

# 1 Definitions

## 1.1 Vector Space

Assume that  $\mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{z}$  are vectors in  $V$ , and  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are scalars in  $\mathbb{R}$ . A **vector space** is a set  $V$  with the following properties:

Commutativity:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

Associativity:

- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
- $(\mathbf{a}\mathbf{b})\mathbf{v} = \mathbf{a}(\mathbf{b}\mathbf{v})$

Additive Identity:

- there exists  $\mathbf{0} \in V$  such that  $\mathbf{v} + \mathbf{0} = \mathbf{v}$  for all  $\mathbf{v} \in V$

Additive Inverse:

- for all  $\mathbf{v} \in V$ , there exists  $\mathbf{x} \in V$  such that  $\mathbf{v} + \mathbf{x} = \mathbf{0}$

Multiplicative Identity:

- $1\mathbf{v} = \mathbf{v}$

Distributive Properties:

- $\mathbf{a}(\mathbf{x} + \mathbf{y}) = \mathbf{a}\mathbf{x} + \mathbf{a}\mathbf{y}$
- $(\mathbf{a} + \mathbf{b})\mathbf{v} = \mathbf{a}\mathbf{v} + \mathbf{b}\mathbf{v}$

## 1.2 Linear Combination

A linear combination of a list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is itself a vector, taking the form:

$$\mathbf{a}_1\mathbf{v}_1 + \dots + \mathbf{a}_n\mathbf{v}_n$$

where each  $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}$

## 1.3 Span

The set of all linear combinations of a list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is called the **span** of  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , and is defined:

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_n) = \{\mathbf{a}_1\mathbf{v}_1 + \dots + \mathbf{a}_n\mathbf{v}_n : \mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}\}$$

If the span is equal to some space  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_n) = V$ , then you could say that  $\mathbf{v}_1, \dots, \mathbf{v}_n$  **spans**  $V$ .

## 1.4 Linearly Independent

For  $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$  and  $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}$  such that:

$$\mathbf{a}_1\mathbf{v}_1 + \dots + \mathbf{a}_n\mathbf{v}_n = \mathbf{0}$$

The list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is called **linearly independent** when

$$\mathbf{a}_1 = \dots = \mathbf{a}_n = 0$$

for all possible values of  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

## 1.5 Basis

A **basis** of  $V$  is a list of vectors in  $V$  that is both linearly independent and spans  $V$ .

The **Standard Basis** of the vector space  $\mathbb{R}^n$  is

$$(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$$

which could also be written, using matrix bracket notation, as:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

## 1.6 Dimension

The dimension of a vector space is the length of any basis of the vector space. For example,

$$\dim \mathbb{R}^n = n$$

## 1.7 Inner Product

For a pair of vectors  $\mathbf{u}, \mathbf{v} \in V$  in the same vector space (they are both in  $\mathbb{R}^n$  for example), the Inner Product is defined as:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}_1\mathbf{v}_1 + \dots + \mathbf{u}_n\mathbf{v}_n$$

which is also sometimes written using angular brackets:

$$\langle \mathbf{u}, \mathbf{v} \rangle$$

Keep in mind that the dimension of  $\mathbf{u}$  and  $\mathbf{v}$  must be the same. Using matrix dimension notation:

$$\mathbf{u}_{\{n \times 1\}} \cdot \mathbf{v}_{\{n \times 1\}}$$

The **Inner Product** is also a function  $f : (\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ . The input is an ordered pair of vectors, and the output is a number. Inner products have the following properties:

Positivity:

- $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$  for all  $\mathbf{v} \in V$

Definiteness:

- $\langle \mathbf{v}, \mathbf{v} \rangle = 0$  if and only if  $\mathbf{v} = \mathbf{0}$

Additivity in First Slot:

- $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$  for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

Homogeneity in First Slot:

- $\langle \mathbf{a}\mathbf{u}, \mathbf{v} \rangle = \mathbf{a}\langle \mathbf{u}, \mathbf{v} \rangle$  for all  $\mathbf{a} \in \mathbb{R}$  and all  $\mathbf{u}, \mathbf{v} \in V$

In another definition of the Inner Product, the concepts of “additivity” and “homogeneity” are combined into a concept called “linearity”. **Bilinearity** is when there is linearity in both the First and Second slots. Additionally, there is a concept called **Symmetry** for all real numbers.

For  $x, y, z \in V$  and  $a, b \in \mathbb{R}$ :

Bilinearity:

- Additivity and Homogeneity in First and Second Slot:
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
- $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$

Symmetry:

- $\langle x, y \rangle = \langle y, x \rangle$

## 1.8 Norm

The Norm of a vector  $x$  is defined as the square root inner product of  $x$  with itself:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

The Euclidean Norm, also called 2-norm, is defined:

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$$

which has the following properties:

Positivity:

- $\|x\| \geq 0$
- $\|x\| = 0$  if and only if  $x = 0$

Homogeneity:

- $\|ax\| = |a|\|x\|$  for all  $a \in \mathbb{R}$

Triangle Inequality:

- $\|x + y\| \leq \|x\| + \|y\|$

## 1.9 Orthogonal

Two vectors  $u, v \in V$  are called **orthogonal** if the inner product between them is 0,

$$\langle u, v \rangle = 0$$

you could also say “ $u$  is orthogonal to  $v$ ”. Orthogonal is another way of saying “at right angles to each other”, or “perpendicular”.

## 1.10 Linear Map

A linear map from vector space  $V$  to vector space  $W$  is a function  $T : V \rightarrow W$  with the following properties:

Additivity:

- $T(u + v) = Tu + Tv$  for all vectors  $u, v \in V$

Homogeneity:

- $T(av) = a(Tv)$  for all  $a \in \mathbb{R}$  and all  $v \in V$

## 1.11 Linear Maps and Matrices

Suppose  $M$  is a linear map  $f : \mathbb{R}^a \rightarrow \mathbb{R}^b$ , then  $M$  can be written as  $b$ -by- $a$  matrix:

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,a} \\ \vdots & \vdots & \vdots \\ x_{b,1} & \cdots & x_{b,a} \end{bmatrix}$$

## 2 Proofs

### 2.1 Cosine Formula for Inner Product

For two non-zero vectors  $x, y \in V$ ,

$$\langle x, y \rangle = \|x\|\|y\| \cos \theta$$

where the angle  $\angle xy = \theta$ .

*Proof:*

There are two cases we need to write a proof for.

- Case 1: when  $x$  and  $y$  are not scalar multiples of each other.
- Case 2: when  $x$  and  $y$  are scalar multiples.

*Case 1:*

For any triangle with sides  $a, b, c$ , The Law of Cosines states,

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

where the angle  $\angle ab = \theta$ . For vectors  $x, y \in V$ , we can treat them as sides of the triangle. Let:

$$a = \|x\|$$

$$b = \|y\|$$

$$c = \|x - y\|$$

Which allows us to rewrite the Law of Cosines:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \theta$$

Start with the definition of Inner Product, and apply its algebraic properties (notably the Bilinearity property), to show that Law of Cosines for Inner Products is correct.

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x - y \rangle - \langle y, x - y \rangle \\ &= (\langle x, x \rangle - \langle x, y \rangle) - (\langle y, x \rangle - \langle y, y \rangle) \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \end{aligned}$$

Returning to the Law of Cosines,

$$\begin{aligned}\|x - y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \\ \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \\ -2\langle x, y \rangle &= -2\|x\|\|y\|\cos\theta \\ \langle x, y \rangle &= \|x\|\|y\|\cos\theta\end{aligned}$$

And this is what we were trying to show.

□

Case 2:

Since  $x$  and  $y$  are scalar multiples of each other, we can write,

$$y = cx$$

for some scalar  $c \in \mathbb{R}$  where  $c \neq 0$  (since the theorem statement says that  $x$  and  $y$  are “nonzero vectors”). Now, to find the value of  $\theta$ , we look at the value of  $c$ :

- If  $c > 0$ , then  $\theta = 0$ , and  $\cos\theta = 1$
- If  $c < 0$ , then  $\theta = \pi$ , and  $\cos\theta = -1$

Define the sign of  $c$ , so that we can use it in our proof:

$$\text{sign}(c) = \{s \in \{-1, 1\} : s = \cos\theta\}$$

And here's the proof:

$$\begin{aligned}\langle x, y \rangle &= \langle cx, x \rangle \\ &= c\langle x, x \rangle \\ &= c\|x\|^2 \\ &= c\|x\|\|x\| \\ &= c\sqrt{(x_1^2 + \dots + x_n^2)}\|x\| \\ &= \text{sign}(c)\sqrt{c^2(x_1^2 + \dots + x_n^2)}\|x\| \\ &= \text{sign}(c)\sqrt{(c^2x_1^2 + \dots + c^2x_n^2)}\|x\| \\ &= \text{sign}(c)\sqrt{(y_1^2 + \dots + y_n^2)}\|x\| \\ &= \text{sign}(c)\|y\|\|x\| \\ &= \|x\|\|y\|\cos\theta\end{aligned}$$

□

## 2.4 Other

For vectors  $u, v \in V$  such that:

$$\langle u, v \rangle = \|u\|\|v\|\cos\theta$$

Show that  $u, v$  are orthogonal when  $\theta = 0$ .

## 2.2 Triangle Inequality

TODO

## 2.3 Cauchy-Schwartz Inequality

TODO