CP 9 *Chris Achenbach*

Find the Laplace transform of $\sin(\beta t)$,

$$\begin{aligned}\mathcal{L}\{\sin(\beta t)\} &= \int_0^\infty e^{-st}\sin(\beta t)\,dt \\ &= \lim_{N\to\infty}\int_0^N e^{-st}\sin(\beta t)\,dt \\ &= \lim_{N\to\infty}\Big[F\Big]_0^N\end{aligned}$$

First let's find the integral $\int e^{-st}\sin(\beta t)$ so that we will be able to solve the rest of the problem,

$$\text{Let} \quad F = \int e^{-st}\sin(\beta t)$$

Let's use *Integration by Parts* where

$$\begin{aligned} u &= e^{-st} & dv &= \sin(\beta t) \\ du &= (-s)e^{-st} & v &= \left(-\tfrac{1}{\beta}\right)\cos(\beta t)\end{aligned}$$

so the integral F now looks like

$$\begin{aligned} F &= \int u\,dv \\ &= uv - \int v\ du \\ &= e^{-st}\left(-\tfrac{1}{\beta}\right)\cos(\beta t) - \int \left(-\tfrac{1}{\beta}\right)\cos(\beta t)(-s)e^{-st}\ dt \\ &= \left(-\tfrac{1}{\beta}\right)e^{-st}\cos(\beta t) - \left(\tfrac{s}{\beta}\right)\int e^{-st}\cos(\beta t)\ dt \\ &= \left(-\tfrac{1}{\beta}\right)e^{-st}\cos(\beta t) - \left(\tfrac{s}{\beta}\right)G\end{aligned}$$

Well now we have got another integral to deal with,

$$\text{Let} \quad G = \int e^{-st}\cos(\beta t)$$

`THINGS WENT BAD AFTER THIS POINT... LOL`

And we want to do integration by parts again, but switch it up a bit,

$$\begin{aligned} u &= \cos(\beta t) & dv &= e^{-st} \\ du &= -\beta\sin(\beta t) & v &= \left(-\tfrac{1}{s}\right)e^{-st}\end{aligned}$$

so the integral G now looks like,

$$
\begin{aligned}
G &= \int u \; dv \\
&= uv - \int v \; du \\
&= \cos(\beta t)\left(-\tfrac{1}{s}\right)e^{-st} - \int \left(\tfrac{1}{s}\right)e^{-st}\beta\sin(\beta t) \\
&= \left(-\tfrac{1}{s}\right)e^{-st}\cos(\beta t) + \left(\tfrac{\beta}{s}\right)\int e^{-st}\sin(\beta t)
\end{aligned}
$$

Now what's interesting is that we have ended up with a term that is identical to F from our earlier integral, so we can substitute that,

$$
G = \left(-\tfrac{1}{s}\right)e^{-st}\cos(\beta t) + \left(\tfrac{\beta}{s}\right)F
$$

And now we can go back to our definition of F and plug in our newly discovered expression for G and simplify,

$$
\begin{aligned}
F &= \left(-\tfrac{1}{\beta}\right)e^{-st}\cos(\beta t) - \left(\tfrac{s}{\beta}\right)G \\
&= \left(-\tfrac{1}{\beta}\right)e^{-st}\cos(\beta t) - \left(\tfrac{s}{\beta}\right)\Big(\left(-\tfrac{1}{s}\right)e^{-st}\cos(\beta t) + \left(\tfrac{\beta}{s}\right)F\Big) \\
&= \left(-\tfrac{1}{\beta}\right)e^{-st}\cos(\beta t) + \left(\tfrac{1}{\beta}\right)e^{-st}\cos(\beta t) - F \\
2F &= \left(\tfrac{1}{\beta}\right)\Big(-e^{-st}\cos(\beta t) + e^{-st}\cos(\beta t)\Big)
\end{aligned}
$$