for the sake of the proof recall '~' signifies asymptotic equality meaning $f(x) \sim g(x)$ implies $\lim_{x\to\infty} f(x)/g(x) = 1$

$\pi(x)$ denotes the number of primes $\leq x$

The Prime number theorem states $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$

↳ this will take 6 steps to prove:

1. $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ for $Re(s) > 1$

recall this fact of the Riemann zeta function from Khanh

refresher proof: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \underbrace{\sum_{r_2, r_3 \ldots \geq 0}}_{\text{prime factorization}} (2^{r_2} 3^{r_3} \ldots)^{-s} = \prod_p \left(\sum_{r\geq 0} p^{-rs} \right)$

$\underset{\text{geom. series}}{=} \prod_p \frac{1}{1-p^{-s}}$

2. $\zeta(s) - \frac{1}{s-1}$ extends holomorphically to $Re(s) > 0$

3. let $\theta(x) = \sum_{p\leq x} \log p$, then $\theta(x) = O(x)$ where O represents an asymptotic upper bound

4. $\zeta(s) \neq 0$ and $\phi(s) - 1/(s-1)$ is holomorphic for $Re(s) \geq 1$ where $\phi(s) = \sum_p \frac{\log p}{p^s}$

proof: since $\zeta(s)$ has a simple pole at $s=1$ with $Res_1 = 1$

we can write $\zeta(s) = \frac{1}{s-1} + \underbrace{\sum_{n=0}^{\infty} c_n (s-1)^n}_{\text{analytic}}$ and $\zeta'(s) = \frac{-1}{(s-1)^2} + \underbrace{\sum_{m=1}^{\infty} m c_m (s-1)^m}_{\text{analytic}}$

so $-\zeta'(s)/\zeta(s) = \dfrac{\frac{1}{s-1} - \sum_{m=1}^{\infty} m c_m (s-1)^{m+1}}{1 + \sum_{n=0}^{\infty} c_n (s-1)^{n+1}} = \frac{1}{s-1} + \underbrace{\cdots}_{\text{analytic}}$

note $-\frac{\zeta'(s)}{\zeta(s)} = \phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}$, so $\phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \underbrace{\sum_p \frac{\log p}{p^s(p^s-1)}}_{\text{convergent sum}}$

$= \frac{1}{s-1} + \underbrace{\cdots}_{} \to \text{analytic}$

so $\phi(s)$ has a pole at 1, $\therefore \phi(s) - 1/(s-1)$ is analytic for $Re(s) \geq 1$

5. $\int_1^{\infty} \frac{\theta(x) - x}{x^2} dx$ is a convergent integral

$\phi(s) = \sum_p \frac{\log p}{p^s} \underset{\text{stieltjes integral}}{=} \int_1^{\infty} \frac{d\theta(x)}{x^s} = s \int_1^{\infty} \frac{\theta(x)}{x^{s+1}} dx \overset{(*)}{=} s \int_0^{\infty} e^{-st} \theta(e^t) dt$

analytic thm: let $f(t)$ $(t \geq 0)$ be a bounded and locally integrable function and suppose $g(z) = \int_0^{\infty} f(t) e^{-zt} dt$ $(Re(z) > 0)$. ~~Then~~ g Extends analytically to $Re(z) \geq 0$ then $\int_0^{\infty} f(t)$ exists

let $f(z) = \theta(e^t)e^{-t} - 1$ and $g(z) = \phi(z+1)/(z+1) - 1/z$ then by the analytic thm we have

(V)

6. $\theta(x) \sim x$

proof: assume for some $\lambda > 1$ $\exists$ x s.t. $\theta(x) \geq \lambda x$

then we have $\int_x^{\lambda x} \frac{\theta(t)-t}{t^2} dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt = \int_1^{\lambda} \frac{\lambda - t}{t^2} dt > 0$ contradicting (V)

similarly $\theta(x) \leq \lambda x$ with $\lambda < 1$ yields a contradiction

$\Longrightarrow$ now $\theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log(x)$ $(*)$

and $\theta(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log x^{1-\varepsilon} = \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) \log x$ $(**)$

so from $(*)$ and 6 $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} \geq \lim_{x \to \infty} \frac{\theta(x)}{x} = 1$ ▲

from $(**)$ we know $(**) \geq (1-\varepsilon) \log x \, (\pi(x) - \pi(x^{1-\varepsilon})) \geq (1-\varepsilon) \log x \, (\pi(x) + O(x^{1-\varepsilon}))$

and so $\frac{\pi(x) \log x}{x} \leq \frac{\theta(x)}{(1-\varepsilon)x} + \underbrace{O\left(\frac{\log x}{x^{\varepsilon}}\right)}_{\to 0 \text{ as } x \to \infty}$

so $\frac{\pi(x) \log x}{x} \leq \frac{\theta(x)}{(1-\varepsilon)x}$ so $\varepsilon \to 0$ then $\frac{\pi(x) \log x}{x} \leq \frac{\theta(x)}{x} = 1$ ●

By ▲ and ● $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$ ■