Midterm Study Guide

Brent A. Thorne

brentathorne@gmail.com

Topics

1. Example of a finite dimensional vector space and an example of an infinite dimensional vector space (why is it infinite dim?),
2. Spanning vectors,
3. Independent vectors.
4. Dimension of a vector space.
5. What is a direct sum? Keep an example in mind. When is a sum a direct sum?
6. Illustrate the matrix for a particular linear map, say for $T(x, y, z) = (2x + z, 4y + 1)$
7. State the Fundamental Theorem of Algebra. Apply it to a simple example.
8. Define product of two spaces, its dimension,
9. Define the quotient of two vector spaces. Give an example of a linear quotient space.
10. Define a linear functional and the dual map.
11. Compute a dual basis of a vector space spanned by a basis.

1. Give an example of a finite dimensional vector space and an example of an infinite dimensional vector space. Why is it infinite dim?

Recall:

A vector space is finite-dimensional is some list of vector spans the space. Thus V is a finite vector space if span $(v_1, \ldots, v_m) = \{a_1 v_1 + \ldots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}$ with $\{0\}$ being the simpliest finite-dimensional space. An infinite-dimension space has no list that spans the vector space.

Ex 1.1 Finite dimensional vector space

$V = \{a + b\sqrt{5}; a, b \in \mathbb{Q}\}$ is a vector space over $\mathbb{Q}$.

$0 + 0\sqrt{5} = 0 \in V$

$\underbrace{a + b\sqrt{5}}_{\in V} + \underbrace{c + d\sqrt{5}}_{\in V} = \underbrace{(a + c)}_{\in Q} + \underbrace{(b + d)}_{\in Q}\sqrt{5} \in V$ closed under addition

$\underbrace{\lambda}_{\in Q}(a + b\sqrt{5}) = \underbrace{\lambda \ a}_{\in Q} + \underbrace{\lambda \ b}_{\in Q}\sqrt{5} \in V$ closed under multiplication

note: $\mathbb{Q}$ is Rational Space.

Ex 1.2 Finite dimensional vector space

$\mathcal{P}_m(\mathbb{F})$ is finite dimensional with degree at most m.

Ex 1.2 Infinite dimensional vector space, and why.

$\mathcal{P}(\mathbb{F})$ is infinite dimensional as every polymonial is span has degree at most m thus z^{m+1} is not in span, hence no list spans $\mathcal{P}(\mathbb{F})$ thus infinate dimensional.

2. Give an example of Spanning vectors.

Recall:

A span is the smallest list containing a subspace.

Ex 2.1 Spanning list

Suppose $(x_1, \ldots, x_n) \in \mathbb{F}$ then, $(x_1, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, \ldots, 0) + \ldots + x_n(0, \ldots, 0, 1)$.

Thus, $(x_1, \ldots, x_n) \in$ span$((1, 0, \ldots, 0) + (0, 1, \ldots, 0) + \ldots + (0, \ldots, 0, 1))$.

3. Provide an example of Independent vectors.

Recall:

A list $v_1, \ldots, v_m \in V$ is linearly-independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes $a_1 v_1 + \ldots + a_m v_m = 0$ is $a_1 = \ldots = a_m = 0$.

Ex3.1 $\mathbb{F}^n$

$(1,0,0,0),(0,1,0,0),(0,0,1,0)$ is linearly-independent in $\mathbb{F}^n$.

Ex3.2 $\mathcal{P}(\mathbb{F})$

$1, \zeta, \ldots, \zeta^m$ is linearly-independent in $\mathcal{P}(\mathbb{F})$ for each non-negative interger m.

4. Demonstrate the dimension of a vector space.

Recall:

The dimension of a vector space is the length of its basis.

Ex 4.1 $\mathbb{F}^n$

dim$\mathbb{F}^n = n$

Ex 4.2 $\mathcal{P}_m(\mathbb{F})$

dim
$P_m(\mathbb{F}) = m + 1$

5. Explain what a direct sum is. Keep an example in mind. When is a sum a direct sum?

Recall:

The sum $U_1 + \cdots + U_2$ is a direct sum is each element can only be written on way.

Ex5.1 F^3

$U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F})$ and

$W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F})$, then

$\mathbb{F}^3 = U \oplus W$.

Ex5.2 $\mathbb{F}^n$

Suppose U_j is a subspace on $\mathbb{F}^n$ of vectors with coordinated all 0 except possibly the j^{th} slot

(ie. $U_2 = \{0, x, 0, \ldots, 0)\} \in \mathbb{F}^n$), then

$\mathbb{F}^n = U_1 \oplus \cdots \oplus U_n$

Ex5.3 Non-example

$U_1 = (x, y, 0)$

$U_2 = (0, 0, z)$

$U_3 = (0, y, y)$

Note there are two ways to write $(0, 0, 0)$...

$(0, 1, 0) + (0, 0, 1) + (0, -1, -1) = (0, 0, 0)$ and

$(0, 0, 0) + (0, 0, 0) + (0, 0, 0) = (0, 0, 0)$

Things to consider:

The direct sum as a projection on some subspace if and only if it is idempotent, $E^2 = E$. * Where E is U along W.

$z = Ez + (1 - E)z$

Consider $Ez = x$ and $(1 - E)z = y$ then

$Ex = E^2 z = Ez = x$ and $Ey = E(1 - E)z = Ez - E^2 z = 0$

so that x is in U and y is in W. Thus,

$V = U \oplus W$, and that the projection on U along W is precisely E.

see also 41 FDVS Halmos

ask Arek for some insight on this

6. Illustrate the matrix for a particular linear map

Recall:

$Tv_k = A_{1,k}w_1 + \ldots + A_{m,k}w_m = \sum_{j=1}^{m} A_{j,k}w_j$ or when context is not clear,

$M(\mathcal{T}, (v_1, \ldots, v_m), (w_1, \ldots, w_m)$ is used.

Ex.6.1 $T(x, y, z) = (2x + z, 4y + 1) \leftarrow$ Arek said this is a typo (1 ought to be z).

$\mathcal{T} \in \mathcal{L}(\mathbb{F}^2, \mathcal{P}_2(\mathbb{F})) \leftarrow$ We made it work anyway.

$$M(\mathcal{T}) = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 4 & 0 \end{pmatrix}$$

Ex.6.2 $T(x, y, z) = (2x + z, 4y + z) \leftarrow *$Note correction

$\mathcal{T} \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

$$M(\mathcal{T}) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

7. State the Fundamental Theorem of *Linear Algebra. Apply it to a simple example.

Recall: Fundamental Theorem of Linear Maps

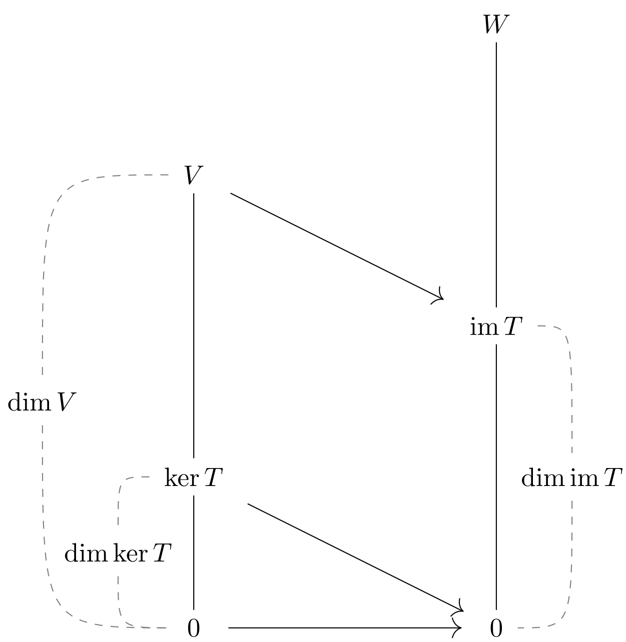
Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\mathrm{range}T$ is finite-dimensional and

$$\mathrm{dim}V = \mathrm{dim}\ \mathrm{null}T + \mathrm{dim}\ \mathrm{range}T. *$$

*Lame... Axler fumbles here and misses an opportunity to enlighten the student.

Recall: Rank-nullity Theorem (Strang)

$$\mathrm{dim}V = \mathrm{dim}\ \mathrm{ker}T + \mathrm{dim}\ \mathrm{im}T$$



Recall: Fundamental Theorem of Linear Algebra

Let f be a linear map between two finite-dimensional vector spaces, representing a $m \times n$ matrix M of rank r then,

- r is the dimension of column space of M, which represent the image of f.
- $n - r$ is the dimension of the null space of M, which represent the kernel of f.
- $m - r$ is the dimension of the cokernel of f.

The transpose M^T of M is the matrix of the dual f^* of f, and it follows,

- r is the dimension of row space of M, which represent the image of f^*.
- $n - r$ is the dimension of the left null space of M, which represent the kernel of f^*.
- $m - r$ is the dimension of the cokernel of f^*.

FIXME!!! Make use of Axler's notation...

Recall: Fundamental Theorem of Linear Algebra (Strang*)

*Strang is way clearer. Probably because he is presenting to Engineers.

Let $A \in \mathbb{R}^{m\times n}$, then onsider Four subspaces, $\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^T), \mathcal{N}(A^T)$

- $\mathbb{R}^n = \mathcal{R}(A^T) \oplus \mathcal{N}(A)$, also $\mathbb{R}^n = \mathcal{R}(A^T) \perp \mathcal{N}(A)$
    - dim $\mathcal{R}(A^T) = r$
    - dim $\mathcal{N}(A) = n - r$
- $\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(AT)$, also $\mathbb{R}^n = \mathcal{R}(A) \perp \mathcal{N}(A^T)$
    - dim $\mathcal{R}(A) = r$
    - dim $\mathcal{N}(A^T) = m - r$

Where r is called the $rank\ of\ A$. Also note that $r \leq n$ and $r \leq m$.

Ex7.1 Basis

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$ is a bais for $\mathcal{R}(A)$, $\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$ is a basis for $\mathcal{N}(A)$.

$\left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right]$ is a bais for $\mathcal{R}(A)$, $[]$ is a basis for $\mathcal{N}(A^T)$.

8. Define product of two spaces, its dimension,

Recall: Dimension of a product is the sum of dimensions

Suppose $V_1, \ldots, V_m$ are finite-dimensional vector space. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

Recall: Products and direct sums $*$

Suppose the $U_1, \ldots, U_m$ are subspaces of V. Define a linear map $\Gamma : U_1 \times \cdots \times U_m \mapsto U_1 + \cdots + U_m$ by

$$\Gamma(u_1, \ldots, u_m) = u_1 + \ldots + u_m.$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective*.

$*$ Lame... Axler fails to make the connection to biliear maps.

Recall: A sum is a direct sum if and only if dimensions add up

Suppose V is finite-diemnsional and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \ldots + U_m$ is a direct sum if and only if

$$dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m.$$

9. Define the quotient of two vector spaces. Give an example of a linear quotient space.

Recall: Sum of vector and subspace

Suppose $v \in V$ and U is a subspace of V, then $v + U$ is a subset of V defined by,

$$v + U = \{v + u : u \in U\}.$$

Recall: Affine subset, parrallel

- subset of V of form $v + U$ for some $v \in V$ and some subspace U of V.
- for $v \in V$ and U a subspace of V, the affine subset $v + U$ is said to be $parrallel$ to U.

Recall: Quotient Space, V/U

Suppose U is a subspace of V, then V/U is the set of all affine subsets of V parrallel to U.

$$V/U = \{v + U : v \in V\}.$$

Recall: Two affine subsets parallel to U are equal or disjoint

Suppose U is a subspace of V and $v, w \in V$. Thne the following are equivalent:

(a) $v - w \in U$

(b) $v + U = w + U$

(c) $(v + U) \cap (w + U) \neq \{0\}$

Recall: Quotient map, π

Suppose U is a subspace of V. Then π is the linear map $\pi : V \mapsto V/U$ defined by,

$$\pi(v) = v + U$$

for $v \in V$.

Recall: Dimension of a quotient space

$$\text{dim} V/U = \text{dim} V - \text{dim} U.$$

Ex 9.1

Suppose $U = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then U is the line in R^2 through the orgin with slope 2. Thus $p + U$ is the line in R^2 that contains the point p and has slope 2.

Recall: $\tilde{T}$

$*$ Axler makes a lame attempt to demonstrate the Rank-Nullity Theorem using the definition of $\tilde{T}$. He's not wrong but why? I suppose it gets us thinking but in no way it this 'Done Right'.

Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : V/(\text{null} T) \mapsto W$ by,

$$\tilde{T}(v + \text{null} T) = Tv.$$

Recall: Null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Then,

(a) $T \in \mathcal{L}(V, W)$ is a linear map from $V/(\text{null}T)$ to W

(b) $T \in \mathcal{L}(V, W)$ is injective

(c) range$T \in \mathcal{L}(V, W)$=range T

(d) $V/(\text{null}T)$ is isomporphic to range$T$

10. Define a linear functional and the dual map.

Recall: Linear functional

A $\mathit{linear\ functional}$ on V is a linear map from $V \mapsto \mathbb{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

Ex 10.1 Linear functional

Define $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ by $\varphi(x, y, z) = 4x - 5y + 2z$. Then φ is a linear functional on R^3.

Fix $(c_1, \ldots, c_n) \in \mathbb{F}^n$. Define $\varphi : \mathbb{F}^n \mapsto \mathbb{F}$ by $\varphi(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$. Then φ is a linear functional on F^3.

Define $\varphi : \mathbb{R} \mapsto \mathbb{R}$ by $\varphi(p) = 3p''(5) + 7p(4)$. Then φ is a linear functional on $\mathcal{P}(\mathbb{R})$.

Define $\varphi : \mathbb{R} \mapsto \mathbb{R}$ by $\varphi(p) = \int_0^1 p(x)dx$. Then φ is a linear functional on $\mathcal{P}(\mathbb{R})$.

FIXME!!! Add more example if time allows...

Recall: Dual map, T'

If $T \in \mathcal{L}(V, W)$, then the $\mathit{dual\ map}$ of T is the linear map $T' \in (W', V')*$ defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.

$*$ note swapping of order due to nature of coordinate change

Ex 10.2 Dual space

Define $D : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Dp = p'$. $**$ Such an awful mixture of notation. WTF, Axler!?

Suppose φ is the linear functional on $\mathcal{P}(\mathbb{R})$ defined by $\varphi(p) = p(3)$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = p'(3).$$

Note: $D'(\varphi)$ is the linear functional on $\mathcal{P}(R)$ that takes p to $p'(3)$.

Ex 10.3 Dual space

Define $D : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Dp = p'$.

Suppose φ is the linear functional on $\mathcal{P}(\mathbb{R})$ defined by $\varphi(p) = \int_0^1 p$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = \int_0^1 p' = p(1) - p(0).$$

Note: $D'(\varphi)$ is the linear functional on $\mathcal{P}(R)$ that takes p to p(1)-p(0)$.

11. Compute a dual basis of a vector space spanned by a basis.

Recall: Dual basis is a basis of the dual space

Suppose V is finite-dimensional. Then the dual basis of the basis of V is the basis of V'

Ex. 11.1 Proof that dual basis is a basis of the dual space

Suppose $v_1, \ldots, v_n$ is a basis of V. Let $\varphi_1, \ldots, \varphi_n$ denote the dual basis.

Consider that $\varphi_1, \ldots, \varphi_n$ is linearly indendent list of element of V' such that

$$a_1\varphi_1 + \ldots + a_n\varphi = 0.$$

Consider $(a_1\varphi_1 + \ldots + a_n\varphi_n)(v_j) = a_j$ for $j = 1, \ldots, n$. Thus, $a_1 = \ldots = a_n = 0$. Hence $\varphi_1, \ldots, \varphi_n$ is linearly independent, thus a basis of V'.

Recall: We already computed this...

Ex 7.1 Basis

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$ is a bais for $\mathcal{R}(A)$, $\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$ is a basis for $\mathcal{N}(A)$.

$\left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right]$ is a bais for $\mathcal{R}(A^T)$, [] is a basis for $\mathcal{N}(A^T)$.

Appendix 1. State the Fundamental Theorem of Algebra. Apply it to a simple example.

Ex A.1

$p(z) = 0$ has a solution in $\mathbb{C}$.

$p(z) = a_n z^n + \ldots + a_0$ for $a_0, \ldots, a_n \in \mathbb{C}$.

Consider $M(z) = \frac{1}{p(z)}$.

Suppose $p(z) \neq 0$ then converges to a_0.

In other words, every nonconstant polynomial with complex coefficents has a zero.

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0:a
1:b
2:c
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