Dirichlet, Schur, and Octopus

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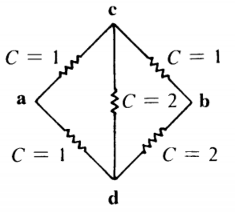
Weighted graph = Electric network

- Let $G = (V, E)$ be a finite connected undirected graph (assuming no self-edges).
- To each edge $xy \in E$, assign a positive number c_{xy} (weight, edge conductance). Put $c_x := \sum_y c_{xy}$ for all $x \in V$.
- The pair $(G, \mathbf{c} = \{c_{xy}\}_{xy \in E})$ is called a weighted graph.
- Now construct the **stochastic matrix P** whose entries are $p_{xy} = \dfrac{c_{xy}}{c_x} \in [0, 1]$.
- Symmetric random walk on $(G, \mathbf{c})$: At each time step, the random (drunk, memoryless) walker at $x \in V$ chooses a nearest neighbor $y \in V$ with probability p_{xy} and moves there.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{2}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}$$

(The entries along each row must add up to 1.)

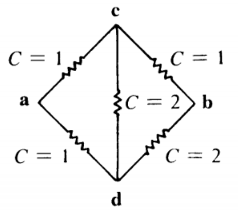


Conductances

Laplacian and Dirichlet energy

- Define the Laplacian associated to the weighted graph $(G, \mathbf{c})$ by $\mathbf{L} = \mathbf{I} - \mathbf{P}$. This is a nonnegative definite matrix.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{2}{4} \\ -\frac{1}{5} & -\frac{2}{5} & -\frac{2}{5} & 1 \end{bmatrix}$$



Conductances

- The associated quadratic form is called the Dirichlet energy: for every $f : V \to \mathbb{R}$,

$$\mathcal{E}(f) = \sum_{xy \in E} c_{xy} [f(x) - f(y)]^2 \overset{\text{or}}{=} \sum_{x \in V} c_x f(x) (\mathbf{L} f)(x),$$

where $(\mathbf{L}f)(x) = \sum_y \mathbf{L}_{xy} f(y)$.

Connection to electric networks

- Given two vertices $x, y \in V$, consider the class of "voltage" functions $f : V \to \mathbb{R}$ with the property that $f(x) = 1$ and $f(y) = 0$ (think a voltage drop of 1 is imposed across x and y). Does there exist a unique minimizer of the energy $\mathcal{E}(f)$?
- **Answer:** Yes! The minimizer is the function h which is the unique solution to

$$\begin{cases} (\mathbf{L}h)(z) = 0 & \text{if } z \in V \setminus \{x, y\}, \\ h(x) = 1, & h(y) = 0. \qquad (*) \end{cases}$$

  h is said to be the **harmonic** function satisfying the (Dirichlet) boundary condition $(*)$.
- In the language of electric networks, $\mathcal{E}(h)$ gives the **effective conductance** between x and y, denoted $c_{\mathrm{eff}}(x, y)$. Its reciprocal is the **effective resistance**, $R_{\mathrm{eff}}(x, y) = [c_{\mathrm{eff}}(x, y)]^{-1}$.

Theorem (Dirichlet's principle† 1867). For any two vertices $x, y \in V$ and any function $f : V \to \mathbb{R}$,

$$\mathcal{E}(f) \geq c_{\mathrm{eff}}(x, y)[f(x) - f(y)]^2.$$

Equality is attained iff f is harmonic on $V \setminus \{x, y\}$.

(†According to Doyle & Snell, it was actually Thomson (Lord Kelvin) who proved this inequality.)

There are multiple ways to prove Dirichlet's principle. I will introduce one which uses the idea of **network reduction**: Remove vertices (and edges attached to them) without changing the effective conductance between any of the non-removed vertices.

- Suppose we remove the vertex $x \in V$ from $(G, \mathbf{c})$, as well as the edges attached to x. Call the reduced graph $G_x = (V_x, E_x)$. In the linear algebra language, we will reduce the Laplacian $\mathbf{L}$ to a new Laplacian $\mathbf{L}'$ (of one fewer dimension). This is attained by taking the Schur complement of the (x, x) block in $\mathbf{L}$:

$$\text{If } \mathbf{L} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{L}_{xx} \end{bmatrix}, \quad \text{then } \mathbf{L}' = \mathbf{X} - \mathbf{Y}(\mathbf{L}_{xx})^{-1}\mathbf{Z} = \mathbf{X} - \mathbf{YZ}. \qquad (\text{Recall } \mathbf{L}_{xx} = -1.)$$

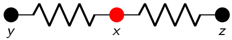
- In component form, $\mathbf{L}'_{yz} = \mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}$ for $y, z \in V_x$. Since $\mathbf{L}^{(')}_{yz} = -p^{(')}_{yz} = -\frac{c^{(')}_{yz}}{c_y}$ whenever $y \neq z$, we see that the new conductances on E_x become

$$c'_{yz} = -c_y \mathbf{L}'_{yz} = -c_y(\mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}) = c_{yz} + \frac{c_{yx}c_{xz}}{c_x}.$$

Proposition. Upon network reduction (by removing x from $(G, \mathbf{c})$), the conductance on each edge in E_x increases by

$$\tilde{c}_{yz} := c'_{yz} - c_{yz} = \frac{c_{yx}c_{xz}}{c_x}.$$

Example 1: Series Law



Let $c_{xy} = \alpha$ and $c_{xz} = \beta$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} {\color{blue}1} & 0 & -1 \\ 0 & {\color{blue}1} & -1 \\ -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} & {\color{red}1} \end{bmatrix}.$$

Let $\mathbf{L}'$ be the Schur complement of the 1 block in $\mathbf{L}$:

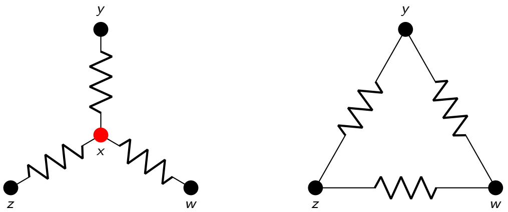
$$\mathbf{L}' = \begin{bmatrix} {\color{blue}1} & 0 \\ 0 & {\color{blue}1} \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \\ -\frac{\alpha}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

So $\mathbf{L}'_{yz} = -\frac{\beta}{\alpha+\beta}$. Since $c_y = \alpha$, we get $c'_{yz} = -c_y \mathbf{L}_{yz} = \dfrac{\alpha\beta}{\alpha+\beta}$, *i.e.*,

$$R'_{yz} = \frac{1}{c'_{yz}} = \frac{1}{\alpha} + \frac{1}{\beta} = R_{xy} + R_{xz}.$$

(Resistors in series ADD!)

Example 2: Y-Δ transform



Let $c_{xy} = \alpha$, $c_{xz} = \beta$, $c_{xw} = \gamma$, and $\sigma = \alpha + \beta + \gamma$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \alpha/\sigma & \beta/\sigma & \gamma/\sigma & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma & 1 \end{bmatrix}.$$

$$\mathbf{L}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma \end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix} \beta+\gamma & -\beta & -\gamma \\ -\alpha & \alpha+\gamma & -\gamma \\ -\alpha & -\beta & \alpha+\beta \end{bmatrix}.$$

After a little more algebra we get

$$c'_{yz} = \frac{\alpha\beta}{\sigma}, \quad c'_{zw} = \frac{\beta\gamma}{\sigma}, \quad c'_{wy} = \frac{\gamma\alpha}{\sigma}.$$

(Anyone who studied electric circuits would find this familiar!)

Proof of Dirichlet's principle via network reduction

$$\mathcal{E}(f) = \sum_{xy \in E} c_{xy}[f(x) - f(y)]^2.$$

In going from G to the reduced graph G_x, energy is

- lost due to the removal of edges attached to x: amount $\sum_{y \in V_x} c_{xy}[f(x) - f(y)]^2$.
- gained due to the increased conductance on the non-removed edges: amount $\sum_{yz \in E_x} \tilde{c}_{yz}[f(y) - f(z)]^2$.

Proposition ("Octopus inequality" for electric network). For all $f : V \to \mathbb{R}$,

$$\sum_{y \in V_x} c_{xy}[f(x) - f(y)]^2 \geq \sum_{yz \in E_x} \tilde{c}_{yz}[f(y) - f(z)]^2,$$

Energy lost from removed edges $\geq$ Energy gained from increased conductances

where equality is attained iff $(\mathbf{L}f)(x) = 0$.

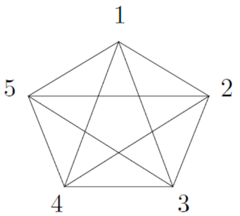
Proof. An exercise in high school algebra.

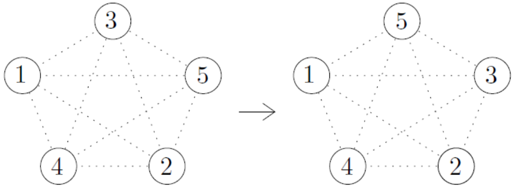
Corollary. The Dirichlet energy is monotone non-increasing upon successive network reductions.

By carrying out network reduction one vertex at a time until two vertices z and y are left, we recover Dirichlet's principle: $\mathcal{E}(f) \geq c_{\rm eff}(z, y)[f(z) - f(y)]^2$.

Why the name "octopus"? The tentacular nature of removing of a vertex and its edges may remind you of an octopus.

- What I stated on the last slide is very classical, and probably does not deserve the cute name.
- The genesis of the real "octopus inequality" comes from the study of the interchange process.





The corresponding energy on an n-vertex weighted graph is

$$\mathcal{E}^{\mathrm{IP}}(f) = \sum_{xy \in E} c_{xy} \sum_{\eta \in \mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2,$$

where $\mathcal{X}_n$ is the space of permutations on $\{1, 2, \cdots, n\}$, id'ed with V, and η^{xy} is obtained from η by transposing x and y.

The (remarkable) octopus inequality

Theorem (Caputo, Liggett, Richthammer 2009, published in *JAMS* 2010).

For all $f : \mathcal{X}_n \to \mathbb{R}$,

$$\sum_{y \in V_x} c_{xy} \sum_{\eta \in \mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2 \geq \sum_{yz \in E_x} \tilde{c}_{yz} \sum_{\eta \in \mathcal{X}_n} [f(\eta^{yz}) - f(\eta)]^2.$$

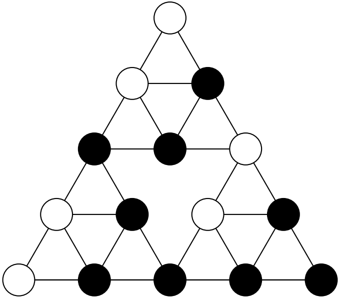
Energy lost from removed edges $\geq$ Energy gained from increased conductances

(Their) Proof. A careful linear algebra analysis using Schur complements.

- **Why remarkable?** Using this theorem they were able to solve a famous open problem in discrete probability (Aldous' spectral gap conjecture, posed in 1992).
- **MathSciNet review of this work:** "One leaves this beautiful paper with the dream that maybe a simpler proof could be found." MR2629990
- Applying successive network reductions till two vertices z and y are left, we see that the octopus inequality implies the following "moving particle lemma" for the interchange process

$$\mathcal{E}^{\rm IP}(f) \geq c_{\rm eff}(x, y) \sum_{\eta \in \mathcal{X}_n} [f(\eta^{xy}) - f(\eta)]^2.$$

- The exclusion process involves k random walkers on a graph subject to the constraint that no 2 walkers can occupy the same vertex. This process has been long studied by many probabilists, since the model is very simple to state. But many open problems remain.



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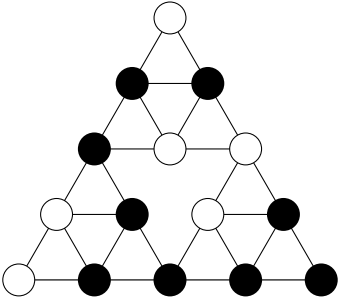

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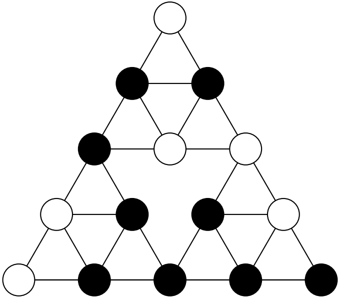

The exclusion process

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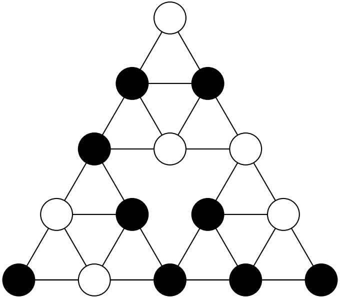

Theorem (C. 2016). The "moving particle lemma" for the exclusion process [arXiv:1606.01577, forthcoming in *Electron. Comm. Probab.*]

For all $f : \{0,1\}^V \to \mathbb{R}$,

$$\underbrace{\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [f(\eta^{zw}) - f(\eta)]^2 \nu(d\eta)}_{=\mathcal{E}^{\mathrm{EX}}(f)} \geq c_{\mathrm{eff}}(x,y) \int_{\{0,1\}^V} [f(\eta^{xy}) - f(\eta)]^2 \nu(d\eta),$$

where ν is the product Bernoulli measure on $\{0,1\}^V$ with constant one-site marginal $\nu\{\eta : \eta(x) = 1\} = \alpha$ for all $x \in V$. (When $\alpha = \frac{1}{2}$, ν is the uniform probability measure.)

(My) Proof. Network reduction + octopus inequality + projection from the interchange process to the exclusion process.

- This theorem overcomes a technical obstacle in proving the scaling limit of the exclusion processes on (non-translationally-invariant) weighted graphs.

  The (empirical) particle density (measure) converges upon suitable space-time scaling to a measure whose density solves a (weakly nonlinear) heat equation $\partial_t \rho = \Delta\rho + (\text{nonlinearity})$.

  [Known results on $\mathbb{Z}^d$: Guo, Kipnis, Papanicolaou, Varadhan, Landim, *et al.* 1987∼199x. Recent work involves extensions to the boundary-driven exclusion process, exclusion with asymmetric jump rates, etc.]

  [NEW results on fractals & strongly recurrent graphs: C., Hinz, Teplyaev 2017+. A short summary already appears in arXiv:1702.03376.]

Thank you!