NATURAL LOG

$e^{\ln(x)} = x$
$\ln(e^x) = x$
$\ln(x \cdot y) = \ln(x) + \ln(y)$
$\ln(x^y) = y \cdot \ln(x)$
$\ln(\frac{x}{y}) = \ln(x) - \ln(y)$
$\ln(\frac{1}{x}) = -\ln(x)$
$\ln(1) = 0$
$\ln(0) = \text{undefined}$

TRIGONOMETRY

$\tan x = \sin x / \cos x$
$\cot x = 1/\tan x$
$\sec x = 1/\cos x$
$\csc x = 1/\sin x$
$\sin^2 x + \cos^2 x = 1$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\cos(A-B) = \cos A cos B + \sin A \sin B$
$\sin 2x = 2 \sin x \cos x$

DERIVATIVES

Product: $(fg)' = fg' + f'g$
Quotient: $(f/g)' = (f'g - g'f)/g^2$
Reciprocal: $(1/f) = -f'/f^2$
Chain: $f(g)' = f'(g) \cdot g'$

INTEGRALS

$\int u\,dv = uv - \int v\,du$
$\int \frac{1}{x} = \ln|x|$
$\int \ln ax = x \ln ax - x$
$\int \tan ax = -\ln|\cos ax|/a$
$\int \cot ax = \ln|\sin ax|/a$
$\int \sec x = \ln|\sec x + \tan x|$
$\int \sec^2 x = \tan x$
$\int \csc x = -\ln|\csc x + \cot x|$
$\int \csc^2 x = -\cot x$
$\int \frac{1}{1+x^2} = \arctan x$
$\int \frac{1}{\sqrt{1-x^2}} = \arcsin x$

LINEAR EQUATION : $\frac{dy}{dx} + P(x)y = Q(x)$

`Step1` :: $\mu = e^{(\int P\,dx)}$
`Step2` :: $y = \frac{1}{\mu}(\int \mu Q\,dx + C)$

EXACT EQUATIONS : $M\,dx + N\,dy = 0$

`Exact?` :: $\partial_y M = \partial_x N$
`MyTrick` :: $F = \int M\,dx + C(y) = \int N\,dy + C(x)$
`Solution` :: $F(x,y) = C$

SPECIAL INTEGRATING FACTOR

`Step1` :: $\begin{cases} A(x) = \left(\dfrac{\partial_y M - \partial_x N}{N}\right) \\ B(y) = \left(\dfrac{\partial_x N - \partial_y M}{M}\right) \end{cases}$

`Step2` :: $Q = \begin{cases} A(x) \text{ if it only depends on } x \\ B(y) \text{ if it only depends on } y \end{cases}$
`Step3` :: $\mu = \exp(\int Q)$
`ExactEq` :: $\mu\,M\,dx + \mu\,N\,dy = 0$

HOMOGENEOUS EQUATIONS $\frac{dy}{dx} = G\left(\frac{y}{x}\right)$

`test` :: $f(tx, ty) = f(x,y)$
`step1` :: Let $v = y/x$,
`step2` :: $\frac{dy}{dx} = v + x\frac{dv}{dx}$
`step3` :: Solve separable.

SUBSTITUTION $\frac{dy}{dx} = G(ax+by)$

`Step1` :: Let $z = ax + by$
`Step2` :: $\frac{dz}{dx} = a + b\frac{dy}{dx}$
`Step3` :: $\frac{dy}{dx} = (\frac{dz}{dx} - a)/b$
`Step4` :: Substitute $\frac{dy}{dx}$ into original.
`Step5` :: Solve separable.

BERNOULLI EQUATIONS

$\frac{dy}{dx} = P(x)y = Q(x)y^n$
`Req` :: $n \neq 0, 1$
`Step1` :: Let $v = y^{(1-n)}$
`Step2` :: $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$,
`Step3` :: Solve Linear Eq with v and x.

LINEAR COEFFICIENTS

$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$
`Req` :: $a_1b_2 \neq a_2b_1$
`Step1` :: Let $x = u + h$
`Step2` :: Let $y = v + k$
`Step3` :: Solve $\begin{Bmatrix} a_1\boldsymbol{h} + b_1\boldsymbol{k} + c_1 = 0 \\ a_2\boldsymbol{h} + b_2\boldsymbol{k} + c_2 = 0 \end{Bmatrix}$
`Step4` :: Solve homogeneous eq with u and v.

EULERS METHOD

$x_{n+1} = x_n + h$
$y_{n+1} = y_n + hf(x_n, y_n)$

IMPROVED_EULER(x_0, y_0, c, N) :

$x = x_0$
$y = y_0$
$h = (c - x_0)/N$
`FOR` i `FROM 0 TO` N :
  $F = f(x,y)$
  $G = f((x+h),\ (y + h * F))$
  $x = x + h$
  $y = y + h(F+G)/2$

NEWTONS MOTION

$x(t) = \frac{mg}{b}t + \frac{m}{b}\left(v_0 - \frac{mg}{b}\right)\left(1 - e^{-bt/m}\right)$

NEWTONS LAW OF COOLING

$\frac{dT}{dt} = K[M(t) - T(t)] + H(t) + U(t)$
$M(t)$: outside temp , $T(t)$: inside temp
$H(t)$: additional heating , $U(t)$: furnace rate

SPRING-MASS OSCILLATOR
$my'' + by' + ky = F_{\text{ext}}(t)$
$y(t) = A\cos(\Omega t) + B\sin(\Omega t)$

TWO REAL ROOTS:
$y_h = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}$
$y_h' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$

REPEATED ROOTS
$y_h = c_1 \cdot e^{rt} + c_2 \cdot t \cdot e^{rt}$
$y_h' = (c_1 r + c_2)e^{rt} + (c_2 r)te^{rt}$

COMPLEX ROOTS:
$y_h = c_1 \cdot e^{\alpha t} \cdot \sin(\beta t) + c_2 \cdot e^{\alpha t} \cdot \cos(\beta t)$
$y_h' = (c_1\alpha - c_2\beta)ES + (c_1\beta + c_2\alpha)EC$

UNDETERMINED COEFFICIENTS
CASE: $f == Ct^m e^{rt}$
SOLU: $y_p = t^s P^m e^{rt}$
$s = 0$ if r is not a root of the equation.
$s = 1$ if r is a simple root.
$s = 2$ if r is a double root.

CASE: $f = Ct^m e^{\alpha t}\left\{\begin{smallmatrix}\sin\beta t\\ \cos\beta t\end{smallmatrix}\right\}$
SOLU: $y_p = t^s P_1^m e^{\alpha t}\cos(\beta t) + t^s P_2^m e^{\alpha t}\sin(\beta t)$
$s = 0$ if $\alpha + i\beta$ is not a root.
$s = 1$ if $\alpha + i\beta$ is a root.

SUPERPOSITION
LET: $F = ay'' + by' + cy$
IF: y_1 is a solution to $F = f_1$
AND: y_2 is a solution to $F = f_2$
THEN: $k_1 y_1 + k_2 y_2$ is a solution to $F = k_1 f_1 + k_2 f_2$

VARIATION OF PARAMETERS
`Step 1` :: Find y_h, Let $\{y_1, y_2\} = y_h$
`Step 2` :: $W[y_1, y_2] = y_1 y_2' - y_1' y_2$
`Step 3` :: $\left[\begin{matrix} v_1 = \displaystyle\int \frac{-fy_2}{aW[y_1,y_2]} \\ v_2 = \displaystyle\int \frac{+fy_1}{aW[y_1,y_2]} \end{matrix}\right]$
`Step 4` :: $y_p = v_1 y_1 + v_2 y_2$
`Step 5` :: $y = y_h + y_p$

CAUCHY-EULER (Oily Couch)
$at^2 y'' + bty' + cy = f$
`CHAR` :: $ar^2 + (b-a)r + c = 0$
`REAL` :: $y_h = t^{r_1} + t^{r_2}$
`DOUB` :: $y_h = t^r + t^r \ln(t)$
`PLEX` :: $y_h = t^\alpha \sin(\beta \ln t) + t^\alpha \cos(\beta \ln t)$
`NOTE` :: If solving for $t < 0$, Set $t \leftarrow (-t)$

REDUCTION OF ORDER
$y'' + Py' + Qy = 0$; given y_1
$y_2 = y_1 \displaystyle\int \frac{e^{-\int P}}{(y_1)^2}$
$y_h = y_1 + y_2$

LINEAR INDEPENDENCE TEST
For whole interval: $W[y_1, y_2] = 0$

PHASE PLANE
$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y)$
Trajectory:
$x(t), y(t)$ that solve the system.
Related Phase Plane Eq:
$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{dy/dt}{dx/dt}$
Critical Points:
$\Big\{(x_0, y_0) \ : \ f(x_0,y_0) = g(x_0,y_0) = 0\Big\}$
Critical Endpoints:
(x^*, y^*) if limits exist and finite:
$x^* = \lim_{t\to+\infty} x(t)$
$y^* = \lim_{t\to+\infty} y(t)$

ANNIHILATION
$(D - r)\left[e^{rx}\right] = 0$
$(D - r)^{k+1}\left[x^k e^{rx}\right] = 0$
$(D^2 + \beta^2)\left[\left\{\begin{smallmatrix}\sin\beta t\\ \cos\beta t\end{smallmatrix}\right\}\right] = 0$
$\left((D-\alpha)^2 + \beta^2\right)^{k+1}\left[x^k e^{\alpha x}\left\{\begin{smallmatrix}\sin\beta t\\ \cos\beta t\end{smallmatrix}\right\}\right] = 0$
$(D)^{k+1}\left[x^k + x^{k-1} + \dots\right] = 0$

LAPLACE TRANSFORM
$F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$
$\displaystyle\lim_{N\to\infty} Ne^{-sN} = 0, \quad s > 0$

CONVOLUTION
$(f * g) = \displaystyle\int_0^t f(t-v) \cdot g(v) \cdot dv$
Properties:
$f * g = g * f$
$f * (g + h) = (f * g) + (f * h)$
$(f * g) * h = f * (g * h)$
$f * 0 = 0$

IMPULSE RESPONSE
`Given` :: $ay'' + by' + cy = g, \quad y(0) = y_0, \quad y'(0) = y_0'$
`Step 1` :: $H(s) = \frac{1}{as^2+bs+c}$ $\longleftarrow$ Transfer Func
`Step 2` :: $h(s) = \mathscr{L}^{-1}\{H(s)\}$ $\longleftarrow$ Impulse Response
`Step 3` :: $y_k \longleftarrow$ [homogenous solution when $g = 0$]
`Solution` :: $\left[\begin{matrix} y = (y_k + (h * g)) \\ \quad = y_k + \displaystyle\int_0^t h(t-v) \cdot g(v) \cdot dv \end{matrix}\right]$

POWER SERIES
$y \ = \ \displaystyle\sum_{n=0} a_n x^n$
$y' \ = \ \displaystyle\sum_{n=1} n a_n x^{n-1}$
$y'' \ = \ \displaystyle\sum_{n=2} n(n-1) a_n x^{n-2}$
…when given Initial Values:
$y_0 = a_0, \quad y_0' = a_1$