NATURAL LOG
$e^{\ln(x)} = x$
$\ln(x \cdot y) = \ln(x) + \ln(y)$
$\ln(x/y) = \ln(x) + \ln(y)$
$\ln(x^y) = y \cdot \ln(x)$
$\ln(0) = \text{undefined}$

TRIGONOMETRY
$\tan x = \sin x / \cos x$
$\cot x = 1/\tan x$
$\sec x = 1/\cos x$
$\csc x = 1/\sin x$
$\sin^2 x + \cos^2 x = 1$
$\sin 2x = 2 \sin x \cos x$

DERIVATIVES
Product: $(fg)' = fg' + f'g$
Quotient: $(f/g)' = (f'g - g'f)/g^2$
Reciprocal: $(1/f) = -f'/f^2$
Chain: $f(g)' = f'(g) \cdot g'$

INTEGRALS
$\int u\,dv = uv - \int v\,du$
$\int (1/x) = \ln|x|$
$\int (1/(ax+b)) = \ln|ax+b|/a$
$\int \tan ax = -\ln|\cos ax|/a = \ln|\sec ax|/a$
$\int \cot ax = \ln|\sin ax|/a$
$\int \sec x = \ln|\sec x + \tan x|$
$\int \sec^2 x = \tan x$
$\int \csc x = -\ln|\csc x + \cot x|$
$\int \csc^2 x = -\cot x$
$\int (1/(1+x^2)) = \arctan x$
$\int (1/\sqrt{1-x^2}) = \arcsin x$
$\int (x/e^{ax}) = e^{-ax}(ax-1)/a^2$

THE QUADRATIC EQUATION
$x = \left(-b \pm \sqrt{b^2 - 4ac}\,\right)/2a$

HOMOGENEOUS LINEAR EQUATIONS:
$ay'' + by' + cy = 0, \quad a \neq 0$

TWO REAL ROOTS:
$y_h = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}$
$y_h' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$

REPEATED ROOTS
$y_h = c_1 \cdot e^{rt} + c_2 \cdot t \cdot e^{rt}$
$y_h' = (c_1 r + c_2)e^{rt} + (c_2 r)te^{rt}$

COMPLEX ROOTS:
$y_h = c_1 \cdot e^{\alpha t} \cdot \sin(\beta t) + c_2 \cdot e^{\alpha t} \cdot \cos(\beta t)$
$y_h' = (c_1\alpha - c_2\beta)ES + (c_1\beta + c_2\alpha)EC$

COMPLEX ROOTS (init value):
$c_1 = \frac{y_0' - \alpha y_0}{\beta}, \quad c_2 = y_0$

UNDETERMINED COEFFICIENTS

CASE: $f == C\sin(\beta t) \;||\; C\cos(\beta t)$
SOLU: $y_p = A_1 \sin(\beta t) + A_2 \cos(\beta t)$

CASE: $f == Ct^m e^{rt}$
SOLU: $y_p = t^s P^m e^{rt}$
$s = 0$ if r is not a root of the equation.
$s = 1$ if r is a simple root.
$s = 2$ if r is a double root.

CASE: $f = Ct^m e^{\alpha t}\left\{\begin{smallmatrix}\sin\beta t\\ \cos\beta t\end{smallmatrix}\right\}$
SOLU: $y_p = t^s P_1^m e^{\alpha t}\cos(\beta t) + t^s P_2^m e^{\alpha t}\sin(\beta t)$
$s = 0$ if $\alpha + i\beta$ is not a root.
$s = 1$ if $\alpha + i\beta$ is a root.

NON-HOM GENERAL SOLUTION
$y = y_h + y_p$

SUPERPOSITION
LET: $F = ay'' + by' + cy$
IF: y_1 is a solution to $F = f_1$
AND: y_2 is a solution to $F = f_2$
THEN: $k_1 y_1 + k_2 y_2$ is a solution to $F = k_1 f_1 + k_2 f_2$

VARIATION OF PARAMETERS
$y_p = v_1 y_1 + v_2 y_2$
$W[y_1, y_2] = y_1 y_2' - y_1' y_2$
$$v_1 = \int \frac{-fy_2}{aW[y_1,y_2]}, \quad v_2 = \int \frac{+fy_1}{aW[y_1,y_2]}$$
$y_1 v_1' + y_2 v_2' = 0,$
$y_1' v_1' + y_2' v_2' = f/a$

CAUCHY-EULER (Oily Couch)
$at^2 y'' + bty' + cy = f$
CHARACTERISTIC: $ar^2 + (b-a)r + c = 0$
REAL: $y_h = t^{r_1} + t^{r_2}$
DOUBLE: $y_h = t^r + t^r \ln(t)$
COMPLEX: $y_h = t^\alpha \sin(\beta \ln t) + t^\alpha \cos(\beta \ln t)$

REDUCTION OF ORDER
$y'' + Py' + Qy = 0$
Given: y_1
$$y_2 = y_1 \int \frac{e^{-\int P}}{(y_1)^2}$$
$y_h = y_1 + y_2$

LINEAR INDEPENDENCE TEST
For whole interval: $W[y_1, y_2] = 0$

IMPROVED UNDETERMINED COEF

Given: $L[y] = Cx^m e^{rx}$
$y_p = x^s [A_m x^m + \ldots + A_1 x + A_0]\, e^{rx}$
with $s = 0$ if r is not a root of characteristic, otherwise s is equal to multiplicity of r.

Given: $L[y] = Cx^m e^{rx} \left\{ \begin{smallmatrix} \sin \beta t \\ \cos \beta t \end{smallmatrix} \right\}$
$y_p = x^s P_1^m e^{\alpha x} \sin(\beta x) + x^s P_2^m e^{\alpha x} \cos(\beta x)$
with $s = 0$ if $\alpha + i\beta$ is not a root of characteristic, otherwise s is equal to multiplicity of r.

Phase Plane

Given Autonomous System:
$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$

Trajectory:
$x(t), y(t)$ that solve the system.

Related Phase Plane Diff Eq:
$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{dy/dt}{dx/dt}$

Critical Points:
$\Big\{ (x_0, y_0) \,:\, f(x_0, y_0) = g(x_0, y_0) = 0 \Big\}$

Critical Endpoints:
(x^*, y^*) is critical if limits exist and finite:
$x^* = \lim_{t \to +\infty} x(t)$
$y^* = \lim_{t \to +\infty} y(t)$

Annihilation

Common Nonhomogenous Annihilators
$(D - r)\, [e^{rx}] = 0$
$(D - r)^{k+1} \left[x^k e^{rx} \right] = 0$
$(D^2 + \beta^2) \left[\left\{ \begin{smallmatrix} \sin \beta t \\ \cos \beta t \end{smallmatrix} \right\} \right] = 0$
$\left((D - \alpha)^2 + \beta^2 \right)^{k+1} \left[x^k e^{\alpha x} \left\{ \begin{smallmatrix} \sin \beta t \\ \cos \beta t \end{smallmatrix} \right\} \right] = 0$
$(D)^{k+1} \left[x^k + x^{k-1} + \ldots \right] = 0$

Laplace Transform

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt = \lim_{N \to \infty} \int_0^N f(t)\ dt$$

$$\lim_{N \to \infty} e^{-sN} = 0, \quad s > 0$$

$\mathcal{L}\left\{1\right\} = \frac{1}{s}, \quad s > 0$
$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad s > a$
$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad s > 0$
$\mathcal{L}\left\{\sin(bt)\right\} = \frac{b}{s^2+b^2}, \quad s > 0$
$\mathcal{L}\left\{\cos(bt)\right\} = \frac{s}{s^2+b^2}, \quad s > 0$

Laplace of Piecewise
$f = \begin{cases} f_1 \text{ for } t \in (0, a) \\ f_2 \text{ for } t \in (a, \infty) \end{cases}$
$\mathcal{L}(f) = \int_0^a (e^{-st} f_1) + \int_a^\infty (e^{-st} f_2)$

Jump Discontinuity
$\lim_{t \to a^+} f(t) \ \neq\ \lim_{t \to a^-} f(t)$

Exponential Order
$f(t)$ is of exponential order α,
if $\exists$ positive constants T and M,
such that: $\left| f(t) \right| \leq M e^{\alpha t}$ for all $t \geq T$

Condition for Existence of Laplace
if f is piecewise continuous on $[0, \infty]$
and of exponential order α, then
$\mathcal{L}\{f\} = F(s)$ exists for $s > \alpha$

Translation in s
If $\mathcal{L}\{f\}$ exists for $s > \alpha$, then
$\mathcal{L}\left\{e^{at} f(t)\right\} = F(s - a)$
for $s > \alpha + a$