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1 Definitions

1.1 Vector Space

A **vector space** is a set V with the following properties: (Assume that v, x, y, z are in V, and a, b, c are scalars in $\mathbb{R}$)

Commutativity:

- $x + y = y + x$

Associativity:

- $(x + y) + z = x + (y + z)$
- $(ab)v = a(bv)$

Additive Identity:

- there exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$

Additive Inverse:

- for all $v \in V$, there exists $x \in V$ such that $v + x = 0$

Multiplicative Identity:

- $1v = v$

Distributive Properties:

- $a(x + y) = ax + ay$
- $(a + b)v = av + bv$

1.2 Linear Combination

A linear combination of a list of vectors $v_1, \ldots, v_n$ is itself a vector, taking the form:

$$a_1 v_1 + \ldots + a_m v_m$$

where each $a_1, \ldots a_n \in \mathbb{R}$

1.3 Span

The set of all linear combinations of a list of vectors $v_1, \ldots, v_n$ is called the **span** of $v_1, \ldots, v_n$, and is defined:

$$\text{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n \ : \ a_1, \ldots, a_m \in \mathbb{R}\}$$

If the span is equal to some space $\text{span}(v_1, \ldots, v_n) = V$, then you could say that $v_1, \ldots, v_n$ **spans** V.

1.4 Linearly Independent

For $v_1, \dots, v_n \in V$ and $a_1, \dots, a_n \in \mathbb{R}$ such that:

$$a_1 v_1 + \dots + a_n v_n = 0$$

The list of vectors $v_1, \dots, v_n$ is called **linearly independent** when

$$a_1 = \dots = a_n = 0$$

for all possible values of $v_1, \dots, v_n$.

1.5 Basis

A **basis** of V is a list of vectors in V that is both linearly independent and spans V.

The **Standard Basis** of the vector space $\mathbb{R}^n$ is

$$(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$$

which could also be written, using matrix bracket notation, as:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

1.6 Dimension

The dimension of a vector space is the length of any basis of the vector space. For example,

$$\dim \mathbb{R}^n = n$$

1.7 Inner Product

For a pair of vectors $u, v \in V$ in the same vector space (they are both in $\mathbb{R}^n$ for example), the Inner Product is defined as:

$$u \cdot v = u_1 v_1 + ... + u_n v_n$$

which is also sometimes written using angular brackets:

$$\langle u, v \rangle$$

Keep in mind that the dimension of u and v must be the same. Using matrix dimension notation:

$$u_{\{n \times 1\}} \cdot v_{\{n \times 1\}}$$

The **Inner Product** is also a function $f : (\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$. The input is an ordered pair of vectors, and the output is a number. Inner products have the following properties:

Positivity:

- $\langle v, v \rangle \geq 0$ for all $v \in V$

Definiteness:

- $\langle v, v \rangle = 0$ if and only if $v = 0$

Additivity in First Slot:

- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$

Homogeneity in First Slot:

- $\langle au, v \rangle = a\langle u, v \rangle$ for all $a \in \mathbb{R}$ and all $u, v \in V$

In another definition of the Inner Product, the concepts of "additivity" and "homogeneity" are combined into a concept called "linearity". **Bilinearity** is when there is linearity in both the First and Second slots. Additionally, there is a concept called **Symmetry** for all real numbers.

For $x, y, z \in V$ and $a, b \in \mathbb{R}$:

Bilinearity:

- Additivity and Homogeneity in First and Second Slot:
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
- $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$

Symmetry:

- $\langle x, y \rangle = \langle y, x \rangle$

1.8 Norm

The Norm of a vector x is defined as the square root inner product of x with itself:

$$\|x\| = \sqrt{\langle x, x \rangle}$$

The Euclidean Norm, also called 2-norm, is defined:

$$\|x\|_2 = \sqrt{{x_1}^2 + \ldots + {x_n}^2}$$

which has the following properties:

Positivity:

- $\|x\| \geq 0$
- $\|x\| = 0$ if and only if $x = 0$

Homogeneity:

- $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{R}$

Triangle Inequality:

- $\|x + y\| \leq \|x\| + \|y\|$

1.9 Orthogonal

Two vectors $u, v \in V$ are called **orthogonal** if the inner product between them is 0,

$$\langle u, v \rangle = 0$$

you could also say "u is orthogonal to v". Orthogonal is another way of saying "at right angles to each other", or "perpendicular".

1.10 Linear Map

A linear map from vector space V to vector space W is a function $T : V \to W$ with the following properties:

Additivity:

- $T(u+v) = Tu + Tv$ for all vectors $u, w \in V$

Homogeneity:

- $T(av) = a(Tv)$ for all $a \in \mathbb{R}$ and all $v \in V$

1.11 Linear Maps and Matrices

Suppose M is a linear map $f : \mathbb{R}^a \to \mathbb{R}^b$, then M can be written as b-by-a matrix:

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,a} \\ \vdots & \vdots & \vdots \\ x_{b,1} & \cdots & x_{b,a} \end{bmatrix}$$

2 Proofs

2.1 Cosine Formula for Inner Product

For two non-zero vectors $x, y \in V$,

$$\langle x, y \rangle = \|x\|\|y\| \cos\theta$$

where the angle $\angle xy = \theta$.

Proof:

There are two cases we need to write a proof for.

- Case 1: when x and y are not scalar multiples of each other.
- Case 2: when x and y are scalar multiples.

Case 1:

For any triangle with sides a, b, c, The Law of Cosines states,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where the angle $\angle ab = \theta$. For vectors $x, y \in V$, we can treat them as sides of the triangle. Let:

$$\begin{aligned} a &= \|x\| \\ b &= \|y\| \\ c &= \|x - y\| \end{aligned}$$

Which allows us to rewrite the Law of Cosines:

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$$

Start with the definition of Inner Product, and apply its algebraic properties (notably the Bilinearity property), to show that Law of Cosines for Inner Products is correct.

$$\begin{aligned} &\|x-y\|^2 \\ &= \langle x-y, x-y \rangle \\ &= \langle x, x-y \rangle - \langle y, x-y \rangle \\ &= \Big(\langle x, x \rangle - \langle x, y \rangle\Big) - \Big(\langle y, x \rangle - \langle y, y \rangle\Big) \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \end{aligned}$$

Returning to the Law of Cosines,

$$\begin{aligned} \|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta \\ \cancel{\|x\|^2} - 2\langle x, y \rangle + \cancel{\|y\|^2} &= \cancel{\|x\|^2} + \cancel{\|y\|^2} - 2\|x\|\|y\|\cos\theta \\ \cancel{-2}\langle x, y \rangle &= \cancel{-2}\|x\|\|y\|\cos\theta \\ \langle x, y \rangle &= \|x\|\|y\|\cos\theta \end{aligned}$$

□

Case 2:

Since x and y are scalar multiples of each other, we can write,

$$y = cx$$

for some scalar $c \in \mathbb{R}$ where $c \neq 0$ (since the theorem statement says that x and y are "nonzero vectors"). Now, to find the value of θ, we look at the value of c:

- If $c > 0$, then $\theta = 0$, and $\cos\theta = 1$
- If $c < 0$, then $\theta = \pi$, and $\cos\theta = -1$

Define the sign of c, so that we can use it in our proof:

$$\operatorname{sign}(c) = \cos\theta$$

And here's the proof:

$$\begin{aligned} \langle x, y \rangle &= \langle cx, x \rangle \\ &= c\langle x, x \rangle \\ &= c\|x\|^2 \\ &= c\|x\|\|x\| \\ &= c\sqrt{(x_1{}^2 + \ldots + x_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\sqrt{c^2(x_1{}^2 + \ldots + x_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\sqrt{(c^2x_1{}^2 + \ldots + c^2x_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\sqrt{(y_1{}^2 + \ldots + y_n{}^2)}\ \|x\| \\ &= \operatorname{sign}(c)\|y\|\|x\| \\ &= \|x\|\|y\|\cos\theta \end{aligned}$$

□

2.2 Orthogonal Decomposition

Suppose $u, v \in V$ with $v \neq 0$, Set:

$$c = \frac{\langle u, v \rangle}{\|v\|^2}$$

and

$$w = u - \frac{\langle u, v \rangle}{\|v\|^2} v$$

Then

$$\langle w, v \rangle = 0 \quad \text{and} \quad u = cv + w$$

2.3 Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Proof:

If $v = 0$, then both sides of the inequality are equal to 0. If $v \neq 0$, then consider the orthogonal decomposition:

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + w$$

where w is orthogonal to v. By the Pythagorean Theorem:

$$\|u\|^2 = \left\| \frac{\langle u, v \rangle}{\|v\|^2} \right\| + \|w\|^2$$

$$= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2$$

$$\geq \frac{|\langle u, v \rangle|^2}{\|v\|^2}$$

2.4 Triangle Inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof:

$$\begin{aligned} \|u + v\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle y.y \rangle + \langle x, y \rangle + \langle y, x \rangle \\ &= \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \end{aligned}$$

3 Matrices

3.1 Algebraic Properties of Matrices

Compare these with the properties of Vector Space.

Protip: Matrices are in Vector Space.

Commutativity:

- $A + B = B + A$

Associativity:

- $A + (B + C) = (A + B) + C$

Additive Identity:

- $A + 0 = A$

Additive Inverse:

- $A + (-A) = 0$

Distributivity of matrix addition:

- $a(A + B) = aA + aB$

Distributivity of scalar addition:

- $(a + b)A = aA + bA$

Associativity of scalar multiplication

- $a(bA) = (ab)A$

Multiplicative Identity of scalar multiplication

- $1A = A$

3.2 Algebraic Properties of Matrix Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

3.3 Leading Entry

The Leading Entry of a row in a matrix the is first non-zero element in that row (from left-to-right).

3.4 Special Notations

Entries

- A_{ij} of matrix A is the entry in the i^{th} row and j^{th} column.
- I like to use $A_{i,j}$ or $A_{[i,j]}$ depending on the situation.

The $1 - \star - \times - 0$ notation:

- 1 : must be a 1
- $\star$: Non-zero numbers, $= \{c \in \mathbb{R} : c \neq 0\}$
- $\times$: any number $= \{c \in \mathbb{R}\}$
- 0 : must be a 0

MATLAB Syntax and Commands:

- `A(i,k)` returns the entry $A_{i,k}$
- `A(i,:)` returns the i^{th} row
- `A(:,k)` returns the k^{th} column
- `numel(A)` returns the number of elements in matrix A
- `nnz(A)` returns the number of non-zero elements in A

3.5 Main Diagonal

For a matrix entry $a_{i,k}$, the main diagonal entries would be defined as the set:

$$\{a_{i,k} : i = k\}$$

In the following example, the Main Diagonal would be the 1s:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Non-Diagonal entries are all values that are *not* in the main diagonal:

$$\{a_{i,k} : i \neq k\}$$

3.6 Diagonal Matrix

Diagonal Matrix is a matrix where all non-diagonal entries are 0.

For example, the following is a Diagonal Matrix:

$$\begin{bmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$$

3.7 Identity Matrix

An Identity Matrix, denoted I_n or just I, is a square matrix in $\mathbb{R}^{n \times n}$ where all diagonal entries are 1, and all non-diagonal entries are 0. For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When used in Matrix Multiplication, for some matrix $A \in \mathbb{R}^{m \times n}$, the Identity Matrix has the property:

$$I_m A = A I_n = A$$

3.8 Lower-Triangular Entries

Lower-Triangular Entries of a matrix are either: on the diagonal, or below the diagonal.

$$\{L_{i,k} : i \geq k\}$$

Strictly Lower-Triangular Entries of a matrix are only the values below the diagonal:

$$\{L_{i,k} : i > k\}$$

3.9 Lower-Triangular Matrix

A Lower-Triangular Matrix, $L \in \mathbb{R}^{n \times n}$, is a square matrix such that

$$L_{i,k} = 0 \quad \text{for all } i < k$$

For example, in this Lower-Triangular Matrix, $L \in \mathbb{R}^{3 \times 3}$,

$$\begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{bmatrix}$$

the lower-triangular entries can be anything, and the rest must be 0.

3.10 Unit Lower-Triangular Matrix

The Unit Lower-Triangular Matrix, $L \in \mathbb{R}^{n \times n}$ is both:

$$\begin{aligned} L_{i,k} &= 1 \qquad \text{for all } i = k \\ L_{i,k} &= 0 \qquad \text{for all } i < k \end{aligned}$$

An example of a Unit Lower-Triangular Matrix, $L \in \mathbb{R}^{3 \times 3}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & \times & 1 \end{bmatrix}$$

3.11 Upper-Triangular Matrix

Upper-Triangular Entries are defined as:

$$\{U_{i,k} : i \leq k\}$$

Strictly-Upper-Triangular Entries are defined as:

$$\{U_{i,k} : i < k\}$$

Upper-Triangular Matrix example:

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

Unit Upper-Triangular Matrix example:

$$\begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}$$

Quite similar to the Lower-Triangular Matrix definitions and examples.

3.12 Bands of a Matrix

Diagonal Band

The d^{th}-diagonal-band of a matrix A is the set of entries:

$$d^{th} \text{ diagonal band} = \{A_{i,k} : i - k = d\}$$

For example, the 0-diagonal-band is the main diagonal, and the 2-band of $A \in \mathbb{R}^3$ would be:

$$\left\{A_{[0,2]}, A_{[1,1]}, A_{[2,0]}\right\}$$

Upper-Triangular Bands :

- Set of entries
- $\{A_{i,k} : i - k \leq 0\}$

Lower-Triangular Bands :

- Set of entries
- $\{A_{i,k} : k \geq 0\}$

Lower Bandwidth :

- Number
- d such that $A_{i,k} = 0$ for $(i - k > d)$.
- The lowest band before everything becomes 0s.

Upper Bandwidth:

- Number
- d such that $A_{i,k} = 0$ for $(i - k < d)$.
- The highest band before everything becomes 0s.

3.13 Outer Product of Vectors

For $x \in \mathbb{R}^m, y \in \mathbb{R}^n$ the outer product is defined,

$$x \otimes y = xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & \vdots & \vdots \\ x_my_1 & \cdots & x_my_n \end{bmatrix}$$

you could also say that the outer product is a function:

$$(\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}^{m \times n}$$

3.14 Rank-one Updates

For $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^m, y \in \mathbb{R}^n$, the rank-one update is defined,

$$A + xy^T$$

3.15 Shear

Shear matrices are Rank-One Updates to the Identity matrix. An $n \times n$ shear matrix is:

$$S_{[i,k]}(c) = I_n + c\ \mathbf{e}_i(\mathbf{e}_k{}^T)$$

where $\mathbf{e}$ is the standard basis, and $i \neq k$.

Example: where $i = 3, k = 1, c = -5, n = 3$,

$$S_{[3,1]}(-5) = I_3 + -5\mathbf{e}_3(\mathbf{e}_1{}^T)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + -5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

3.16 Dilation

For $n \in \mathbb{N},\ j \in \mathbb{N},\ j \leq n$, define an $n \times n$ dilation matrix to be:

$$D_j(c) = I_n + (c - 1)e_j(e_j{}^T)$$

3.17 Transposition

For $n \in \mathbb{N},\ i \neq k$, define the $n \times n$ transposition matrix to be:

$$P_{[i,k]} = e_i(e_k{}^T) + e_k(e_i{}^T) + \sum_{\substack{j=1 \\ j \neq k}}^{n} e_j(e_j{}^T)$$

Example:

$$P_{[2,4]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3.18 Gauss Transform

Let $n, k \in \mathbb{N},\ k < n,\ T \in \mathbb{R}^n$, where T is a vector whose first k components are zero.

$$T^T = \begin{bmatrix} 0 & \cdots & T_{k+1} & \cdots & T_n \end{bmatrix}$$

The Gauss Transformation is a matrix

$$L_k = I_n - T(e_k{}^T)$$

4 Applications

Examples of applying Linear Algebra to other things. Includes models made with Vectors and Matrices.

4.1 Incidence Matrix of a Graph

- Rows = Edges
- Columns = Nodes

Undirected Incidence Matrix:

- 1 if an edge connects a node
- 0 otherwise

Directed Incident Matrix

- 1 if edge pointing away from node. (leaves)
- -1 if edge is pointing inward to node. (enters)
- 0 otherwise.

4.2 3D Wireframe

The Wireframe model has a list of vectors in $\mathbb{R}^3$, where each corresponds to a position in the space. They are put together into a vertex table in $\mathbb{R}^{3\times n}$ like so:

$$\begin{bmatrix} v_{1_x} & \cdots & v_{n_x} \\ v_{1_y} & \cdots & v_{n_y} \\ v_{1_z} & \cdots & v_{n_z} \end{bmatrix}$$

They can be connected using an Wireframe Edgetable, where the vertex number corresponds to the column number from the table above.

$$\begin{bmatrix} \texttt{edge}_1 & \texttt{StartVertex}_1 & \texttt{EndVertex}_1 \\ \vdots & \vdots & \vdots \\ \texttt{edge}_m & \texttt{StartVertex}_m & \texttt{EndVertex}_m \end{bmatrix}$$

This is enough information to create wireframes.

4.3 3D Polygons

To create polygons, you also need to include a polygon face table

$$\begin{bmatrix} \texttt{face} & \texttt{vertex1} & \texttt{vertex2} & \texttt{vertex3} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

with a matrix of faces and the vertices that generate them.

4.4 Spring-Mass Problem

- $x_1(t)$: position of mass 1 at time t
- x_0 : vector of positions at time 0
- $x(t)$: vector of positions at time t.
- $u(t) = x(t) - x_0$: vector of Δx between time 0 and t.
- Let $e(t)$ be the "elongation" of the spring, the change in length of the spring between time 0 and t.
- $e(0)$ would be set to 0 by default, since that is our point of reference for checking the change in length of the spring.
- To find the length of the spring, measure the distance between the two masses that hold the spring together (one of those might just be the ceiling). For your own sanity, try finding the initial length: l_0 and the final length: l_f, then set $e(t) = l_f - l_0$.
- Try to set $e(t)$ in terms of $u(t)$.

Finding $e(t)$:

However, it will be useful for this problem to define each $e_i(t)$ in terms of $u_i(t)$, where i represents one of the masses.

$$e_1(t) = a_1u_1(t) + \ldots + a_nu_n(t)$$
$$e_2(t) = a_1u_1(t) + \ldots + a_nu_n(t)$$

and so on, one equation for each of the springs. (With different values of a in each equation). There are n masses total, and each of them is an argument to the function. Thus, we are looking at a linear map here:

- $a_1, \ldots, a_n \in \mathbb{R}$
- Let m be the number of masses
- Let n is the number of springs
- Notice that $n = m + 1$ in this problem.
- Our linear map is from $\mathbb{R}^m \to \mathbb{R}^n$

Which means we can write it as matrix:

$$e(t) = \begin{bmatrix} e_1(t) \\ \vdots \\ e_n(t) \end{bmatrix} = A_1u_m(t) + \ldots A_mu_m(t) = Au(t)$$

where $A_1, \ldots, A_m \in \mathbb{R}^{\ltimes}$ and $A \in \mathbb{R}^{\gtrdot\times\ltimes}$

Finding Forces:

- Each mass has 3 forces acting upon it
- f_s is restoring force (going up) given by Hooke's law.

Find the restoring force

$$f_s(t) = \begin{bmatrix} f_{s_1}(t) \\ \vdots \\ f_{s_n}(t) \end{bmatrix} = \begin{bmatrix} k_1e_1(t) \\ \vdots \\ k_ne_n(t) \end{bmatrix} = e_1(t)\begin{bmatrix} k_1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + e_n(t)\begin{bmatrix} 0 \\ \vdots \\ k_n \end{bmatrix}$$

Find net internal forces

$$y_1(t) = f_{s_2}(t) - f_{s_1}(t)$$
$$\vdots$$
$$y_m(t) = f_{s_n}(t) - f_{s_{(n-1)}}(t)$$

Notice again that $n = m + 1$ in this problem. Write each $y_i(t)$ for $i \in [1, m]$ as a combination of all forces $f_s(t)$:

$$y_{1(t)} = (-1)f_{s_{1(t)}} + (1)f_{s_{2(t)}} + \ldots + (0)f_{s_{n(t)}}$$
$$\vdots$$
$$y_{m(t)} = (0)f_{s_{1(t)}} + \ldots + (-1)f_{s_{(n-1)(t)}} + (1)f_{s_{n(t)}}$$

Write the matrix version of $y(t)$, (see the Trick: Matrix as a Function):

$$y(t) = \begin{bmatrix} -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} f_{s_1(t)} \\ \vdots \\ f_{s_{(n-1)}(t)} \\ f_{s_n(t)} \end{bmatrix} = -A^T f_s(t)$$

And,

$$\begin{aligned} y(t) &= -A^T f_s(t) \\ &= -A^T Ce(t) \\ &= -A^T CAu(t) \\ &= -Ku(t) \end{aligned}$$

where

$$K = A^T CA$$

and where

$$C = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix}$$

And eventually we get the net external forces,

$$Ku(t) = F_e(t)$$

which is a function we can use to calculate the force $F_e(t)$ given the displacement vector $u(t)$.

5 Tips and Tricks

Extra things that are useful as a reference.

5.1 Dimensions of Nine Different Products

$$\begin{aligned} \text{Scalar , Scalar}: &\quad \mathbb{R} \times \mathbb{R} && \to \mathbb{R} \\ \text{Scalar , Column Vector}: &\quad \mathbb{R} \times \mathbb{R}^n && \to \mathbb{R}^n \\ \text{Scalar , Row Vector}: &\quad \mathbb{R} \times \mathbb{R}^{1\times n} && \to \mathbb{R}^{1\times n} \\ \text{Inner Product on } \mathbb{R}^n: &\quad \mathbb{R}^n \times \mathbb{R}^n && \to \mathbb{R} \\ \text{Inner Product on } \mathbb{R}^{1\times n}: &\quad \mathbb{R}^{1\times n} \times \mathbb{R}^{1\times n} && \to \mathbb{R} \\ \text{Outer Product}: &\quad \mathbb{R}^{m\times 1} \times \mathbb{R}^{n\times 1} && \to \mathbb{R}^{m\times n} \\ \text{Scalar, Matrix}: &\quad \mathbb{R} \times \mathbb{R}^{m\times n} && \to \mathbb{R}^{m\times n} \\ \text{Matrix, Column Vector}: &\quad \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times 1} && \to \mathbb{R}^{m\times 1} \\ \text{Row Vector, Matrix}: &\quad \mathbb{R}^{1\times m} \times \mathbb{R}^{m\times n} && \to \mathbb{R}^{1\times n} \end{aligned}$$

5.2 The Matrix as a Function

Let f be a function:

$$f(x, y) = \big(x + 2y,\ \ 3x + 4y,\ \ 5x + 6y\big)$$

The function takes 2 elements as input and gives 3 elements as output,

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

Suppose $x = 1$ and $y = 2$,

$$\begin{aligned} &f(1, 2) \\ &= \Big(1(1) + 2(2),\ \ 3(1) + 4(2),\ \ 5(1) + 6(2)\Big) \\ &= (5, 11, 17) \end{aligned}$$

We could rewrite the input list $(1, 2)$ and output list $(5, 11, 17)$ as vectors, which reveals:

$$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

Now, let's say that function f is a Linear Map, A, from $\mathbb{R}^2$ to $\mathbb{R}^3$, and rewrite this in an algebraic form.

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

Looking back above to the function f, we can use this to rewrite A in a matrix notation.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$$

At this point, compare and contrast the dimensions of the matrix with the function definition,

$$\begin{aligned} f &: \mathbb{R}^2 \to \mathbb{R}^3 \\ A &\in \mathbb{R}^{3\times 2} \end{aligned}$$

and compare and contrast the input and output:

$$\begin{aligned} x &\in \mathbb{R}^2 \\ Ax &\in \mathbb{R}^3 \end{aligned}$$

We can rewrite the function again. This time, let's use our matrices to gain a new perspective of the nature of Linear Algebra:

$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + 2y \\ 3x + 4y \\ 5x + 6y \end{bmatrix}$$

6 Matrix Multiplication as the Composition of Linear Functions

Suppose there are two functions, f and g.

Let function f be defined as $f : R^2 \to R^3$ such that,

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0x + 1y \\ 2x + 3y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for some values $x, y \in R$.

Let function g be defined as $g : R^3 \to R^2$ such that,

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for some different values of $x, y, z \in R$.

You could write these as a composition of functions,

$$f \circ g : R^2 \to R^3 \to R^2$$

which just simplifies to:

$$f \circ g : R^2 \to R^2$$

So let's create some example input so we can see what's going on here. When composing functions, we don't actually care what the input is, but it helps for understanding.

Let our example input be a vector $a \in R^2$. Let's rewrite our composition of functions in a way that looks more familiar,

$$(f \circ g)(a) = g(f(a))$$

Take a look at what happens as the input is mapped to different sets through each function,

$$\begin{aligned} a &\in R^2 \\ f(a) &\in R^3 \\ g(f(a)) &\in R^2 \end{aligned}$$

Looking back at the definition of f, let's plug in the input a,

$$f(a) = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Looking back at the definition of g, let's put in $f(a)$,

$$g(f(a)) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

We have now created a new function!

To understand why multiplying matrices is still the composition of functions, consider how it looks when we remove a, and just focus on the matrices,

$$f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$$

And finally, let's say that A represents the first matrix, and B represents the second matrix,

$$f \circ g = AB$$

7 Linear Maps

7.1 Linear Maps Definitions

A Linear Map from vector spaces V to W is a function $T : V \to W$ with the following properties:

Additivity

- $T(u + v) = Tu + Tv$ for all $u, v \in V$

Homogeneity

- $T(cv) = c(Tv)$ for all $c \in F$ and all $v \in V$

Scalar Addition

- $(S + T)(v) = Sv + Tv$

Scalar Multiplication

- $(cT)(v) = c(Tv)$

Product of Linear Map

- $(ST)(u) = ST(u)$

7.2 Algebraic Properties

These properties can be derived from the Linear Map Definitions.

Associativity

- $(T_1T_2)T_3 = T_1(T_2T_3)$

Identity

- $TI = IT = T$

Distributive Properties

- $(S_1 + S_2)T = S_1T + S_2T$
- $S(T_1 + T_2) = ST_1 + ST_2$

7.3 Linear Maps and Basis of domain

Suppose some list of vectors $v_1, \ldots v_n$ is a basis V.

Therefore, there exists a unique linear map $T : V \to W$ such that

$$Tv_i = w_i$$

for $i = 1, \ldots, n$ and w is a list of vectors $w_i, \ldots, w_n$ in a different vector space W.

7.4 Matrix Multiplication as the Composition of Linear Functions

Suppose there are two functions, f and g.

Let function f be defined as $f : R^2 \to R^3$ such that,

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0x + 1y \\ 2x + 3y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for some values $x, y \in R$.

Let function g be defined as $g : R^3 \to R^2$ such that,

$$g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for some different values of $x, y, z \in R$.

You could write these as a composition of functions,

$$f \circ g : R^2 \to R^3 \to R^2$$

which just simplifies to:

$$f \circ g : R^2 \to R^2$$

So let's create some example input so we can see what's going on here. When composing functions, we don't actually care what the input is, but it helps for understanding.

Let our example input be a vector $a \in R^2$. Let's rewrite our composition of functions in a way that looks more familiar,

$$(f \circ g)(a) = g(f(a))$$

Take a look at what happens as the input is mapped to different sets through each function,

$$a \in R^2$$
$$f(a) \in R^3$$
$$g(f(a)) \in R^2$$

Looking back at the definition of f, let's plug in the input a,

$$f(a) = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Looking back at the definition of g, let's put in $f(a)$,

$$g(f(a)) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

We have now created a new function!

To understand why multiplying matrices is still the composition of functions, consider how it looks when we remove a, and just focus on the matrices,

$$f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$$

And finally, let's say that A represents the first matrix, and B represents the second matrix,

$$f \circ g = AB$$

Hardcore Way:

Look back up at the definitions for f and g. Let's try evaluating the input a.

$$f\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 0a_1 + 1a_2 \\ 2a_1 + 3a_2 \\ 4a_1 + 5a_2 \end{bmatrix}$$

We can use this as input to the function g,

$$g\left(\begin{bmatrix} 0a_1 + 1a_2 \\ 2a_1 + 3a_2 \\ 4a_1 + 5a_2 \end{bmatrix}\right) = \begin{bmatrix} 1(0a_1 + 1a_2) + 2(2a_1 + 3a_2) + 3(4a_1 + 5a_2) \\ 4(0a_1 + 1a_2) + 5(2a_1 + 3a_2) + 6(4a_1 + 5a_2) \end{bmatrix}$$

Applying the distributive property of real numbers, we can expand:

$$g(f(a)) = \begin{bmatrix} (1 \cdot 0)a_1 + (1 \cdot 1)a_2 + (2 \cdot 2)a_1 + (2 \cdot 3)a_2 + (3 \cdot 4)a_1 + (3 \\ (4 \cdot 0)a_1 + (4 \cdot 1)a_2 + (5 \cdot 2)a_1 + (5 \cdot 3)a_2) + (6 \cdot 4)a_1 + (6 \end{bmatrix}$$

And now simplify by combining similar terms together,

$$g(f(a)) = \begin{bmatrix} (1 \cdot 0 + 2 \cdot 2 + 3 \cdot 4)a_1 + (1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5)a_2 \\ (4 \cdot 0 + 5 \cdot 2 + 6 \cdot 4)a_1 + (4 \cdot 1 + 5 \cdot 3 + 6 \cdot 5)a_2 \end{bmatrix}$$

Now, separate out the a_1 and a_2 terms,

$$g(f(a)) = \begin{bmatrix} (1 \cdot 0 + 2 \cdot 2 + 3 \cdot 4) & (1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5) \\ (4 \cdot 0 + 5 \cdot 2 + 6 \cdot 4) & (4 \cdot 1 + 5 \cdot 3 + 6 \cdot 5) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Now, let's remove the input entirely. After all, we don't actually care what it is right now. Instead, let's call our functions something like A and B. Each of A and B is a matrix.

$$f \circ g = A \cdot B$$

7.5 TITLE

TODO

7.6 TITLE

TODO