CUTOFFS FOR EXCLUSION PROCESSES ON GRAPHS WITH OPEN BOUNDARIES

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Abstract. We prove a general theorem on cutoffs for symmetric simple exclusion processes on graphs with open boundaries, under the natural assumption that the graphs converge geometrically and spectrally to a compact metric measure space with Dirichlet boundary condition. Our theorem is valid on a variety of settings including, but not limited to: the d-dimensional grid for every integer dimension d; and self-similar fractal graphs and products thereof. Our method of proof is to identify a rescaled version of the density fluctuation field—the cutoff martingale—which allows us to prove the mixing time upper bound that matches the lower bound obtained via Wilson's method.

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1. Introduction

1.1. **Exclusion processes with open boundaries.** The symmetric simple exclusion process (SSEP) on a connected graph $G = (V, E)$ is a well-known interacting particle system used to model the dynamics of some fluid. In this process, particles interact as nearest-neighbor random walks on G, subject to the exclusion rule which forbids vertices to be occupied by more than one particle. By construction this process conserves the total number of particles. More recently, there has been interest in studying a variant of SSEP where an "open boundary" mechanism is added: namely, declare a nonempty "boundary" subset ∂V of V, and to each $a \in \partial V$ we add a birth-and-death chain (Glauber dynamics) of bounded rates. This models a set of reservoirs through which particles can enter, or exit from, the graph G.

A physical motivation behind this model comes from non-equilibrium statistical mechanics; see [ELS90, ELS91] for the first mathematical analysis of this model. Imagine, in the 1D discrete interval with reservoirs at the two ends, we tune the Glauber rates so that the mean density at the two reservoirs differ. This process has a unique invariant measure which is not product Bernoulli, and is generally not easy to characterize. Nevertheless, one can still study the asymptotic behavior of the particle density, in the form of a law of large numbers (LLN, or the *hydrodynamic limit*) and a central limit theorem (CLT, to characterize the fluctuations about the LLN) [KL99]. The LLN is given by the heat equation on the unit interval, and the CLT follows a generalized Ornstein-Uhlenbeck process. We refer the reader to [LMO08, BMNS17, Gon19, GJMN20] for a full account.

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In this paper, we study exclusion processes $\{\{\eta_t^N\,;\,t\geq 0\}\}_{N\in\mathbb{N}}$ on a sequence of connected graphs $G_N=(V_N,E_N)$ with boundaries, which form discrete approximations of a compact connceted metric measure space. Our goal is to establish sharp convergence to stationarity for $\{\{\eta_t^N\,;\,t\geq 0\}\}_{N\in\mathbb{N}}$, a phenomenon known as *cutoff*.

1.2. **The cutoff problem.** Given the process $\{\eta_t^N\,;\,t\geq 0\}$, we introduce the *ε-mixing time* $t_{\mathrm{mix}}^N(\varepsilon)$, the first time t that the total variation distance between the law of η_t^N and its invariant measure is smaller than a fixed threshold $\varepsilon>0$. If there exists a sequence of times $\{t_N\}_{N\in\mathbb{N}}$ such that for any $\varepsilon>0$,

$$\lim_{N\to\infty}\frac{t_{\mathrm{mix}}^N(\varepsilon)}{t_N}=1,$$

we say that the sequence $\{\{\eta_t^N\,;\,t\geq 0\}\}_{N\in\mathbb{N}}$ exhibits cutoff.

For the SSEP with open boundaries, the only result on cutoff that has appeared is from Gantert, Nestoridi, and Schmid [GNS20], where the graph is the discrete 1D interval with an open boundary at one end and a closed (reflecting) boundary at the other end. They also showed pre-cutoff in the case where both ends are open, as well as related results for the 1D asymmetric exclusion processes with open boundaries. For SSEP with open boundaries on higher-dimensional graphs, the cutoff problem has not been resolved until this work.

In contrast, the cutoff problem for SSEP without boundaries has been studied extensively; see the seminal works of Lacoin [Lac16a, Lac16b, Lac17]. In [CM20] the first-named and the last-named authors report progress on this problem for higher-dimensional graphs.

1.3. **Our contribution.** In this work we establish cutoff for symmetric exclusion processes on graphs with open boundaries, under the assumption that the graphs converge geometrically and spectrally to a compact connected metric measure space. Our result works in *any* integer dimension d, or more generally, on the Cartesian product of d copies of the same graphs with open boundaries (under mild assumptions on the asymptotics of the Laplacian eigensolutions).

The main ideas of our proof are as follows. We first construct a coupling of two independent copies of the process such that these copies meet at some time T_N. Then we show that this time is exactly the first time such that the exclusion process with absorbing reservoirs (which is the dual in the sense of [Lig05]) has all its particles absorbed, starting from the slowest scenario. To prove the lower bound for the mixing time, we work in this slowest scenario, and use an adaptation of Wilson's method [Wil04]. To prove the upper bound, we identify an observable of the process—the ***cutoff martingale***—which, on the one hand, has almost-sure magnitude of order unity at time T_N, and on the other hand has variance tending to ∞ as $N\to\infty$ at all times κt_N, $\kappa>1$. This then yields the matching upper bound.

Organization of the paper. In §2 we lay out our assumptions and state our main result, Theorem 1. This is followed by §3 where we provide examples to which Theorem 1 applies. The proof of Theorem 1 begins in §4, where we describe the coupling scheme that allows us to pass the mixing time analysis to the "worst-case" scenario, namely, where the processes are initialized from the all occupied configuration, and all the Glauber dynamics are purely absorbing. We then apply Wilson's method to this model in §5, and obtain lower bounds for the mixing times. Our main technical novelty—the cutoff martingale—is introduced in §6; we shall explain how it is used to yield the matching upper bounds for the mixing times, thereby implying the cutoff Theorem 1. The required computations on the variance of the cutoff martingale are given in §7. In §8 we discuss the connection between a technical condition in Theorem 1 and Fourier analysis. Finally, we conclude the paper with some open questions in §9.

Asymptotic notation used in the paper. Given two sequences of positive numbers $\{f_N\,;\,N\in\mathbb{N}\}$ and $\{g_N\,;\,N\in\mathbb{N}\}$, we use the notation

- $f_N=\mathcal{O}(g_N)$ (or $f_N\lesssim g_N$ when it suits better) if and only if there exists a positive constant C such that $f_N\leq Cg_N$ for all sufficiently large N.
- $f_N=\Theta(g_N)$ if and only if there exist positive constants $C_1<C_2$ such that $C_1\,g_N\leq f_N\leq C_2\,g_N$ for all sufficiently large N.
- $f_N\sim g_N$ if and only if $\lim_{N\to\infty}\frac{f_N}{g_N}=1$. (This should not be confused with the adjacency of vertices.)
- $f_N=o(g_N)$ if and only if $\lim_{N\to\infty}\frac{f_N}{g_N}=0$.

2. Notation and main result

2.1. **Exclusion process with open boundaries.** Let $\{G_N\}_{N\in\mathbb{N}}$, $G_N = (V_N, E_N)$, be a sequence of finite connected graphs with $|V_N| \to \infty$ as $N \to \infty$. For every N we declare a nonempty subset ∂V_N of V_N. We call V_N the *bulk* and ∂V_N the *boundary* of G_N. A configuration of particles in V_N is a function $\eta : V_N \to \{0,1\}$ where, for any vertex $x \in V_N$, $\eta(x) = 1$ if x is occupied with a particle, and $\eta(x) = 0$ if x is empty. Let $\Omega_N = \{0,1\}^{V_N}$ denote the set of all possible configurations in V_N. The symmetric simple exclusion process on G_N with open boundary ∂V_N is the continuous-time Markov process with state space Ω_N and whose infinitesimal generator $\mathcal{L}_N$ acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$\mathcal{L}_N = \mathcal{L}_N^{\text{bulk}} + \mathcal{L}_N^{\text{boundary}}. \tag{2.1}$$

where

$$(\mathcal{L}_N^{\text{bulk}} f)(\eta) = \sum_{x\in V_N} \sum_{\substack{y\in V_N \\ y\sim x}} \eta(x)(1-\eta(y))[f(\eta^{xy}) - f(\eta)] \tag{2.2}$$

and

$$(\mathcal{L}_N^{\text{boundary}} f)(\eta) = \sum_{a\in\partial V_N} [\lambda_{-,N}(a)\eta(a) + \lambda_{+,N}(a)(1-\eta(a))][f(\eta^a) - f(\eta)], \tag{2.3}$$

Above, $x \sim y$ means that $\{x,y\} \in E_N$; η^{xy} stands for the configuration obtained from η after exchanging the occupations of vertices x and y; and η^a stands for the configuration obtained from η after changing the occupation at vertex a, that is:

$$\eta^{xy}(z) = \begin{cases} \eta(y), & \text{if } z = x; \\ \eta(x), & \text{if } z = y; \\ \eta(z), & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta^a(z) = \begin{cases} 1-\eta(z), & \text{if } z = a; \\ \eta(z), & \text{otherwise.} \end{cases} \tag{2.4}$$

The dynamics generated by $\mathcal{L}_N$ is described as follows: In the bulk, particles interact as rate-1 nearest-neighbor random walks in the bulk, subject to the *exclusion rule* which forbids a particle to jump to an occupied vertex. Furthermore, a particle at any vertex $a \in \partial V_N$ can be annihilated with rate $\lambda_{-,N}(a) > 0$, and a particle can be created at any empty vertex $a \in \partial V_N$ with rate $\lambda_{+,N}(a) > 0$. For convenience we record the following identity:

$$\mathcal{L}_N\eta(x) = \sum_{y\sim x}[\eta(y) - \eta(x)] + \lambda_{\Sigma,N}(x)(\bar{\rho}_N(x) - \eta(x))\mathbb{1}_{\{x\in\partial V_N\}}, \tag{2.5}$$

where $\lambda_{\Sigma,N}(a) = \lambda_{+,N}(a) + \lambda_{-,N}(a)$ and $\bar{\rho}_N(a) = \lambda_{+,N}(a)/\lambda_{\Sigma,N}(a)$.

Since the cutoff problem involves taking space-time scaling limits, we will accelerate the exclusion process on G_N by $\mathcal{T}_N$, where $\{\mathcal{T}_N\}_N$ is an increasing sequence of time scales with $\mathcal{T}_N \uparrow \infty$. (There will be additional assumptions on $\{\mathcal{T}_N\}_N$ to come, see Assumptions 2 and 3 below.) From now on, we denote by $\{\eta_t^N\,;\, t \geq 0\}$ the exclusion process with generator $\mathcal{T}_N\,\mathcal{L}_N$. Let μ_{ss}^N denote the invariant measure of $\{\eta_t^N\,;\, t \geq 0\}$, which is not explicitly known for most of the choices of the Glauber rates; μ_N be the initial measure of $\{\eta_t^N\,;\, t \geq 0\}$; $\mathbb{P}_{\mu_N}$ denote the law of the process with initial distribution μ_N; and $\mathbb{E}_{\mu_N}$ the corresponding expectation. When μ_N is concentrated on a single configuration $\eta \in \Omega_N$, we write $\mathbb{P}_\eta$ instead of $\mathbb{P}_{\delta_\eta}$.

Recall that for any two probability measures μ, ν on a finite state space Ω, the *total variation distance* between μ and ν is defined as

$$\|\mu - \nu\|_{\text{TV}} = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)|. \tag{2.6}$$

The distance to equilibrium of the process $\{\eta_t^N\,;\, t \geq 0\}$ is defined as

$$d_N(t) = \sup_{\eta\in\Omega_N} \|\mathbb{P}_\eta(\eta_t^N \in \cdot) - \mu_{\text{ss}}^N\|_{\text{TV}}. \tag{2.7}$$

For each $\varepsilon \in (0,1)$, the ε-mixing time of the process $\{\eta_t^N\,;\, t \geq 0\}$ is defined as

$$t_{\text{mix}}^N(\varepsilon) = \inf\{t \geq 0\,;\, d_N(t) \leq \varepsilon\}. \tag{2.8}$$

Since the process is sped up in time by the factor $\mathcal{T}_N$, the ε-mixing time of original Markov chain, with generator $\mathcal{L}_N$, is $\mathcal{T}_N t^N_{\text{mix}}(\varepsilon)$.

2.2. **Geometric and analytic assumptions.**

Assumption 1 (Geometric convergence)**.** Let $\{(G_N, \partial V_N)\}_N$ be a sequence of connected, bounded-degree graphs with boundaries; in particular the degree bound is assumed to be uniform in N. We say that the sequence of graphs $\{(G_N, \partial V_N)\}_N$ ***converges geometrically*** to a compact connected metric measure space $(K, d, \mathfrak{m})$ with boundary ∂K and boundary measure $\mathfrak{s}$ if:

(1) For every $N \in \mathbb{N}$, $V_N \subseteq K$ and $\partial V_N \subseteq \partial K$.

Moreover, as $N \to \infty$:

(2) $|\partial V_N|/|V_N| \to 0$.
(3) $\mathfrak{m}_N := \frac{1}{|V_N|}\sum_{x\in V_N}\delta_x$ converges weakly to $\mathfrak{m}$.
(4) $\mathfrak{s}_N := \frac{1}{|\partial V_N|}\sum_{a\in\partial V_N}\delta_a$ converges weakly to $\mathfrak{s}$.

Above δ_x is the Dirac delta measure at x. Without loss of generality, we assume that $\mathfrak{m}$ and $\mathfrak{s}$ have full support on K and ∂K, respectively.

2.2.1. *Laplacian, Dirichlet form, and the spectral problem.* Let us introduce the following operators on $f : V_N \to \mathbb{R}$. Motivated by the generator identity (2.5), we define the *Laplacian* rescaled by the time factor $\mathcal{T}_N$:

$$(\Delta_N f)(x) = \mathcal{T}_N \sum_{\substack{y\in V_N\\ y\sim x}} (f(y) - f(x)) - \mathcal{T}_N \lambda_{\Sigma,N}(x) f(x) \mathbb{1}_{\{x\in\partial V_N\}}, \qquad x \in V_N. \tag{2.9}$$

Note that (2.9) is defined also at $x \in \partial V_N$, and Δ_N is a negative (semi)definite operator. In order to isolate the action on the boundary, we separately define the **outward normal derivative**

$$(\partial_N^{\perp} f)(a) = -\frac{\mathcal{T}_N}{|V_N|/|\partial V_N|}\sum_{\substack{y\in V_N\\ y\sim a}} [f(y) - f(a)], \qquad a \in \partial V_N. \tag{2.10}$$

Next we introduce the *Dirichlet form* associated with Δ_N: for any $f, g : V_N \to \mathbb{R}$,

$$\begin{aligned}\mathcal{E}_N(f,g) &= \int_{V_N} f(-\Delta_N g)\, d\mathfrak{m}_N = \int_{V_N} (-\Delta_N f) g\, d\mathfrak{m}_N \\ &= \frac{1}{2}\frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}} [f(x) - f(y)][g(x) - g(y)] + \frac{\mathcal{T}_N}{|V_N|}\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a) f(a) g(a).\end{aligned} \tag{2.11}$$

It is direct to check that $-\Delta_N$ is a nonnegative self-adjoint operator on $L^2(K, \mathfrak{m}_N)$. Using the summation-by-parts identity

$$\begin{aligned}&\frac{1}{2}\frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}} [f(x) - f(y)][g(x) - g(y)] \\ &= \frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}} f(x)[g(x) - g(y)] \\ &= \frac{1}{|V_N|}\sum_{x\in V_N\setminus\partial V_N} f(x)(-\Delta_N g)(x) + \frac{1}{|\partial V_N|}\sum_{a\in\partial V_N} f(a)(\partial_N^{\perp} g)(a).\end{aligned} \tag{2.12}$$

we can rewrite the Dirichlet form as

$$\mathcal{E}_N(f,g) = \int_{V_N\setminus\partial V_N} f(x)(-\Delta_N g)(x)\, d\mathfrak{m}_N(x) + \int_{\partial V_N} f(a)\left[\mathcal{T}_N\frac{|\partial V_N|}{|V_N|}\lambda_{\Sigma,N}(a) g(a) + (\partial_N^{\perp} g)(a)\right] d\mathfrak{s}_N(a). \tag{2.13}$$

As usual, we adopt the shorthand $\mathcal{E}_N(f) := \mathcal{E}_N(f,f)$, and remind the reader the polarization formula $\mathcal{E}_N(f,g) = \frac{1}{4}[\mathcal{E}_N(f+g) - \mathcal{E}_N(f-g)]$.

Assumption 2. If $a_N := \mathcal{T}_N \frac{|\partial V_N|}{|V_N|}$ and $b_N := \left(\inf_{a\in\partial V_N} \lambda_{\Sigma,N}(a)\right)^{-1}$, then $\liminf_{N\to\infty} \frac{\log a_N}{\log b_N} > 1$.

Assumption 2 forces, in the limit $N \to \infty$, zero boundary condition on functions of finite energy, which can be seen from the second integral term on the RHS of (2.13). More precisely, if $\mathcal{F} := \{f : K \to \mathbb{R} : \sup_N \mathcal{E}_N(f) < \infty\}$, then Assumption 2 implies that for every $f \in \mathcal{F}$, $f(a) = 0$ for s-a.e. $a \in \partial K$.

Last but not least, we discuss the *spectral problem*. A function $\psi : V_N \to \mathbb{R}$ is an eigenfunction of $-\Delta_N$ if there exists an eigenvalue $\lambda \in \mathbb{C}$ such that

$$\begin{cases} -\Delta_N \psi = \lambda\psi, & \text{on } V_N \setminus \partial V_N, \\ \lambda\psi(a) = \frac{|V_N|}{|\partial V_N|}(\partial_N^\perp \psi)(a) + \mathcal{T}_N \lambda_{\Sigma,N}(a)\psi(a), & a \in \partial V_N. \end{cases} \tag{2.14}$$

To explain the boundary condition (2.14), we first use the eigenvalue equation and (2.13) to obtain

$$\mathcal{E}_N(f,\psi) = \frac{1}{|V_N|}\sum_{x\in V_N\setminus\partial V_N} f(x)(\lambda\psi(x)) + \frac{1}{|V_N|}\sum_{a\in\partial V_N} f(a)\left[\frac{|V_N|}{|\partial V_N|}(\partial_N^\perp\psi)(a) + \mathcal{T}_N\lambda_{\Sigma,N}(a)\psi(a)\right] \tag{2.15}$$

for any $f : V_N \to \mathbb{R}$. On the other hand, by definition we have

$$\mathcal{E}_N(f,\psi) = \frac{1}{|V_N|}\sum_{x\in V_N} f(x)(\lambda\psi(x)). \tag{2.16}$$

Matching the RHS of (2.15) and (2.16) and then setting $f = \mathbb{1}_a$ for each $a \in \partial V_N$ yields (2.14).

Since $-\Delta_N$ is a nonnegative self-adjoint operator on $L^2(\mathfrak{m}_N)$, the eigenvalue problem (2.14) has $|V_N|$ solutions, each of which is associated with a $\mathbb{R}_+$-valued eigenvalue λ_i^N and an eigenfunction ψ_i^N (unique up to normalization). We choose the index i such that $\{\lambda_i^N\}_{i=1}^{|V_N|}$ forms an increasing sequence. Moreover, we normalize the eigenfunctions such that

$$\int |\psi_i^N|^2\, d\mathfrak{m}_N = \frac{1}{|V_N|}\sum_{x\in V_N} |\psi_i^N(x)|^2 = 1 \qquad \text{for all } N \text{ and } i. \tag{2.17}$$

Obviously, if $\lambda_i^N \neq \lambda_j^N$, then $\langle \psi_i^N, \psi_j^N\rangle_{L^2(\mathfrak{m}_N)} = 0$. In the case where the eigenvalue λ_i^N is repeated with multiplicity k, the corresponding eigenspace is k-dimensional, and we apply Gram-Schmidt to generate k orthonormal eigenfunctions; see Remark 2.1 below. In this way we can construct an orthonormal basis $\{\psi_i^N\}_{i=1}^{|V_N|}$ for $L^2(\mathfrak{m}_N)$. Note that even though ψ_i^N is defined on V_N only, we shall extend its domain to K via continuous interpolation, and call the extension ψ_i^N still.

By the Perron-Frobenius theorem (see [BLS07] or [Fri93]), $\lambda_1^N > 0$ is the unique lowest eigenvalue, and the eigenfunction ψ_1^N takes the same sign everywhere on V_N. We shall adopt the positive convention, $\psi_1^N \geq 0$.

Assumption 3 (Spectral convergence)**.** The collection of discrete Laplacian eigensolutions

$$\left\{(\lambda_i^N, \psi_i^N) : 1 \leq i \leq |V_N|\right\}_{N\in\mathbb{N}}$$

defined above satisfies the following conditions:

(1) There exist $\{\lambda_i\}_{i=1}^\infty \subseteq \mathbb{R}_+$ such that $\lim_{N\to\infty} \lambda_i^N = \lambda_i$ for each $i \in \mathbb{N}$, and in particular $0 < \lambda_1 < \lambda_2$.

(2) There exist $\{\psi_i\}_{i=1}^\infty \subseteq C(K)$ such that $\psi_i^N \to \psi_i$ in $C(K)$ for each $i \in \mathbb{N}$.

(3) The sequence of first eigenfunctions $\{\psi_1^N\}_{N\in\mathbb{N}}$ satisfies

$$\lim_{N\to\infty} \sup_{\substack{x,y\in V_N \\ x\sim y}} |\psi_1^N(x) - \psi_1^N(y)| = 0.$$

(4) For every $i \in \mathbb{N}$, $\sup_N \sup_{a\in\partial V_N} \left|(\partial_N^\perp \psi_i)(a)\right| < \infty$.

(5) Let $p(t,x,y) := \sum_{i=1}^\infty e^{-t\lambda_i}\psi_i(x)\psi_i(y)$, $(t,x,y) \in [0,\infty)\times K\times K$, be the *Dirichlet heat kernel*. Then:

(I) (Continuity.) $(t,x,y) \mapsto p(t,x,y)$ is continuous on $(0,\infty)\times K\times K$.

(II) (Positivity.) For any x, y which are in the same connected component of $K \setminus \partial K$, we have $p(t,x,y) > 0$ for all $t > 0$.

(III) (Exponential decay for large times.) $\sup_{(t,x,y)\in[1,\infty)\times K\times K} e^{t\lambda_1} p(t,x,y) < \infty$.

Remark 2.1 (Eigenfunctions corresponding to a multiple eigenvalue). If λ_i is a simple eigenvalue (*i.e.*, has multiplicity 1), then the statements in Items (2) and (4) of Assumption 3 are clear. If λ_i is a multiple eigenvalue, then strictly speaking Item (2) should be replaced by the convergence of the corresponding eigenspaces. That said, for the examples to be described in §3 below, there is no loss of generality in assuming that an orthonormal basis has been chosen for each eigenspace such that Item (2) holds.

Remark 2.2. In Assumption 3-(4), the uniform-in-N bound on the normal derivative is for each fixed $i \in \mathbb{N}$, but the bound does not hold uniformly for all i. See Remark 3.2 below.

Remark 2.3. Let $p_N(t,x,y) := \sum_{i=1}^{|V_N|} e^{-t\lambda_i^N} \psi_i^N(x)\psi_i^N(y)$, $(t,x,y) \in [0,\infty)\times K\times K$, be the discrete version of the heat kernel. By Assumptions 3-(1) and 3-(2), $p_N(t,x,y) \to p(t,x,y)$ pointwise. By Assumption 3-(5)(I), we can upgrade the pointwise convergence to uniform convergence on $(0,\infty)\times K\times K$. This result, as well as the other two items in Assumption 3-(5), are needed for the proof of Lemma 7.1 below.

Throughout the paper we denote by

$$\mathsf{c}_i^N := \int_K \psi_i^N \, d\mathfrak{m}_N \tag{2.18}$$

the ith Fourier coefficient of the constant function 1 with respect to the $L^2(\mathfrak{m}_N)$-orthonormal basis $\{\psi_i^N\}_{i=1}^{|V_N|}$. By Plancherel's identity, $\sum_{i=1}^{|V_N|} |\mathsf{c}_i^N|^2 = \int_K 1 \, d\mathfrak{m}_N = 1$.

2.3. **Main result.** Set the following figures of merit, $t_N := \dfrac{\log|V_N|}{2\lambda_1^N}$ and

$$\kappa^* := \sup\left\{\kappa > 0 : \limsup_{N\to\infty} \frac{|V_N|^{\kappa-1}}{|\partial V_N|} \sum_{i=1}^{|V_N|} \left(\sup_{b\in\partial V_N} |\partial_N^\perp \psi_i^N(b)|\right) \frac{|\mathsf{c}_i^N|}{\lambda_i^N} \sum_{a\in\partial V_N} (\psi_1^N(a))^2 = 0\right\}. \tag{2.19}$$

Our main theorem is

Theorem 1 (Cutoff)**.** *Suppose Assumptions 1, 2, and 3 hold, and furthermore assume that $\kappa^* > 1$. Then for any $\varepsilon \in (0,1)$, $\lim_{N\to\infty} \dfrac{t_{\mathrm{mix}}^N(\varepsilon)}{t_N} = 1$.*

The above result asserts that the distance to equilibrium of $\{\eta_t^N ; t \geq 0\}$ converges abruptly from 1 to 0 at times t_N as $N \to \infty$, a phenomenon known as *cutoff*. Equivalently, a sequence of Markov chains presents cutoff at times t_N if and only if

$$\lim_{n\to\infty} d_N(\kappa\, t_N) = \begin{cases} 1, & \text{if } \kappa < 1, \\ 0, & \text{if } \kappa > 1. \end{cases} \tag{2.20}$$

Remark 2.4. The seemingly technical condition $\kappa^* > 1$ comes from the analysis of the boundary term in a specific martingale (§7.1). We will show in Proposition 8.1 below that under assumptions which are slightly stronger than Assumption 3, the sum over i is bounded uniformly in N, so that (2.19) can be reduced to

$$\kappa^* = \sup\left\{\kappa > 0 : \limsup_{N\to\infty} \frac{|V_N|^{\kappa-1}}{|\partial V_N|} \sum_{a\in\partial V_N} (\psi_1^N(a))^2 = 0\right\}. \tag{2.21}$$

Remark 2.5. Using the boundary condition in (2.14) we find that

$$(\mathcal{T}_N \lambda_{\Sigma,N}(a) - \lambda_i^N)\psi_i^N(a) = -\frac{|V_N|}{|\partial V_N|}(\partial_N^\perp \psi_i^N)(a), \qquad a \in \partial V_N. \tag{2.22}$$

Fix $i \in \mathbb{N}$. By Assumption 3-(1), λ_i^N is of order unity as $N \to \infty$, so by Assumption 2, $\lambda_i^N = o(\mathcal{T}_N \lambda_{\Sigma,N}(a))$ for every $a \in \partial V_N$. This leads to the estimate

$$|\psi_i^N(a)| \leq C \left(\mathcal{T}_N \frac{|\partial V_N|}{|V_N|} \lambda_{\Sigma,N}(a)\right)^{-1} |(\partial_N^\perp \psi_i^N)(a)|, \tag{2.23}$$

for a constant C independent of everything but i. By Assumption 3-(4), $|(\partial_N^{\perp}\psi_i^N)(a)| = \mathcal{O}(1)$, so we obtain that for all $a \in \partial V_N$, $|\psi_i^N(a)|$ decays at a rate no more slowly than $\left(\inf_{a\in\partial V_N} \mathcal{T}_N \frac{|\partial V_N|}{|V_N|}\lambda_{\Sigma,N}(a)\right)^{-1} =: b_N/a_N$.

At this point we can explain why we imposed Assumption 2 instead of the weaker condition $\liminf_{N\to\infty} \frac{a_N}{b_N} = +\infty$. It is because we need a genuine difference between the two growth rates, $\log a_N$ and $\log b_N$, to obtain quantitative decay of $|\psi_1^N(a)|$ as $N \to \infty$, which then allows us to show $\kappa^* > 1$ based on (2.21). Were we to assume that $\inf_{a\in\partial V_N} |\psi_1^N(a)|$ be $o(1)$ or $\Theta((\log |V_N|)^{-1})$, it would not be possible to deduce that $\kappa^* > 1$.

3. Examples

In this section we reset to the microscopic time scale, *i.e.,* without accelerating the exclusion process on G_N by $\mathcal{T}_N$. Here we abuse the notation introduced before, and write $t_{\text{mix}}^N(\varepsilon)$ for the ε-mixing time of the process with generator $\mathcal{L}_N$. We show examples of sequences of symmetric simple exclusion processes on G_N with open boundaries ∂V_N that present cutoff at times

$$\mathcal{T}_N\, t_N = \frac{\mathcal{T}_N \log |V_N|}{2\lambda_1^N}. \tag{3.1}$$

3.1. The d-dimensional discrete grid.

Fix $d \in \mathbb{N}$. For each $N \in \mathbb{N}$, let

$$V_N = \{N^{-1}\mathbf{x} : \mathbf{x} = (x_1, x_2, \dots, x_d),\ x_i \in \{0, 1, \dots, N\} \text{ for } 1 \le i \le d\}$$

be the vertex set embedded in $[0,1]^d$. Two vertices $x, y \in V_N$ are connected by an edge if their Euclidean distance $|x - y| = N^{-1}$; the resultant edge set is denoted E_N. We also declare the boundary set of vertices $\partial V_N = \{x \in V_N : x \in \partial([0,1]^d)\}$. Altogether, $(G_N = (V_N, E_N), \partial V_N)$ represents the Nth discretization of the unit hypercube $[0,1]^d$ with boundary $\partial([0,1]^d)$. In particular, $|V_N| = (N+1)^d$, $\mathcal{T}_N = N^2$, and $|\partial V_N| = 2dN^{d-1}$. Assumption 1 is thus easily verified.

We next assume that the boundary rates $\lambda_{\Sigma,N}(0) = N^{-\theta}\lambda_\Sigma(0)$ and $\lambda_{\Sigma,N}(1) = N^{-\theta}\lambda_\Sigma(1)$, where $\lambda_\Sigma(0)$ and $\lambda_\Sigma(1)$ are positive numbers independent of N, and $\theta \in (0,1)$ is also independent of N. This includes not only the case where all the $\lambda_{N,\Sigma}(a)$ are of order unity [LMO08], but also the variable-speed (fast/slow) boundary regime as investigated in [BMNS17, GJMN20]. The reason why we assume $\theta < 1$ is to stay within the Dirichlet regime, namely, that Assumption 2 holds: $\frac{\log a_N}{\log b_N} \sim \frac{\log N}{\log N^\theta} = \frac{1}{\theta} > 1$.

Remark 3.1. If $\theta = 1$, the system is in the Robin regime, and it is possible to prove cutoffs in this case by modifying the proofs slightly. We do not elaborate on the details. On the other hand, if $\theta > 1$, the system is in the Neumann regime, the analysis of which properly belongs to the scope of [CM20].

We now verify that Assumption 3 holds on $\{(G_N, \partial V_N)\}_{N\in\mathbb{N}}$. On $[0,1]^d$ with Dirichlet boundary condition on $\partial([0,1]^d)$, we can index the eigensolutions of the Dirichlet Laplacian $-\Delta = -\sum_{j=1}^d \partial_{x_j}^2$ by the multi-index $\mathbf{k} = (k_1, k_2, \cdots, k_d)$, $k_j \in \mathbb{N} \cup \{0\}$ for all $j \in \{1, 2, \cdots, d\}$:

$$\boldsymbol{\psi}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^d \sqrt{2}\sin(k_j \pi x_j), \quad \boldsymbol{\lambda}_{\mathbf{k}} = \pi^2 \sum_{j=1}^d (k_j)^2. \tag{3.2}$$

In particular, the first eigensolution is $\psi_1(\mathbf{x}) = \prod_{j=1}^d \sqrt{2}\sin(\pi x_j)$ and $\lambda_1 = d\pi^2$, and we have $0 < \lambda_1 < \lambda_2$.

It is more difficult to write down the eigensolutions of $-\Delta_N$ on the discrete lattice $\{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}^d$. Fortunately, we know that under Assumption 2,

$$\mathcal{E}_N(f,g) \xrightarrow[N\to\infty]{} \mathcal{E}(f,g) := \int_{[0,1]^d} \nabla f \cdot \nabla g \, dx$$

for all $f, g \in C^2([0,1]^d)$ which vanish on the boundary $\partial([0,1]^d)$. The convergence of Dirichlet forms implies that the eigenvalues converge, and the eigenfunctions converge in $C([0,1]^d)$, thereby verifying the first two items of Assumption 3. It remains to verify the last two items of Assumption 3, and for these we can use WLOG the discretization of (3.2): for $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}^d$,

$$\boldsymbol{\psi}_{\mathbf{k}}^N(\mathbf{x}) = \prod_{j=1}^d \psi_{k_j}^N(x_j) \quad \text{where} \quad \psi_k^N(x) := \sqrt{2}\sin(k\pi x); \quad \boldsymbol{\lambda}_{\mathbf{k}}^N = \sum_{j=1}^d 2\left(1 - \cos\left(\frac{k_j\pi}{N}\right)\right). \tag{3.3}$$

Then Item (3) follows from the fact that for every $x \in [0,1]$,

$$\left|\sin\left(\pi\left(x+\frac{1}{N}\right)\right)-\sin(\pi x)\right| = 2\left|\sin\left(\frac{\pi}{2N}\right)\cos\left(\pi\left(x+\frac{1}{2N}\right)\right)\right| \le 2\sin\left(\frac{\pi}{2N}\right) \xrightarrow[N\to\infty]{} 0$$

To verify Item (4) we note that the normal derivative at $\mathbf{b} = (b_1,\dots,b_{j-1},0,b_{j+1},\dots,b_d)$ reads

$$\begin{aligned}(\partial_N^\perp \boldsymbol{\psi}_{\mathbf{k}}^N)(b_1,\dots,b_{j-1},0,b_{j+1},\dots,b_d) &= \frac{N^2}{N^d}2dN^{d-1}\sum_{j=1}^d\prod_{\substack{i=1\\i\neq j}}^d \psi_{k_i}^N(b_i)\sum_{y\sim b_j}\left[\psi_{k_j}^N(b_j)-\psi_{k_j}^N(y)\right]\\ &= 2d\sum_{j=1}^d\prod_{\substack{i=1\\i\neq j}}^d \psi_{k_i}^N(b_i)\cdot N\sum_{y\sim b_i}\left[\psi_{k_j}^N(b_j)-\psi_{k_j}^N(y)\right].\end{aligned} \tag{3.4}$$

(An analogous expression holds when $b_j = 1$.) Observe that for any $f \in C^2([0,1])$,

$$N\sum_{y\sim b_i}[f(b_j)-f(y)] = \begin{cases}(\partial_N^\perp f)(b_j) & \text{if } b_j \in \{0,1\},\\ N^{-1}(-\Delta_N f)(b_j) & \text{if } b_j \notin \{0,1\}.\end{cases} \tag{3.5}$$

Using this fact, and the explicit form of the eigenfunction ψ_k^N, we deduce that for each fixed multi-index $\mathbf{k}$, $\sup_{\mathbf{b}\in\partial V_N}|(\partial_N^\perp \boldsymbol{\psi}_{\mathbf{k}}^N)(\mathbf{b})| \lesssim 1$.

Remark 3.2 (Nonvanishing of the normal derivatives of the discrete eigenfunctions on the boundary). Recall that the bound in Assumption 3-(4) need not hold uniformly in $i \in \mathbb{N}$. Here we have a good example: take $d = 1$, $\psi_k^N(x) = \sqrt{2}\sin(k\pi x)$, $k \in \{1,2,\dots,N\}$. Then given $k \in \mathbb{N}$, we have for all $N \ge k$ that

$$(\partial_N^\perp \psi_k^N)(0) = N\left[\psi_k^N(0)-\psi_k^N\left(\frac{1}{N}\right)\right] = -\sqrt{2}N\sin\left(\frac{k\pi}{N}\right) \xrightarrow[N\to\infty]{} -\sqrt{2}k\pi.$$

For Item (5): (I) follows from plugging the eigensolutions $\{(\lambda_i,\psi_i)\}$ into the spectral representation $p(t,x,y) = \sum_i e^{-t\lambda_i}\psi_i(x)\psi_i(y)$; using that $\sup_{x\in[0,1]^d}|\psi_i(x)| \lesssim 1$ for all i, and $\lambda_i \gtrsim i^{2/d}$ for all large i; and finally checking that the series converges uniformly on $[T,\infty)\times[0,1]^d\times[0,1]^d$ for every $T > 0$. (III) follows once we write $p(t,x,y) = e^{-\lambda_1 t}\sum_i e^{-t(\lambda_i-\lambda_1)}\psi_i(x)\psi_i(y)$ and then estimate the right-hand series using the same approach to prove (I). Last but not least, (II) is a consequence of the *strong maximum principle* for the solution $u \in C([0,\infty)\times[0,1]^d)$ of the parabolic problem

$$\begin{cases}\partial_t u = \Delta u & \text{in } (0,\infty)\times[0,1]^d,\\ u = 0 & \text{on } [0,\infty)\times\partial([0,1]^d).\end{cases}$$

Finally we wish to show that $\kappa^* > 1$. Recall the identity $\int_0^1 \sin(j\pi x)\,dx = \frac{1-\cos(j\pi)}{j\pi}$ for $j \in \mathbb{N}$. So a quick calculation shows that for all large enough i (say, $i \ge \frac{1}{2}N^d$), $|\mathsf{c}_i^N| \lesssim i^{-d}$. We also have the spectral asymptotics $\lambda_i^N \gtrsim i^{2/d}$. As for the boundary normal derivative, we use the full strength of (3.4) to find that $\sup_{b\in\partial V_N}|(\partial_N^\perp \psi_i^N)(b)| \lesssim i^{1/d}$ for all sufficiently large i. Putting everything together we see that

$$\sum_{i=1}^{|V_N|}\left(\sup_{b\in\partial V_N}|\partial_N^\perp \psi_i^N(b)|\right)\frac{|\mathsf{c}_i^N|}{\lambda_i^N} \lesssim \sum_{i=1}^{|V_N|} i^{-d-\frac{1}{d}}$$

which is bounded uniformly in N. Thus in (2.19) we need not worry about the sum over i. Using (2.23) we deduce that $\sup_{a\in\partial V_N}|\psi_1^N(a)| \lesssim N^{\theta-1}$, and putting everything together we obtain that $\kappa^* = 1 + \frac{2(1-\theta)}{d} > 1$.

Having verified all the conditions for Theorem 1, we have thus proved

Theorem 2 (Cutoff for SSEP on the d-dimensional grid)**.** *Let $t_{\text{mix}}^{N,d}(\varepsilon)$ be the ε-mixing time of the SSEP on the graph G_N uniformly approximating $[0,1]^d$ with open boundaries $\partial([0,1]^d)$. For every dimension $d \in \mathbb{N}$ and every $\varepsilon \in (0,1)$,*

$$\lim_{N\to\infty}\frac{2\pi^2 t_{\text{mix}}^{N,d}(\varepsilon)}{N^2\log N} = 1. \tag{3.6}$$

Boundary condition	*Eigenvalue* λ_k^N	*Eigenfunction* $\psi_k^N(x)$ $(x \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\})$	*Index & range*
(P)	$2N^2\left(1-\cos\left(\frac{2k\pi}{N}\right)\right)$	$\cos(2k\pi x),\ \sin(2k\pi x)$	$k \in \{0, 1, \dots, N-1\}$ (*Note:* $\lambda_k^N = \lambda_{N-k}^N$)
(C)	$2N^2\left(1-\cos\left(\frac{k\pi}{N}\right)\right)$	$\cos(k\pi x)$	$k \in \{0, 1, \dots, N\}$
(O)	$2N^2\left(1-\cos\left(\frac{k\pi}{N}\right)\right)$	$\sin(k\pi x)$	$k \in \{1, 2, \dots, N\}$
(OC)	$2N^2\left(1-\cos\left(\frac{k\pi}{2N}\right)\right)$	$\sin\left(\frac{k\pi x}{2}\right)$	$k \in \{1, 3, \dots, 2N-1\}$

TABLE 1. Eigenvalues and (unnormalized) eigenfunctions associated with the Laplacian on the discrete interval $\{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}$ with boundary condition: periodic, closed (reflecting), and open. These results come respectively from the following discrete transform methods: the discrete Fourier transform (DFT); the discrete cosine transform (DCT); and for the last two, the discrete sine transform (DST).

We can adapt the above analysis to the grid where part of the boundary is attached to reservoirs (open), while the remaining part is closed. The basic idea comes from elementary analysis of 1D Sturm-Liouville problems. Given the unit interval with endpoints $\{0, 1\}$, consider the (nonnegative definite) Laplacian $-\Delta = -d_x^2$ with the following boundary conditions:

- **(P)** If the two ends are identified to create a torus, then $-\Delta$ has eigenvalues $4k^2\pi^2$, $k \in \mathbb{N} \cup \{0\}$, and the corresponding eigenfunctions are $\cos(2\pi kx)$ and $\sin(2\pi kx)$ (note that $\sin(2\pi kx)$ degenerates to 0 if $k = 0$).
- **(C)** If both ends are closed (Neumann), then $-\Delta$ has eigenvalues $k^2\pi^2$, $k \in \mathbb{N} \cup \{0\}$, and the corresponding eigenfunctions are $\cos(k\pi x)$.
- **(O)** If both ends are open (Dirichlet), then $-\Delta$ has eigenvalues $k^2\pi^2$, $k \in \mathbb{N}$, and the corresponding eigenfunctions are $\sin(k\pi x)$.
- **(OC)** If $\{0\}$ is open (Dirichlet) and $\{1\}$ is closed (Neumann), then $-\Delta$ has eigenvalues $k^2\pi^2/4$, $k \in 2\mathbb{N} - 1$, and the corresponding eigenfunctions are $\sin(\pi kx/2)$.

Analogs of these results for the discrete interval $\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$ are given in Table 1, where for consistency we have accelerated the Laplacian by N^2. In fact, each set of eigenfunctions $\{\psi_k^N\}_k$ forms the basis for one of the famous discrete transform methods—DFT, DCT, and DST—used widely in signal processing and data compression. (For an accessible introduction see [Str99].) We have written the eigenfunctions in a way that is manifestly independent of N, and it is direct to verify that $\lim_{N\to\infty} \lambda_k^N = \lambda_k$, and $\psi_k^N \to \psi_k$ in $C([0, 1])$.

To generalize the 1d results to higher dimensions, consider the d-dimensional hypercube formed by the product of d unit intervals, $I_1 \times \ldots \times I_d$, where each $I_j \in \{\mathbf{P}, \mathbf{C}, \mathbf{O}, \mathbf{OC}\}$. Using separation of variables, we deduce that each eigenfunction $\boldsymbol{\psi}_{\mathbf{k}}(\mathbf{x})$ is of the form $\prod_{j=1}^d \psi_{k_j}(x_j)$, with corresponding eigenvalue $\boldsymbol{\lambda}_{\mathbf{k}} = \sum_{j=1}^k \lambda_{k_j}$, where $(\lambda_{k_j}, \psi_{k_j})$ is the k_jth eigensolution of the Laplacian on I_j with the chosen boundary condition. Similar analogs hold for the discrete d-dimensional grid, *i.e.,* the product of d discrete intervals.

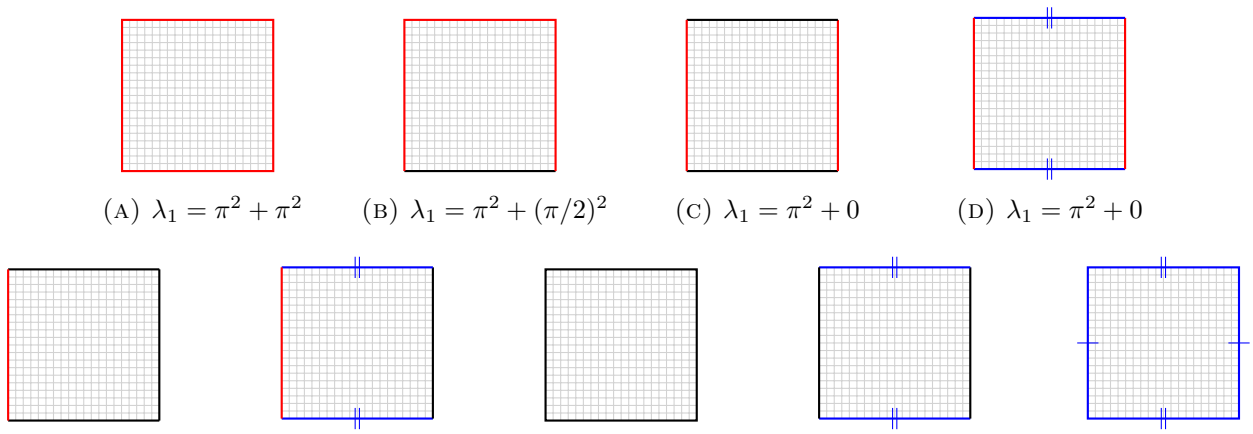
Caveat: In the context of this paper, we insist that at least one of the I_j must belong to either **O** or **OC**; that is, there must be an open codimension-1 boundary somewhere. Cutoffs for exclusion processes on the discrete grids or tori with no open boundaries are established in [CM20].

A word about verification of the assumptions: Assumptions 1 through 3 can be checked exactly as in the above case where the entire boundary is open. We leave the details to the reader. To check the condition $\kappa^* > 1$ is more delicate, and this follows from the systematic discussion in §8 below.

In any case, we illustrate the cutoff results in Figure 1 ($d = 1$) and Figure 2 ($d = 2$). There we also provide references for known cutoff results in $d = 1$ [Lac16a, Lac16b, Lac17, GNS20], and indicate the examples without open boundaries which are resolved by [CM20, Theorem 1].

(A) $\lambda_1 = \pi^2$ (this paper) (B) $\lambda_1 = (\pi/2)^2$ [GNS20] (C) $\lambda_1 = \pi^2$ [Lac16a] (D) $\lambda_1 = (2\pi)^2$ [Lac16b, Lac17]

FIGURE 1. A summary of results on cutoffs for exclusion processes on the discrete interval. The boundary condition can be: open; closed (reflecting); or periodic. In each case where λ_1 is given, total variation cutoff is established in the indicated reference: for every $\varepsilon \in (0,1)$, $\lim_{N\to\infty} \frac{2\lambda_1 t^N_{\text{mix}}(\varepsilon)}{N^2 \log N} = 1$. Our main contribution is the proof of Example (A).



(A) $\lambda_1 = \pi^2 + \pi^2$ (B) $\lambda_1 = \pi^2 + (\pi/2)^2$ (C) $\lambda_1 = \pi^2 + 0$ (D) $\lambda_1 = \pi^2 + 0$

(E) $\lambda_1 = (\pi/2)^2 + 0$ (F) $\lambda_1 = (\pi/2)^2 + 0$ (G) $\lambda_1 = \pi^2 + 0$ (H) $\lambda_1 = \pi^2 + 0$ (I) $\lambda_1 = (2\pi)^2 + 0$

FIGURE 2. A summary of results on cutoffs for exclusion processes on the discrete 2D grid. The boundary condition can be one of the following: open; closed (reflecting); or periodic. In each case where λ_1 is given, cutoff is established: for every $\varepsilon \in (0,1)$, $\lim_{N\to\infty} \frac{\lambda_1 t^N_{\text{mix}}(\varepsilon)}{N^2 \log N} = 1$. The first 6 examples, (A) through (F), follow from Theorem 1. The last 3 examples, (G) through (I), follow from [CM20, Theorem 1].

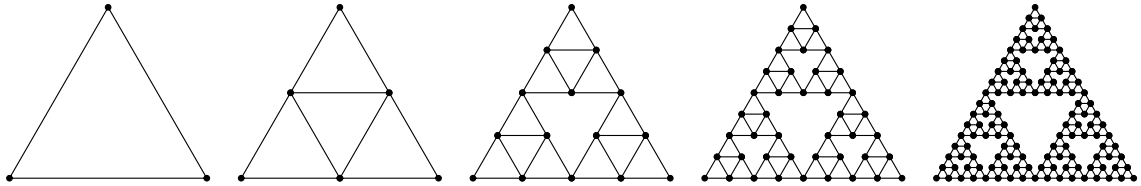


FIGURE 3. The level-N approximating graph $\mathcal{G}_N$ of the Sierpinski gasket, for $N = 0, 1, 2, 3, 4$ (from left to right). For every $N \in \mathbb{N}$, the boundary set ∂V_N is the set of 3 corner vertices of the outer triangle.

3.2. **Self-similar fractals.** Our assumptions also apply to a larger family of unweighted graphs which are known to converge geometrically and spectrally to a compact metric measure space. A concrete example is the Sierpinski gasket, whose graph approximations shown in Fig. 3, and has parameters $|V_N| = \frac{3}{2}(3^N + 1)$ and $\mathcal{T}_N = 5^N$ [BP88, Kig01, Str06]. In this case the boundary set is always the three corner vertices of the outer triangle, so $|\partial V_N| = 3$. We assume furthermore that the boundary rates $\lambda_{N,\Sigma}(a) = \mathsf{b}^{-N}\lambda_\Sigma(a)$, where each $\lambda_\Sigma(a) > 0$ is independent of N, and $\mathsf{b} \in (0, \frac{5}{3})$ is independent of N.

Theorem 3 (Cutoff for SSEP on the Sierpinski gasket)**.** *Let $t^{N,SG}_{\text{mix}}(\varepsilon)$ be the ε-mixing time of the SSEP on G_N approximating the Sierpinski gasket with 3 boundary reservoirs. For every $\varepsilon \in (0,1)$,*

$$\lim_{N\to\infty} \frac{2\lambda_1^{SG} t^{N,SG}_{\text{mix}}(\varepsilon)}{N 5^N \log 3} = 1. \tag{3.7}$$

Proof. Assumption 1 is easily verified. To verify Assumption 2, note that $\frac{\log a_N}{\log b_N} \sim \frac{\log((5/3)^N)}{\log \mathsf{b}^N} = \frac{\log(5/3)}{\log \mathsf{b}} > 1$ as long as $\mathsf{b} < 5/3$. This is the asymptotically Dirichlet regime for which the scaling limits of the density observable are established in [CG19, CFGM20].

We now check Assumption 3. By virtue of the consistency of the renormalized graph energies, one has $\lim_{N\to\infty} \mathcal{E}_N(f) = \mathcal{E}(f)$, and this implies that for each $i \in \mathbb{N}$, $\lambda_i^N \to \lambda_i$, and $\psi_i^N \to \psi_i$ in $C(K)$ [Kig01]. In particular Perron-Frobenius holds: $0 < \lambda_1 < \lambda_2$. (As an aside, we have from [Str06, §3.2 and §3.3] that $\lambda_1^{SG} = \frac{3}{2} \lim_{k\to\infty} 5^k \phi^{\circ k}(2)$ where $\phi(t) := \frac{5-\sqrt{25-4t}}{2}$.) This verifies Items (1), (2), and (3). For Item (4), the existence of the limit $\lim_{N\to\infty}(\partial_N^{\perp}\psi_i^N)(a)$ is provided by Kigami's theory [Kig01, Definition 3.7.6]. For Item (5), (I) follows from [Kig01, Proposition 5.1.2(1)], (II) follows from [Kig01, Proposition 5.1.10(2)], and (III) can be derived as a consequence of [Kig01, Lemmas 5.1.3 and 5.1.4].

To verify $\kappa^* > 1$ we will use Proposition 8.1 to reduce (2.19) to (2.21). Assuming that this reduction is valid—details are given in §8.2—we conclude that $\kappa^* = 1 + \frac{2\log(5/(3\mathsf{b}))}{\log 3} > 1$ by further noting that

$$\frac{|V_N|^{\kappa-1}}{|\partial V_N|} \sum_{a\in\partial V_N} (\psi_1^N(a))^2 = \Theta\left(3^{N(\kappa-1)} \left(\frac{3\mathsf{b}}{5}\right)^{2N}\right)$$

using the argument described in Remark 2.5. This completes the proof. □

Generalizations of the cutoff result to the d-dimensional Sierpinski simplex; the Vicsek tree; and other post-critically finite fractals are more or less immediate. Furthermore, we also can deal with products of fractals thanks to Proposition 8.1. Here is an example. (Again, the verification of $\kappa^* > 1$ follows from §8.2.)

Corollary 3.3. *Let $t_{\text{mix}}^{N,SG^D}(\varepsilon)$ be the ε-mixing time of the SSEP on the Cartesian product of D Sierpinski gaskets ($D \geq 2$), each having 3 boundary reservoirs. For every $\varepsilon \in (0,1)$,*

$$\lim_{N\to\infty} \frac{2\lambda_1^{SG} t_{\text{mix}}^{N,SG^D}(\varepsilon)}{N 5^N \log 3} = 1. \tag{3.8}$$

4. Coupling dual processes

We now commence the proof of Theorem 1. This section describes a coupling scheme which justifies the subsequent analysis of mixing times in the worst-case scenario.

Consider the following trio of Markov chains: $\{\eta_t^i\,;\, t \geq 0\}$, $i = 1,2$, are two copies of $\{\eta_t^N\,;\, t \geq 0\}$ starting from different configurations η and η'; and $\{\xi_t^N\,;\, t \geq 0\}$ is a third copy started from the same initial configuration η as the first copy, but where the boundary vertices only annihilate particles. More precisely, for each $a \in \partial V_N$ in $\{\xi_t^N\,;\, t \geq 0\}$, we have $\lambda_{+,N}(a) = 0$ and $\lambda_{-,N}(a) = \lambda_{\Sigma,N}(a)$, the total rate at a in the first two chains.

Let $\{(\eta_t^{1,N}, \eta_t^{2,N}, \xi_t^N)\,;\, t \geq 0\}$ be a coupling of these Markov chains driven by the following rules:

- A rate-1 exponential clock is attached to each edge $xy \in E_N$, and a rate-$\lambda_{\Sigma,N}(a)$ exponential clock is attached to each vertex $a \in \partial V_N$. All clocks are independent.
- If a clock associated with an edge of E_N rings, then for all three chains we flip that edge, exchanging the occupations of its incident vertices.
- If a clock associated with a vertex $a \in \partial V_N$ rings, then we empty the vertex a for all three chains, and flip a coin with probability of heads $\bar{\rho}_N(a)$. If the coin turns up heads, then we reoccupy a in the first two chains $\{\eta_t^i\,;\, t \geq 0\}$, $i = 1,2$, and keep a empty in $\{\xi_t^N\,;\, t \geq 0\}$. If the coin turns up tails, we do nothing.

Observe that since particles are indistinguishable, flipping an edge whose incident vertices have the same occupation does not affect the mixing of the process. Moreover, the above coin flipping rates are chosen so that the boundary vertex occupations agree with the generator $\mathcal{L}_N^{\text{boundary}}$.

Let η_{full} and η_{empty} be given by $\sum_{x\in V_N} \eta_{\text{full}}(x) = |V_N|$ and $\sum_{x\in V_N} \eta_{\text{empty}}(x) = 0$. We described the above coupling with these three Markov chains for the following reasons:

(1) Assume that $\eta = \eta_{\text{full}}$. The pair $\{(\eta_t^{1,N}, \eta_t^{2,N})\,;\, t \geq 0\}$ induces the projection $\eta_t^{1,N} \mapsto \eta_t^{2,N}$ of $\{\eta_t^{1,N}\,;\, t \geq 0\}$ onto $\{\eta_t^{2,N}\,;\, t \geq 0\}$. In order to see that, one just have to annihilate some of the particles of η_{full}. More

precisely, for every $\eta' \in \Omega_N$ there exists a set $S \subset V_N$ (which depends on η') such that

$$\eta'(x) = \begin{cases} \eta_{\text{full}}(x) = 1, & \text{if } x \in V_N \setminus S; \\ 1 - \eta_{\text{full}}(x) = 0, & \text{if } x \in S. \end{cases} \tag{4.1}$$

Since projections can not increase distance, we have

$$\|\mathbb{P}_{\eta'}(\eta_t \in \cdot) - \mu_{\text{ss}}\|_{TV} \leq \|\mathbb{P}_{\eta_{\text{full}}}(\eta_t \in \cdot) - \mu_{\text{ss}}\|_{TV} \quad \text{for any } \eta' \in \Omega_N \text{ and any } t \geq 0. \tag{4.2}$$

By the above inequality and the definition of distance to equilibrium given in (2.7) we conclude that

$$d_N(t) = \|\mathbb{P}_{\eta_{\text{full}}}(\eta_t \in \cdot) - \mu_{\text{ss}}\|_{TV} \quad \text{for any } t \geq 0. \tag{4.3}$$

(2) Let $T_N = \inf\{t \geq 0\,;\, \eta_t^{1,N} = \eta_t^{2,N}\}$. By [LP17, Corollary 5.3] we can bound the distance to equilibrium from above by

$$d_N(t) \leq \max_{\eta,\eta' \in \Omega_N} \mathbb{P}\left(T_N > t \,|\, \eta_0^1 = \eta\,,\, \eta_0^2 = \eta'\right). \tag{4.4}$$

(3) The pair $\{(\eta_t^{1,N}, \xi_t^N)\,;\, t \geq 0\}$ induces the projection of the copy $\{\eta_t^{1,N}\,;\, t \geq 0\}$ of the simple exclusion process with generator $\mathcal{T}_N\mathcal{L}_N$ onto the simple exclusion process with absorbing reservoirs $\{\xi_t^N\,;\, t \geq 0\}$. Recall that the unique absorbing state of this second process is the configuration η_{empty}. Let δ_{empty} denote the Dirac measure on this state. Again, since projections can not increase distance, we have

$$\left\|\mathbb{P}_\eta\left(\xi_t^N \in \cdot\right) - \delta_{\eta_{\text{empty}}}\right\|_{TV} \leq \left\|\mathbb{P}_\eta\left(\eta_t^{1,N} \in \cdot\right) - \mu_{\text{ss}}\right\|_{TV} \quad \text{for any } t \geq 0. \tag{4.5}$$

In particular, by identity (4.3),

$$d_N(t) \geq \left\|\mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N \in \cdot\right) - \delta_{\eta_{\text{empty}}}\right\|_{TV} \quad \text{for any } t \geq 0. \tag{4.6}$$

Combining the aforementioned results, we can prove the following:

Proposition 4.1. *Let T_N be the first time that all particles of η_{full} have been absorbed in the process $\{\xi_t^N\,;\, t \geq 0\}$. For every $t \geq 0$ we have*

$$d_N(t) = \left\|\mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N \in \cdot\right) - \delta_{\eta_{\text{empty}}}\right\|_{TV} = \mathbb{P}_{\eta_{\text{full}}}(T_N > t)\,. \tag{4.7}$$

Proof. By inequality (4.6) and the definition of the total variation distance, for any $t \geq 0$ we have

$$\begin{aligned} d_N(t) &\geq \left\|\mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N \in \cdot\right) - \delta_{\eta_{\text{empty}}}\right\|_{TV} \\ &= \frac{1}{2}\sum_{\xi \in \Omega_n} \left|\mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N = \xi\right) - \delta_{\eta_{\text{empty}}}(\xi)\right| \\ &= \frac{1}{2}\left[1 - \mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N = \eta_{\text{empty}}\right)\right] + \frac{1}{2}\sum_{\xi \in \Omega_N \setminus \{\eta_{\text{empty}}\}} \mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N = \xi\right) \\ &= 1 - \mathbb{P}_{\eta_{\text{full}}}\left(\xi_t^N = \eta_{\text{empty}}\right) \qquad (4.9) \\ &= 1 - \mathbb{P}_{\eta_{\text{full}}}(T_N \leq t) \\ &= \mathbb{P}_{\eta_{\text{full}}}(T_N > t). \qquad (4.10) \end{aligned} \tag{4.8}$$

In order to obtain the converse inequality of (4.8), we use the coupling $\{(\eta_t^{1,N}, \eta_t^{2,N})\,;\, t \geq 0\}$ again. We will show that T_N is a stopping time, so that at $\eta_t^{1,N} = \eta_t^{2,N}$ for any $t \geq T_N$, and then use inequality (4.4).

Indeed, let $x \in V_N$ and let $t \geq 0$. For each $i = 1, 2$ denote by V_N^i the copy of V_N associated with the the processes $\{\eta_t^i\,;\, t \geq 0\}$. Let x^i be the copy of x in V_N^i. Observe that if there is a particle at both x^1 and x^2 then, after edge flips, these particles will walk together (on their respective graph) until they are annihilated at one of the boundary vertices. When that happens both boundary vertices will be either occupied or empty. Analogously, if both x^1 and x^2 are empty, then those empty occupations will walk together until they are changed to two new particles at a boundary vertex. However, if there is particle at x^1 and a hole at x^2 (resp. a hole and a particle), then this pair of occupation variables $(1, 0)$ (resp. $(0, 1)$) will walk together until they reach a boundary vertex and the clock associated with that vertex rings. After that update, those occupations will be equal, that is, $(0, 0)$ or $(1, 1)$. Therefore, we must wait until every pair of different occupations changes at a boundary vertex. In order to enter the worst scenario, we must assume

that $\eta_0^{1,N}(x) = 1 - \eta_0^{2,N}(x)$ for every $x \in V_N$. Without loss of generality, we may assume that $\eta_0^{1,N} = \eta_{\text{full}}$ and $\eta_0^{2,N} = \eta_{\text{empty}}$. Thus, we must wait until every initial particle of $\eta_0^{1,N}$ is annihilated at the boundary, that is, T_N. Inequality (4.4) finishes the proof. □

Remark 4.2. It is not surprising that we can write the distance to equilibrium of $\{\eta_t^N\,;\,t \geq 0\}$ in terms of the process $\{\xi_t^N\,;\,t \geq 0\}$ with absorbing boundary. Actually, it can be verified that, the processes $\{\eta_t^N\,;\,t \geq 0\}$ and $\{\xi_t^N\,;\,t \geq 0\}$ are dual in the sense of [Lig05, Chapter II, Section 3]. Duality is a powerful tool that can be used to obtain quantitative behavior of one process from the behavior of its dual process.

Remark 4.3. The expression in (4.9) defines the *separation distance* $s_N(t)$ of the absorbing Markov chain $\{\xi_t^N\,;\,t \geq 0\}$ at time t. From (4.9) and (4.10), we have $s_N(t) = \mathbb{P}_{\eta_{\text{full}}}(T_N > t)$. In [AD87] Aldous and Diaconis introduced the *strong stationary times* for irreducible Markov chains, which are times for which the Markov chain has reached its stationary state, and its position at those times are independent of them. Furthermore, Aldous and Diaconis proved that for any irreducible finite Markov chain with state space Ω, given a state $x \in \Omega$, there always exists a strong stationary time T such that the separation distance of the process is equal to $\mathbb{P}_x(T > t)$. The stopping time T_N introduced in Proposition 4.1 is the absorption time of $\{\xi_t^N\,;\,t \geq 0\}$, which generalizes the notion of strong stationary times from irreducible chains to absorbing ones. Moreover, in (4.9), we have shown that the separation distance of $\{\xi_t^N\,;\,t \geq 0\}$ is actually equal to the total variation distance between its time marginal law and the Dirac measure concentrated at η_{empty}.

Remark 4.4. The coupling explained above was also used in [GNS20]. In order to estimate the time T_N, we used duality, whereas the authors of [GNS20] used an argument based on second-class particles.

5. Lower bounds on mixing times

From now on, we assume that $\{\eta_t^N; t \geq 0\}$ is the SSEP with complete absorption at the boundary vertices, *i.e.*, $\lambda_{+,N}(a) = 0$ for all $a \in \partial V_N$, starting from the initial configuration η_{full} with every vertex occupied with a particle. We denote the law of this process by $\mathbb{P}_{\mu_N}$, and the corresponding expectation by $\mathbb{E}_{\mu_N}$. Recall that Proposition 4.1 allows us to choose between a tail probability and a total variation distance characterization for the distance to equilibrium, whenever one suits better or is simpler to be estimated. Indeed, a sharp lower bound can be obtained using the total variation distance between the time distribution of the SSEP with absorbing reservoirs and the measure $\delta_{\eta_{\text{empty}}}$, using Wilson's method [Wil04]. The method suggests that we find a good observable of the process and that we distinguish its statistics with respect to the two measures. Wilson's claim that the first eigenfunctions of the generator are good observables.

To this end, consider the *empirical density measure*

$$\pi_t^N := \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \delta_{\{x\}}. \tag{5.1}$$

We denote the integral of a test function $f : V_N \to \mathbb{R}$ with respect to π_t^N by

$$\pi_t^N(f) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) f(x). \tag{5.2}$$

Replacing f by ψ_1^N in (5.2), applying the generator $\mathcal{T}_N \mathcal{L}_N$ to the function $\pi_s^N(\psi_1^N)$ and summing by parts, we obtain that

$$\mathcal{T}_N \mathcal{L}_N \pi_s^N(\psi_1^N) = \pi_s^N(\Delta_N \psi_1^N) = -\lambda_1^N \pi_s^N(\psi_1^N). \tag{5.3}$$

Therefore, the function $\pi_s^N(\psi_1^N)$ in an eigenfunction of $\mathcal{T}_N \mathcal{L}_N$ associated with $-\lambda_1^N$. Now, observe that for any $s \geq 0$

$$\partial_s \pi_s^N(\psi_1^N) e^{\lambda_1^N s} = \lambda_1^N \pi_s^N(\psi_1^N) e^{\lambda_1^N s}. \tag{5.4}$$

Hence, by Dynkin's formula (see [KL99, Appendix 1, Lemma 5.1]) applied to the function $\pi_t^N(\psi_1^N) e^{\lambda_1^N t}$, we obtain

$$\pi_t^N(\psi_1^N) e^{\lambda_1^N t} = \pi_0^N(\psi_1^N) + \mathcal{M}_t^N, \tag{5.5}$$

where $\mathcal{M}_t^N$ is a mean-zero martingale with respect to the natural filtration, and its quadratic variation is given by

$$\begin{aligned}\left\langle \mathcal{M}_t^N(\psi_1^N)\right\rangle_t &= \frac{\mathcal{T}_N}{|V_N|^2}\int_0^t \sum_{xy\in E_N} e^{2\lambda_1^N s}\left(\eta_s^N(x)-\eta_s^N(y)\right)^2\left(\psi_1^N(x)-\psi_1^N(y)\right)^2 ds \\ &+ \frac{\mathcal{T}_N}{|V_N|^2}\int_0^t e^{2\lambda_1^N s}\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\eta_s^N(a)(\psi_1^N(a))^2\,ds.\end{aligned} \tag{5.6}$$

Now we estimate the mean and variance of $\pi_t^N(\psi_1^N)$ with respect to $\mathbb{P}\mu_N$ and $\mu_{\rm ss}$. Recall that $\eta_0^N=\eta_{\rm full}$. From (5.5) and since $\mathcal{M}_t^N$ is a mean-zero martingale, we have

$$\mathbb{E}_{\mu_N}[\pi_t^N(\psi_1^N)] = \mathbb{E}_{\mu_N}[\pi_0^N(\psi_1^N)]e^{-\lambda_1^N t} = c_1^N e^{-\lambda_1^N t}. \tag{5.7}$$

By (5.5) and (5.6), the variance can be estimated as

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{\mu_N}}(\pi_t^N(\psi_1^N)) &= e^{-2\lambda_1^N t}\mathbb{E}_{\mu_N}\left[\left\langle \mathcal{M}_t^N(\psi_1^N)\right\rangle_t\right] \\ &\le \frac{\mathcal{T}_N}{|V_N|^2}e^{-2\lambda_1^N t}\int_0^t\sum_{xy\in E_N} e^{2\lambda_1^N s}\left(\psi_1^N(x)-\psi_1^N(y)\right)^2 ds \\ &+ \frac{\mathcal{T}_N}{|V_N|^2}e^{-2\lambda_1^N t}\int_0^t\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)e^{2\lambda_1^N s}\rho_s^N(a)(\psi_1^N(a))^2\,ds.\end{aligned} \tag{5.8}$$

We argue that the first term on the RHS of (5.8) is the dominant term, whereas the second term is $o(1)$. To see the latter, we can bound the second term from above by

$$\begin{aligned}&\frac{\mathcal{T}_N}{|V_N|^2}\int_0^t\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\rho_s^N(a)(\psi_1^N(a))^2\,ds \\ &\underset{(6.3)}{=} \frac{\mathcal{T}_N}{|V_N|^2}\int_0^t\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\left(\sum_{i=1}^{|V_N|}e^{-\lambda_i^N s}\mathsf{c}_i^N\psi_i^N(a)\right)(\psi_1^N(a))^2\,ds \\ &= \frac{\mathcal{T}_N}{|V_N|^2}\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\sum_{i=1}^{|V_N|}\frac{\mathsf{c}_i^N}{\lambda_i^N}(1-e^{-\lambda_i^N t})\psi_i^N(a)(\psi_1^N(a))^2.\end{aligned} \tag{5.9}$$

To make further progress, let us rewrite the boundary condition in (2.14) as

$$\psi_i^N(a) = -\frac{|V_N|}{|\partial V_N|}\frac{\partial_N^\perp\psi_i^N(a)}{\mathcal{T}_N\lambda_{\Sigma,N}(a)}\left(1-\frac{\lambda_i^N}{\mathcal{T}_N\lambda_{\Sigma,N}(a)}\right). \tag{5.10}$$

By Assumptions 2, 3-(1), and 3-(4), we arrive at

$$|\psi_i^N(a)| = \frac{|V_N|}{|\partial V_N|}\frac{1}{\mathcal{T}_N\lambda_{\Sigma,N}(a)}\cdot|\partial_N^\perp\psi_i^N(a)|\cdot(1-o(1)) \qquad \text{as } N\to\infty. \tag{5.11}$$

Therefore (5.9) is bounded from above by

$$\frac{|V_N|^{-1}}{|\partial V_N|}\sum_{i=1}^{|V_N|}\frac{|\mathsf{c}_i^N|}{\lambda_i^N}\sum_{a\in\partial V_N}(\psi_1^N(a))^2|\partial_N^\perp\psi_i^N(a)| \le \frac{1}{|V_N|}\sum_{i=1}^{|V_N|}\left(\sup_{b\in\partial V_N}|\partial_N^\perp\psi_i^N(b)|\right)\frac{|\mathsf{c}_i^N|}{\lambda_i^N}\frac{1}{|\partial V_N|}\sum_{a\in\partial V_N}(\psi_1^N(a))^2. \tag{5.12}$$

Compare the RHS expression with the functional in (2.19): Since we assume $\kappa^*>1$ in the statement of Theorem 1, it follows that this expression is $o(|V_N|^{-1})$. Taking (5.8), (5.9), and (5.12) altogether we find that

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_{\mu_N}}(\pi_t^N(\psi_1^N)) &\le \frac{e^{-2\lambda_1^N t}}{|V_N|}(e^{2\lambda_1^N t}-1)\frac{1}{\lambda_1^N}\mathcal{E}_N(\psi_1^N,\psi_1^N)+o(|V_N|^{-1}) \\ &= \frac{1}{|V_N|}\left(1-e^{-2\lambda_1^N t}\right)+o(|V_N|^{-1}) = \Theta\left(|V_N|^{-1}\right).\end{aligned} \tag{5.13}$$

Finally, with respect to the measure $\delta_{\eta_{\text{empty}}}$, the mean and the variance of $\pi_t^N(\psi_1^N)$ are equal to 0. Hence, by [LP17, Proposition 7.9] and the above estimates, there exists a positive constant c such that for every $t \geq 0$ and every $N \in \mathbb{N}$, we have

$$d_N(t) \geq 1 - \frac{ce^{2\lambda_1^N t}}{|V_N|}. \tag{5.14}$$

This implies that for any $\varepsilon > 0$, there exists $\beta = \beta(\varepsilon) > 0$ such that $d_N(t_N - \beta) \geq 1 - \varepsilon$. In particular, for any $\kappa < 1$, we have $d_N(\kappa t_N) \to 1$ as $N \to \infty$.

6. The cutoff martingale and upper bounds on mixing times

Continuing from the previous section, we focus on SSEP with complete absorption at the boundary vertices. As we argued in §4, the symmetric simple exclusion process with open boundary (at any uniformly bounded rates) becomes "close" to equilibrium at the first time that all particles in its dual process exit from V_N to the reservoirs.

Using the empirical density measure paired with the first Laplacian eigenfunction ψ_1^N, we obtained a lower bound on the mixing time. In order to obtain a matching upper bound, we shall use a rescaled version of the *density fluctuation field* paired with ψ_1^N. For $\kappa \geq 0$ we define

$$\mathcal{X}_\kappa^N(\psi_1^N) = |V_N|^{\frac{\kappa}{2}-1} \sum_{x \in V_N} \left(\eta_{\kappa t_N}^N(x) - \rho_{\kappa t_N}^N(x)\right) \psi_1^N(x), \tag{6.1}$$

where $\rho_t^N(x) = \mathbb{E}_{\mu_N}[\eta_t^N(x)]$. (By construction $\mathcal{X}_0^N(\psi_1^N) = 0$.) Observe the unusual prefactor $|V_N|^{\frac{\kappa}{2}-1}$ which replaces the central limit theorem scaling $|V_N|^{-1/2}$. To see why this scaling makes sense, let us apply the forward Kolmogorov's equation $\partial_t \mathbb{E}_{\mu_N}[\eta_t^N(x)] = \mathbb{E}_{\mu_N}[\mathcal{T}_N \mathcal{L}_N \eta_t^N(x)]$, recall (2.5), and verify that ρ_t^N solves the discrete heat equation

$$\begin{cases} \partial_t \rho_t^N(x) = \Delta_N \rho_t^N(x), & x \in V_N \setminus \partial V_N,\ t > 0, \\ \partial_t \rho_t^N(a) = -\frac{|V_N|}{|\partial V_N|}(\partial_N^\perp \rho_t^N)(a) - \mathcal{T}_N \lambda_{\Sigma,N}(a) \rho_t^N(a), & a \in \partial V_N,\ t > 0, \\ \rho_0^N(x) = 1, & x \in V_N. \end{cases} \tag{6.2}$$

Using Definition 2.2.1 we can express ρ_t^N as the linear combination of the eigenfunctions $\{\psi_i^N\}_i$:

$$\rho_t^N(x) = \sum_{i=1}^{|V_N|} \mathsf{c}_i^N e^{-\lambda_i^N t} \psi_i^N(x) \tag{6.3}$$

where the Fourier coefficients c_i^N were defined in (2.18). By this and the orthonormality of $\{\psi_i^N\}_i$, we can replace $\rho_{\kappa t_N}^N(x)$ in (6.1) and obtain

$$\mathcal{X}_\kappa^N(\psi_1^N) = |V_N|^{\frac{\kappa}{2}-1} \sum_{x \in V_N} \eta_{\kappa t_N}^N(x) \psi_1^N(x) - \mathsf{c}_1^N. \tag{6.4}$$

Now by our geometric and spectral assumption, $\mathsf{c}_1^N \to \int_K \psi_1 \, d\mathfrak{m}$, a positive $\Theta(1)$ constant. On the other hand, using Dynkin's formula and martingale calculations, we can estimate the second moment of $\mathcal{X}_\kappa^N(\psi_1^N)$ for $\kappa \in [0, \kappa^*)$. The precise statement is

Proposition 6.1 (Cutoff martingale)**.** *For every N, the stochastic process $\{\mathcal{X}_\kappa^N(\psi_1^N) : \kappa \in [0, \kappa^*)\}$ is a mean-zero martingale. Moreover, if $\kappa > 1$ then its variance satisfies*

$$\lim_{N \to \infty} \mathbb{E}_{\mu_N}[\mathcal{X}_\kappa^N(\psi_1^N)]^2 = \infty. \tag{6.5}$$

Assuming $\kappa^* > 1$, we will observe the cutoff martingale over the macroscopic time scale $\kappa \in [0, \kappa^*)$.

Proof of Theorem 1, assuming Proposition 6.1. By (2.20), Proposition 4.1 and (5.14) it suffices to show that for any $\kappa > 1$, $\mathbb{P}_{\mu_N}(T_N > \kappa t_N)$ converges to zero as $N \to \infty$. Indeed, starting from the initial distribution μ_N, suppose at time κt_N all particles have exited to the reservoirs, for all N. Then $\mathbb{P}_{\mu_N}$-a.s., $\eta_{\kappa t_N}^N(x) = 0$ for all $x \in V_N$, and so

$$\chi_\kappa^N(\psi_1^N) = -\mathsf{c}_1^N = \Theta(1). \tag{6.6}$$

(This holds for all larger κ as well, since particles cannot reenter the system after exiting.) On the other hand, by Proposition 6.1, if $\kappa \in (1, \kappa^*)$, then $\mathbb{E}_{\mu_N}[\mathcal{X}_\kappa^N(\psi_1^N)]^2 \to \infty$ as $N \to \infty$, which contradicts (6.6). □

7. Variance of the cutoff martingale

This section is devoted to the proof of Proposition 6.1. We apply Dynkin's formula to $\mathcal{X}_\kappa^N(\psi_1^N)$, which yields the martingale

$$\mathcal{M}_\kappa^N = \mathcal{X}_\kappa^N(\psi_1^N) - \mathcal{X}_0^N(\psi_1^N) - \int_0^\kappa (\partial_s + t_N \mathcal{T}_N \mathcal{L}_N)\mathcal{X}_s^N(\psi_1^N)\, ds. \tag{7.1}$$

We remind again that $\mathcal{X}_0^N(\psi_1^N) = 0$. Differentiating (6.4) with respect to time yields

$$\partial_s \mathcal{X}_s^N(\psi_1^N) = \frac{\log|V_N|}{2}|V_N|^{\frac{\kappa}{2}-1}\sum_{x\in V_N}\eta_{st_N}^N(x)\psi_1^N(x) = t_N\lambda_1^N|V_N|^{\frac{\kappa}{2}-1}\sum_{x\in V_N}\eta_{st_N}^N(x)\psi_1^N(x). \tag{7.2}$$

Meanwhile, using (2.5) and (2.12), we find that $t_N\mathcal{T}_N\mathcal{L}_N\chi_s^N(\psi_1^N)$ equals

$$\begin{aligned} &t_N|V_N|^{\frac{\kappa}{2}-1}\left(\sum_{x\in V_N\setminus\partial V_N}\eta_{st_N}^N(x)(\Delta_N\psi_1^N)(x) - \sum_{a\in\partial V_N}\eta_{st_N}^N(a)\left[\frac{|V_N|}{|\partial V_N|}(\partial_N^\perp\psi_1^N)(a) + \mathcal{T}_N\lambda_{\Sigma,N}(a)\psi_1^N(a)\right]\right) \\ &\underset{(2.14)}{=} t_N|V_N|^{\frac{\kappa}{2}-1}\left((-\lambda_1^N)\sum_{x\in V_N\setminus\partial V_N}\eta_{st_N}^N(x)\psi_1^N(x) - \lambda_1^N\sum_{a\in\partial V_N}\eta_{st_N}^N(a)\psi_1^N(a)\right). \end{aligned} \tag{7.3}$$

Therefore $(\partial_s + t_N\mathcal{T}_N\mathcal{L}_N)\mathcal{X}_s^N(\psi_1^N) = 0$, which implies that $\mathcal{X}_\kappa^N(\psi_1^N) = \mathcal{M}_\kappa^N$.

To compute the second moment of $\mathcal{X}_\kappa^N(\psi_1^N)$, we first find the quadratic variation

$$\langle\mathcal{M}^N\rangle_\kappa = \int_0^\kappa t_N\mathcal{T}_N\left(\mathcal{L}_N[\mathcal{X}_s^N(\psi_1^N)]^2 - 2\mathcal{X}_s^N(\psi_1^N)\mathcal{L}_N\mathcal{X}_s^N(\psi_1^N)\right)ds. \tag{7.4}$$

A tedious but straightforward computation shows that the last display equals the sum of

$$\mathcal{Q}_\kappa^{N,(1)} := t_N\mathcal{T}_N|V_N|^{\kappa-2}\int_0^\kappa\sum_{x\in V_N}\sum_{y\sim x}(\eta_{st_N}^N(x) - \eta_{st_N}^N(y))^2(\psi_1^N(x) - \psi_1^N(y))^2\, ds \tag{7.5}$$

and

$$\mathcal{Q}_\kappa^{N,(2)} := t_N\mathcal{T}_N|V_N|^{\kappa-2}\int_0^\kappa\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\eta_{st_N}^N(a)(\psi_1^N(a))^2\, ds. \tag{7.6}$$

Then

$$\mathbb{E}_{\mu_N}[\mathcal{X}_\kappa^N(\psi_1^N)]^2 = \mathbb{E}_{\mu_N}[\langle\mathcal{M}^N\rangle_\kappa] = \mathbb{E}_{\mu_N}[\mathcal{Q}_\kappa^{N,(1)}] + \mathbb{E}_{\mu_N}[\mathcal{Q}_\kappa^{N,(2)}]. \tag{7.7}$$

The main contribution comes from $\mathbb{E}_{\mu_N}[\mathcal{Q}_\kappa^{N,(1)}]$. We remind the reader the elementary identity

$$\mathbb{E}_{\mu_N}[\eta_t^N(x) - \eta_t^N(y)]^2 = 2\chi(\rho_t^N, xy) - 2\varphi_t^N(x,y), \tag{7.8}$$

where the mobility across the edge xy and the two-point correlation function read, respectively,

$$\chi(\rho, xy) := \frac{1}{2}\left[\rho(x)(1-\rho(y)) + \rho(y)(1-\rho(x))\right], \tag{7.9}$$

$$\varphi_t^N(x,y) := \mathbb{E}_{\mu_N}[(\eta_t^N(x) - \rho_t^N(x))(\eta_t^N(y) - \rho_t^N(y))]. \tag{7.10}$$

Therefore

$$\mathbb{E}_{\mu_N}[\mathcal{Q}_\kappa^{N,(1)}] = t_N|V_N|^{\kappa-1}\int_0^\kappa 2\frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{y\sim x}\left(\chi(\rho_{st_N}^N, xy) - \varphi_{st_N}^N(x,y)\right)(\psi_1^N(x) - \psi_1^N(y))^2\, ds. \tag{7.11}$$

The analysis of (7.11) is lengthy; details are given in §7.2 and §7.3 below.

7.1. **Boundary estimate.** Meanwhile, $\mathbb{E}_{\mu_N}[\mathcal{Q}_\kappa^{N,(2)}]$ equals

$$t_N\mathcal{T}_N|V_N|^{\kappa-2}\int_0^\kappa \sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\rho^N_{st_N}(a)(\psi_1^N(a))^2\,ds$$

$$\underset{(6.3)}{=} t_N\mathcal{T}_N|V_N|^{\kappa-2}\int_0^\kappa \sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\left(\sum_{i=1}^{|V_N|}e^{-\lambda_i^N st_N}\mathsf{c}_i^N\psi_i^N(a)\right)(\psi_1^N(a))^2\,ds \tag{7.12}$$

$$=\mathcal{T}_N|V_N|^{\kappa-2}\sum_{a\in\partial V_N}\lambda_{\Sigma,N}(a)\sum_{i=1}^{|V_N|}\frac{\mathsf{c}_i^N}{\lambda_i^N}(1-e^{-\lambda_i^N\kappa t_N})\psi_i^N(a)(\psi_1^N(a))^2$$

Using (5.11) and elementary estimates, we can bound the magnitude of (7.12) from above by

$$\frac{|V_N|^{\kappa-1}}{|\partial V_N|}\sum_{i=1}^{|V_N|}\frac{|\mathsf{c}_i^N|}{\lambda_i^N}\sum_{a\in\partial V_N}(\psi_1^N(a))^2|\partial_N^\perp\psi_i^N(a)|\le\frac{|V_N|^{\kappa-1}}{|\partial V_N|}\sum_{i=1}^{|V_N|}\left(\sup_{b\in\partial V_N}|\partial_N^\perp\psi_i^N(b)|\right)\frac{|\mathsf{c}_i^N|}{\lambda_i^N}\sum_{a\in\partial V_N}(\psi_1^N(a))^2. \tag{7.13}$$

The RHS expression was used to define κ^* in (2.19): For every $\kappa<\kappa^*$, (7.12) vanishes as $N\to\infty$.

7.2. **Estimate of the mobility term.** We pick up the analysis of (7.11), beginning with the integral over the mobility term χ. By exploiting the symmetry of the integrand under the exchange of x and y, we can write

$$t_N|V_N|^{\kappa-1}\int_0^\kappa 2\frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{y\sim x}\chi(\rho^N_{st_N},xy)(\psi_1^N(x)-\psi_1^N(y))^2\,ds \tag{7.14}$$

$$=t_N|V_N|^{\kappa-1}\int_0^\kappa 2\frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{y\sim x}\rho^N_{st_N}(x)(1-\rho^N_{st_N}(x))(\psi_1^N(x)-\psi_1^N(y))^2\,ds \tag{7.15}$$

$$+t_N|V_N|^{\kappa-1}\int_0^\kappa \frac{\mathcal{T}_N}{|V_N|}\sum_{x\in V_N}\sum_{y\sim x}(\rho^N_{st_N}(x)-\rho^N_{st_N}(y))^2(\psi_1^N(x)-\psi_1^N(y))^2\,ds. \tag{7.16}$$

Before proceeding further, let us introduce the notion of an *energy measure* Γ_N. For each $f:V_N\to\mathbb{R}$, we define $\Gamma_N(f)$ via the identity

$$\mathcal{E}_N(f,f)=\sum_{x\in V_N}\Gamma_N(f)(x). \tag{7.17}$$

It is easy to see that

$$\Gamma_N(f)(x)=\frac{1}{2}\frac{\mathcal{T}_N}{|V_N|}\sum_{y\sim x}[f(x)-f(y)]^2+\frac{\mathcal{T}_N}{|V_N|}\lambda_{\Sigma,N}(x)[f(x)]^2\mathbb{1}_{\{x\in\partial V_N\}}. \tag{7.18}$$

Using the summation-by-parts formula (2.12), the reader can verify the Γ-*calculus identity*

$$\int h\,d\Gamma_N(f)=\mathcal{E}_N(f,fh)-\frac{1}{2}\mathcal{E}_N(h,f^2). \tag{7.19}$$

Using (7.18) and (7.19) we find that

$$(7.15)=4|V_N|^{\kappa-1}t_N\int_0^\kappa\int_K\chi(\rho^N_{st_N}(x))\,d\Gamma_N(\psi_1^N)(x)\,ds+o(1),$$

where $\chi(\alpha)=\alpha(1-\alpha)$ for $\alpha\in[0,1]$. The $o(1)$ term in the last display comes from adding and subtracting the boundary reservoir term $\frac{\mathcal{T}_N}{|V_N|}\lambda_{\Sigma,N}(x)[\psi_1^N(x)]^2\mathbb{1}_{\{x\in\partial V_N\}}$ in the energy measure, and arguing that it vanishes asymptotically à la (7.12). This again requires $\kappa<\kappa^*$.

Let us define the (macroscopic-)time-dependent function

$$\Xi_N(\kappa):=t_N\int_0^\kappa\int_K\chi(\rho^N_{st_N}(x))\,d\Gamma_N(\psi_1^N)(x)\,ds\overset{\text{or}}{=}\int_0^{\kappa t_N}\int_K\chi(\rho_t^N(x))\,d\Gamma_N(\psi_1^N)(x)\,dt, \tag{7.20}$$

so that

$$(7.15)=4|V_N|^{\kappa-1}\Xi_N(\kappa)+o(1)\quad\text{for }\kappa\in[0,\kappa^*).$$

Lemma 7.1. $\Xi_N(\kappa) = \Theta(1)$ *independent of* κ.

This follows if we can prove that $\Xi_N(\kappa) \geq \Theta(1)$ and $\Xi_N(\kappa) \leq \Theta(1)$.

Proof that $\Xi_N(\kappa) \geq \Theta(1)$. Fix $\kappa > 0$ and set

$$\mathcal{L}_N^+ := \{(t,x) \in (0, \kappa t_N] \times K \,:\, \chi(\rho_t^N(x)) > 0\} = \{(t,x) \in (0, \kappa t_N] \times K \,:\, \rho_t^N(x) \in (0,1)\}$$

to be the support of $(t,x) \mapsto \chi(\rho_t^N(x))$ in $(0, \kappa t_N] \times K$. It suffices to show that the product measure on the support $\big(dt \times \Gamma_N(\psi_1^N)\big)\big(\mathcal{L}_N^+\big) \geq \Theta(1)$.

Observe that by (6.3) we have that

$$\rho_t^N(x) = \int_K p_N(t,x,y)\, d\mathfrak{m}_N(y) \qquad \text{where} \quad p_N(t,x,y) := \sum_{j=1}^{|V_N|} e^{-t\lambda_j^N} \psi_j^N(x)\psi_j^N(y).$$

Recall Remark 2.3, which says that $p_N(t,x,y) \to p(t,x,y)$ uniformly on $(0,\infty) \times K \times K$. In conjunction with the weak convergence $\mathfrak{m}_N \rightharpoonup \mathfrak{m}$, Assumption 1-(3), we deduce that

$$\rho_t^N(x) = \int_K p_N(t,x,y)\, d\mathfrak{m}_N(y) \xrightarrow[N\to\infty]{} \int_K p(t,x,y)\, d\mathfrak{m}(y) =: \rho_t(x)$$

pointwise for all $(t,x) \in (0,\infty) \times K$. Moreover, from Assumption 3-(5)(II) (resp. (III)), we deduce that $\rho_t(x) > 0$ for all $x \in K \setminus \partial K$ and $t > 0$ (resp. $\sup_{(t,x)\in[1,\infty)\times K} e^{t\lambda_1}\rho_t(x) =: M < \infty$). The last result implies that $\rho_t(x) < 1$ for all $t > \lambda_1^{-1} \log M =: t_1$.

Now consider the support set

$$\mathcal{L}_T^+ := \{(t,x) \in (0,T] \times K \,:\, \chi(\rho_t(x)) > 0\} = \{(t,x) \in (0,T] \times K \,:\, \rho_t(x) \in (0,1)\}.$$

Based on the last paragraph, we see that for any $T > t_1$, $\mathcal{L}_T^+ \supseteq (t_1, T] \times (K \setminus \partial K)$. Clearly $dt((t_1,T]) = T - t_1$. Meanwhile we claim that $\Gamma_N(\psi_1^N)(\partial K) = o(1)$, so that $\Gamma_N(\psi_1^N)(K \setminus \partial K) = \mathcal{E}_N(\psi_1^N) - o(1) = \lambda_1^N - o(1)$.

To verify the claim, observe from (7.18) that

$$\Gamma_N(\psi_1^N)(a) = \frac{1}{2}\frac{\mathcal{T}_N}{|V_N|}\sum_{y\sim a}(\psi_1^N(y) - \psi_1^N(a))^2 + \frac{\mathcal{T}_N}{|V_N|}\lambda_{\Sigma,N}(a)[\psi_1^N(a)]^2, \qquad a \in \partial V_N. \tag{7.21}$$

On the one hand, since $\psi_1^N \to \psi_1$ in $C(K)$, $\psi_1 \geq 0$, and $\psi_1(a) = 0$, we may assume WLOG that for all sufficiently large N that $\psi_1^N(y) - \psi_1^N(a) \geq 0$ for all $y \sim a$. Then

$$(\psi_1^N(y) - \psi_1^N(a))^2 \leq (\psi_1^N(y) - \psi_1^N(a)) \cdot \sup_{x\sim a}\big(\psi_1^N(x) - \psi_1^N(a)\big) \tag{7.22}$$

so that

$$\frac{1}{2}\frac{\mathcal{T}_N}{|V_N|}\sum_{y\sim a}(\psi_1^N(y) - \psi_1^N(a))^2 \leq \frac{1}{2|\partial V_N|}\sup_{x\sim a}\big(\psi_1^N(x) - \psi_1^N(a)\big)\big(-\partial_N^\perp \psi_1^N(a)\big) = \frac{o(1)}{|\partial V_N|} \tag{7.23}$$

by Assumptions 3-(3) and 3-(4). On the other hand, using Assumption 2 and Remark 2.5 we see that

$$\frac{\mathcal{T}_N}{|V_N|}\lambda_{\Sigma,N}(a)[\psi_1^N(a)]^2 \lesssim \frac{1}{|\partial V_N|}\frac{b_N}{a_N} = \frac{o(1)}{|\partial V_N|}. \tag{7.24}$$

Combining (7.23) and (7.24) we find that $\Gamma_N(\psi_1^N)(a) \leq |\partial V_N|^{-1} o(1)$. Summing over all $a \in \partial V_N$ yields the claim.

Thus $\big(dt \times \Gamma_N(\psi_1^N)\big)\big(\mathcal{L}_T^+\big) > 0$ for every $T > t_1$. By the pointwise convergence $\rho_t^N(x) \to \rho_t(x)$, conclude that $\big(dt \times \Gamma_N(\psi_1^N)\big)\big(\mathcal{L}_N^+\big) \geq \Theta(1)$. □

Next we show that $\Xi_N(\kappa) \leq \Theta(1)$. We give two versions of the proof. Version 1 is shorter, and uses Assumption 3-(5). Version 2 is longer, but does not require Assumption 3-(5).

Proof that $\Xi_N(\kappa) \leq \Theta(1)$, *Version 1.* WLOG we estimate $\Xi_N(\kappa)$ from above by

$$\Xi_N(\infty) = \int_0^1 \int_K \chi(\rho_t^N(x))\, d\Gamma_N(\psi_1^N)(x)\, dt + \int_1^\infty \int_K \chi(\rho_t^N(x))\, d\Gamma_N(\psi_1^N)(x)\, dt.$$

For the first integral, we replace $\chi(\rho_t^N(x))$ by the maximum value $\frac{1}{4}$, and then integrate in t and in x to find the upper bound $\frac{1}{4}\mathcal{E}_N(\psi_1^N) = \frac{\lambda_1^N}{4}$. For the second integral, we use Assumptions 3-(1) and 3-(5)(III) to bound $\chi(\rho_t^N(x))$ from above by $(1+o(1))e^{-t\lambda_1}M$, and then integrate in t and in x to find the upper bound $(1+o(1))\lambda_1^{-1}e^{-\lambda_1}M\mathcal{E}_N(\psi_1^N) = (1+o(1))e^{-\lambda_1}M$. Thus $\Xi_N(\infty)$ is bounded in N. □

Proof that $\Xi_N(\kappa) \le \Theta(1)$, Version 2. We use the spectral resolution of $\rho_{st_N}^N$ and the convergence of the resultant series. We plug (6.3) into (7.20) to get

$$\Xi_N(\kappa) = t_N \int_0^\kappa \int_K \left(\sum_{i=1}^{|V_N|} e^{-\lambda_i^N s t_N} \mathsf{c}_i^N \psi_i^N(x)\right)\left(1 - \sum_{j=1}^{|V_N|} e^{-\lambda_j^N s t_N} \mathsf{c}_j^N \psi_j^N(x)\right) d\Gamma_N(\psi_1^N)(x)\, ds. \tag{7.25}$$

Using Fubini's theorem and carrying out the integral over s, we find the last display equals

$$\sum_{i=1}^{|V_N|} \frac{1-e^{-\lambda_i^N \kappa t_N}}{\lambda_i^N} \mathsf{c}_i^N \int_K \psi_i^N\, d\Gamma_N(\psi_1^N) - \sum_{i,j=1}^{|V_N|} \frac{1-e^{-(\lambda_i^N+\lambda_j^N)\kappa t_N}}{\lambda_i^N+\lambda_j^N} \mathsf{c}_i^N \mathsf{c}_j^N \int_K \psi_i^N \psi_j^N\, d\Gamma_N(\psi_1^N). \tag{7.26}$$

We use the Γ-calculus identity (7.19) to simplify the spatial integrals:

$$\int_K \psi_i^N\, d\Gamma_N(\psi_1^N) = \mathcal{E}_N(\psi_1^N, \psi_1^N\psi_i^N) - \frac{1}{2}\mathcal{E}_N(\psi_i^N, (\psi_1^N)^2) = \left(\lambda_1^N - \frac{\lambda_i^N}{2}\right)\int_K (\psi_1^N)^2 \psi_i^N\, d\mathfrak{m}_N, \tag{7.27}$$

$$\int_K \psi_i^N \psi_j^N\, d\Gamma_N(\psi_1^N) = \mathcal{E}_N(\psi_1^N, \psi_1^N\psi_i^N\psi_j^N) - \frac{1}{2}\mathcal{E}_N(\psi_i^N\psi_j^N, (\psi_1^N)^2). \tag{7.28}$$

The first term on the RHS of (7.28) poses no difficulty. As for the second term on the RHS of (7.28), we employ a small generalization of (7.19),

$$\int h\, d\Gamma_N(f,g) = \frac{1}{2}\left[\mathcal{E}_N(f, gh) + \mathcal{E}_N(g, fh) - \mathcal{E}_N(h, fg)\right], \tag{7.29}$$

to obtain

$$-\frac{1}{2}\mathcal{E}_N(\psi_i^N\psi_j^N, (\psi_1^N)^2) = \int_K (\psi_1^N)^2\, d\Gamma_N(\psi_i^N, \psi_j^N) - \frac{1}{2}\mathcal{E}_N(\psi_i^N, \psi_j^N(\psi_1^N)^2) - \frac{1}{2}\mathcal{E}_N(\psi_j^N, \psi_i^N(\psi_1^N)^2). \tag{7.30}$$

Altogether (7.28) equals

$$\left[\lambda_1^N - \frac{1}{2}(\lambda_i^N + \lambda_j^N)\right]\int_K (\psi_1^N)^2 \psi_j^N \psi_k^N\, d\mathfrak{m}_N + \int_K (\psi_1^N)^2\, d\Gamma_N(\psi_i^N, \psi_j^N). \tag{7.31}$$

Plugging (7.27) and (7.28) into (7.26) we obtain

$$\begin{aligned}\Xi_N(\kappa) = &\sum_{i=1}^{|V_N|}(1-e^{-\lambda_i^N\kappa t_N})\left(\frac{\lambda_1^N}{\lambda_i^N} - \frac{1}{2}\right)\mathsf{c}_i^N \int_K (\psi_1^N)^2\psi_i^N\, d\mathfrak{m}_N \\ &- \sum_{i,j=1}^{|V_N|}(1-e^{-(\lambda_i^N+\lambda_j^N)\kappa t_N})\left(\frac{\lambda_1^N}{\lambda_i^N+\lambda_j^N} - \frac{1}{2}\right)\mathsf{c}_i^N\mathsf{c}_j^N \int_K (\psi_1^N)^2\psi_i^N\psi_j^N\, d\mathfrak{m}_N \\ &- \sum_{i,j=1}^{|V_N|}\frac{1-e^{-(\lambda_i^N+\lambda_j^N)\kappa t_N}}{\lambda_i^N+\lambda_j^N}\mathsf{c}_i^N\mathsf{c}_j^N \int_K (\psi_1^N)^2\, d\Gamma_N(\psi_i^N, \psi_j^N).\end{aligned} \tag{7.32}$$

Observe there are two dependences on $|V_N|$ in the last display: the number of terms in the sums, and the decaying exponentials such as $e^{-\lambda_i^N\kappa t_N}$. As $1-e^{-\lambda_i^N\kappa t_N}$ converges exponentially fast to 1, we can bound this uniformly in $|V_N|$.

Therefore the main thing to check is the summability of each series, and we will use the following elementary fact: Given two sequences of real numbers $\{c_n\}_n$ and $\{f_n\}_n$, suppose $0 \le \alpha \le |c_n| \le \beta$ for all

sufficiently large n. Then setting $\Sigma_+ = \{n : f_n \geq 0\}$ and $\Sigma_- = \{n : f_n < 0\}$, we find

$$\left|\sum_n c_n f_n\right| = \left|\sum_{n\in\Sigma_+} c_n f_n + \sum_{n\in\Sigma_-} c_n f_n\right| \leq \sum_{n\in\Sigma_+} |c_n f_n| + \sum_{n\in\Sigma_-} |c_n f_n| \leq \beta \sum_{n\in\Sigma_+} f_n - \alpha \sum_{n\in\Sigma_-} f_n. \tag{7.33}$$

Thus summability of $\sum f_n$ implies summability of $\sum c_n f_n$.

Let us start with the first term of (7.32). Observe that $1 - e^{-\lambda_i^N \kappa t_N}$ converges exponentially to 1 at rate $\lambda_i^N t_N$. Meanwhile, by Assumption 3-(1), $\frac{\lambda_1^N}{\lambda_i^N} \to \frac{\lambda_1}{\lambda_i}$, and the RHS tends to 0 as $i \to \infty$. Therefore given $\epsilon > 0$ we can find $I \in \mathbb{N}$ such that

$$-\frac{1}{2} \leq (1 - e^{-\lambda_i^N \kappa t_N}) \left(\frac{\lambda_1^N}{\lambda_i^N} - \frac{1}{2}\right) \leq -\frac{1}{2} + \epsilon \qquad \text{for all sufficiently large } N \text{ and all } i \geq I. \tag{7.34}$$

By the preceding paragraph, we have boiled the problem down to checking the summability of

$$\sum_i \mathsf{c}_i^N \int_K (\psi_1^N)^2 \psi_i^N \, d\mathfrak{m}_N. \tag{7.35}$$

By the inner product identity

$$\int_K fg \, d\mathfrak{m}_N = \sum_i \left(\int_K f\psi_i^N \, d\mathfrak{m}_N\right)\left(\int_K g\psi_i^N \, d\mathfrak{m}_N\right), \quad f, g \in L^2(\mathfrak{m}_N), \tag{7.36}$$

(7.35) equals $\int_K (\psi_1^N)^2 \, d\mathfrak{m}_N = 1$, which is what we want.

The above strategy can be repeated to show that the second term of (7.32) is $\Theta(1)$. We only mention the reduction of the sum of spatial integrals, which involves applying (7.36) twice:

$$\sum_{i,j} \mathsf{c}_i^N \mathsf{c}_j^N \int_K (\psi_1^N)^2 \psi_i^N \psi_j^N \, d\mathfrak{m}_N = \sum_i \mathsf{c}_i^N \int_K (\psi_1^N)^2 \psi_i^N \, d\mathfrak{m}_N = \int_K (\psi_1^N)^2 \, d\mathfrak{m}_N = 1. \tag{7.37}$$

The third term of (7.32) requires a different treatment, because of the integral with respect to the energy measure. We start by separating the $1 - e^{-(\lambda_i^N + \lambda_j^N)\kappa t_N}$ term from the rest: given $\epsilon > 0$ there exists $I \in \mathbb{N}$ such that

$$1 - \epsilon \leq 1 - e^{-(\lambda_i^N + \lambda_j^N)\kappa t_N} \leq 1 \qquad \text{for all sufficiently large } N \text{ and all } i \geq I. \tag{7.38}$$

So it is enough to establish the summability of

$$\sum_{i,j} \frac{\mathsf{c}_i^N \mathsf{c}_j^N}{\lambda_i^N + \lambda_j^N} \int_K (\psi_1^N)^2 \, d\Gamma_N(\psi_i^N, \psi_j^N). \tag{7.39}$$

As will become obvious shortly, it is easier to work with $2\sqrt{\lambda_i^N \lambda_j^N}$ in place of $\lambda_i^N + \lambda_j^N$ in the denominator. So we rewrite (7.39) as

$$\sum_{i,j} \frac{2\sqrt{\lambda_i^N \lambda_j^N}}{\lambda_i^N + \lambda_j^N} \frac{\mathsf{c}_i^N \mathsf{c}_j^N}{2\sqrt{\lambda_i^N \lambda_j^N}} \int_K (\psi_1^N)^2 \, d\Gamma_N(\psi_i^N, \psi_j^N). \tag{7.40}$$

Now recall the elementary AM-GM inequality

$$0 \leq \frac{2\sqrt{\alpha\beta}}{\alpha + \beta} \leq 1 \qquad \text{for all } \alpha, \beta > 0. \tag{7.41}$$

This allows us to bound $\frac{2\sqrt{\lambda_i^N \lambda_j^N}}{\lambda_i^N + \lambda_j^N}$ between 0 and 1 uniformly in N, i, and j, and reduce the summability problem to

$$\sum_{i,j} \frac{\mathsf{c}_i^N \mathsf{c}_j^N}{2\sqrt{\lambda_i^N \lambda_j^N}} \int_K (\psi_1^N)^2 \, d\Gamma_N(\psi_i^N, \psi_j^N) = \frac{1}{2} \int_K (\psi_1^N)^2 \, d\Gamma_N \left(\sum_i \frac{\mathsf{c}_i^N \psi_i^N}{\sqrt{\lambda_i^N}}, \sum_j \frac{\mathsf{c}_j^N \psi_j^N}{\sqrt{\lambda_j^N}}\right) \tag{7.42}$$

where the bilinearity of the energy measure was used in the last step. Notice that both functions in the energy measure are identical, so the measure is nonnegative. Thus we can bound the last display by

$$\frac{1}{2}\|\psi_1^N\|_\infty^2 \mathcal{E}_N\left(\sum_i \frac{\mathsf{c}_i^N \psi_i^N}{\sqrt{\lambda_i^N}}, \sum_j \frac{\mathsf{c}_j^N \psi_j^N}{\sqrt{\lambda_j^N}}\right) = \frac{1}{2}\|\psi_1^N\|_\infty^2 \sum_i (\mathsf{c}_i^N)^2 \underset{(7.36)}{=} \frac{1}{2}\|\psi_1^N\|_\infty^2. \tag{7.43}$$

This shows that (7.20) is at most $\Theta(1)$. □

We turn now to (7.16) and show that it decays faster than (7.15). Using Assumption 3-(3), we see that

$$(7.16) = t_N |V_N|^{\kappa-1} o(1) \int_0^\kappa \mathcal{E}_N(\rho_{st_N}^N)\, ds$$

Since $\rho_\cdot^N$ is the solution of the discrete heat equation, $\partial_t \rho_t^N = \Delta_N \rho_t^N$ with initial condition $\rho_0^N \equiv 1$, we find

$$\mathcal{E}_N(\rho_r^N) = \langle \rho_r^N, -\Delta_N \rho_r^N\rangle_{\mathfrak{m}_N} = \langle \rho_r^N, -\partial_r \rho_r^N\rangle_{\mathfrak{m}_N} = -\frac{1}{2}\partial_r \langle \rho_r^N, \rho_r^N\rangle_{\mathfrak{m}_N}. \tag{7.44}$$

Thus

$$(7.16) = |V_N|^{\kappa-1} o(1) \frac{1}{2}\left[1 - \langle \rho_{\kappa t_N}^N, \rho_{\kappa t_N}^N\rangle_{\mathfrak{m}_N}\right] \le C|V_N|^{\kappa-1} o(1),$$

which is what we wanted.

7.3. **Estimate of the correlation term.** We now proceed to show that the dynamical two-point correlation

$$\varphi_t^N(x,y) := \mathbb{E}_{\mu_N}[(\eta_t^N(x) - \rho_t^N(x))(\eta_t^N(y) - \rho_t^N(y))]$$

is nonpositive for any $x \neq y$ and any $t > 0$. While the negative correlation in the exclusion process seems to be well known to the experts, we are unable to find an exact reference that applies to our model, symmetric exclusion with boundary Glauber dynamics on a general graph. The closest results we found are [And88, Lig02], which apply to symmetric exclusion on a general graph without boundary dynamics.

To state our result, we need to introduce a process called the *diagonal-reflected random walk* on the Cartesian product of two copies of the same graph. (For more details see [Che20].) Throughout this discussion we fix $G_N = (V_N, E_N)$ with boundary set ∂V_N. The Cartesian product graph $G_N \square G_N$ is defined as the graph with vertex set

$$V(G_N \square G_N) = \{(x_1, x_2) : x_i \in V_N,\ i \in \{1,2\}\}$$

and edge set

$$E(G_N \square G_N) = \{\{(x,y_1),(x,y_2)\} : x \in V_N,\ \{y_1,y_2\} \in E_N\} \cup \{\{(x_1,y),(x_2,y)\} : \{x_1,x_2\} \in E_N,\ y \in V_N\}.$$

(For instance, the Cayley graph $\mathbb{Z}^2 = \mathbb{Z}\square\mathbb{Z}$.) We now introduce the product graph $G_N \boxslash G_N$, constructed from $G_N \square G_N$ by removing the vertices on the diagonal, as well as the edges connecting the diagonal: that is,

$$V(G_N \boxslash G_N) = V(G_N \square G_N) \setminus \{(x,x) : x \in V_N\},$$
$$E(G_N \boxslash G_N) = E(G_N \square G_N) \setminus (\{\{(x,x),(x,y)\} : x \in V_N,\ \{x,y\} \in E_N\} \cup \{\{(x,x),(y,x)\} : x \in V_N,\ \{x,y\} \in E_N\})$$

Recall the diffusively scaled SSEP-induced Laplacian Δ_N on G_N, defined in (2.9). Its generalization to the product graph $G_N \square G_N$ is the product Laplacian

$$(\boldsymbol{\Delta}_N^\square f)(x,y) := (\Delta_N f(\cdot,y))(x) + (\Delta_N f(x,\cdot))(y), \quad f : V(G_N \square G_N) \to \mathbb{R}; \tag{7.45}$$

and to the graph $G_N \boxslash G_N$,

$$(\boldsymbol{\Delta}_N^{\boxslash} f)(x,y) := (\boldsymbol{\Delta}_N^\square f)(x,y) - \mathcal{T}_N \mathbb{1}_{\{x\sim y\}}[f(x,x) + f(y,y) - 2f(x,y)], \quad f : V(G_N \boxslash G_N) \to \mathbb{R}. \tag{7.46}$$

We call $\boldsymbol{\Delta}_N^{\boxslash}$ the *diagonal-reflected Laplacian* on $G_N \boxslash G_N$. (Observe that the term $f(x,x)$ for any $x \in V_N$ is absent from (7.46)).

The Markov process $(\mathbf{X}_t^{N,\boxslash})_{t\ge 0}$ generated by $\boldsymbol{\Delta}_N^{\boxslash}$ is a variable-speed random walk process, accelerated by $\mathcal{T}_N$, on $G_N \boxslash G_N$, with an appropriate boundary condition on $\partial(G_N \boxslash G_N) := \{(x,y) \in V(G_N \boxslash G_N) : x \in \partial V_N \text{ or } y \in \partial V_N\}$. By construction, $\mathbf{X}_\cdot^{N,\boxslash}$ can visit a vertex which is at distance 1 from the diagonal, but then must jump to a vertex which is at distance 2 away. We call this phenomenon "reflection off the

diagonal"; thus, for a lack of a better name, we call $(\mathbf{X}_t^{N,\boxslash})_{t\geq 0}$ the *diagonal-reflected random walk* process on $G_N \boxslash G_N$, accelerated by $\mathcal{T}_N$.

Lemma 7.2. *Suppose $\varphi_0^N(x,y)=0$ for all $x,y\in V_N$ with $x\neq y$. Then $\varphi_t^N(x,y)\leq 0$ for all $x,y\in V_N$ with $x\neq y$ and all $t>0$.*

Proof. We use the fact that the Kolmogorov equation applied to $\varphi_t^N(x,y)$, $x\neq y$,

$$\partial_t\varphi_t^N(x,y)=\mathbb{E}_{\mu_N}\left[\mathcal{T}_N\mathcal{L}_N(\bar\eta_t^N(x)\bar\eta_t^N(y))\right],$$

yields the inhomogeneous heat equation

$$(\partial_t-\mathbf{\Delta}_N^{\boxslash})\varphi_t^N(x,y)=-\mathcal{T}_N\mathbb{1}_{\{x\sim y\}}(\rho_t^N(x)-\rho_t^N(y))^2. \tag{7.47}$$

By Duhamel's principle,

$$\varphi_t^N(x,y)=\mathbf{E}_{(x,y)}^{N,\boxslash}[\varphi_0^N(\mathbf{X}_t^{N,\boxslash})]+\int_0^t\sum_{\substack{z,w\in V_N\\ z\sim w}}\mathbf{P}_{(x,y)}^{N,\boxslash}[\mathbf{X}_s^{N,\boxslash}=(z,w)]\left(-\mathcal{T}_N(\rho_{t-s}^N(z)-\rho_{t-s}^N(w))^2\right)ds, \tag{7.48}$$

where $(\mathbf{X}_t^{N,\boxslash})_{t\geq 0}$ is the diagonal-reflected random walk on $G_N\boxslash G_N$ defined above; $\mathbf{P}_{(x,y)}^{N,\boxslash}$ denotes the law of $(\mathbf{X}_t^{N,\boxslash})_{t\geq 0}$ started from (x,y); and $\mathbf{E}_{(x,y)}^{N,\boxslash}$ is the corresponding expectation. Since $\mathbf{E}_{(x,y)}^{N,\boxslash}[\varphi_0^N(\mathbf{X}_t^{N,\boxslash})]=\sum_{\substack{z,w\in V_N\\ z\neq w}}\varphi_0^N(z,w)\mathbf{P}_{(x,y)}^{N,\boxslash}[\mathbf{X}_t^{N,\boxslash}=(z,w)]=0$ by assumption, we have that $\varphi_t^N(x,y)$ equals the integral term in (7.48), which is nonpositive for all $t>0$. □

It remains to note that under the initial measure $\delta_{\eta_{\text{full}}^N}$, $\eta_0^N(x)=1$ and $\rho_0^N(x)=1$ for all $x\in V_N$, and so $\varphi_0^N(x,y)=0$ for all $x\neq y$. The negativity of $\varphi_t^N(x,y)$ for any $x\neq y$ and $t>0$ follows from Lemma 7.2.

8. The $\kappa^*>1$ condition and Fourier analysis

This section discusses the analytic estimates needed to verify the $\kappa^*>1$ condition in Theorem 1. Note the sum $\displaystyle\sum_{i=1}^{|V_N|}\left(\sup_{b\in\partial V_N}|\partial_N^\perp\psi_i^N(b)|\right)\frac{|\mathsf{c}_i^N|}{\lambda_i^N}$ appearing in the definition (2.19) of κ^*. In the case of the d-dimensional grid in §3.1, we checked that this sum is bounded uniformly in N. Our objective here is to show that under reasonable assumptions on the approximating graphs, the above sum is bounded in N, thereby justifying the reduction from (2.19) to (2.21). Moreover this reduction can be made on the Cartesian product of D copies of the said graph.

8.1. **Reduction from (2.19) to (2.21).** First of all, recall from Assumption 3-(2) that for fixed $i\in\mathbb{N}$, $\mathsf{c}_i^N\to\int_K\psi_i\,d\mathfrak{m}$. From Fourier analysis we know that the sequence of Fourier coefficients $\{\int_K f\psi_i\,d\mathfrak{m}\}_{i\in\mathbb{N}}$ of $f\in L^2(K,\mathfrak{m})$ decays to 0 as $i\to\infty$. By Plancherel's identity, this sequence belongs to ℓ^2, which implies that $\int_K f\psi_i\,dm$ must decay faster than $i^{-1/2}$. If the function f is more regular than just in L^2, then one can exhibit a faster decay of its Fourier coefficients. For instance, on the unit interval with $f\equiv 1$, we have $|\int_{[0,1]}\psi_i\,d\mathfrak{m}|\sim|\int_0^1\sin(i\pi x)\,dx|\sim i^{-1}$ for large i.

To state our general result, we fix a family of graphs with boundaries $\{(G_N,\partial G_N)\}_N$ satisfying the three assumptions stated in §2.2, and converging to K with boundary ∂K. Suppose there exist constants $C_1,C_2,C_3>0$, $\beta,\gamma\geq 0$, and $d_S\geq 1$ such that

$$\limsup_{N\in\mathbb{N}}\sup_{x\in V_N}|\psi_i^N(x)|\leq C_1 i^\beta,\quad \limsup_{N\in\mathbb{N}}\sup_{b\in\partial V_N}|\partial_N^\perp\psi_i^N(b)|\leq C_2 i^\gamma,\quad\text{and}\quad \lambda_i\geq C_3 i^{2/d_S} \tag{8.1}$$

for all sufficiently large i. Let $G_N^{\square D}$ be the Cartesian product of D copies of G_N ($D\geq 2$) with boundary $\partial(G_N^{\square D})$, again satisfying the three assumptions stated in §2.2, and converging to K^D with boundary $\partial(K^D)$.

Proposition 8.1. *Assume $\beta\leq\frac12$ and $\gamma-\frac{2}{d_S}<-\frac12$. Then the sum $\displaystyle\sum_{i=1}^{|V_N|}\left(\sup_{b\in\partial V_N}|\partial_N^\perp\psi_i^N(b)|\right)\frac{|\mathsf{c}_i^N|}{\lambda_i^N}$ is bounded uniformly in N if we choose $\{(\psi_i^N,\lambda_i^N)\}_i$ to be the Laplacian eigensolutions on $(G_N^{\square D},\partial(G_N^{\square D}))$, where D*

can be any positive integer. Consequently, on $(G_N^{\square D}, \partial(G_N^{\square D}))$ *we have*

$$\kappa^* = \kappa^*(D) := \sup\left\{\kappa > 0 : \limsup_{N\to\infty} \frac{|V_N|^{D(\kappa-1)}}{|\partial(G_N^{\square D})|} \sum_{a\in\partial(G_N^{\square D})} (\psi_1^N(a))^2 = 0\right\}. \tag{8.2}$$

This proposition says κ^* is determined by two rates: the growth rate of $|V_N|$, and the decay rate of the L^2-norm of the restriction of the first eigenfunction ψ_1^N to the boundary.

Proof of Proposition 8.1. Let $\delta := -\frac{1}{2} - \gamma + \frac{2}{d_S} > 0$. We begin with the case $D = 1$. Using (8.1) and that $|\mathsf{c}_i^N|$ decays faster than $i^{-1/2}$, we deduce that

$$\sum_{i=1}^{|V_N|} \left(\sup_{b\in\partial V_N} |\partial_N^\perp \psi_i^N(b)|\right) \frac{|\mathsf{c}_i^N|}{\lambda_i^N} \le C \sum_{i=1}^{|V_N|} i^{\gamma-\frac{1}{2}-\frac{2}{d_S}} = C \sum_{i=1}^{|V_N|} i^{-(1+\delta)}.$$

where the constant $C > 0$ is independent of i and N. The sum on the RHS is bounded uniformly in N.

For $D \ge 2$, we note that the eigenfunctions are of the form $\boldsymbol{\psi}_{\mathbf{k}}^N(\mathbf{x}) = \prod_{i=1}^D \psi_{k_i}^N(x_i)$ with corresponding eigenvalues $\boldsymbol{\lambda}_{\mathbf{k}}^N = \sum_{i=1}^D \lambda_{k_i}^N$, where $\{(\psi_i^N, \lambda_i^N)\}_i$ are the eigensolutions on $(G_N, \partial G_N)$ (We use the **bold** face to distinguish quantities on the Cartesian product graph from those on a single graph.). As a result, the Fourier coefficients $|\mathbf{c}_i^N| \lesssim i^{-D/2}$, and the eigenvalues $\boldsymbol{\lambda}_i^N \gtrsim i^{2/(Dd_S)}$. If $\mathbf{b} = (b_1, b_2, \dots, b_D) \in \partial(G_N^{\square D})$, then at least one of the coordinates belongs to ∂G_N. Setting $\mathcal{I}(\mathbf{b}) := \{1 \le d \le D : b_d \in \partial G_N)$, we find

$$\begin{aligned}
(\partial_N^\perp \boldsymbol{\psi}_{\mathbf{k}}^N)(\mathbf{b}) &= \frac{\mathcal{T}_N}{|V_N|^D} |\partial(G_N^{\square D})| \sum_{\mathbf{y}\sim\mathbf{b}} [\boldsymbol{\psi}_{\mathbf{k}}^N(\mathbf{b}) - \boldsymbol{\psi}_{\mathbf{k}}^N(\mathbf{y})] \\
&= \mathcal{T}_N \left[1 - \left(1 - \frac{|\partial V_N|}{|V_N|}\right)^D\right] \sum_{d=1}^D \left(\prod_{\substack{j=1\\ j\neq d}}^D \psi_{k_j}^N(b_j) \sum_{y\sim b_d} [\psi_{k_d}^N(b_d) - \psi_{k_d}^N(y)]\right) \\
&= D(1+o(1)) \left[\left(\sum_{d\in\mathcal{I}(\mathbf{b})} + \sum_{d\notin\mathcal{I}(\mathbf{b})}\right) \frac{\mathcal{T}_N |\partial V_N|}{|V_N|} \left(\sum_{y\sim b_d} [\psi_{k_d}^N(b_d) - \psi_{k_d}^N(y)]\right) \left(\prod_{\substack{j=1\\ j\neq d}}^D \psi_{k_j}^N(b_j)\right)\right] \\
&= D(1+o(1)) \left[\sum_{d\in\mathcal{I}(\mathbf{b})} \left((\partial_N^\perp \psi_{k_d}^N)(b_d)\right) \left(\prod_{\substack{j=1\\ j\neq d}}^D \psi_{k_j}^N(b_j)\right) + \sum_{d\notin\mathcal{I}(\mathbf{b})} \left(\frac{-\Delta_N \psi_{k_d}^N(b_d)}{|V_N|/|\partial V_N|}\right) \left(\prod_{\substack{j=1\\ j\neq d}}^D \psi_{k_j}^N(b_j)\right)\right].
\end{aligned} \tag{8.3}$$

By Assumption 1-(2), the first sum dominates over the second sum in the last display. Therefore

$$\sup_{\mathbf{b}\in\partial(G_N^{\square D})} |(\partial_N^\perp \boldsymbol{\psi}_{\mathbf{k}}^N)(\mathbf{b})| \lesssim \sum_{d\in\mathcal{I}(\mathbf{b})} \left(\sup_{b\in\partial V_N} |(\partial_N^\perp \psi_{k_d}^N)(b)|\right) \prod_{\substack{j=1\\ j\neq d}}^D \left(\sup_{x\in V_N} |\psi_{k_j}^N(x)|\right), \tag{8.4}$$

which implies that for all sufficiently large i,

$$\limsup_{N\to\infty} \sup_{b\in\partial(G_N^{\square D})} |\partial_N^\perp \psi_i^N(b)| \lesssim i^{(\beta(D-1)+\gamma)/D}. \tag{8.5}$$

It follows that

$$\sum_{i=1}^{|V_N|^D} \left(\sup_{b\in\partial(G_N^{\square D})} |\partial_N^\perp \psi_i^N(b)|\right) \frac{|\mathbf{c}_i^N|}{\boldsymbol{\lambda}_i^N} \lesssim \sum_{i=1}^{|V_N|^D} i^{\frac{\beta(D-1)+\gamma}{D} - \frac{D}{2} - \frac{2}{Dd_S}} \le \sum_{i=1}^{|V_N|^D} i^{\frac{1}{2}(1-D-\frac{2}{D}) - \frac{\delta}{D}}$$

which is bounded uniformly in N. The claim follows. □

Remark 8.2. If $K = [0,1]$, we have that $\psi_i(x) = \sqrt{2}\sin(i\pi x)$, $i \in \mathbb{N}$, and therefore $\beta = 0$, $\gamma = 1$, and $d_S = 1$. Thus the conditions of Proposition 8.1 are satisfied.

Note also that Proposition 8.1 may be extended to treat the Cartesian product of the same graph with different boundary conditions (Dirichlet, Neumann, or periodic), provided that the exponents β, γ, and d_S are independent of the boundary conditions. We leave the extension to the reader.

8.2. **Verifying the $\kappa^* > 1$ condition on the Sierpinski gasket.** We now tie up the loose end in the proofs of Theorem 3 and Corollary 3.3, by showing that on the Sierpinski gasket (SG) we can pick parameters $\beta = \frac{1}{2}$ and $\gamma < \frac{2}{d_S} - \frac{1}{2}$, matching the conditions of Proposition 8.1. (For SG, $d_S = 2\log 3/\log 5 = 1.365...$ is the spectral dimension.) Here we shall invoke the fact that the *Nash inequality* holds, *cf.* [Kig01, Theorem 5.3.3], which implies that for any eigenfunction ψ with eigenvalue $\lambda \geq 1$, $\|\psi\|_{L^\infty(\mathfrak{m})} \leq C\lambda^{d_S/4}\|\psi\|_{L^2(\mathfrak{m})}$, where C is independent of ψ and λ [Kig01, Corollary B.3.9]. Using that $\lambda_i \lesssim i^{2/d_S}$ for large i and that ψ_i is L^2-normalized, we find $\|\psi_i\|_{L^\infty(\mathfrak{m})} \lesssim i^{1/2}$, *i.e.*, $\beta = \frac{1}{2}$. Meanwhile, to estimate the magnitude of the discrete normal derivative $(\partial_N^\perp \psi_i^N)(a)$, $a \in \partial V_N$, we use the integration by parts formula:

$$\mathcal{E}_N(\psi_i^N, h) = \langle -\Delta_N \psi_i^N, h\rangle + \sum_{a\in\partial V_N} (\partial_N^\perp \psi_i^N)(a)h(a) \tag{8.6}$$

Fix an $a \in \partial V_N$, and set $h = h_N := h_N(a)$ to be the unique discrete harmonic function satisfying

$$\begin{cases} \Delta_N h_N = 0 & \text{on } V_N \setminus \partial V_N, \\ h_N = \delta_a & \text{on } \partial V_N. \end{cases}$$

Then $\mathcal{E}_N(\psi_i^N, h_N) = 0$, $-\Delta_N \psi_i^N = \lambda_i^N \psi_i^N$, and so $(\partial_N^\perp \psi_i^N)(a) = -\lambda_i^N \langle \psi_i^N, h_N\rangle$. Furthermore, both $\psi_i^N \to \psi_i$ and $h_N \to h$ in $C(K)$, where h is the unique harmonic function with boundary value δ_a on K [Kig01, Theorem 3.2.4]. So $\langle \psi_i^N, h_N\rangle \to \int_K \psi_i h \, d\mathfrak{m}$. Recognizing that this latter integral is the ith Fourier coefficient of $h \in L^2(K, \mathfrak{m})$, we deduce that $\limsup_{N\to\infty} \sup_{a\in\partial V_N} |(\partial_N^\perp \psi_i^N)(a)|$ decays faster than $i^{\frac{2}{d_S}-\frac{1}{2}}$, as claimed.

Remark 8.3 (Optimizing β and γ). In the preceding proof we only used the fact that $1 \in L^2(K, \mathfrak{m})$ and $h \in L^2(K, \mathfrak{m})$ to arrive at the "worst-case" decay of the Fourier coefficients, namely, the parameters β and γ. On the one hand, this L^2-argument is robust enough that it should be generalizable to many other domains where the Nash inequality holds. On the other hand, it is an interesting and challenging problem in analysis to identify sharp decay of the Fourier coefficients on non-Euclidean domains. Indeed, a related phenomenon concerning convergence of Fourier series has been shown on SG, namely: Fourier series on SG along a "hierarchical" subsequence converges faster than what is expected for Fourier series on the interval. This is due to the presence of large gaps in the Laplacian eigenvalue counting function $s \mapsto \#\{\lambda \leq s : \lambda \text{ is an eigenvalue of } -\Delta\}$[1] [Str05].

9. Open questions

9.1. **Cutoff window.** The cutoff martingale method presented in this paper established the sharp cutoff result on many graphs with boundaries. The natural follow-up would be to characterize the *cutoff window*—the time interval $t \in [t_N - \beta_N,\ t_N + \beta_N]$ with $\beta_N = o(t_N)$ such that $d_N(t_N - \beta_N) = 1 - \delta$ and $d_N(t_N + \beta_N) = \delta$. This will require several quantitative refinements of the arguments presented here, and we are unsure about their efficacy.

9.2. **Cutoffs for symmetric exclusion processes on weighted graphs.** In this paper we deal solely with exclusion processes on unweighted graphs. To extend the analysis to weighted graphs will require several modifications including, but not limited to, the invariant measures of the processes. We leave the extension to future work.

A key issue here is "spectral consistency" in the convergence of the random walk processes and of the exclusion processes *simultaneously*. The discrete Euclidean grid and the Sierpinski gasket, as unweighted graphs, satisfy spectral consistency. For domains which have curvature, or arise from scaling limits of random graphs, it is not always possible to find a spectrally consistent discretization via unweighted graphs. An example of the former is a compact Riemannian manifold [vGR19, vGR20], where a "good" discretization is

[1] This feature is not to be confused with the *spectral gap* of the Markov generator.

realized as a family of weighted graphs wherein the weight of the edge connecting vertices x and y decreases exponentially with the Riemannian distance $d(x, y)$. An example of the latter is the random self-similar dendrite construction of the continuum random tree, due to [CH08].

9.3. **(Pre-)cutoff in ASEP with open boundaries.** Finally, it is an open question whether the cutoff martingale method can be applied to exhibit (pre-)cutoff for the asymmetric exclusion process (ASEP) with open boundaries in any dimension. In 1D the scaling limit for the density fluctuation field has been proven to follow a stochastic Burgers' equation, with a quadratic nonlinear term whose strength is determined by the weak asymmetry. The challenge will be to address the nonlinear effect, as well as control the two-point correlations in ASEP.

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