# Midterm Study Guide

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## Topics

1. Example of a finite dimensional vector space and an example of an infinite dimensional vector space (why is it infinite dim?),
2. Spanning vectors,
3. Independent vectors.
4. Dimension of a vector space.
5. What is a direct sum? Keep an example in mind. When is a sum a direct sum?
6. Illustrate the matrix for a particular linear map, say for $T(x, y, z) = (2x + z, 4y + 1)$
7. State the Fundamental Theorem of Algebra. Apply it to a simple example.
8. Define product of two spaces, its dimension,
9. Define the quotient of two vector spaces. Give an example of a linear quotient space.
10. Define a linear functional and the dual map.
11. Compute a dual basis of a vector space spanned by a basis.

## 1. Give an example of a finite dimensional vector space and an example of an infinite dimensional vector space. Why is it infinite dim?

Recall:

A vector space is finite-dimensional is some list of vector spans the space. Thus $V$ is a finite vector space if span $(v_1, \ldots, v_m) = \{a_1v_1 + \ldots + a_mv_m : a_1, \ldots, a_m \in \mathbb{F}\}$ with $\{0\}$ being the simpliest finite-dimensional space. An infinite-dimension space has no list that spans the vector space.

### Ex 1.1 Finite dimensional vector space

$V = \{a + b\sqrt{5}; a, b \in \mathbb{Q}\}$ is a vector space over $\mathbb{Q}$.

$0 + 0\sqrt{5} = 0 \in V$

$\underbrace{a + b\sqrt{5}}_{\in V} + \underbrace{c + d\sqrt{5}}_{\in V} = \underbrace{(a + c)}_{\in Q} + \underbrace{(b + d)}_{\in Q}\sqrt{5} \in V$ closed under addition

$\underbrace{\lambda}_{\in Q}(a + b\sqrt{5}) = \underbrace{\lambda}_{\in Q} a + \lambda \underbrace{b}_{\in Q} \sqrt{5} \in V$ closed under multiplication

note: $\mathbb{Q}$ is Rational Space.

### Ex 1.2 Finite dimensional vector space

$\mathcal{P}_m(\mathbb{F})$ is finite dimensional with degree at most $m$.

### Ex 1.2 Infinite dimensional vector space, and why.

$\mathcal{P}(\mathbb{F})$ is infinite dimensional as every polymonial is span has degree at most $m$ thus $z^{m+1}$ is not in span, hence no list spans $\mathcal{P}(\mathbb{F})$ thus infinate dimensional.

## 2. Give an example of Spanning vectors.

Recall:

A span is the smallest list containing a subspace.

### Ex 2.1 Spanning list

Suppose $(x_1, \ldots, x_n) \in \mathbb{F}$ then, $(x_1, \ldots, x_n) = x_1(1, 0, \ldots, 0) + x_2(0, 1, \ldots, 0) + \ldots + x_n(0, \ldots, 0, 1)$.

Thus, $(x_1, \ldots, x_n) \in$ span$((1, 0, \ldots, 0) + (0, 1, \ldots, 0) + \ldots + (0, \ldots, 0, 1))$.

## 3. Provide an example of Independent vectors.

Recall:

A list $v_1, \ldots, v_m \in V$ is linearly-independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes $a_1 v_1 + \ldots + a_m v_m = 0$ is $a_1 = \ldots = a_m = 0$.

### Ex3.1 $\mathbb{F}^n$

$(1,0,0,0),(0,1,0,0),(0,0,1,0)$ is linearly-independent in $\mathbb{F}^n$.

### Ex3.2 $\mathcal{P}(\mathbb{F})$

$1, \zeta, \ldots, \zeta^m$ is linearly-independent in $\mathcal{P}(\mathbb{F})$ for each non-negative interger $m$.

## 4. Demonstrate the dimension of a vector space.

Recall:

The dimension of a vector space is the length of its basis.

### Ex 4.1 $\mathbb{F}^n$

dim$\mathbb{F}^n = n$

### Ex 4.2 $\mathcal{P}_m(\mathbb{F})$

dim
$P_m(\mathbb{F}) = m + 1$

# 5. Explain what a direct sum is. Keep an example in mind. When is a sum a direct sum?

Recall:

The sum $U_1 + \cdots + U_2$ is a direct sum is each element can only be written on way.

### Ex5.1 $F^3$

$U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F})$ and

$W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F})$, then

$\mathbb{F}^3 = U \oplus W$.

### Ex5.2 $\mathbb{F}^n$

Suppose $U_j$ is a subspace on $\mathbb{F}^n$ of vectors with coordinated all 0 except possibly the $j^{th}$ slot

(ie. $U_2 = \{0, x, 0, \ldots, 0)\} \in \mathbb{F}^n$), then

$\mathbb{F}^n = U_1 \oplus \cdots \oplus U_n$

### Ex5.3 Non-example

$U_1 = (x, y, 0)$

$U_2 = (0, 0, z)$

$U_3 = (0, y, y)$

Note there are two ways to write $(0, 0, 0)$...

$(0, 1, 0) + (0, 0, 1) + (0, -1, -1) = (0, 0, 0)$ and

$(0, 0, 0) + (0, 0, 0) + (0, 0, 0) = (0, 0, 0)$

Things to consider:

The direct sum as a projection on some subspace if and only if it is idempotent, $E^2 = E$. * Where $E$ is $U$ along $W$.

$z = Ez + (1 - E)z$

Consider $Ez = x$ and $(1 - E)z = y$ then

$Ex = E^2z = Ez = x$ and $Ey = E(1 - E)z = Ez - E^2z = 0$

so that $x$ is in $U$ and $y$ is in $W$. Thus,

$V = U \oplus W$, and that the projection on $U$ along $W$ is precisely $E$.

see also 41 FDVS Halmos

ask Arek for some insight on this

# 6. Illustrate the matrix for a particular linear map

Recall:

$Tv_k = A_{1,k}w_1 + \ldots + A_{m,k}w_m = \sum_{j=1}^{m} A_{j,k}w_j$ or when context is not clear,

$M(\mathcal{T}, (v_1, \ldots, v_m), (w_1, \ldots, w_m)$ is used.

Ex.6.1 $T(x, y, z) = (2x + z, 4y + 1) \leftarrow$ Arek said this is a typo ($1$ ought to be $z$).

$\mathcal{T} \in \mathcal{L}(\mathbb{F}^3, \mathcal{P}_3(\mathbb{F})) \leftarrow$ We made it work anyway.

$$M(\mathcal{T}) = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 4 & 0 \end{pmatrix}$$

Ex.6.2 $T(x, y, z) = (2x + z, 4y + z) \leftarrow *$Note correction

$\mathcal{T} \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

$$M(\mathcal{T}) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

# 7. State the Fundamental Theorem of *Linear Algebra. Apply it to a simple example.

## Recall: Fundamental Theorem of Linear Maps

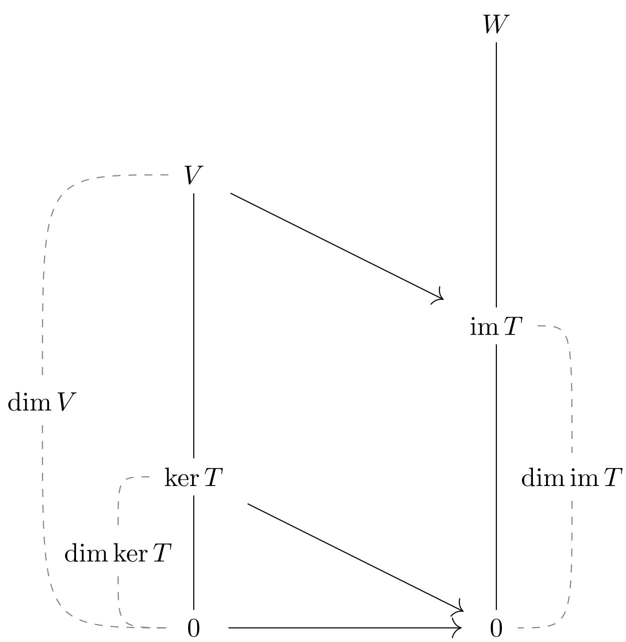
Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\mathrm{range} T$ is finite-dimensional and

$$\dim V = \dim \mathrm{null} T + \dim \mathrm{range} T. *$$

*Lame... Axler fumbles here and misses an opportunity to enlighten the student.

## Recall: Rank-nullity Theorem (Strang)

$$\dim V = \dim \ker T + \dim \mathrm{im} T$$




## Recall: Fundamental Theorem of Linear Algebra

Let $f$ be a linear map between two finite-dimensional vector spaces, representing a $m \times n$ matrix M of rank $r$ then,

- $r$ is the dimension of column space of $M$, which represent the image of $f$.
- $n - r$ is the dimension of the null space of $M$, which represent the kernel of $f$.
- $m - r$ is the dimension of the cokernel of $f$.

The transpose $M^T$ of $M$ is the matrix of the dual $f^*$ of $f$, and it follows,

- $r$ is the dimension of row space of $M$, which represent the image of $f^*$.
- $n - r$ is the dimension of the left null space of $M$, which represent the kernel of $f^*$.
- $m - r$ is the dimension of the cokernel of $f^*$.

FIXME!!! Make use of Axler's notation...

### Recall: Fundamental Theorem of Linear Algebra (Strang*)

*Strang is way clearer. Probably because he is presenting to Engineers.

Let $A \in \mathbb{R}^{m \times n}$, then onsider Four subspaces, $\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^T), \mathcal{N}(A^T)$

- $\mathbb{R}^n = \mathcal{R}(A^T) \oplus \mathcal{N}(A)$, also $\mathbb{R}^n = \mathcal{R}(A^T) \perp \mathcal{N}(A)$
    - dim $\mathcal{R}(A^T) = r$
    - dim $\mathcal{N}(A) = n - r$
- $\mathbb{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(AT)$, also $\mathbb{R}^n = \mathcal{R}(A) \perp \mathcal{N}(A^T)$
    - dim $\mathcal{R}(A) = r$
    - dim $\mathcal{N}(A^T) = m - r$

Where $r$ is called the $rank\ of\ A$. Also note that $r \leq n$ and $r \leq m$.

### Ex7.1 Basis

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$ is a basis for $\mathcal{R}(A)$, $\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$ is a basis for $\mathcal{N}(A)$.

$\left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right]$ is a basis for $\mathcal{R}(A)$, $[]$ is a basis for $\mathcal{N}(A^T)$.

# 8. Define product of two spaces, its dimension,

Recall: Dimension of a product is the sum of dimensions

Suppose $V_1, \ldots, V_m$ are finite-dimensional vector space. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$dim(V_1 \times \cdots \times V_m) = \text{dim} V_1 + \cdots + \text{dim} V_m.$$

Recall: Products and direct sums $*$

Suppose the $U_1, \ldots, U_m$ are subspaces of $V$. Define a linear map $\Gamma : U_1 \times \cdots \times U_m \mapsto U_1 + \cdots + U_m$ by

$$\Gamma(u_1, \ldots, u_m) = u_1 + \ldots + u_m.$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if $\Gamma$ is injective*.

$*$ Lame... Axler fails to make the connection to biliear maps.

### Recall: A sum is a direct sum if and only if dimensions add up

Suppose $V$ is finite-diemnsional and $U_1, \ldots, U_m$ are subspaces of $V$. Then $U_1 + \ldots + U_m$ is a direct sum if and only if

$$dim(U_1 + \cdots + U_m) = \text{dim} U_1 + \cdots + \text{dim} U_m.$$

# 9. Define the quotient of two vector spaces. Give an example of a linear quotient space.

Recall: Sum of vector and subspace

Suppose $v \in V$ and $U$ is a subspace of $V$, then $v+U$ is a subset of $V$ defined by,

$$v+U=\{v+u : u \in U\}.$$

### Recall: Affine subset, parrallel

- subset of $V$ of form $v+U$ for some $v \in V$ and some subspace $U$ of $V$.
- for $v \in V$ and $U$ a subspace of $V$, the affine subset $v+U$ is said to be $parrallel$ to $U$.

Recall: Quotient Space, $V/U$

Suppose $U$ is a subspace of $V$, then $V/U$ is the set of all affine subsets of $V$ parrallel to $U$.

$$V/U=\{v+U : v \in V\}.$$

Recall: Two affine subsets parallel to $U$ are equal or disjoint

Suppose $U$ is a subspace of $V$ and $v, w \in V$. Then the following are equivalent:

(a) $v-w \in U$

(b) $v+U=w+U$

(c) $(v+U) \cap (w+U) \neq \{0\}$

Recall: Quotient map, $\pi$

Suppose $U$ is a subspace of $V$. Then $\pi$ is the linear map $\pi : V \mapsto V/U$ defined by,

$$\pi(v)=v+U$$

for $v \in V$.

Recall: Dimension of a quotient space

$$\text{dim}V/U=\text{dim}V-\text{dim}U.$$

### Ex 9.1

Suppose $U=\{(x,2x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, then $U$ is the line in $R^2$ through the orgin with slope 2. Thus $p+U$ is the line in $R^2$ that contains the point $p$ and has slope 2.

Recall: $\tilde{T}$

$*$ Axler makes a lame attempt to demonstrate the Rank-Nullity Theorem using the definition of $\tilde{T}$. He's not wrong but why? I suppose it gets us thinking but in no way it this 'Done Right'.

Suppose $T \in \mathcal{L}(V,W)$. Define $\tilde{T} : V/(\text{null}T) \mapsto W$ by,

$$\tilde{T}(v+\text{null}T)=Tv.$$

Recall: Null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Then,

(a) $T \in \mathcal{L}(V, W)$ is a linear map from $V/(\mathrm{null} T)$ to $W$

(b) $T \in \mathcal{L}(V, W)$ is injective

(c) range$T \in \mathcal{L}(V, W)$=range T

(d) $V/(\mathrm{null} T)$ is isomporphic to range$T$

# 10. Define a linear functional and the dual map.

Recall: Linear functional

A $linear\ functional$ on $V$ is a linear map from $V \mapsto \mathbb{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

## Ex 10.1 Linear functional

Define $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ by $\varphi(x, y, z) = 4x - 5y + 2z$. Then $\varphi$ is a linear functional on $R^3$.

Fix $(c_1, \ldots, c_n) \in \mathbb{F}^n$. Define $\varphi : \mathbb{F}^n \mapsto \mathbb{F}$ by $\varphi(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$. Then $\varphi$ is a linear functional on $F^3$.

Define $\varphi : \mathbb{R} \mapsto \mathbb{R}$ by $\varphi(p) = 3p''(5) + 7p(4)$. Then $\varphi$ is a linear functional on $\mathcal{P}(\mathbb{R})$.

Define $\varphi : \mathbb{R} \mapsto \mathbb{R}$ by $\varphi(p) = \int_0^1 p(x)dx$. Then $\varphi$ is a linear functional on $\mathcal{P}(\mathbb{R})$.

FIXME!!! Add more example if time allows...

Recall: Dual map, $T'$

If $T \in \mathcal{L}(V, W)$, then the $dual\ map$ of $T$ is the linear map $T' \in (W', V')*$ defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.

$*$ note swapping of order due to nature of coordinate change

## Ex 10.2 Dual space

Define $D : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Dp = p'$. $**$ Such an awful mixture of notation. WTF, Axler!?

Suppose $\varphi$ is the linear functional on $\mathcal{P}(\mathbb{R})$ defined by $\varphi(p) = p(3)$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = p'(3).$$

Note: $D'(\varphi)$ is the linear functional on $\mathcal{P}(R)$ that takes $p$ to $p'(3)$.

## Ex 10.3 Dual space

Define $D : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ by $Dp = p'$.

Suppose $\varphi$ is the linear functional on $\mathcal{P}(\mathbb{R})$ defined by $\varphi(p) = \int_0^1 p$. Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(p') = \int_0^1 p' = p(1) - p(0).$$

Note: $D'(\varphi)$ is the linear functional on $\mathcal{P}(R)$ that takes $p$ to p(1)-p(0)$.

# 11. Compute a dual basis of a vector space spanned by a basis.

Recall: Dual basis is a basis of the dual space

Suppose $V$ is finite-dimensional. Then the dual basis of the basis of $V$ is the basis of $V'$

## Ex. 11.1 Proof that dual basis is a basis of the dual space

Suppose $v_1, \ldots, v_n$ is a basis of $V$. Let $\varphi_1, \ldots, \varphi_n$ denote the dual basis.

Consider that $\varphi_1, \ldots, \varphi_n$ is linearly indendent list of element of $V'$ such that

$$a_1\varphi_1 + \ldots + a_n\varphi = 0.$$

Consider $(a_1\varphi_1 + \ldots + a_n\varphi_n)(v_j) = a_j$ for $j = 1, \ldots, n$. Thus, $a_1 = \ldots = a_n = 0$. Hence $\varphi_1, \ldots, \varphi_n$ is linearly independent, thus a basis of $V'$.

Recall: We already computed this...

## Ex 7.1 Basis

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$ is a bais for $\mathcal{R}(A)$, $\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$ is a basis for $\mathcal{N}(A)$.

$\left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right]$ is a bais for $\mathcal{R}(A^T)$, [] is a basis for $\mathcal{N}(A^T)$.

# Appendix 1. State the Fundamental Theorem of Algebra. Apply it to a simple example.

## Ex A.1

$p(z) = 0$ has a solution in $\mathbb{C}$.

$p(z) = a_n z^n + \ldots + a_0$ for $a_0, \ldots, a_n \in \mathbb{C}$.

Consider $M(z) = \frac{1}{p(z)}$.

Suppose $p(z) \neq 0$ then converges to $a_0$.

In other words, every nonconstant polynomial with complex coefficents has a zero.

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0:a
1:b
2:c
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