Definitions

Vector Space

Assume that u, v, w are vectors in V, and a, b, c are scalars in $\mathbb{R}$. A **vector space** is a set V with the following properties:

Commutativity:

- $u + v = v + u$

Associativity:

- $(u + v) + w = u + (v + w)$
- $(ab)v = a(bv)$

Additive Identity:

- there exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$

Multiplicative Identity:

- for all $v \in V$, there exists $w \in V$ such that $v + w = 0$

Distributive Properties:

- $a(u + v) = au + av$
- $(a + b)v = av + bv$

Linear Combination

A linear combination of a list of vectors $v_1, \ldots, v_n$ is itself a vector, taking the form:

$$a_1 v_1 + \ldots + a_m v_m$$

where each $a_1, \ldots a_n \in \mathbb{R}$

Span

The set of all linear combinations of a list of vectors $v_1, \ldots, v_n$ is called the **span** of $v_1, \ldots, v_n$, or $\text{Span}(v_1, \ldots, v_n)$. Defined as:

$$\text{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n \ : \ a_1, \ldots, a_m \in \mathbb{R}\}$$

If the span is equal to some space $\text{span}(v_1, \ldots, v_n) = V$, then you could say that $v_1, \ldots, v_n$ **spans** V.

Linearly Independent

For $v_1, \ldots, v_n \in V$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that:

$$a_1 v_1 + \cdots + a_n v_n = 0$$

The list of vectors $v_1, \ldots, v_n$ is called **linearly independent** when

$$a_1 = \cdots = a_n = 0$$

for all possible values of $v_1, \ldots, v_n$.

Basis

A **basis** of V is a list of vectors in V that is both linearly independent and spans V.

The **Standard Basis** of the vector space $\mathbb{R}^{\ltimes}$ is

$$(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$$

which could also be written, using matrix bracket notation, as:

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Dimension

The dimension of a vector space is the length of any basis of the vector space. For example,

$$\dim \mathbb{R}^{\ltimes} = n$$

Linear Map

A linear map from vector space V to vector space W is a function $T : V \to W$ with the following properties:

Additivity

- $T(u + v) = Tu + Tv$ for all vectors $u, w \in V$

Homogeneity

- $T(av) = a(Tv)$ for all $a \in \mathbb{R}$ and all $v \in V$

Inner Product

For a pair of vectors $u, v \in V$ in the same vector space (they are both in $\mathbb{R}^n$ for example), the Inner Product is defined as:

$$u \cdot v = u_1 v_1 + \ldots + u_n v_n$$

which is also sometimes written using angular brackets:

$$\langle u, v \rangle$$

Keep in mind that the dimension of u and v must be the same. Using matrix dimension notation:

$$u_{\{n \times 1\}} \cdot v_{\{n \times 1\}}$$

Norm

The norm of a vector v is defined:

$$\|x\| = \sqrt{{x_1}^2 + \ldots + {x_n}^2}$$

Linear Maps and Matrices

Suppose M is a linear map $f : \mathbb{R}^a \to \mathbb{R}^b$, then M can be written as b-by-a matrix:

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,a} \\ \vdots & \vdots & \vdots \\ x_{b,1} & \cdots & x_{b,a} \end{bmatrix}$$

Proofs

Law of Cosines

TODO

Triangle Inequality

TODO

Cauchy-Schwartz Inequality

TODO