(f, 1)$ is the residue of f at $z = 1$ which can be found via the following limit approaching the pole

$$
\begin{aligned}
\operatorname{Res}(f,1) &= \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \cdot \frac{sin^2 z}{(z-1)^2}\right] && (2)\\
&= \lim_{z\to 1} 2\sin z \cos z && (3)\\
&= 2\sin 1 \cos 1 && (4)\\
&= \sin 2. && (5)
\end{aligned}
$$

Now the integral from the problem can be evaluated

$$
\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{2\pi i}{2\pi i}\operatorname{Res}(f,1) = \operatorname{Res}(f,1) = \sin(2) \approx 0.708. \qquad (6)
$$