Homework 8

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Problem 1

Proposition. For $n \in \mathbb{Z}$, If n^2 is even, then n is even.

Proof. (Contrapositive). Suppose n is not even.
Thus, n is odd, and can be expressed by the equation $n = 2x + 1$ for some integer x. Squaring n and substituting, we find that

$$\begin{aligned} n^2 &= (2x+1)^2 \\ &= 4x^2 + 4x + 1 \\ &= 2(2x^2 + 2x) + 1 \end{aligned}$$

Let z be an integer such that $z = 2x^2 + 2x$, which allows us to rewrite the equation above as

$$n^2 = 2z + 1$$

which reveals that n^2 is odd.
Therefore, n^2 is not even. ■

Problem 3

Proposition. For $a, b \in \mathbb{Z}$, If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Proof. (Contrapositive). Suppose a is not odd or b is not odd.
Let x, y and z be integers.

Case 1. a is not odd, b is not odd.
We can represent a and b using the equations $a = 2x$ and $b = 2y$.
Using these definitions of a and b, we can write the equality,

$$\begin{aligned} a^2(b^2 - 2b) &= (2x)^2\big((2y)^2 - 2(2y)\big) \\ &= 4x^2(4y^2 - 4y) \\ &= 2(8x^2y^2 - 8x^2y) \end{aligned}$$

Let integer $z = 8x^2y^2 - 8x^2y$, and substitute into the equality above,

$$a^2(b^2 - 2b) = 2z$$

Therefore, $a^2(b^2 - 2b)$ is not odd.

Case 2. a is not odd, b is odd.
We can represent a and b using the equations $a = 2x$, and $b = 2y + 1$. Using these definitions of a and b, we can write the equality,

$$\begin{aligned} a^2(b^2 - 2b) &= (2x)^2\big((2y+1)^2 - 2(2y+1)\big) \\ &= 4x^2(4y^2) \\ &= 2(8x^2y^2) \end{aligned}$$

Let integer $z = 8x^2y^2$ and substitute into the equality above,

$$a^2(b^2 - 2b) = 2z$$

Therefore, $a^2(b^2 - 2b)$ is not odd.

Case 3. a is odd, b is not odd.
We can represent a and b using the equations $a = 2x + 1$, and $b = 2y$. Using these definitions of a and b, we can write the equality,

$$\begin{aligned} a^2(b^2 - 2b) &= (2x+1)^2\big((2y)^2 - 2(2y)\big) \\ &= (4x^2 + 4x + 1)(4y^2 - 4y) \\ &= 2 \cdot (4x^2 + 4x + 1)(2y^2 - 2y) \end{aligned}$$

Let integer $z = (4x^2 + 4x + 1)(2y^2 - 2y)$ and substitute into the equality above,

$$a^2(b^2 - 2b) = 2z$$

Therefore, $a^2(b^2 - 2b)$ is not odd. ■

Problem 7

Proposition. For $a, b \in \mathbb{Z}$, If both ab and $a+b$ are even, then both a and b are even.

Proof. (Contrapositive). Suppose either a is not even, or b is not even.
For each of these cases, it will be sufficient to prove that either ab is not even, or $a+b$ is not even.

Case 1. a is not even, b is even.
Let $a = 2x+1$ and $b = 2y$ for some $x, y \in \mathbb{Z}$. So,

$$\begin{aligned} a+b &= 2x+1+2y \\ &= 2(x+y)+1 \end{aligned}$$

Let some integer $z = x+y$. Thus, $a+b = 2z+1$. Therefore, $a+b$ is not even.

Case 2. a is even, b is not even.
This case is actually identical to Case 1, because we can let $a = 2y$ and $b = 2x+1$ for some $x, y \in \mathbb{Z}$, and due to the Commutative Law of Addition,

$$a+b = b+a$$

Hence, we can apply Case 1. Therefore, $a+b$ is not even.

Case 3. a is not even, b is not even.
Let $a = 2x+1$ and $b = 2y+1$ for some $x, y \in \mathbb{Z}$. So,

$$\begin{aligned} ab &= (2x+1)(2y+1) \\ &= 4xy+2x+2y+1 \\ &= 2(2xy+x+y)+1 \end{aligned}$$

Let some integer $z = 2xy+x+y$. Thus, $ab = 2z+1$. Therefore, ab is not even.

Therefore, in summary of all cases, either ab or $a+b$ is not even. ■

Problem 13

Proposition. For $a \in \mathbb{Z}$, If $a^2 \nmid 4$, then a is odd.

Proof. (Contrapositive). Suppose a is not odd.
Let $a = 2x$ for some $x \in \mathbb{Z}$. So,

$$\begin{aligned} a^2 &= (2x)^2 \\ &= 4(x^2) \end{aligned}$$

Let some integer $z = x^2$. Thus $a^2 = 4z$. Therefore a^2 is divisible by 4. ■