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# 1 Definitions

## 1.1 Vector Space

A **vector space** is a set  $V$  with the following properties:  
(Assume that  $v, x, y, z$  are in  $V$ , and  $a, b, c$  are scalars in  $\mathbb{R}$ )

Commutativity:

$$\bullet \quad x + y = y + x$$

Associativity:

$$\bullet \quad (x + y) + z = x + (y + z)$$

$$\bullet \quad (ab)v = a(bv)$$

Additive Identity:

$$\bullet \quad \text{there exists } 0 \in V \text{ such that } v + 0 = v \text{ for all } v \in V$$

Additive Inverse:

$$\bullet \quad \text{for all } v \in V, \text{ there exists } x \in V \text{ such that } v + x = 0$$

Multiplicative Identity:

$$\bullet \quad 1v = v$$

Distributive Properties:

$$\bullet \quad a(x + y) = ax + ay$$

$$\bullet \quad (a + b)v = av + bv$$

## 1.2 Linear Combination

A linear combination of a list of vectors  $v_1, \dots, v_n$  is itself a vector, taking the form:

$$a_1 v_1 + \dots + a_n v_n$$

where each  $a_1, \dots, a_n \in \mathbb{R}$

## 1.3 Span

The set of all linear combinations of a list of vectors  $v_1, \dots, v_n$  is called the **span** of  $v_1, \dots, v_n$ , and is defined:

$$\text{span}(v_1, \dots, v_n) = \{a_1 v_1 + \dots + a_n v_n : a_1, \dots, a_n \in \mathbb{R}\}$$

If the span is equal to some space  $\text{span}(v_1, \dots, v_n) = V$ , then you could say that  $v_1, \dots, v_n$  **spans**  $V$ .

## 1.4 Linearly Independent

For  $v_1, \dots, v_n \in V$  and  $a_1, \dots, a_n \in \mathbb{R}$  such that:

$$a_1 v_1 + \dots + a_n v_n = 0$$

The list of vectors  $v_1, \dots, v_n$  is called **linearly independent** when

$$a_1 = \dots = a_n = 0$$

for all possible values of  $v_1, \dots, v_n$ .

## 1.5 Basis

A **basis** of  $V$  is a list of vectors in  $V$  that is both linearly independent and spans  $V$ .

The **Standard Basis** of the vector space  $\mathbb{R}^n$  is

$$(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$$

which could also be written, using matrix bracket notation, as:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

## 1.6 Dimension

The dimension of a vector space is the length of any basis of the vector space. For example,

$$\dim \mathbb{R}^n = n$$

## 1.7 Inner Product

For a pair of vectors  $u, v \in V$  in the same vector space (they are both in  $\mathbb{R}^n$  for example), the Inner Product is defined as:

$$u \cdot v = u_1 v_1 + \dots + u_n v_n$$

which is also sometimes written using angular brackets:

$$\langle u, v \rangle$$

Keep in mind that the dimension of  $u$  and  $v$  must be the same. Using matrix dimension notation:

$$u_{\{n \times 1\}} \cdot v_{\{n \times 1\}}$$

The **Inner Product** is also a function  $f : (\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ . The input is an ordered pair of vectors, and the output is a number. Inner products have the following properties:

Positivity:

- $\langle v, v \rangle \geq 0$  for all  $v \in V$

Definiteness:

- $\langle v, v \rangle = 0$  if and only if  $v = 0$

Additivity in First Slot:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$  for all  $u, v, w \in V$

Homogeneity in First Slot:

- $\langle au, v \rangle = a \langle u, v \rangle$  for all  $a \in \mathbb{R}$  and all  $u, v \in V$

In another definition of the Inner Product, the concepts of “additivity” and “homogeneity” are combined into a concept called “linearity”. **Bilinearity** is when there is linearity in both the First and Second slots. Additionally, there is a concept called **Symmetry** for all real numbers.

For  $x, y, z \in V$  and  $a, b \in \mathbb{R}$ :

Bilinearity:

- Additivity and Homogeneity in First and Second Slot:
- $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$
- $\langle x, ay + bz \rangle = a \langle x, y \rangle + b \langle x, z \rangle$

Symmetry:

- $\langle x, y \rangle = \langle y, x \rangle$

## 1.8 Norm

The Norm of a vector  $x$  is defined as the square root inner product of  $x$  with itself:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

The Euclidean Norm, also called 2-norm, is defined:

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$$

which has the following properties:

Positivity:

- $\|x\| \geq 0$
- $\|x\| = 0$  if and only if  $x = 0$

Homogeneity:

- $\|ax\| = |a| \|x\|$  for all  $a \in \mathbb{R}$

Triangle Inequality:

- $\|x + y\| \leq \|x\| + \|y\|$

## 1.9 Orthogonal

Two vectors  $u, v \in V$  are called **orthogonal** if the inner product between them is 0,

$$\langle u, v \rangle = 0$$

you could also say “ $u$  is orthogonal to  $v$ ”. Orthogonal is another way of saying “at right angles to each other”, or “perpendicular”.

## 1.10 Linear Map

A linear map from vector space  $V$  to vector space  $W$  is a function  $T : V \rightarrow W$  with the following properties:

Additivity:

- $T(u + v) = Tu + Tv$  for all vectors  $u, v \in V$

Homogeneity:

- $T(av) = a(Tv)$  for all  $a \in \mathbb{R}$  and all  $v \in V$

## 1.11 Linear Maps and Matrices

Suppose  $M$  is a linear map  $f : \mathbb{R}^a \rightarrow \mathbb{R}^b$ , then  $M$  can be written as  $b$ -by- $a$  matrix:

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,a} \\ \vdots & \vdots & \vdots \\ x_{b,1} & \cdots & x_{b,a} \end{bmatrix}$$

## 2 Proofs

### 2.1 Cosine Formula for Inner Product

For two non-zero vectors  $x, y \in V$ ,

$$\langle x, y \rangle = \|x\| \|y\| \cos \theta$$

where the angle  $\angle xy = \theta$ .

*Proof:*

There are two cases we need to write a proof for.

- Case 1: when  $x$  and  $y$  are not scalar multiples of each other.
- Case 2: when  $x$  and  $y$  are scalar multiples.

*Case 1:*

For any triangle with sides  $a, b, c$ , The Law of Cosines states,

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

where the angle  $\angle ab = \theta$ . For vectors  $x, y \in V$ , we can treat them as sides of the triangle. Let:

$$\begin{aligned} a &= \|x\| \\ b &= \|y\| \\ c &= \|x - y\| \end{aligned}$$

Which allows us to rewrite the Law of Cosines:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\| \|y\| \cos \theta$$

Start with the definition of Inner Product, and apply its algebraic properties (notably the Bilinearity property), to show that Law of Cosines for Inner Products is correct.

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x - y \rangle - \langle y, x - y \rangle \\ &= (\langle x, x \rangle - \langle x, y \rangle) - (\langle y, x \rangle - \langle y, y \rangle) \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \end{aligned}$$

Returning to the Law of Cosines,

$$\begin{aligned} \|x - y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\| \|y\| \cos \theta \\ \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\| \|y\| \cos \theta \\ -2\langle x, y \rangle &= -2\|x\| \|y\| \cos \theta \\ \langle x, y \rangle &= \|x\| \|y\| \cos \theta \end{aligned}$$

□

*Case 2:*

Since  $x$  and  $y$  are scalar multiples of each other, we can write,

$$y = cx$$

for some scalar  $c \in \mathbb{R}$  where  $c \neq 0$  (since the theorem statement says that  $x$  and  $y$  are “nonzero vectors”). Now, to find the value of  $\theta$ , we look at the value of  $c$ :

- If  $c > 0$ , then  $\theta = 0$ , and  $\cos \theta = 1$
- If  $c < 0$ , then  $\theta = \pi$ , and  $\cos \theta = -1$

Define the sign of  $c$ , so that we can use it in our proof:

$$\text{sign}(c) = \cos \theta$$

And here's the proof:

$$\begin{aligned} \langle x, y \rangle &= \langle cx, x \rangle \\ &= c \langle x, x \rangle \\ &= c \|x\|^2 \\ &= c \|x\| \|x\| \\ &= c \sqrt{(x_1^2 + \dots + x_n^2)} \|x\| \\ &= \text{sign}(c) \sqrt{c^2(x_1^2 + \dots + x_n^2)} \|x\| \\ &= \text{sign}(c) \sqrt{c^2 x_1^2 + \dots + c^2 x_n^2} \|x\| \\ &= \text{sign}(c) \sqrt{(y_1^2 + \dots + y_n^2)} \|x\| \\ &= \text{sign}(c) \|y\| \|x\| \\ &= \|x\| \|y\| \cos \theta \end{aligned}$$

□

### 2.2 Triangle Inequality

TODO

### 2.3 Cauchy-Schwartz Inequality

TODO

## 3 Matrices

### 3.1 Algebraic Properties of Matrices

Compare these with the properties of Vector Space.

*Protip:* Matrices are in Vector Space.

Commutativity:

- $A + B = B + A$

Associativity:

- $A + (B + C) = (A + B) + C$

Additive Identity:

- $A + 0 = A$

Additive Inverse:

- $A + (-A) = 0$

Distributivity of matrix addition:

- $a(A + B) = aA + aB$

Distributivity of scalar addition:

- $(a + b)A = aA + bA$

Associativity of scalar multiplication

- $a(bA) = (ab)A$

Multiplicative Identity of scalar multiplication

- $1A = A$

### 3.2 Algebraic Properties of Matrix Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

### 3.3 Leading Entry

The Leading Entry of a row in a matrix is the first non-zero element in that row (from left-to-right).

### 3.4 Special Notations

Entries

- $A_{ij}$  of matrix  $A$  is the entry in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.
- I like to use  $A_{i,j}$  or  $A_{[i,j]}$  depending on the situation.

The  $1 - \star - \times - 0$  notation:

- $1$  : must be a 1
- $\star$  : Non-zero numbers,  $= \{c \in \mathbb{R} : c \neq 0\}$
- $\times$  : any number  $= \{c \in \mathbb{R}\}$
- $0$  : must be a 0

MATLAB Syntax and Commands:

- $A(i,k)$  returns the entry  $A_{i,k}$
- $A(i,:)$  returns the  $i^{\text{th}}$  row
- $A(:,k)$  returns the  $k^{\text{th}}$  column
- $\text{numel}(A)$  returns the number of elements in matrix  $A$
- $\text{nnz}(A)$  returns the number of non-zero elements in  $A$

### 3.5 Main Diagonal

For a matrix entry  $a_{i,k}$ , the main diagonal entries would be defined as the set:

$$\{a_{i,k} : i = k\}$$

In the following example, the Main Diagonal would be the 1s:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Non-Diagonal* entries are all values that are *not* in the main diagonal:

$$\{a_{i,k} : i \neq k\}$$

### 3.6 Diagonal Matrix

Diagonal Matrix is a matrix where all non-diagonal entries are 0.

For example, the following is a Diagonal Matrix:

$$\begin{bmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$$

### 3.7 Identity Matrix

An Identity Matrix, denoted  $I_n$  or just  $I$ , is a square matrix in  $\mathbb{R}^{n \times n}$  where all diagonal entries are 1, and all non-diagonal entries are 0. For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When used in Matrix Multiplication, for some matrix  $A \in \mathbb{R}^{m \times n}$ , the Identity Matrix has the property:

$$I_m A = A I_n = A$$

### 3.8 Lower-Triangular Entries

*Lower-Triangular Entries* of a matrix are either: on the diagonal, or below the diagonal.

$$\{L_{i,k} : i \geq k\}$$

*Strictly Lower-Triangular Entries* of a matrix are only the values below the diagonal:

$$\{L_{i,k} : i > k\}$$

### 3.9 Lower-Triangular Matrix

A Lower-Triangular Matrix,  $L \in \mathbb{R}^{n \times n}$ , is a square matrix such that

$$L_{i,k} = 0 \quad \text{for all } i < k$$

For example, in this Lower-Triangular Matrix,  $L \in \mathbb{R}^{3 \times 3}$ ,

$$\begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}$$

the lower-triangular entries can be anything, and the rest must be 0.

### 3.10 Unit Lower-Triangular Matrix

The Unit Lower-Triangular Matrix,  $L \in \mathbb{R}^{n \times n}$  is both:

$$L_{i,k} = 1 \quad \text{for all } i = k$$

$$L_{i,k} = 0 \quad \text{for all } i < k$$

An example of a Unit Lower-Triangular Matrix,  $L \in \mathbb{R}^{3 \times 3}$ ,

$$\begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & \times & 1 \end{bmatrix}$$

### 3.11 Upper-Triangular Matrix

Upper-Triangular Entries are defined as:

$$\{U_{i,k} : i \leq k\}$$

Strictly-Upper-Triangular Entries are defined as:

$$\{U_{i,k} : i < k\}$$

Upper-Triangular Matrix example:

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

Unit Upper-Triangular Matrix example:

$$\begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}$$

Quite similar to the Lower-Triangular Matrix definitions and examples.

### 3.12 Bands of a Matrix

*Diagonal Band*

The  $d^{\text{th}}$ -diagonal-band of a matrix  $A$  is the set of entries:

$$d^{\text{th}} \text{ diagonal band} = \{A_{i,k} : i - k = d\}$$

For example, the 0-diagonal-band is the main diagonal, and the 2-band of  $A \in \mathbb{R}^3$  would be:

$$\{A_{[0,2]}, A_{[1,1]}, A_{[2,0]}\}$$

Upper-Triangular Bands :

- Set of entries
- $\{A_{i,k} : i - k \leq 0\}$

Lower-Triangular Bands :

- Set of entries
- $\{A_{i,k} : k \geq 0\}$

Lower Bandwidth :

- Number
- $d$  such that  $A_{i,k} = 0$  for  $(i - k > d)$ .
- The lowest band before everything becomes 0s.

Upper Bandwidth:

- Number
- $d$  such that  $A_{i,k} = 0$  for  $(i - k < d)$ .
- The highest band before everything becomes 0s.

### 3.13 Outer Product of Vectors

For  $x \in \mathbb{R}^m, y \in \mathbb{R}^n$  the outer product is defined,

$$x \otimes y = xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \vdots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix}$$

you could also say that the outer product is a function:

$$(\mathbb{R}^m, \mathbb{R}^n) \rightarrow \mathbb{R}^{m \times n}$$

### 3.14 Rank-one Updates

TODO

### 3.15 Shear

TODO

### 3.16 Dilation

TODO

### 3.17 Transposition

TODO

### 3.18 Givens Rotation

TODO

### 3.19 Gauss Transform

TODO

## 4 Applications

Examples of applying Linear Algebra to other things. Includes models made with Vectors and Matrices.

### 4.1 Incidence Matrix of a Graph

TODO

### 4.2 3D Wireframe

TODO

### 4.3 3D Polygons

TODO

### 4.4 Spring-Mass Problem

TODO

## 5 Tips and Tricks

Extra things that are useful as a reference.

### 5.1 Dimensions of Nine Different Products

Scalar , Scalar :	$\mathbb{R} \times \mathbb{R}$	$\rightarrow \mathbb{R}$
Scalar , Column Vector :	$\mathbb{R} \times \mathbb{R}^n$	$\rightarrow \mathbb{R}^n$
Scalar , Row Vector :	$\mathbb{R} \times \mathbb{R}^{1 \times n}$	$\rightarrow \mathbb{R}^{1 \times n}$
Inner Product on $\mathbb{R}^n$ :	$\mathbb{R}^n \times \mathbb{R}^n$	$\rightarrow \mathbb{R}$
Inner Product on $\mathbb{R}^{1 \times n}$ :	$\mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times n}$	$\rightarrow \mathbb{R}$
Outer Product :	$\mathbb{R}^{m \times 1} \times \mathbb{R}^{n \times 1}$	$\rightarrow \mathbb{R}^{m \times n}$
Scalar, Matrix :	$\mathbb{R} \times \mathbb{R}^{m \times n}$	$\rightarrow \mathbb{R}^{m \times n}$
Matrix, Column Vector :	$\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times 1}$	$\rightarrow \mathbb{R}^{m \times 1}$
Row Vector, Matrix :	$\mathbb{R}^{1 \times m} \times \mathbb{R}^{m \times n}$	$\rightarrow \mathbb{R}^{1 \times n}$

### 5.2 Matrix Operations

TODO

### 5.3 The Matrix as a Function

Let  $f$  be a function:

$$f(x, y) = (x + 2y, 3x + 4y, 5x + 6y)$$

The function takes 2 elements as input and gives 3 elements as output,

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

Suppose  $x = 1$  and  $y = 2$ ,

$$\begin{aligned} f(1, 2) &= (1(1) + 2(2), 3(1) + 4(2), 5(1) + 6(2)) \\ &= (5, 11, 17) \end{aligned}$$

We could rewrite the input list  $(1, 2)$  and output list  $(5, 11, 17)$  as vectors, which reveals:

$$f \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

Now, let's say that function  $f$  is a Linear Map,  $A$ , from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ , and rewrite this in an algebraic form.

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

Looking back above to the function  $f$ , we can use this to rewrite  $A$  in a matrix notation.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

At this point, compare and contrast the dimensions of the matrix with the function definition,

$$\begin{aligned}f &: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \\ A &\in \mathbb{R}^{3 \times 2}\end{aligned}$$

and compare and contrast the input and output:

$$\begin{aligned}\mathbf{x} &\in \mathbb{R}^2 \\ A\mathbf{x} &\in \mathbb{R}^3\end{aligned}$$

We can rewrite the function again. This time, let's use our matrices to gain a new perspective of the nature of Linear Algebra:

$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + 2y \\ 3x + 4y \\ 5x + 6y \end{bmatrix}$$