1 Definitions

1.1 Vector Space

A **vector space** is a set V with the following properties:

(Assume that v, x, y, z are in V, and a, b, c are scalars in $\mathbb{R}$)

Commutativity:

- $x + y = y + x$

Associativity:

- $(x + y) + z = x + (y + z)$
- $(ab)v = a(bv)$

Additive Identity:

- there exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$

Additive Inverse:

- for all $v \in V$, there exists $x \in V$ such that $v + x = 0$

Multiplicative Identity:

- $1v = v$

Distributive Properties:

- $a(x + y) = ax + ay$
- $(a + b)v = av + bv$

1.2 Linear Combination

A linear combination of a list of vectors $v_1, \ldots, v_n$ is itself a vector, taking the form:

$$a_1v_1 + \ldots + a_mv_m$$

where each $a_1, \ldots a_n \in \mathbb{R}$

1.3 Span

The set of all linear combinations of a list of vectors $v_1, \ldots, v_n$ is called the **span** of $v_1, \ldots, v_n$, and is defined:

$$\text{span}(v_1, \ldots, v_n) = \{a_1v_1 + \cdots + a_nv_n \ : \ a_1, \ldots, a_m \in \mathbb{R}\}$$

If the span is equal to some space $\text{span}(v_1, \ldots, v_n) = V$, then you could say that $v_1, \ldots, v_n$ **spans** V.

1.4 Linearly Independent

For $v_1, \ldots, v_n \in V$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that:

$$a_1v_1 + \cdots + a_nv_n = 0$$

The list of vectors $v_1, \ldots, v_n$ is called **linearly independent** when

$$a_1 = \cdots = a_n = 0$$

for all possible values of $v_1, \ldots, v_n$.

1.5 Basis

A **basis** of V is a list of vectors in V that is both linearly independent and spans V.

The **Standard Basis** of the vector space $\mathbb{R}^n$ is

$$(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$$

which could also be written, using matrix bracket notation, as:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

1.6 Dimension

The dimension of a vector space is the length of any basis of the vector space. For example,

$$\dim \mathbb{R}^n = n$$

1.7 Inner Product

For a pair of vectors $u, v \in V$ in the same vector space (they are both in $\mathbb{R}^n$ for example), the Inner Product is defined as:

$$u \cdot v = u_1v_1 + ... + u_nv_n$$

which is also sometimes written using angular brackets:

$$\langle u, v \rangle$$

Keep in mind that the dimension of u and v must be the same. Using matrix dimension notation:

$$u_{\{n \times 1\}} \cdot v_{\{n \times 1\}}$$

The **Inner Product** is also a function $f : (\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$. The input is an ordered pair of vectors, and the output is a number. Inner products have the following properties:

Positivity:

- $\langle v, v \rangle \geq 0$ for all $v \in V$

Definiteness:

- $\langle v, v \rangle = 0$ if and only if $v = 0$

Additivity in First Slot:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$

Homogeneity in First Slot:

- $\langle au, v \rangle = a\langle u, v \rangle$ for all $a \in \mathbb{R}$ and all $u, v \in V$

In another definition of the Inner Product, the concepts of "additivity" and "homogeneity" are combined into a concept called "linearity". **Bilinearity** is when there is linearity in both the First and Second slots. Additionally, there is a concept called **Symmetry** for all real numbers.

For $x, y, z \in V$ and $a, b \in \mathbb{R}$:

Bilinearity:

- Additivity and Homogeneity in First and Second Slot:
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$
- $\langle x, ay + bz\rangle = a\langle x, y\rangle + b\langle x, z\rangle$

Symmetry:

- $\langle x, y\rangle = \langle y, x\rangle$

1.8 Norm

The Norm of a vector x is defined as the square root inner product of x with itself:

$$\|x\| = \sqrt{\langle x, x\rangle}$$

The Euclidean Norm, also called 2-norm, is defined:

$$\|x\|_2 = \sqrt{{x_1}^2 + \ldots + {x_n}^2}$$

which has the following properties:

Positivity:

- $\|x\| \geq 0$
- $\|x\| = 0$ if and only if $x = 0$

Homogeneity:

- $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{R}$

Triangle Inequality:

- $\|x + y\| \leq \|x\| + \|y\|$

1.9 Orthogonal

Two vectors $u, v \in V$ are called **orthogonal** if the inner product between them is 0,

$$\langle u, v\rangle = 0$$

you could also say "u is orthogonal to v". Orthogonal is another way of saying "at right angles to each other", or "perpendicular".

1.10 Linear Map

A linear map from vector space V to vector space W is a function $T : V \to W$ with the following properties:

Additivity:

- $T(u + v) = Tu + Tv$ for all vectors $u, w \in V$

Homogeneity:

- $T(av) = a(Tv)$ for all $a \in \mathbb{R}$ and all $v \in V$

1.11 Linear Maps and Matrices

Suppose M is a linear map $f : \mathbb{R}^a \to \mathbb{R}^b$, then M can be written as b-by-a matrix:

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,a} \\ \vdots & \vdots & \vdots \\ x_{b,1} & \cdots & x_{b,a} \end{bmatrix}$$

2 Proofs

2.1 Cosine Formula for Inner Product

For two non-zero vectors $x, y \in V$,

$$\langle x, y\rangle = \|x\|\|y\| \cos\theta$$

where the angle $\angle xy = \theta$.

Proof:

There are two cases we need to write a proof for.

- Case 1: when x and y are not scalar multiples of each other.
- Case 2: when x and y are scalar multiples.

Case 1:

For any triangle with sides a, b, c, The Law of Cosines states,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where the angle $\angle ab = \theta$. For vectors $x, y \in V$, we can treat them as sides of the triangle. Let:

$$\begin{aligned} a &= \|x\| \\ b &= \|y\| \\ c &= \|x - y\| \end{aligned}$$

Which allows us to rewrite the Law of Cosines:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta$$

Start with the definition of Inner Product, and apply its algebraic properties (notably the Bilinearity property), to show that Law of Cosines for Inner Products is correct.

$$\begin{aligned} &\|x - y\|^2 \\ &= \langle x - y, x - y\rangle \\ &= \langle x, x - y\rangle - \langle y, x - y\rangle \\ &= \Big(\langle x, x\rangle - \langle x, y\rangle\Big) - \Big(\langle y, x\rangle - \langle y, y\rangle\Big) \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \end{aligned}$$

Returning to the Law of Cosines,

$$\begin{aligned}\|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \\ \cancel{\|x\|^2} - 2\langle x, y\rangle + \cancel{\|y\|^2} &= \cancel{\|x\|^2} + \cancel{\|y\|^2} - 2\|x\|\|y\|\cos\theta \\ \cancel{-2}\langle x, y\rangle &= \cancel{-2}\|x\|\|y\|\cos\theta \\ \langle x, y\rangle &= \|x\|\|y\|\cos\theta\end{aligned}$$

□

Case 2:

Since x and y are scalar multiples of each other, we can write,

$$y = cx$$

for some scalar $c \in \mathbb{R}$ where $c \neq 0$ (since the theorem statement says that x and y are "nonzero vectors"). Now, to find the value of θ, we look at the value of c:

- If $c > 0$, then $\theta = 0$, and $\cos\theta = 1$
- If $c < 0$, then $\theta = \pi$, and $\cos\theta = -1$

Define the sign of c, so that we can use it in our proof:

$$\operatorname{sign}(c) = \cos\theta$$

And here's the proof:

$$\begin{aligned}\langle x, y\rangle &= \langle cx, x\rangle \\ &= c\langle x, x\rangle \\ &= c\|x\|^2 \\ &= c\|x\|\|x\| \\ &= c\sqrt{(x_1{}^2 + \ldots + x_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\sqrt{c^2(x_1{}^2 + \ldots + x_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\sqrt{(c^2x_1{}^2 + \ldots + c^2x_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\sqrt{(y_1{}^2 + \ldots + y_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\|y\|\|x\| \\ &= \|x\|\|y\|\cos\theta\end{aligned}$$

□

2.2 Triangle Inequality

TODO

2.3 Cauchy-Schwartz Inequality

TODO

3 Matrices

3.1 Algebraic Properties of Matrices

Compare these with the properties of Vector Space.

Protip: Matrices are in Vector Space.

Commutativity:

- $A + B = B + A$

Associativity:

- $A + (B + C) = (A + B) + C$

Additive Identity:

- $A + 0 = A$

Additive Inverse:

- $A + (-A) = 0$

Distributivity of matrix addition:

- $a(A + B) = aA + aB$

Distributivity of scalar addition:

- $(a + b)A = aA + bA$

Associativity of scalar multiplication

- $a(bA) = (ab)A$

Multiplicative Identity of scalar multiplication

- $1A = A$

3.2 Algebraic Properties of Matrix Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^TA^T$

3.3 Leading Entry

The Leading Entry of a row in a matrix the is first non-zero element in that row (from left-to-right).

3.4 Special Notations

Entries

- A_{ij} of matrix A is the entry in the i^{th} row and j^{th} column.
- I like to use $A_{i,j}$ or $A_{[i,j]}$ depending on the situation.

The $1 - \star - \times - 0$ notation:

- 1 : must be a 1
- $\star$: Non-zero numbers, $= \{c \in \mathbb{R} : c \neq 0\}$
- $\times$: any number $= \{c \in \mathbb{R}\}$
- 0 : must be a 0

MATLAB Syntax and Commands:

- `A(i,k)` returns the entry $A_{i,k}$
- `A(i,:)` returns the i^{th} row
- `A(:,k)` returns the k^{th} column
- `numel(A)` returns the number of elements in matrix A
- `nnz(A)` returns the number of non-zero elements in A

3.5 Main Diagonal

For a matrix entry $a_{i,k}$, the main diagonal entries would be defined as the set:

$$\{a_{i,k} : i = k\}$$

In the following example, the Main Diagonal would be the 1s:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Non-Diagonal entries are all values that are *not* in the main diagonal:

$$\{a_{i,k} : i \neq k\}$$

3.6 Diagonal Matrix

Diagonal Matrix is a matrix where all non-diagonal entries are 0.

For example, the following is a Diagonal Matrix:

$$\begin{bmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$$

3.7 Identity Matrix

An Identity Matrix, denoted I_n or just I, is a square matrix in $\mathbb{R}^{n \times n}$ where all diagonal entries are 1, and all non-diagonal entries are 0. For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When used in Matrix Multiplication, for some matrix $A \in \mathbb{R}^{m \times n}$, the Identity Matrix has the property:

$$I_m A = A I_n = A$$

3.8 Lower-Triangular Entries

Lower-Triangular Entries of a matrix are either: on the diagonal, or below the diagonal.

$$\{L_{i,k} : i \geq k\}$$

Strictly Lower-Triangular Entries of a matrix are only the values below the diagonal:

$$\{L_{i,k} : i > k\}$$

3.9 Lower-Triangular Matrix

A Lower-Triangular Matrix, $L \in \mathbb{R}^{n \times n}$, is a square matrix such that

$$L_{i,k} = 0 \quad \text{for all } i < k$$

For example, in this Lower-Triangular Matrix, $L \in \mathbb{R}^{3 \times 3}$,

$$\begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}$$

the lower-triangular entries can be anything, and the rest must be 0.

3.10 Unit Lower-Triangular Matrix

The Unit Lower-Triangular Matrix, $L \in \mathbb{R}^{n \times n}$ is both:

$$L_{i,k} = 1 \qquad \text{for all } i = k$$
$$L_{i,k} = 0 \qquad \text{for all } i < k$$

An example of a Unit Lower-Triangular Matrix, $L \in \mathbb{R}^{3 \times 3}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & \times & 1 \end{bmatrix}$$

3.11 Upper-Triangular Matrix

Upper-Triangular Entries are defined as:

$$\{U_{i,k} : i \leq k\}$$

Strictly-Upper-Triangular Entries are defined as:

$$\{U_{i,k} : i < k\}$$

Upper-Triangular Matrix example:

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

Unit Upper-Triangular Matrix example:

$$\begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}$$

Quite similar to the Lower-Triangular Matrix definitions and examples.

3.12 Bands of a Matrix

Diagonal Band

The d^{th}-diagonal-band of a matrix A is the set of entries:

$$d^{\text{th}} \text{ diagonal band} = \{A_{i,k} : i - k = d\}$$

For example, the 0-diagonal-band is the main diagonal, and the 2-band of $A \in \mathbb{R}^3$ would be:

$$\left\{A_{[0,2]}, A_{[1,1]}, A_{[2,0]}\right\}$$

Upper-Triangular Bands :

- Set of entries
- $\{A_{i,k} : i - k \leq 0\}$

Lower-Triangular Bands :

- Set of entries
- $\{A_{i,k} : k \geq 0\}$

Lower Bandwidth :

- Number
- d such that $A_{i,k} = 0$ for $(i - k > d)$.
- The lowest band before everything becomes 0s.

Upper Bandwidth:

- Number
- d such that $A_{i,k} = 0$ for $(i - k < d)$.
- The highest band before everything becomes 0s.

3.13 Outer Product of Vectors

For $x \in \mathbb{R}^m, y \in \mathbb{R}^n$ the outer product is defined,

$$x \otimes y = xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & \vdots & \vdots \\ x_my_1 & \cdots & x_my_n \end{bmatrix}$$

you could also say that the outer product is a function:

$$(\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}^{m \times n}$$

3.14 Rank-one Updates

TODO

3.15 Shear

TODO

3.16 Dilation

TODO

3.17 Transposition

TODO

3.18 Givens Rotation

TODO

3.19 Gauss Transform

TODO

4 Applications

Examples of applying Linear Algebra to other things. Includes models made with Vectors and Matrices.

4.1 Incidence Matrix of a Graph

TODO

4.2 3D Wireframe

TODO

4.3 3D Polygons

TODO

4.4 Spring-Mass Problem

TODO

5 Tips and Tricks

Extra things that are useful as a reference.

5.1 Dimensions of Nine Different Products

$$\begin{aligned}
\text{Scalar , Scalar}: &\quad \mathbb{R} \times \mathbb{R} &&\to \mathbb{R} \\
\text{Scalar , Column Vector}: &\quad \mathbb{R} \times \mathbb{R}^n &&\to \mathbb{R}^n \\
\text{Scalar , Row Vector}: &\quad \mathbb{R} \times \mathbb{R}^{1\times n} &&\to \mathbb{R}^{1\times n} \\
\text{Inner Product on } \mathbb{R}^n: &\quad \mathbb{R}^n \times \mathbb{R}^n &&\to \mathbb{R} \\
\text{Inner Product on } \mathbb{R}^{1\times n}: &\quad \mathbb{R}^{1\times n} \times \mathbb{R}^{1\times n} &&\to \mathbb{R} \\
\text{Outer Product}: &\quad \mathbb{R}^{m\times 1} \times \mathbb{R}^{n\times 1} &&\to \mathbb{R}^{m\times n} \\
\text{Scalar, Matrix}: &\quad \mathbb{R} \times \mathbb{R}^{m\times n} &&\to \mathbb{R}^{m\times n} \\
\text{Matrix, Column Vector}: &\quad \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times 1} &&\to \mathbb{R}^{m\times 1} \\
\text{Row Vector, Matrix}: &\quad \mathbb{R}^{1\times m} \times \mathbb{R}^{m\times n} &&\to \mathbb{R}^{1\times n}
\end{aligned}$$

5.2 Matrix Operations

TODO

5.3 The Matrix as a Function

Let f be a function:

$$f(x, y) = \Big(x + 2y,\ \ 3x + 4y,\ \ 5x + 6y\Big)$$

The function takes 2 elements as input and gives 3 elements as output,

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

Suppose $x = 1$ and $y = 2$,

$$\begin{aligned}
&f(1, 2) \\
&= \Big(1(1) + 2(2),\ \ 3(1) + 4(2),\ \ 5(1) + 6(2)\Big) \\
&= (5, 11, 17)
\end{aligned}$$

We could rewrite the input list $(1, 2)$ and output list $(5, 11, 17)$ as vectors, which reveals:

$$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

Now, let's say that function f is a Linear Map, A, from $\mathbb{R}^2$ to $\mathbb{R}^3$, and rewrite this in an algebraic form.

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

Looking back above to the function f, we can use this to rewrite A in a matrix notation.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

At this point, compare and contrast the dimensions of the matrix with the function definition,

$$\begin{aligned} f &: \mathbb{R}^2 \to \mathbb{R}^3 \\ A &\in \mathbb{R}^{3\times 2} \end{aligned}$$

and compare and contrast the input and output:

$$\begin{aligned} x &\in \mathbb{R}^2 \\ Ax &\in \mathbb{R}^3 \end{aligned}$$

We can rewrite the function again. This time, let's use our matrices to gain a new perspective of the nature of Linear Algebra:

$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + 2y \\ 3x + 4y \\ 5x + 6y \end{bmatrix}$$