Taming Divergent Series with Zeta Functions: Lecture Notes

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I. INTRODUCTION

There are situations in math and physics in which divergent series appear, and yet finite results are required.[1]. There are a number of regularization techniques that can be used to circumvent the issues presented by divergent series; one of which is zeta function regularization.

II. THE TRACE

The technique of zeta function regularization is best understood by examining the trace of linear operators. In finite dimensions, a linear operator can be expressed as a matrix in some basis. The trace is then defined as the sum of the diagonal components of that matrix. That is,

$$Tr(A) = \sum_i a_{ii} \tag{1}$$

where matrix A has components a_{ij}. It is straightforward to show that the trace of a matrix is invariant under similarity transformations[2]:

$$tr(A') = tr(A) \tag{2}$$

if

$$A' = BAB^{-1} \tag{3}$$

for some invertible matrix B. It follows that the trace of a diagonalizable matrix may be equivalently defined as the sum of its eigenvalues, λ_n:

$$Tr(A) = \sum_n \lambda_n. \tag{4}$$

Thus, there is no issue in calculating the trace of a finite dimensional linear operator.

The trouble arises in defining the trace for an infinite dimensional linear operator. An infinite dimensional linear operator L maps functions to other functions:

$$L[\theta(x)] = \psi(x). \tag{5}$$

The definitions for eigenfunctions (or eigenvectors) and eigenvalues carry over to the infinite dimension case. $\phi(x)$ is an eigenfunction of L with eigenvalue λ if

$$L[\phi(x)] = \lambda\phi(x), \tag{6}$$

and the spectrum of L is defined as the set of eigenvalues, $\{\lambda_n\}$. One may naively try to carry over the definition of the finite dimensional trace (equation 4) as well. However, the operator L may have infinitely many eigenvalues, and therefore the series may diverge. In fact, this sum only converges if the ordered sequence λ_n is bounded and tends to zero sufficiently quickly. Many linear operators, including the important class of Sturm-Liouville operators, do not satisfy this stringent condition.[3] Thus, a broader definition of the trace is necessary to properly extend the notion of a trace to infinite dimensions.

The example of the Dirichlet Laplacian can be used to motivate a new definition of the trace. The Dirichlet Laplacian is defined as

$$-\Delta_D = -\frac{d^2}{dx^2} \tag{7}$$

and the corresponding eigenvalue problem is

$$\frac{d^2\phi}{dx^2} = -\lambda\phi(x) \tag{8}$$

with boundary conditions

$$\begin{cases} \phi(0) = 0 \\ \phi(L) = 0. \end{cases} \tag{9}$$

The eigenvalues are all positive[4], so the general solutions of equation 8 take the form

$$\phi_n(x) = A_n sin(\sqrt{\lambda_n}x) + B_n cos(\sqrt{\lambda_n}x). \tag{10}$$

The cosines do not satisfy the boundary condition at $x = 0$, while the boundary condition at $x = L$ constrains the eigenvalues. The resultant eigenfunctions and eigenvalues are

$$\begin{cases} \phi_n(x) = sin\big(\frac{n\pi}{L}x\big) \\ \lambda_n = \big(\frac{n\pi}{L}\big)^2 \end{cases} \tag{11}$$

where $n \in \mathbb{N}$. Attempting to calculate the trace of $-\Delta_D$ using equation (1) yields the divergent series,

$$Tr(-\Delta_D) \stackrel{?}{=} \sum_{n=1}^{\infty} \big(\frac{n\pi}{L}\big)^2 \tag{12}$$

Clearly, this definition of the trace is ill-defined for $-\Delta_D$.

III. ZETA FUNCTION REGULARIZATION

With the aim of defining a meaningful trace of an infinite dimensional linear operator, we define the spectral zeta function associated with L,

$$\zeta_L(s) = A.C. \sum_{n=1}^{\infty} \lambda_n^{-s}. \tag{13}$$

Assuming $\lim_{n\to\infty} \lambda_n = \infty$, the series in equation 13 converges for $Re(s)$ sufficiently large, that is, for $Re(s) > c$ for some $c > 0$. Since the series is well-behaved for $Re(s) > c$, it can be differentiated term by term, and therefore the series is analytic on that domain. It follows that the series can then be (uniquely) analytically continued into the region $Re(s) < c$[5]. ζ_L is defined as this analytic continuation. The trace is then redefined to be

$$Tr(L) = \zeta_L(-1). \tag{14}$$

This avoids the evaluation of a divergent series because

$$Tr(L) \neq \sum_{n=1}^{\infty} \lambda_n \tag{15}$$

but rather the trace is the analytic continuation of the series $\sum_{n=1}^{\infty} \lambda_n^{-s}$ at $s = -1$. Replacing the divergent series with this analytic continuation is known as zeta function regularization. We now demonstrate how this technique is practically used in a well-known physical application.

IV. THE CASIMIR EFFECT

The Casimir effect refers to the phenomenon that two neutral infinite conducting plates existing in a vacuum and separated by some distance L will attract each other. This effect can be understood by examining the zero-point energy of the electromagnetic field in the vacuum subject to the relevant boundary conditions.[6] The electric field $\mathcal{E}$in free space satisfies the homogeneous wave equation (in natural units):

$$\frac{\partial^2 \mathcal{E}}{\partial t^2} - \Delta\mathcal{E} = \frac{\partial^2 \mathcal{E}}{\partial t^2} - \frac{\partial^2 \mathcal{E}}{\partial x^2} - \frac{\partial^2 \mathcal{E}}{\partial y^2} - \frac{\partial^2 \mathcal{E}}{\partial z^2} = 0 \tag{16}$$

Suppose the conducting plates are positioned at $z = 0$ and $z = L$ and extend infinitely in the x and y directions. Electric fields cannot exist inside of perfect conductors, so the plates impose the Dirichlet boundary conditions

$$\begin{cases} \mathcal{E}(t,x,y,0) = 0 \\ \mathcal{E}(t,x,y,L) = 0. \end{cases} \tag{17}$$

The PDE can be solved using separation of variables, yielding oscillatory solutions in all variables. As in the case of the 1D Laplacian, the Dirichlet boundary conditions restrict the oscillations in z to sine functions with discrete spatial frequencies. Thus, the solutions are linear combinations of the following modes:

$$\mathcal{E}_{k_x,k_y,n} = e^{-i\omega_n t} e^{-(ik_x x + ik_y y)} sin(\frac{n\pi}{L} z) \tag{18}$$

where the temporal frequencies ω_n are given by

$$\omega_n = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{L}\right)^2} \tag{19}$$

where $k_x, k_y \in \mathbb{R}$ and $n \in \mathbb{N}$. The zero-point energy of each mode stores an energy equal to[7]

$$E_{\omega_n} = \hbar\omega_n \tag{20}$$

Thus, one may expect to obtain the total energy by summing over all modes (which amounts to integrating over k_x and k_y since they are continuous):

$$\frac{E}{A} \stackrel{?}{=} \frac{1}{4\pi^2} \int_{\mathcal{K}_{(x,y)}} \sum_{n=1}^{\infty} \omega_n dk_x dk_y \tag{21}$$

where the area of the plates A and the factor of $4\pi^2$ are geometric factors from the integration. The ω_n are unbounded, so the summation and integration clearly diverge. However, a finite result can be obtained by using zeta function regularization. We define the zeta function associated with the 3D Dirichlet Laplacian as

$$\zeta_\Delta(s) = A.C.\Big[\frac{1}{4\pi^2} \int_{\mathcal{K}_{(x,y)}} \sum_{n=1}^{\infty} \omega_n^{-s} dk_x dk_y\Big]$$
$$= \frac{1}{4\pi^2} A.C.\left[\sum_{n=1}^{\infty} \int_{\mathcal{K}_{(x,y)}} \left(k_x^2 + k_y^2 + \left(\frac{n\pi}{L}\right)^2\right)^{-s/2} dk_x dk_y\right] \tag{22}$$

and identify

$$\frac{E}{A} = \zeta_\Delta(-1). \tag{23}$$

The integral can be evaluated in closed form by switching to polar coordinates. If we let $q = k_x^2 + k_y^2$ then

$$\zeta_\Delta(s) = \frac{1}{4\pi^2} A.C.\left[\sum_{n=1}^{\infty} \int_0^{\infty} \left(q^2 + \left(\frac{n\pi}{L}\right)^2\right)^{-s/2} 2\pi q dq\right]$$
$$= \frac{\pi}{4(1-s/2)} A.C.\Big[\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^{2-s}\Big]. \tag{24}$$

The analytic continuation of the series in the last line is simply a constant multiple of the Riemann Zeta function of $s - 2$, thus the spectral zeta function is

$$\zeta_\Delta(s) = -\frac{\pi^{1-s}}{2(2-s)L^{2-s}} \zeta(s-2) \tag{25}$$

From equations 23 and 25 we see

$$\frac{E}{A} = -\frac{\hbar\pi^2}{6L^3}\zeta(-3) = -\frac{\hbar\pi^2}{720L^3}. \tag{26}$$

Note that $\lim_{L\to 0}\frac{E}{A} = -\infty$. If we allow the plates to move in the z in direction, the system will evolve towards states of lower energy; that is, the plates will move together. This is also apparent by calculating the pressure (force per area) on the plates:

$$\frac{F}{A} = -\frac{d}{dL}\left(\frac{E}{A}\right) = -\frac{\hbar\pi^2}{240L^4} \tag{27}$$

Thus, we see explicitly that there is an attractive force between the plates that scales like L^{-4}. Although the zeta-function regularization may appear to be some dubious slight-of-hand, these predictions have been observed experimentally.[8]

This is a remarkable result, although it was not appreciated as such at the time of its discovery.[9] The Casimir effect for parallel plates (including the L^{-4} dependence) was first predicted by Hendrik Casimir in 1948 using a more classical argument rather than the one based on quantum field theory presented here.[10] Casimir and his contemporaries believed the attractive force was due to Van der Waals forces. These are attractive forces between neutral, yet polarization objects. However, the attractive Van der Waals forces fail to explain the true complexity of the Casimir force, namely, the fact that the Casimir force can be either attractive or repulsive. For example, if a spherical conductor is used instead of parallel plates, one finds an outward pressure on the sphere. The sign of the pressure depends in a highly non-trivial way on the geometry, dimension, and curvature of the system at hand. Currently, the only known way to resolve the sign of the pressure is to go through the procedure of zeta function regularization (which becomes increasingly difficult with less trivial configurations than the one presented here). Emilio Elizalde dubs this poorly-understood sign dependence as "the mystery of the Casimir force."[11]

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[1] Such situations frequently arise in quantum field theory calculations. See E. Zeidler's *Quantum Field Theory II: Quantum Electrodynamics* (2008) Ch. 7 for examples from the analysis of the free particle and harmonic oscillator.

[2] A short proof is contributed by A. Zimmer in the open source book *A First Course in Linear Algebra* (2004). See http://linear.ups.edu/jsmath/0200/fcla-jsmath-2.00li100.html

[3] Sturm-Liouville operators are those that take the form $L = \frac{d}{dx}\big(p(x)\frac{d}{dx}\big) + q(x)$ where $p(x)$, $q(x) > 0$ and the boundary conditions are homogeneous. These operators always have unbounded spectra (see R. Haberman *Applied Partial Differential Equations* (2012) p. 155-57.

[4] This can be shown either by using the Reyleigh quotient, or by simply solving the ODE for $\lambda \leq 0$ and showing that $\phi(x)$ cannot satisfy the boundary conditions.

[5] See the theorem in *Complex Variables and Applications* §28 (Brown and Churchill). Technically, this theorem does not preclude singularities in the analytic continuation. In fact, most zeta functions will have a pole at $z = c$.

[6] The derivation of the Casimir energy and the Casimir force in this section are based on the derivation given in the Wikipedia article on the Casimir effect. See *https://en.wikipedia.org/wiki/Casimir_effect*.

[7] Each mode actually contributes $\frac{1}{2}\hbar\omega_n$, but there are two polarizations for each frequency so the total energy is $\hbar\omega_n$.

[8] See S. K. Lamoreaux, "Demonstration of the Casimir Force in the 0.6 to 6 m Range" (1997). For practical reasons, this experiment used a geometrical variant of the parallel plate configuration discussed in these notes.

[9] This paragraph is based on information presented in E. Elizalde's book, *Ten Applications of Spectral Zeta Functions* (2012), Ch. 5.

[10] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. B 51, 793 (1948).

[11] E. Elizalde, *Ten Applications of Spectral Zeta Functions* (2012)p. 101