Contest Sloutions

Ibrahim Abou Elenein

December 19, 2021

1. Given two integers, define an operation $*$ such that if a and b are integers, then $a * b$ is an integer. The operation $*$ has the following properties:

   - $a * a = 0$ for all integers a;
   - $(ka + b) * a = b * a$ for all integers a, b, and k;
   - $0 \leq b * a \leq a$.
   - $0 \leq b \leq a$ then $b * a = b$.

   Find $2021 * 16$

   Sol. $2021 = 16(126) + 5$ so $2021 * 16 = (16(126) + 5) * 16 = 5 * 16 = \boxed{5}$

2. It's currently $6 : 00$ on a 12 hour clock. What time will be shown on the clock 100 hours from now? Express your answer in the form $hh : mm$.

   Sol. We note that adding any multiple of 12 hours does not change the time on the clock. We see that 100 divided by 12 has remainder 4. So that means the time 100 hours from now is equal to the time 4 hours from now, which is $\boxed{10:00}$.

3. We call a positive integer *binary-okay* if at least half of the digits in its binary (base 2) representation are 1's, but no two 1s are consecutive. For example, $10_{10} = 1010_2$ and $5_{10} = 101_2$ are both binary-okay, but $16_{10} = 10000_2$ and $11_{10} = 1011_2$ are not. Compute the number of binary-okay positive integers less than or equal to 2020 (in base 10).

> *Sol.* Begin by noting that 2020_{10} has 11 digits in binary, as $2^{10} < 2020 < 2^{11}$. There are two cases: either the positive integer has an odd number of digits, or the positive integer has an even number of digits.
> **Case 1**: Odd number of digits: If a positive integer has an odd number of binary digits, it must begin with a 1 and alternate between 0's and 1's, since it will necessarily have one more 1 than 0s. (If it were to have any more 1s, then by Pigeonhole Principle, at least two of them must be adjacent.)
> **Case 2**: Even number of digits: If it has an even number of binary digits, then exactly half of them must be 1s by a similar argument. Precisely, if it has 2n digits, then there are n ways to arrange the 1s with no two consecutive. (If there are n 1s, there must be n - 1 0s inserted between the 1s to prevent the 1s from being adjacent.) The final 0 can go anywhere except at the beginning, so there are n positions for the final 0. Thus, there are 1 + 1 + 1 + 1 + 1 + 1 = 6 binary-okay numbers with an odd number of digits (as $101010101012 < 1024 + 512 = 1536 < 2020$) and 5 + 4 + 3 + 2 + 1 = 15 binary-okay numbers with an even number of digits. Altogether, we have $\boxed{21}$ binary-okay numbers under 2020.

4. Let n be an integer such that $n^4 - 2n^3 - n^2 + 2n + 2$ is a prime number. What's the sum of all possible n?

> *Sol.* We can see that this expression must be an even number since n^2 and n^4 must be the same parity. since The only even prime is 2, so the sum is $\boxed{2}$(by Viete's theorem)

5. Find the number of subsets S of $\{1, 2, ..., 10\}$ such that no two of the elements in S are consecutive

> *Sol.* The subsets can be interpted as $n - words$ from The alphapet $\{0, 1\}'$. Let a_n be the number of words with no consecutive ones. Then a word can start from 0 and proceed with in a_{n-1} ways or start with 10 and proceed with in a_{n-2} ways. Therefore $a_n = a_{n-1} + a_{n-2}$ So $a_1 0 = \boxed{144}$

6. Can an 8×8 board be coverd by 15 1×4 rectangles and only one 2×2 square without overlaping? prove your answer.

> *Sol.* No.

7. A 3×3 magic square is a grid of distinct numbers whose rows, columns, and diagonals all add to the same integer sum. Connie creates a magic square whose sum is N, but her keyboard is broken so that when she types a number, one of the digits $(0 - 9)$ always appears as a different digit (e.g. if the digit 8 always appears as 5, the number 18 will appear as 15). The altered square is shown below. Find N.

9	11	10
18	17	6
14	11	15

Sol. First, notice that 11 appears in the square twice. This is not possible in the original magic square, because the numbers must be distinct, so we conclude that the output of the broken key is 1. Next, we see that the digits 0, 1, 4, 5, 6, 7, 8, 9 are all present in the square. Therefore, the broken key must be 2 or 3. We then calculate the sum of the rows and columns as shown:

9	11	10	**30**
18	17	6	**41**
14	11	15	**40**
41	**39**	**31**	

each row has a distinct sum, so there must be at least two altered squares. the middle row is $18 + 17 + 6 = 41$. since none of the ones digits are 1, the ones digit of n is the same as the ones digit of 41. the ones digits in the $9 + 11 + 10 = 30$ row needs to sum up to end in 1, so the broken key must be 2. similarly, the ones digit in $14 + 11 + 15 = 40$ needs to be 1, so both of the 11's actually end with a 2

9	**12**	10
18	17	6
14	**12**	15

They cannot both be 12's, so one of them is a 22. This means that the sum of the middle column is either $12 + 22 + 17 = 51$ or $12 + 22 + 27 = 61$. Looking at the top row, even if all of the 1's are changed into 2's, the maximum value of the row is $9 + 22 + 20 = 51$. Therefore, the sum of each row and column is $\boxed{N = 51}$. The original square is shown below, with the modified digits in bold:

9	**22**	**2**0
28	17	6
14	1**2**	**2**5

8. Compute

$$\sum_{k\geq 0}\binom{1000}{3k}$$

Sol. We can rewrite the sum as

$$\sum_{k\geq 0}\binom{1000}{n}f(n)$$

where

$$f(n)=\begin{cases}1 & n\equiv 0 \pmod 3\\ 0 & otherwise\end{cases}$$

now we can have

$$f(n)=\frac{1}{3}(1^n+\omega^n+\omega^{2n})$$

where $\omega=e^{\frac{2}{3}\pi i}$ a cubic root of unity. sastifying the equation $\omega^2+\omega+1=0$. Thus we have

$$\sum_{n\geq 0}\binom{1000}{n}f(n)=\frac{1}{3}\sum_{n\geq 0}\binom{1000}{n}(1^n+\omega^n+\omega^{2n})=$$

$$\frac{1}{3}\sum_{n\geq 0}\binom{1000}{n}+\frac{1}{3}\sum_{n\geq 0}\binom{1000}{n}\omega^n+\frac{1}{3}\sum_{n\geq 0}\binom{1000}{n}\omega^{2n}=$$

$$\sum_{n\geq 0}\binom{1000}{n}f(n)=\frac{1}{3}[(1+1)^{1000}+(1+\omega)^{1000}+(1+\omega^2)^{1000}]=\frac{1}{3}(2^{1000}-1)$$