

# Problems from 2024

April 5, 2024

Please enjoy this collection of math problems that I encountered and found interesting throughout the year.

## 4 April

### 4.1 Problem 1

This problem comes from the 2023 GRE Mathematics Test Practice Book available from ETS.

In the complex plane, let  $C$  be the circle  $\{z = 1 + e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ , oriented counterclockwise. What is the value of  $\frac{1}{2\pi i} \int_C \left(\frac{\sin z}{z-1}\right)^2 dz$ ?

#### Solution

Let  $f(z) = \left(\frac{\sin z}{z-1}\right)^2$  which is a holomorphic function with a non-essential singularity (or pole) at  $z = 1$  with order 2. The contour integral over the circle can be calculated by leveraging the *residue theorem* which in our case for one singularity states

$$\int_C f(z) dz = 2\pi i I(C, 1) \operatorname{Res}(f, 1) = 2\pi i \operatorname{Res}(f, 1). \quad (1)$$

$I(C, 1)$  is the winding number at  $z = 1$ , which is equal to 1 since the curve  $C$  winds around the point counterclockwise a single time.  $\operatorname{Res}(f, 1)$  is the residue of  $f$  at  $z = 1$  which can be found via the following limit approaching the pole

$$\operatorname{Res}(f, 1) = \lim_{z \rightarrow 1} \frac{d}{dz} \left[ (z - 1)^2 \cdot \frac{\sin^2 z}{(z - 1)^2} \right] \quad (2)$$

$$= \lim_{z \rightarrow 1} 2 \sin z \cos z \quad (3)$$

$$= 2 \sin 1 \cos 1 \quad (4)$$

$$= \sin 2. \quad (5)$$

Now the integral from the problem can be evaluated

$$\frac{1}{2\pi i} \int_C f(z) dz = \frac{2\pi i}{2\pi i} \operatorname{Res}(f, 1) = \operatorname{Res}(f, 1) = \sin(2) \approx 0.708. \quad (6)$$