Math 725 Advanced Linear Algebra

HW7

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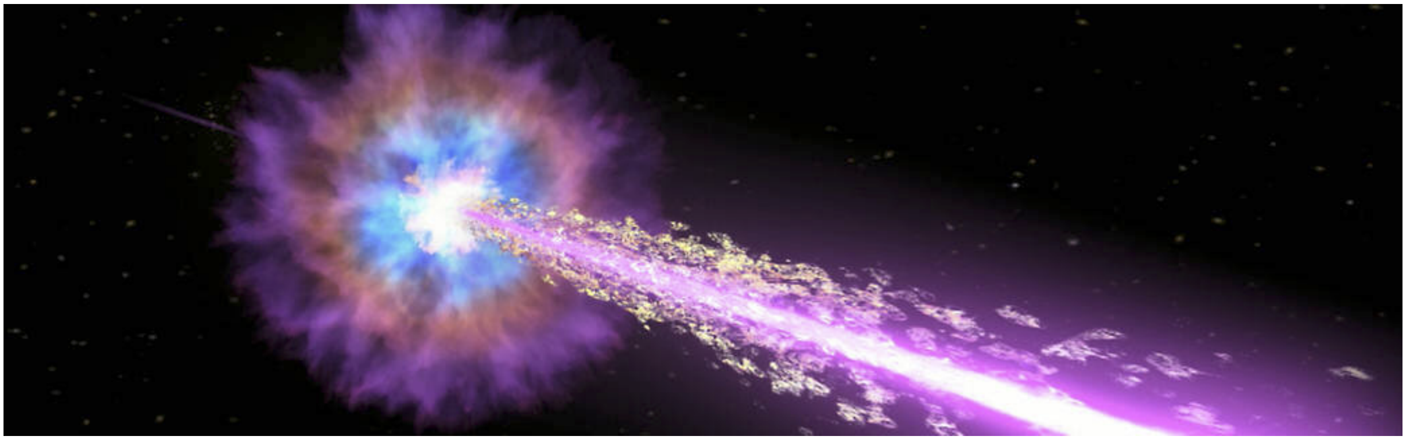
Inner Product Spaces

This final assigment explores problems based on the central theme of inner-product spaces. Problems were selected from Chapters 3, 5, 6, and 7.



Gamma Ray Source 221009A, NASA Artist

Additional resources:

```
Linear Algebra Done Right; Axler
Finite-Dimensional Vector Spaces; Halmos
Advanced Linear and Matrix Algebra; Nathaniel Johnston
The Linear Algebra a beginning Gradute Student Ought to Know; Golan.

https://github.com/celiopassos/linear-algebra-done-right-solutions; Passos, Ashrafian, Fangyuan
https://linearalgebras.com/; Rashidi, et al.
https://toanqpham.github.io/writing/; Pham
```

Each solutions was worked independently then validated against the aforementioned texts and solution guides. Passos' Solutions were used as a style guide for formatting. Passos' work is largely based on Rashidi and has been widely forked. Unfortunately Pham's notes are nolinger available, however I acknowledge his influence on my approach and methodology.

Instructions:

Your final assignment/exam is this:

For each of the following sections: 2C, 3A, 3B, 3C, 3D, 3E, 3F, 5A, 5B, 5C, 6A, 6B, 6C choose zero or one problem from the text [that had not been assigned]. In total please choose 6 problems. Please copy the text of the problem together with your solution.

Notice that you may (but do not have to) choose also problems from chapter 7*, only partially covered during my absence. This is a beautiful chapter.

Notes:

Let's pick some questions that show we care, because we do. Also, this is fun so let's show that too.

Choose 6 from...

```
2C Dimension
3A Vector Spaces of Linear Maps
3B Null Spaces of Linear Maps
3C Matrices
3D Invertiblity and Isomorpic Vector Spaces
3E Products and Quotients of Vector Spaces
+3F Duality
5A Invariant Subspaces
5B Eigenvectors and Upper-Triagular Matrices
+5C Eigenspaces and Diagonal Matrices
6A Inner Products and Norms
+6B Orthonormal Basis
+6C Orthogonal Complements and Minimization Problems
```

...or optionally from:

```
+7A Self-Adjoint and Normal Operators
+7B Spectral Theorem
7C Positive Operators and Isomeries
7D Polar Decomposition and Singular Value Decomposition
```

In [1]:
```python
# import libraries
import numpy as np
import sympy as sym
from sympy.matrices import Matrix
from sympy import I
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

from sympy import init_printing
init_printing()
```

1. *3F Duality* (3Fx: $x \notin 1, 13, 22, 34$)

(3F14)

Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x) + p''(x)$ for $x \in \mathbb{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbb{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.

Recall:

Example 3.100.

Solution:

(a) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbb{R})$.

Define

$$(Tp)(x) = x^2 p(x) + p''(x)$$

for $x \in \mathbb{R}$, then

$$(T'(\varphi))(p) = (\varphi \circ T)(p) = \varphi(Tp) = \varphi p'$$

$$\begin{aligned} \implies \varphi p' &= \varphi(x^2 p(x) + p''(x)) \\ &= (x^2 p(x) + p''(x))'(4) \\ &= (2xp(x) + x^2 p'(x) + p'''(x))(4) \text{ (by product rule)} \\ &= 8p(4) + 16p'(4) + p'''(4). \end{aligned}$$

□

Solution:

(b) Suppose $\varphi \in \mathcal{P}(\mathbb{R})'$ is defined by $\varphi(p) = \int_0^1 p(x)dx$. Evaluate $(T'(\varphi))(x^3)$.

Define

$$(Tp) = x^2 p + p'' \leftarrow \text{(drop the x's as the context is clear)}$$

for $x \in \mathbb{R}$, then

$$(T'(\varphi))(p) = (\varphi \circ T)(p) = \varphi(Tp)$$

$$\begin{aligned} \implies \varphi(Tp) &= \varphi(x^2 p + p'') \\ &= \varphi(x^5 + 6x) \\ &= \int_0^1 (x^5 + 6x)\, dx \\ &= \left(\frac{x^6}{6} + 3x^2\right)\Bigg|_0^1 = \frac{1}{6} + 3 \end{aligned}$$

□

Notes:

The idea here is simple but Axler's (Halmos') notation (T' rather than T^*) is somewhat unnecessary. There is no confusion with the self-adjoint operator as the context is clear (foreshawdowing). Anyway, we selected this problem based on the theme of inner-product spaces.

Going deeper:

Consider the difference between self-dual and self-adjoint. This is where we might draw some conclusion about projection, prespective, and spectrum.

2. *5C Eigenspaces and Diagonal Matrices* (5Cx: $x \notin 3, 8, 10, 16$)

(5C5)

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \text{null}\,(T - \lambda I) \oplus \text{range}\,(T - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

Recall: 5.41 Conditions equivalent to to diagonalizablity

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ denote the distinct eigen values of T. Then the following are equivalent:

a) T is diagonalizable.

b) V has a basis consisting of eigenvectors of T.

c) There exist 1-dimensional subspaces $U_1, \cdots, U_n$ of V, each invarient under T such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

d) $V = E(\lambda_1) \oplus \cdots \oplus E(\lambda_m)$.

Recall: 5.44 Enough eigenvalues implies diagonalizablity

If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Recall: 5.27 Over $\mathbb{C}$, every operator has an upper-triangular matrix

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Solution:

Prove that T is diagonalizable if and only if

$$V = \text{null } (T - \lambda I) \oplus \text{range } (T - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

$\Leftarrow$

Assume that T has a diagonal matrix

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with respect to a basis $v_1, \cdots, v_n$ of V if and only if $Tv_j = \lambda_j v_j$ for each j.

Let $\lambda \in \mathbb{C}$. Then

$$\mathcal{M}(T - \lambda I) = \begin{pmatrix} \lambda_1 - \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda_n - \lambda \end{pmatrix}.$$

$\Rightarrow$

Hence $T - \lambda I$ is diagonalizable if and only if λ does not equal one of the numbers $\lambda_1, \cdots, \lambda_n$. In other words, if $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Thus, $\dim \text{null } (T - \lambda I) = 0$ and $\dim \text{range } (T - \lambda I) = \dim V$. Thus we conclude that with respect to this basis consisting of eigenvectors, T has a diagonal matrix.

$\square$

Scratch Ideas...

$\Rightarrow$

Let $\mathcal{T}$ defined by $(T - \lambda I)$ and suppose $\mathcal{T} \in \mathcal{L}(V)$ is diagonalizable.

An operator $\mathcal{T} \in \mathcal{L}(V)$ has a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with respect to a basis $v_1, \cdots, v_n$ of V if and only if $\mathcal{T} v_j = \lambda_j v_j$ for each j.

Thus $\lambda_j = 0$ is the basis of $\operatorname{null} \mathcal{T}$ and all v_j such that $\lambda_j \neq 0$ are a basis of $\operatorname{range} \mathcal{T}$.

[It's worth noting here that this is a basis with respect to $\mathcal{T}$ rather than T.]

Hence if T is diagonalizable then $\mathcal{T} = T - \lambda I$ is also diagonalizable for $\lambda \in \mathbb{C}$.

This is obvious due to linearity, however is this a strong argument? Think about this.

$\square$

$\Leftarrow$

Consider a contrary case where $\mathcal{T}$ is invertible thus not diagonalizable. Recall

$$V = \operatorname{null}(\mathcal{T}) \oplus \operatorname{range}(\mathcal{T}).$$

For this to be true $\lambda \notin \operatorname{span} \mathcal{T}$, in other words $\lambda_1, \cdots, \lambda_m$ are not distinct eigenvalues of $\mathcal{T}$.

Hence if T is diagonalizable then $\lambda \neq \lambda_j$.

$\square\square$

Going deeper:

Our scratch use of $\mathcal{T}$ implies duality/adjoint. Consider reworking to employ linear functionals to simplify argument.

Every diagonal matrix is an upper trianglar matrix, thus cast as operator and restate proof...

Let $\mathcal{T} \in \mathcal{L}(T)$ be a linear operator defined by $(T - \lambda I)$, thus $\mathcal{L}(\mathcal{T})$ has an upper-triangular matrix with respect to some basis of V. Apply proof of 5.27 Over $\mathbb{C}$, every operator has an upper-triangular matrix

Also, consider Cayley–Hamilton theorem...

Define a linear recursive sequence with characteristic polynomial

$$p(t) = c_0 + c_1 t + \cdots + c_n - 1t_n - 1 + t_n p(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + t^n.$$

What was I thinking here?

Something about... Matrix A defined by $p_A(\lambda) = \det(\lambda I_n - A) p_A(\lambda) = \det(\lambda I_n - A)$

It might have been Sylvester's law of Nullity...

That's not really the Axler way though.

Consider Symmetric relationship...

Let $A : V \to v$ be a linear map, then A is symmetric (with respect to the scalar production if

$$\langle Av, w \rangle = \langle v, Aw \rangle$$

for all $v, w \in V$. Also let V be a finite dimensional vector space with a positive define scalar product.

Suppose that v is an eigenvector of A. Then

$$\langle Aw, v \rangle = \langle w, Av \rangle = \langle w, \lambda v \rangle = \lambda \langle w, v \rangle = 0.$$

Hense Aw is perpendicular to v.

Now suppose W is stable under A. Let $u \in W^{\perp}$. Then for all $w \in W$ implies $Aw \in W$, thus

$$\langle Au, w \rangle = \langle u, Aw \rangle = 0.$$

Hence $Au \in W^{\perp}$, thus A has a nonzero eigenvector.

See also: Finite-Dimensional Vector Spaces, Halmos.

Going deep:

I really like this example and considered using this to show the relationship between diagonal matrix and the innerproduct of the dual basis and ajoint basis. Anyway, we'll pin this here for inspiration...

Consider if $F \in \mathbb{R} \text{ or } \mathbb{C}$ and is $D = \left[d_{ij}\right]$ is nonsingular matrix in $\mathcal{M}_{n \times n}(F)$ then inner product on F^n is defined by,

$$\left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix} \right\rangle = [a_1 \quad \cdots \quad a_n] D D^H \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix}, \text{ where } D^H = \left[\overline{d_{ij}}\right]^T.$$

Weighted dot product is defined by, $\left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix} \right\rangle = \sum_{i=1}^{n} c_i a_i \overline{b_i}.$

See also: The Linear Algebra a beginning Gradute Student Ought to Know, Golan.

3. *6B Orthonormal Basis* (6Bx: $x \notin 2, 5, 15$)

(6B17)

For $u \in V$, let Φu denote the linear functional on V defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $v \in V$.

(a) Show that is $\mathbb{F} = \mathbb{R}$, the Φ is a linear map from V to V'. (Recall from Section 3.F that $V' = \mathcal{L}(V, \mathbb{F})$ and that V' is called the dual space of V.)

(b) Show that if $\mathbb{F} = \mathbb{C}$ and $V \neq \{0\}$, then Φ is not a linear map.

(c) Show that Φ is injective.

(d) Suppose $\mathbb{F} = \mathbb{R}$ and V is finite-dimensional. Use parts (a) and (c) and a dimension-counting argument (but without using 6.42) to show that Φ is an isomorphism from V onto V'.

[Part (d) gives and alternative proof of the Riesz Representation Theorem (6.42) when $\mathbb{F} = \mathbb{R}$. Part (d) also gives a natural isomorphism (meaning that it does not depend on a choise of basis) from a finite-dimensional real inner product space onto its dual space.]

Recall: Linear Map

A linear map from V to W is a function $T : V \to W$ with the following properties:

Additivity: $T(u + v) = Tu + Tv$ for all $u, v \in V$.

Homogeneity: $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$.

Solution:

(a) Show that is $\mathbb{F} = \mathbb{R}$, the Φ is a linear map from V to V'. (Recall from Section 3.F that $V' = \mathcal{L}(V, \mathbb{F})$ and that V' is called the dual space of V.)

$\Rightarrow$ Additiviy:

Suppose $u_1, u_2 \in V$ and let Φu denote the linear functional on V defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $u, v \in V$.

Then,

$$\begin{aligned}(\Phi(u_1 + u_2))(v) &= \langle v, u_1 + u_2 \rangle \\ &= \langle v, u_1 \rangle + \langle v, u_2 \rangle \\ &= (\Phi u_1)(v) + (\Phi u_2)(v).\end{aligned}$$

$\Leftarrow$ Homogeneity:

Suppose $u, v \in V$, $c \in \mathbb{R}$ and let Φu denote the linear functional on V defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $u, v \in V$.

Then,

$$\begin{aligned}(\Phi(cu))(v) &= \langle v, cu \rangle \\ &= c\langle v, u \rangle \\ &= c(\Phi u)(v).\end{aligned}$$

Thus Φ is a linear map from V to V'

$\square$

Solution:

(b) Show that if $\mathbb{F} = \mathbb{C}$ and $V \neq \{0\}$, then Φ is not a linear map.

Consider $\mathbb{F} = \mathbb{C}$, then

$$(\Phi(cu))(v) = \bar{c}(\Phi u)(v)$$

$$\implies c = \bar{c}.$$

However this can only be true if $c = 0$ or $c \in \mathbb{R}$, thus Φ is not a linear map.

$\square$

Recall: Injective

A function $T : V \to W$ is injective is $Tu = Tv \implies u = v$.

Solution:

(c) Show that Φ is injective.

Let Φu denote the linear functional on V defined by

$$(\Phi u)(v) = \langle v, u \rangle$$

for $u, v \in V$.

Suppose there are u_1 and u_2 in V such that $\Phi u_1 = \Phi u_2$. Then

$$0 = (\Phi u_1 - \Phi u_2)(v) = (\Phi(u_1 - u_2))(v) = \langle v, u_1 - u_2 \rangle,$$

thus Φ is injective.

□

See also:

6.42 Riesz Representation Theorem and 6.30 Linear Combination of Orthonormal Basis.

Recall: Fundamental Theorem of Linear Maps

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\operatorname{range} T$ is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Recall: T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Recall: T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then T is injective if and only if T' is sujective.

Recall: Dimension shows whether vector spaces are isomorphic

Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Solution:

(d) Suppose $\mathbb{F} = \mathbb{R}$ and V is finite-dimensional. Use parts (a) and (c) and a dimension-counting argument (but without using 6.42) to show that Φ is an isomorphism from V onto V'.

By (a), we know Φ is a linear map from V to V' and by (c), we showed that $\dim \operatorname{null} \Phi = 0$.

Then, by the Fundamental Theorem of Linear Maps,

$$\begin{aligned} \dim V &= \dim \operatorname{null} \Phi + \dim \operatorname{range} \Phi \\ &= \dim \operatorname{range} \Phi \\ &= \dim V'. \end{aligned}$$

Thus, Φ isomorphism from V to V'.

□

Notes:

This one was some more inner-product space warm up. Here we've attempted to predicate each solution with the fundumental ideas used therein. Hopefully that's made the answers more readible. I really liked this problem how it expands on diagonalizable matrices explored in the previous question.

4. *6C Orthogonal Complements and Minimization Problems* (6Cx: $\forall x$)

(6C13)

Find $p \in \mathcal{P}_5(\mathbb{R})$ that make

$$\int_{-\pi}^{\pi} \left| \sin x - p(x) \right|^2 dx$$

as small as possible.

[The polynomial 6.60 is an excellent approximation to the answer to this exericise, but here you are asked to find the exact solution, which involves powers of π. A computer that can perform symbolic integration will be useful.]

Recall: Gram-Schmidt Procedure

Suppose $v_1, \cdots, v_m$ is linearly independent list of vectors in V. Let $e_e = v_1/\|v_1\|$. For $j = 2, \ldots m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}.$$

Then $e_1, \cdots, e_m$ is an orthonormal list of vector in V such that

$$\text{span}\,(v_1, \cdots, v_j) = \text{span}\,(e_1, \cdots, e_j)$$

for $j = 1, \ldots, m$.

In [18]:

```
# find our basis
n = 5
x = sym.symbols('x', real=True)
v = [x**i for i in range(n+1)]
lim0 = -sym.pi
lim1 = sym.pi

# inner-product lambda function
ip = lambda f,g: sym.integrate(f*g, (x,lim0,lim1))*g

# orthonomal basis procedure (GramSchmidt)
def basis(v):
    e = []
    for i, vi in enumerate(v):
        ei_nom = (vi - sum([ip(vi,ej) for ej in e]))
        #ei_denom = sym.sqrt(ip(ei_nom,ei_nom)) # very slow
        ei_denom = sym.sqrt(sym.integrate(ei_nom**2, (x,lim0,lim1))) # much better
        ei = ei_nom/ei_denom
        e.append(ei)
    return e

# show our basis
e = basis(v)
display(e) # Our polynominal basis...
```

$$\left[\frac{\sqrt{2}}{2\sqrt{\pi}}, \frac{\sqrt{6}x}{2\pi^{\frac{3}{2}}}, \frac{3\sqrt{10}\left(x^2 - \frac{\pi^2}{3}\right)}{4\pi^{\frac{5}{2}}}, \frac{5\sqrt{14}\left(x^3 - \frac{3\pi^2 x}{5}\right)}{4\pi^{\frac{7}{2}}}, \frac{105\sqrt{2}\left(x^4 - \frac{6\pi^2\left(x^2 - \frac{\pi^2}{3}\right)}{7} - \frac{\pi^4}{5}\right)}{16\pi^{\frac{9}{2}}}, \frac{63\sqrt{22}\left(x^5 - \frac{3\pi^4 x}{7} - \frac{10\pi^2\left(x^3 - \frac{3\pi^2 x}{5}\right)}{9}\right)}{16\pi^{\frac{11}{2}}} \right]$$

Recall: *orthongonal projection P_U

Suppose $u \in V$ with $u \neq 0$ and $U = \text{span}\,(u)$. Then

$$P_U v = \frac{\langle v, u\rangle}{\langle u, u\rangle} u = \frac{\langle v, u\rangle}{\|u\|^2} u$$

for every $v \in V$.

* adapted from Example 6.54.

In [19]:
```
# project V onto U
Pu = sum([ip(sym.sin(x), e_j) for e_j in e]) # not exactly obvious from example
p = sym.collect(sym.expand(Pu), x) # make pretty
```

In [20]:
```
# compare with text
p.n(8) # checks out, matches text
```

Out[20]: $0.0056431191x^5 - 0.15527143x^3 + 0.98786214x$

In [21]:
```
print(sym.latex(p))
display(p)
```

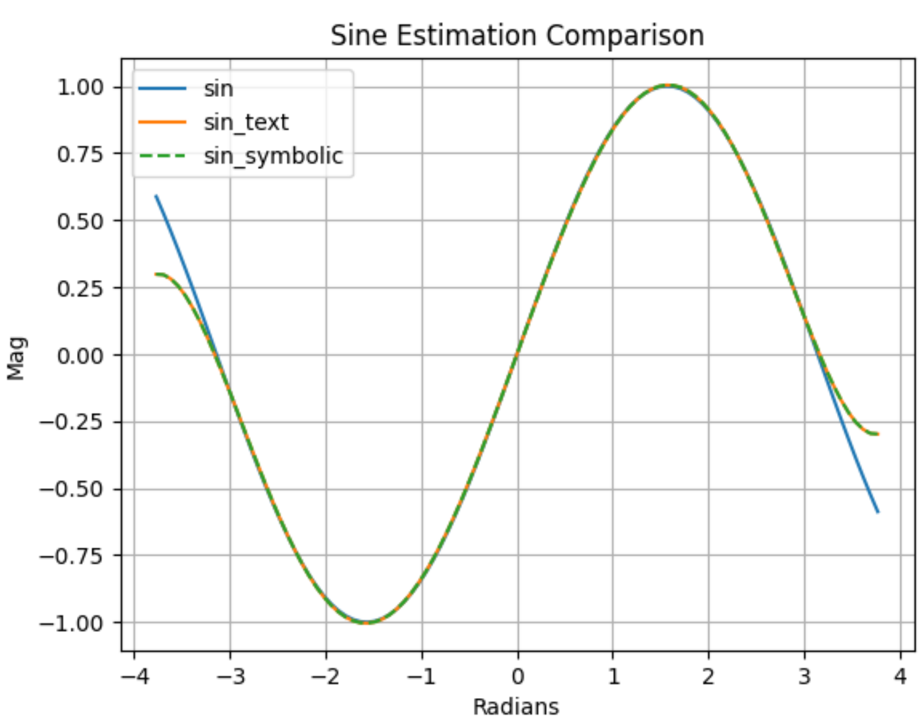
```
x^{5} \left(- \frac{72765}{8 \pi^{8}} + \frac{693}{8 \pi^{6}} + \frac{654885}{8 \pi^{10}}\right) + x^{3} \left(- \frac{363825}{4 \pi^{8}} - \frac{315}{4 \pi^{4}} + \frac{39375}{4 \pi^{6}}\right) + x \left(- \frac{16065}{8 \pi^{4}} + \frac{105}{8 \pi^{2}} + \frac{155925}{8 \pi^{6}}\right)
```

$$x^{5} \left(- \frac{72765}{8 \pi^{8}} + \frac{693}{8 \pi^{6}} + \frac{654885}{8 \pi^{10}}\right) + x^{3} \left(- \frac{363825}{4 \pi^{8}} - \frac{315}{4 \pi^{4}} + \frac{39375}{4 \pi^{6}}\right) + x \left(- \frac{16065}{8 \pi^{4}} + \frac{105}{8 \pi^{2}} + \frac{155925}{8 \pi^{6}}\right)$$

In [22]:
```
xx = np.linspace(-1.2*np.pi, 1.2*np.pi, 100)
sin_est0 = [p.subs(x,v) for v in xx]
sin_est1 = [(p.n(8)).subs(x,v).n(8) for v in xx] # rounded
sin_axler = lambda x: 0.987862*x   - 0.155271*x**3   + 0.00564312*x**5 # text
```

In [7]:
```
plt.plot(xx,np.sin(xx), label='sin')
plt.plot(xx,sin_axler(xx), label='sin_text')
plt.plot(xx,sin_est0, label='sin_symbolic', linestyle='dashed')
#plt.plot(xx,sin_est1, label='sin_rounded')
plt.legend()
plt.grid()
plt.title('Sine Estimation Comparison')
plt.xlabel('Radians')
plt.ylabel('Mag')
```

Out[7]: `Text(0, 0.5, 'Mag')`



Solution:

We applied the Gram-Schmidt procedure to the basis $1, x, x^2, x^3, x^4, x^5$ then computed $P_U v$ as in Example 6.59.

Thus by direct calculation, we have

$$u(x) = x^5\left(-\frac{72765}{8\pi^8} + \frac{693}{8\pi^6} + \frac{654885}{8\pi^{10}}\right) + x^3\left(-\frac{363825}{4\pi^8} - \frac{315}{4\pi^4} + \frac{39375}{4\pi^6}\right) + x\left(-\frac{16065}{8\pi^4} + \frac{105}{8\pi^2} + \frac{155925}{8\pi^6}\right).$$

We then plotted our results comparing the polynominal estimation presenting in the text, our symbolical compututed results, and the actaul value of sine. Our symbolic calculation closely matched the reference from the text.

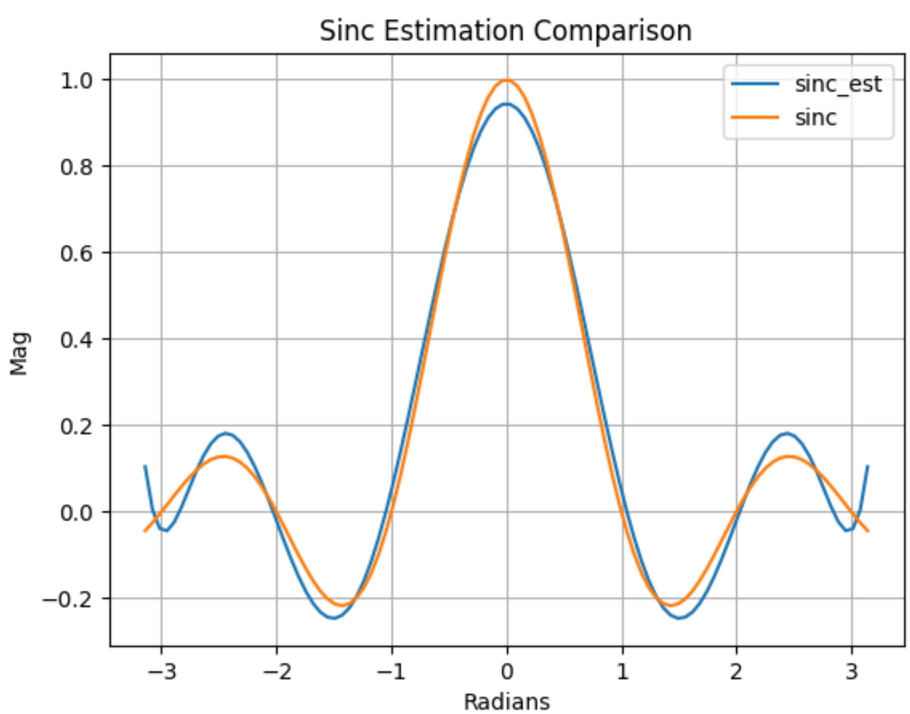
□

Going deeper:

Abstract this for other functions. Actually this already pretty abstracted.

In [57]:
```
# do the sinc function

# find our basis
n = 9
x = sym.symbols('x', real=True)
v = [x**i for i in range(n+1)]
lim0 = -sym.pi
lim1 = sym.pi
e2 = basis(v) # we needed some more terms to find a good fit

# projection
Pu2 = sum([ip(sym.sin(x*sym.pi)/(x*sym.pi), e_j) for e_j in e2]) # rather slow
```

In [58]:
```
#display(sym.collect(sym.expand(Pu2), x)) # no making this pretty...
xx = np.linspace(-np.pi, np.pi, 100)
sinc_est = [Pu2.subs(x,v) for v in xx]
sinc_exact= np.sinc(xx) # normalize sinc

plt.plot(xx,sinc_est, label='sinc_est')
plt.plot(xx,sinc_exact, label='sinc')
plt.legend()
plt.grid()
plt.title('Sinc Estimation Comparison')
plt.xlabel('Radians')
plt.ylabel('Mag');
```



Notes:

This problem is playing to my strength of being more machine than man.

```
I'm not a man or machine
I'm just something in between
(Whoa-oh, whoa-whoa)
I'm all love, a dynamo
So push the button and let me go
(Whoa-oh, whoa-whoa)

--'Lovin’ Every Minute of It', Loverboy
```

Besides that silliness, it demonstrates the Gram-Schmidt Proceedure and how the innerproduct can be used to minimize the distance to a function, thus motivating functional analysis.

5. *7A Self-Adjoint and Normal Operators* (7Ax: $\forall x$)

(7A20)

Suppose $T\mathcal{L}(V, W)$ and $\mathbb{F} = \mathbb{R}$. Let Φ_V be the isomorphisim from V onto the dual space V' given by Exercise 17 in Section 6.B, and let Φ_W be the corresponding isomorphism from W onto W'. Show that if Φ_V and Φ_W are used to identify V and W with V' and W', then T^* is identified with the dual map T'. More precisely, show that $\Phi_V \circ T^* = T' \circ \Phi_W$.

Recall: Riesz Representation Theorem

Suppose V is finite-dimensional and φ is a linear functional on V. Then there is a unique vector $u \in V$ such that

$$\varphi = \langle v, u \rangle$$

for every $v \in V$.

Solution:

Let $v \in V$ and $w \in W$, show that

$$\Phi_V \circ T^* = T' \circ \Phi_W.$$

$\Leftarrow$ LHS

$$\begin{aligned}((\Phi_V \circ T^*)(w))(v) &= (\Phi_V(T^* w))(v) \\ &= \langle v, T^* v\rangle \\ &= \langle Tv, w\rangle.\end{aligned}$$

$\Rightarrow$ RHS

$$\begin{aligned}((T' \circ \Phi_W)(w))(v) &= (T' \circ \Phi_W(w))(v) \\ &= (\Phi_W(w) \circ T)(v) \\ &= (\Phi_W(w))(Tv) \\ &= \langle Tv, w\rangle.\end{aligned}$$

Thus

$$\begin{gathered}((\Phi_V \circ T^*)(w))(v) = (T' \circ \Phi_W)(w))(v) \\ \implies ((\Phi_V \circ T^*) = (T' \circ \Phi_W),\end{gathered}$$

as desired.

□

Notes:

Well we can't say we didn't see that coming. Indeed this is the bee in our bonnet about Axler's dual notation.

Going deeper:

The interesting thing here is that the isomorphism seems to depend on the particular choice of basis that we make on V. This suggests that some (many) vector spaces don't have a standard basis.

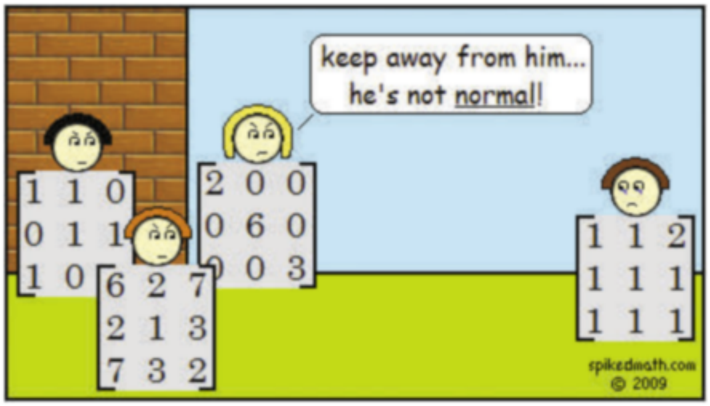
Consider defining $\Phi_v \in V$ by $\Phi_V(f) = f(v)$. Is Double-dual space any better?

Hum... Think about this one.

6. *7B Spectral Theorem* (7Bx: $\forall x$)

(7B15)

Find the matrix entry below that is covered up.



In [8]:
```python
x=sym.symbols('x')

M1=Matrix([[1,1,0],[0,1,1],[1,0,x]])
M2=Matrix([[2,0,0],[0,6,0],[0,0,3]])
M3=Matrix([[6,2,7],[2,1,3],[7,3,2]])
M4=Matrix([[1,1,2],[1,1,1],[1,1,1]])
```

In [9]:
```python
is_normal = lambda M: M*sym.conjugate(M).T == sym.conjugate(M).T * M
dis_normal = lambda M: display(Latex(f'${sym.latex(M*sym.conjugate(M).T)}={sym.latex(sym.conjugate(M).T * M)}$'))
text_dis_normal = lambda M: print(f'${sym.latex(M*sym.conjugate(M).T)}={sym.latex(sym.conjugate(M).T * M)}$')

display(M1)
dis_normal(M1)
text_dis_normal(M1)

print(sym.latex(M1))
```

$$\left[\begin{matrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & x\end{matrix}\right]$$

$$\left[\begin{matrix}2 & 1 & 1\\1 & 2 & \overline{x}\\1 & x & x \overline{x} + 1\end{matrix}\right]=\left[\begin{matrix}2 & 1 & x\\1 & 2 & 1\\\overline{x} & 1 & x \overline{x} + 1\end{matrix}\right]$$

```
$\left[\begin{matrix}2 & 1 & 1\\1 & 2 & \overline{x}\\1 & x & x \overline{x} + 1\end{matrix}\right]=\left[\begin{matrix}2 & 1 & x\\1 & 2 & 1\\\overline{x} & 1 & x \overline{x} + 1\end{matrix}\right]$
\left[\begin{matrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & x\end{matrix}\right]
```

In [10]:
```python
display(M1.subs(x,1))
is_normal(M1.subs(x,1)), is_normal(M2), is_normal(M3), is_normal(M4)
```

$$\left[\begin{matrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{matrix}\right]$$

Out[10]:
```
(True, True, True, False)
```

Recall: Definition of Normal

- an operator on an inner production is called normal if it commutes with its adjoint
- in other words, $T \in \mathcal{L}(V)$ is normal if

$$TT^* = T^*T.$$

Solution:

Let $T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & x \end{bmatrix}$. By direct computation,

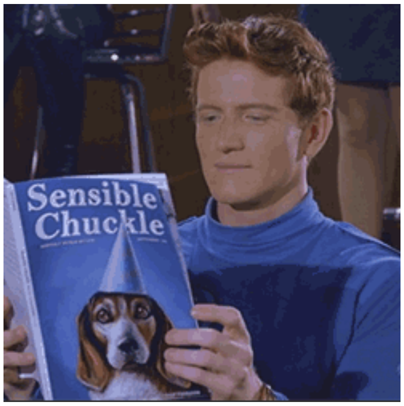
$$TT^* = T^*T.$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & \overline{x} \\ 1 & x & x\overline{x}+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & x \\ 1 & 2 & 1 \\ \overline{x} & 1 & x\overline{x}+1 \end{bmatrix}$$

$$\implies \overline{x} = 1 \text{ and } x = 1.$$

Hence the covered number is 1.

$\square$



Sensible Chuckle, Danger 5 S01E02

Going deeper:

Think about vector spaces don't have a standard basis... closest standard basis.

Appendix 0. DFT

Exploring the Fourier Basis using visulizatons.

See also:

https://python.plainenglish.io/visualizing-vector-fields-in-matplotlib-33d8ced0c6bb

In [11]:
```
n = 16
omega = -sym.I * sym.pi/n
sym.exp(omega)
V = Matrix([sym.exp(omega)**i for i in range(n)])
M = sym.diag(V.tolist(), unpack=True)
M
```

Out[11]:

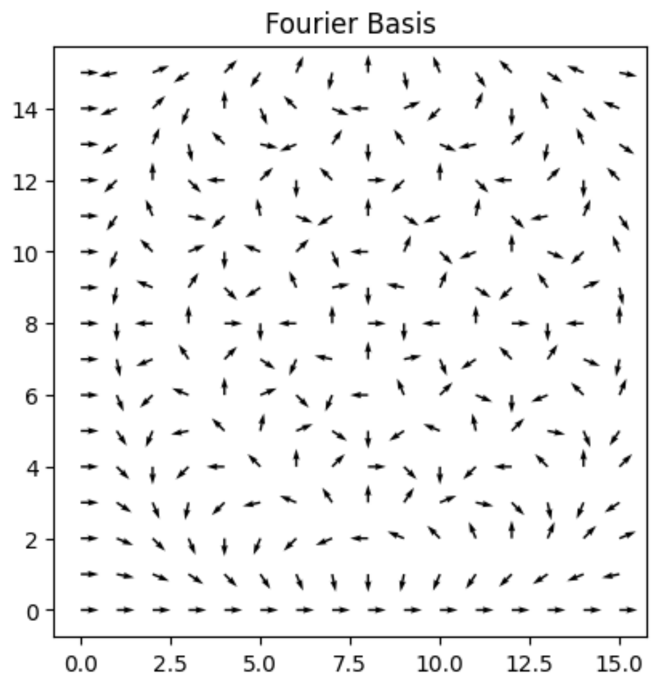
$$\left[\begin{matrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & e^{-\frac{i\pi}{16}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & e^{-\frac{i\pi}{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & e^{-\frac{3i\pi}{16}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & e^{-\frac{i\pi}{4}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & e^{-\frac{5i\pi}{16}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{3i\pi}{8}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{7i\pi}{16}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{9i\pi}{16}} & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{5i\pi}{8}} & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{11i\pi}{16}} & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{3i\pi}{4}} & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{13i\pi}{16}} & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{7i\pi}{8}} & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{15i\pi}{16}}\end{matrix}\right]$$

In [12]:

```
# Build a Matrix for our basis using permutations M^k
B = Matrix([[(M**k)*sym.ones(n,1) for k in range(n)]]) # be weird
display(B) #notice these are our polynomial bases
```

$$\left[\begin{matrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\1 & e^{-\frac{i\pi}{16}} & e^{-\frac{i\pi}{8}} & e^{-\frac{3i\pi}{16}} & e^{-\frac{i\pi}{4}} & e^{-\frac{5i\pi}{16}} & e^{-\frac{3i\pi}{8}} & e^{-\frac{7i\pi}{16}} & -i & e^{-\frac{9i\pi}{16}} & e^{-\frac{5i\pi}{8}} & e^{-\frac{11i\pi}{16}} & e^{-\frac{3i\pi}{4}} & e^{-\frac{13i\pi}{16}} & e^{-\frac{7i\pi}{8}} & e^{-\frac{15i\pi}{16}}\\1 & e^{-\frac{i\pi}{8}} & e^{-\frac{i\pi}{4}} & e^{-\frac{3i\pi}{8}} & -i & e^{-\frac{5i\pi}{8}} & e^{-\frac{3i\pi}{4}} & e^{-\frac{7i\pi}{8}} & -1 & e^{\frac{7i\pi}{8}} & e^{\frac{3i\pi}{4}} & e^{\frac{5i\pi}{8}} & i & e^{\frac{3i\pi}{8}} & e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{8}}\\1 & e^{-\frac{3i\pi}{16}} & e^{-\frac{3i\pi}{8}} & e^{-\frac{9i\pi}{16}} & e^{-\frac{3i\pi}{4}} & e^{-\frac{15i\pi}{16}} & e^{\frac{7i\pi}{8}} & e^{\frac{11i\pi}{16}} & i & e^{\frac{5i\pi}{16}} & e^{\frac{i\pi}{8}} & e^{-\frac{i\pi}{16}} & e^{-\frac{i\pi}{4}} & e^{-\frac{7i\pi}{16}} & e^{-\frac{5i\pi}{8}} & e^{-\frac{13i\pi}{16}}\\1 & e^{-\frac{i\pi}{4}} & -i & e^{-\frac{3i\pi}{4}} & -1 & e^{\frac{3i\pi}{4}} & i & e^{\frac{i\pi}{4}} & 1 & e^{-\frac{i\pi}{4}} & -i & e^{-\frac{3i\pi}{4}} & -1 & e^{\frac{3i\pi}{4}} & i & e^{\frac{i\pi}{4}}\\1 & e^{-\frac{5i\pi}{16}} & e^{-\frac{5i\pi}{8}} & e^{-\frac{15i\pi}{16}} & e^{\frac{3i\pi}{4}} & e^{\frac{7i\pi}{16}} & e^{\frac{i\pi}{8}} & e^{-\frac{3i\pi}{16}} & -i & e^{-\frac{13i\pi}{16}} & e^{\frac{7i\pi}{8}} & e^{\frac{9i\pi}{16}} & e^{\frac{i\pi}{4}} & e^{-\frac{i\pi}{16}} & e^{-\frac{3i\pi}{8}} & e^{-\frac{11i\pi}{16}}\\1 & e^{-\frac{3i\pi}{8}} & e^{-\frac{3i\pi}{4}} & e^{\frac{7i\pi}{8}} & i & e^{\frac{i\pi}{8}} & e^{-\frac{i\pi}{4}} & e^{-\frac{5i\pi}{8}} & -1 & e^{\frac{5i\pi}{8}} & e^{\frac{i\pi}{4}} & e^{-\frac{i\pi}{8}} & -i & e^{-\frac{7i\pi}{8}} & e^{\frac{3i\pi}{4}} & e^{\frac{3i\pi}{8}}\\1 & e^{-\frac{7i\pi}{16}} & e^{-\frac{7i\pi}{8}} & e^{\frac{11i\pi}{16}} & e^{\frac{i\pi}{4}} & e^{-\frac{3i\pi}{16}} & e^{-\frac{5i\pi}{8}} & e^{\frac{15i\pi}{16}} & i & e^{\frac{i\pi}{16}} & e^{-\frac{3i\pi}{8}} & e^{-\frac{13i\pi}{16}} & e^{\frac{3i\pi}{4}} & e^{\frac{5i\pi}{16}} & e^{-\frac{i\pi}{8}} & e^{-\frac{9i\pi}{16}}\\1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i & 1 & -i & -1 & i\\1 & e^{-\frac{9i\pi}{16}} & e^{\frac{7i\pi}{8}} & e^{\frac{5i\pi}{16}} & e^{-\frac{i\pi}{4}} & e^{-\frac{13i\pi}{16}} & e^{\frac{5i\pi}{8}} & e^{\frac{i\pi}{16}} & -i & e^{\frac{15i\pi}{16}} & e^{\frac{3i\pi}{8}} & e^{-\frac{3i\pi}{16}} & e^{-\frac{3i\pi}{4}} & e^{\frac{11i\pi}{16}} & e^{\frac{i\pi}{8}} & e^{-\frac{7i\pi}{16}}\\1 & e^{-\frac{5i\pi}{8}} & e^{\frac{3i\pi}{4}} & e^{\frac{i\pi}{8}} & -i & e^{\frac{7i\pi}{8}} & e^{\frac{i\pi}{4}} & e^{-\frac{3i\pi}{8}} & -1 & e^{\frac{3i\pi}{8}} & e^{-\frac{i\pi}{4}} & e^{-\frac{7i\pi}{8}} & i & e^{-\frac{i\pi}{8}} & e^{-\frac{3i\pi}{4}} & e^{\frac{5i\pi}{8}}\\1 & e^{-\frac{11i\pi}{16}} & e^{\frac{5i\pi}{8}} & e^{-\frac{i\pi}{16}} & e^{-\frac{3i\pi}{4}} & e^{\frac{9i\pi}{16}} & e^{-\frac{i\pi}{8}} & e^{-\frac{13i\pi}{16}} & i & e^{-\frac{3i\pi}{16}} & e^{-\frac{7i\pi}{8}} & e^{\frac{7i\pi}{16}} & e^{-\frac{i\pi}{4}} & e^{-\frac{15i\pi}{16}} & e^{\frac{3i\pi}{8}} & e^{-\frac{5i\pi}{16}}\\1 & e^{-\frac{3i\pi}{4}} & i & e^{-\frac{i\pi}{4}} & -1 & e^{\frac{i\pi}{4}} & -i & e^{\frac{3i\pi}{4}} & 1 & e^{-\frac{3i\pi}{4}} & i & e^{-\frac{i\pi}{4}} & -1 & e^{\frac{i\pi}{4}} & -i & e^{\frac{3i\pi}{4}}\\1 & e^{-\frac{13i\pi}{16}} & e^{\frac{3i\pi}{8}} & e^{-\frac{7i\pi}{16}} & e^{\frac{3i\pi}{4}} & e^{-\frac{i\pi}{16}} & e^{-\frac{7i\pi}{8}} & e^{\frac{5i\pi}{16}} & -i & e^{\frac{11i\pi}{16}} & e^{-\frac{i\pi}{8}} & e^{-\frac{15i\pi}{16}} & e^{\frac{i\pi}{4}} & e^{-\frac{9i\pi}{16}} & e^{\frac{5i\pi}{8}} & e^{-\frac{3i\pi}{16}}\\1 & e^{-\frac{7i\pi}{8}} & e^{\frac{i\pi}{4}} & e^{-\frac{5i\pi}{8}} & i & e^{-\frac{3i\pi}{8}} & e^{\frac{3i\pi}{4}} & e^{-\frac{i\pi}{8}} & -1 & e^{\frac{i\pi}{8}} & e^{-\frac{3i\pi}{4}} & e^{\frac{3i\pi}{8}} & -i & e^{\frac{5i\pi}{8}} & e^{-\frac{i\pi}{4}} & e^{\frac{7i\pi}{8}}\\1 & e^{-\frac{15i\pi}{16}} & e^{\frac{i\pi}{8}} & e^{-\frac{13i\pi}{16}} & e^{\frac{i\pi}{4}} & e^{-\frac{11i\pi}{16}} & e^{\frac{3i\pi}{8}} & e^{-\frac{9i\pi}{16}} & i & e^{-\frac{7i\pi}{16}} & e^{\frac{5i\pi}{8}} & e^{-\frac{5i\pi}{16}} & e^{\frac{3i\pi}{4}} & e^{-\frac{3i\pi}{16}} & e^{\frac{7i\pi}{8}} & e^{-\frac{i\pi}{16}}\end{matrix}\right]$$

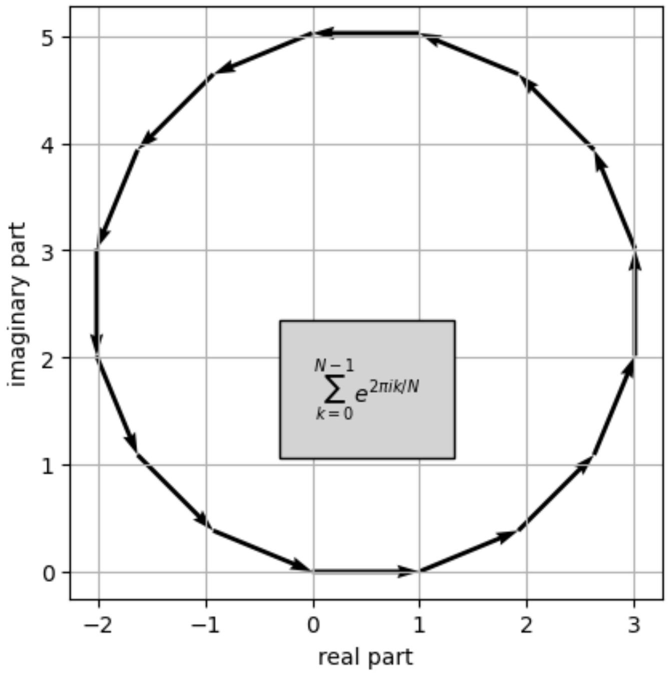
In [13]:
```
#plt.quiver(X, Y, U, V, color='b', units='xy', scale=1)
points = []
arrows = []

for x in range(n):
    for y in range(n):
        z = complex(B[x,y])
        points.append([x,y])
        arrows.append([z.real,z.imag])

points = np.array(points)
arrows = np.array(arrows)

fig = plt.figure()
ax = fig.gca()
ax.quiver(points[:, 0], points[:, 1],
          arrows[:, 0], arrows[:, 1],
          angles='xy', scale_units='xy', scale=2)
#ax.grid()
ax.set_title('Fourier Basis')
ax.set_aspect('equal')
plt.show()
display(Latex("Whoa Dude, it's full of Polynominals.  ;-)"))
```



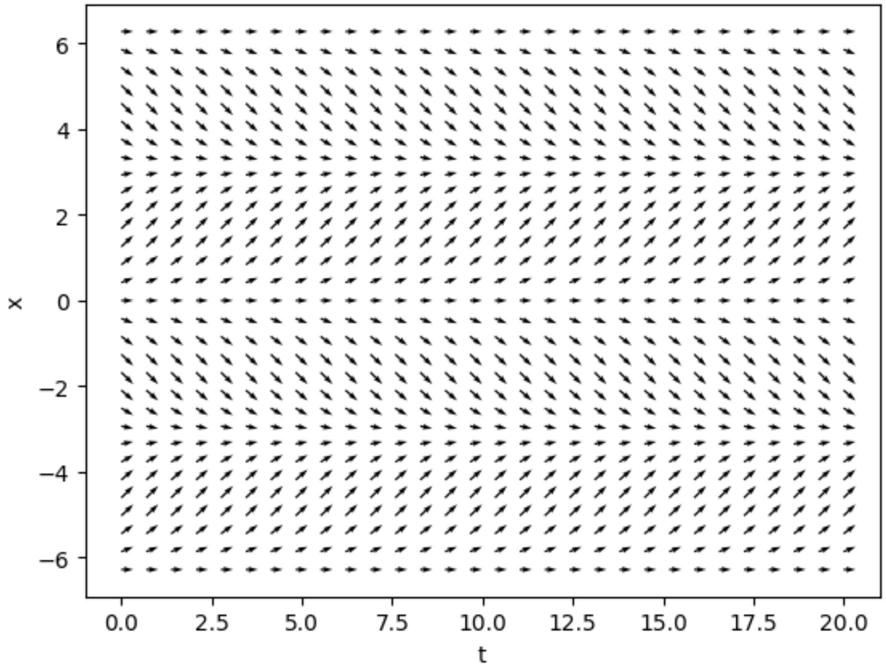
Whoa Dude, it's full of Polynominals. ;-)

In [14]:
```
import numpy as np

N = 16
e = np.exp(1j*2*np.pi/N)

s = 0
points = [[0, 0]]
arrows = []
for k in range(N):
    s += e**k
    point = [s.real, s.imag]
    arrow = [(e**k).real, (e**k).imag]
    points.append(point)
    arrows.append(arrow)

points = np.array(points[:-1])
arrows = np.array(arrows)
```

In [15]:
```python
import matplotlib.pyplot as plt
fig = plt.figure()
ax = fig.gca()

ax.quiver(points[:, 0], points[:, 1],
          arrows[:, 0], arrows[:, 1],
          angles='xy', scale_units='xy', scale=1)

ax.text(0, N/10, r'$\sum_{k=0}^{N-1} e^{2\pi i k / N}$',
        bbox=dict(boxstyle='square,pad=1.5', fc='lightgrey'))

ax.set_aspect('equal')
ax.grid()
ax.set_xlabel('real part')
ax.set_ylabel('imaginary part');
```



In [16]:
```python
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return np.sin(x)

t = np.linspace(0, 20, 30)
x = np.linspace(-2*np.pi, 2*np.pi, 31)
T, X = np.meshgrid(t, x)

DT = np.ones_like(X)
DX = f(X)

fig = plt.figure()
ax = fig.gca()

ax.quiver(T, X, DT, DX)

ax.set_xlabel('t')
ax.set_ylabel('x');
```



In [17]:
```
x =  y = np.linspace(-5, 5, 40)
q=1

X, Y = np.meshgrid(x, y)

a = np.array([1.5, 0, 0])

rma = np.array([X - a[0], Y + a[1]])
rpa = np.array([X + a[0], Y + a[1]])

E_dip = q * rma / np.sqrt(rma[0, ...]**2 + rma[0, ...]**2)**3
E_dip -= q * rpa / np.sqrt(rpa[0, ...]**2 + rpa[0, ...]**2)**3

# Normalize the arrows because we have singularities where the
# charges are located:

norms = np.sqrt(E_dip[0, ...]**2 + E_dip[1, ...]**2)
E_dip /= norms

plt.quiver(X, Y, E_dip[0, ...], E_dip[1, ...], minshaft=2);
```