PHASE TRANSITION IN THE EXCLUSION PROCESS ON THE SIERPINSKI GASKET WITH SLOWED BOUNDARY RESERVOIRS

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Abstract. We derive the macroscopic laws that govern the evolution of the density of particles in the exclusion process evolving on the Sierpinski gasket in the presence of a slow boundary. Depending on the slowness of the boundary we obtain, at the hydrodynamics level, the heat equation evolving on the Sierpinski gasket with either Dirichlet or Neumann boundary conditions, depending on whether the reservoirs are fast or slow. For a particular strength of the boundary dynamics we obtain linear Robin boundary conditions. As for the fluctuations, we also prove that, when starting from the stationary measure, namely the product Bernoulli measure in the equilibrium setting, they are governed by Ornstein-Uhlenbeck processes with the respective boundary conditions.

1. Introduction

The purpose of this article is to derive the macroscopic laws that govern the evolution of the thermodynamical quantities of an interacting particle system (IPS) evolving on a non-lattice, non-translationally-invariant state space. The IPS were introduced in the mathematics community by Spitzer in [Spi70] (but were already known to physicists) as microscopic stochastic systems, whose dynamics conserves a certain number of thermodynamical quantities of interest. Depending on whether one is looking at the Law of Large Numbers or the Central Limit Theorem, the macroscopic laws can have different nature, either partial differential equations (PDEs) or stochastic PDEs. Over the last years there have been many studies around microscopic models whose dynamics conserves one or more quantity of interest, and the goal in the so-called *hydrodynamic limit* is to make rigorous the derivation of these PDEs by means of a scaling argument procedure.

One of the intriguing questions in the field of IPS is to understand how a local microscopic perturbation of the system has an impact at the level of its macroscopic behavior. In recent years, many articles have been devoted to the study of 1D microscopic systems in presence of a slow bond [FGN13], a slow site [FGS16], or a slow/fast boundary [BMNS17, BGJO17, BGJO18]; see also references therein.

In this article, we analyze the same type of problems when the microscopic system evolves on a fractal which has spatial dimension > 1. Our chosen fractal is the Sierpinski gasket, and the microscopic stochastic dynamics is the classical exclusion process that we describe as follows. Consider the exclusion process evolving on a discretization of the gasket, that is on a level-N approximating graph denoted by $\mathcal{G}_N = (V_N, E_N)$, where V_N is the set of vertices and E_N denotes the set of edges; see Figure 1. The exclusion process on $\mathcal{G}_N$ is a continuous-time Markov process denoted by $\{\eta_t^N : t \geq 0\}$ with state space $\Omega_N = \{0,1\}^{V_N}$. Its dynamics is defined as follows. For every two pair of vertices $x, y \in V_N$ which are connected by an edge, we place Poisson processes of rate 1, whose role is to exchange the occupation variables at the sites x and y. Nevertheless, the exclusion rule dictates that exchanges between x and y only occur when one of the sites is empty and the other one is occupied. Otherwise nothing happens.

On the vertices of $V_0 = \{a_0, a_1, a_2\}$, we attach three extra vertices $\{A_i\}_{i=0}^2$ whose role is to mimic the reservoirs' action. This means that each one of these extra vertices A_i can inject (resp. remove) particles into (resp. from) the corresponding vertex a_i at a rate $\lambda_+(a_i)$ (resp. $\lambda_-(a_i)$); see Figure 2. In order to have a nontrivial limit, we speed up the process in the time scale 5^N. Furthermore, to analyze the impact of

Date: April 17, 2019.

2010 *Mathematics Subject Classification.* (Primary) 35K05; 35R60; 60H15; 60K35; 82C22; (Secondary) 28A80; 60J27.

Key words and phrases. Exclusion process, hydrodynamic limit, equilibrium fluctuations, heat equation, Ornstein-Uhlenbeck equation, analysis on fractals, Dirichlet forms, moving particle lemma, phase transition.

JPC thanks the Simons Foundation (Collaboration Grant for Mathematicians #523544) and the Research Council of Colgate University for support. PG thanks FCT/Portugal for support through the project UID/MAT/04459/2013. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovative programme (grant agreement No 715734).

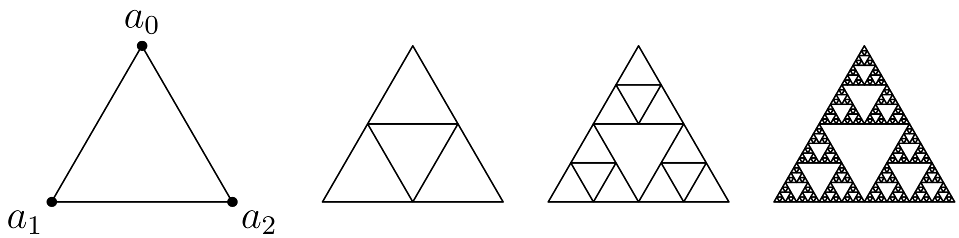

FIGURE 1. The level-N approximating graph $\mathcal{G}_N$ of the Sierpinski gasket, for $N = 0, 1, 2, 5$ (from left to right).

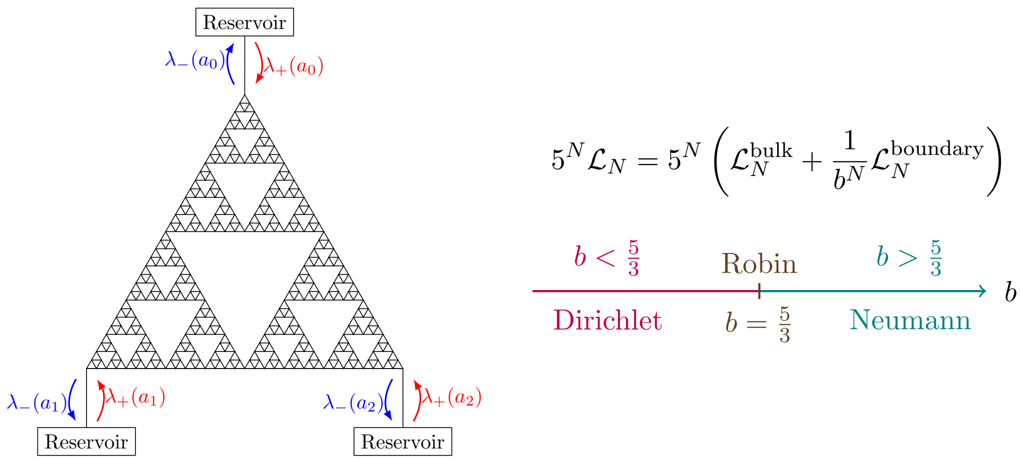

FIGURE 2. A schematic of the boundary-driven exclusion process on the Sierpinski gasket (left); and the scaling regimes determined by the inverse strength b of the reservoirs' dynamics (right).

changing the reservoirs' dynamics, we scale it by a factor $1/b^N$ for some $b > 0$. The precise definition of the infinitesimal generator of this Markov process is given in (2.4). If the reservoirs $\{A_i\}_i$ did not exist, then the exclusion dynamics would conserve the total number of particles; therefore the quantity that we want to analyze is the density of particles. Putting the system in contact with reservoirs, the form of the hydrodynamic equation is unaffected in the bulk, but the reservoirs' dynamics will manifest itself in the boundary conditions on V_0.

Our aim is to analyze the hydrodynamic limit, the fluctuations of this process, and their dependence on the parameter b which governs the strength of the reservoirs. As we are working with an exclusion process whose jump rates are equal to 1 between connected vertices, we expect to obtain the heat equation on the Sierpinski gasket, but with certain types of boundary conditions.

In general terms, the goal in the *hydrodynamic limit* is to show that starting the process from a collection of measures $\{\mu_N\}_N$ for which the Law of Large Numbers holds—that is, the random measure $\pi_0^N(\eta^N) = \frac{1}{|V_N|}\sum_{x\in V_N}\eta_0^N(x)\delta_{\{x\}}$ converges, in probability with respect to μ_N and when $N\to\infty$, to the deterministic measure $\varrho(x)\,m(dx)$, where $\varrho(\cdot)$ is a function defined on the Sierpinksi gasket, and m is the standard self-similar measure on the gasket—then, the same holds at later times $t > 0$—that is, the random measure $\pi_t^N(\eta^N) = \frac{1}{|V_N|}\sum_{x\in V_N}\eta_{t5^N}^N(x)\delta_{\{x\}}$ converges, in probability with respect to $\mu_N(t)$, the distribution of η_{t5^N}, and as $N\to\infty$, to $\rho_t(x)\,m(dx)$, where $\rho_t(\cdot)$ is the solution (in the weak sense) of the hydrodynamic equation of the system.

For the model that we consider here, we obtain as hydrodynamic equations the heat equation with Dirichlet, Robin, or Neumann boundary conditions, depending on whether $b < 5/3$, $b = 5/3$, and $b > 5/3$, respectively; see again Figure 2. Our method of proof is the classical entropy method of [GPV88], which relies on showing tightness of the sequence $\{\pi_\cdot^N\}_N$ and to characterize uniquely the limit point $\pi_\cdot$. Once uniqueness is proved, the convergence follows. To prove uniqueness we need to associate to the random measure π_t^N a collection of martingales M_t^N which correspond to a random discretization of the solution of the PDE. By means of several replacement lemmas, we are able to recognize the limit of the sequence M_t^N as a weak solution to the corresponding hydrodynamic equation.

What we just mentioned is a Law of Large Numbers for the random measure π_t^N, the *empirical density measure*. Another question we address in this article is related to the Central Limit Theorem. To wit, consider the system starting from the stationary measure. We observe that when the reservoirs' rates are all identical, *i.e.,* $\lambda_+(a_i) = \lambda_+$ and $\lambda_-(a_i) = \lambda_-$ for all $i = 0, 1, 2$, then the product Bernoulli measures ν_ρ^N with $\rho := \lambda_+/(\lambda_+ + \lambda_-)$ are reversible for $\{\eta_t^N : t \geq 0\}$. Without the identical rates condition ν_ρ^N are no longer invariant. Nevertheless, since we work with an irreducible Markov process on a finite state space, we know that the invariant measure is unique. The characterization of this measure is so far out of reach, and we leave this issue for a future work. That said, we observe that, from the result on our hydrodynamic limit, we cannot say that it holds when we start from the stationary measure (the so called *hydrostatic limit*) in the case when $\lambda_-(a) \neq \lambda_+(a)$, $a \in V_0$. Since we do not know that the stationary destiny correlations vanish as $N \to \infty$, we cannot prove that starting from the stationary measure, the empirical measure converges to $\bar{\rho}(x)dx$, where $\bar{\rho}(\cdot)$ is the stationary solution of our hydrodynamic equations.

Given the outstanding technical obstacles, we decide for the moment to analyze the Central Limit Theorem for π_t^N only in the case when $\lambda_+(a_i) = \lambda_+$ and $\lambda_-(a_i) = \lambda_-$ for all $i = 0, 1, 2$. Then we start from the product Bernoulli measure ν_ρ^N where $\rho = \lambda_+/(\lambda_+ + \lambda_-)$. We define the *density fluctuation field* $\mathcal{Y}_t^N$ which acts on test functions F as $\mathcal{Y}_t^N(F) = |V_N|^{-\frac{1}{2}}(\pi_t^N(F) - \mathbb{E}_{\nu_\rho^N}[\pi_t^N(F)])$, where $\pi_t^N(F)$ denotes the integral of F with respect to the random measure $\pi_t^N(\eta)$. We prove that for a suitable space of test functions, the density fluctuation field converges to the unique solution of the Ornstein-Uhlenbeck process on the gasket. The method of proof goes by showing tightness of the sequence $\{\mathcal{Y}_\cdot^N\}_N$ and to characterize uniquely the limit point $\mathcal{Y}_\cdot$ as the solution of an Ornstein-Uhlenbeck equation with the respective boundary conditions.

Now we comment on our chosen fractal, the Sierpinski gasket. In §9 we describe possible generalizations of our work to other fractals. More precisely, the results that we have obtained here can be adapted to other post-critically finite self-similar fractals as defined in [Bar98, Kig01], and more generally, to *resistance spaces* introduced by Kigami [Kig03]. What is most important for our proof to work is to have discrete analogues of the Laplacians and of energy forms on the underlying graph, and good rates of convergence of discrete operators to their continuous versions. Meanwhile, we also need a method to perform local averaging of the particle density on a graph which lacks translational invariance. This is made possible through a functional inequality called the *moving particle lemma*, which holds on any graph approximation of a resistance space. See [Che17] for the proof, as well as its connection to the *octopus inequality* of Caputo, Liggett, and Richthammer [CLR10], which was key to the positive resolution of Aldous' spectral gap conjecture.

Regarding our choice of the interacting particle system, the exclusion process, we believe that our proof can be carried out to more general dynamics with asymmetric rates or long-range interactions. Due to the length of the present paper, we leave the details of these generalizations to a future work. On a historical note, Jara [Jar09] had studied the boundary-driven zero-range process on the Sierpinski gasket, and obtained the density hydrodynamic limit using the H_{-1}-norm method [CY92, GQ00].

We also point out that a natural extension of the fluctuations result to the non-equilibrium setting is being studied [CFGM19], and the main issue is to have a good decay of the space-time correlations. As a consequence of the study of the correlations, we will be able to prove the hydrostatic limit, which we leave here as an open problem. Another work in the non-equilibrium setting, concerning a large deviations principle for the empirical density, appears as [CH19].

The rest of the paper is organized as follows. In §2 we formally define the boundary-driven exclusion process on the Sierpinski gasket. In §3 we state the hydrodynamic limit theorem for the empirical density (Theorem 1), exhibiting the three limit regimes: Neumann, Robin, and Dirichlet. In §4 we state the convergence of the equilibrium density fluctuation field to the Ornstein-Uhlenbeck equation with appropriate boundary condition (Theorem 2). In §5 we establish several replacement lemmas on the Sierpinski gasket, which form the technical core of the paper. We then prove Theorem 1 in §6 and §7, and Theorem 2 in §8. Generalizations to mixed boundary conditions on SG, as well as to other state spaces, are described in §9. Appendices A and B summarize several key results from analysis on fractals which are needed for this article.

2. Model

2.1. **Sierpinski gasket.** Consider the iterated function system (IFS) consisting of three contractive similitudes $\mathfrak{F}_i : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathfrak{F}_i(x) = \frac{1}{2}(x - a_i) + a_i$, $i \in \{0, 1, 2\}$, where $\{a_i\}_{i=0}^2$ are the three vertices of an equilateral triangle of side 1. The Sierpinski gasket K is the unique fixed point under this IFS: $K = \bigcup_{i=0}^2 \mathfrak{F}_i(K)$. Set $V_0 = \{a_0, a_1, a_2\}$. Given a word $w = w_1 w_2 \dots w_j$ of length $|w| = j$ drawn from the alphabet $\{0, 1, 2\}$, we define $\mathfrak{F}_w := \mathfrak{F}_{w_1} \circ \mathfrak{F}_{w_2} \circ \cdots \circ \mathfrak{F}_{w_j}$. Set $K_w := \mathfrak{F}_w(K)$, which we call a j-cell if $|w| = j$. Also set $V_N := \bigcup_{|w|=N} \mathfrak{F}_w(V_0)$, and $V_* := \bigcup_{N \geq 0} V_N$. We then introduce the approximating Sierpinski gasket graph of level N, $\mathcal{G}_N = (V_N, E_N)$, where two vertices x and y are connected by an edge (denoted $xy \in E_N$ or $x \sim y$) iff there exists a word w of length N such that $x, y \in \mathfrak{F}_w(V_0)$.

Let m_N be the uniform measure on V_N, charging each vertex $x \in V_N$ a mass $|V_N|^{-1} = (\frac{3}{2}(3^N + 1))^{-1}$: this explains the appearance of the prefactor $\frac{2}{3}$ in the results to follow. It is a standard argument that m_N converges weakly to m, the self-similar probability measure on K, which is a constant multiple of the d_H-dimensional Hausdorff measure with $d_H = \log_2 3$ in the Euclidean metric. From now on we fix our measure space (K, m).

The starting point of analysis on fractals is the construction of Dirichlet forms on $L^2(K, m)$. Define the renormalized graph energy

$$\mathcal{E}_N(f) = \frac{5^N}{3^N} \frac{1}{2} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} (f(x) - f(y))^2, \quad f \in L^2(K, m). \tag{2.1}$$

It can be shown that for each fixed $f \in L^2(K, m)$, the sequence $\{\mathcal{E}_N(f)\}_N$ is monotone increasing, so it either converges to a finite limit or diverges to $+\infty$. We thus define

$$\mathcal{E}(f) = \lim_{N \to \infty} \mathcal{E}_N(f) \tag{2.2}$$

with natural domain

$$\mathcal{F} := \{f \in L^2(K, m) : \mathcal{E}(f) < +\infty\}. \tag{2.3}$$

To produce a quadratic form we use the polarization formula $\mathcal{E}(f, g) = \frac{1}{4} (\mathcal{E}(f + g) + \mathcal{E}(f - g))$. In fact, $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(K, m)$, and by the general theory [FOT11] it has an associated Feller diffusion process on K.

A word about the prefactor $\frac{5}{3}$ in (2.1): the 3 derives from the volume (measure) scaling, while the 5 comes from the time scale of random walks on $\mathcal{G}_N$. To be precise, the expected time for a random walk $\{X_t^N : t \geq 0\}$ started from a_0 to hit $\{a_1, a_2\}$ equals 5^N on $\mathcal{G}_N$: this is a simple one-step Markov chain calculation which can be found in *e.g.* [Bar98, Lemma 2.16]. It is by now a well-known result [BP88] that the sequence of rescaled random walks $\{X_{t5^N}^N : t \geq 0\}_N$ is tight in law and in resolvent, and converges to a unique (up to deterministic time change) Markov process $\{X_t : t \geq 0\}$, which agrees with the aforementioned Feller diffusion (or "Brownian motion") on K.

2.2. **Exclusion process on the Sierpinski gasket.** The *boundary-driven symmetric simple exclusion process* (SSEP) on $\mathcal{G}_N$ is a continuous-time Markov process on $\Omega_N := \{0, 1\}^{V_N}$ with generator

$$5^N \mathcal{L}_N = 5^N \left(\mathcal{L}_N^{\text{bulk}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boundary}} \right), \tag{2.4}$$

where for all functions $f : \Omega_N \to \mathbb{R}$,

$$\left(\mathcal{L}_N^{\text{bulk}} f \right)(\eta) = \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} \eta(x)[1 - \eta(y)] \left[f(\eta^{xy}) - f(\eta) \right], \tag{2.5}$$

$$\left(\mathcal{L}_N^{\text{boundary}} f \right)(\eta) = \sum_{a \in V_0} \left[\lambda_-(a)\eta(a) + \lambda_+(a)(1 - \eta(a)) \right] \left[f(\eta^a) - f(\eta) \right]. \tag{2.6}$$

Here 5^N is the aforementioned diffusive time scaling on SG; $b > 0$ is a scaling parameter which indicates the inverse strength of the reservoirs' dynamics relative to the bulk dynamics; $\lambda_+(a) > 0$ (resp. $\lambda_-(a) > 0$)

is the birth (resp. death) rate of particles at the boundary vertex $a \in V_0$, and is fixed for all N;

$$\eta^{xy}(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases} \quad \text{and} \quad \eta^{a}(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases} \tag{2.7}$$

See Figure 2 for a schematic. For convenience, we denote the sum of the boundary birth and death rates at $a \in V_0$ by $\lambda_\Sigma(a) := \lambda_+(a) + \lambda_-(a)$.

3. Hydrodynamic limits: Statement of results

In this section we state the hydrodynamic limit for our model. For that purpose, we first introduce all the partial differential equations that will be derived.

3.1. Laplacian & integration by parts.

Definition 3.1 (Laplacian)**.** Let $\mathrm{dom}\Delta$ denote the space of functions $u \in \mathcal{F}$ for which there exists $f \in C(K)$ such that

$$\mathcal{E}(u,\varphi) = \int_K f\varphi\, dm \quad \text{for all } \varphi \in \mathcal{F}_0 := \{g \in \mathcal{F} : g|_{V_0} = 0\}, \tag{3.1}$$

We then write $-\Delta u = f$, and call $\mathrm{dom}\Delta$ the domain of the Laplacian associated with $(\mathcal{E}, \mathcal{F})$ on $L^2(K,m)$. Also we define the domain of the Laplacian with Dirichlet boundary condition on V_0, $\mathrm{dom}\Delta_0 = \{u \in \mathrm{dom}\Delta : u|_{V_0} = 0\}$.

Definition 3.1 is the weak formulation of the Laplacian. A pointwise formulation is also available, and the reader is referred to [Kig01, §3.7] or [Str06, §2.2] for details. The following Lemma will be used repeatedly in this paper:

Lemma 3.2 ([Kig01, Str06])**.** *If $u \in \mathrm{dom}\Delta$, then:*

(1) $\frac{3}{2}\Delta_N u \to \Delta u$ uniformly on $K \setminus V_0$.
(2) For every $a \in V_0$, $(\partial^\perp u)(a) := \lim_{N\to\infty}(\partial^\perp_N u)(a)$ exists.
(3) (Integration by parts formula)

$$\mathcal{E}(u,\varphi) = \int_K (-\Delta u)\varphi\, dm + \sum_{a\in V_0} (\partial^\perp u)(a)\varphi(a) \quad \text{for all } \varphi \in \mathcal{F}. \tag{3.2}$$

Above the discrete Laplacian on the bulk and the outward discrete normal derivative on the boundary read as follows:

$$(\Delta_N F)(x) = 5^N \sum_{\substack{y\in V_N \\ y\sim x}} (F(y) - F(x)) \quad \text{for } x \in V_N \setminus V_0, \tag{3.3}$$

$$(\partial^\perp_N F)(a) = \frac{5^N}{3^N} \sum_{\substack{y\in V_N\setminus V_0 \\ y\sim a}} (F(a) - F(y)) \quad \text{for } a \in V_0. \tag{3.4}$$

For $f, g : K \to \mathbb{R}$ we define

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + \langle f, g\rangle_{L^2(K,m)}. \tag{3.5}$$

Then $\mathcal{F}$ endowed with the inner product $\mathcal{E}_1(\cdot,\cdot)$ is a Hilbert space. This allows us to further define the space $L^2(0,T,\mathcal{F})$, which is the space where our solutions will live. For $F, G : [0,T] \times K \to \mathbb{R}$ define

$$\langle F, G\rangle_{L^2(0,T,\mathcal{F})} := \int_0^T \mathcal{E}_1(F_s, G_s)\, ds, \quad \|F\|_{L^2(0,T,\mathcal{F})} := \left(\int_0^T \mathcal{E}_1(F_s)\, ds\right)^{1/2}. \tag{3.6}$$

Then $L^2(0,T,\mathcal{F})$ with the inner product $\langle\cdot,\cdot\rangle_{L^2(0,T,\mathcal{F})}$ is a Hilbert space.

3.2. Weak formulation of the heat equations.

Definition 3.3 (Heat equation with Dirichlet boundary condition)**.** We say that ρ is a weak solution to the heat equation with *Dirichlet* boundary conditions started from a measurable function $\varrho : K \to [0,1]$,

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ \rho(t,a) = \bar\rho(a), & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K, \end{cases} \tag{3.7}$$

if the following conditions are satisfied:

(1) $\rho \in L^2(0,T,\mathcal{F})$.

(2) ρ satisfies the weak formulation of (3.7): for any $t \in [0,T]$ and $F \in C([0,T], \mathrm{dom}\Delta_0) \cap C^1((0,T), \mathrm{dom}\Delta_0)$,

$$\begin{aligned} \Theta_{\mathrm{Dir}} := &\int_K \rho_t(x) F_t(x)\, dm(x) - \int_K \varrho(x) F_0(x)\, dm(x) \\ &- \int_0^t \int_K \rho_s(x) \left(\frac{2}{3}\Delta + \partial_s\right) F_s(x)\, dm(x)\, ds + \frac{2}{3}\int_0^t \sum_{a\in V_0} \bar\rho(a)(\partial^\perp F_s)(a)\, ds = 0. \end{aligned} \tag{3.8}$$

(3) $\rho(t,a) = \bar\rho(a)$ for a.e. $t \in (0,T]$ and for all $a \in V_0$.

Definition 3.4 (Heat equation with Robin boundary condition)**.** We say that ρ is a weak solution to the heat equation with *Robin* boundary condition started from a measurable function $\varrho : K \to [0,1]$,

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ \partial^\perp \rho(t,a) = -\bar\lambda_\Sigma(a)(\rho(t,a) - \bar\rho(a)), & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K, \end{cases} \tag{3.9}$$

if the following conditions are satisfied:

(1) $\rho \in L^2(0,T,\mathcal{F})$.

(2) ρ satisfies the weak formulation of (3.9): for any $t \in [0,T]$ and $F \in C([0,T], \mathrm{dom}\Delta) \cap C^1((0,T), \mathrm{dom}\Delta)$,

$$\begin{aligned} \Theta_{\mathrm{Rob}} := &\int_K \rho_t(x) F_t(x)\, dm(x) - \int_K \varrho(x) F_0(x)\, dm(x) - \int_0^t \int_K \rho_s(x) \left(\frac{2}{3}\Delta + \partial_s\right) F_s(x)\, dm(x)\, ds \\ &+ \frac{2}{3}\int_0^t \sum_{a\in V_0} \left[\rho_s(a)(\partial^\perp F_s)(a) + \bar\lambda_\Sigma(a)(\rho_s(a) - \bar\rho(a))F_s(a)\right] ds = 0. \end{aligned} \tag{3.10}$$

Definition 3.5 (Heat equation with Neumann boundary condition)**.** We say that ρ is a weak solution to the heat equation with *Neumann* boundary condition started from a measurable function $\varrho : K \to [0,1]$ if ρ satisfies Definition 3.4 with $\bar\lambda_\Sigma(a) = 0$ for all $a \in V_0$.

Lemma 3.6. *There exists a unique weak solution of* (3.7) *(resp.* (3.9)*) in the sense of Definition 3.3 (resp. Definition 3.4).*

Proof. See §7. □

3.3. Hydrodynamic limits.

Definition 3.7. We say that a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on Ω_N is associated to a density profile $\varrho : K \to [0,1]$ if for any continuous function $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \Omega_N \ :\ \left| \frac{1}{|V_N|} \sum_{x\in V_N} F(x)\eta(x) - \int_K F(x)\varrho(x)\, dm(x) \right| > \delta \right\} = 0. \tag{3.11}$$

We now state our first main theorem. Given the process $\{\eta_t^N : t \geq 0\}$ generated by $5^N \mathcal{L}_N$, we define the empirical density measure π_t^N given by

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) \delta_{\{x\}} \tag{3.12}$$

and for any test function $F : K \to \mathbb{R}$ whose domain will be specified later on, we denote the integral of F with respect to π_t^N by $\pi_t^N(F)$ which equals

$$\pi_t^N(F) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x). \tag{3.13}$$

Theorem 1 (Hydrodynamic limits)**.** *Let $\varrho : K \to [0,1]$ be measurable, and $\{\mu_N\}_N$ be a sequence of probability measures on Ω_N which is associated to ϱ. Then for any $t \in [0,T]$, any continuous function $F : K \to \mathbb{R}$, and any $\delta > 0$, we have*

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \;:\; \left| \frac{1}{|V_N|} \sum_{x \in V_N} F(x)\eta_t^N(x) - \int_K F(x)\rho(t,x)\,dm(x) \right| > \delta \right\} = 0, \tag{3.14}$$

where ρ is the unique weak solution of:

- *the heat equation with Dirichlet boundary condition (Definition 3.3), if $b < 5/3$;*
- *the heat equation with Robin boundary condition (Definition 3.4) with $\bar\lambda_\Sigma(a) = \lambda_\Sigma(a)$, if $b = 5/3$;*
- *the heat equation with Neumann boundary condition (Definition 3.5), if $b > 5/3$.*

3.4. **Heuristics for hydrodynamic equations.** In this subsection we present a heuristic argument, based on martingales associated to the empirical measure, to deduce the aforementioned weak solutions. In order to simplify the exposition, let us fix a time-independent function $F : K \to \mathbb{R}$. By Dynkin's formula, see, for example, Lemma A.1.5.1 of [KL99], the process

$$M_t^N(F) = \pi_t^N(F) - \pi_0^N(F) - \int_0^t 5^N \mathcal{L}_N \pi_s^N(F)\,ds \tag{3.15}$$

is a martingale with quadratic variation

$$\langle M^N(F)\rangle_t = \int_0^t 5^N \left[\mathcal{L}_N \left(\pi_s^N(F)\right)^2 - 2\pi_s^N(F)\mathcal{L}_N \pi_s^N(F) \right] ds. \tag{3.16}$$

An elementary calculation shows that

$$\begin{aligned} 5^N \mathcal{L}_N \pi_t^N(F) &= \frac{1}{|V_N|} \sum_{x \in V_N \setminus V_0} \eta_t^N(x)(\Delta_N F)(x) \\ &\quad - \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_t^N(a)(\partial_N^\perp F)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_t^N(a) - \bar\rho(a))F(a) \right], \end{aligned} \tag{3.17}$$

where the stationary density on the boundary reads as follows:

$$\bar\rho(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)} \quad \text{for } a \in V_0. \tag{3.18}$$

Another simple computation shows that the quadratic variation writes as

$$\begin{aligned} \langle M^N(F)\rangle_t &= \int_0^t \frac{5^N}{|V_N|^2} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} (\eta_s^N(x) - \eta_s^N(y))^2 (F(x) - F(y))^2 ds \\ &\quad + \int_0^t \sum_{a \in V_0} \frac{5^N}{b^N |V_N|^2} \{\lambda_-(a)\eta_s^N(a) + \lambda_+(a)(1 - \eta_s^N(a))\} F^2(a) ds. \end{aligned} \tag{3.19}$$

Let us take a moment to discuss the boundary term in (3.17). If $b \geq 5/3$, the second term in the square bracket is at most of order unity, regardless of the value of $F(a)$. On the other hand, if $b < 5/3$, the scaling parameter $5^N/(3^N b^N)$ diverges as $N \to \infty$. The only way to get around this is to impose $F(a) = 0$ for all $a \in V_0$. This analysis will inform us of the function space from which F is drawn.

With Lemma 3.2 in mind, we will now insist that the test function F belong to $\mathrm{dom}\Delta$. By Part (1) of the Lemma, $\Delta F := \lim_{N\to\infty} \frac{3}{2}\Delta_N F$ is uniformly continuous on $K \setminus V_0$, a precompact set. Therefore we can extend ΔF continuously from $K \setminus V_0$ to K, and we denote the continuous extension by ΔF still. As a consequence, we can rewrite the first term on the RHS of (3.17) as

(3.20)
$$\frac{1}{|V_N|}\sum_{x\in V_N\setminus V_0}\eta_t^N(x)(\Delta_N F)(x) = \frac{1}{|V_N|}\sum_{x\in V_N\setminus V_0}\eta_t^N(x)\left(\frac{2}{3}\Delta F\right)(x) + \frac{1}{|V_N|}\sum_{x\in V_N\setminus V_0}\eta_t^N(x)\left(\Delta_N F - \frac{2}{3}\Delta F\right)(x)$$
$$= \left(\frac{1}{|V_N|}\sum_{x\in V_N}\eta_t^N(x)\left(\frac{2}{3}\Delta F\right)(x) - \frac{1}{|V_N|}\sum_{a\in V_0}\eta_t^N(a)\left(\frac{2}{3}\Delta F\right)(a)\right) + o_N(1) = \pi_t^N\left(\frac{2}{3}\Delta F\right) + o_N(1)$$

as $N\to\infty$. The second term in the penultimate expression is $o_N(1)$ because $|V_N| = \mathcal{O}(3^N)$ overwhelms the finite sum.

Now suppose the test function is time-dependent: take $F \in C([0,T], \mathrm{dom}\Delta) \cap C^1((0,T), \mathrm{dom}\Delta)$, and denote $F_t = F(t,\cdot)$. Then by Dynkin's formula and the aforementioned arguments, we obtain that

$$\begin{aligned} M_t^N(F) &:= \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N\left(\left(\frac{2}{3}\Delta + \partial_s\right)F_s\right)ds \\ &\quad + \int_0^t \frac{3^N}{|V_N|}\sum_{a\in V_0}\left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a) - \bar\rho(a))F_s(a)\right]ds + o_N(1) \end{aligned} \tag{3.21}$$

is a martingale with quadratic variation

$$\begin{aligned} \langle M^N(F)\rangle_t &= \int_0^t \frac{5^N}{|V_N|^2}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}}(\eta_s^N(x) - \eta_s^N(y))^2(F_s(x) - F_s(y))^2\,ds \\ &\quad + \int_0^t \sum_{a\in V_0}\frac{5^N}{b^N|V_N|^2}\left(\lambda_-(a)\eta_s^N(a) + \lambda_+(a)(1-\eta_s^N(a))\right)F_s^2(a)\,ds. \end{aligned} \tag{3.22}$$

To deduce heuristically from the previous decompositions the notion of weak solutions that appear in (3.8) and (3.10) for the corresponding regime of b, we argue as follows. From the computations of §6.1, we will see that the martingale that appears in (3.21) vanishes in $L^2(\mu_N)$ as $N\to\infty$. The third term on the RHS of (3.21) will correspond to the third term on the RHS of both $\Theta_{\rm Dir}$ and $\Theta_{\rm Rob}$. Now we argue for boundary terms for each regime of b. In the case $b < 5/3$, $F(a) = 0$ for all $a\in V_0$, so from Lemma 5.4 we easily obtain the remaining term in the definition of $\Theta_{\rm Dir}$. In the case $b > 5/3$, we easily see that the term on the RHS inside the square brackets vanishes as $N\to\infty$. To treat the remaining term it is enough to recall the replacement Lemma 5.3. Finally, in the Robin case $b = 5/3$, one repeats exactly the same procedure as in the two previous cases. All the details can be found in §6.

4. Equilibrium density fluctutations: Statement of results

4.1. **Equilibrium density fluctuations and heuristics.** To study the exclusion process at equilibrium, we set $\lambda_+(a) = \lambda_+ > 0$ and $\lambda_-(a) = \lambda_- > 0$ for all $a\in V_0$, and $\lambda_\Sigma = \lambda_+ + \lambda_-$. Then it is easy to check that the product Bernoulli measure ν_ρ^N with constant density $\rho = \lambda_+/\lambda_\Sigma$, *i.e.*, $\nu_\rho^N\{\eta\in\Omega_N : \eta(x) = 1\} = \rho$ for every $x\in V_N$, is reversible for the process $\{\eta_t^N : t\geq 0\}$. In particular, $\mathbb{E}_{\nu_\rho^N}[\eta_t^N(x)] = \rho$ for all $x\in V_N$ and all $t\geq 0$. Therefore the interesting problem is to study fluctuations about this equilibrium density profile ρ.

We define the equilibrium **density fluctuation field (DFF)** $\mathcal{Y}^N_\cdot$ given by

$$\mathcal{Y}_t^N(F) = \frac{1}{\sqrt{|V_N|}}\sum_{x\in V_N}\bar\eta_t^N(x)F(x), \qquad \bar\eta_t^N(x) := \eta_t^N(x) - \rho, \tag{4.1}$$

where the space of test functions F will be specified shortly. Note that the prefactor $1/\sqrt{|V_N|}$ is consistent with the Central Limit Theorem scaling. Our goal now is to show that the DFF converges, in a proper topology to be defined later on, to an Ornstein-Uhlenbeck process $\mathcal{Y}_t$ on K, with suitable boundary conditions which depend on the regime of b.

Before formally stating our results, we give a heuristic explanation for the choice of the space of test functions. To do that, we fix a time-independent function F, and apply Dynkin's formula to find that

$$\mathcal{M}_t^N(F) := \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t 5^N \mathcal{L}_N \mathcal{Y}_s^N(F)\, ds \tag{4.2}$$

is a martingale with quadratic variation

$$\langle \mathcal{M}^N(F)\rangle_t = \int_0^t 5^N \left(\mathcal{L}_N[\mathcal{Y}_s^N(F)]^2 - 2\mathcal{Y}_s^N(F)\mathcal{L}_N\mathcal{Y}_s^N(F)\right) ds. \tag{4.3}$$

We directly compute the generator term which gives

$$5^N\mathcal{L}_N\mathcal{Y}_t^N(F) = \frac{5^N}{\sqrt{|V_N|}} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (\eta_t^N(y) - \eta_t^N(x))F(x) + \frac{5^N}{b^N\sqrt{|V_N|}} \sum_{a\in V_0} \left(-\lambda_+\eta_t^N(a) + \lambda_-(1-\eta_t^N(a))F(a)\right). \tag{4.4}$$

By making a change of variables and centering with respect to ν_ρ^N, we obtain:

$$\begin{aligned} 5^N\mathcal{L}_N\mathcal{Y}_t^N(F) = & \frac{5^N}{\sqrt{|V_N|}} \sum_{x\in V_N\setminus V_0} \sum_{\substack{y\in V_N\\ y\sim x}} (F(y)-F(x))\bar\eta_t^N(x) \\ & + \frac{5^N}{\sqrt{|V_N|}} \sum_{a\in V_0} \sum_{\substack{y\in V_N\\ y\sim a}} (F(y)-F(a))\bar\eta_t^N(a) - \frac{5^N}{b^N\sqrt{|V_N|}}\lambda_\Sigma \sum_{a\in V_0}\bar\eta_t^N(a)F(a) \\ & = \mathcal{Y}_t^N(\Delta_N F) + o_N(1) - \frac{3^N}{\sqrt{|V_N|}} \sum_{a\in V_0} \bar\eta_t^N(a)\left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N}\lambda_\Sigma F(a)\right]. \end{aligned} \tag{4.5}$$

This gives

$$\begin{aligned} \mathcal{M}_t^N(F) = & \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1) \\ & + \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a\in V_0} \bar\eta_s^N(a)\left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N}\lambda_\Sigma F(a)\right] ds. \end{aligned} \tag{4.6}$$

Looking back at the previous display, we need to show that the last integral vanishes in some topology. Observe that, as in the hydrodynamics setting, for $b < 5/3$, the test functions satisfy $F(a) = 0$ for all $a \in V_0$. With this condition we still need to control the term with the normal derivative in the last integral. At this point we use the replacement Lemma 5.6, thereby closing the equation for the DFF as

$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1). \tag{4.7}$$

In all regimes of b, our goal is to choose suitable boundary conditions for the test functions so that the previous equality holds. In the case $b > 5/3$, the test functions satisfy $(\partial^\perp F)(a) = 0$ for all $a \in V_0$. Therefore, by Lemma 3.2 and by controlling the rate of convergence of the discrete normal derivative to the continous normal derivative, we can just bound the variables $\eta(x)$ by 1, and to achieve our goal we just need to control the term with $F(a)$. This last term can be estimated from the replacement Lemma 5.5. Finally, in the Robin case $b = 5/3$, the test function must satisfy $(\partial^\perp F)(a) = -\lambda_\Sigma F(a)$ for all $a \in V_0$. In this case the term inside the time integral in Dynkin's martingale vanishes as a consequence of Lemma 3.2, the convergence of the discrete normal derivative to the continuous normal derivative, and the replacement Lemma 5.7.

Let us make a technical remark here. In order to prove tightness of the sequence $\{\mathcal{Y}_\cdot^N\}_N$, we will have to impose extra boundary conditions on the test functions. This point will be explained in the next subsection §4.2. But for the purpose of closing the equation for Dynkin's martingale, the boundary conditions mentioned in the last paragraph are sufficient.

Next we analyze the quadratic variation of Dynkin's martingale. Another straightforward calculation yields that the martingale's quadratic variation is given by

$$\begin{aligned}\langle \mathcal{M}^N(F)\rangle_t =& \frac{5^N}{|V_N|}\int_0^t \sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}} \eta_s^N(x)(1-\eta_s^N(y))(F(x)-F(y))^2\,ds \\ &+ \frac{5^N}{b^N|V_N|}\int_0^t \sum_{a\in V_0}\left(\lambda_-\eta_s^N(a)+\lambda_+(1-\eta_s^N(a))\right)F^2(a)\,ds\end{aligned} \tag{4.8}$$

As in the last section, let $\mathbb{P}_{\mu_N}$ be the probability measure on $D([0,T],\Omega_N)$ induced by the process $\{\eta_t^N : t\in[0,T]\}$ geneated by $5^N\mathcal{L}_N$ and started from the initial measure μ_N. In the current setting, we take $\mu_N=\nu_\rho^N$ and write $\mathbb{P}_\rho^N := \mathbb{P}_{\nu_\rho^N}$, and denote the corresponding expectation by $\mathbb{E}_\rho^N$. It follows from a direct computation of (4.8) that

$$\mathbb{E}_\rho^N\left[|\mathcal{M}_t^N(F)|^2\right] = \frac{3^N}{|V_N|}2\chi(\rho)t\left[\mathcal{E}_N(F)+\frac{5^N}{3^Nb^N}\lambda_\Sigma\sum_{a\in V_0}F^2(a)\right], \tag{4.9}$$

where $\chi(\rho)=\rho(1-\rho)$ is the conductivity in the exclusion process. The martingale equation (4.7) together with (4.9) suggests that the density fluctuation field $\mathcal{Y}_\cdot^N$ satisfies a discrete Ornstein-Uhlenbeck equation. Indeed, as mentioned previously, the second goal of our work is to show that $\{\mathcal{Y}_\cdot^N\}_N$ converges to an Ornstein-Uhlenbeck process on K with suitable boundary condition.

4.2. **Function spaces.** Having provided the heuristics, we now set up all the definitions and introduce the necessary background from analysis on fractals. Recall Definition 3.1 and Lemma 3.2.

Definition 4.1 (Laplacian with b.c.)**.** Let $\Delta_{\rm b}$, ${\rm b}\in\{\text{Dir, Neu, Rob}\}$, denote the Laplacian with Dirichlet (resp. Neumann, Robin) boundary condition on V_0 with domain

$$\text{dom}\Delta_{\rm b} := \left\{\begin{array}{ll}\{F\in\text{dom}\Delta : F|_{V_0}=0\}\ (=\text{dom}\Delta_0), & \text{if b =Dir},\\ \{F\in\text{dom}\Delta : (\partial^\perp F)|_{V_0}=0\}, & \text{if b =Neu},\\ \{F\in\text{dom}\Delta : (\partial^\perp F)|_{V_0}=-\lambda_\Sigma F|_{V_0}\}, & \text{if b =Rob}.\end{array}\right. \tag{4.10}$$

According to our classification of the scaling regimes, we set

$$\Delta_b = \left\{\begin{array}{ll}\Delta_{\rm Dir}, & \text{if } b<5/3,\\ \Delta_{\rm Rob}, & \text{if } b=5/3,\\ \Delta_{\rm Neu}, & \text{if } b>5/3.\end{array}\right. \tag{4.11}$$

Definition 4.2 (Space of test functions)**.** For a fixed value of $b>0$, set

$$\mathcal{S}_b := \{F\in\text{dom}\Delta_b : \Delta_bF\in\text{dom}\Delta_b\} \overset{\text{or}}{=} \bigcap_{j=1}^\infty \text{dom}\left((-\Delta_b)^j\right). \tag{4.12}$$

Then define on $\mathcal{S}_b$ the seminorms

$$\|F\|_0 := \sup_{x\in K}|F(x)|; \tag{4.13}$$

$$\|F\|_j := \sup_{x\in K}\left|(-\Delta)^jF(x)\right| + \sqrt{\mathcal{E}((-\Delta)^{j-1}F)},\quad j\in\mathbb{N} \tag{4.14}$$

where $\mathcal{E}$ was defined in (2.2).

Proposition 4.3. *$\mathcal{S}_b$ endowed with the family of seminorms $\{\|\cdot\|_j : j\in\mathbb{N}_0\}$ is a Fréchet space.*

Proof. We follow the definition of a Fréchet space in [RS80, §V.2], *i.e.*, a complete metrizable locally convex space. Using the natural topology generated by the countably many seminorms, $\mathcal{S}_b$ is locally convex and metrizable. Therefore it remains to show that $\mathcal{S}_b$ is complete.

Suppose $\{F_i\}_{i=1}^\infty\subset\mathcal{S}_b$ is Cauchy in each $\|\cdot\|_j$. Then $F_i\to G_0$ and $(-\Delta)^jF_i\to G_j$ in $(C(K),\|\cdot\|_{\sup})$ and in $(\mathcal{F},\sqrt{\mathcal{E}})$. We want to show that $(-\Delta)^jG_0=G_j$. This will be achieved using integration by parts. We verify the case $j=1$; the proof for $j \geq 2$ is obtained via induction. Start with Definition 3.1 applied to F_i, namely:

$$\mathcal{E}(F_i, \varphi) = \int_K (-\Delta F_i)\varphi \, dm \quad \text{for all } \varphi \in \mathcal{F}_0 \tag{4.15}$$

Since $F_i \to G_0$ in $\sqrt{\mathcal{E}}$, and $-\Delta F_i \to G_1$ in $C(K)$, we obtain that

$$\mathcal{E}(G_0, \varphi) = \int_K G_1 \varphi \, dm \quad \text{for all } \varphi \in \mathcal{F}_0. \tag{4.16}$$

Referring back to Definition 3.1 this says that $G_1 = -\Delta G_0$. □

Remark 4.4. (1) In the analysis of exclusion processes on the 1D interval $[0,1]$ [GPS17, FGN17] (resp. the real line $\mathbb{R}$ [FGN13]), the Fréchet space of choice is the completion of $C^\infty([0,1])$ (resp. $C_c^\infty(\mathbb{R})$) with respect to the seminorms

$$\|f\|_k := \sup_{x\in[0,1]} \left|f^{(k)}(x)\right| \quad \left(\text{resp. } \|f\|_k := \sup_{x\in\mathbb{R}} \left|f^{(k)}(x)\right|\right), \quad k \in \mathbb{N}_0$$

Our Definition 4.2 generalizes this idea to SG, using powers of the Laplacian and the Dirichlet energy. More precisely, a function F with finite $\|\cdot\|_j$ seminorm has a continuous $(2j)$th derivative and a weak $(2j-1)$th derivative.

Observe also that once $(-\Delta)^j F$ and $(-\Delta)^{j-1}F$ are known, then $\partial^\perp(-\Delta)^{j-1}F|_{V_0}$ can be obtained from the integration by parts formula (Lemma 3.2-(3)).

(2) We introduce the Fréchet space $(S_b, \{\|\cdot\|_j : j \in \mathbb{N}_0\})$ in order to verify tightness of the density fluctuation fields $\{\mathcal{Y}^N_\cdot\}_N$ via Mitoma's criterion (Lemma 8.2 below).

4.3. **Dirichlet forms, Laplacians, and heat semigroups.** We now explain the connection between the Laplacian Δ_b, the Dirichlet form, and the heat semigroup. Details can be found in Appendix A.

Define the quadratic form

$$\mathscr{E}_b(F,G) = \mathcal{E}(F,G) + \mathbf{1}_{\{b=5/3\}} \sum_{a\in V_0} \lambda_\Sigma F(a)G(a), \quad \forall F, G \in \mathcal{F}_b \tag{4.17}$$

where

$$\mathcal{F}_b = \begin{cases} \mathcal{F}, & \text{if } b \geq 5/3, \\ \mathcal{F}_0 \ (:= \{g \in \mathcal{F} : g|_{V_0} = 0\}), & \text{if } b < 5/3. \end{cases} \tag{4.18}$$

Lemma 4.5.

(1) $(\mathscr{E}_b, \mathcal{F}_b)$ is a local regular Dirichlet form on $L^2(K,m)$, and the corresponding non-negative self-adjoint operator H_b on $L^2(K,m)$ has compact resolvent.

(2) The operator H_b and the Laplacian $-\Delta$ agree on $\mathrm{dom}\Delta_b$: $H_b|_{\mathrm{dom}\Delta_b} = -\Delta|_{\mathrm{dom}\Delta_b} =: -\Delta_b$. In fact, H_b is the Friedrichs extension of $-\Delta$ on $\mathrm{dom}\Delta_b$.

Proof. When b ∈ {Neu, Dir} this is already established in [Kig01]. We indicate the statements, and supply the corresponding proofs for b = Rob, on page 40 in Appendix A. □

See [Kig01, Appendix B] for a quick set of definitions on Dirichlet forms, and [FOT11] for more information about Dirichlet forms. The distinction between H_b and $-\Delta_b$ lies in their respective domains: the former has a larger domain (form domain) than the latter (operator domain). That H_b has compact resolvent implies that H_b has pure point spectrum, and spectral asymptotics of H_b is a well studied problem; see Lemmas A.1 through A.4 in Appendix A.

From standard arguments in functional analysis, H_b is associated to a unique strongly continuous **heat semigroup** $\{\mathsf{T}^b_t : t > 0\}$ on $L^2(K,m)$, satisfying $\mathsf{T}^b_t\mathsf{T}^b_s = \mathsf{T}^b_{t+s}$ for any $t,s>0$, which is given by

$$H_b f = \lim_{t\downarrow 0} \frac{\mathsf{T}^b_t f - f}{t}, \quad \forall f \in \mathrm{dom}(H_b).$$

In this sense H_b is the infinitesimal generator of the semigroup $\{\mathsf{T}^b_t : t > 0\}$. In particular,

$$\mathsf{T}^b_t H_b f = H_b \mathsf{T}^b_t f = \lim_{h\downarrow 0} \frac{\mathsf{T}^b_{t+h} f - \mathsf{T}^b_t f}{h}, \quad \forall f \in \mathrm{dom}(H_b), \tag{4.19}$$

where the limit is the strong limit in the Hilbert space $L^2(K,m)$.

To summarize, we have the following 1-to-1 correspondence:

$$(\mathscr{E}_b, \mathcal{F}_b) \longleftrightarrow H_b \longleftrightarrow \{\mathsf{T}^b_t : t > 0\}. \tag{4.20}$$

For the proofs to come, we will need two key properties of the heat semigroup.

Lemma 4.6. *The following hold for* $\{\mathsf{T}^b_t : t > 0\}$*:*

(1) $\mathsf{T}^b_t : L^p(K,m) \to C(K)$ *is a bounded operator for any* $t > 0$ *and any* $p \in [1,\infty]$.

(2) $\mathsf{T}^b_t(L^1(K,m)) \subset \mathrm{dom}\Delta_b$ *for any* $t > 0$.

(3) Let $u \in L^1(K,m)$*, and set* $u(t,x) = (\mathsf{T}^b_t u)(x)$*. Then* $u(\cdot,x) \in C^\infty((0,\infty))$ *for any* $x \in K$*. Moreover,* $\partial_t u(t,x) = \Delta u(t,x)$ *for any* $(t,x) \in (0,\infty)\times K$.

Proof. Our proof is based on the use of heat kernels and spectral asymptotics on SG. See page 42 in Appendix A. Note that when $\mathrm{b} \in \{\mathrm{Neu}, \mathrm{Dir}\}$ this is established in [Kig01]. □

Recall that we work on the Fréchet space $\mathcal{S}_b$. The following corollary will be invoked in §8.2.

Corollary 4.7. *If* $f \in \mathcal{S}_b$*, then for any* $t > 0$*,* $\mathsf{T}^b_t f \in \mathcal{S}_b$ *and* $\Delta_b \mathsf{T}^b_t f \in \mathcal{S}_b$.

Proof. Since $\mathrm{dom}\Delta_b \subset L^1(K,m)$, by Lemma 4.6-(2), we deduce that $f \in \mathrm{dom}\Delta_b$ implies $\mathsf{T}^b_t f \in \mathrm{dom}\Delta_b$. Similarly, for any $j \in \mathbb{N}$, $(-\Delta_b)^j f \in \mathrm{dom}\Delta_b$ implies that $(-\Delta_b)^j \mathsf{T}^b_t f = \mathsf{T}^b_t (-\Delta_b)^j f \in \mathrm{dom}\Delta_b$, where we used Lemma 4.5-(2) and (4.19) to commute powers of Δ_b with T^b_t on $\mathrm{dom}\Delta_b$. The claim follows. □

Lastly, we should mention that due to our scaling convention, we will use $\frac{2}{3}H_b$ to generate the heat semigroup, which we denote as $\tilde{\mathsf{T}}^b_t$. All the above results still hold modulo the substitution of H_b (resp. Δ) by $\frac{2}{3}H_b$ (resp. $\frac{2}{3}\Delta$).

4.4. **Ornstein-Uhlenbeck equations.** Let $\mathcal{S}'_b$ be the topological dual of $\mathcal{S}_b$ with respect to the topology generated by the seminorms $\{\|\cdot\|_j : j \in \mathbb{N}_0\}$.

Definition 4.8 (Ornstein-Uhlenbeck equation)**.** We say that a random element $\mathcal{Y}$ taking values in $C([0,T],\mathcal{S}'_b)$ is a solution to the Ornstein-Uhlenbeck equation on K with parameter b if:

(**OU1**) For every $F \in \mathcal{S}_b$,

$$\mathcal{M}_t(F) = \mathcal{Y}_t(F) - \mathcal{Y}_0(F) - \int_0^t \mathcal{Y}_s\Big(\frac{2}{3}\Delta_b F\Big)\,ds \tag{4.21}$$

$$\text{and} \quad \mathcal{N}_t(F) = (\mathcal{M}_t(F))^2 - \frac{2}{3}\cdot 2\chi(\rho) t \mathscr{E}_b(F) \tag{4.22}$$

are $\mathscr{F}_t$-martingales, where $\mathscr{F}_t := \sigma\{\mathcal{Y}_s(F) : s \le t\}$ for each $t \in [0,T]$, and $\mathscr{E}_b$ was defined in (4.17).

(**OU2**) $\mathcal{Y}_0$ is a centered Gaussian $\mathcal{S}'_b$-valued random variable with covariance

$$\mathbb{E}^b_\rho\left[\mathcal{Y}_0(F)\mathcal{Y}_0(G)\right] = \chi(\rho)\int_K F(x)G(x)\,dm(x), \quad \forall F,G \in \mathcal{S}_b. \tag{4.23}$$

Moreover, for every $F \in \mathcal{S}_b$, the process $\{\mathcal{Y}_t(F) : t \ge 0\}$ is Gaussian: the distribution of $\mathcal{Y}_t(F)$ conditional upon $\mathscr{F}_s$, $s < t$, is Gaussian with mean $\mathcal{Y}_s(\tilde{\mathsf{T}}^b_{t-s}F)$ and variance $\int_0^{t-s} \frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}^b_r F)\,dr$, where $\{\tilde{\mathsf{T}}^b_t : t > 0\}$ is the heat semigroup generated by $\frac{2}{3}H_b$.

For notational simplicity, we have suppressed the dependence of $\mathcal{Y}$ on b.[1]

[1] In the Euclidean setting the Ornstein-Uhlenbeck equation can be formally written as

$$d\mathcal{Y}_t = \Delta_{\mathrm{b}}\mathcal{Y}_t\,dt + \sqrt{2\chi(\rho)}\nabla_{\mathrm{b}}\,d\mathcal{W}_t,$$

where Δ_{b} (resp. ∇_{b}) stands respectively for the Laplacian (resp. the gradient) with boundary condition $\mathrm{b} \in \{\mathrm{Neu}, \mathrm{Rob}, \mathrm{Dir}\}$; $\{\mathcal{W}_t : t \ge 0\}$ is a cylindrical standard Wiener process; and $\nabla_{\mathrm{b}}\mathcal{W}_t$ is a space-time white noise of unit quadratic variation.

4.5. **Convergence of density fluctuations to the Ornstein-Uhlenbeck equations.** Fox a fixed value of b, let $\mathbb{Q}_\rho^{N,\mathrm{b}}$ be the probability measure on $D([0,T],\mathcal{S}_b')$ induced by the density fluctuation field $\mathcal{Y}^N_\cdot$ and by $\mathbb{P}_\rho^{N,b}$. We are ready to state the second main theorem of this paper.

Theorem 2 (Ornstein-Uhlenbeck limit of density fluctuations)**.** *The sequence $\{\mathbb{Q}_\rho^{N,b}\}_N$ converges in distribution, as $N\to\infty$, to a unique solution of the Ornstein-Uhlenbeck equation with parameter b, in the sense of Definition 4.8.*

5. Replacement lemmas

In this section we prove all the replacement lemmas that we need in this article. We divide it into four sections. §5.1 deals with some inequalities that will be used in subsequent proofs. §5.2 is concerned with the relation between the Dirichlet form and the carré du champ operator in the exclusion process, to be defined ahead. In §5.3 and §5.4 we present the replacement lemmas needed for the hydrodynamics and density fluctuations, respectively.

5.1. **Functional inequalities.** Given a finite set Λ and a function $g:\Lambda\to\mathbb{R}$, we denote the average of g over Λ by

$$\mathrm{Av}_\Lambda[g]=|\Lambda|^{-1}\sum_{x\in\Lambda}g(x).$$

A key functional inequality we will need is the *moving particle lemma*, stated and proved in [Che17, Theorem 1.1]. On SG this replaces the telescoping sum and Cauchy-Schwarz arguments in the 1D case. For a discussion of the rationale behind the moving particle lemma, see [Che17, §1.1].

Lemma 5.1 (Moving particle lemma)**.** *Let $G=(V,E)$ be a finite connected graph endowed with positive edge weights $\{c_{xy}\}_{xy\in E}$. Then for any $f:\{0,1\}^V\to\mathbb{R}$ and any product Bernoulli measure ν_ρ with constant density $\rho\in[0,1]$ on $\{0,1\}^V$,*

$$\frac{1}{2}\int(f(\eta^{xy})-f(\eta))^2\,d\nu_\rho(\eta)\le R_{\mathrm{eff}}(x,y)\frac{1}{2}\int\sum_{zw\in E}c_{zw}(f(\eta^{zw})-f(\eta))^2\,d\nu_\rho(\eta),\tag{5.1}$$

where

$$R_{\mathrm{eff}}(x,y):=\sup\left\{\frac{(h(x)-h(y))^2}{\sum_{zw\in E}c_{zw}(h(z)-h(w))^2}\;\middle|\;h:V\to\mathbb{R}\right\}\tag{5.2}$$

is the effective resistance between x and y.

We will employ Lemma 5.1 in the special case where $G=\mathcal{G}_N$ and the edge weights $c_{xy}=1$ for all $xy\in E_N$.

The other tools are known to practitioners of 1D exclusion processes. For the density replacement lemmas (§5.3), the main inequality used is the Feynman-Kac formula.

We will also use the following estimate which is stated and proved in [BGJO17, Lemma 5.1]. The *support* of a function $g:\{0,1\}^V\to\mathbb{R}$, denoted $\mathrm{supp}(g)$, is the smallest subset Λ of V such that g is measurable with respect to $\{\eta(x):x\in\Lambda\}$. A function $g:\{0,1\}^{V_N}\to\mathbb{R}$ is called *local* if there exist $x\in V_N$ and $r\ge 0$, independent of N, such that $\mathrm{supp}(g)\subset B_d(x,r)$, the ball of radius r centered at x in the graph metric d.

Lemma 5.2. *Let $T:\Omega_N\to\Omega_N$ be a transformation, $\omega:\Omega_N\to\mathbb{R}_+$ be a positive local function, and f be a density with respect to a probability measure μ on Ω_N. Then*

$$\begin{aligned}\left\langle\omega(\eta)\left(\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\right),\sqrt{f(\eta)}\right\rangle_\mu&\le-\frac{1}{4}\int\omega(\eta)\left(\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\right)^2d\mu\\&+\frac{1}{16}\int\frac{1}{\omega(\eta)}\left(\omega(\eta)-\omega(T(\eta))\frac{d\mu(T(\eta))}{d\mu(\eta)}\right)^2\left(\sqrt{f(T(\eta))}+\sqrt{f(\eta)}\right)^2d\mu.\end{aligned}\tag{5.3}$$

As is well-known to experts in analysis on fractals, it is *a priori* unclear how to define a *gradient* ∇_{b} on fractals. (A pointwise formulation based on locally harmonic functions is given in [Str00, Tep00], while an abstract formulation based on Dirichlet forms can be found in [CS03, IRT12, HRT13].) That said, to define a white noise on a fractal does not require using the gradient. It suffices to invoke the Dirichlet form $\mathcal{E}$ (or, more generally, $\mathscr{E}_b$ to incorporate boundary conditions) to define the covariance of a standard white noise.

5.2. **Exclusion process Dirichlet form estimates.** Given a function $f:\Omega_N\to\mathbb{R}$ and a measure μ on Ω_N, we define the carré du champ operator by

$$\Gamma_N(f,\mu):=\int \frac{1}{2}\sum_{xy\in E_N}(f(\eta^{xy})-f(\eta))^2\,d\mu(\eta) \tag{5.4}$$

Also, given a function $f:\Omega_N\to\mathbb{R}$ and a measure μ on Ω_N, we define the Dirichlet form $\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_\mu$ by

$$\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_\mu=\langle\sqrt{f},-\mathcal{L}_N^{\text{bulk}}\sqrt{f}\rangle_\mu+\frac{1}{b^N}\langle\sqrt{f},-\mathcal{L}_N^{\text{boundary}}\sqrt{f}\rangle_\mu.$$

In this subsection we first take $\mu=\nu_\rho^N$ and we compare the carré du champ operator $\Gamma_N(\sqrt{f},\nu_\rho)$ with the Dirichlet form $\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_{\nu_\rho^N}$. A simple computation shows that

$$\begin{aligned}\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_{\nu_\rho^N}&=\int\sum_{x\in V_N}\sum_{y\sim x}\eta(x)(1-\eta(y))\sqrt{f(\eta)}(\sqrt{f(\eta)}-\sqrt{f(\eta^{xy})})\,d\nu_\rho^N(\eta)\\&+\frac{1}{b^N}\int\sum_{a'\in V_0}[\lambda_-(a')\eta(a')+\lambda_+(a')(1-\eta(a'))]\sqrt{f(\eta)}\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{a'})}\right)d\nu_\rho^N(\eta).\end{aligned} \tag{5.5}$$

It is direct to verify that the bulk term equals the carré du champ $\Gamma_N(\sqrt{f},\nu_\rho^N)$. Then we use Lemma 5.2 to bound the boundary term from below by

$$\begin{aligned}&\frac{1}{b^N}\sum_{a'\in V_0}\left(\frac{1}{4}\int\omega_{a'}(\eta)\left(\sqrt{f(\eta^{a'})}-\sqrt{f(\eta)}\right)^2d\nu_\rho^N(\eta)\right.\\&\left.\qquad-\frac{1}{16}\int\frac{1}{\omega_{a'}(\eta)}\left(\omega_{a'}(\eta)-\omega_{a'}(\eta^{a'})\frac{d\nu_\rho^N(\eta^{a'})}{d\nu_\rho^N(\eta)}\right)^2\left(\sqrt{f(\eta^{a'})}+\sqrt{f(\eta)}\right)^2d\nu_\rho^N(\eta)\right)\end{aligned} \tag{5.6}$$

with $\omega_{a'}(\eta)=\lambda_-(a')\eta(a')+\lambda_+(a')(1-\eta(a'))$. The second term in the last expression represents the error of replacing the boundary Dirichlet form by the boundary carré du champ (the first term). We simplify it as follows. For each $a\in V_0$, denote $\eta=(\eta(a);\tilde\eta)$ where $\tilde\eta$ represents the configuration except at a. Then

$$\begin{aligned}&\frac{1}{16}\int\frac{1}{\omega_a(\eta)}\left(\omega_a(\eta)-\omega_a(\eta^a)\frac{d\nu_\rho^N(\eta^a)}{d\nu_\rho^N(\eta)}\right)^2\left(\sqrt{f(\eta^a)}+\sqrt{f(\eta)}\right)^2d\nu_\rho^N(\eta)\\&\le\frac{1}{8}\int\frac{1}{\omega_a(\eta)}\left(\omega_a(\eta)-\omega_a(\eta^a)\frac{d\nu_\rho^N(\eta^a)}{d\nu_\rho^N(\eta)}\right)^2(f(\eta^a)+f(\eta))\,d\nu_\rho^N(\eta)\\&=\frac{1}{8}\int\frac{1}{\lambda_-(a)}\left(\frac{\lambda_\Sigma(a)(\rho-\bar\rho(a))}{\rho}\right)^2(f(0;\tilde\eta)+f(1;\tilde\eta))\,\rho\,d\nu_\rho^N(\tilde\eta)\\&\quad+\frac{1}{8}\int\frac{1}{\lambda_+(a)}\left(\frac{\lambda_\Sigma(a)(\bar\rho(a)-\rho)}{1-\rho}\right)^2(f(1;\tilde\eta)+f(0;\tilde\eta))(1-\rho)d\nu_\rho^N(\tilde\eta)\\&=\frac{1}{8}(\lambda_\Sigma(a))^2\left(\frac{1}{\lambda_-(a)\rho}+\frac{1}{\lambda_+(a)(1-\rho)}\right)(\rho-\bar\rho(a))^2\int(f(0;\tilde\eta)+f(1;\tilde\eta))\,d\nu_\rho^N(\tilde\eta)\\&\le C'(\rho,\lambda_\pm(a))(\rho-\bar\rho(a))^2.\end{aligned} \tag{5.7}$$

for a bounded constant $C'(\rho,\lambda_\pm(a))$. Putting these altogether we see that

$$\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_{\nu_\rho^N}\ge\Gamma_N(\sqrt{f},\nu_\rho^N)-\frac{1}{b^N}\sum_{a'\in V_0}C'(\rho,\lambda_\pm(a'))(\rho-\bar\rho(a'))^2=\Gamma_N(\sqrt{f},\nu_\rho^N)-\frac{1}{b^N}C''(\rho). \tag{5.8}$$

We remark that as a consequence of the previous estimate, we will be able to use the Bernoulli product measure with constant parameter ρ when $b\ge 5/3$ but not in the case $b<5/3$, since in that case, as we will see below, the error term blows up when $N\to+\infty$. In the later case, we will have to use the Bernoulli product measure $\nu_{\rho(\cdot)}^N$ for a suitable chosen profile. Therefore, we need now to compare the carré du champ operator wih the Dirichlet form in the case $\mu=\nu_{\rho(\cdot)}^N$ for a profile $\rho(\cdot)\in\mathcal{F}$ with $\rho(a)=\bar\rho(a)$ for all $a\in V_0$ and this is done in the same fashion as (5.5). Note that with our choice of ρ, the boundary part of the Dirichlet form equals a carré du champ:

$$\frac{1}{b^N}\langle\sqrt{f},-\mathcal{L}_N^{\rm boun}\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}=\frac{1}{b^N}\frac{1}{2}\int\sum_{a'\in V_0}\omega_{a'}(\eta)\left(\sqrt{f(\eta^{a'})}-\sqrt{f(\eta)}\right)^2\,d\nu^N_{\rho(\cdot)}(\eta)\tag{5.9}$$

where $\omega_{a'}(\eta)=\lambda_-(a')\eta(a')+\lambda_+(a')(1-\eta(a'))$. We estimate this from below by retaining only the contribution from a:

$$\frac{1}{b^N}\langle\sqrt{f},-\mathcal{L}_N^{\rm boun}\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\geq\frac{1}{b^N}\frac{1}{2}\int\omega_a(\eta)(\sqrt{f(\eta^a)}-\sqrt{f(\eta)})^2\,d\nu^N_{\rho(\cdot)}(\eta).\tag{5.10}$$

For the bulk part of the Dirichlet form, we apply Lemma 5.2 to find

$$\begin{aligned}&\langle\sqrt{f},-\mathcal{L}_N^{\rm bulk}\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}=2\int\sum_{xy\in E_N}(\eta(x)-\eta(y))^2\sqrt{f(\eta)}\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{xy})}\right)\,d\nu^N_{\rho(\cdot)}(\eta)\\&\geq\frac{1}{2}\int\sum_{xy\in E_N}(\eta(x)-\eta(y))^2\left(\sqrt{f(\eta^{xy})}-\sqrt{f(\eta)}\right)^2\,d\nu^N_{\rho(\cdot)}(\eta)\\&\quad-\frac{1}{8}\int\sum_{xy\in E_N}(\eta(x)-\eta(y))^2\left(1-\frac{\rho(y)(1-\rho(x))}{\rho(x)(1-\rho(y))}\mathbf{1}_{\{\eta(x)=1,\eta(y)=0\}}-\frac{\rho(x)(1-\rho(y))}{\rho(y)(1-\rho(x))}\mathbf{1}_{\{\eta(x)=0,\eta(y)=1\}}\right)^2\\&\quad\times\left(\sqrt{f(\eta^{xy})}+\sqrt{f(\eta)}\right)^2\,d\nu^N_{\rho(\cdot)}(\eta)\end{aligned}\tag{5.11}$$

The first term in the RHS of (5.11) equals $\Gamma_N(\sqrt{f},\nu^N_{\rho(\cdot)})$. For the second term, or the error, in the RHS of (5.11), observe that for each $xy\in E_N$, the integrand is nonzero if and only if $\eta(x)\neq\eta(y)$, and that

$$1-\frac{\rho(y)(1-\rho(x))}{\rho(x)(1-\rho(y))}=\frac{\rho(x)-\rho(y)}{\rho(x)(1-\rho(y))}.\tag{5.12}$$

Therefore we can bound the second term in the RHS of (5.11) from below by

$$\begin{aligned}&-\frac{1}{8}\int\sum_{xy\in E_N}(\eta(x)-\eta(y))^2\frac{(\rho(x)-\rho(y))^2}{(\min(\rho(x)(1-\rho(y)),\rho(y)(1-\rho(x))))^2}\left(\sqrt{f(\eta^{xy})}+\sqrt{f(\eta)}\right)^2\,d\nu^N_{\rho(\cdot)}(\eta)\\&\geq-\frac{C(\rho)}{4}\int\sum_{xy\in E_N}(\eta(x)-\eta(y))^2(\rho(x)-\rho(y))^2(f(\eta^{xy})+f(\eta))\,d\nu^N_{\rho(\cdot)}(\eta)\\&\geq-C'(\rho)\sum_{xy\in E_N}(\rho(x)-\rho(y))^2.\end{aligned}\tag{5.13}$$

Altogether

$$\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\geq\Gamma_N(\sqrt{f},\nu^N_{\rho(\cdot)})-C'(\rho)\sum_{xy\in E_N}(\rho(x)-\rho(y))^2+\frac{1}{b^N}\frac{1}{2}\int\omega_a(\eta)(\sqrt{f(\eta^a)}-\sqrt{f(\eta)})^2\,d\nu^N_{\rho(\cdot)}(\eta).\tag{5.14}$$

5.3. **Density replacement lemmas.** In this subsection the initial measure μ_N is arbitrary.

Lemma 5.3 (Boundary replacement for the empirical density, $b\geq 5/3$)**.** *For every $a\in V_0$, let $K_j(a)$ denote the unique j-cell K_w, $|w|=j$, which contains a. Then*

$$\varlimsup_{j\to\infty}\varlimsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t\left(\eta_s^N(a)-{\rm Av}_{K_j(a)\cap V_N}[\eta_s^N]\right)\,ds\right|\right]=0\tag{5.15}$$

Proof. Consider a bounded function $g:\Omega_N\to\mathbb{R}$. Since we are in the regime $b\geq 5/3$ we can use the entropy inequality, and, thanks to (5.8), transfer the initial measure from μ_N to the product Bernoulli measure $\nu^N_{\rho(\cdot)}$ with $\rho(\cdot)=\rho$:

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t g(\eta_s^N)\,ds\right|\right]\leq\frac{H(\mu_N|\nu^N_\rho)}{\kappa|V_N|}+\frac{1}{\kappa|V_N|}\log\mathbb{E}_{\nu^N_\rho}\left[\exp\left(\kappa|V_N|\left|\int_0^t g(\eta_s^N)\,ds\right|\right)\right]\tag{5.16}$$

for every $\kappa > 0$. For the first term on the RHS, we use the estimate $H(\mu_N|\nu_\rho^N) \lesssim |V_N|$. Then we use the inequality $e^{|z|} \leq \max(e^z, e^{-z})$ and the Feynman-Kac formula to bound the logarithm in the second term on the RHS by the largest eigenvalue of $\max(-5^N\mathcal{L}_N + \kappa|V_N|g(\cdot), -5^N\mathcal{L}_N - \kappa|V_N|g(\cdot))$. Thus

$$\text{RHS}_{(5.16)} \leq \frac{C}{\kappa} + \sup_f \left\{ \int \pm g(\eta)f(\eta)\,d\nu_\rho^N(\eta) - \frac{5^N}{\kappa|V_N|}\langle \sqrt{f}, -\mathcal{L}_N\sqrt{f}\rangle_{\nu_\rho^N} \right\}, \tag{5.17}$$

where the supremum is taken over all probability densities f with respect to ν_ρ^N. WLOG we estimate the $+$ case.

Let us now specialize to

$$g(\eta) = \eta(a) - \text{Av}_B[\eta] = \frac{1}{|B|}\sum_{z\in B}(\eta(a) - \eta(z)). \tag{5.18}$$

with $B = K_j(a) \cap V_N$, and $\rho(\cdot) = \rho \in (0,1)$. Using a change of variables, followed by the identity $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$ and Young's inequality, we obtain

$$\begin{aligned}
\int g(\eta)f(\eta)\,d\nu_\rho(\eta) &= \frac{1}{|B|}\sum_{z\in B}\int (\eta(a)-\eta(z))f(\eta)\,d\nu_\rho^N(\eta) = \frac{1}{|B|}\sum_{z\in B}\int (\eta(z)-\eta(a))f(\eta^{za})\,d\nu_\rho^N(\eta)\\
&= \frac{1}{|B|}\sum_{z\in B}\frac{1}{2}\int (\eta(z)-\eta(a))(f(\eta^{za}) - f(\eta))\,d\nu_\rho^N(\eta)\\
&= \frac{1}{2|B|}\sum_{z\in B}\int (\eta(z)-\eta(a))(\sqrt{f(\eta^{za})}+\sqrt{f(\eta)})(\sqrt{f(\eta^{za})}-\sqrt{f(\eta)})\,d\nu_\rho^N(\eta)\\
&\leq \frac{1}{2|B|}\sum_{z\in B}\left(\frac{A_z}{2}\int (\eta(z)-\eta(a))^2(\sqrt{f(\eta^{za})}+\sqrt{f(\eta)})^2\,d\nu_\rho^N(\eta) + \frac{1}{2A_z}\int (\sqrt{f(\eta^{za})}-\sqrt{f(\eta)})^2\,d\nu_\rho^N(\eta)\right)
\end{aligned} \tag{5.19}$$

for any family of positive numbers $\{A_z\}_{z\in B}$.

For the first term in the bracket in (5.19), we obtain an upper bound by using $(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$ and the fact that f is a probability density:

$$\frac{1}{2}\int \int (\eta(z)-\eta(a))^2(\sqrt{f(\eta^{za})}+\sqrt{f(\eta)})^2\,d\nu_\rho^N(\eta) \leq \int (\eta(z)-\eta(a))^2(f(\eta^{za})+f(\eta))\,d\nu_\rho^N(\eta) \leq 2. \tag{5.20}$$

For the second term in the bracket in (5.19), we use the moving particle Lemma 5.1 to get

$$\frac{1}{2}\int (\sqrt{f(\eta^{za})}-\sqrt{f(\eta)})^2\,d\nu_\rho^N(\eta) \leq R_{\text{eff}}(z,a)\Gamma_N(\sqrt{f},\nu_\rho^N). \tag{5.21}$$

Since $z, a \in B$, we may bound $R_{\text{eff}}(z,a)$ from above by the diameter of B in the effective resistance metric, $\text{diam}_R(B)$. Altogether the expression (5.19) is bounded above by

$$\frac{1}{2|B|}\sum_{z\in B}\left(A_zC(\rho) + \frac{1}{A_z}\text{diam}_R(B)\Gamma_N(\sqrt{f},\nu_\rho^N)\right). \tag{5.22}$$

We then set $2A_z = \frac{\kappa|V_N|}{5^N}\text{diam}_R(B)$ for all $z \in B$ to bound the last expression from above by

$$\frac{\kappa|V_N|}{5^N}\text{diam}_R(B)C(\rho) + \frac{5^N}{\kappa|V_N|}\Gamma_N(\sqrt{f},\nu_\rho^N). \tag{5.23}$$

It is known (*cf.* [Str06, Lemma 1.6.1]) that $\text{diam}_R(K_j(a)\cap V_N) \lesssim (5/3)^{N-j}$, so the first term tends to 0 in the limit $N\to\infty$ followed by $j\to\infty$.

Recalling (5.8) and harkening back to (5.17) and (5.23), we have that the LHS of (5.16) is bounded above by

$$\begin{aligned}
&\frac{C}{\kappa} + \sup_f\left\{\frac{\kappa|V_N|}{5^N}\text{diam}_R(K_j(a)\cap V_N)C(\rho) + \frac{5^N}{\kappa|V_N|}\Gamma_N(\sqrt{f},\nu_\rho^N) - \frac{5^N}{\kappa|V_N|}\left(\Gamma_N(\sqrt{f},\nu_\rho^N) - \frac{C''(\rho)}{b^N}\right)\right\}\\
&\quad\leq \frac{C}{\kappa} + \frac{\kappa|V_N|}{5^N}\text{diam}_R(K_j(a)\cap V_N)C(\rho) + \frac{1}{\kappa}\frac{5^N}{|V_N|b^N}C''(\rho).
\end{aligned} \tag{5.24}$$

When $b > 5/3$, the final term goes to 0 as $N \to \infty$. When $b = 5/3$, the final term tends to κ^{-1} times a constant as $N \to \infty$. In any case, taking the limit $N \to \infty$, then $j \to \infty$, and finally $\kappa \to \infty$, the RHS of (5.24) tends to 0. This proves the Lemma. □

Lemma 5.4 (Boundary replacement for the empirical density, $b < 5/3$)**.** *For every $a \in V_0$,*

$$\varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \left(\eta_s^N(a) - \bar\rho(a)\right)\, ds\right|\right] = 0 \tag{5.25}$$

Proof. As in the proof of Lemma 5.3, we use the entropy inequality to transfer the initial measure from μ_N to $\nu^N_{\rho(\cdot)}$, where not only $\rho(\cdot) \in \mathcal{F}$ but also $\rho(a) = \bar\rho(a)$ for all $a \in V_0$.[2] This is necessary since we are in the regime $b < 5/3$ and the bounds obtained in (5.8) are not good enough to control the error term as $N \to +\infty$. Then we use the Feynman-Kac formula and the variational characterization of the largest eigenvalue to obtain the estimate

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_s^N(a) - \bar\rho(a))\, ds\right|\right] \le \frac{C}{\kappa} + \sup_f \left\{ \int \pm(\eta(a) - \bar\rho(a)) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) - \frac{5^N}{\kappa |V_N|}\langle \sqrt{f}, -\mathcal{L}_N \sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\right\}, \tag{5.26}$$

where the supremum is taken over all probability densities with respect to $\nu^N_{\rho(\cdot)}$.

The first term in the variational functional reads

$$\begin{aligned}
\int (\eta(a) - \bar\rho(a)) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) &= \int (1 - \eta(a) - \bar\rho(a)) f(\eta^a) \frac{d\nu^N_{\rho(\cdot)}(\eta^a)}{d\nu^N_{\rho(\cdot)}(\eta)}\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= \int \left(-\bar\rho(a)\frac{1-\bar\rho(a)}{\bar\rho(a)}\mathbf{1}_{\{\eta(a)=1\}} + (1-\bar\rho(a))\frac{\bar\rho(a)}{1-\bar\rho(a)}\mathbf{1}_{\{\eta(a)=0\}}\right) f(\eta^a)\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= -\int (\eta(a) - \bar\rho(a)) f(\eta^a)\, d\nu^N_{\rho(\cdot)}(\eta) = \int (\eta(a) - \bar\rho(a))(f(\eta) - f(\eta^a))\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= \int (\eta(a) - \bar\rho(a))(\sqrt{f(\eta)} + \sqrt{f(\eta^a)})(\sqrt{f(\eta)} - \sqrt{f(\eta^a)})\, d\nu^N_{\rho(\cdot)}(\eta) \\
&\le \frac{A}{2}\int (\eta(a) - \bar\rho(a))^2 (\sqrt{f(\eta)} + \sqrt{f(\eta^a)})^2\, d\nu^N_{\rho(\cdot)}(\eta) + \frac{1}{2A}\int (\sqrt{f(\eta)} - \sqrt{f(\eta^a)})^2\, d\nu^N_{\rho(\cdot)}(\eta)
\end{aligned} \tag{5.27}$$

for any $A > 0$, using Young's inequality at the end. The first term on the last expression can be bounded above by $AC(\bar\rho(a))$, using the inequality $(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2)$ and that f is a density with respect to $\nu^N_{\rho(\cdot)}$. Indeed, let us write $\eta = (\eta(a); \tilde\eta)$ where $\tilde\eta$ denotes the configuration except at a. Then

$$\begin{aligned}
&\frac{A}{2}\int (\eta(a) - \bar\rho(a))^2(\sqrt{f(\eta)} + \sqrt{f(\eta^a)})^2\, d\nu^N_{\rho(\cdot)}(\eta) \le A\int (\eta(a) - \bar\rho(a))^2 (f(\eta) + f(\eta^a))\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= A\left(\int (1-\bar\rho(a))^2 (f(0;\tilde\eta) + f(1;\tilde\eta))\bar\rho(a)\, d\nu^N_{\rho(\cdot)}(\tilde\eta) + \int \bar\rho(a)^2 (f(1;\tilde\eta) + f(0;\tilde\eta))(1-\bar\rho(a))\, d\nu^N_{\rho(\cdot)}(\tilde\eta)\right) \\
&= A\chi(\bar\rho(a))\int (f(0;\tilde\eta) + f(1;\tilde\eta))\, d\nu^N_{\rho(\cdot)}(\tilde\eta) \le AC(\tilde\rho(a)).
\end{aligned} \tag{5.28}$$

[2] In the proof of the replacement lemma for the 1D interval analogous to our Lemma 5.4, *cf.* [Gon18, Lemma 9 in Appendix A.4], it is assumed that the profile $\rho(\cdot)$ is Lipschitz. Here we point out that it is enough to assume the weaker condition that $\rho(\cdot) \in \mathcal{F}$. Indeed, on a compact resistance space (K, R) equipped with the effective resistance metric R, we have the inequality $|g(x) - g(y)|^2 \le R(x,y)\mathcal{E}(g) \le \mathrm{diam}_R(K)\mathcal{E}(g)$ for all $g \in \mathcal{F}$. So any function in $\mathcal{F}$ is $\frac{1}{2}$-Hölder continuous with respect to R. When K is the closed unit interval, R agrees with the Euclidean metric, so we recover the well-known result that functions in $H^1([0,1])$ have $\frac{1}{2}$-Hölder regularity with respect to the Euclidean distance.

Recalling (5.14), we can estimate (5.26) from above by

(5.29)

$$\frac{C}{\kappa}+\sup_f\left\{AC(\bar\rho(a))+\frac{1}{2A}\int(\sqrt{f(\eta)}-\sqrt{f(\eta^a)})^2\,d\nu^N_{\rho(\cdot)}(\eta)\right.$$
$$\left.-\frac{5^N}{\kappa|V_N|}\left(\Gamma_N(\sqrt{f},\nu^N_{\rho(\cdot)})-C'(\rho)\sum_{xy\in E_N}(\rho(x)-\rho(y))^2+\frac{1}{b^N}\frac{1}{2}\int\omega_a(\eta)\left(\sqrt{f(\eta^a)}-\sqrt{f(\eta)}\right)^2d\nu^N_{\rho(\cdot)}(\eta)\right)\right\}$$

To obtain a further upper bound on (5.29), set $A=\frac{1}{\min(\lambda_+(a),\lambda_-(a))}\frac{\kappa|V_N|b^N}{5^N}$ to eliminate the boundary carré du champ, and replace $\Gamma_N(\sqrt{f},\nu^N_{\rho(\cdot)})$ with 0; that is, (5.29) is bounded above by

$$\frac{C}{\kappa}+\frac{1}{\min(\lambda_+(a),\lambda_-(a))}\frac{\kappa|V_N|b^N}{5^N}C(\bar\rho(a))+\frac{1}{\kappa}\frac{5^N}{|V_N|}\sum_{xy\in E_N}(\rho(x)-\rho(y))^2. \tag{5.30}$$

Since $b<5/3$, the second term tends to 0 as $N\to\infty$. On the other hand, $\rho\in\mathcal{F}$ implies that $\sup_N\dfrac{5^N}{3^N}\displaystyle\sum_{xy\in E_N}(\rho(x)-\rho(y))^2<\infty$, so the final term is bounded above by κ^{-1} times a constant as $N\to\infty$. Therefore (5.30) tends to 0 in the limit $N\to\infty$ followed by $\kappa\to\infty$. This proves the Lemma. □

5.4. Density fluctuation replacement lemmas.

Lemma 5.5 (Boundary replacement for the DFF, $b>5/3$)**.** *For every $a\in V_0$,*

$$\varlimsup_{N\to\infty}\mathbb{E}^{N,b}_\rho\left[\left(\int_0^t\frac{5^N}{b^N\sqrt{|V_N|}}\bar\eta^N_s(a)\,ds\right)^2\right]=0. \tag{5.31}$$

Proof. By the Kipnis-Varadhan inequality, the expectation on the LHS can be bounded above by

$$Ct\sup_{f\in L^2(\nu^N_\rho)}\left\{2\int\frac{5^N}{b^N\sqrt{|V_N|}}\bar\eta(a)f(\eta)\,d\nu^N_\rho(\eta)-5^N\langle f,-\mathcal{L}_Nf\rangle_{\nu^N_\rho}\right\} \tag{5.32}$$

Note that ν^N_ρ is reversible for both the bulk generator $\mathcal{L}^{\rm bulk}_N$ and the boundary generator $\mathcal{L}^{\rm boundary}_N$. So we may rewrite the second term in the variational functional in terms of carrés du champ with no error:

$$\langle f,-\mathcal{L}_Nf\rangle_{\nu^N_\rho}=\Gamma_N(f,\nu^N_\rho)+\frac{1}{b^N}\sum_{a'\in V_0}\frac{1}{2}\int\omega_{a'}(\eta)(f(\eta^{a'})-f(\eta))^2\,d\nu^N_\rho(\eta) \tag{5.33}$$

where $\omega_{a'}(\eta)=\lambda_-\eta(a')+\lambda_+(1-\eta(a'))$. For the ensuing estimate we discard the bulk carré du champ and the boundary carré du champ except at a, that is:

$$\langle f,-\mathcal{L}_Nf\rangle_{\nu^N_\rho}\geq\frac{1}{b^N}\frac{1}{2}\int\omega_a(\eta)(f(\eta^a)-f(\eta))^2\,d\nu^N_\rho(\eta). \tag{5.34}$$

On the other hand, we may write the first term in the variational functional as $\frac{5^N}{b^N\sqrt{|V_N|}}$ times

$$\begin{aligned}2\int(\eta(a)-\rho)f(\eta)\,d\nu^N_\rho(\eta)&=2\int(1-\eta(a)-\rho)f(\eta^a)\frac{d\nu^N_\rho(\eta^a)}{d\nu^N_\rho(\eta)}\,d\nu^N_\rho(\eta)\\&=2\int\left(-\rho\frac{1-\rho}{\rho}\mathbf{1}_{\{\eta(a)=1\}}+(1-\rho)\frac{\rho}{1-\rho}\mathbf{1}_{\{\eta(a)=0\}}\right)f(\eta^a)\,d\nu^N_\rho(\eta)\\&=-2\int(\eta(a)-\rho)f(\eta^a)\,d\nu^N_\rho(\eta)=\int(\eta(a)-\rho)(f(\eta)-f(\eta^a))\,d\nu^N_\rho(\eta)\\&\leq\frac{A}{2}\int(\eta(a)-\rho)^2\,d\nu^N_\rho(\eta)+\frac{1}{2A}\int(f(\eta)-f(\eta^a))^2\,d\nu^N_\rho(\eta)\end{aligned} \tag{5.35}$$

for any $A > 0$. Now implement the estimates (5.34) and (5.35) into the variational functional in (5.32). To eliminate the boundary carré du champ at a, we set $A = \frac{1}{\min(\lambda_+,\lambda_-)}\frac{1}{\sqrt{|V_N|}}$, and this yields an upper bound on the variational functional in (5.32):

$$\begin{aligned}(5.36)\qquad &\frac{1}{\min(\lambda_+,\lambda_-)}\frac{5^N}{2b^N|V_N|}\chi(\rho) + \min(\lambda_+,\lambda_-)\frac{5^N}{b^N}\frac{1}{2}\int (f(\eta)-f(\eta^a))^2\,d\nu_\rho^N(\eta)\\ &- \frac{5^N}{b^N}\frac{1}{2}\int \omega_a(\eta)(f(\eta^a)-f(\eta))^2\,d\nu_\rho^N(\eta) \leq \frac{1}{\min(\lambda_+,\lambda_-)}\frac{5^N}{2b^N|V_N|}\chi(\rho).\end{aligned}$$

Since $b > 5/3$, the RHS goes to 0 as $N \to \infty$. This proves the Lemma. ☐

Lemma 5.6 (Boundary replacement for the DFF, $b < 5/3$)**.** *For every $a \in V_0$,*

$$(5.37)\qquad \varlimsup_{N\to\infty} \mathbb{E}_\rho^{N,b}\left[\left(\int_0^t \frac{3^N}{\sqrt{|V_N|}}\bar{\eta}_s^N(a)\,ds\right)^2\right] = 0.$$

Proof. The proof is virtually identical to that of Lemma 5.5. The only difference is in the scaling parameter, which is $\frac{3^N}{\sqrt{|V_N|}}$ instead of $\frac{5^N}{b^N\sqrt{|V_N|}}$. We follow the proof up to (5.35), and then set $A = \frac{1}{\min(\lambda_+,\lambda_-)}\frac{3^N b^N}{5^N\sqrt{|V_N|}}$ to eliminate the boundary carré du champ at a. This yields

$$(5.38)\qquad \frac{1}{\min(\lambda_+,\lambda_-)}\frac{3^{2N}b^N}{2|V_N|5^N}\chi(\rho),$$

as an upper bound on the variational functional. Since $b < 5/3$, the last expression goes to 0 as $N \to \infty$. ☐

Lemma 5.7 (Boundary replacement for the DFF, $b = 5/3$)**.** *Let $\{\beta_N\}_N$ be a sequence of numbers tending to 0 as $N \to \infty$. For every $a \in V_0$,*

$$(5.39)\qquad \varlimsup_{N\to\infty} \mathbb{E}_\rho^{N,b}\left[\left(\int_0^t \frac{3^N}{\sqrt{|V_N|}}\bar{\eta}_s^N(a)\beta_N\,ds\right)^2\right] = 0.$$

Proof. Follow the proof of Lemma 5.6 and set the same A. Then we obtain on the variational functional an upper bound

$$(5.40)\qquad \frac{1}{\min(\lambda_+,\lambda_-)}\frac{3^N}{2|V_N|}\beta_N^2\chi(\rho),$$

which tends to 0 as $N \to \infty$. ☐

6. Hydrodynamic limits of the empirical density

In this section we rigorously prove Theorem 1. Throughout the proof, we fix a time horizon $T > 0$, the boundary scaling parameter b, the initial density profile ϱ, and a sequence of probability measures $\{\mu_N\}_N$ on Ω_N associated to ϱ (*cf.* Definition 3.7). Recall that $\mathbb{P}_{\mu_N}$ is the probability measure on the Skorokhod space $D([0,T],\Omega_N)$ induced by the Markov process $\{\eta_t^N : t \geq 0\}$ with infinitesimal generator $5^N\mathcal{L}_N$. Expectation with respect to $\mathbb{P}_{\mu_N}$ is written $\mathbb{E}_{\mu_N}$.

Let $\mathcal{M}_+$ be the space of nonnegative measures on K with total mass bounded by 1. Then we denote by $\mathbb{Q}_N$ the probability measure on the Skorokhod space $D([0,T],\mathcal{M}_+)$ induced by $\{\pi_t^N : t \geq 0\}$ and by $\mathbb{P}_{\mu_N}$. The proof proceeds as follows: we show tightness of the sequence $\mathbb{Q}_N$, and then we characterize uniquely the limit point, by showing that it is a Dirac measure on the trajectory of measures $\pi_t(du) = \rho(t,u)du$, where $\rho(t,u)$ is the unique weak solution of the corresponding hydrodynamic equation.

6.1. Tightness.

In this subsection we show that $\{\mathbb{Q}_N\}_N$ is tight via the application of Aldous' criterion.

Lemma 6.1 (Aldous' criterion)**.** *Let (E,d) be a complete separable metric space. A sequence $\{P_N\}_N$ of probability measures on $D([0,T],E)$ is tight if the following hold:*

(A1) For every $t \in [0,T]$ and every $\epsilon > 0$, there exists $K_\epsilon^t \subset E$ compact such that

$$\sup_N P_N\left(X_t \notin K_\epsilon^t\right) \leq \epsilon.$$

(A2) For every $\epsilon > 0$,

$$\lim_{\gamma\to 0}\varlimsup_{N\to\infty}\sup_{\substack{\tau\in\mathcal{T}_T\\ \theta\le\gamma}} P_N\left(d(X_{(\tau+\theta)\wedge T}, X_\tau) > \epsilon\right) = 0,$$

where $\mathcal{T}_T$ denotes the family of stopping times (with respect to the canonical filtration) bounded by T.

By [KL99, Proposition 4.1.7], it suffices to show that for every F in a dense subset of $C(K)$, with respect to the uniform topology, the sequence of measures on $D([0,T],\mathbb{R})$ that correspond to the $\mathbb{R}$-valued processes $\pi_t^N(F)$ is tight. Part (A1) of Aldous' criterion says that

$$\lim_{M\to\infty}\sup_N \mathbb{P}_{\mu_N}\left[\eta_\cdot^N : |\pi_t^N(F)| > M\right] = 0. \tag{6.1}$$

This is directly verified using Chebyshev's inequality and the exclusion dynamics fact that the total mass of $\pi_\cdot^N$ is bounded above by 1. As for Part (A2) of Aldous' criterion, we need to verify that for every $\epsilon > 0$,

$$\lim_{\gamma\to 0}\varlimsup_{N\to\infty}\sup_{\substack{\tau\in\mathcal{T}_T\\ \theta\le\gamma}} \mathbb{P}_{\mu_N}\left[\eta_\cdot^N : \left|\pi_{(\tau+\theta)\wedge T}^N(F) - \pi_\tau^N(F)\right| > \epsilon\right] = 0 \tag{6.2}$$

To avoid an overcharged notation we shall write $\tau+\theta$ for $(\tau+\theta)\wedge T$ in what follows. By (3.21) we have

$$\begin{aligned}\pi_{\tau+\theta}^N(F) - \pi_\tau^N(F) &= \left(M_{\tau+\theta}^N(F) - M_\tau^N(F)\right) + \int_\tau^{\tau+\theta}\pi_s^N\left(\frac{2}{3}\Delta F\right)ds\\ &\quad - \int_\tau^{\tau+\theta}\frac{3^N}{|V_N|}\sum_{a\in V_0}\left[\eta_s^N(a)(\partial^\perp F)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a)-\bar\rho(a))F(a)\right]ds + o_N(1)\end{aligned} \tag{6.3}$$

Denoting the last integral term as $\mathcal{B}_{\tau,\tau+\theta}^N(F)$, it follows that

$$\begin{aligned}&\mathbb{P}_{\mu_N}\left[\left|\pi_{\tau+\theta}^N(F) - \pi_\tau^N(F)\right| > \epsilon\right] \le \mathbb{P}_{\mu_N}\left[\left|M_{\tau+\theta}^N(F) - M_\tau^N(F)\right| > \frac{\epsilon}{3}\right]\\ &\quad + \mathbb{P}_{\mu_N}\left[\left|\int_\tau^{\tau+\theta}\pi_s^N\left(\frac{2}{3}\Delta F\right)ds\right| > \frac{\epsilon}{3}\right] + \mathbb{P}_{\mu_N}\left[\left|\mathcal{B}_{\tau,\tau+\theta}^N(F)\right| > \frac{\epsilon}{3}\right]\\ &\quad \le \frac{9}{\epsilon^2}\left(\mathbb{E}_{\mu_N}\left[\left|M_{\tau+\theta}^N(F) - M_\tau^N(F)\right|^2\right] + \mathbb{E}_{\mu_N}\left[\left|\int_\tau^{\tau+\theta}\pi_s^N\left(\frac{2}{3}\Delta F\right)ds\right|^2\right] + \mathbb{E}_{\mu_N}\left[\left|\mathcal{B}_{\tau,\tau+\theta}^N(F)\right|^2\right]\right)\end{aligned} \tag{6.4}$$

where we used Chebyshev's inequality at the end. Our goal is to show that all three terms on the RHS of last display —the martingale term, the Laplacian term, and the boundary term—vanish in the limit stated in (6.2).

Before carrying out the estimates, we comment on the space of test functions F. When $b \ge 5/3$, we take F from $\mathrm{dom}\Delta$, which is dense in $C(K)$. When $b < 5/3$, we take F from $\mathrm{dom}\Delta_0$, which however is not dense in $C(K)$. This will be addressed at the end of the subsection.

The martingale term. We have

$$
\begin{aligned}
\mathbb{E}_{\mu_N}\left[\left|M^N_{\tau+\theta}(F)-M^N_\tau(F)\right|^2\right] &= \mathbb{E}_{\mu_N}\left[\langle M^N(F)\rangle_{\tau+\theta}-\langle M^N(F)\rangle_\tau\right]\\
&\underset{(3.19)}{=} \mathbb{E}_{\mu_N}\left[\int_\tau^{\tau+\theta}\frac{5^N}{|V_N|^2}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}}(\eta^N_s(x)-\eta^N_s(y))^2(F(x)-F(y))^2\,ds\right]\\
&\quad+\mathbb{E}_{\mu_N}\left[\int_\tau^{\tau+\theta}\sum_{a\in V_0}\frac{5^N}{b^N|V_N|^2}\{\lambda_-(a)\eta^N_s(a)+\lambda_+(a)(1-\eta^N_s(a))\}F^2(a)\,ds\right]\\
&\le C\theta\left(\frac{1}{3^N}\frac{5^N}{3^N}\sum_{\substack{x,y\in V_N\\ x\sim y}}(F(x)-F(y))^2+\frac{5^N}{b^N3^{2N}}\sum_{a\in V_0}\max(\lambda_+(a),\lambda_-(a))F^2(a)\right)\\
&\le C\theta\left(\frac{1}{3^N}\mathcal{E}_N(F)+\frac{5^N}{b^N3^{2N}}\sum_{a\in V_0}F^2(a)\right).
\end{aligned}
\tag{6.5}
$$

Since $\sup_N\mathcal{E}_N(F)<\infty$, the first term is $o_N(1)$. As for the second term, it is $o_N(1)$ when $b>5/9$. When $b\le 5/9$, we are in the Dirichlet regime and $F(a)=0$ for all $a\in V_0$, so the term vanishes anyway.

The Laplacian term. By Cauchy-Schwarz, that $\pi^N_\cdot$ has total mass bounded by 1, and that $F\in\mathrm{dom}\Delta$, we obtain

$$
\mathbb{E}_{\mu_N}\left[\left|\int_\tau^{\tau+\theta}\pi^N_s\left(\frac{2}{3}\Delta F\right)ds\right|^2\right]\le\mathbb{E}_{\mu_N}\left[\theta\int_\tau^{\tau+\theta}\left|\pi^N_s\left(\frac{2}{3}\Delta F\right)\right|^2ds\right]\le C\theta^2\left(\sup_{x\in K}|\Delta F(x)|\right)^2\le C\theta^2.
\tag{6.6}
$$

The RHS vanishes as $\theta\to 0$, so tightness of the Laplacian term follows.

The boundary term. When $b>5/3$, the second term of the integrand of $\mathcal{B}^N_{\tau,\tau+\theta}(F)$ is $o_N(1)$, and

$$
\mathbb{E}_{\mu_N}\left[\left|\mathcal{B}^N_{\tau,\tau+\theta}(F)\right|^2\right]\le C\theta^2\left(\sum_{a\in V_0}(\partial^\perp F)(a)\right)^2+o_N(1).
\tag{6.7}
$$

When $b=5/3$, both terms in the integrand of $\mathcal{B}^N_{\tau,\tau+\theta}(F)$ contribute equally:

$$
\mathbb{E}_{\mu_N}\left[\left|\mathcal{B}^N_{\tau,\tau+\theta}(F)\right|^2\right]\le C\theta^2\left(\sum_{a\in V_0}\Big((\partial^\perp F)(a)+F(a)\Big)\right)^2.
\tag{6.8}
$$

When $b<5/3$, the second term vanishes since $F(a)=0$ for all $a\in V_0$, and we have the same estimate as (6.7) without the additive $o_N(1)$. In all cases the RHS estimate vanishes as $\theta\to 0$, from which we obtain tightness of the boundary term.

We have thus far proved tightness of $\{\mathbb{Q}_N\}_N$ for $b\ge 5/3$. That said, there remains a loose end in the case $b<5/3$, since our test function space $\mathrm{dom}\Delta_0$ is not dense in $C(K)$. To tackle this issue, we follow the L^1-approximation scheme given in [Gon18, §2.9]. Note that $\mathrm{dom}\Delta_0\subset\mathcal{F}\subset L^2(K,m)\subset L^1(K,m)$, and that $\mathcal{F}$ is dense in $C(K)$. So it suffices to show that for any $F\in\mathcal{F}$ and any $\epsilon>0$,

$$
\lim_{\gamma\to 0}\varlimsup_{N\to\infty}\sup_{\substack{\tau\in\mathcal{T}_T\\ \theta\le\gamma}}\mathbb{P}_{\mu_N}\left[\eta^N_\cdot:\left|\pi^N_{\tau+\theta}(F)-\pi^N_\tau(F)\right|>\epsilon\right]=0.
\tag{6.9}
$$

Given $F\in\mathcal{F}$, let $F_k{}_k$ be a sequence in $\mathrm{dom}\Delta_0$ which converges to F in $L^1(K,m)$. Then

$$
\begin{aligned}
\mathbb{P}_{\mu_N}\left[\eta^N_\cdot:\left|\pi^N_{\tau+\theta}(F)-\pi^N_\tau(F)\right|>\epsilon\right]&\le\mathbb{P}_{\mu_N}\left[\eta^N_\cdot:\left|\pi^N_{\tau+\theta}(F-F_k)-\pi^N_\tau(F-F_k)\right|>\frac{\epsilon}{2}\right]\\
&\quad+\mathbb{P}_{\mu_N}\left[\eta^N_\cdot:\left|\pi^N_{\tau+\theta}(F_k)-\pi^N_\tau(F_k)\right|>\frac{\epsilon}{2}\right].
\end{aligned}
\tag{6.10}
$$

We have already shown that the second term on the RHS goes to 0 in the stated limit. As for the first term, we use the triangle inequality, that $\pi^N_\cdot$ is bounded above by the uniform measure on V_N, and the weak convergence of the latter measure to the self-similar measure m on K, to get

$$\left|\pi^N_{\tau+\theta}(F-F_k) - \pi^N_\tau(F-F_k)\right| \leq \frac{2}{|V_N|}\sum_{x\in V_N} |F-F_k|(x) \leq 2\|F-F_k\|_{L^1(K,m)} + o_N(1). \tag{6.11}$$

The RHS vanishes in the limit $N\to\infty$ followed by $k\to\infty$. This proves (6.9) and hence completes the proof of tightness.

6.2. **Identification of limit points.** Now that we have proved tightness of $\{\mathbb{Q}_N\}_N$, let $\mathbb{Q}$ be a limit point of $\{\mathbb{Q}_N\}_N$, *i.e.,* there exists a subsequence $\{\mathbb{Q}_{N_k}\}_k$ which converges weakly to $\mathbb{Q}$. The goal of this subsection is to prove:

Proposition 6.2. *For any limit point* $\mathbb{Q}$,

$$\mathbb{Q}\{\pi_\cdot : \pi_t(dx) = \rho_t(x)\, dm(x),\ \forall t\in[0,T]\} = 1, \tag{6.12}$$

where $\rho\in L^2(0,T,\mathcal{F})$ *is a weak solution of the heat equation with the appropriate boundary condition.*

In what follows we will fix one such limit point $\mathbb{Q}$. For ease of notation, we will suppress the subsequence subscript k from the notation. Alternatively one can assume WLOG that $\mathbb{Q}_N$ converges to $\mathbb{Q}$.

6.2.1. *Characterization of absolute continuity.* We first show that $\mathbb{Q}$ is concentrated on trajectories which are absolutely continuous with respect to the self-similar measure m on K:

$$\mathbb{Q}\{\pi_\cdot : \pi_t(dx) = \pi(t,x)\, dm(x),\ \forall t\in[0,T]\} = 1. \tag{6.13}$$

To see this, fix a $F\in C(K)$. Since there is at most one particle per site, we have that

$$\sup_{t\in[0,T]} |\pi^N_t(F)| \leq \frac{1}{|V_N|}\sum_{x\in V_N}|F(x)|.$$

It follows that the map $\pi_\cdot \mapsto \sup_{t\in[0,T]}|\pi_t(F)|$ is continuous. Consequently, all limit points are concentrated on trajectories $\pi_\cdot$ such that

$$|\pi_t(F)| \leq \int_K |F(x)|\, dm(x). \tag{6.14}$$

To see that π_t is absolutely continuous with respect to m, we will show that for any set $A\subset K$, $m(A)=0$ implies $\pi_t(A)=0$. Indeed, let $\{F_j\}_j$ be a sequence in $C(K)$ which converges to the indicator function $\mathbf{1}_A$. Then the estimate (6.14) gives $|\pi_t(A)|\leq m(A)$, which is what we need to deduce (6.13).

6.2.2. *Characterization of the initial measure.* Next we show that $\mathbb{Q}$ is concentrated on a Dirac measure equal to $\varrho(x)\,dm(x)$ at time 0. Fix $\epsilon>0$ and $F\in C(K)$. By the tightness result in the previous subsection §6.1 and Portmanteau's lemma, we have

$$\begin{aligned}
&\mathbb{Q}\left(\left|\pi_0(F) - \int_K F(x)\varrho(x)\,dm(x)\right| > \epsilon\right) \leq \varlimsup_{N\to\infty}\mathbb{Q}_N\left(\left|\pi^N_0(F) - \int_K F(x)\varrho(x)\,dm(x)\right| > \epsilon\right)\\
&= \varlimsup_{N\to\infty}\mu_N\left(\eta\in\Omega_N : \left|\frac{1}{|V_N|}\sum_{x\in V_N}F(x)\eta(x) - \int_K F(x)\varrho(x)\,dm(x)\right| > \epsilon\right) = 0,
\end{aligned} \tag{6.15}$$

since we assumed that $\{\mu_N\}_N$ is associated to ϱ, *cf.* Definition 3.7. This holds for any $\epsilon>0$ and $F\in C(K)$, so we obtain the desired claim.

6.2.3. *Characterization of the limit density.* Now we want to show that $\mathbb{Q}$ is concentrated on trajectories whose m-density equals ρ_t, a weak solution of the heat equation. Recall the definition of $\Theta_{\rm Dir}$ and $\Theta_{\rm Rob}$ from (3.8) and (3.10).

Proposition 6.3. $\mathbb{Q}\{\pi_\cdot : \Theta_b = 0,\ \forall t \in [0,T],\ \forall F \in \mathrm{dom}_b\} = 1,$ *where*

$$\Theta_b = \begin{cases} \Theta_{\rm Dir}, & \text{if } b < 5/3, \\ \Theta_{\rm Rob}, & \text{if } b = 5/3, \\ \Theta_{\rm Rob} \text{ with } \lambda_\Sigma(a) = 0,\ \forall a \in V_0, & \text{if } b > 5/3, \end{cases} \tag{6.16}$$

and

$$\mathrm{dom}_b = \begin{cases} C([0,T], \mathrm{dom}\Delta_0) \cap C^1((0,T), \mathrm{dom}\Delta_0), & \text{if } b < 5/3, \\ C([0,T], \mathrm{dom}\Delta) \cap C^1((0,T), \mathrm{dom}\Delta), & \text{if } b \geq 5/3. \end{cases} \tag{6.17}$$

Proof. We present the full proof for the case $b \geq 5/3$, which consists of several approximation and replacement steps. The proof for the case $b < 5/3$ is simpler and will be sketched at the end.

We want to show that for every $\delta > 0$,

$$\mathbb{Q}\left\{\pi \in D([0,T], \mathcal{M}_+) : \sup_{t\in[0,T]} |\Theta_{\rm Rob}| > \delta\right\} = 0, \tag{6.18}$$

where ρ in $\Theta_{\rm Rob}$ should be understood as the m-density of π. There is however a problem: the boundary terms involving $\rho_\cdot(a)$ are not direct functions of π, so the event in question is not an open set in the Skorokhod space. Therefore we cannot apply Portmanteau's lemma right away. To address the issue we use two ideas from analysis. The first idea is local averaging, that is, to replace $\rho_\cdot(a)$ by $\pi_\cdot(\iota_j^a)$, the pairing of the limit measure $\pi_\cdot$ with the approximate identity $\iota_j^a : K \to \mathbb{R}_+$ given by

$$\iota_j^a(x) = \frac{1}{m(K_j(a))} \mathbf{1}_{K_j(a)}(x) \tag{6.19}$$

where $K_j(a)$ is the unique j-cell containing a. This almost achieves what we want, except that ι_j^a is not a continuous function. Thus comes the second idea, which is to approximate $\iota_j^a \in L^1(K,m)$ by a sequence of bump functions $\{\tilde{\iota}_{j,\epsilon}^a\}_{\epsilon>0} \subset C(K)$ in $L^1(K,m)$. Here we use the standard fact that $C(K)$ is dense in $L^1(K,m)$.

With these two ideas we can use Portmanteau's Lemma and pass from $\pi_\cdot$ to the discrete empirical measure $\pi_\cdot^N$. Note that

$$\pi_\cdot^N(\iota_j^a) = \frac{1}{|V_N|}\frac{1}{m(K_j(a))} \sum_{x\in K_j(a)\cap V_N} \eta_\cdot^N(x) = \frac{1}{|K_j(a)\cap V_N|} \sum_{x\in K_j(a)\cap V_N} \eta_\cdot^N(x) = \mathrm{Av}_{K_j(a)\cap V_N}[\eta_\cdot^N]. \tag{6.20}$$

The density replacement Lemma 5.3 states that (6.20) replaces $\eta_\cdot^N(a)$ in $L^1(\mathbb{P}_{\mu_N})$ as $N \to \infty$ then $j \to \infty$.

To summarize, the order in which we perform the replacements is

$$\rho_\cdot(a) \xrightarrow[\substack{\epsilon\downarrow 0 \\ j\to\infty}]{\rho = \frac{d\pi}{dm}} \pi_\cdot(\tilde{\iota}_{j,\epsilon}^a) \xrightarrow[\text{Portmanteau}]{\text{in } D([0,T],\mathcal{M}_+)} \pi_\cdot^N(\tilde{\iota}_{j,\epsilon}^a) \xrightarrow[\substack{\text{approx. in} \\ L^1(K,m)}]{\text{in } L^1(\mathbb{P}_{\mu_{\rm N}})} \pi_\cdot^N(\iota_j^a) = \mathrm{Av}_{K_j(a)\cap V_N}[\eta_\cdot^N] \xrightarrow[\substack{\text{Replacement} \\ \text{Lemma 5.3}}]{\text{in } L^1(\mathbb{P}_{\mu_{\rm N}})} \eta_\cdot^N(a). \tag{6.21}$$

Starting with the first step in the replacement diagram (6.21), we subtract and add $\pi_s(\tilde{\iota}^a_{j,\epsilon})$ to each $\rho_s(a)$ in $\Theta_{\rm Rob}$, which then leads to the following upper bound on the probability in (6.18) (recall $|V_0|=3$):

$$\mathbb{Q}\left(\sup_{t\in[0,T]}\left|\int_K \rho_t(x)F_t(x)\,dm(x) - \int_K \rho_0(x)F_0(x)\,dm(x) - \int_0^t\int_K \rho_s(x)\left(\frac{2}{3}\Delta+\partial_s\right)F_s(x)\,dm(x)\,ds\right.\right.$$
$$\left.\left.+\frac{2}{3}\int_0^t \sum_{a\in V_0}\left[\pi_s(\tilde{\iota}^a_{j,\epsilon})(\partial^\perp F_s)(a)+\lambda_\Sigma(a)(\pi_s(\tilde{\iota}^a_{j,\epsilon})-\bar{\rho}(a))F_s(a)\right]ds\right|>\frac{\delta}{5}\right) \tag{6.22}$$

$$+\mathbb{Q}\left(\left|\int_K(\rho_0(x)-\varrho(x))F_0(x)\,dm(x)\right|>\frac{\delta}{5}\right) \tag{6.23}$$

$$+\sum_{a\in V_0}\mathbb{Q}\left(\sup_{t\in[0,T]}\left|\frac{2}{3}\int_0^t(\rho_s(a)-\pi_s(\tilde{\iota}^a_{j,\epsilon}))\left((\partial^\perp F_s)(a)+\lambda_\Sigma(a)F_s(a)\right)ds\right|>\frac{\delta}{5}\right). \tag{6.24}$$

The second term (6.23) vanishes by (6.15). For the last term (6.24), note that $\pi_\cdot(\tilde{\iota}^{j,\epsilon}_a)\to\pi_\cdot(\iota^j_a)$ as $\epsilon\downarrow 0$, and $\pi_\cdot(\iota^j_a)$, being the average density over $K_j(a)$, converges to $\rho_\cdot(a)$ as $j\to\infty$. So (6.24) vanishes in the limit $\epsilon\downarrow 0$ followed by $j\to\infty$. That leaves us with the first term (6.22): we note that the supremum of the long expression is a continuous function of $\pi\in D([0,T],\mathcal{M}_+)$, so the event is an open set in the Skorokhod space. Therefore by Portmanteau's Lemma, (6.22) is bounded above by

$$\varlimsup_{N\to\infty}\mathbb{Q}_N\left(\sup_{t\in[0,T]}\left|\pi^N_t(F_t)-\pi^N_0(F_0)-\int_0^t\pi^N_s\left(\left(\frac{2}{3}\Delta+\partial_s\right)F_s\right)ds\right.\right.$$
$$\left.\left.+\frac{2}{3}\int_0^t\sum_{a\in V_0}\left[\pi^N_s(\tilde{\iota}^a_{j,\epsilon})(\partial^\perp F_s)(a)+\lambda_\Sigma(a)(\pi^N_s(\tilde{\iota}^a_{j,\epsilon})-\bar{\rho}(a))F_s(a)\right]ds\right|>\frac{\delta}{5}\right). \tag{6.25}$$

We now apply the last two steps of the replacement diagram (6.21) by writing

$$\pi^N_\cdot(\tilde{\iota}^a_{j,\epsilon})=\eta^N_\cdot(a)+(\pi^N_\cdot(\iota^a_j)-\eta^N_\cdot(a))+(\pi^N_\cdot(\tilde{\iota}^a_{j,\epsilon})-\pi^N_\cdot(\iota^a_j)),$$

and thereby bounding (6.25) from above by

$$\varlimsup_{N\to\infty}\mathbb{Q}_N\left(\sup_{t\in[0,T]}\left|\pi^N_t(F_t)-\pi^N_0(F_0)-\int_0^t\pi^N_s\left(\left(\frac{2}{3}\Delta+\partial_s\right)F_s\right)ds\right.\right.$$
$$\left.\left.+\frac{2}{3}\int_0^t\sum_{a\in V_0}\left[\eta^N_s(a)(\partial^\perp F_s)(a)+\lambda_\Sigma(a)(\eta^N_s(a)-\bar{\rho}(a))F_s(a)\right]ds\right|>\frac{\delta}{35}\right) \tag{6.26}$$

$$+\sum_{a\in V_0}\varlimsup_{N\to\infty}\mathbb{Q}_N\left(\sup_{t\in[0,T]}\left|\frac{2}{3}\int_0^t(\pi^N_s(\iota^a_j)-\eta^N_s(a))\left((\partial^\perp F_s)(a)+\lambda_\Sigma(a)F_s(a)\right)ds\right|>\frac{\delta}{35}\right) \tag{6.27}$$

$$+\sum_{a\in V_0}\varlimsup_{N\to\infty}\mathbb{Q}_N\left(\sup_{t\in[0,T]}\left|\frac{2}{3}\int_0^t(\pi^N_s(\tilde{\iota}^a_{j,\epsilon})-\pi^N_s(\iota^a_j))\left((\partial^\perp F_s)(a)+\lambda_\Sigma(a)F_s(a)\right)ds\right|>\frac{\delta}{35}\right). \tag{6.28}$$

The second term (6.27) vanishes as $j\to\infty$ thanks to Lemma 5.3. The last term (6.28) vanishes as $\epsilon\downarrow 0$ due to the convergence $\tilde{\iota}^a_{j,\epsilon}\to\iota^a_j$ in $L^1(K,m)$. As for the first term (6.26), observe that the expression inside the absolute value matches $M^N_t(F)$ (3.21) up to an additional $o_N(1)$ function. Thus it remains to show that

$$\varlimsup_{N\to\infty}\mathbb{Q}_N\left(\sup_{t\in[0,T]}|M^N_t(F)|>\frac{\delta}{70}\right)=0. \tag{6.29}$$

By Doob's inequality,

$$\mathbb{Q}_N\left(\sup_{t\in[0,T]}|M^N_t(F)|>\delta\right)\le\frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left[|M^N_T(F)|^2\right]=\frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left[\langle M^N(F)\rangle_T\right]. \tag{6.30}$$

By a similar computation as in (6.5) we find that the last term goes to 0 as $N \to \infty$. This proves (6.18).

For the case $b < 5/3$, observe that Θ_{Dir} does not have the boundary term issue of Θ_{Rob}, and is already a continuous function of $\pi \in D([0,T], \mathcal{M}_+)$. Therefore the proof goes through provided that we can replace $\bar{\rho}(a)$ by $\eta^N_\cdot(a)$ in $\mathbb{P}_{\mu_N}$-probability as $N \to \infty$, which follows from the replacement Lemma 5.4. □

6.2.4. *Characterization of the limit density in $L^2(0,T,\mathcal{F})$.* Last but not least, we show that $\mathbb{Q}$ is concentrated on trajectories $\pi_\cdot$ whose density ρ is in $L^2(0,T,\mathcal{F})$. This is to verify the notion of weak solutions introduced in Definitions 3.3 through 3.5.

Proposition 6.4. $\mathbb{Q}\{\pi_\cdot : \rho \in L^2(0,T,\mathcal{F})\} = 1$.

To prove Proposition 6.4, we use a variational approach which is reminiscent of the quadratic minimization principle in PDE theory.

Lemma 6.5. *There exists $\kappa > 0$ such that*

(6.31)
$$\mathbb{E}_{\mathbb{Q}}\left[\sup_F \left\{\int_0^T \int_K (-\Delta F_s)(x)\rho_s(x)\,dm(x)\,ds + \int_0^T \sum_{a\in V_0} (\partial^\perp F_s)(a)\rho_s(a)\,ds - \kappa\int_0^T \mathcal{E}(F_s)\,ds\right\}\right] < \infty,$$

where the supremum is taken over all $F \in C([0,T], \text{dom}\Delta)$.

Remark 6.6. We invite the reader to compare the linear functional in (6.31) to the one used in the 1D setting, *e.g.* [BMNS17, Lemma 5.11]. A key difference is that on SG we do not have an easy notion of a 1st derivative (gradient); instead we appeal to the integration by parts formula (Lemma 3.2-(3)).

Before proving Lemma 6.5, let us observe how Proposition 6.4 follows from the Lemma and the Riesz representation theorem.

Proof of Proposition 6.4 assuming Lemma 6.5. Given a density $\rho : [0,T] \times K \to [0,1]$, define the linear functional $\ell_\rho : C([0,T], \text{dom}\Delta) \to \mathbb{R}$ by

$$\ell_\rho(F) = \int_0^T \int_K (-\Delta F_s)(x)\rho_s(x)\,dm(x)\,ds + \int_0^T \sum_{a\in V_0} (\partial^\perp F_s)(a)\rho_s(a)\,ds. \tag{6.32}$$

Observe that if we had known in advance that $\rho \in L^2(0,T,\mathcal{F})$, then $\ell_\rho(F) = \int_0^T \mathcal{E}(F_s,\rho_s)\,ds$ by the integration by parts formula (Lemma 3.2-(3)). In fact we will prove the reverse implication. For the rest of the proof all statements hold $\mathbb{Q}$-a.s. On the one hand, Lemma 6.5 implies that there exists a constant $C = C(\rho)$ independent of F such that

$$\ell_\rho(F) - \kappa\int_0^T \mathcal{E}(F_s)\,ds \le C. \tag{6.33}$$

On the other hand, by (6.14) we have $\|\rho_t\|_{L^2(K,m)} \le 1$ for every $t \in [0,T]$. So by Cauchy-Schwarz, for any $\kappa > 0$,

$$\int_0^T \langle F_s,\rho_s\rangle_{L^2(K,m)}\,ds - \kappa\int_0^T \|F_s\|^2_{L^2(K,m)}\,ds \le -\kappa\int_0^T \left(\|F_s\|_{L^2(K,m)} - \frac{1}{2\kappa}\right)^2 ds + \frac{T}{4\kappa} \le \frac{T}{4\kappa}. \tag{6.34}$$

Adding (6.33) and (6.34) together, we see that

$$\left(\ell_\rho(F) + \int_0^T \langle F_s,\rho_s\rangle_{L^2}\,ds\right) - \kappa\|F\|^2_{L^2(0,T,\mathcal{F})} \le C' := C + \frac{T}{4\kappa}, \tag{6.35}$$

the RHS being independent of F. Let us denote $\ell^1_\rho(F) := \ell_\rho(F) + \int_0^T \langle F_s,\rho_s\rangle_{L^2(K,m)}\,ds$. Observe that we can apply the transformation $F \to \alpha F$ for any number α to get

$$\alpha\ell^1_\rho(F) - \alpha^2\kappa\|F\|^2_{L^2(0,T,\mathcal{F})} \le C'. \tag{6.36}$$

Making the square on the LHS we obtain that

$$-\kappa\|F\|^2_{L^2(0,T,\mathcal{F})}\left(\alpha-\frac{\ell^1_\rho(F)}{2\kappa\|F\|^2_{L^2(0,T,\mathcal{F})}}\right)^2+\frac{(\ell^1_\rho(F))^2}{4\kappa\|F\|^2_{L^2(0,T,\mathcal{F})}}\leq C'. \tag{6.37}$$

Minimizing the LHS we find

$$\ell^1_\rho(F)\leq(4\kappa C')^{1/2}\|F\|_{L^2(0,T,\mathcal{F})}, \tag{6.38}$$

which shows that ℓ^1_ρ is a bounded linear functional on $C([0,T],\text{dom}\Delta)$. Moreover, since $\text{dom}\Delta$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and $C([0,T])$ is dense in $L^2([0,T])$, we can extend ℓ_ρ via density to a bounded linear functional on the Hilbert space $L^2(0,T,\mathcal{F})$. By the Riesz representation theorem, there exists $\mathfrak{R}\in L^2(0,T,\mathcal{F})$ such that

$$\ell^1_\rho(F)=\langle F,\mathfrak{R}\rangle_{L^2(0,T,\mathcal{F})}=\int_0^T\mathcal{E}_1(F_s,\mathfrak{R}_s)\,ds,\quad\forall F\in L^2(0,T,\mathcal{F}). \tag{6.39}$$

By (6.32) and the integration by parts formula, deduce that for all $F\in C([0,T],\text{dom}\Delta)$

$$\begin{aligned}&\int_0^T\int_K[(-\Delta F_s)(x)+F_s(x)]\rho_s(x)\,dm(x)\,ds+\int_0^T\sum_{a\in V_0}(\partial^\perp F_s)(a)\rho_s(a)\,ds\\&=\int_0^T\int_K[(-\Delta F_s)(x)+F_s(x)]\mathfrak{R}_s(x)\,dm(x)\,ds+\int_0^T\sum_{a\in V_0}(\partial^\perp F_s)(a)\mathfrak{R}_s(a)\,ds.\end{aligned} \tag{6.40}$$

Infer that $\rho=\mathfrak{R}$ $(m\times dt)$-a.e. on $K\times[0,T]$, and also on V_0 for a.e. $t\in[0,T]$. This implies in particular that $\rho=\mathfrak{R}$ in $L^2(0,T,\mathcal{F})$. □

Proof of Lemma 6.5. We focus on the case $b\geq 5/3$. Given $F\in C([0,T],\text{dom}\Delta)$, construct a sequence $\{F^i\}_{i\in\mathbb{N}}$ which converges to F in the $C([0,T],\text{dom}\Delta)$-norm. It then suffices to verify that there exists a constant C such that for every $n\in\mathbb{N}$,

$$\mathbb{E}_{\mathbb{Q}}\left[\max_{1\leq i\leq n}\left\{\int_0^T\int_K(-\Delta F^i_s)(x)\rho_s(x)\,dm(x)\,ds+\int_0^T\sum_{a\in V_0}(\partial^\perp F^i_s)(a)\rho_s(a)\,ds-\kappa\int_0^T\mathcal{E}(F^i_s)\,ds\right\}\right]\leq C \tag{6.41}$$

We would like to pass from $\mathbb{Q}$ to $\mathbb{Q}_N$ using Portmanteau's lemma, but due to the appearance of $\rho_\cdot(a)$ in the boundary term, the functional in (6.31) is not a continuous function of π. To fix this issue we use the same replacement ideas as in the previous subsection, *cf.* (6.21). First we replace $\rho_\cdot(a)$ by $\pi_\cdot(\tilde{\iota}^a_{j,\epsilon})$: the LHS of (6.41) is bounded above by

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\max_{1\leq i\leq n}\left\{\int_0^T\int_K(-\Delta F^i_s)(x)\,\pi_s(dx)\,ds+\int_0^T\sum_{a\in V_0}(\partial^\perp F^i_s)(a)\pi_s(\tilde{\iota}^a_{j,\epsilon})\,ds-\kappa\int_0^T\mathcal{E}(F^i_s)\,ds\right\}\right]\\&+\mathbb{E}_{\mathbb{Q}}\left[\max_{1\leq i\leq n}\left|\int_0^T\sum_{a\in V_0}(\partial^\perp F^i_s)(a)\left(\rho_s(a)-\pi_s(\tilde{\iota}^a_{j,\epsilon})\right)\,ds\right|\right],\end{aligned} \tag{6.42}$$

and the last term tends to 0, $\mathbb{Q}$-a.s., as $\epsilon \downarrow 0$ then $j \to \infty$. We can then apply Portmanteau's Lemma to the first term and rewrite it as

(6.43)
$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}_N}\left[\max_{1\le i\le n}\left\{\int_0^T \pi_s^N(-\Delta F_s^i)\,ds + \int_0^T \sum_{a\in V_0}(\partial^\perp F_s^i)(a)\pi_s^N(\tilde{\iota}_{j,\epsilon}^a)\,ds - \kappa\int_0^T \mathcal{E}(F_s^i)\,ds\right\}\right]\\
&\le \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\max_{1\le i\le n}\left\{\int_0^T -\frac{3}{2}\frac{1}{|V_N|}\sum_{x\in V_N\setminus V_0}\eta_s^N(x)(\Delta_N F_s^i)(x)\,ds + \int_0^T \sum_{a\in V_0}(\partial_N^\perp F_s^i)(a)\eta_s^N(a)\,ds - \kappa\int_0^T \mathcal{E}(F_s^i)\,ds\right\}\right]\\
&+ \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\max_{1\le i\le n}\left|\int_0^T \left(\frac{3}{2}\frac{1}{|V_N|}\sum_{x\in V_N\setminus V_0}\eta_s^N(x)(\Delta_N F_s^i)(x) - \pi_s^N(\Delta F_s^i)\right)ds\right|\right]\\
&+ \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\max_{1\le i\le n}\left|\int_0^T \sum_{a\in V_0}(\partial^\perp F_s^i)(a)(\pi_s^N(\tilde{\iota}_{j,\epsilon}^a) - \eta_s^N(a))\,ds\right|\right]\\
&+ \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\max_{1\le i\le n}\left|\int_0^T \sum_{a\in V_0}((\partial^\perp F_s^i)(a) - (\partial_N^\perp F_s^i)(a))\eta_s^N(a)\,ds\right|\right].
\end{aligned}$$

On the RHS, the second term vanishes by the convergence $\frac{3}{2}\Delta_N F^i \to \Delta F^i$ in $C([0,T]\times(K\setminus V_0))$ and the argument in (3.20). The penultimate term vanishes using the last two steps of the replacement (6.21) in $L^1(\mathbb{P}_{\mu_N})$. The last term vanishes by Lemma 3.2-(2). This leaves us with the first term: we can combine the first two integrals to get

$$\int_0^T \frac{5^N}{3^N}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}}(F_s^i(x) - F_s^i(y))\eta_s^N(x)\,ds + o_N(1). \tag{6.44}$$

Then we apply the entropy inequality, Jensen's inequality, and the inequality $\exp(\max_i a_i) \le \sum_i e^{a_i}$ to bound it from above by

$$\begin{aligned}
&\frac{H(\mu_N|\nu_{\rho(\cdot)}^N)}{|V_N|} + \frac{1}{|V_N|}\log\left(\sum_{1\le i\le j}\mathbb{E}_{\nu_{\rho(\cdot)}^N}\left[\exp\left(|V_N|\int_0^T \frac{5^N}{3^N}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}}(F_s^i(x) - F_s^i(y))\eta_s^N(x)\,ds\right.\right.\right.\\
&\qquad\left.\left.\left. +|V_N|o_N(1) - \kappa|V_N|\int_0^T \mathcal{E}(F_s^i)\,ds\right)\right]\right),
\end{aligned} \tag{6.45}$$

where the density $\rho(\cdot)$ is taken to be constant ρ. On the one hand, the first term of (6.45) is bounded by a constant C independent of N. On the other hand, we also need to bound the second term by a constant independent of N and F. This will be proved in Lemma 6.7 below.

Now we give a sketch of the argument for the case $b < 5/3$. Here the condition $\rho_\cdot(a) = \bar{\rho}(a)$ is imposed for all $a \in V_0$, so the passage from $\mathbb{Q}$ to $\mathbb{Q}_N$ via Portmanteau's Lemma is straightforward. After that we make two changes in the preceding proof to arrive at (6.45): replace $\bar{\rho}(a)$ by $\eta_\cdot^N(a)$ in $L^1(\mathbb{P}_{\mu_N})$; and use a density profile $\rho(\cdot)$ such that $\rho(\cdot) \in \mathcal{F}$ **and** $\rho(a) = \bar{\rho}(a)$ for all $a \in V_0$ in (6.45). □

Lemma 6.7. *Choose $\rho(\cdot) = \rho$ constant if $b \ge 5/3$, whereas if $b < 5/3$, choose $\rho(\cdot) \in \mathcal{F}$ such that it is bounded away from 0 and from 1, and $\rho(a) = \bar{\rho}(a)$ for all $a \in V_0$. Then there exists a positive constant C such that for all $F \in C([0,T], \mathrm{dom}\Delta)$,*

(6.46)
$$\varlimsup_{N\to\infty}\frac{1}{|V_N|}\log\mathbb{E}_{\nu_{\rho(\cdot)}^N}\left[\exp\left(|V_N|\left(\int_0^T \frac{5^N}{3^N}\sum_{x\in V_N}\sum_{\substack{y\in V_N\\ y\sim x}}(F_s(x) - F_s(y))\eta_s^N(x)\,ds - \kappa\int_0^T \mathcal{E}(F_s)\,ds\right)\right)\right] \le C,$$

Proof. By the Feynman-Kac formula, the expression under the limit in the LHS of (6.46) equals

$$\int_0^T \sup_f \left\{ \int \frac{5^N}{3^N} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (F_s(x)-F_s(y))\eta(x) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) - \kappa\mathcal{E}(F_s) - \frac{5^N}{|V_N|}\left\langle \sqrt{f}, -\mathcal{L}_N\sqrt{f}\right\rangle_{\nu^N_{\rho(\cdot)}} \right\} ds, \tag{6.47}$$

where the supremum is taken over all probability densities f with respect to $\nu^N_{\rho(\cdot)}$. Observe that

$$\begin{aligned}
&\int \frac{5^N}{3^N} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (F_s(x)-F_s(y))\eta(x) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= \int \frac{5^N}{3^N}\frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (F_s(x)-F_s(y))(\eta(x)-\eta(y)) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= \frac{5^N}{3^N}\frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (F_s(x)-F_s(y)) \left(\int \eta(x) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) - \int \eta(x) f(\eta^{xy})\, d\nu^N_{\rho(\cdot)}(\eta^{xy}) \right),
\end{aligned} \tag{6.48}$$

where we apply a change of variable $\eta \to \eta^{xy}$ in the last line.

Suppose $b \geq 5/3$, so we choose $\rho(\cdot) = \rho$ constant. Then $\nu^N_\rho(\eta^{xy}) = \nu^N_\rho(\eta)$, and (6.48) rewrites as

$$\begin{aligned}
&\frac{5^N}{3^N}\frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (F_s(x)-F_s(y)) \int \eta(x)(f(\eta)-f(\eta^{xy}))\, d\nu^N_\rho(\eta) \\
&= \frac{5^N}{3^N}\frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y\in V_N\\ y\sim x}} (F_s(x)-F_s(y)) \int \eta(x)(\sqrt{f(\eta)}+\sqrt{f(\eta^{xy})})(\sqrt{f(\eta)}-\sqrt{f(\eta^{xy})})\, d\nu^N_\rho(\eta),
\end{aligned} \tag{6.49}$$

which, by Young's inequality and $(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$, can be bounded above by

$$\begin{aligned}
&\frac{5^N}{3^N}\left(\sum_{xy\in E_N} \int \frac{A}{2}(\eta(x))^2 (F_s(x)-F_s(y))^2 \cdot 2(f(\eta)+f(\eta^{xy}))\, d\nu^N_\rho(\eta) + \sum_{xy\in E_N} \int \frac{1}{2A}(\sqrt{f(\eta)}-\sqrt{f(\eta^{xy})})^2\, d\nu^N_\rho(\eta) \right) \\
&\leq \frac{5^N}{3^N}\left(2A \sum_{xy\in E_N} (F_s(x)-F_s(y))^2 + \frac{1}{A}\Gamma_N(\sqrt{f}, \nu^N_\rho) \right) = 2A\mathcal{E}_N(F_s) + \frac{5^N}{3^N A}\Gamma_N(\sqrt{f}, \nu^N_\rho)
\end{aligned} \tag{6.50}$$

for any $A > 0$. Combine with the lower estimate (5.8) of the Dirichlet form $\langle \sqrt{f}, -\mathcal{L}_N\sqrt{f}\rangle_{\nu^N_\rho}$, and we find that (6.47) is bounded above by

$$\int_0^T \sup_f \left\{ 2A\mathcal{E}_N(F_s) + \frac{5^N}{3^N A}\Gamma_N(\sqrt{f}, \nu^N_\rho) - \kappa\mathcal{E}(F_s) - \frac{5^N}{|V_N|}\Gamma_N(\sqrt{f}, \nu^N_\rho) + \frac{5^N}{b^N |V_N|} C''(\rho) \right\} ds \tag{6.51}$$

To eliminate the dependence on f and F of the variational functional, we choose $A = 3^{-N}|V_N| \to \frac{3}{2}$ and $\kappa = 2A$ (recall that $\mathcal{E}_N(F) \uparrow \mathcal{E}(F)$). This allows us to further bound from above by the time integral of the last term, which is at most of order unity.

Now suppose $b < 5/3$, so we choose $\rho(\cdot) \in \mathcal{F}$ such that $\rho(\cdot) \in [\delta, 1-\delta]$ for some $\delta > 0$, and that $\rho(a) = \bar{\rho}(a)$ for all $a \in V_0$. Due to the nonconstancy of $\rho(\cdot)$, the argument following the change of variables performed in (6.48) has to be modified:

$$\begin{aligned}
&\int \eta(x) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) - \int \eta(x) f(\eta^{xy})\, d\nu^N_{\rho(\cdot)}(\eta^{xy}) = \int \eta(x) \left[f(\eta) - f(\eta^{xy}) \frac{d\nu^N_{\rho(\cdot)}(\eta^{xy})}{d\nu^N_{\rho(\cdot)}(\eta)} \right] d\nu^N_{\rho(\cdot)}(\eta) \\
&= \int \eta(x) \left[f(\eta) - f(\eta^{xy}) \frac{\rho(y)(1-\rho(x))}{\rho(x)(1-\rho(y))} \right] d\nu^N_{\rho(\cdot)}(\eta) \\
&= \int \eta(x)(f(\eta) - f(\eta^{xy}))\, d\nu^N_{\rho(\cdot)}(\eta) + \frac{\rho(x)-\rho(y)}{\rho(x)(1-\rho(y))} \int \eta(x) f(\eta^{xy})\, d\nu^N_{\rho(\cdot)}(\eta).
\end{aligned} \tag{6.52}$$

We implement (6.52) into (6.48) and rewrite the latter as

$$\frac{5^N}{3^N} \frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y \in V_N \\ y\sim x}} (F_s(x) - F_s(y)) \int \eta(x) (\sqrt{f(\eta)} + \sqrt{f(\eta^{xy})})(\sqrt{f(\eta)} - \sqrt{f(\eta^{xy})})\, d\nu^N_{\rho(\cdot)}(\eta) \tag{6.53}$$

$$+\frac{5^N}{3^N} \frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y \in V_N \\ y\sim x}} (F_s(x) - F_s(y)) \frac{\rho(x)-\rho(y)}{\rho(x)(1-\rho(y))} \int \eta(x) f(\eta^{xy})\, d\nu^N_{\rho(\cdot)}(\eta). \tag{6.54}$$

The first term (6.53) is treated as in (6.49) through the first line of (6.50). Then we will come across an integral which admits the estimate

$$\begin{aligned}
\int (\eta(x))^2 f(\eta^{xy})\, d\nu^N_{\rho(\cdot)}(\eta) &= \int (\eta(y))^2\, f(\eta) \frac{d\nu^N_{\rho(\cdot)}(\eta^{xy})}{d\nu^N_{\rho(\cdot)}(\eta)}\, d\nu^N_{\rho(\cdot)}(\eta) \\
&= \frac{\rho(x)(1-\rho(y))}{\rho(y)(1-\rho(x))} \int \eta(y) f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) \leq \delta^{-2}.
\end{aligned} \tag{6.55}$$

In the above inequality we bound the numerator $\rho(x)(1-\rho(y))$ from above by 1, the denominator $\rho(y)(1-\rho(x))$ from below by δ^2, and the integral from above by $\int f(\eta)\, d\nu^N_{\rho(\cdot)}(\eta) = 1$. Consequently

$$(6.53) \leq A(1+\delta^{-2}) \mathcal{E}_N(F_s) + \frac{5^N}{3^N A} \Gamma_N(\sqrt{f}, \nu^N_{\rho(\cdot)}). \tag{6.56}$$

As for the second term (6.54), we use again that $\rho(\cdot) \in [\delta, 1-\delta]$, that the integral $\int \eta(x) f(\eta^{xy})\, d\nu^N_{\rho(\cdot)}(\eta)$ is bounded above by δ^{-2}, and Young's inequality to obtain

$$\begin{aligned}
(6.54) &\leq \delta^{-4} \frac{5^N}{3^N} \frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y \in V_N \\ y\sim x}} |F_s(x) - F_s(y)| |\rho(x) - \rho(y)| \\
&\leq \frac{\delta^{-4}}{2} \frac{5^N}{3^N} \left(\frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y \in V_N \\ y\sim x}} |F_s(x) - F_s(y)|^2 + \frac{1}{2} \sum_{x\in V_N} \sum_{\substack{y \in V_N \\ y\sim x}} |\rho(x) - \rho(y)|^2 \right) = \frac{\delta^{-4}}{2} \left(\mathcal{E}_N(F_s) + \mathcal{E}_N(\rho) \right).
\end{aligned} \tag{6.57}$$

Finally recall the lower estimate (5.14) of the Dirichlet form $\langle \sqrt{f}, -\mathcal{L}_N \sqrt{f} \rangle_{\nu^N_{\rho(\cdot)}}$, except that we will discard the final boundary contribution. Putting everything together, we bound (6.47) from above by

$$\begin{aligned}
\int_0^T \sup_f \Bigg\{ & A(1+\delta^{-2}) \mathcal{E}_N(F_s) + \frac{5^N}{3^N A} \Gamma_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + \frac{\delta^{-4}}{2} (\mathcal{E}_N(F_s) + \mathcal{E}_N(\rho)) - \kappa \mathcal{E}(F_s) \\
& - \frac{5^N}{|V_N|} \Gamma_N(\sqrt{f}, \nu^N_{\rho(\cdot)}) + C'(\rho) \frac{5^N}{|V_N|} \sum_{xy \in E_N} (\rho(x) - \rho(y))^2 \Bigg\}\, ds.
\end{aligned} \tag{6.58}$$

To eliminate the dependence on f and F of the variational functional, we choose $A = 3^{-N}|V_N| \to \frac{3}{2}$ and $\kappa = A(1+\delta^{-2}) + \frac{\delta^{-4}}{2}$. This gives a further upper bound in the form of the time integral of a constant multiple of $\mathcal{E}(\rho)$, which is finite because $\rho(\cdot) \in \mathcal{F}$. □

6.2.5. *Characterization of the density at the boundary when $b < 5/3$.* To finish the characterization of limit points in the case $b < 5/3$, it remains to show condition (3) in Definition 3.3. We observe however that, showing that the profile has the value $\bar\rho(a)$ at a is quite standard now (for details on this the reader can check Section 5.3 of [BGJO17]) and it is a consequence of the next lemma.

Lemma 6.8 (Fixing the profile at the boundary)**.** *For every $a \in V_0$, let $K_j(a)$ denote the unique j-cell K_w, $|w| = j$, which contains a. Then*

$$\varlimsup_{j\to\infty} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \left(\bar\rho(a) - \mathrm{Av}_{K_j(a)\cap V_N}[\eta_s^N]\right) ds\right|\right] = 0$$

The proof of this lemma is a consequence of both Lemmas 5.4 and 5.3. We also remark that Lemma 5.3 is proved in the regime $b \geq 5/3$, but if fact it holds for any b. The only difference in the proof is that one has to use the reference measure $\nu^N_{\rho(\cdot)}$ with a suitable profile as the one in the proof of Lemma 5.4. We leave the details to the reader on the adaptation of the arguments to this case.

7. Existence & uniqueness of weak solutions to the heat equation

To conclude the proof of Theorem 1 it remains to establish Lemma 3.6. Our main result is:

Proposition 7.1. *The unique weak solution of the heat equation with boundary condition* b *is*

$$\rho^{\rm b}(t,\cdot) = \rho^{\rm b}_{\rm ss} + \tilde{\mathsf{T}}^b_t\left(\varrho - \rho^{\rm b}_{\rm ss}\right), \tag{7.1}$$

where $\rho^{\rm b}_{\rm ss}$ is the steady-state solution satisfying Laplace's equation $\Delta_{\rm b}\rho^{\rm b}_{\rm ss} = 0$ on $K \setminus V_0$, and boundary condition (for all $a \in V_0$)

$$\begin{cases} \rho^{\rm b}_{\rm ss}(a) = \bar\rho(a), & \text{if } {\rm b} = \text{Dir},\\ \partial^\perp \rho^{\rm b}_{\rm ss}(a) = 0, & \text{if } {\rm b} = \text{Neu},\\ \partial^\perp \rho^{\rm b}_{\rm ss}(a) = -\bar\lambda_\Sigma(a)(\rho^{\rm b}_{ss}(a) - \bar\rho(a)), & \text{if } {\rm b} = \text{Rob}.\end{cases} \tag{7.2}$$

In particular, we have the following long-time limit:

$$\lim_{t\to\infty} \rho^{\rm b}(t,\cdot) = \begin{cases} \rho^{\rm b}_{\rm ss}, & \text{if } {\rm b} \in \{\text{Dir}, \text{Rob}\},\\ \int_K \varrho\, dm, & \text{if } {\rm b} = \text{Neu}.\end{cases} \tag{7.3}$$

Actually (7.1) is a strong solution of the heat equation. Upon multiplying (7.1) by a test function $F(t,x)$ and integrating over $[0,T] \times K$, and performing integration by parts, one can verify that a strong solution is a weak solution. It thus remains to show that weak solutions are unique, which we verify in the next subsection.

The underlying ideas of this section are standard from the PDE perspective, and are well known to analysts on fractals; see *e.g.* [Jar13, Chapter 4] for an exposition in the Dirichlet case. Nevertheless, we decide to spell out the arguments for completeness, especially for the Robin boundary case.

7.1. **Strong solution.** It is readily verified that $\rho^{\rm b} = \rho^{\rm b}_{\rm ss} + u^{\rm b}$ is a strong solution of (7.1), where $u^{\rm b}$ satisfies

$$\begin{cases} \partial_t u^{\rm b} = \frac{2}{3}\Delta u^{\rm b}, & t \in [0,T],\ x \in K \setminus V_0\\ u^{\rm b}(0,x) = \varrho(x) - \rho^{\rm b}_{\rm ss}(x), & x \in K\end{cases} \tag{7.4}$$

along with boundary condition (for all $a \in V_0$)

$$\begin{cases} u^{\rm b}(a) = 0, & \text{if b} = \text{Dir},\\ \partial^\perp u^{\rm b}(a) = 0, & \text{if b} = \text{Neu},\\ \partial^\perp u^{\rm b}(a) = -\bar\lambda_\Sigma(a) u^{\rm b}(a), & \text{if b} = \text{Rob}.\end{cases} \tag{7.5}$$

Solution to the homogeneous heat equation, $u^{\rm b}$. By the functional calculus, the solution to (7.4) is uniquely given by

$$u^{\rm b}(t,x) = \tilde{\mathsf{T}}^b_t(\varrho - \rho^{\rm b}_{\rm ss}) = \sum_{n=1}^\infty \alpha^{\rm b}_n[\varrho - \rho^{\rm b}_{\rm ss}] e^{-(2/3)\lambda^{\rm b}_n t} \varphi^{\rm b}_n(x), \tag{7.6}$$

where $\alpha^{\rm b}_n[f] = \int_K f \varphi^{\rm b}_n\, dm$ are the Fourier coefficients. By Lemma 4.6-(2), $u^{\rm b} \in L^2(0,T,\mathcal{F})$.

Steady-state solution, $\rho_{\rm ss}^{\rm b}$. If $\rm b = Dir$, we have

$$\begin{cases} \Delta\rho_{\rm ss}^{\rm Dir}(x) = 0, & x \in K \setminus V_0, \\ \rho_{\rm ss}^{\rm Dir}(a) = \bar\rho(a), & a \in V_0, \end{cases} \tag{7.7}$$

namely, $\rho_{\rm ss}^{\rm Dir}$ is the unique harmonic extension of the boundary data $\bar\rho$ from V_0 to K. We remind the reader the explicit harmonic extension algorithm known as the "$\frac{1}{5}$-$\frac{2}{5}$ rule" [Str06, §1.3]. In particular, the algorithm implies that the space of harmonic functions on K is 3-dimensional.

If $\rm b = Neu$, we have

$$\begin{cases} \Delta\rho_{\rm ss}^{\rm Neu}(x) = 0, & x \in K \setminus V_0, \\ \partial^\perp\rho_{\rm ss}^{\rm Neu}(a) = 0, & a \in V_0. \end{cases} \tag{7.8}$$

Note that $\rho_{\rm ss}^{\rm Neu}$ is non-unique: any constant function is a solution.

Finally, if $\rm b = Rob$, we have

$$\begin{cases} \Delta\rho_{\rm ss}^{\rm Rob}(x) = 0, & x \in K \setminus V_0, \\ \partial^\perp\rho_{\rm ss}^{\rm Rob}(a) = -\bar\lambda_\Sigma(a)(\rho_{\rm ss}^{\rm Rob}(a) - \bar\rho(a)), & a \in V_0. \end{cases} \tag{7.9}$$

We can convert this to a Dirichlet problem and solve for the unique solution using the Dirichlet-to-Neumann map. The outcome is that $\rho_{\rm ss}^{\rm Rob}$ is the harmonic extension of $\bar\rho^{\rm R}$ from V_0 to K, where

$$\begin{bmatrix} \bar\rho^{\rm R}(a_0) \\ \bar\rho^{\rm R}(a_1) \\ \bar\rho^{\rm R}(a_2) \end{bmatrix} = \frac{1}{\boldsymbol\Delta} \begin{bmatrix} 3+2(\kappa_1+\kappa_2)+\kappa_1\kappa_2 & 3+\kappa_2 & 3+\kappa_1 \\ 3+\kappa_2 & 3+2(\kappa_2+\kappa_0)+\kappa_2\kappa_0 & 3+\kappa_0 \\ 3+\kappa_1 & 3+\kappa_0 & 3+2(\kappa_0+\kappa_1)+\kappa_0\kappa_1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}, \tag{7.10}$$

$$\boldsymbol\Delta := 3(\kappa_0+\kappa_1+\kappa_2) + 2(\kappa_0\kappa_1+\kappa_1\kappa_2+\kappa_2\kappa_0) + \kappa_0\kappa_1\kappa_2, \tag{7.11}$$

$\kappa_i = \bar\lambda_\Sigma(a_i)$, and $\gamma_i = \bar\lambda_\Sigma(a_i)\bar\rho(a_i)$, $i \in \{0,1,2\}$. Note that $\boldsymbol\Delta \neq 0$ if not all of the boundary rates $\lambda_\Sigma(a_i)$ are zero, and that $\bar\rho^{\rm R} \neq \bar\rho$. The interested reader is referred to Appendix B for the computations.

It follows that $\rho^{\rm b}_\cdot = \rho_{\rm ss}^{\rm b} + \tilde{\mathsf T}^b_\cdot\left(\varrho - \rho_{\rm ss}^{\rm b}\right) \in L^2(0,T,\mathcal F)$.

At this point we have addressed all but the uniqueness question when $\rm b = Neu$. Since $\rho_{\rm ss}^{\rm Neu} = c$ for any constant c, we write $\rho^{\rm Neu}(t,\cdot) = c + \tilde{\mathsf T}_t^{\rm Neu}(\varrho - c)$. But by the fact that $\Delta c = 0$ and functional calculus, we find that $c + \tilde{\mathsf T}_t^{\rm Neu}(\varrho - c) = c + \tilde{\mathsf T}_t^{\rm Neu}\varrho - c = \tilde{\mathsf T}_t^{\rm Neu}\varrho$. So $\rho^{\rm Neu}$ is uniquely determined by ϱ.

We leave the reader to verify (7.3), and that $\rho^{\rm b}$ is also a weak solution in the sense of Definition 3.3, 3.4, or 3.5.

7.2. **Uniqueness of weak solutions.** In this subsection we prove uniqueness of the weak solution. For this purpose, let $\rho^1, \rho^2 \in L^2(0,T,\mathcal F)$ be two weak solutions of the heat equation. Set $u := \rho^1 - \rho^2 \in L^2(0,T,\mathcal F)$. From the initial condition we have $u(0,\cdot) \equiv 0$. We want to show that $u \equiv 0$.

<u>If $\rm b = Dir$:</u> Recall (3) of Definition 3.3. Then, for a.e. $t \in (0,T]$ and all $a \in V_0$, $\rho^1(t,a) = \rho^2(t,a) = \bar\rho(a)$, so that $u(t,a) = 0$, *i.e.*, $u \in L^2(0,T,\mathcal F_0)$. Using (3.8) we find that

$$\int_K u_T(x)F_T(x)\,dm(x) - \int_0^T \int_K u_s(x)\left(\frac{2}{3}\Delta + \partial_s\right)F_s(x)\,dm(x)\,ds = 0 \tag{7.12}$$

for all $F \in C([0,T],{\rm dom}\Delta_0) \cap C^1((0,T),{\rm dom}\Delta_0)$. Furthermore, by the integration by parts formula (Definition 3.1) we may rewrite this as

$$\int_K u_T(x)F_T(x)\,dm(x) - \int_0^t \int_K u_s(x)(\partial_s F_s)(x)\,dm(x)\,ds + \frac{2}{3}\int_0^T \mathcal E(u_s, F_s)\,ds = 0 \tag{7.13}$$

Since ${\rm dom}\Delta_0$ is $\mathcal E_1$-dense in $\mathcal F_0$, and $C([0,T]) \cap C^1((0,T))$ is dense in $L^2(0,T)$, we can find a sequence $\{u_j\}_{j\in\mathbb N}$ in $C([0,T],{\rm dom}\Delta_0) \cap C^1((0,T),{\rm dom}\Delta_0)$ which converges to u in $L^2(0,T,\mathcal F_0)$. Let

$$v_j(t,x) = \int_t^T u_j(s,x)\,ds \qquad \forall j \in \mathbb N,\ \forall t \in [0,T],\ \forall x \in K. \tag{7.14}$$

By plugging v_j into F in (7.13), and taking the limit $j \to \infty$ using Lemma 7.2 below, we obtain

$$\int_0^T \int_K |u_s(x)|^2\,dm(x)\,ds + \frac{1}{3}\mathcal E\left(\int_0^T u_s\,ds\right) = 0. \tag{7.15}$$

Both terms on the LHS being nonnegative, we deduce that $u \equiv 0$ in $L^2([0,T]\times K, ds\times m)$, and hence also in $L^2(0,T,\mathcal{F}_0)$.

It remains to prove:

Lemma 7.2. *Let $\{v_j\}_{j\in\mathbb{N}}$ be defined as in (7.14). Then:*

(1) $\displaystyle\lim_{j\to\infty}\int_0^T\int_K u_s(x)(\partial_s v_j)(s,x)\,dm(x)\,ds = -\int_0^T\int_K |u_s(x)|^2\,dm(x)\,ds.$

(2) $\displaystyle\lim_{j\to\infty}\int_0^T \mathcal{E}(u_s, v_j(s,\cdot))\,ds = \frac{1}{2}\mathcal{E}\left(\int_0^T u_s\,ds\right).$

Proof. For Item (1), we use that $(\partial_s v_j)(s,x) = -u_j(s,x)$ to write

$$\begin{aligned}\int_0^T\int_K u_s(x)(\partial_s v_j)(s,x)\,dm(x)\,ds &= -\int_0^T\int_K u_s(x)u_j(s,x)\,dm(x)\,ds\\ &= -\int_0^T\int_K |u_s(x)|^2\,dm(x)\,ds + \int_0^T\int_K u_s(x)(u_s(x)-u_j(s,x))\,dm(x)\,ds.\end{aligned}\tag{7.16}$$

We then use Cauchy-Schwarz to argue that the second term vanishes as $j\to\infty$:

$$\begin{aligned}&\left|\int_0^T\int_K u_s(x)(u_s(x)-u_j(s,x))\,dm(x)\,ds\right|\\ &\leq \left(\int_0^T\int_K |u_s(x)|^2\,dm(x)\,ds\right)^{1/2}\left(\int_0^T\int_K |u_s(x)-u_j(s,x)|^2\,dm(x)\,ds\right)^{1/2}\xrightarrow[j\to\infty]{} 0\end{aligned}\tag{7.17}$$

since $u_j \to u$ in $L^2(0,T,\mathcal{F}_0)$.

For Item (2), we use the bilinearity of the Dirichlet form to write

$$\begin{aligned}\int_0^T \mathcal{E}(u_s, v_j(s,\cdot))\,ds &= \int_0^T \mathcal{E}\left(u_s, \int_s^T u_r\,dr\right)ds + \int_0^T \mathcal{E}\left(u_s, v_j(s,\cdot) - \int_s^T u_r\,dr\right)ds\\ &= \int_0^T\int_s^T \mathcal{E}(u_s,u_r)\,dr\,ds + \int_0^T \mathcal{E}\left(u_s, \int_s^T (u_j(r,\cdot)-u_r)\,dr\right)ds.\end{aligned}\tag{7.18}$$

We further exploit the bilinearity and symmetry of the Dirichlet from to rewrite the first term of (7.18):

$$\begin{aligned}\int_0^T\int_s^T \mathcal{E}(u_s,u_r)\,dr\,ds &= \int_{0\leq s\leq r\leq T}\mathcal{E}(u_s,u_r)\,dr\,ds\\ &= \frac{1}{2}\int_{[0,T]^2}\mathcal{E}(u_s,u_r)\,dr\,ds = \frac{1}{2}\mathcal{E}\left(\int_0^T u_s\,ds, \int_0^T u_r\,dr\right).\end{aligned}\tag{7.19}$$

Meanwhile, for the second term of (7.18), we apply Cauchy-Schwarz and Hölder's inequalities in succession to show that it vanishes as $j\to\infty$:

$$\begin{aligned}&\left|\int_0^T \mathcal{E}\left(u_s, \int_s^T (u_j(r,\cdot)-u_r)\,dr\right)ds\right| \leq \int_0^T \sqrt{\mathcal{E}(u_s)}\sqrt{\mathcal{E}\left(\int_s^T (u_j(r,\cdot)-u_r)\,dr\right)}\,ds\\ &\underset{(7.21)}{\leq} \int_0^T \sqrt{\mathcal{E}(u_s)}\left(\int_s^T \sqrt{\mathcal{E}(u_j(r,\cdot)-u_r)}\,dr\right)ds\\ &\leq \left(\int_0^T \sqrt{\mathcal{E}(u_s)}\,ds\right)\cdot\sup_{s\in[0,T]}\left(\int_s^T \sqrt{\mathcal{E}(u_j(r,\cdot)-u_r)}\,dr\right) \qquad \text{(Hölder's inequality)}\\ &\leq \left(\int_0^T \sqrt{\mathcal{E}(u_s)}\,ds\right)\left(\int_0^T \sqrt{\mathcal{E}(u_j(r,\cdot)-u_r)}\,dr\right)\\ &\leq T\left(\int_0^T \mathcal{E}(u_s)\,ds\right)^{1/2}\left(\int_0^T \mathcal{E}(u_j(r,\cdot)-u_r)\,dr\right)^{1/2}\xrightarrow[j\to\infty]{} 0\end{aligned}\tag{7.20}$$

since $u_j \to u$ in $L^2(0,T,\mathcal{F}_0)$. In the second inequality above we used

$$\begin{aligned}\mathcal{E}\left(\int_s^T f(r)\,dr, \int_s^T f(r')\,dr'\right) &= \int_s^T \int_s^T \mathcal{E}(f(r), f(r'))\,dr'\,dr \\ &\le \int_s^T \int_s^T \sqrt{\mathcal{E}(f(r))}\sqrt{\mathcal{E}(f(r'))}\,dr'\,dr = \left(\int_s^T \sqrt{\mathcal{E}(f(r))}\,dr\right)^2.\end{aligned} \tag{7.21}$$

for any $f \in L^2(0,T,\mathcal{F})$. □

<u>If b $\in \{\text{Rob}, \text{Neu}\}$:</u> Using (3.10) we have

$$\begin{aligned}&\int_K u_T(x)F_T(x)\,dm(x) - \int_0^T \int_K u_s(x)\left(\frac{2}{3}\Delta + \partial_s\right)F_s(x)\,dm(x)\,ds \\ &\quad + \frac{2}{3}\int_0^T \sum_{a\in V_0}\left[u_s(a)(\partial^\perp F_s)(a) + \bar{\lambda}_\Sigma(a)u_s(a)F_s(a)\right]\,ds = 0\end{aligned} \tag{7.22}$$

for all $F \in C([0,T], \text{dom}\Delta) \cap C^1((0,T), \text{dom}\Delta)$. Again using integration by parts (Lemma 3.1-(3)), we can rewrite this as

$$\int_K u_T(x)F_T(x)\,dm(x) - \int_0^T \int_K u_s(x)(\partial_s F_s)(x)\,dm(x)\,ds + \frac{2}{3}\int_0^T \mathscr{E}_b(u_s, F_s)\,ds = 0, \tag{7.23}$$

where $\mathscr{E}_b$ was defined in (4.17). Now we follow the same strategy as in the Dirichlet case. Let $L^2(0,T,\mathcal{F}_b)$ be the Hilbert space with norm

$$\|f\|_{L^2(0,T,\mathcal{F}_b)} := \left(\int_0^T \left(\mathscr{E}_b(f_s) + \|f_s\|^2_{L^2(K,m)}\right)\,ds\right)^{1/2}. \tag{7.24}$$

On the one hand, $L^2(0,T,\mathcal{F}_{\text{Rob}})$ contains $C([0,T], \text{dom}\Delta) \cap C^1((0,T), \text{dom}\Delta)$. On the other hand, any function $f \in L^2(0,T,\mathcal{F})$ also belongs to $L^2(0,T,\mathcal{F}_{\text{Rob}})$, with $\|f\|_{L^2(0,T,\mathcal{F}_{\text{Rob}})} \ge \|f\|_{L^2(0,T,\mathcal{F})}$. Since $C([0,T], \text{dom}\Delta) \cap C^1((0,T), \text{dom}\Delta)$ is dense in $L^2(0,T,\mathcal{F})$, it follows that $C([0,T], \text{dom}\Delta) \cap C^1((0,T), \text{dom}\Delta)$ is dense in $L^2(0,T,\mathcal{F}_{\text{Rob}})$. Let $\{u_j\}_{j\in\mathbb{N}} \subset C([0,T], \text{dom}\Delta) \cap C^1((0,T), \text{dom}\Delta)$ be a sequence converging to u in $L^2(0,T,\mathcal{F}_b)$, $b \in \{\text{Rob}, \text{Neu}\}$. Define v_j exactly as in (7.14). Then we plug v_j into F in (7.23), and note that we have the exact analogs of Lemma 7.2, except that $\mathcal{E}$ is replaced by $\mathscr{E}_b$ in Item (2). Applying the analogs and taking the limit $j \to \infty$, we obtain

$$\int_0^T \int_K |u_s(x)|^2\,dm(x)\,ds + \frac{1}{3}\mathscr{E}_b\left(\int_0^T u_s\,ds\right) = 0. \tag{7.25}$$

Each term on the LHS being nonnegative, we infer that $u \equiv 0$ in $L^2(0,T,\mathcal{F}_b)$ (whence in $L^2(0,T,\mathcal{F})$).

8. Ornstein-Uhlenbeck limits of equilibrium density fluctuations

In this section we prove Theorem 2.

8.1. Tightness and identification of limit points.

The main result of this subsection is

Proposition 8.1. *The sequence $\{\mathbb{Q}_\rho^{N,b}\}_N$ is tight with respect to the uniform topology on $C([0,T],\mathcal{S}_b')$. Under any limit point $\mathbb{Q}_\rho^b$ of the sequence, the process $\{\mathcal{Y}_t(F) : t \in [0,T],\ F \in \mathcal{S}_b\}$ satisfies **(OU1)** of Definition 4.8.*

First of all, we invoke Mitoma's criterion in order to establish tightness of the $\mathcal{S}_b'$-valued processes $\{\mathcal{Y}_t^N : t \in [0,T]\}_N$ from tightness of the $\mathbb{R}$-valued processes $\{\mathcal{Y}_t^N(F) : t \in [0,T]\}_N$ for $F \in \mathcal{S}_b$.

Lemma 8.2 (Mitoma's criterion [Mit83, Theorems 3.1 & 4.1])**.** *Let $\mathcal{S}$ be a Fréchet space and $\mathcal{S}'$ be its topological dual. A sequence of processes $\{X_t^N : t \in [0,T]\}_N$ is tight with respect to the Skorokhod topology on $D([0,T],\mathcal{S}')$ (resp. the uniform topology on $C([0,T],\mathcal{S}')$) if and only if the sequence $\{X_t^N(f) : t \in [0,T]\}_N$ of $\mathbb{R}$-valued processes is tight with respect to the Skorokhod topology on $D([0,T],\mathbb{R})$ (resp. the uniform topology on $C([0,T],\mathbb{R})$) for any $f \in \mathcal{S}$.*

To check for tightness of $\{\mathcal{Y}_t^N(F) : t \in [0,T]\}_N$, we verify (A1) of Aldous' criterion, and the following condition for an $\mathbb{R}$-valued process in place of (A2):

(AC2) For every $\epsilon > 0$,

$$\overline{\lim_{N\to\infty}} P_N\left(\sup_{t\in[0,T]} |X_t - X_{t-}| > \epsilon\right) = 0. \tag{8.1}$$

Recall from (4.6) that for any $F \in \mathcal{S}_b$,

$$\mathcal{Y}_t^N(F) = \mathcal{Y}_0^N(F) + \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds - \mathcal{B}_t^N(F) + \mathcal{M}_t^N(F) + o_N(1), \tag{8.2}$$

$$\text{where } \mathcal{B}_t^N(F) = \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a\in V_0} \bar{\eta}_s^N(a)\left[(\partial^\perp F)(a) + \frac{5^N}{b^N 3^N}\lambda_\Sigma F(a) + \left((\partial_N^\perp F)(a) - (\partial^\perp F)(a)\right)\right] ds. \tag{8.3}$$

To show tightness of $\{\mathcal{Y}_t^N(F) : t \in [0,T]\}_N$, it suffices to check, up to extraction of a common subsequence, tightness of each of the four terms on the RHS of (8.2)—the initial measure, the Laplacian term, the boundary term, and the martingale term—using either Aldous' criterion or a direct proof of convergence. To avoid an overcharged notation, we suppress the subsequence index in what follows.

Convergence of the initial measure. We want to prove that $\mathcal{Y}_0^N \xrightarrow{d} \mathcal{Y}_0$, where $\mathcal{Y}_0$ is a centered $\mathcal{S}_b'$-valued Gaussian random variable with covariance given by (4.23). This relies on computing the characteristic function of $\mathcal{Y}_0^N(F)$, which is possible thanks to the product Bernoulli measure $\mathbb{P}_\rho^{N,b}$ (below $i = \sqrt{-1}$):

$$\begin{aligned}
\log\mathbb{E}_\rho^{N,b}\left[\exp\left(i\lambda\mathcal{Y}_0^N(F)\right)\right] &= \log\mathbb{E}_\rho^{N,b}\left[\exp\left(i\lambda\frac{1}{\sqrt{|V_N|}}\sum_{x\in V_N}(\eta_0^N(x)-\rho)F(x)\right)\right]\\
&= \log\prod_{x\in V_N}\left(1 - \frac{\lambda^2}{2|V_N|}\chi(\rho)F^2(x) + \mathcal{O}\left(\frac{1}{|V_N|^{3/2}}\right)\right)\\
&= -\frac{\lambda^2}{2}\chi(\rho)\frac{1}{|V_N|}\sum_{x\in V_N}F^2(x) + \mathcal{O}\left(\frac{1}{|V_N|^{1/2}}\right) \xrightarrow[N\to\infty]{} -\frac{\lambda^2}{2}\chi(\rho)\int_K F^2(x)\,dm(x).
\end{aligned} \tag{8.4}$$

For the convergence in the last line we used that $F \in C(K)$ to pass from the discrete sum to the integral. To conclude the proof, we replace F by a linear combination of functions and use the Crámer-Wold device.

Remark 8.3. The above proof can be repeated to show that for each $t \in [0,T]$, $\mathcal{Y}_t^N \xrightarrow{d} \mathcal{Y}_t$, where $\mathcal{Y}_t$ is a centered $\mathcal{S}_b'$-valued Gaussian random variable. In particular, $\{\mathcal{Y}_t : t \in [0,T]\}$ is a stationary solution to the Ornstein-Uhlenbeck equation, Definition 4.8, for any $b > 0$.

The Laplacian term. We verify the tightness criterion. To check (A1), we estimate using Cauchy-Schwarz and the stationarity of the product measure $\mathbb{P}_\rho^{N,b}$ to find that for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}_\rho^{N,b}\left[\left(\int_0^t \mathcal{Y}_s^N(\Delta_N F)\,ds\right)^2\right] &= \mathbb{E}_\rho^{N,b}\left[\left(\int_0^t \frac{1}{\sqrt{|V_N|}}\sum_{x\in V_N}\bar{\eta}_s^N(x)(\Delta_N F)(a)\,ds\right)^2\right]\\
&\le CT\int_0^T \mathbb{E}_\rho^{N,b}\left[\left(\frac{1}{\sqrt{|V_N|}}\sum_{x\in V_N}\bar{\eta}^N(x)(\Delta_N F)(x)\right)^2\right] ds = CT^2\chi(\rho)\left(\frac{1}{|V_N|}\sum_{x\in V_N}(\Delta_N F)^2(x)\right).
\end{aligned} \tag{8.5}$$

Since $F \in \mathcal{S}_b$ implies that $\Delta F \in C(K)$, we see that the last expression is bounded. To check (AC2), we use Chebyshev's inequality and (8.5) to find that for every pair of times $t - \theta < t \in [0,T]$ and every $\epsilon > 0$,

$$\mathbb{Q}_\rho^{N,b}\left[\left|\int_{t-\theta}^t \mathcal{Y}_s(\Delta_N F)\,ds\right| > \epsilon\right] \le \frac{1}{\epsilon^2}\mathbb{E}_\rho^{N,b}\left[\left|\int_{t-\theta}^t \mathcal{Y}_s(\Delta_N F)\,ds\right|^2\right] \le \frac{C\theta^2}{\epsilon^2} \xrightarrow[\theta\downarrow 0]{} 0. \tag{8.6}$$

We can perform one more substitution, by replacing $\Delta_N F$ with $\frac{2}{3}\Delta F$. Apply an estimate similar to (8.5), and we get

(8.7)
$$\mathbb{E}_\rho^{N,b}\left[\left(\int_0^t \left(\mathcal{Y}_s^N\left(\frac{2}{3}\Delta F\right) - \mathcal{Y}_s^N(\Delta_N F)\right) ds\right)^2\right] \leq CT^2\chi(\rho)\left(\frac{1}{|V_N|}\sum_{x\in V_N}\left(\left(\frac{2}{3}\Delta F\right)(x) - (\Delta_N F)(x)\right)^2\right).$$

Since $\frac{3}{2}\Delta_N F \to \Delta F$ in $C(K)$, *cf.* Lemma 3.2-(1) and the remark following Lemma 3.2, the bracketed sum tends to 0 as $N \to \infty$. This proves

$$\int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds \to \int_0^t \mathcal{Y}_s^N\left(\frac{2}{3}\Delta F\right) ds \quad \text{in } L^2(\mathbb{Q}_\rho^{N,b}) \tag{8.8}$$

and shows that any limit point of the Laplacian term takes on the form $\int_0^t \mathcal{Y}_s(\frac{2}{3}\Delta F)\, ds$.

The boundary term. We claim that for any regime of b,

$$\varlimsup_{N\to\infty} \mathbb{E}_\rho^{N,b}\left[|\mathcal{B}_t^N(F)|^2\right] = 0, \quad \forall F \in \mathcal{S}_b, \tag{8.9}$$

where $\mathcal{B}_t^N(F)$ was defined in (8.3). We verify this case by case.

The case $b > 5/3$: Then $(\partial^\perp F)|_{V_0} = 0$ for any $F \in \mathcal{S}_b$. There are two contributions to $\mathcal{B}_t^N(F)$:

$$\int_0^t \sum_{a\in V_0} \frac{3^N}{\sqrt{|V_N|}} \bar{\eta}_s^N(a)(\partial_N^\perp F)(a)\, ds + \int_0^t \sum_{a\in V_0} \frac{5^N}{b^N\sqrt{|V_N|}} \bar{\eta}_s^N(a)\lambda_\Sigma F(a)\, ds \tag{8.10}$$

We argue that both terms vanish as $N \to \infty$, using different arguments.

For the first term of (8.10), we upper bound $|\bar{\eta}_s^N(a)|$ by 1, and aim to show that $3^{N/2}|(\partial_N^\perp F)(a_0)| \to 0$ for every $a \in V_0$. By [Str06, Lemma 2.7.4(b)], if $F \in \text{dom}\Delta$ and $(\partial^\perp F)(a_0) = 0$, then there exists $C > 0$ such that for all N,

$$\sup_{x\in\mathfrak{F}_0^N(K)} |F(x) - F(a_0)| \leq CN5^{-N}. \tag{8.11}$$

(In addition, if ΔF satisfies a Hölder condition—which holds for $F \in \mathcal{S}_{\text{Neu}}$—then the RHS estimate can be improved to $C5^{-N}$.) Therefore

(8.12)
$$3^{N/2}|(\partial_N^\perp F)(a_0)| = 3^{N/2}\left|\frac{5^N}{3^N}\sum_{\substack{y\in V_N\\ y\sim a_0}}(F(a_0) - F(y))\right| \leq \frac{5^N}{3^{N/2}}\sum_{\substack{y\in V_N\\ y\sim a_0}}|F(a_0) - F(y)| \underset{(8.11)}{\leq} 2C\frac{(N)}{3^{N/2}} \xrightarrow[N\to\infty]{} 0,$$

proving the desired claim at a_0. The same argument applies to the other boundary points a_1 and a_2.

The second term of (8.10) vanishes as $N \to \infty$ by virtue of the replacement Lemma 5.5.

The case $b < 5/3$: Then $F|_{V_0} = 0$ for any $F \in \mathcal{S}_b$. Thus $\mathcal{B}_t^N(F)$ equals

$$\int_0^t \sum_{a\in V_0} \frac{3^N}{\sqrt{|V_N|}} \bar{\eta}_s^N(a)(\partial_N^\perp F)(a)\, ds,$$

which vanishes as $N \to \infty$ by virtue of the replacement Lemma 5.6.

The case $b = 5/3$: Then $(\partial^\perp F)|_{V_0} = -\lambda_\Sigma F|_{V_0}$ for any $F \in \mathcal{S}_b$. Thus $\mathcal{B}_t^N(F)$ equals

$$\frac{3^N}{\sqrt{|V_N|}}\int_0^t \sum_{a\in V_0} \bar{\eta}_s^N(a)\left((\partial_N^\perp F)(a) - (\partial^\perp F)(a)\right) ds.$$

This vanishes as $N \to \infty$ by virtue of $(\partial_N^\perp F)(a) - (\partial^\perp F)(a) = o_N(1)$ and the replacement Lemma 5.7.

The martingale term. Recall the computation (4.9), and note that for any $F \in \mathcal{S}_b$,

$$\lim_{N\to\infty} \mathbb{E}_\rho^{N,b}\left[|\mathcal{M}_t^N(F)|^2\right] = \frac{2}{3}\cdot 2\chi(\rho)t\mathscr{E}_b(F) < \infty, \tag{8.13}$$

where $\mathscr{E}_b$ was defined in (4.17). This estimate is enough to verify tightness. In fact, it shows that $\{\mathcal{M}_t^N(F) : t \in [0,T]\}_N$ is a uniformly integrable (UI) family of martingales, so by the martingale convergence theorem it converges in distribution to a martingale $\{\mathcal{M}_t(F) : t \in [0,T]\}$.

Identification of limit points. At this point we have shown that any limit point $\mathcal{Y}_\cdot(F) \in C([0,T],\mathbb{R})$ of $\{\mathcal{Y}_\cdot^N(F)\}_N$, whose law we denote by $\mathbb{Q}_b^\rho$, satisfies that $\mathcal{Y}_t(F)$ is Gaussian for each t, and that

$$\mathcal{M}_t(F) = \mathcal{Y}_t(F) - \mathcal{Y}_0(F) - \int_0^t \mathcal{Y}_s\left(\frac{2}{3}\Delta_b F\right) ds \tag{8.14}$$

is a martingale. It remains to show that the quadratic variation of $\mathcal{M}_t(F)$ equals $\frac{2}{3}\cdot 2\chi(\rho)t\mathscr{E}_b(F)$.

Recall that each term of the sequence

$$\left\{(\mathcal{M}_t^N(F))^2 - \langle \mathcal{M}^N(F)\rangle_t : t \in [0,T]\right\}_N \tag{8.15}$$

is a martingale. Using tightness of $\{\mathcal{M}_t^N(F)\}_N$ and (8.13), we see that as $N \to \infty$, the limit in distribution of this sequence is

$$\left\{\mathcal{N}_t(F) := (\mathcal{M}_t(F))^2 - \frac{2}{3}\cdot 2\chi(\rho)t\mathscr{E}_b(F) : t \in [0,T]\right\}. \tag{8.16}$$

The quadratic variation claim follows once we show that $\mathcal{N}_\cdot(F)$ is a martingale. This is done by checking both $\{(\mathcal{M}_t^N(F))^2\}_N$ and $\{\langle \mathcal{M}^N(F)\rangle_t\}_N$ are UI families, and then applying the martingale convergence theorem to the sequence (8.15).

By (4.9) (or (8.13)), $\mathbb{E}_\rho^{N,b}\left[\langle \mathcal{M}^N(F)\rangle_t\right]$ is bounded for all N, which is enough to imply that $\{\langle \mathcal{M}^N(F)\rangle_t\}_N$ is UI. To show that $\{(\mathcal{M}_t^N(F))^2\}_N$ is UI, it suffices to show that $\mathbb{E}_\rho^{N,b}\left[(\mathcal{M}_t^N(F))^4\right]$ is uniformly bounded in N. By [DG91, Lemma 3], which is a consequence of the Burkholder-Davis-Gundy inequality, there exists $C > 0$ such that for all N,

$$\mathbb{E}_\rho^{N,b}\left[(\mathcal{M}_t^N(F))^4\right] \le C\left(\mathbb{E}_\rho^{N,b}[(\mathcal{M}_t^N(F))^2] + \mathbb{E}_\rho^{N,b}\left[\sup_{t\in[0,T]} |\mathcal{M}_t^N(F) - \mathcal{M}_{t^-}^N(F)|^4\right]\right). \tag{8.17}$$

On the RHS we already showed that the first term is bounded in N. For the second term, observe that

$$\begin{aligned}\sup_{t\in[0,T]} |\mathcal{M}_t^N(F) - \mathcal{M}_{t^-}^N(F)| &= \sup_{t\in[0,T]} |\mathcal{Y}_t^N(F) - \mathcal{Y}_{t^-}^N(F)| \\ &\le \sup_{t\in[0,T]} \frac{1}{\sqrt{|V_N|}} \sum_{x\in V_N} \left|(\bar\eta_t^N(x) - \bar\eta_{t^-}^N(x))F(x)\right| \le \frac{C(F)}{\sqrt{|V_N|}},\end{aligned} \tag{8.18}$$

since in a single jump in the exclusion process, at most 2 points in V_N change configuration, and almost surely no two jumps occur at the same time.

We have thus proved Proposition 8.1, and in particular, verified condition **(OU1)** of Definition 4.8.

8.2. **Uniqueness of the limit point.** To prove uniqueness of $\mathcal{Y}_\cdot$, we follow the strategy described in [KL99, §11.4], which is based on the analysis of martingales. Throughout this subsection, $i = \sqrt{-1}$ and $\mathscr{F}_s := \sigma\{\mathcal{Y}_t(F) : t \in [0,s]\}$.

Lemma 8.4. *Fix $s \ge 0$ and $F \in \mathcal{S}_b$. The process $\{\mathcal{X}_t^s(F) : t \ge s\}$ under $\mathbb{Q}_\rho^b$ given by*

$$\mathcal{X}_t^s(F) := \exp\left[i\left(\mathcal{Y}_t(F) - \mathcal{Y}_s(F) - \int_s^t \mathcal{Y}_r\left(\frac{2}{3}\Delta_b F\right) dr\right) + \frac{1}{2}\left(\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(F)\right)(t-s)\right] \tag{8.19}$$

is a martingale.

Proof. Using (8.14) we see that

$$\mathcal{X}_t^s(F) = \exp\left[i\left(\mathcal{M}_t(F) - \mathcal{M}_s(F)\right) + \frac{1}{2}\left(\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(F)\right)(t-s)\right] \tag{8.20}$$

So to prove the desired claim, it suffices to show that for $t \ge s$,

$$\mathbb{E}_\rho^b\left[e^{i(\mathcal{M}_t(F)-\mathcal{M}_s(F))}\middle|\mathscr{F}_s\right] = \exp\left[-\frac{1}{2}\left(\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(F)\right)(t-s)\right]. \tag{8.21}$$

Indeed, since the distribution of $\mathcal{M}_t(F)-\mathcal{M}_s(F)$ conditional upon $\mathscr{F}_s$ is a centered Gaussian, the LHS of (8.21) equals

(8.22)
$$\exp\left(-\frac{1}{2}\mathbb{E}^b_\rho\left[(\mathcal{M}_t(F)-\mathcal{M}_s(F))^2|\mathscr{F}_s\right]\right)=\exp\left(-\frac{1}{2}\mathbb{E}^b_\rho\left[(\mathcal{M}_t(F))^2-(\mathcal{M}_s(F))^2|\mathscr{F}_s\right]\right)$$
$$\underset{(8.16)}{=}\exp\left(-\frac{1}{2}\left(\mathbb{E}^b_\rho[\mathcal{N}_t(F)-\mathcal{N}_s(F)|\mathscr{F}_s]+\frac{2}{3}\cdot 2\chi(\rho)(t-s)\mathscr{E}_b(F)\right)\right)=\exp\left(-\frac{1}{2}\left(\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(F)\right)(t-s)\right).$$

□

Lemma 8.5. *Fix $S\ge 0$ and $F\in\mathcal{S}_b$. The process $\{\mathcal{Z}^S_t(F):t\in[0,S]\}$ under $\mathbb{Q}^b_\rho$ given by*

$$\mathcal{Z}^S_t(F):=\exp\left[i\mathcal{Y}_t(\tilde{\mathsf{T}}^b_{S-t}F)+\frac{1}{2}\int_0^t\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}^b_{S-r}F)\,dr\right] \tag{8.23}$$

is a martingale.

Proof. Recall that for any $F\in\mathcal{S}_b$, $t\mapsto\mathcal{Y}_t(F)$ is continuous (Proposition 8.1) and $t\mapsto\tilde{\mathsf{T}}^b_tF$ is continuous (Lemma 4.6-(3)). Also recall Corollary 4.7. With these in mind, we fix $0\le t_1<t_2\le S$, and consider the partition of $[t_1,t_2]$ into n equal subintervals, namely, $t_1=s_0<s_1<\cdots<s_n=t_2$ with $s_j=t_1+j\left(\frac{t_2-t_1}{n}\right)$. A direct computation shows that

$$\begin{aligned}\prod_{j=0}^{n-1}\mathcal{X}^{s_j}_{s_{j+1}}\left(\tilde{\mathsf{T}}^b_{S-s_j}F\right)=\exp\Bigg[&i\sum_{j=0}^{n-1}\left(\mathcal{Y}_{s_{j+1}}(\tilde{\mathsf{T}}^b_{S-s_j}F)-\mathcal{Y}_{s_j}(\tilde{\mathsf{T}}^b_{S-s_j}F)-\int_{s_j}^{s_{j+1}}\mathcal{Y}_r\left(\frac{2}{3}\Delta_b\tilde{\mathsf{T}}^b_{S-s_j}F\right)dr\right)\\&+\frac{1}{2}\cdot\frac{2}{3}\cdot 2\chi(\rho)\frac{t_2-t_1}{n}\sum_{j=0}^{n-1}\mathscr{E}_b(\tilde{\mathsf{T}}^b_{S-s_j}F)\Bigg]\end{aligned} \tag{8.24}$$

Using the continuity of $t\mapsto\tilde{\mathsf{T}}^b_tF$ and Riemann sum approximation, we see that

$$\frac{t_2-t_1}{n}\sum_{j=0}^{n-1}\mathscr{E}_b(\tilde{\mathsf{T}}^b_{S-s_j}F)\xrightarrow[n\to\infty]{}\int_{t_1}^{t_2}\mathscr{E}_b(\tilde{\mathsf{T}}^b_{S-r}F)\,dr. \tag{8.25}$$

Meanwhile, we can rewrite the sum in the first term on the RHS of (8.24) as

(8.26)
$$\mathcal{Y}_{t_2}(\tilde{\mathsf{T}}^b_{S-t_2-\frac{t_2-t_1}{n}}F)-\mathcal{Y}_{t_1}(\tilde{\mathsf{T}}^b_{S-t_1}F)+\sum_{j=1}^{n-1}\mathcal{Y}_{s_j}(\tilde{\mathsf{T}}^b_{S-s_{j-1}}F-\tilde{\mathsf{T}}^b_{S-s_j}F)-\sum_{j=0}^{n-1}\int_{s_j}^{s_{j+1}}\mathcal{Y}_r\left(\frac{2}{3}\Delta_b\tilde{\mathsf{T}}^b_{S-s_j}F\right)dr$$

By Lemma 4.6-(3), $\tilde{\mathsf{T}}^b_{t+\epsilon}F-\tilde{\mathsf{T}}^b_tF=\epsilon\frac{2}{3}\Delta_bF+o(\epsilon)$ as $\epsilon\downarrow 0$, so we get

(8.27)
$$\sum_{j=1}^{n-1}\mathcal{Y}_{s_j}(\tilde{\mathsf{T}}^b_{S-s_{j-1}}F-\tilde{\mathsf{T}}^b_{S-s_j}F)=\sum_{j=1}^{n-1}\frac{t_2-t_1}{n}\mathcal{Y}_{s_j}\left(\frac{2}{3}\Delta_b\tilde{\mathsf{T}}^b_{S-s_j}F\right)+o\left(\frac{1}{n}\right)\xrightarrow[n\to\infty]{}\int_{t_1}^{t_2}\mathcal{Y}_r\left(\frac{2}{3}\Delta_b\tilde{\mathsf{T}}^b_{S-r}F\right)dr,$$

which cancels with the $n\to\infty$ limit of the last term of (8.26). Altogether we have

$$\lim_{n\to\infty}\prod_{j=0}^{n-1}\mathcal{X}^{s_j}_{s_{j+1}}\left(\tilde{\mathsf{T}}^b_{S-s_j}F\right)=\exp\left[i\left(\mathcal{Y}_{t_2}(\tilde{\mathsf{T}}^b_{S-t_2}F)-\mathcal{Y}_{t_1}(\tilde{\mathsf{T}}^b_{S-t_1}F)\right)+\frac{1}{2}\int_{t_1}^{t_2}\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}^b_{S-r}F)\,dr\right], \tag{8.28}$$

the RHS being equal to $\mathcal{Z}^S_{t_2}(F)/\mathcal{Z}^s_{t_1}(F)$, $\mathbb{Q}^b_\rho$-a.s. Moreover, since the complex exponential is bounded, the dominated convergence theorem implies the limit (8.28) also takes place in $L^1(\mathbb{Q}^b_\rho)$. So for any bounded random variable G,

$$\mathbb{E}^b_\rho\left[G\frac{\mathcal{Z}^S_{t_2}(F)}{\mathcal{Z}^S_{t_1}(F)}\right]=\lim_{n\to\infty}\mathbb{E}^b_\rho\left[G\prod_{j=0}^{n-1}\mathcal{X}^{s_j}_{s_{j+1}}\left(\tilde{\mathsf{T}}^b_{S-s_j}F\right)\right]. \tag{8.29}$$

Suppose further that G is $\mathscr{F}_{t_1}$-measurable. Since $\{\mathcal{X}_t^s(F) : t \geq s\}$ is a martingale by Lemma 8.4, we have

$$\begin{aligned}
\mathbb{E}_\rho^b\left[G\prod_{j=0}^{n-1}\mathcal{X}_{s_{j+1}}^{s_j}\left(\tilde{\mathsf{T}}_{S-s_j}^b F\right)\right] &= \mathbb{E}_\rho^b\left[\mathbb{E}_\rho^b\left[G\prod_{j=0}^{n-1}\mathcal{X}_{s_{j+1}}^{s_j}\left(\tilde{\mathsf{T}}_{S-s_j}^b F\right)\middle|\mathscr{F}_{s_{n-1}}\right]\right]\\
&= \mathbb{E}_\rho^b\left[G\prod_{j=0}^{n-2}\mathcal{X}_{s_{j+1}}^{s_j}\left(\tilde{\mathsf{T}}_{S-s_j}^b F\right)\mathbb{E}_\rho^b\left[\mathcal{X}_{s_n}^{s_{n-1}}\left(\tilde{\mathsf{T}}_{S-s_{n-1}}^b F\right)\middle|\mathscr{F}_{s_{n-1}}\right]\right]\\
&= \mathbb{E}_\rho^b\left[G\prod_{j=0}^{n-2}\mathcal{X}_{s_{j+1}}^{s_j}\left(\tilde{\mathsf{T}}_{S-s_j}^b F\right)\mathcal{X}_{s_{n-1}}^{s_{n-1}}\left(\tilde{\mathsf{T}}_{S-s_{n-1}}^b F\right)\right] = \mathbb{E}_\rho^b\left[G\prod_{j=0}^{n-2}\mathcal{X}_{s_{j+1}}^{s_j}\left(\tilde{\mathsf{T}}_{S-s_j}^b F\right)\right].
\end{aligned} \tag{8.30}$$

By induction we can boil the last expression down to $\mathbb{E}_\rho^b[G]$. Combine this result with (8.29) to obtain

$$\mathbb{E}_\rho^b\left[G\frac{\mathcal{Z}_{t_2}^S(F)}{\mathcal{Z}_{t_1}^S(F)}\right] = \mathbb{E}_\rho^b[G] \tag{8.31}$$

for any bounded $\mathscr{F}_{t_1}$-measurable random variable G. This shows that $\{\mathcal{Z}_t^S(F) : t \in [0,S]\}$ is a martingale. □

We can now finish the uniqueness proof. From the martingale identity $\mathbb{E}_\rho^b[\mathcal{Z}_t^S(F)|\mathscr{F}_s] = \mathcal{Z}_s^S(F)$ for $S \geq t \geq s$, we get

$$\mathbb{E}_\rho^b\left[\exp\left(i\mathcal{Y}_t(\tilde{\mathsf{T}}_{S-t}^b F) + \frac{1}{2}\int_0^t \frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}_{S-r}^b F)\,dr\right)\middle|\mathscr{F}_s\right] = \exp\left(i\mathcal{Y}_t(\tilde{\mathsf{T}}_{S-s}^b F) + \frac{1}{2}\int_0^s \frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}_{S-r}^b F)\,dr\right). \tag{8.32}$$

This can be rearranged to give

$$\mathbb{E}_\rho^b\left[\exp\left(i\mathcal{Y}_t(\tilde{\mathsf{T}}_{S-t}^b F)\right)\middle|\mathscr{F}_s\right] = \exp\left(i\mathcal{Y}_t(\tilde{\mathsf{T}}_{S-s}^b F) - \frac{1}{2}\int_s^t \frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}_{S-r}^b F)\,dr\right). \tag{8.33}$$

By a change of variables and the semigroup definition $\tilde{\mathsf{T}}_{S-s}^b = \tilde{\mathsf{T}}_{t-s}^b\tilde{\mathsf{T}}_{S-t}^b$ for $S > t > s$, the last expression can be rewritten as

$$\mathbb{E}_\rho^b\left[\exp\left(i\mathcal{Y}_t(F)\right)\middle|\mathscr{F}_s\right] = \exp\left(i\mathcal{Y}_t(\tilde{\mathsf{T}}_{t-s}^b F) - \frac{1}{2}\int_0^{t-s} \frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}_r^b F)\,dr\right). \tag{8.34}$$

Changing F to λF, $\lambda \in \mathbb{R}$, we see that the distribution of $\mathcal{Y}_t(F)$ conditional upon $\mathscr{F}_s$ is Gaussian with mean $\mathcal{Y}_t(\tilde{\mathsf{T}}_{t-s}^b F)$ and variance $\int_0^{t-s}\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_b(\tilde{\mathsf{T}}_r^b F)\,dr$, matching the condition **(OU2)** in Definition 4.8. Successive conditioning at different times implies uniqueness of the finite-dimensional distributions of the process $\{\mathcal{Y}_t(F) : t \in [0,T]\}$, which then implies uniqueness of the law of $\mathcal{Y}_\cdot$. This completes the proof of Theorem 2.

9. Generalizations

9.1. **Mixed boundary conditions.** With minor tweaks to the preceding proofs, it is straightforward to establish limit theorems for the exclusion process with different boundary scaling parameters $b_a > 0$ at each $a \in V_0$. Let $\mathbf{b} = \{b_a > 0 : a \in V_0\}$, and consider the process $\{\eta_t^N : t \in [0,T]\}$ on Ω_N generated by $5^N\mathcal{L}_N^{\mathbf{b}}$, where

$$\mathcal{L}_N^{\mathbf{b}} = \mathcal{L}_N^{\text{bulk}} + \sum_{a\in V_0}\frac{1}{(b_a)^N}\mathcal{L}_N^a, \tag{9.1}$$

$\mathcal{L}_N^{\text{bulk}}$ is as in (2.5), and

$$\left(\mathcal{L}_N^a f\right)(\eta) = \left[\lambda_-(a)\eta(a) + \lambda_+(a)(1-\eta(a))\right]\left[f(\eta^a) - f(\eta)\right]. \tag{9.2}$$

We then obtain generalization of Theorem 1, where the boundary condition at a is Neumann (resp. Robin, Dirichlet) if $b_a > 5/3$ (resp. $b_a = 5/3$, $b_a < 5/3$).

Theorem 1M (Hydrodynamic limit of the empirical density)**.** *Let $\varrho : K \to [0,1]$ be measurable, and $\{\mu_N\}_N$ be a sequence of probability measures on Ω_N which is associated to ϱ. Then for any $t \in [0,T]$, any continuous function $F : K \to \mathbb{R}$, and any $\delta > 0$, we have*

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N : \left| \frac{1}{|V_N|} \sum_{x\in V_N} F(x)\eta_t^N(x) - \int_K F(x)\rho^{\mathbf{b}}(t,x)\,dm(x) \right| > \delta \right\} = 0, \tag{9.3}$$

where $\rho^{\mathbf{b}}$ is the unique weak solution of the heat equation with mixed boundary condition

$$\begin{cases} \partial_t \rho^{\mathbf{b}}(t,x) = \frac{2}{3}\Delta\rho^{\mathbf{b}}(t,x), & t\in[0,T],\ x\in K\setminus V_0, \\ \rho^{\mathbf{b}}(t,a) = \bar\rho(a), & t\in(0,T],\ a\in V_0,\ b_a<5/3, \\ \partial^\perp \rho^{\mathbf{b}}(t,a) = -\lambda_\Sigma(a)(\rho^{\mathbf{b}}(t,a)-\bar\rho(a))\mathbf{1}_{\{b_a=5/3\}}, & t\in(0,T],\ a\in V_0,\ b_a\geq 5/3, \\ \rho^{\mathbf{b}}(0,x) = \varrho(x), & x\in K, \end{cases} \tag{9.4}$$

that is:

(1) $\rho^{\mathbf{b}} \in L^2(0,T,\mathcal{F})$.

(2) $\rho^{\mathbf{b}}$ satisfies the weak formulation of (9.4): For any $t\in[0,T]$ and $F \in C([0,T],\mathscr{D}_{\mathbf{b}}) \cap C^1((0,T),\mathscr{D}_{\mathbf{b}})$,

$$\begin{aligned} &\int_K \rho_t(x)F_t(x)\,dm(x) - \int_K \varrho(x)F_0(x)\,dm(x) - \int_0^t \int_K \rho_s(x)\left(\frac{2}{3}\Delta + \partial_s\right)F_s(x)\,dm(x)\,ds \\ &+\frac{2}{3}\int_0^t \sum_{a\in V_0}\left[\bar\rho(a)(\partial^\perp F_s)(a)\mathbf{1}_{\{b_a<5/3\}} + \left(\rho_s(a)(\partial^\perp F_s)(a) + \lambda_\Sigma(a)(\rho_s(a)-\bar\rho(a))F_s(a)\mathbf{1}_{\{b_a=5/3\}}\right)\mathbf{1}_{\{b_a\geq 5/3\}}\right] ds = 0, \end{aligned} \tag{9.5}$$

where $\mathscr{D}_{\mathbf{b}} := \{F \in \mathrm{dom}\Delta : F(a) = 0$ whenever $b_a < 5/3\}$.

(3) $\rho(t,a) = \bar\rho(a)$ for a.e. $t\in(0,T]$ and all $a\in V_0$ with $b_a<5/3$.

Next assume $\lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$, and set $\bar\rho = \lambda_+/\lambda_\Sigma$. We introduce the Laplacian $\Delta_{\mathbf{b}}$ with boundary parameter $\mathbf{b}$, whose domain is

$$\mathrm{dom}\Delta_{\mathbf{b}} = \left\{ F\in\mathrm{dom}\Delta : \left\{ \begin{array}{ll} F(a)=0, & \text{if } b_a<5/3, \\ (\partial^\perp F)(a) = 0, & \text{if } b_a>5/3, \\ (\partial^\perp F)(a) = -\lambda_\Sigma F(a), & \text{if } b_a = 5/3. \end{array}\right\}\right\} \tag{9.6}$$

We then define the Fréchet space $\mathcal{S}_{\mathbf{b}} := \{F \in \mathrm{dom}\Delta_{\mathbf{b}} : \Delta_{\mathbf{b}}F \in \mathrm{dom}\Delta_{\mathbf{b}}\}$ equipped with the family of seminorms $\{\|\cdot\|_j : j\in\mathbb{N}_0\}$ in the same way as in Definition 4.2. The closed quadratic energy form will be

$$\mathscr{E}_{\mathbf{b}}(F,G) = \mathcal{E}(F,G) + \lambda_\Sigma \sum_{a\in V_0}\mathbf{1}_{\{b_a=5/3\}}F(a)G(a). \tag{9.7}$$

with domain $\mathcal{F}_{\mathbf{b}} = \{F\in\mathcal{F} : F(a)=0 \text{ whenever } b_a<5/3\}$.

Let $\mathcal{S}'_{\mathbf{b}}$ be the topological dual of $\mathcal{S}_{\mathbf{b}}$ with respect to the topology generated by $\{\|\cdot\|_j : j\in\mathbb{N}_0\}$.

Definition 9.1 (Ornstein-Uhlenbeck equation, mixed boundary condition)**.** We say that a random element $\mathcal{Y}$ taking values in $C([0,T],\mathcal{S}'_{\mathbf{b}})$ is a solution to the Ornstein-Uhlenbeck equation on K with boundary parameter $\mathbf{b}$ if:

(**OUM1**) For every $F\in\mathcal{S}_{\mathbf{b}}$,

$$\mathcal{M}_t(F) = \mathcal{Y}_t(F) - \mathcal{Y}_0(F) - \int_0^t \mathcal{Y}_s\left(\frac{2}{3}\Delta_{\mathbf{b}}F\right)ds \tag{9.8}$$

$$\text{and}\quad \mathcal{N}_t(F) = (\mathcal{M}_t(F))^2 - \frac{2}{3}\cdot 2\chi(\rho)t\mathscr{E}_{\mathbf{b}}(F) \tag{9.9}$$

are $\mathscr{F}_t$-martingales, where $\mathscr{F}_t := \sigma\{\mathcal{Y}_s(F) : s\leq t\}$ for each $t\in[0,T]$.

(**OUM2**) $\mathcal{Y}_0$ is a centered Gaussian $\mathcal{S}'_{\mathbf{b}}$-valued random variable with covariance

$$\mathbb{E}_\rho^{\mathbf{b}}[\mathcal{Y}_0(F)\mathcal{Y}_0(G)] = \chi(\rho)\int_K F(x)G(x)\,dm(x), \quad \forall F,G\in\mathcal{S}_{\mathbf{b}}. \tag{9.10}$$

Moreover, for every $F\in\mathcal{S}_{\mathbf{b}}$, the process $\{\mathcal{Y}_t(F) : t\geq 0\}$ is Gaussian: the distribution of $\mathcal{Y}_t(F)$ conditional upon $\mathscr{F}_s$, $s<t$, is Gaussian with mean $\mathcal{Y}_s(\tilde{\mathsf{T}}^{\mathbf{b}}_{t-s}F)$ and variance $\int_0^{t-s}\frac{2}{3}\cdot 2\chi(\rho)\mathscr{E}_{\mathbf{b}}(\tilde{\mathsf{T}}^{\mathbf{b}}_r F)\,dr$, where $\{\tilde{\mathsf{T}}^{\mathbf{b}}_t : t>0\}$

is the heat semigroup generated by $\frac{2}{3}H_{\mathbf{b}}$, and $H_{\mathbf{b}}$ is the nonnegative self-adjoint operator on $L^2(K,m)$ associated to the closed form $(\mathscr{E}_{\mathbf{b}}, \mathcal{F}_{\mathbf{b}})$.

Recall the density fluctuation field $\mathcal{Y}_\cdot^N$ defined in (4.1), which we regard as an element of $D([0,T], \mathcal{S}'_{\mathbf{b}})$. Let $\mathbb{P}_\rho^{N,\mathbf{b}}$ denote the law of $\{\eta_t^N : t \in [0,T]\}$ started from the reversible measure ν_ρ^N, and $\mathbb{Q}_\rho^{N,\mathbf{b}}$ be the probability measure on $D([0,T], \mathcal{S}'_{\mathbf{b}})$ induced by $\mathcal{Y}_\cdot^N$ and by $\mathbb{P}_\rho^{N,\mathbf{b}}$.

Theorem 2M (Ornstein-Uhlenbeck limit of equilibrium density fluctuations)**.** *As $N \to \infty$, the sequence $\{\mathbb{Q}_\rho^{N,\mathbf{b}}\}_N$ converges in distribution to a unique solution of the Ornstein-Uhlenbeck equation with parameter* $\mathbf{b}$*, in the sense of Definition 9.1.*

9.2. **Other post-critically finite self-similar fractals and resistance spaces.** In order to make the paper readable with minimal prerequisites, we have decided to work on the Sierpinski gasket only. That said, the results in this paper can be generalized to other post-critically finite self-similar (p.c.f.s.s.) fractals as defined in [Bar98, Kig01], and more generally, to *resistance spaces* introduced by Kigami [Kig03], which also include 1D random walks with long-range jumps; trees; and random graphs arising from critical percolation.

In some sense there is very little "fractal" involved in our proofs; rather, the most important ingredient is a good notion of calculus, including: convergence of discrete Laplacians and of the discrete energy forms (with respect to the reference measure), and a robust theory of boundary-value elliptic and parabolic problems. It is also important that that the space be bounded in the resistance metric. Otherwise the moving particle Lemma 5.1 becomes ineffective, and we would not have been able to prove the replacement Lemma 5.3 in light of the lack of translational invariance.

It is an open problem to prove hydrodynamic limits of exclusion processes on non-translationally-invariant spaces whose spectral dimension ≥ 2; see [vGR18] for recent progress towards this goal. Due to the length of the present paper, we leave the details of these generalizations to future work.

Appendix A. Spectral analysis on SG

The purpose of this appendix is to describe spectral properties of the Laplacian on SG which are needed to prove the heat semigroup Lemma 4.6. Most results in this section are known to analysts on fractals. In fact, when $\mathrm{b} \in \{\mathrm{Dir}, \mathrm{Neu}\}$, the statements and proofs can be found in [Kig01], and their appearances will be indicated. When $\mathrm{b} = \mathrm{Rob}$, we will supply the necessary proofs.

Recall the definition of $(\mathscr{E}_b, \mathcal{F}_b)$ from (4.17). Our starting point is to present the proof of Lemma 4.5.

Proof of Lemma 4.5.

(1) If $\mathrm{b} = \mathrm{Neu}$ this is [Kig01, Theorem 3.4.6], with complete proof given. For the case $\mathrm{b} = \mathrm{Dir}$ it is stated as a corollary, [Kig01, Corollary 3.4.7].

We now sketch the proof for $\mathrm{b} = \mathrm{Rob}$. The first, and only, property we will check is that $(\mathscr{E}_{\mathrm{Rob}}, \mathcal{F}_{\mathrm{Rob}})$ is a closed form. To each $f \in \mathcal{F}_{\mathrm{Rob}} = \mathcal{F}$ we associate the harmonic function $h_{f|_{V_0}} : K \to \mathbb{R}$, which has the property that $\Delta(h_{f|_{V_0}}) = 0$ on $K \setminus V_0$ and $h_{f|_{V_0}} = f$ on V_0. (The uniqueness of the harmonic extension $h_{f|_{V_0}}$ is ensured by the maximum principle.) Then define $\tilde{f} = f - h_{f|_{V_0}} \in \mathcal{F}_0 := \{f \in \mathcal{F} : f|_{V_0} = 0\}$. Recall that $\mathcal{E}$ is an inner product on $\mathcal{F}_0$, and that $(\mathcal{F}_0, \mathcal{E})$ is complete.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{E}_{\mathrm{Rob},1} := \mathscr{E}_{\mathrm{Rob}} + \|\cdot\|^2_{L^2(K,m)}$. Then $\{u_n\}_n$ is Cauchy in $\mathcal{E}_1 := \mathcal{E} + \|\cdot\|^2_{L^2(K,m)}$, and $\{u_n|_{V_0}\}_n$ is Cauchy. This last statement and the maximum principle imply that $\{h_{u_n|_{V_0}}\}_n$ is $\mathcal{E}_1$-Cauchy. It follows that $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, where $\tilde{u}_n = u_n - h_{u_n|_{V_0}}$, is $\mathcal{E}_1$-Cauchy. Therefore there exists $v \in \mathcal{F}_0$ such that $\tilde{u}_n \to v$ in $\mathcal{E}_1$.

Now let $\bar{u} = \lim_{n\to\infty} u_n|_{V_0}$, $h_{\bar{u}}$ be the unique harmonic extension of $\bar{u}$ from V_0 to K, and $u = v + h_{\bar{u}}$. Then $u_n - u = (\tilde{u}_n - v) + (h_{u_n|_{V_0}} - h_{\bar{u}})$, and note that each of the two terms on the right converges to 0 in $\mathscr{E}_{\mathrm{Rob},1}$ as $n \to \infty$. Thus $u_n \to u$ in $\mathcal{E}_{\mathrm{Rob},1}$, proving the closed property of $(\mathscr{E}_{\mathrm{Rob}}, \mathcal{F}_{\mathrm{Rob}})$.

The other properties—regularity, locality, the Markov property, and compact resolvent—follow nearly word for word the proof of [Kig01, Theorem 3.4.6].

(2) If $\mathrm{b} \in \{\mathrm{Dir}, \mathrm{Neu}\}$ this is proved as [Kig01, Theorem 3.7.9] (see also [Kig01, Corollary 3.7.13]). If $\mathrm{b} = \mathrm{Rob}$, we follow the proof of Theorem 3.7.9, Part 2 in [Kig01], except to replace $\mathcal{E}(f,g)$ by $\mathscr{E}_{\mathrm{Rob}}(f,g)$. This replacement is also needed when invoking Lemma 3.7.12 there.

□

Let $E_b(\lambda) = \{\varphi \in \mathrm{dom}\Delta_b : -\Delta\varphi = \lambda\varphi\}$. If $\mathrm{dim}E_b(\lambda) \geq 1$, then any nontrivial $\varphi \in E_b(\lambda)$ is called a b-eigenfunction of Δ with associated b-eigenvalue λ. Following [Kig01, Proposition 4.1.2] we have equivalence of the following statements:

(Spec1) $\varphi \in \mathrm{dom}H_b$ and $H_b\varphi = \lambda\varphi$.
(Spec2) $\varphi \in \mathcal{F}_b$ and $\mathscr{E}_b(\varphi, u) = \lambda\langle\varphi, u\rangle_\mu$ for any $u \in \mathcal{F}_b$.
(Spec3) $\varphi \in E_b(\lambda)$.

Since H_b has compact resolvent, by standard functional analysis arguments, there exist $\{\lambda_n^b\}_{n=1}^\infty$ and $\varphi_n^b \in E_b(\lambda_n^b)$ such that

$$0 \leq \lambda_1^b < \lambda_2^b \leq \lambda_3^b \leq \cdots \leq \lambda_n^b \leq \cdots \uparrow +\infty$$

and $\{\varphi_n^b\}_n$ forms a complete orthonormal basis for $L^2(K,m)$. Note that $\lambda_1^{\mathrm{b}} = 0$ iff $\mathrm{b} = \mathrm{Neu}$, in which case $\varphi_1^{\mathrm{Neu}} = 1$. It then makes sense to define the b-eigenvalue counting function

$$\#_b(s) = \sum_{\lambda \leq s} \mathrm{dim}E_b(\lambda).$$

Lemma A.1 (Weyl asymptotics)**.** *There exists a nonconstant periodic function G, bounded away from 0 and from ∞, and independent of the boundary condition b, such that*

$$\#_b(s) = s^{\frac{d}{d+1}}(G(\log s) + o(1)) \quad \text{as } s \to \infty, \tag{A.1}$$

where $d = \frac{\log 3}{\log(5/3)}$ is the Hausdorff dimension of K with respect to the resistance metric R. It follows that the asymptotic growth rate of the eigenvalues is $1 + \frac{1}{d}$: there exist constants $C_3, C_4 > 0$ such that

$$C_3 n^{1+\frac{1}{d}} \leq \lambda_n^b \leq C_4 n^{1+\frac{1}{d}} \tag{A.2}$$

for all $n \geq 1$ (resp. $n \geq 2$) if $\mathrm{b} \in \{\mathrm{Dir}, \mathrm{Rob}\}$ (resp. $\mathrm{b} = \mathrm{Neu}$).

Remark A.2. For the purposes of this paper, we only need to know that the function G is bounded away from 0 and from ∞, *i.e.*, $\#_b(s) \asymp s^{d_S/2}$ for some positive exponent d_S known as the **spectral dimension**. For SG, $d_S := \frac{2d}{d+1}$.

Proof of Lemma A.1. If $\mathrm{b} \in \{\mathrm{Dir}, \mathrm{Neu}\}$, (A.1) is established by [FS92, KL93], *cf.* [Kig01, Theorem 4.1.5]. So it suffices to show the same for $\mathrm{b} = \mathrm{Rob}$. We recall the Rayleigh variational characteization of the nth eigenvalue of H_b, *cf.* [Kig01, Theorem B.1.14]:

$$\lambda_n^b = \inf\{\lambda(L) : L \subseteq \mathcal{F}_b,\ \mathrm{dim}L = n\}, \quad \text{where } \lambda(L) = \sup\{\mathscr{E}_b(F) : F \in L,\ \|F\|_{L^2(K,m)} = 1\}. \tag{A.3}$$

On the one hand,

$$\mathscr{E}_{\mathrm{Neu}}(F) = \mathcal{E}(F) \leq \mathcal{E}(F) + \sum_{a \in V_0} \lambda_\Sigma(a)(F(a))^2 = \mathscr{E}_{\mathrm{Rob}}(F), \quad \forall F \in \mathcal{F} \tag{A.4}$$

so by (A.3) $\lambda_n^{\mathrm{Neu}} \leq \lambda_n^{\mathrm{Rob}}$ for all n. On the other hand, $(\mathcal{E}, \mathcal{F}_0)$ is the same as $(\mathscr{E}_{\mathrm{Rob}}, \mathcal{F}_0)$, and since $\mathcal{F}_0 \subset \mathcal{F}$, by (A.3) again we have $\lambda_n^{\mathrm{Rob}} \leq \lambda_n^{\mathrm{Dir}}$ for all n. Therefore $\#_{\mathrm{Neu}}(s) \leq \#_{\mathrm{Rob}}(s) \leq \#_{\mathrm{Dir}}(s)$. Apply (A.1) for $\mathrm{b} \in \{\mathrm{Dir}, \mathrm{Neu}\}$ to obtain (A.1) for $\mathrm{b} = \mathrm{Rob}$. Estimate (A.2) follows immediately. □

We also need an L^∞ bound on the eigenfunctions φ_n^b, which will be achieved via Dirichlet form theory.

Lemma A.3. *The Dirichlet form $(\mathscr{E}_b, \mathcal{F}_b)$ on $L^2(K,m)$ satisfies the **Nash inequality**: there exists a constant $c_1 > 0$ such that for all $f \in \mathcal{F}_b$,*

$$\|f\|_{L^2}^{2+4/d_S} \leq c_1 \left(\mathscr{E}_b(f) + \|f\|_{L^2}^2\right) \|f\|_{L^1}^{4/d_S}. \tag{NI$_b$}$$

Proof. When $\mathrm{b} \in \{\mathrm{Dir}, \mathrm{Neu}\}$ this is proved as [Kig01, Theorem 5.3.3] with $\theta = d_S$ and $\bar{\delta} = 1$. To get (NI$_b$) for $\mathrm{b} = \mathrm{Rob}$, implement the inequality $\mathcal{E}(f) \leq \mathscr{E}_{\mathrm{Rob}}(f)$ for all $f \in \mathcal{F}$ into (NI$_b$) for $\mathrm{b} = \mathrm{Neu}$. □

Lemma A.4. *Let φ be an eigenfunction of $-\Delta_b$ with corresponding eigenvalue $\lambda \geq 1$. Then*

$$\|\varphi\|_{L^\infty} \leq C\lambda^{d_S/4}\|\varphi\|_{L^2} \tag{A.5}$$

for a constant $C > 0$ independent of φ and λ.

Proof. This follows from Lemma A.3 and [Kig01, Corollary B.3.9]. There are two key elements of the proof: $\mathsf{T}_t^b\varphi = e^{-\lambda t}\varphi$ by functional calculus, and that (NI$_b$) implies that $\|\mathsf{T}_t^b\|_{L^2\to L^\infty} \le ct^{-d_S/4}$ for $t\in(0,1]$. □

For the ensuing results we use following series convergence statement [Kig01, Lemma 5.1.4]:

$$\text{For any } \alpha,\beta,T>0,\quad t\mapsto \sum_{n=1}^\infty n^\alpha e^{-n^\beta t}\quad \text{is uniformly convergent on } [T,\infty). \tag{A.6}$$

The **heat kernel** associated with $H_{\rm b}$ is given formally by the infinite series

$$p_t^{\rm b}(x,y) = \sum_{n=1}^\infty e^{-\lambda_n^{\rm b} t}\varphi_n^{\rm b}(x)\varphi_n^{\rm b}(y),\quad t\in[0,\infty),\ x,y\in K. \tag{A.7}$$

Lemma A.5. *$(t,x,y)\mapsto p_t^b(x,y)$ is continuous on $(0,\infty)\times K\times K$.*

Proof. For the reader's benefit we reproduce the proof from [Kig01, Proposition 5.1.2 (1)]. By Lemmas A.1 and A.4, there exists $C>0$ such that $\|\varphi_n^b\|_{L^\infty}\le Cn^{1/2}$ for all sufficiently large n. Therefore $|e^{-\lambda_n^b t}\varphi_n^b(x)\varphi_n^b(y)| \le C^2 n e^{-C' t n^{1/d_S}}$ for all sufficiently large n. Now apply (A.6) to deduce the uniform convergence of the series (A.7) on $[T,\infty)\times K\times K$ for every $T>0$. □

The heat semigroup $\{\mathsf{T}_t^{\rm b} : t>0\}$, which has the heat kernel as its integral kernel, enjoys the spectral representation

$$(\mathsf{T}_t^{\rm b} f)(x) = \int_K p_t^{\rm b}(x,y)f(y)\,dm(y) = \sum_{n=1}^\infty \alpha_n^{\rm b}[f]e^{-\lambda_n^{\rm b}t}\varphi_n^{\rm b}(x),\quad f\in L^1(K,m), \tag{A.8}$$

where $\alpha_n^{\rm b}[f] = \int_K f\varphi_n^{\rm b}\,dm$ are the Fourier coefficients.

We are now ready to prove Lemma 4.6.

Proof of Lemma 4.6. When ${\rm b}\in\{{\rm Dir},{\rm Neu}\}$ this is already established in [Kig01, Theorem 5.1.7]. The proofs for ${\rm b}={\rm Rob}$ use the same ideas. For the reader's convenience we reproduce the proofs.

Item (1) follows directly from Lemma A.5.

To see Item (2), take $u\in L^1(K,m)$ and $t_1\in(0,t)$, and set $s=t-t_1$. By Item (1), $u_1 := \mathsf{T}_{t_1}^b u\in C(K)\subset L^2(K,m)$, so there exist coefficients $\{\alpha_n^b\}_n$ such that $u_1=\sum_{n=1}^\infty \alpha_n^b\varphi_n^b$ with $\sum_{n=1}^\infty|\alpha_n^b|^2<\infty$. Then by the semigroup definition, $\mathsf{T}_t^b u = \mathsf{T}_s^b u_1 = \sum_{n=1}^\infty \alpha_n^b e^{-s\lambda_n^b}\varphi_n^b$. By Lemma A.1 and (A.6), we verify that $\sum_{n=1}^\infty\left(\alpha_n^b e^{-s\lambda_n^b}\lambda_n^b\right)^2<\infty$, which implies that $\mathsf{T}_t^b u\in\mathcal{F}_b$ and $H_b(\mathsf{T}_t^b u) = \sum_{n=1}^\infty \lambda_n^b\alpha_n^b e^{-\lambda_n^b s}\varphi_n^b$. Using this spectral representation along with Lemmas A.1 and A.4 and (A.6), we see that $H_b(\mathsf{T}_t^b u)\in C(K)$. Lemma 4.5, Item (2) implies then that $\mathsf{T}_t^b u\in{\rm dom}\Delta_b$.

Last but not least, $\int_{t_1}^t(-H_b(\mathsf{T}_t^b u))(x)\,ds = (\mathsf{T}_t^b u)(x) - (\mathsf{T}_{t_1}^b u)(x)$, so Item (3) follows. □

Appendix B. Dirichlet-to-Neumann map on SG

In this appendix we characterize the harmonic function which satisfies the Robin boundary condition

$$\begin{cases} \Delta h(x)=0, & x\in K\setminus V_0,\\ \partial^\perp h(a)+\kappa(a)h(a)=\gamma(a), & a\in V_0,\end{cases} \tag{B.1}$$

where $\{\kappa(a):a\in V_0\}$ and $\{\gamma(a):a\in V_0\}$ are given coefficients.

Let $h^i:K\to\mathbb{R}$, $i\in\{0,1,2\}$, denote the harmonic function with Dirichlet boundary condition $h^i(a_j)=\delta_{ij}$, $j\in\{0,1,2\}$. By the harmonic extension algorithm described in [Str06, §1.3], $\{h^i\}_{i=0}^2$ is a basis for the space of harmonic functions on K, so we may express the solution h of (B.1) as a linear combination $h=\sum_{i=0}^2\mathbf{c}_i h^i$, where the coefficients $\{\mathbf{c}_i\}_i$ are determined by the boundary condition in (B.1):

$$\sum_{i=0}^2 \mathbf{c}_i(\partial^\perp h^i)(a_j)+\kappa(a_j)\mathbf{c}_j=\gamma(a_j),\quad j\in\{0,1,2\}. \tag{B.2}$$

We can then conclude that h is a harmonic function satisfying the Dirichlet boundary condition $h(a_i)=\mathbf{c}_i$, $i\in\{0,1,2\}$.

So it suffices to find $\{\mathbf{c}_i\}_i$. The harmonic extension algorithm [Str06, §1.3] yields

$$(\partial^\perp h^i)(a_j) = \begin{cases} 2, & \text{if } j = i, \\ -1, & \text{if } j \neq i. \end{cases} \tag{B.3}$$

Thus we arrive at the matrix problem

$$\begin{bmatrix} 2+\kappa_0 & -1 & -1 \\ -1 & 2+\kappa_1 & -1 \\ -1 & -1 & 2+\kappa_2 \end{bmatrix} \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}. \tag{B.4}$$

where κ_j and γ_j are shorthands for $\kappa(a_j)$ and $\gamma(a_j)$. It can be checked that the LHS matrix is invertible iff its determinant

$$\mathbf{\Delta} := 3(\kappa_0+\kappa_1+\kappa_2) + 2(\kappa_0\kappa_1+\kappa_1\kappa_2+\kappa_2\kappa_0) + \kappa_0\kappa_1\kappa_2 \tag{B.5}$$

is nonzero. Assuming invertibility, we find

$$\begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} = \frac{1}{\mathbf{\Delta}} \begin{bmatrix} 3+2(\kappa_1+\kappa_2)+\kappa_1\kappa_2 & 3+\kappa_2 & 3+\kappa_1 \\ 3+\kappa_2 & 3+2(\kappa_2+\kappa_0)+\kappa_2\kappa_0 & 3+\kappa_0 \\ 3+\kappa_1 & 3+\kappa_0 & 3+2(\kappa_0+\kappa_1)+\kappa_0\kappa_1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}. \tag{B.6}$$

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