Ch 4.3 Online Problem *Chris Achenbach*

Solve the Initial Value Problem for

$$y''' - 4y'' + 8y' - 8y = 0$$

given the Initial Values,

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$$

Step 1. Classify the Differential Equation

- Third Order
- Linear
- Homogeneous
- Constant Coefficients
- Ordinary Differential Equation

Even though this is a Third Order Equation, we can apply our techniques from Second Order Equations if we can factor it and find the roots.

Since it's a Homogeneous Linear Equation, we already can predict what our general solution will look,

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

So our first task will be to discovery the functions of y_1, y_2, y_3 and then we will be able to use our initial value conditions to discover the constants c_1, c_2, c_3.

Step 2. Factor and Find Roots

The characteristic of the differential equation is

$$r^3 - 4r^2 + 8r - 8 = 0$$

Which we can factor into the form

$$(r - 2)(r^2 - 2r + 4) = 0$$

The first term $(r - 2)$ is a simple real root, so right away we have one of our roots.

$$r_1 = 2$$

Now we move our attention to the second term $(r^2 - 2r + 4)$ which is called an *irreducible quadratic equation* because it can't be factored into terms with real numbers. Regardless of what you want to call it, we can still use the Quadratic Equation to find the roots, which gives us,

$$r_2, r_3 = 1 \pm \sqrt{3}i$$

We now have all three roots for the characteristic equation!

Step 3. Find the General Solution

So looking at our three roots for the characteristic, we have

$$r_1 = 2 \quad \text{and} \quad r_2, r_3 = 1 \pm \sqrt{3}i$$

and our goal to is figure out what the y_1, y_2, y_3 terms are in the general solution for this equation,

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

Looking back at the previous chapters, we can make use of our tricks for simple roots, which gives us

$$y_1 = e^{r_1 t} = e^{2t}$$

and then we can make use of our tricks for complex conjugates to give us the remaining two solutions y_2, y_3, making use of the complex roots r_2, r_3 that we found.

$$y_2 = e^{\alpha t} \sin(\beta t) = e^t \sin(\sqrt{3}t)$$
$$y_3 = e^{\alpha t} \cos(\beta t) = e^t \cos(\sqrt{3}t)$$

Now we have all of the pieces of the solution! Woot! Putting them all together, we have found the general solution to the homogeneous equation!

$$\boxed{y = c_1 e^{2t} + c_2 e^t \sin(\sqrt{3}t) + c_3 e^t \cos(\sqrt{3}t)}$$

Step 4. Solve the Initial Value Problem

We have a general solution to the homogeneous equation, and we want to figure out what those constants c_1, c_2, c_3 are. Since there are three constants, we need three initial values. We will create a *System of Linear Equations* with three equations and three unknowns, and solve that to find the what the constants.

We have the initial values,

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$$

and we can find the derivatives of our general solution,

$$\begin{aligned}
y = \quad & c_1 e^{2t} \\
& + c_2 e^t \sin(\sqrt{3}t) \\
& + c_3 e^t \cos(\sqrt{3}t) \\
\\
y' = \quad & 2c_1 e^{2t} \\
& + (c_2 - \sqrt{3}c_3) \cdot e^t \sin(\sqrt{3}t) \\
& + (\sqrt{3}c_2 + c_3) \cdot e^t \cos(\sqrt{3}t) \\
\\
y'' = \quad & 4c_1 e^{2t} \\
& + (-2c_2 - 2\sqrt{3}c_3) \cdot e^t \sin(\sqrt{3}t) \\
& + (2\sqrt{3}c_2 - 2c_3) \cdot e^t \cos(\sqrt{3}t)
\end{aligned}$$

This is where the initial values being set at $t = 0$ becomes extremely useful. Most of the terms get canceled out to either 1 or 0, and we are left with super simple linear equations that still have the unknown constants left in them.

$$\begin{aligned} y(0) &= c_1 + c_3 &&= 1 \\ y'(0) &= 2c_1 + \sqrt{3}c_2 + c_3 &&= 0 \\ y''(0) &= 4c_1 + 2\sqrt{3}c_2 - 2c_3 &&= 0 \end{aligned}$$

Which, we could put into an *Augmented Matrix* and use our awesome linear algebra skillz to solve!

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & \sqrt{3} & 1 & 0 \\ 4 & 2\sqrt{3} & -2 & 0 \end{array}\right)$$

And after you solve for this, you end up with the constants,

$$\begin{aligned} c_1 &= 1 \\ c_2 &= \tfrac{-2}{\sqrt{3}} \\ c_3 &= 0 \end{aligned}$$

which we plug into our original general solution, finally giving us the answer to the initial value problem!

$$\boxed{\boxed{\boxed{\boxed{y = e^{2t} - \tfrac{2}{\sqrt{3}} e^t \sin(\sqrt{3}t)}}}}$$