Math 725 Advanced Linear Algebra

HW4

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Quotent Space and Dual Basis

In [1]:
```
# import libraries
import numpy as np
import sympy as sym
from sympy.matrices import Matrix
from sympy import I
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

from sympy import init_printing
init_printing()
```

1. (3E2)

Suppose $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_j is finite-dimenstional for each $j = 1, \cdots, m$.

Solution:

Consider a basis of each V_j such that the j^{th} slot is 1 and all other slots is 0. The list of all such vector is linearly independent and spans $V_1 \times \cdots \times V_m$ and has lenght of the basis is dim$V + \cdots +$dimV_m, which is finite dimensional as desired.

see also Axler 3.76 proof

Another way to see this is to consider the contrary...

Consider $(v_{1,1}, v_{1,2}, \cdots, v_{1,m}), (v_{2,1}, v_{2,2}, \cdots, v_{2,m}), \cdots, (v_{k,1}, v_{k,2}, \cdots, v_{k,m})$ as a baisis for $V_1 \times \cdots \times V_m$ and there exits $v \in V$ so the $v \notin$ span$(v_{1,1}, v_{2,1}, \ldots v_{k,1})$. Thus $u_1, u_2, \ldots, u_m$ for any $u_i \in V$ is not a linear combination of $v_{i,1}, \ldots, v_{i,m}$, $1 \leq i \leq k$, thus contradiction. V_i must be finite dimensional as desired.

2. (3E3)

Give an example of a vector space V and subspaces U_1, U_2 of V such that $U_1 \times U_2$ is isomorphic to $U_1 + U_2$ but $U_1 + U_2$ is not a direct sum.

Hint: The vector space V must be infinite-dimensional.

Solution:

Suppose polynominal vector spaces, $V = \mathcal{P}(\mathbb{R})$, $U_1 =$span$(1, x, x^2)$, and $U_2 =${$p \in \mathcal{P}(\mathbb{R})$: deg $p \geq 2$}.

$U_1 + U_2 \neq U_1 \oplus U_2$ since,

$1 + x + 2x^2 + x^3 = (1 + x + x^2) + (x^2 + x^3) = (1 + x) + (2x^2 + x^3)$.

Let T: $U_1 \times U_2 \mapsto U_1 + U_2$ such that $T(u_1, u_2) = u_1, xu_2$.

If $T(u_1, u_2) = T(v_1, v_2)$ then $u_1 + xu_2 = v1 + xv_2$, thus surjective.

If deg $u_1 \leq$ deg xu_2 for any $u_1 \in U_1$ and $u_2 \in U_2$ then $u_1 = v_1$ and $u_2 = v_2$ must be True. Thus T is injective and is an isomorphism.

3. (3E6)

For n a positive integer, define V^n by

$$V^n = \underbrace{V \times \cdots \times V}_{n \text{ times}}$$

Prove that V^n and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic vector spaces.

Solution:

Define $T : V^n \mapsto \mathcal{L}(\mathbb{F}^n, V)$ such that $T(v_1 \ldots v_n) = S \in \mathcal{L}(\mathbb{F}^n, V)$ and $S(0, \ldots, 0, 1_i, 0, \ldots, 0) = v_i$. Thus by construction T is surjective.

If $T(v_1 \ldots v_n) = T(u_1 \ldots u_n)$ the $S_v = S_u$ so $S_v(0, \ldots, 0, 1_i, 0, \ldots, 0) = S_u(0, \ldots, 0, 1_i, 0, \ldots, 0)$.

Hence $v_i = u_i \implies v_1 \ldots v_n = u_1 \ldots u_n$ thus T is injective $\implies V^n$ and $\mathcal{L}(\mathbb{F}^n, V)$ are isomorphic.

4. (3E8)

Prove that a nonempty subset A of V is an affine subset of V if and only if $\lambda v + (1 - \lambda)w \in A$ and for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.

Solution:

If A is an affine subspace of V then A is $a + U$ for $a \in V$ and $U \in V$.

Suppose $v, w \in A$ and $u_1, u_2 \in U$ such that $a + u_1 = v$ and $a + u_2 = w$. Hence,

$\lambda v + (1 - \lambda)w = a + \lambda u_1 + (1 - \lambda)u_2 \in A$ for any $\lambda \in \mathbb{F}$.

If $\lambda v + (1 - \lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$, then consider arbitary $w \in A$ so that $U = \{u - w : u \in A\}$ and any $x_1, x_2 \in U$ such that $x_1 = u_1 - w, x_2 = u_2 - w$ for $u_1, u_2 \in A$.

Let $\lambda = 2$ then $2u_1 - w \in A, 2u_2 - w \in A$.

Let $\lambda = \frac{1}{2}$ then $\frac{1}{2}(2u_1 - w) + \frac{1}{2}(2u_2 - w) \in A \implies u_1 + u_2 - 2w \in A$. Thus it follows $u_1 + u_2 - 2w \in u \implies x_1 + x_2 \in U$ for any $x_1, x_2 \in U$.

If $\lambda(2u_1 - w) + (1 - \lambda)w \in A$ or $2\lambda u_1 \in A$ for any $u_1 \in A \implies U$ is a subspace of V.

Hence $A = w + U$ thus A is an affine subset of V as desired.

see proof of 3.87 Axler

see also Toan Quang Pham's notes for LADR

5. (3F1)

Explain why every linear function is either surjective or the zero map.

Solution:

Consider $\varphi \in \mathcal{L}(V, \mathbb{F})$, if $\dim \operatorname{range} \varphi = 0$, then φ is the zero map.

On the other hand, if φ not a zero map there exists $v \in V$ so $v\varphi = c \neq 0 \implies$ that for any $\lambda \in \mathbb{F}$ that $\varphi(\lambda/c \cdot v) = \lambda$. Thus φ is surjective.

6. (3F13)

Define $\mathcal{T} : \mathbb{R}^3 \mapsto \mathbb{R}^2$ by $\mathcal{T}(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z)$.

Suppose φ_1, φ_2 denotes that dual basis of the standard basis of R^2 and ψ_1, ψ_2, ψ_3 denotes that dual basis of the standard basis of R^3.

(a) Describe the linear functionals $\mathcal{T}'(\varphi_1)$ and $\mathcal{T}'(\varphi_2)$.

(b) Write $\mathcal{T}'(\varphi_1)$ and $\mathcal{T}'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

Solution:

(a) Describe the linear functionals $\mathcal{T}'(\varphi_1)$ and $\mathcal{T}'(\varphi_2)$.

This is a coordinate transform...

$(\mathcal{T}'(\varphi_1))(x, y, z) = (\varphi_1 \circ \mathcal{T})(x, y, z) = \varphi_1(4x + 5y + 6z, 7x + 8y + 9z) = 4x + 5y + 6z.$

Similarly, $(\mathcal{T}'(\varphi_2))(x, y, z) = 7x + 8y + 9z$

(b) Write $\mathcal{T}'(\varphi_1)$ and $\mathcal{T}'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

Recall $\mathcal{T}'(\varphi) \in (\mathbb{R}^3)'$ and ψ_1, ψ_2, ψ_3 is dual basis of $(\mathbb{R}^3)'$..

$(\mathcal{T}'(\varphi_1)) = (\mathcal{T}'(\varphi_1))(1, 0, 0)\psi_1 + (\mathcal{T}'(\varphi_1))(0, 1, 0)\psi_2 + (\mathcal{T}'(\varphi_1))(0, 0, 1)\psi_3 = 4\psi_1 + 5\psi_2 + 6\psi_3,$
and

$(\mathcal{T}'(\varphi_2)) = (\mathcal{T}'(\varphi_2))(1, 0, 0)\psi_1 + (\mathcal{T}'(\varphi_2))(0, 1, 0)\psi_2 + (\mathcal{T}'(\varphi_2))(0, 0, 1)\psi_3 = 7\psi_1 + 8\psi_2 + 9\psi_3,$

7. (3F22)

Suppose U, W are subspaces of V.

Show that $(U + W)^0 = U^0 \cap W^0$.

Solution:

Consider $U \subset U + W \implies U^0 \subset U^0 + W^0 \implies (U + W)^0 \subset U^0$ (by proof of 3.105 Axler).

Similarly, $(U + W)^0 \subset W^0$, thus $(U + W)^0 \subset U^0 \cap W^0$.

Conversely if $\varphi \in U^0 \cap W^0$ then $\varphi(u) = 0$ for any $u \in U$ or $u \in W$. Thus, $\varphi(u + w) = 0$ for any $u \in U, w \in U$ so $\varphi \in (U + W)^0$. (also by proof of 3.105 Axler)

It follows that $U^0 \cap W \subset (U + W)^0 \implies U^0 \cap W = (U + W)^0$ as desired.

8. (3F34)

The *double dual space* of V, denoted V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda : V \mapsto V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $\mathcal{T} \in \mathcal{L}(V)$, then $\mathcal{T}'' \circ \Lambda = \Lambda \circ \mathcal{T}$, where $\mathcal{T}'' = (\mathcal{T}')'$.

(c) Show that is finite-dimensional, then Λ is an isomorphism from V onto V''.

[Suppose V is finite-dimensional. Then V and V' are isomorphic, but find an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.]

Solution:

(a) Show that Λ is a linear map from V to V''.

For any $\varphi \in V'$ then

$$(\Lambda(v_1 + v_2))(\varphi) = \Lambda\varphi(v_1 + v_2) = \Lambda(\varphi(v_1) + \varphi(v_2)) = (\Lambda v_1)(\varphi) + (\Lambda v_2)(\varphi).$$

Thus, $\Lambda(v_1 + v_2) = \Lambda v_1 + \Lambda v_2$. Similarly, $\Lambda(\lambda v) = \lambda(\Lambda v)$. Thus, Λ is a linear map from V to V''.

(b) Show that if $\mathcal{T} \in \mathcal{L}(V)$, then $\mathcal{T}'' \circ \Lambda = \Lambda \circ \mathcal{T}$, where $\mathcal{T}'' = (\mathcal{T}')'$.

Consider $\varphi \in V'$ then,

$$(\Lambda(\mathcal{T} v))(\varphi) = (\mathcal{T}''(\Lambda v)(\varphi).$$

Thus, $(\Lambda(\mathcal{T} v))(\varphi) = \varphi(\mathcal{T} v)$ and $T'' = (T')' \in \mathcal{L}(V'', V'')$ so

$$(T''(\Lambda v))(\varphi) = ((\Lambda v) \circ \mathcal{T}')(\varphi) = (\Lambda v)(\mathcal{T}'(\varphi)) = (\mathcal{T}'(\varphi))(v) = (\varphi \circ \mathcal{T})(v) = \varphi(\mathcal{T} v).$$

Thus $(T''(\Lambda v))(\varphi) = (\Lambda(\mathcal{T} v))(\varphi)$ for any $\varphi \in V' \implies \Lambda(\mathcal{T} v) = T''(\Lambda)$ for any $v \in V$. Thus, $\Lambda \circ \mathcal{T} = \mathcal{T}'' \circ \Lambda$ as desired.

(c) Show that is finite-dimensional, then Λ is an isomorphism from V onto V''.

Suppose $\Lambda v = 0$ then $(\Lambda v)(\varphi) = 0$ for any $\varphi \in V' \implies v = 0$. Thus, Λ is injective.

Consider $v_1 \dots v_n$ to be a basis of V and $\varphi_1 \dots \varphi_n$ to be a dual basis of V'. Also consider $S \in V''$ so that $\Lambda v = S$ where $v = \sum\limits_{i=1}^{n} S(\varphi_i) v_i$.

We can see that $\varphi_i(v) = S(\varphi_i)$ for each $1 \leq i \leq n$. Thus $\varphi(v) = S(\varphi)$ for any $\varphi \in V'$. Thus, $\Lambda v = S \implies \Lambda$ surjective.

Thus, Λ is an isomorphism from V onto V''

see also Toan Quang Pham's notes for LADR

Appendix 0. Extra Problems

x. (3F32)

Suppose $\mathcal{T} \in \mathcal{L}(V)$, and $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V. Prove that the following are equivalent:

(a) $\mathcal{T}$ is invertible.

Surjective

(b) The columns of $\mathcal{M}(\mathcal{T})$ are linearly independent in $\mathbb{F}^{n,1}$.

(c) The columns of $\mathcal{M}(\mathcal{T})$ span $\mathbb{F}^{n,1}$.

(d) The rows $\mathcal{M}(\mathcal{T})$ are linearly independent in $\mathbb{F}^{1,n}$.

(e) The rows $\mathcal{M}(\mathcal{T})$ span $\mathbb{F}^{1,n}$.

Here $\mathcal{M}(\mathcal{T})$ means $\mathcal{M}(\mathcal{T}, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

Appendix 1. Complex

In [2]:

```
# fancy plot
def z_plot(Z, c=None):
    #display(Latex(f'${sym.latex(Z.T)}$'))
    z= np.array(Z.tolist()).astype(np.complex64)
    n = len(z)
    #plt.scatter(z.real, z.imag, c=c)
    if False:
        for i in range(len(z)): # this got a bit fancy
            zz = z[i] + .06 *np.exp(1j*2*np.pi*i/n)  #offset text
            plt.text(zz.real, zz.imag, i, fontsize=12)
    z = np.append(z,z[0]) # close the shape
    plt.plot(z.real, z.imag, c=c)
    plt.grid(visible=True);
    plt.gca().set_aspect("equal") # square grids are pretty
    plt.axhline(0, color='black', alpha = .2, linestyle='--')
    plt.axvline(0, color='black', alpha = .2, linestyle='--')
```

In [3]:

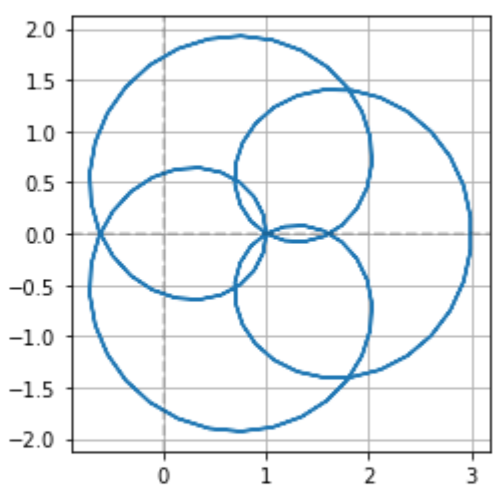
```
# we're replicating Arek's algorithm here
# FIXME!!! we ought to use sympy's plot_implicit() to avoid np.linspace()
z,R, theta =sym.symbols('z R theta')
p = z**4-z+1
eq = R*sym.exp(2*sym.pi*sym.I*theta)

phi = np.linspace(0,1,100)
r = np.linspace(3,0,10)

Z=[p.subs([(z,eq.subs([(R,1),(theta,t)]))]) for t in phi]
z_plot(Matrix([Z]))

display(Latex(f'Recall: $p={sym.latex(p)}$, and $z={sym.latex(eq)}$'))
```

Recall: $p = z^4 - z + 1$, and $z = Re^{2i\pi\theta}$



In [4]:

```
import ipywidgets as widgets
from ipywidgets import HBox, VBox
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display
%matplotlib inline
```

In [5]:

```
@widgets.interact
def f(r=3.1):
    Z=[p.subs([(z,eq.subs([(R,r),(theta,t)]))]) for t in phi]
    z_plot(Matrix([Z]))
```

```
interactive(children=(FloatSlider(value=3.1, description='r', max=9.3, min=-3.1), Output
()), _dom_classes=('wi…
```