Herbie

or: what to do if you can't stop worrying and hate floating point

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Ambiata

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Arithmetic quiz

```
sum :: [Double] -> Double
```

Arithmetic quiz

```haskell
sum :: [Double] -> Double
sum = foldl (+) 0.0
```

Arithmetic quiz

```
sum :: [Double] -> Double
sum = foldl (+) 0.0 . sort
```

Arithmetic quiz

```
sum :: [Double] -> Double
sum [] = 0.0
sum xs =
  uncurry go $ bisect xs
  where
    go [y] [] = y
    go [] [z] = z
    go [y] [z] = y + z
    go ys zs =
      let (y1s, y2s) = bisect ys
          (z1s, z2s) = bisect zs in
      (go y1s y2s) + (go z1s z2s)

    bisect ws =
      let len = length ws `div` 2 in
      (take len ws, drop len ws)
```

Arithmetic quiz

```haskell
sum :: [Double] -> Double
sum = fst . foldl add (0.0, 0.0)
  where
    add (acc, err) x =
      let
        -- Correct for the error from the last iteration.
        y = x - err
        acc' = acc + y
        -- Algebraically, err' should be zero.
        err' = (acc' - acc) - y
      in (acc', err')
```

A cautionary tale

```
stddev :: Double -> Double
stddev variance = sqrt $ abs variance
```

A cautionary tale

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stddev variance = sqrt $ abs variance
```

$$\sigma^2_{1:n} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_{1:n})^2$$

Refresher on IEEE 754

$$F = \{s \times b^e | s, e \in \mathbb{Z}; b \in \mathbb{N}\}$$

- In general, arithmetic operations are commutative but not associative.
- Common causes of error are subtracting very similar values and adding very different values.
- Multiplication, squaring et cetera can compound existing error.
- Rounding contributes at most 0.5 ULPs (units in the last place) of error per operation.

Refresher on IEEE 754

$$n \cdot \frac{1}{n} = 1$$

```
λ let n = 10
λ sum . replicate n $ 1 / n
0.9999999999999999
```

$$x + y - x = y$$

```
λ let x = 10**(-10)
λ let y = 10**20
λ printf "%f, %f\n" y z
0.0000000001, 100000000000000000000.0
λ printf "%f\n" $ y + x - y
0.0
```

What is error?

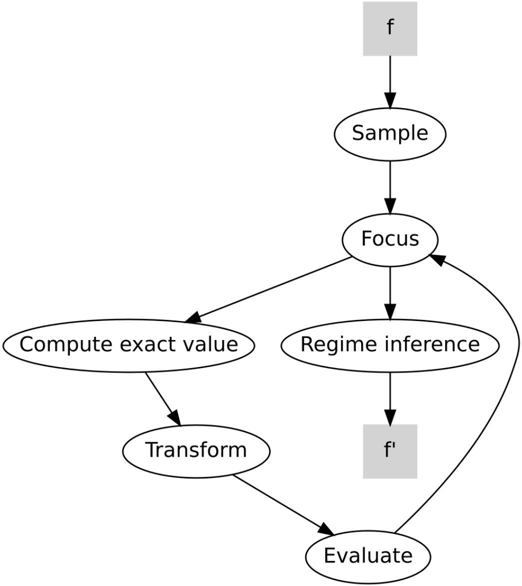
$$\epsilon(x, y) = \log_2 |z \in FP| min(x, y) \leq z \leq max(x, y)|$$

- Error in ULPs: number of floating-point values between the exact result and the approximate result[3].
- Consistent representation of error independent of magnitude.
- The binary log approximates "number of incorrect bits".

Herbie

- Provides automated synthesis of more accurate versions of floating-point computations[2].
- Written by Pavel Panchekha et. al. at the University of Washington.
- Around 10KLOC of Racket.
- Intended for scientists, statisticians, people who don't necessarily have a background in numerical analysis.

Herbie



f
Sample
Focus
Compute exact value
Regime inference
Transform
f'
Evaluate

Sample

- Sample inputs are drawn uniformly from the computation's domain.
- Herbie defaults to 256 samples per iteration.
- More samples leads to greater probability of identifying regions of the domain with differing error behaviour.

Error and input domain

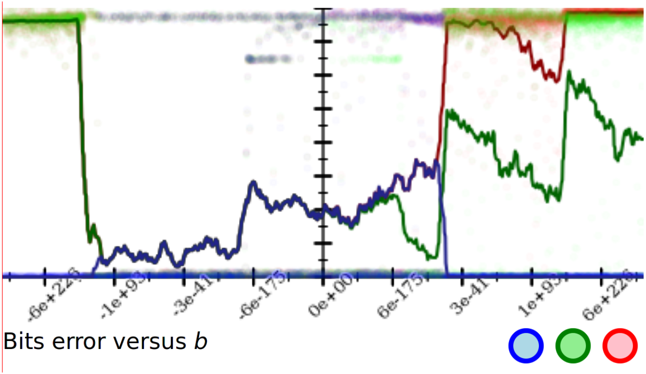


Figure: Herbie's error estimates for the quadratic formula $\frac{-b+\sqrt{b^2-4ac}}{2a}$

Focus

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

- For each operator $\star$, evaluate operands a and b in exact arithmetic for all of the sampled inputs.
- Evaluate $a \star b$ in both exact arithmetic and floating-point arithmetic.
- Focus search on operators which contribute the most error.

“Exact” results

Even with arbitrary precision, how do you know how many bits are “enough”? Herbie guesses:

- Compute the result with n bits of precision over your entire sample.
- Do it again with $2n$ bits.
- If the most significant 64 bits of the results match, this is your exact answer; otherwise continue.

Transform

- Hill-climbing greedy search of a database of rewrite rules.
  - $x^2 - y^2 \rightsquigarrow (x-y)(x+y)$
- Transformations are either mathematical identities or near-identities - sometimes using an approximation can result in a numerical result closer to the true value.
- Followed by a series-expansion pass.
  - $e^x - 1 \rightsquigarrow x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ for $x \approx 0$
- Simplification phase to cancel like terms, et cetera - pattern-match expressions which can be reduced.
  - $\frac{y}{e^{x-x}} \rightsquigarrow y$

Regime inference

- Many computations have error characteristics which vary based on the magnitude of the input within the domain ("regime").
- This necessitates the selection of different implementations at runtime based on the value of the inputs, as no single formula will be accurate in all cases.
- Herbie localises regime boundaries using the segmented least squares dynamic programming algorithm[1].
- To avoid overfitting, a penalty is added when evaluating each segmentation - one bit of error per branch.

A simple example

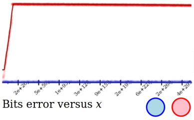
$$\sqrt{x} - \sqrt{x-1}$$

```
(herbie-test (x)
  "Subtracting square roots"
  (- (sqrt x)
     (sqrt (- x 1))))
```

A simple example

$\sqrt{x} - \sqrt{x-1}$

Test: Subtracting square roots
Bits: 128 bits



Bits error versus x

Time:	Input Error:	Output Error:	Log:
10.6 s	59.9	0.3	

$\frac{1}{\sqrt{x}+\sqrt{x-1}}$

- Started with — 59.9 bits

$$\sqrt{x} - \sqrt{x-1}$$

Using strategy `rm` — 59.9 bits

- Applied flip-- to get — 59.4 bits

$$\sqrt{x} - \sqrt{x-1} \rightsquigarrow \frac{(\sqrt{x})^2 - (\sqrt{x-1})^2}{\sqrt{x}+\sqrt{x-1}}$$

- Applied simplify to get — 0.3 bits

$$\frac{(\sqrt{x})^2 - (\sqrt{x-1})^2}{\sqrt{x}+\sqrt{x-1}} \rightsquigarrow \frac{1}{\sqrt{x}+\sqrt{x-1}}$$

Removed slow pow expressions

Original test:

```
(lambda ((x default))
  #:name "Subtracting square roots"
  (- (sqrt x) (sqrt (- x 1))))
```

Figure: Herbie report for difference of square roots.

Combining variance of subsamples

$$\sigma^2_{1:n+m} = \frac{m(\sigma^2_{1:m} + \mu^2_{1:m}) + n(\sigma^2_{m+1:n} + \mu^2_{m+1:n})}{m+n} - \mu^2_{1:n+m}$$

- $\sigma^2_{a:b}$ is the variance of the subsample from values a to b.
- $\mu^2_{a:b}$ is the mean of the subsample from values a to b.

Combining variance of subsamples

```
(herbie-test (mu
              mu1
              [var1 (uniform 0 100000000)]
              [n (< 0 int)]
              mu2
              [var2 (uniform 0 100000000)]
              [m (< 0 int)])
    "Combine variance of subsamples"
    (- (/ (+ (* n (+ var1 (sqr mu1)))
             (* m (+ var2 (sqr mu2))))
          (+ n m))
       (sqr mu)))
```

Combining variance of subsamples

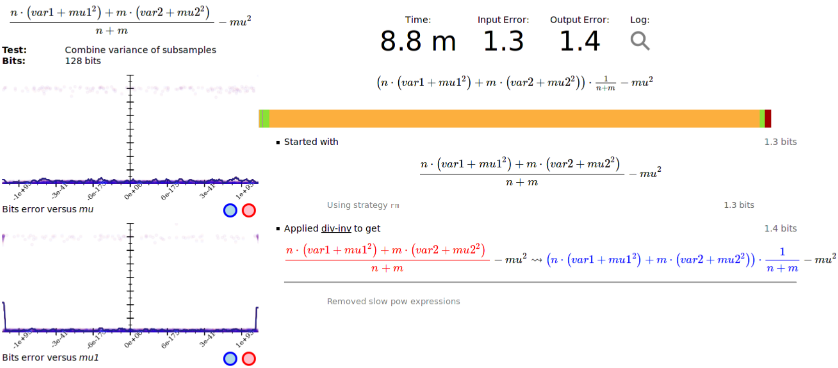


Figure: Herbie report for combining subsample variance.

Miscellanea

- Herbie: `https://github.com/uwplse/herbie`
- GHC plugin: `https://github.com/mikeizbicki/HerbiePlugin`
- Rust plugin: `https://github.com/mcarton/rust-herbie-lint`
- Valgrind plugin: `https://github.com/uwplse/herbgrind`
- These slides: `https://tesser.org/doc/slides/2016-05-25-fp-syd-herbie.pdf`

Bibliography

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