Homework 1

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Problem 1

Show that $|A| = \infty$ if and only if $|\mathcal{P}(A)| = \infty$.

Suppose $|A| = \infty$. The definition of a powerset states that $\mathcal{P}(A) = \{x : x \subseteq A\}$. Since there are an infinite number of elements in A, we could create infinite subsets consisting of only 1 element each,

$$\{a_k\} \in \mathcal{P}(A) \text{ for any } k \in \mathbb{N}$$

Since there will be infinite elements in $\mathcal{P}(A)$, we can conclude that: If $|A| = \infty$, Then $|\mathcal{P}(A)| = \infty$.

Suppose $|\mathcal{P}(A)| = \infty$, and suppose $|A| = n$, such that n is finite. Since $|A| = n$, this would conclude that $|\mathcal{P}(A)| = 2^n$, but this is impossible because 2^n is finite, and our supposition says that $|\mathcal{P}(A)|$ is infinite. Since $|A|$ being finite would lead to a contradition, $|A|$ must be infinite. Thus, If $|P(A)| = \infty$, Then $|A| = \infty$ must also be true.

Problem 2.

Suppose $|A| = m$ and $|B| = n$. Since the definition of union is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

There is a possibility that all elements from A and from B are included in the union:

$$|A \cup B| \leq |A| + |B| = m + n$$

For the intersection,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

each element x must be in both sets, so the cardinality can exceed neither m nor n,

$$|A \cap B| \leq m \quad \text{and} \quad |A \cap B| \leq n$$

Which can be rewritten in the form,

$$|A \cap B| \leq \min(m, n)$$

Other Problems

1.4.4. $\mathcal{P}(\{\mathbb{R}, \mathbb{Q}\}) = \Big\{\emptyset, \{\mathbb{R}\}, \{\mathbb{Q}\}, \{\mathbb{R}, \mathbb{Q}\}\Big\}$

1.4.14. $|\mathcal{P}(\mathcal{P}(A))| = 2^{2^m}$

1.4.15. $|\mathcal{P}(A \times B)| = 2^{m \cdot n}$

1.4.20. $\Big|\{X \subseteq \mathcal{P}(A) : |X| \leq 1\}\Big| = m + 1$

1.5.1.a. $A \cup B = \{1, 3, 4, 5, 6, 7, 8, 9\}$

1.5.1.b. $A \cap B = \{4, 6\}$

1.5.1.c. $A - B = \{1, 3, 9, 7\}$

1.5.3.a. $(A \times B) \cap (B \times B) = \Big\{(1, 2), (1, 1)\Big\}$

1.5.3.g. $\mathcal{P}(A) - \mathcal{P}(B) = \Big\{\{0\}, \{0, 1\}\Big\}$