Math 725 Advanced Linear Algebra

HW6

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Inner Product Spaces

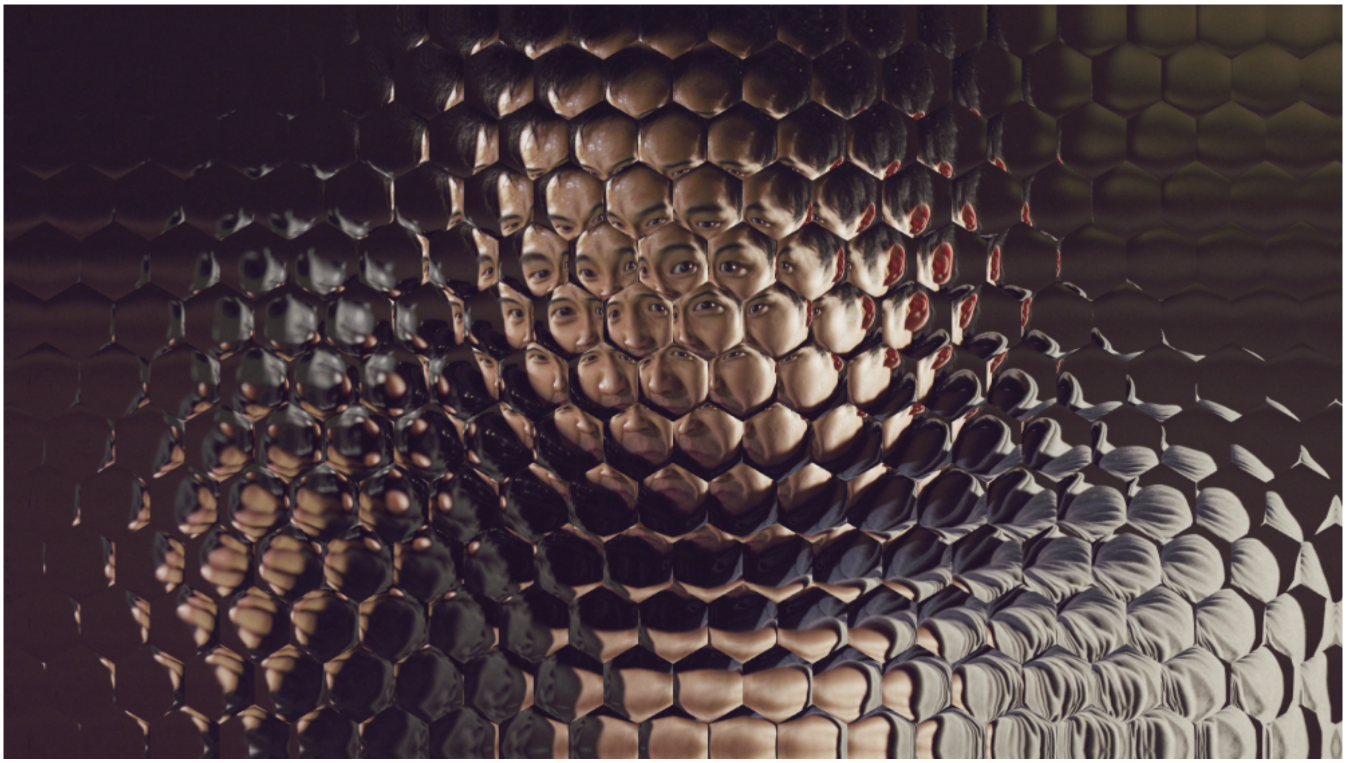


Image from:

Ignite-pro, 'Insect Vision'

In [1]:
```
# import libraries
import numpy as np
import sympy as sym
from sympy.matrices import Matrix
from sympy import I
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

from sympy import init_printing
init_printing()
```

1. (6A12)

Prove that

$$(x_1 + \cdots + x_n)^2 \leq \mathfrak{n}(x_1^2 + \cdots + x_n^2)$$

for all positive integers $\mathfrak{n}$ and all real numbers $x_1, \ldots, x_n$.

Recall: norm, $\|v\|$

For $v \in V$, the norm of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Recall: Euclidean inner product

on $\mathbb{F}^m = n$ is defined by,

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$

Recall: Cauchy-Schwarz Inequality

Suppose $u, v \in V$. Then

$$\|\langle u, v \rangle\| \leq \|u\| \|v\|.$$

This inequality is an equality if and only if one of the u, v is a scalar multiple of the other.

Recall: Cauchy-Schwarz-Bunyakosky Theorem

Restated in a slightly more interpetable manner.

Let V be an inner product space. If $v, w \in V$, the

$$|\langle v, w \rangle|^2 \leq \langle v, v \rangle \langle w, w \rangle.$$

Solution:

Our goal is,

$$(x_1 + \cdots + x_n)^2 \leq \mathfrak{n}(x_1^2 + \cdots + x_n^2).$$

$$(x_1 + \cdots + x_n)^2 \leq \langle x_1, \ldots, x_n, x_1, \ldots, x_n \rangle \langle 1_1, \ldots, 1_n, 1_1, \ldots, 1_n \rangle \text{ (Cauchy-Schwarz)}$$

$$(x_1 + \cdots + x_n)^2 \leq (x_1^2 + \cdots + x_n^2)(1_1^2 + \cdots + 1_n^2)$$

Hence,

$$(x_1 + \cdots + x_n)^2 \leq (x_1^2 + \cdots + x_n^2)\mathfrak{n},$$

as desired.

□

Note: At a more primative level than Cauchy-Schwarz, the Archimedean Property shows that the square of the sum is less than the sum of squares. We may develop this more by considering the circle in the complex plane to be $r_i = \sum_{j \neq i} |x_{ij}|$ and center x_{ij}. This circle will only exist if there is an eigenvalue and thus provides a geometeric intuition into Diagonal Dominance and furthermore the FTLA.

See also:

Appendix 1. Complex Euclidean inner product space $\mathbb{C}^3$, 6) Decompose a

In [2]:
```
# scratch
n = 3
v = sym.ones(n,1)
display(v)
display(v.T*v) # as we know and now demonstrate
```

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[3]$$

Going deeper:

Above we applyied Example 6.17 (a) and reference Caucy-Schwarz, noting that there exists a $v \in V$ such that $\|v\| = 1$ is a unit vector. We can normalize any $v \in V$ to produce a unit vector by dividing it by its norm.

$$(\mathbb{R}^n, \langle , \rangle) : \|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

$$(\mathbb{C}^n, \langle , \rangle) : \|v\| = \sqrt{\sum_{i=1}^{n} v_i \overline{v_i}} = \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

By the parallelogram law we can guarantee that a norm over a vector space will be induced by an inner product, thus the norm is Hilbertian. We've already discovered some interesting properties based on this through experimentation, such as Polarization,

$$\langle v, w \rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right), (\mathbb{F} \in \mathbb{R}), \text{ and}$$

$$\langle v, w \rangle = \frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2 + i\left(\|v+iw\|^2 - \|v-iw\|^2\right)\right], (\mathbb{F} \in \mathbb{C}).$$

See also: From Euclidean to Hilbert Spaces, Edoardo Provenzi

Going deep:

Consider if $F \in \mathbb{R}$ or $\mathbb{C}$ and is $D = \left[d_{ij}\right]$ is nonsingular matrix in $\mathcal{M}_{n\times n}(F)$ then inner production on F^n is defined by,

$$\left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix} \right\rangle = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} D D^H \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix}, \text{ where } D^H = \left[\overline{d}_{ij}\right]^T.$$

Weighted dot product is defined by, $\left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} \overline{b_1} \\ \vdots \\ \overline{b_n} \end{bmatrix} \right\rangle = \sum_{i=1}^{n} c_i a_i \overline{b}_i.$

See also: The Linear Algebra a beginning Gradute Student Ought to Know, Golan.

2. (6A18)

Suppose $p > 0$. Prove that there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in \mathbb{R}^2$ if and only if $p = 2$.

Solution:

Consider polarization/parrallel theorem proof (6.22) and that

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) = 0$$

is orthogonal. Then apply Pythagoras,

$$\sqrt{\|u+v\|^2} = \sqrt{\|u\|^2 + \|v\|^2} = \left(\|u\|^p + \|v\|^p\right)^{1/p}.$$

Now we assume that from some p this is true, thus,

$$\|u\|^2 + \|v\|^2 = \|u\|^p + \|v\|^p,$$

which is true if and only if $p = 2$.

□

See also: "Scratch thoughts" below which were used to refine this solution.

Scratch thoughts:

Consider $u, v \in \mathbb{R}^2$ such that $\langle u, v\rangle = 0$ (othorgonal) and let $u = \begin{bmatrix}1\\1\end{bmatrix}$ and $v = \begin{bmatrix}1\\-1\end{bmatrix}$, then

$$\langle u, v\rangle = \left\langle \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\rangle = 0,$$

Then by Pythagoras,

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$
$$\implies \| \begin{bmatrix}2\\0\end{bmatrix} \|^2 = \| \begin{bmatrix}1\\1\end{bmatrix} \|^2 + \| \begin{bmatrix}1\\-1\end{bmatrix} \|^2$$

Thus for RHS, $\|(x, y)\|$,

$$2^2 = \sqrt{2}^2 + \sqrt{2}^2 = 2 + 2 = 4.$$

Also by some Pythagoras wrangling, the LHS becomes,

$$\sqrt{\|u\|^2 + \|v\|^2} = (< u, u >^p + < v, v >^p)^{1/p},$$
$$\implies \|u\|^2 + \|v\|^2 = \|u\|^p + \|v\|^p,$$
$$\implies \sqrt{2}^2 + \sqrt{2}^2 = \sqrt{2}^p + \sqrt{2}^p,$$
$$4 = (\sqrt{2}\sqrt{2})^p = 2^p,$$

Thus $p = 2$ as desired.

$\square$

In [3]:
```
# scratch the itch
u = Matrix([1,1])
v = Matrix([1,-1])
u.T*v, (u.T*u, v.T*v), u+v, (u+v).norm(), u.norm(),v.norm()
```

Out[3]:

$$\left([0], ([2], [2]), \begin{bmatrix}2\\0\end{bmatrix}, 2, \sqrt{2}, \sqrt{2} \right)$$

In [4]:
```
# think about distance in R^2
x,y,p=sym.symbols('x y p')
eq = (x**p+y**p)**(1/p)
eq
```

Out[4]:

$$(x^p + y^p)^{\frac{1}{p}}$$

In [5]:
```
eq = sym.Eq(u.norm()**p+v.norm()**p,4)
eq, sym.solve(eq,p)
```

Out[5]:

$$\left(2 \cdot 2^{\frac{p}{2}} = 4, [2] \right)$$

3. (6A20)

Suppose V is a complex inner product space. Prove that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$$

for all $u, v \in V$.

Recall: Pythagorean Theorem

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2$$

$$-\|u-v\|^2 = -\langle u-v, u-v\rangle = -\|u\|^2 + \langle u, v\rangle + \langle v, u\rangle - \|v\|^2$$

$$i\|u+v\|^2 = i\langle u+v, u+v\rangle = i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle + i\|v\|^2$$

$$-i\|u+v\|^2 = i\langle u-v, u-v\rangle = -i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle - i\|v\|^2$$

Solution:

By direct computation we have,

$$\frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4} = \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle + i\langle u+iv, u+iv\rangle}{4}$$

$$= \frac{\|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2}{4}$$

$$\frac{-\|u\|^2 + \langle u, v\rangle + \langle v, u\rangle - \|v\|^2}{4}$$

$$\frac{+i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle + i\|v\|^2}{4}$$

$$\frac{-i\|u\|^2 + \langle u, v\rangle - \langle v, u\rangle - i\|v\|^2}{4}$$

$$= \frac{4\langle u, v\rangle}{4} = \langle u, v\rangle,$$

as desired.

□

Going deeper:

Explore by computation...

...apply.

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4} = \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle + i\langle u+iv}{4}$$

Recall: Pythagorean Theorem

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2$$

In [6]:
```
i = sym.I
u,v = sym.symbols('u v')

ip = ((u+v)**2 - (u-v)**2 + i*(u+i*v)**2 - i*(u-i*v)**2)/4
display(sym.expand(ip)) # * zero result is expect for orthogonality
```

0

See also:

Appendix 1. Complex Euclidean inner product space $\mathbb{C}^3$, 4) Orthongonal Projection of v onto S

Notes:

* Based on this result we conclude that the inner product is the matrix multiplication of a vector and a dual vector.

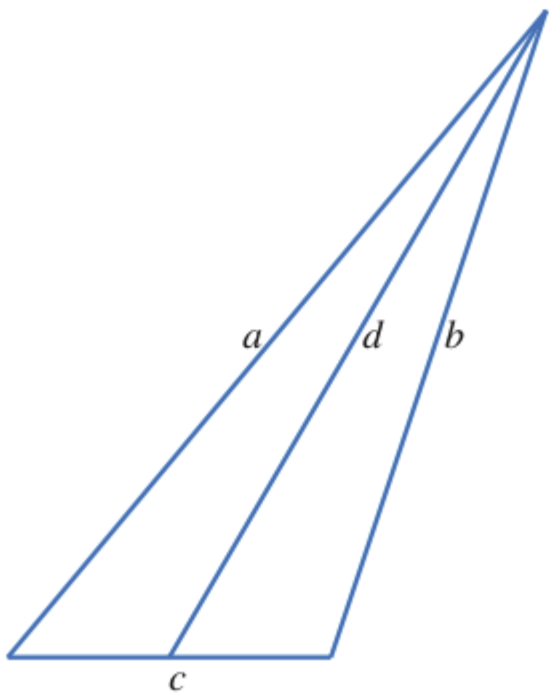
* Consider extending the Sympy library to improve the interpetablity of symbolic inner-products.

* Discuss Gramian matrices of polytopes with Arek

4. (6A31)

Use inner products to prove Apollonius' Identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$



Recall: Euclidean Inner Production

on $\mathbb{F}^n$ is defined by,

$$\langle(w_1,\dots,w_n),(z_1,\dots,z_n)\rangle = w_1\overline{z_1}+\cdots+w_n\overline{z_n}.$$

Recall: Triangle Inequality

Suppose $u, v \in V$. Then

$$\|u+v\| \leq \|u\|+\|v\|.$$

This inequality is an equality if and only if one of the u, v is a nonnegative multiple of the other.

Recall: Cauchy-Schwarz Inequality

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|+\|v\|.$$

This inequality is an equality if and only if one of the u, v is a nonnegative multiple of the other.

Recall: Cauchy-Schwarz-Bunyakosky Theorem

Restated in a slightly more interpetable manner.

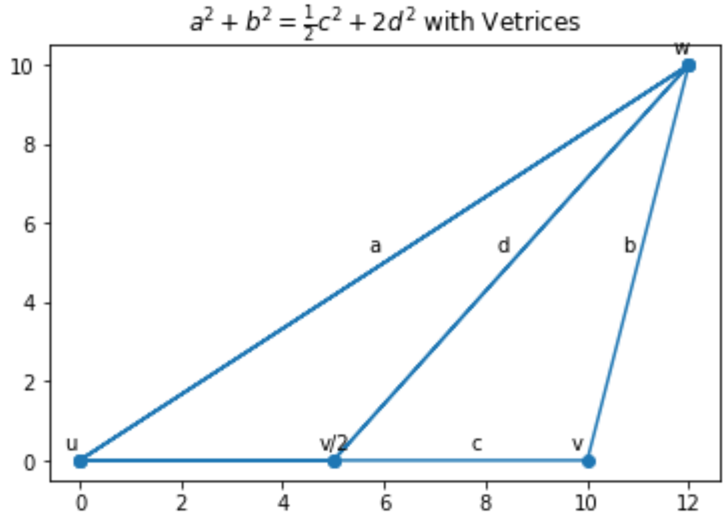
Let V be an inner product space. If $v, w \in V$, the

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle.$$

Scratch and Itch:

In [7]:

```
# let's play about with this a bit in R^2
t1=Matrix([[0,0],[10,0],[12,10]]).T
t2=Matrix([[0,0],[5,0],[12,10]]).T
t3=Matrix([[0,0],[12,10],[5,0]]).T

T=t1
T=(T.row_join(t2).row_join(t3))

x,y=np.array(T)
plt.scatter(x,y)
plt.plot(x,y)

plttext = lambda x,y,t: plt.text(x-.3,y+.3,t)
plttext(12,10,'w')
plttext(0,0,'u')
plttext(10,0,'v')
plttext(5,0,'v/2')

plttext((12+0)/2,10/2,'a')
plttext((12+10)/2,10/2,'b')
plttext(8,0,'c')
plttext((12+5)/2,(10-0)/2,'d')

plt.title('$a^2 + b^2 =\\frac{1}{2}c^2+2d^2$ with Vetrices');
```



Solution:

Goal is,

$$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2.$$

$$\text{Let, } \begin{cases} a = w - u \\ b = w - v \\ c = u - v \\ d = w - \frac{1}{2}(u + v) \end{cases}$$

Consider,

$$\|w - u\|^2 + \|w - v\|^2 = \frac{1}{2}\|u - v\|^2 + 2\Big(\|w\| - \frac{1}{2}\|u + v\|\Big)^2 \text{ is true,}$$

$$\implies a^2 + b^2 = \frac{1}{2}c^2 + 2d^2, \text{ as desired.}$$

$\square$

See also:

Direct calculation below.

See also:

Alternate solution below, however I actually preferred this solution as I only use the definition of the inner product and a couple boundary conditions, by fixing a point(u) at origin and v and $v/2$ on the axis. This is process of selecting the best eigenbasis for the problem. Now that we see this we could have projected w onto the x-axis using the axis as an orthonormal basis.

See also:

Appendix 1. Complex Euclidean inner product space $\mathbb{C}^3$, 4) Orthongonal Projection of v onto S

In [8]:
```
# direct calculation
u,v,w = sym.symbols('u v w')

u=0 # simplify our coordinates by creating boundary condition
a=w-u
b=w-v
c=u-v
d=w-(u+v)/2 # other boundary condition
```

In [9]:
```
display(sym.expand((u-v)**2/2+2*(w-(u+v)/2)**2))
```

$v^2 - 2vw + 2w^2$

In [10]:
```
# now plug and chug on the given formula
display(sym.expand(a**2+b**2))
sym.expand(a**2+b**2)==sym.expand(c**2/2 + 2*d**2)
```

$v^2 - 2vw + 2w^2$

Out[10]:
```
True
```

Alternate Solution:

Our goal is $a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$.

Assume $a, b \in \mathbb{R}^2$ and are scalar multiples of eachother, then by Triangle Inequality,

$$d + \frac{1}{2}c = a \text{ and } b + \frac{1}{2}c = d,$$

$$\implies a = d + \frac{1}{2}c \text{ and } b = d - \frac{1}{2}c.$$

Then by Cauchy-Schwarz (and Pythagoras of Samos),

$$a^2 + b^2 = \|a + b\|^2 = \langle a + b, a + b\rangle,$$

$$= \|a\|^2 + \|b\|^2 = \left\|d + \frac{1}{2}c\right\|^2 + \left\|d - \frac{1}{2}c\right\|^2, \text{ see below: *term expansion}$$

$$= a^2 + b^2 = \frac{1}{2}c^2 + 2d^2,$$

as desired.

$\square$

See also: Axler, proof of 6.18 Triangle Inequality, 6.19 and 6.20.

Note: Alternate solution based on the numberic experiments above and only presented as another view of the same problem less the vertex boundary conditions.

In [11]:
```
# *term expansion
a,b,c,d = sym.symbols('a b c d')
a = (d+c/2)**2
b = (d-c/2)**2
sym.expand(a + b)
```

Out[11]: $\frac{c^2}{2} + 2d^2$

Going deeper:

Consider Cauchy completeness, supremum, Complete metric spaces... Visual Origins of Consciousness. See Appendix. 2

5. (6B2)

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vector in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

Recall 6.30 Writing a vector as linear combinations of otrthonormal basis

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n \text{ and}$$

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

Recall 6.31 Gram-Schmidt Procedure

Suppose $v_1, \ldots, v_m$ is linearly independent list of vectors in V. Let $e_e = v_1/\|v_1\|$. For $j = 2, \ldots m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1\rangle e_1 - \ldots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \ldots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}.$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vector in V such that

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for $j = 1, \ldots, m$.

Solution:

$\Rightarrow$

By Gram-Schimidt $v \in \text{span}\,(e_1, \ldots, e_m)$.

$\square$

$\Leftarrow$

Consider $u = \overline{\langle v, e_1 \rangle} e_1 + \cdots + \overline{\langle v, e_m \rangle} e_m$, then

$$\langle u, v \rangle = \langle v, u \rangle = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 = \|v\|^2.$$

Thus, $\langle u, v \rangle = \|v\|^2$ and $\|u\|^2 = \|v\|^2$,

$$\implies \|u - v\|^2 = \|u\|^2 + \|v\|^2 - \langle u, v \rangle - \langle v, u \rangle = 0.$$

Hence, $v = u \in \text{span}\,(e_1, \ldots, e_m)$.

$\square\square$

6. (6B5)

On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x) q(x) dx.$$

Apply the Gram-Schmidt Procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

Recall: 6.31 Gram-Schmidt Procedure

Suppose $v_1, \ldots, v_m$ is linearly independent list of vectors in V. Let $e_e = v_1 / \|v_1\|$. For $j = 2, \ldots m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}}{\|v_j - \langle v_j, e_1 \rangle e_1 - \ldots - \langle v_j, e_{j-1} \rangle e_{j-1}\|}.$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vector in V such that

$$\text{span}\,(v_1, \ldots, v_j) = \text{span}\,(e_1, \ldots, e_j)$$

for $j = 1, \ldots, m$.

Recall: Example 6.33

Find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, where the inner product is given by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Apply Gram-Schmidt Procedure to the basis $1, x, x^x$.

$$e_1 = \frac{\langle v_1, e_1 \rangle e_1}{\|\langle v_1, e_1 \rangle e_1\|} = \frac{1}{\sqrt{\int_{-1}^{1} 1^2 dx}} = \sqrt{\frac{1}{2}}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \sqrt{\frac{3}{2}}x$$

$$e_3 = \frac{v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2}{\|v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2\|} = \frac{x - \frac{1}{3}}{\sqrt{\int_{-1}^{1} \left(x - \frac{1}{3}\right)^2 dx}} = \frac{3\sqrt{10}\left(x^2 - \frac{1}{3}\right)}{4} = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right)$$

Solution:

Apply Gram-Schmidt as in Example 6.33 but over interval $[0, 1]$ rather than $[-1, 1]$.

$$e_1 = \frac{\langle v_1, e_1 \rangle e_1}{\|\langle v_1, e_1 \rangle e_1\|} = \frac{1}{\sqrt{\int_{0}^{1} 1^2 dx}} = 1$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} = \frac{x - \int_0^1 x dx}{\sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx}} = 2\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$e_3 = \frac{v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2}{\|v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2\|} = \frac{x^2 - \int_0^1 x^2(1)dx(1) - \int_0^1 x^2\sqrt{3}\left(x - \frac{1}{2}\right)dx\sqrt{3}\left(x - \frac{1}{2}\right)}{\|v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2\|} = \frac{x^2 - x + \frac{1}{6}}{\sqrt{\frac{\sqrt{5}}{30}}} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$$

$$(e_1, e_2, e_3) = \left(1, \sqrt{3}\left(2x - 1\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right)$$

□

See also:

'feed the mind' below for direct computation and Example 6.33 above for the algorithm that we implimented.

Note: Valided using Axler's example along with notes from Rashidi and Pham.

In [12]:
```python
# feed the mind
# validated with Example 6.33
x = sym.symbols('x', real=True) #
v1,v2,v3 = [1,x,x**2]
lim0 = 0 # -1 For Example 6.33 and 0 for 6B5
lim1 = 1

# do it
e1_nom = v1
e1_denom = sym.sqrt(sym.integrate(e1_nom**2, (x,lim0,lim1)))
e1 = e1_nom/e1_denom

e2_nom = (v2 - sym.integrate(v2*e1, (x,lim0,lim1))*e1)
e2_denom = sym.sqrt(sym.integrate(e2_nom**2, (x,lim0,lim1)))
e2= e2_nom/e2_denom

e3_nom = (v3 - sym.integrate(v3*e1, (x,lim0,lim1))*e1 - sym.integrate(v3*e2, (x,lim0,lim
e3_denom = sym.sqrt(sym.integrate(e3_nom**2, (x,lim0,lim1))) # why doesn't this agree
e3 = e3_nom/e3_denom

# validation cases
e3_axler = sym.sqrt(sym.Rational(45,8)) * (x**2-sym.Rational(1,3)) # Okay
e3_pham = sym.sqrt(105)*sym.Rational(2,7)* (x**2-x-sym.Rational(1,6)) # Pham made a mist
e3_rashidi = 6*sym.sqrt(5)*(x**2-x+ sym.Rational(1,6))# Rashidi got it right

if (lim0==-1):
    display([e1,e2,e3])
    display(e3==e3_axler) # checks out

if (lim0==0):
    display([e1,e2,e3]) # looks good
    display(e3==e3_rashidi) # checks out
```

$$\left[1,\ 2\sqrt{3}\left(x-\frac{1}{2}\right),\ 6\sqrt{5}\left(x^2-x+\frac{1}{6}\right)\right]$$

True

In [13]:
```python
# put on the latex gloves
print(sym.latex(e1))
print(sym.latex(e2))
print(sym.latex(e3))
```

```
1
2 \sqrt{3} \left(x - \frac{1}{2}\right)
6 \sqrt{5} \left(x^{2} - x + \frac{1}{6}\right)
```

In [14]:
```python
e3_nom, sym.expand(e3_nom**2) # this is where Pham went wrong
```

Out[14]:

$$\left(x^2-x+\frac{1}{6},\ x^4-2x^3+\frac{4x^2}{3}-\frac{x}{3}+\frac{1}{36}\right)$$

7. (6B15)

Suppose $C_R([-1,1])$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

for $f, g \in C_R([-1,1])$. Let φ be the linear functional on $C_R([-1,1])$ defined by $\varphi(f) = f(0)$. Show that there does not exist $g \in C_R([-1,1])$ such that

$$\varphi(f) = \langle f, g\rangle$$

for every $f \in C_R([-1,1])$.

[This exercise shows that the Riesz Representation Theroem (6.42) does not hold on infinte-dimensional vector spaces without additional hypotheses on V and φ.]

Solution:

Consider the contrary and there exists a g such that $\varphi(f) = f(t)$ for all $f \in C_{\mathbb{R}}$ and $t \in [-1, 1]$.

If there exists such a function then,

$$f(x) = (x-t)^2 g(x).$$

Thus $f(t) = 0$ must exist. Hence,

$$\int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} [(x-t)g(x)]^2 dx = f(t) = 0.$$

$$\implies (x-t)g(x) = 0 \text{ and } g(x) = 0,$$

since g(x) is continuous on the interval.

Hence, $\langle f, g \rangle = 0$ for all $f \in C_{\mathbb{R}}([-1, 1])$. This a contradiction if $f(t) \neq 0$, thus no such function $g(x)$ exist.

$\square$

Note:

Using Pham's construction of $(x-t)^2 g(x)$ to create an intergal in the familar form of $\langle f, g \rangle = \int_{-1}^{1} [(x-t)g(x)]^2 dx$. A similar agrument is made by Golan.

See also:

Golan shows that for $C_{\mathbb{C}}([a, b])$ we can define an inner product as $\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$, where continuity is a key property.

In [15]:
```
# Think about another case
v = Matrix([1,sym.I])
display(v)
display(v.T*v) # nope, only works for F \in R
display(sym.conjugate(v.T)*v) # Hermitian transpose works for both
```

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$[0]$$

$$[2]$$

Appendix 0. Being but men

```
Being but men, we walked into the trees
Afraid, letting our syllables be soft
For fear of waking the rooks,
For fear of coming
Noiselessly into a world of wings and cries.

If we were children we might climb,
Catch the rooks sleeping, and break no twig,
And, after the soft ascent,
Thrust out our heads above the branches
To wonder at the unfailing stars.

Out of confusion, as the way is,
And the wonder, that man knows,
Out of the chaos would come bliss.

That, then, is loveliness, we said,
Children in wonder watching the stars,
Is the aim and the end.

Being but men, we walked into the trees.

-Dylan Thomas
```

Appendix 1. Complex Euclidean inner product space $\mathbb{C}^3$

See: Ex 1.1, Euclidean to Hilbert Spaces, Edoardo Provenzi

Consider the complex Euclidean inner product space $\mathbb{C}^3$ and the following three vectors,

$$u = (0, i, 2i), v = (2i, 0, -i), w = \left(0, i, \frac{1}{2}e^{-\frac{\pi}{2}i}\right).$$

1) Determine the orthogonality relationships between vectors u, v, w.

2) Calculate the norm of u, v, m ans the Euclidean distances between them.

3) Verify that (u, v, w) is an (non-orthogonal) basis of $\mathbb{C}^3$.

4) Let S be the vector subspace of $\mathbb{C}^3$ generated by u and w. Calculate $P_s v$, the orthogonal projection of v onto S. Calculate $d(v, P_s v)$, that is, the Euclidean distance between v and its projection onto S, and verify that this minimizes the distance between v and the vector of S ($hint:$ look at the square of the distance).

5) Using the results of the previous questions, determing an orhogonal basis and an orthonormal basis for $\mathbb{C}^3$ without using the Gram-Schmidt orthonormalization process ($hint:$ remember the geometrix relationship between the residual vector r and the subspace S).

6) Given a vector $a = (2i, -1, 0)$, write the decomposition of a and the Plancherel's therom in relation to the orthonormal basis identified in point 5. Use these results to identify the vector from the orthonormal basis which has the heaviest weight in the decomposition of a (and which gives the best "rough approximation" of a). Use a graphics program to draw the progressive vector sum of a, beginning with the rough approximation and adding finer detail supplied by the other vectors.

In [16]:
```
from sympy.physics.quantum.dagger import Dagger
u = Matrix([0,sym.I,2*sym.I])
v = Matrix([2*sym.I, 0, -sym.I])
w = Matrix([0, sym.I, sym.exp(-sym.pi/2*sym.I)/2])

ip = lambda z1, z2: (Dagger(z1)*z2)[0]

display(Latex('1) Determine relationships'))
display([u,v,w])
display([sym.conjugate(v.T), Dagger(v)]) # same, same
display([Dagger(u)*v, Dagger(v)*w, Dagger(w)*u]) # non zero so not normal
display([v.T*u.conjugate(), w.T*v.conjugate(), u.T*w.conjugate()]) # same
display([ip(u,v), ip(v,w), ip(w,u)]) # same again
display(Latex('Note: $u,w$ are orthogonal.'))
```

1) Determine relationships

$$\left[\begin{bmatrix} 0 \\ i \\ 2i \end{bmatrix}, \begin{bmatrix} 2i \\ 0 \\ -i \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ -\frac{i}{2} \end{bmatrix} \right]$$

$$[[-2i \quad 0 \quad i], [-2i \quad 0 \quad i]]$$

$$\left[[-2],\ \left[\frac{1}{2}\right],\ [0] \right]$$

$$\left[[-2],\ \left[\frac{1}{2}\right],\ [0] \right]$$

$$\left[-2,\ \frac{1}{2},\ 0 \right]$$

Note: u, w are orthogonal.

In [17]:
```
display(Latex('2) Calculate Euclidean distances'))
display([ip(u,u), ip(v,v), ip(w,w)])
d = lambda z1,z2: sym.sqrt(ip(z1-z2,z1-z2))
display([d(u,v), d(u,w), d(v,w)])
```

2) Calculate Euclidean distances

$$\left[5,\ 5,\ \frac{5}{4} \right]$$

$$\left[\sqrt{14},\ \frac{5}{2},\ \frac{\sqrt{21}}{2} \right]$$

In [18]:
```
display(Latex('3) Verify Orthogonality'))
A= u.T.col_join(v.T).col_join(w.T) # the dual is in the eye of the i
A, A.rref()
```

3) Verify Orthogonality

Out[18]:

$$\left(\begin{bmatrix} 0 & i & 2i \\ 2i & 0 & -i \\ 0 & i & -\frac{i}{2} \end{bmatrix},\ \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ (0,\ 1,\ 2) \right) \right)$$

In [19]:
```
display(Latex('4) Orthongonal Projection of $v$ onto $S$'))
display(sym.GramSchmidt([u,w,v])) # one normal basis
#P_sv=(ip(v,u)/(ip(u,u)**2)) * u + ip(v,w)/(ip(w,w)**2) * w # Nope, almost.
P_sv=(ip(v,u)/(ip(u,u))) * u + ip(v,w)/(ip(w,w)) * w # Correction.
display(P_sv)
r = v - P_sv
display(r)
d(v,P_sv)**2

# display(Latex('Maybe Maths works differently in France. ;-p.'))
# display(Latex('Looks like Provenzi normalized the vectors.  Think about why Provenzi g
```

4) Orthongonal Projection of v onto S

$$\left[\begin{bmatrix} 0 \\ i \\ 2i \end{bmatrix},\ \begin{bmatrix} 0 \\ i \\ -\frac{i}{2} \end{bmatrix},\ \begin{bmatrix} 2i \\ 0 \\ 0 \end{bmatrix} \right]$$

$$\begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 2i \\ 0 \\ 0 \end{bmatrix}$$

Out[19]: 4

The most general vector in S is $s = \alpha u + \beta w = (0, (\alpha+\beta)i, (2\alpha - \frac{\beta}{2}i)$ and $d(v,s)^2 = \|v - s\|^2 = 4 + (\alpha+\beta)^2 + (2\alpha - \frac{\beta}{2} + 1)^2 \geq 4 = d(v, P_S v)^2$. Thus confirming the minimum distance is $P_S v \in S$.

In [20]:
```
display(Latex('5) Orthonormalize in an abnormal manner.'))
u_= u/sym.sqrt(ip(u,u))
w_= w/sym.sqrt(ip(w,w))
r_ = r/sym.sqrt(ip(r,r))
display([u_, w_, r_])
```

5) Orthonormalize in an abnormal manner.

$$\left[\begin{bmatrix} 0 \\ \frac{\sqrt{5}i}{5} \\ \frac{2\sqrt{5}i}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{2\sqrt{5}i}{5} \\ -\frac{\sqrt{5}i}{5} \end{bmatrix}, \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix} \right]$$

In [21]:
```
display(Latex('6) Decompose $a$'))
a=Matrix([2*sym.I,-1,0])
display(a)
a_ = ip(a,u_)*u_ + ip(a,w_)*w_ + ip(a,r_)*r_
display([a, a_]) # tricky but a very good point about scalar multiples
```

6) Decompose a

$$\begin{bmatrix} 2i \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{bmatrix} 2i \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix} \right]$$

In [22]:
```
ip(a,u_), ip(a,w_), ip(a,r_)
# Provenzi speaks volumes...
# about magnitude, ||a||^2
```

Out[22]: $$\left(-\frac{\sqrt{5}i}{5}, \ -\frac{2\sqrt{5}i}{5}, \ 2 \right)$$

In [23]:
```
display(ip(a,a))
display(sym.Abs(ip(a,u_))**2 + sym.Abs(ip(a,w_))**2 + sym.Abs(ip(a,r_))**2)
```

5

5

Our decomposition is,

$$a = \langle a, \hat{u} \rangle \hat{u} + \langle a, \hat{w} \rangle \hat{w} + \langle a, \hat{r} \rangle \hat{r}.$$

Plancherel's theorem says,

$$\|a\|^2 = |\langle a, \hat{u} \rangle|^2 + |\langle a, \hat{w} \rangle|^2 + |\langle a, \hat{r} \rangle|^2.$$

The vector with the heaveist weight in the reconstruction of a, thus $\hat{r}$ gives the best rough approximation of a.

In [24]:
```
display(sym.Abs(ip(a,u_))**2, sym.Abs(ip(a,w_))**2, sym.Abs(ip(a,r_))**2)
```

$$\frac{1}{5}$$

$$\frac{4}{5}$$

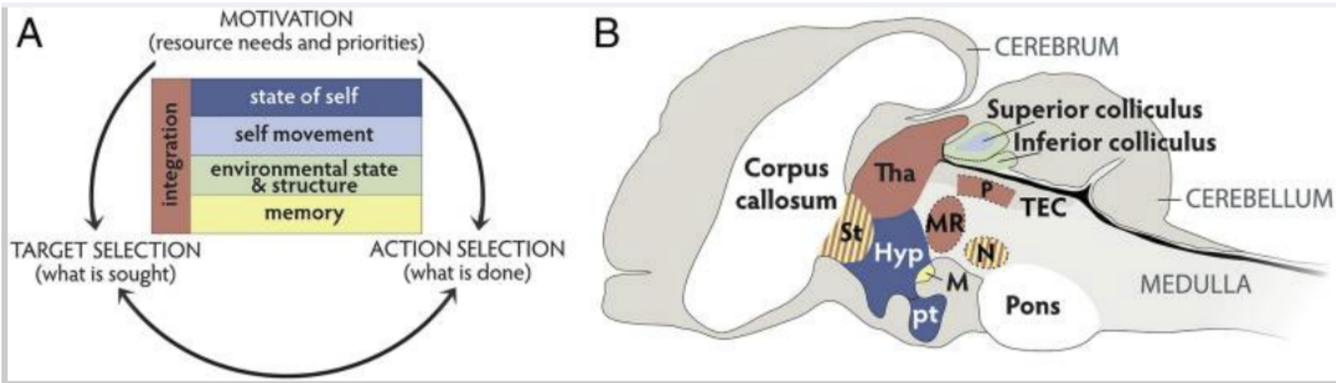
$$4$$

$\square$

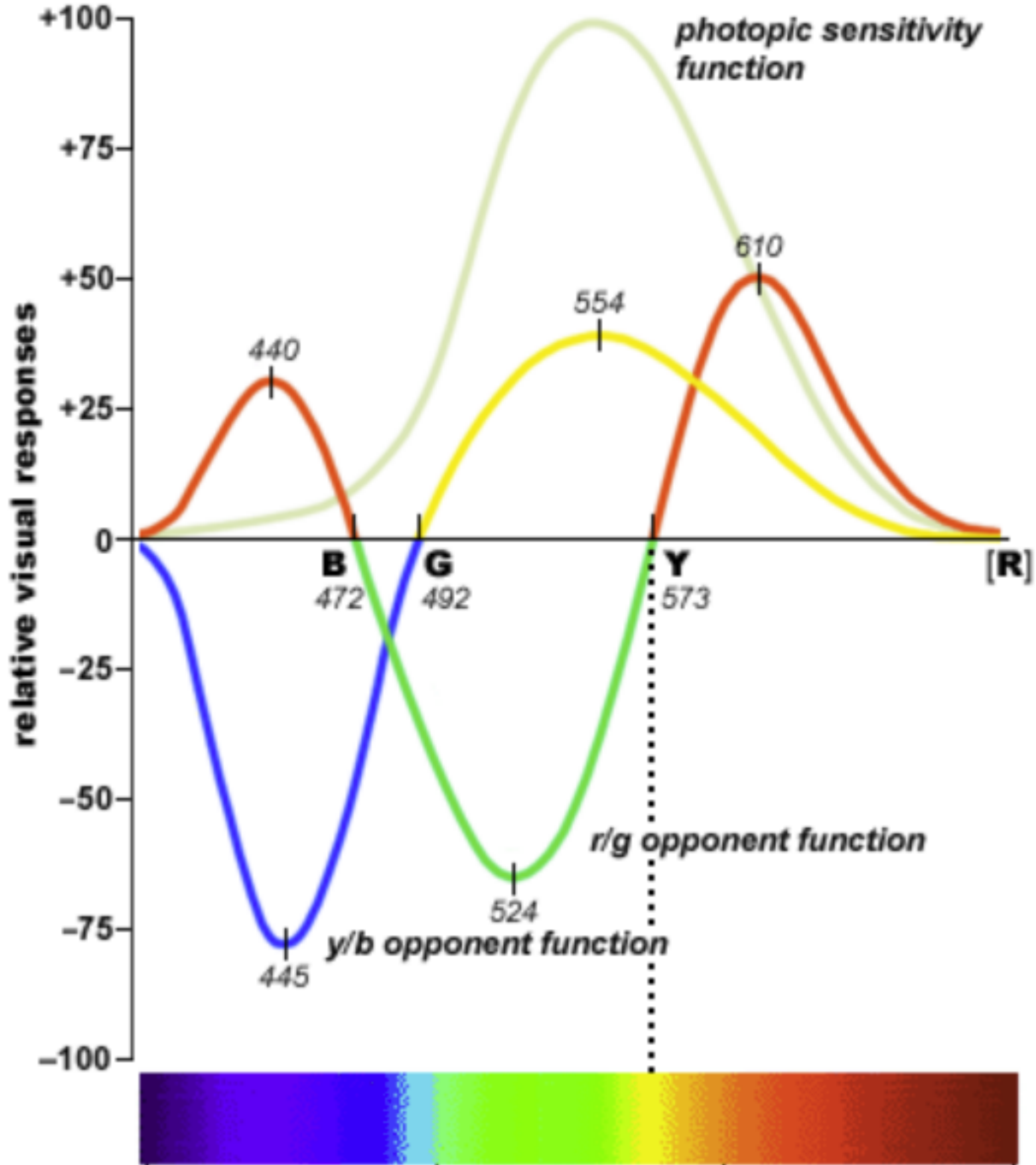
Appendix 2. Stigmergy

Consider Linear Time-Invariant Systems and Convolution of the Visual Origins of Consciousness.

(Seriously, these sort of things come to mind in the early hours of the morning...)



Stigmergy a form of self-organization, producing complex, seemingly intelligent structures, without need for any planning, control, or even direct communication between the agents, yet supports efficient collaboration between extremely simple agents, who may lack memory or individual awareness of each other.



opponent functions in spectral hues

chromatic response functions for the ***CIE 1964 standard observer****; from Hurvich & Jameson (1955); the unique hues are located at B, G and Y (unique red is composed of a mixture of "red" and "violet" light)*

See also:

Mathematics in Civilization, Resnikoff/Wells

What insects can tell us about the origins of consciousness, Barron/Klein

Analysis and Design of Descriptor Linear Systems, Duan

Wax or the Discovery of Television Among the Bees, Blair

Appendix 3. Human Color Vision

The color of light can be represented in a vector $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ where

$R = \text{amount of red}$, $G = \text{amount of green}$, and $B = \text{amount of blue}$.

The human eye and the brain transform the incoming signal into the signal $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$,

where

$$\begin{array}{lcll} \text{intensity} & I & = & \frac{R+G+B}{3} \\ \text{long-wave signal} & L & = & R-G \\ \text{short-wave signal} & S & = & B-\frac{R+G}{2}. \end{array}$$

(a) Find the matrix P representing the transformation from $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ to $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$

Put the transformation into terms of the original basis (RGB).

The basis is $e_{rgb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus, standard form is $P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix}$

(b) Consider a pair of yellow sunglasses for water sports which cuts out all blue light and passes all red and green light. Find the matrix A which represents the transformation incoming light undergoes as it passes through the sunglasses.

No humans involved so this is a this a simple matter of filtering out the blue basis...

The new basis is $e'_{rgb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

thus $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) Find the matrix for the composite transformation which light undergoes as it first passes through the sunglasses and then the eye.

Again simply a matter of writting our P but this time, filtering out the blue base at the eye too.

The basis is $e'_{rgb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, standard form is $P = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 0 \end{bmatrix}$

Appendix 4. Exploring Automated Proof with Python

See also:

https://www.zurab.online/2022/02/lesson-1-python-based-introduction-to.html

In [29]:
```
import sofia
sofia.help()
```

```
================
SOFiA (ver 01200221)
================
General commands.
 ■ Create new proposition: P=sofia.prop("Prop") will create a proposition named "Prop".
 ■ Postulate: P.p("[X]") will turn a proposition into an axiom stating [X].
 ■ Show: P.show() will print proposition P on the screen.
 ■ Show history: P.showh() will print proposition building history for P.
 ■ Axiom builders: A=sofia.nat() and A=sofia.bool() define axioms builders. Call A.help
() to see how to use them.
 ■ Mode: include sofia.showing=False in the code if you do not want to print a proposit
ion every time it is updated.
================
Proof building commands. For a given proposition P, the following proof building comman
ds are available.
 ■ Assumption: P.a("[X]") will assume [X].
 ■ Restate: P.r([[1,1],[2,3]],["x"]) will combine the statements from line 1, position
1, and from line 2, position 3, in a single line. It will in addition rename the first f
ree variable in each of these statements to "x".
 ■ Recall: P.c(Prop) will recall a proposition (axiom/theorem) stored as Prop.
 ■ Selfequate: P.e(2,1) will self-equate the statement at line 2, position 1.
 ■ Synapsis: P.s() will step out from an assumption block.
 ■ Apply: P.d(2,[[1,1],[1,2]],3) will apply an implication at line 2, position 3, to va
riables at line 1, position 1, and line 1, position 2.
 ■ Left substitution: P.ls(1,2,[3,4],5,6) will substitute the left side of equality at
line 1, position 5, in line 2, position 6, replacing occurences 3 and 4 of the right sid
e of the equality.
 ■ Right substitution: P.rs(1,2,[],5,6) will substitute the right side of equality at l
ine 1, position 5, in line 2, position 6, replacing all occurences of the left side of t
he equality.
 ■ Delete: P.x() will delete the last line of the proof.
```

In [60]:
```
P = sofia.prop()
E = "expression"
print(E + ' is a valid SOFiA expression = '+ str(P._valexp(E)))
```

```
expression is a valid SOFiA expression = True
```

In [59]:
```
P = sofia.prop()
E = "[[1]+[1]]=[2]"
print(E + ' is a valid SOFiA expression = '+ str(P._valexp(E)))
```

```
[[1]+[1]]=[2] is a valid SOFiA expression = True
```

In [61]:
```
P = sofia.prop()
S = "[statement]"
print(S + ' is a valid SOFiA statement = '+ str(P._valsta(S)))
```

```
[statement] is a valid SOFiA statement = True
```

In [62]:
```
# Ontology
P = sofia.prop()
F = ["Jabulani likes red flowers","Jabulani likes yellow flowers"]
print('Formulas: '+str(F))
print('Statement: '+P._statfromformulas(F))
```

```
Formulas: ['Jabulani likes red flowers', 'Jabulani likes yellow flowers']
Statement: [Jabulani likes red flowers][Jabulani likes yellow flowers]
```

In [63]:
```
P = sofia.prop()
E = "[[x1]>x2][x3][Did you know that [[x3] is South African]?]"
print(P._vars(E))
```

```
['x1', 'x3', 'x3']
```

In [64]:
```
P = sofia.prop("Equality is Transitive")
P.a("[x][y][z][[x]=[y]][[y]=[z]]")
P.rs(1,1,[1],5,4)
P.s()
```

```
Theorem: Equality is Transitive.
[]
Proof.
 ╓[x][y][z][[x]=[y]][[y]=[z]] /L1: assumption.
QED

Theorem: Equality is Transitive.
[]
Proof.
 ╓[x][y][z][[x]=[y]][[y]=[z]] /L1: assumption.
 ║[[x]=[z]] /L2: right substitution, L1(5) in L1(4).
QED

Theorem: Equality is Transitive.
[[x][y][z][[x]=[y]][[y]=[z]]:[[x]=[z]]]
Proof.
 ╓[x][y][z][[x]=[y]][[y]=[z]] /L1: assumption.
 ╙[[x]=[z]] /L2: right substitution, L1(5) in L1(4).
 [[x][y][z][[x]=[y]][[y]=[z]]:[[x]=[z]]] /L3: synapsis (L1-2).
QED
```