Phase transition in the exclusion process with slowed boundary reservoirs on resistance spaces

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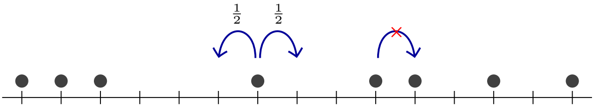
Exclusion process on a finite connected weighted graph

A system of random walkers subject to the **exclusion rule**: no 2 walkers occupy the same vertex at any time.



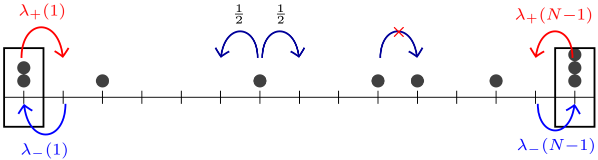
Let $G = (V, E)$ be a finite connected graph endowed with positive edge weights $\mathbf{c} = \{c_{xy}\}_{xy \in E}$.
The **symmetric exclusion process** on $(G, \mathbf{c})$ is a continuous-time Markov process on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} [f(\eta^{xy}) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R},$$

$$\text{where} \quad (\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$$

- Conserved quantity: Total # of particles.
- Each product Bernoulli measure ν_α, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure.
- Dirichlet energy: $\mathcal{E}^{\mathrm{EX}}(f) = \langle f, -\mathcal{L}^{\mathrm{EX}} f \rangle_{\nu_\alpha} = \dfrac{1}{2} \sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta).$

Adding reservoirs (Kawasaki dynamics) to the exclusion process



Designate a finite boundary set $\partial V \subset V$. For each $a \in \partial V$:

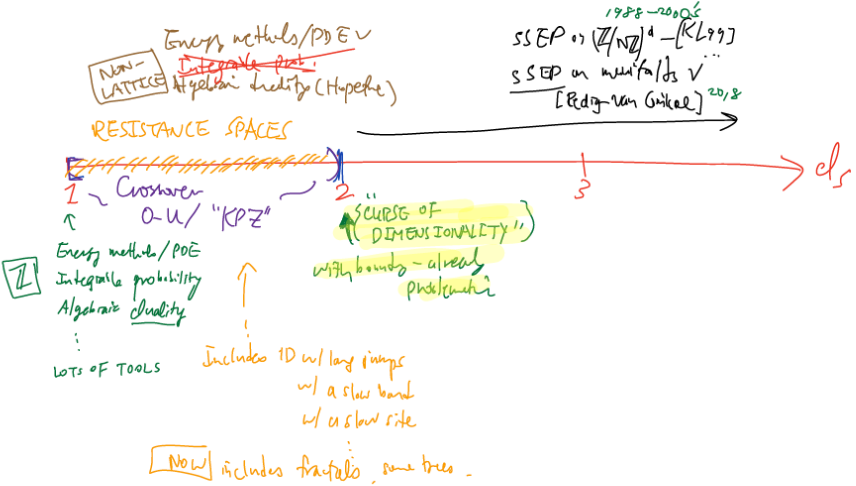
- At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth).
- At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).

Formally, $(\mathcal{L}^{\rm boun}_{\partial V} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1-\eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R}$, where

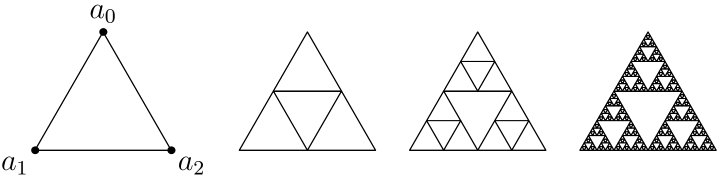
$$\eta^a(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

- 1D boundary-driven simple exclusion process: generator $N^2\left(\mathcal{L}^{\rm EX}_{\{1,2,\cdots,N-1\}} + \mathcal{L}^{\rm boun}_{\{1,N-1\}}\right)$.
- Has been studied extensively for the past ~ 15 years: Hydrodynamic limits, fluctuations, large deviations, etc.
- Difficulties: $\#$ of particles is no longer conserved; the invariant measure is in general not explicit.

Extending the analysis to higher dims & with >2 reservoirs?

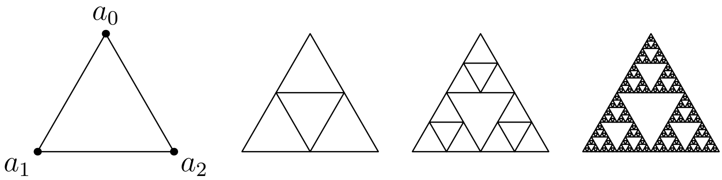


Boundary-driven exclusion process on the Sierpinski gasket



- Construction of Brownian motion with invariant measure m (the standard self-similar measure) as scaling limit of RWs accelerated by $\mathcal{T}_N = 5^N$.
  [Goldstein '87, Kusuoka '88, Barlow-Perkins '88]
- A robust notion of calculus on SG which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.
  [See the books by Kigami '01, Strichartz '06]

  What is the analog of "$\int_K |\nabla f|^2 \, dx$" in the fractal setting?
- A good model for rigorously studying (non)equilibrium stochastic dynamics with ≥ 3 boundary reservoirs.



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$$\mathcal{E}(f) = \lim_{N\to\infty} \frac{5^N}{3^N} \sum_{xy\in E_N} (f(x)-f(y))^2, \quad f \in L^2(K,m).$$

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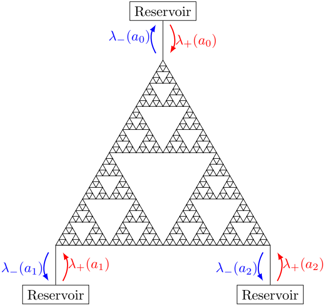
Boundary-driven exclusion process on the Sierpinski gasket



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Parameter $b > 0$ governs the inverse speed (relative to the bulk jump rate) at which the reservoir injects/extracts particles into/from the boundary vertices V_0.

Our main result in a nutshell [C.–Gonçalves '19]

A **phase transition** in the scaling limit of the particle density depending on the value of b, reflected by the different boundary conditions. The critical value of b is $\frac{5}{3}$.

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Hydrodynamic limit: a LLN result

Assume that sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on $\{0,1\}^{V_N}$ is associated to a density profile $\varrho : K \to [0,1]$: for any continuous function $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \{0,1\}^{V_N} \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} F(x)\eta(x) - \int_K F(x)\varrho(x)\, dm(x) \right| > \delta \right\} = 0.$$

Given the process $\{\eta_t^N : t \geq 0\}$ generated by $5^N \mathcal{L}_N^{\mathrm{bEX}}$, the **empirical density measure** π_t^N given by

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \delta_{\{x\}}$$

and for any test function $F : K \to \mathbb{R}$, we denote the integral of F wrt π_t^N by $\pi_t^N(F)$ which equals

$$\pi_t^N(F) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x).$$

Claim. The sequence $\{\pi_\cdot^N\}_N$ converges in the Skorokhod topology on $D([0,T], \mathcal{M}_+)$ to the unique measure $\pi_\cdot$ with $d\pi_\cdot(x) = \rho(\cdot, x)\, dm(x)$.
For any $t \in [0,T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

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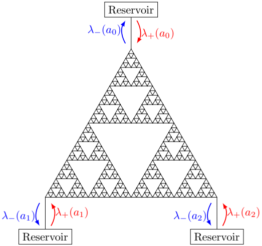


$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$

Theorem (Density hydrodynamic limit)

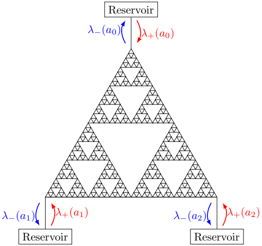
For any $t \in [0, T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t,x) \, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with Dirichlet boundary condition if $b < \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3} \Delta \rho(t,x), & t \in [0,T], \ x \in K \setminus V_0, \\ \rho(t,a) = \bar{\rho}(a), & t \in (0,T], \ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

Hydrodynamic limit: a LLN result



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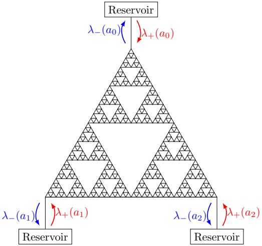
where ρ is the unique weak solution of the heat equation
with Neumann boundary condition if $b > \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t,a) = 0, & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

Hydrodynamic limit: a LLN result



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$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with linear Robin boundary condition if $b = \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t\in[0,T],\ x\in K\setminus V_0, \\ (\partial^\perp \rho)(t,a) = -\lambda_\Sigma(a)(\rho(t,a) - \bar\rho(a)), & t\in(0,T],\ a\in V_0, \\ \rho(0,x) = \varrho(x), & x\in K. \end{cases}$$

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N\left(\left(\frac{2}{3}\Delta + \partial_s\right)F_s\right)\,ds$$
$$+ \int_0^t \frac{3^N}{|V_N|}\sum_{a\in V_0}\left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a))F_s(a)\right]\,ds + o_N(1).$$

[Ingredient #1] **Analysis on fractals**
Convergence of discrete Laplacian to the continuous counterpart; normal derivatives at the boundary; integration by parts formula ... [Kigami '01, Strichartz '06].

Heuristics for hydrodynamics

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[Ingredient #1] **Analysis on fractals**

This part will produce the weak formulation of the heat equation.

Heuristics for hydrodynamics

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[Ingredient #2] **Treatment of the boundary term**

- $b > 5/3$: The first term dominates, should converge to $\int_0^t \frac{2}{3} \sum_{a\in V_0} \rho_s(a)(\partial^\perp F_s)(a)\, ds$
- $b = 5/3$: Both terms contribute equally, should converge to $\int_0^t \frac{2}{3} \sum_{a\in V_0} \left[\rho_s(a)(\partial^\perp F_s)(a) + \lambda_\Sigma(a)(\rho_s(a) - \bar{\rho}(a))F_s(a)\right] ds$
- $b < 5/3$: Impose $\rho_t(a) = \bar{\rho}(a)$ for all $a \in V_0$, should converge to $\int_0^t \frac{2}{3} \sum_{a\in V_0} \bar{\rho}(a)(\partial^\perp F_s)(a)$

Require a series of **replacement lemmas** — not trivial on state spaces without translational invariance!
$\longrightarrow$ I will come back to address this at the end.

Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N\left(\left(\frac{2}{3}\Delta + \partial_s\right) F_s\right) ds$$

$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a))F_s(a)\right] ds + o_N(1).$$

$$\Big\downarrow N \to \infty$$

$$0 = \pi_t(F_t) - \pi_0(F_0) - \int_0^t \pi_s\left(\left(\frac{2}{3}\Delta + \partial_s\right) F_s\right) ds + (\text{boundary term})$$

[Ingredient #3] **Convergence of stochastic processes**

- Show that $\{\pi_\cdot^N\}_N$ is tight in the Skorokhod topology on $D([0,T], \mathcal{M}_+)$ via Aldous' criterion.
- Prove that any limit point $\pi_\cdot$ is absolutely continuous w.r.t. the self-similar measure m, with $\pi_t(dx) = \rho(t,x)\, dm(x)$, and $\rho \in L^2(0,T,\mathrm{dom}\mathcal{E})$.
- Finally, prove ! of the weak solution to the heat equation to conclude ! of the limit point.

Density fluctuation field (at equilibrium): Heuristics

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
The product Bernoulli measure ν_ρ^N with $\rho = \lambda_+/(\lambda_+ + \lambda_-)$ is stationary for the process.

Density fluctuation field (DFF) $\mathcal{Y}_t^N(F) = \dfrac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \left(\eta_t^N(x) - \rho\right) F(x)$

The corresponding Dynkin's martingale is

$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1)$$
$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma F(a) \right] ds,$$

which has QV

$$\langle M^N(F) \rangle_t = \int_0^t \frac{5^N}{|V_N|^2} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} (\eta_s^N(x) - \eta_s^N(y))^2 (F(x) - F(y))^2 ds$$
$$+ \int_0^t \sum_{a \in V_0} \frac{5^N}{b^N |V_N|^2} \{\lambda_-(a)\eta_s^N(a) + \lambda_+(a)(1 - \eta_s^N(a))\} F^2(a) ds.$$

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which, as $N \to \infty$, has the QV of a space-time white noise (with boundary condition)

$$\frac{2}{3} \cdot 2\rho(1-\rho) t \mathscr{E}_b(F), \quad \text{where } \mathscr{E}_b(F) = \mathcal{E}(F) + \lambda_\Sigma \sum_{a \in V_0} F^2(a) \mathbf{1}_{\{b = 5/3\}}$$

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$$\begin{aligned}\mathcal{M}_t^N(F) &= \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1)\\ &+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma F(a) \right] ds,\end{aligned}$$

We then argue that the test function $F \in \mathrm{dom}\Delta_b$ be chosen appropriate to each boundary condition such that the boundary term vanishes as $N \to \infty$.

$$\mathrm{dom}\Delta_b := \begin{cases} \{F \in \mathrm{dom}\Delta : F|_{V_0} = 0\}, & \text{if } b < 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = -\lambda_\Sigma F|_{V_0}\}, & \text{if } b = 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = 0\}, & \text{if } b > 5/3. \end{cases}$$

For technical reasons (in order to use Mitoma's tightness criterion) we use a smaller test function space $\mathcal{S}_b := \{F \in \mathrm{dom}\Delta_b : \Delta_b F \in \mathrm{dom}\Delta_b\}$, which can be made into a Frechét space.
Let S_b' be the topological dual of S_b.

Definition (Ornstein-Uhlenbeck equation)

We say that a random element $\mathcal{Y}$ taking values in $C([0,T],\mathcal{S}_b')$ is a solution to the **Ornstein-Uhlenbeck equation** on K with parameter b if:

1. For every $F \in \mathcal{S}_b$,

$$\mathcal{M}_t(F) = \mathcal{Y}_t(F) - \mathcal{Y}_0(F) - \int_0^t \mathcal{Y}_s(\frac{2}{3}\Delta_b F)\,ds$$

$$\text{and} \quad \mathcal{N}_t(F) = (\mathcal{M}_t(F))^2 - \frac{2}{3}\cdot 2\rho(1-\rho)t\mathscr{E}_b(F)$$

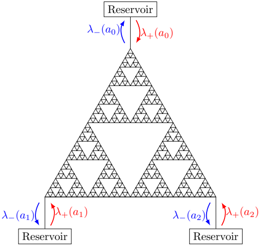
   are $\mathscr{F}_t$-martingales, where $\mathscr{F}_t := \sigma\{\mathcal{Y}_s(F) : s \leq t\}$ for each $t \in [0,T]$.

2. $\mathcal{Y}_0$ is a centered Gaussian $\mathcal{S}_b'$-valued random variable with covariance

$$\mathbb{E}_\rho^b\left[\mathcal{Y}_0(F)\mathcal{Y}_0(G)\right] = \rho(1-\rho)\int_K F(x)G(x)\,dm(x), \quad \forall F, G \in \mathcal{S}_b.$$

   Moreover, for every $F \in \mathcal{S}_b$, the process $\{\mathcal{Y}_t(F) : t \geq 0\}$ is Gaussian: the distribution of $\mathcal{Y}_t(F)$ conditional upon $\mathscr{F}_s$, $s < t$, is Gaussian with mean $\mathcal{Y}_s(\tilde{\mathrm{T}}^b_{t-s}F)$ and variance $\int_0^{t-s} \frac{2}{3}\cdot 2\rho(1-\rho)\mathscr{E}_b(\tilde{\mathrm{T}}^b_r F)\,dr$, where $\{\tilde{\mathrm{T}}^b_t : t > 0\}$ is the heat semigroup associated with $\frac{2}{3}\mathscr{E}_b$.

O-U limit of equilibrium density fluctuations: a CLT result



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Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Equilibrium $\Leftrightarrow$ $\lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$.

Let $\mathbb{Q}_\rho^{N,\mathrm{b}}$ be the probability measure on $D([0,T], \mathcal{S}_b')$ induced by the DFF $\mathcal{Y}_\cdot^N$ started from ν_ρ^N and boundary parameter b.

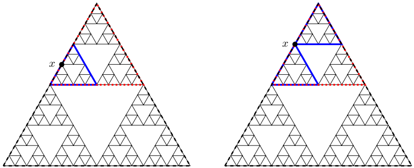
Theorem (CLT)

The sequence $\{\mathbb{Q}_\rho^{N,b}\}_N$ converges in distribution, as $N \to \infty$, to a unique solution of the Ornstein-Uhlenbeck equation with parameter b (as defined previously).

Key Lemma. $\tilde{\mathrm{T}}_t^b(\mathcal{S}_b) \subset \mathcal{S}_b$ for any $t > 0$. Enough to verify that $\tilde{\mathrm{T}}_t^b\left(L^1(K,m)\right) \subset \mathrm{dom}\Delta_b$, which can be shown using *e.g.* the Nash inequality (heat kernel upper bound).

The rest of the argument follows a martingale approach of Kipnis–Landim.

A quick word on the replacement lemma



For finite $\Lambda \subset V$, denote $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z \in \Lambda} \eta(z)$.
We need to prove that for every $t > 0$,

Replacement lemma

$$\overline{\lim_{j\to\infty}}\ \overline{\lim_{N\to\infty}}\ \mathbb{E}_{\mu_N}\left[\left|\int_0^t \left(\eta_s^N(a) - \mathrm{Av}_{V_N \cap K_j(a)}[\eta_s^N]\right) ds\right|\right] = 0, \quad a \in V_0.$$

where $K_j(a)$ is the cell of depth level j which contains a.
Proving the replacement lemma without using translational invariance is difficult!
What other mechanism can be used to carry out this replacement?

A quick word on the replacement lemma

For finite $\Lambda \subset V$, denote $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z\in\Lambda} \eta(z)$.
We need to prove that for every $t > 0$,

Replacement lemma

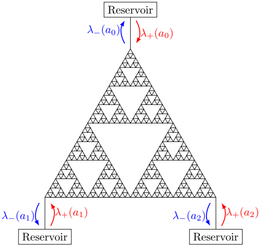
$$\varlimsup_{j\to\infty} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \left(\eta_s^N(a) - \mathrm{Av}_{V_N\cap K_j(a)}[\eta_s^N]\right) ds\right|\right] = 0, \quad a \in V_0.$$

where $K_j(a)$ is the cell of depth level j which contains a.
For resistance spaces, we can use the **moving particle lemma** [C., *ECP* '17]:

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2\, d\nu_\alpha(\eta) \le R_{\rm eff}(x,y)\mathcal{E}^{\rm EX}(f), \quad f : \{0,1\}^V \to \mathbb{R},$$

where $R_{\rm eff}(x,y)$ is the **effective resistance** between x and y in the *random walk process* on the same weighted graph $(G, \mathbf{c})$.
For more details on the MPL and its connection to the octopus inequality of [Caputo–Liggett–Richthammer *JAMS* '10], see my *ECP* paper.

A sneak preview of upcoming series of works, and **Thank you!**



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Symmetric exclusion process with **slowed** boundary on the Sierpinski gasket

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Equilibrium $\Leftrightarrow$ $\lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)

- (Non)equilibrium density hydrodynamic limit (DRN✓) This talk
- Ornstein-Uhlenbeck limit of equilibrium density fluctuations (DRN✓). This talk
- Large deviations principle for the (non)equilibrium density (D✓) [C.–Hinz '19]
- Hydrostatic limit, scaling limit of nonequilibrium density fluctuations (D in progress). [C.–Franceschini–Gonçalves–Menezes '19+] $\rightarrow$ *careful study of two-particle correlations*
- **More in the pipeline:**

  Motion of the tagged particle (a fractional BM on the gasket?).

  Add (suitably rescaled) weak asymmetry to the jump rate, prove that the equilibrium density fluctuations converges (subsequentially) to a stochastic Burgers equation [C. '19+]