1 Definitions

1.1 Vector Space

Assume that v, x, y, z are vectors in V, and a, b, c are scalars in $\mathbb{R}$. A **vector space** is a set V with the following properties:

Commutativity:

- $x + y = y + x$

Associativity:

- $(x + y) + z = x + (y + z)$
- $(ab)v = a(bv)$

Additive Identity:

- there exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$

Additive Inverse:

- for all $v \in V$, there exists $x \in V$ such that $v + x = 0$

Multiplicative Identity:

- $1v = v$

Distributive Properties:

- $a(x + y) = ax + ay$
- $(a + b)v = av + bv$

1.2 Linear Combination

A linear combination of a list of vectors $v_1, \ldots, v_n$ is itself a vector, taking the form:

$$a_1 v_1 + \ldots + a_m v_m$$

where each $a_1, \ldots a_n \in \mathbb{R}$

1.3 Span

The set of all linear combinations of a list of vectors $v_1, \ldots, v_n$ is called the **span** of $v_1, \ldots, v_n$, and is defined:

$$\operatorname{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n \ : \ a_1, \ldots, a_m \in \mathbb{R}\}$$

If the span is equal to some space $\operatorname{span}(v_1, \ldots, v_n) = V$, then you could say that $v_1, \ldots, v_n$ **spans** V.

1.4 Linearly Independent

For $v_1, \ldots, v_n \in V$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that:

$$a_1 v_1 + \cdots + a_n v_n = 0$$

The list of vectors $v_1, \ldots, v_n$ is called **linearly independent** when

$$a_1 = \cdots = a_n = 0$$

for all possible values of $v_1, \ldots, v_n$.

1.5 Basis

A **basis** of V is a list of vectors in V that is both linearly independent and spans V.

The **Standard Basis** of the vector space $\mathbb{R}^{\ltimes}$ is

$$(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$$

which could also be written, using matrix bracket notation, as:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

1.6 Dimension

The dimension of a vector space is the length of any basis of the vector space. For example,

$$\dim \mathbb{R}^n = n$$

1.7 Inner Product

For a pair of vectors $u, v \in V$ in the same vector space (they are both in $\mathbb{R}^n$ for example), the Inner Product is defined as:

$$u \cdot v = u_1 v_1 + ... + u_n v_n$$

which is also sometimes written using angular brackets:

$$\langle u, v \rangle$$

Keep in mind that the dimension of u and v must be the same. Using matrix dimension notation:

$$u_{\{n \times 1\}} \cdot v_{\{n \times 1\}}$$

The **Inner Product** is also a function $f : (\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$. The input is an ordered pair of vectors, and the output is a number. Inner products have the following properties:

Positivity:

- $\langle v, v \rangle \geq 0$ for all $v \in V$

Definiteness:

- $\langle v, v \rangle = 0$ if and only if $v = 0$

Additivity in First Slot:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$

Homogeneity in First Slot:

- $\langle au, v \rangle = a \langle u, v \rangle$ for all $a \in \mathbb{R}$ and all $u, v \in V$

In another definition of the Inner Product, the concepts of "additivity" and "homogeneity" are combined into a concept called "linearity". **Bilinearity** is when there is linearity in both the First and Second slots. Additionally, there is a concept called **Symmetry** for all real numbers.

For $x, y, z \in V$ and $a, b \in \mathbb{R}$:

Bilinearity:

- Additivity and Homogeneity in First and Second Slot:
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$
- $\langle x, ay + bz\rangle = a\langle x, y\rangle + b\langle x, z\rangle$

Symmetry:

- $\langle x, y\rangle = \langle y, x\rangle$

1.8 Norm

The Norm of a vector x is defined as the square root inner product of x with itself:

$$\|x\| = \sqrt{\langle x, x\rangle}$$

The Euclidean Norm, also called 2-norm, is defined:

$$\|x\|_2 = \sqrt{{x_1}^2 + \ldots + {x_n}^2}$$

which has the following properties:

Positivity:

- $\|x\| \geq 0$
- $\|x\| = 0$ if and only if $x = 0$

Homogeneity:

- $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{R}$

Triangle Inequality:

- $\|x + y\| \leq \|x\| + \|y\|$

1.9 Orthogonal

Two vectors $u, v \in V$ are called **orthogonal** if the inner product between them is 0,

$$\langle u, v\rangle = 0$$

you could also say "u is orthogonal to v". Orthogonal is another way of saying "at right angles to each other", or "perpendicular".

1.10 Linear Map

A linear map from vector space V to vector space W is a function $T : V \to W$ with the following properties:

Additivity:

- $T(u + v) = Tu + Tv$ for all vectors $u, w \in V$

Homogeneity:

- $T(av) = a(Tv)$ for all $a \in \mathbb{R}$ and all $v \in V$

1.11 Linear Maps and Matrices

Suppose M is a linear map $f : \mathbb{R}^a \to \mathbb{R}^b$, then M can be written as b-by-a matrix:

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,a} \\ \vdots & \vdots & \vdots \\ x_{b,1} & \cdots & x_{b,a} \end{bmatrix}$$

2 Proofs

2.1 Cosine Formula for Inner Product

For two non-zero vectors $x, y \in V$,

$$\langle x, y\rangle = \|x\|\|y\| \cos\theta$$

where the angle $\angle xy = \theta$.

Proof:

There are two cases we need to write a proof for.

- Case 1: when x and y are not scalar multiples of each other.
- Case 2: when x and y are scalar multiples.

Case 1:

For any triangle with sides a, b, c, The Law of Cosines states,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where the angle $\angle ab = \theta$. For vectors $x, y \in V$, we can treat them as sides of the triangle. Let:

$$a = \|x\|$$
$$b = \|y\|$$
$$c = \|x - y\|$$

Which allows us to rewrite the Law of Cosines:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta$$

Start with the definition of Inner Product, and apply its algebraic properties (notably the Bilinearity property), to show that Law of Cosines for Inner Products is correct.

$$\begin{aligned} &\|x - y\|^2 \\ &= \langle x - y, x - y\rangle \\ &= \langle x, x - y\rangle - \langle y, x - y\rangle \\ &= \Big(\langle x, x\rangle - \langle x, y\rangle\Big) - \Big(\langle y, x\rangle - \langle y, y\rangle\Big) \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \end{aligned}$$

Returning to the Law of Cosines,

$$\begin{aligned}
\|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \\
\cancel{\|x\|^2} - 2\langle x, y\rangle + \cancel{\|y\|^2} &= \cancel{\|x\|^2} + \cancel{\|y\|^2} - 2\|x\|\|y\|\cos\theta \\
\cancel{-2}\langle x, y\rangle &= \cancel{-2}\|x\|\|y\|\cos\theta \\
\langle x, y\rangle &= \|x\|\|y\|\cos\theta
\end{aligned}$$

And this is what we were trying to show.

$\square$

Case 2:

Since x and y are scalar multiples of each other, we can write,

$$y = cx$$

for some scalar $c \in \mathbb{R}$ where $c \neq 0$ (since the theorem statement says that x and y are "nonzero vectors"). Now, to find the value of θ, we look at the value of c:

- If $c > 0$, then $\theta = 0$, and $\cos\theta = 1$
- If $c < 0$, then $\theta = \pi$, and $\cos\theta = -1$

Define the sign of c, so that we can use it in our proof:

$$\operatorname{sign}(c) = \left\{ s \in \{-1, 1\} \; : \; s = \cos\theta \right\}$$

And here's the proof:

$$\begin{aligned}
\langle x, y\rangle &= \langle cx, x\rangle \\
&= c\langle x, x\rangle \\
&= c\|x\|^2 \\
&= c\|x\|\|x\| \\
&= c\sqrt{(x_1{}^2 + \ldots + x_n{}^2)}\;\|x\| \\
&= \operatorname{sign}(c)\sqrt{c^2(x_1{}^2 + \ldots + x_n{}^2)}\;\|x\| \\
&= \operatorname{sign}(c)\sqrt{(c^2x_1{}^2 + \ldots + c^2x_n{}^2)}\;\|x\| \\
&= \operatorname{sign}(c)\sqrt{(y_1{}^2 + \ldots + y_n{}^2)}\;\|x\| \\
&= \operatorname{sign}(c)\|y\|\|x\| \\
&= \|x\|\|y\|\cos\theta
\end{aligned}$$

$\square$

2.2 Triangle Inequality

TODO

2.3 Cauchy-Schwartz Inequality

TODO

2.4 Other

For vectors $u, v \in V$ such that:

$$\langle u, v\rangle = \|u\|\|v\|\cos\theta$$

Show that u, v are orthogonal when $\theta = 0$.