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Example 22: Solve the initial value problem using Laplace transforms:

$$p'' - 2p' + p = 6t - 2, \quad p(-1) = 3, \quad p'(-1) = 7$$

Step 1: Shift Initial Value

We are given values of p and p' when $t = -1$, but that isn't very helpful for solving second-order differential equations. It's much nicer when we have definitions at $t = 0$. This can be achieved by making a substitution.

$$\text{Let } y(t) = p(t-1)$$

By doing this, we can now redefine our initial values,

$$y(0) = 3, \quad y'(0) = 7$$

and consequently we will have to redefine the differential equation by replacing t with $t - 1$,

$$p''(t-1) - 2p'(t-1) + p(t-1) = 6(t-1) - 2$$

Which now leaves us with the differential equation and initial values that we will use our Lagrange Transform techniques on,

$$y'' - 2y' + y = 6t - 8, \quad y(0) = 3, \quad y'(0) = 7 \tag{1}$$

Step 2: Laplace transform of each side

For this part we will make a shorthand that will help us solve things,

$$\text{Let } Y = \mathcal{L}\{y\}, \quad Y_0 = y(0) = 3, \quad Y_0' = y'(0) = 7$$

Now using our differential equation (1), take the Laplace Transform on each side,

$$\begin{aligned}
\mathcal{L}\{y'' - 2y' + y\} &= \mathcal{L}\{6t - 8\} \\
\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} &= 6\mathcal{L}\{t\} - 8\mathcal{L}\{1\} \\
(s^2Y - sY_0 - Y_0') - 2(sY - Y_0) + (Y) &= 6\left(\frac{1}{s^2}\right) - 8\left(\frac{1}{s}\right)
\end{aligned}$$

Now we can plug in our initial values, and rearrange things until we have Y alone on one side,

$$\begin{aligned}
s^2Y - 3s - 7 - 2sY - 6 + Y &= \frac{6}{s^2} - \frac{8}{s} \\
(s^2 - 2s + 1)Y &= \frac{6}{s^2} - \frac{8}{s} + 3s + 13 \\
Y &= \left(\frac{6}{s^2} - \frac{8}{s} + 3s + 13\right)(s^2 - 2s + 1)
\end{aligned}$$

Notice here that the term $(s^2 - 2s + 1)$ can be factored into a squared polynomial $(s-1)^2$, and so we can expand Y to look like this,

$$Y = \frac{6}{s^2(s-1)^2} - \frac{8}{s(s-1)^2} + \frac{3s}{(s-1)^2} + \frac{13}{(s-1)^2} \tag{2}$$

Now that we have Y we *want* to just take the Inverse Laplace, because that would spit out our final answer: $\mathcal{L}^{-1}\{Y\} = y$, but we can't do that easily yet because of the nasty polynomials.

Step 3: Partial Fraction Decomposition

Take equation (2) and add the fractions together so that they form one big fraction.

$$Y = \frac{3s^3 + 13s^2 - 8s + 6}{s^2(s-1)^2} \tag{3}$$

Now we can do our Partial Fraction Decomposition with 4 unknowns, because the order of the polynomial on the bottom of the fraction is 4.

$$Y = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2} \tag{4}$$

Adding these fractions together into one big fraction gives us,

$$Y = \frac{As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2}{s^2(s-1)^2} \tag{5}$$

By comparing equations (3) and (5), we can create an equality that can be used to find the unknowns. I personally like to take a shortcut at this point by using a vector to represent a polynomial.

$$A\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + B\begin{bmatrix}0\\1\\-2\\1\end{bmatrix} + C\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + D\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\13\\-8\\6\end{bmatrix} \tag{6}$$

Step 4: Solve the System of Unknowns

We can take equation (6) and put this into an augmented matrix to help us solve for all of the unknowns at the same time. Then we can *Row Reduce* until we reach *Echelon Form*, giving us the solutions we want.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ -2 & 1 & 1 & 1 & 13 \\ 1 & -2 & 0 & 0 & -8 \\ 0 & 1 & 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 16 \end{bmatrix}$$

Which means our unknowns are,

$$\begin{aligned} A &= 4 \\ B &= 6 \\ C &- 1 \\ D &= 16 \end{aligned}$$

And now we can plug these unknowns into equation (4) to get a useful representation of Y,

$$Y = \frac{4}{s} + \frac{6}{s^2} + \frac{-1}{(s-1)} + \frac{16}{(s-1)^2} \tag{7}$$

Step 5: Inverse Laplace

Now we have a solid form for Y, we can take the Inverse Laplace Transform of equation (7), giving us the solution for y.

$$y = 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 6\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 1\,\mathcal{L}^{1}\left\{\frac{1}{(s-1)}\right\} + 16\,\mathcal{L}^{1}\left\{\frac{1}{(s-1)^2}\right\}$$

Which we can check our table of Laplace Transforms and convert each of them into an actual function, giving us,

$$y = 4 + 6t - e^t + 16te^t \tag{8}$$

Step 6: Shift Back the Initial Values

Now the final thing we do is shift back our initial values that we shifted in the very first step. Remember how we replaced t with $t-1$? Now we are going to do the opposite of that by replacing t with $t+1$.

$$p(t) = y(t+1)$$

So here we go, replacing the t values in equation (8) gives us,

$$p = 4 + 6(t+1) - e^{t+1} + 16(t+1)e^{t+1}$$