Gamma Function

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The Gamma function is typically described by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t} t^{s-1}\, dt \quad \text{for } Re(s) > 0$

Claim: $\forall n \in \mathbb{N},\ \Gamma(n) = (n-1)!$

Proof:

<u>Base Case</u>: $\Gamma(1) = \displaystyle\int_0^\infty e^{-t}\, dt = 1 = 0!$ which is very straightforward

<u>Inductive Hypothesis</u>: Assume $\Gamma(k) = \displaystyle\int_0^\infty e^{-t} t^{k-1}\, dt = (k-1)!$

<u>Now consider</u>:
$$\Gamma(k+1) = \int_0^\infty e^{-t} t^k\, dt = -e^{-t} t^k \Big|_0^\infty - \int_0^\infty -k e^{-t} t^{k-1}\, dt$$
$$= 0 + k \int_0^\infty e^{-t} t^{k-1}\, dt$$
$$= k(k-1)! \text{ by the Inductive Hypothesis}$$
$$= ((k+1)-1)! \quad \blacksquare$$

The functional equation for Γ is given as $\Gamma(s+1) = s\Gamma(s)$. Hence $\Gamma(s) = \dfrac{\Gamma(s+1)}{s}$, and consequently $\Gamma(s) = \dfrac{\Gamma(s+n)}{s(s+1)(s+2)\ldots(s+n-1)}$ which illuminates a method of analytic continuation from $Re(s) > 0$ to $Re(s) > -n$. From this, we also see clearly where the poles of Γ lie: at the non-positive integers.

In fact, we plainly see that Γ has simple poles at $s = 0,\ -1,\ -2,\ -3,\ \ldots$ and we can easily compute the residue at those points. Let's do that now:

$$Res_{-n}\,\Gamma = \lim_{s \to -n} (s+n)\Gamma(s) = \lim_{s \to -n} \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\ldots(s+n-1)}$$

$$= \frac{\Gamma(1)}{-1 \cdot -2 \cdot -3 \cdot \ldots \cdot -n} = \frac{(-1)^n}{n!}$$

Euler's Product Formula: Euler's product formula converges for all s that are not non-positive integers and can be written in terms of Γ as $\Gamma(s) = \dfrac{1}{s} \displaystyle\prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^n}{1+\frac{n}{k}}$

Reflection formula: $\forall z \notin \mathbb{Z} \quad \Gamma(z)\Gamma(1-z) = \dfrac{\pi}{sin(z\pi)}$

Gauss Representation: The Gauss representation for Γ is valid for all s not equal to the non-positive integers and is given by: $\Gamma(s) = \lim_{n\to\infty} \dfrac{n^s}{s} \prod_{k=1}^{n}(1+\dfrac{s}{k})^{-1}$

Weierstrass Representation: Finally, the Weierstrass representation for Γ is valid for all s not equal to the non-positive integers, and is given by the beautiful: $\Gamma(s) = \dfrac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty}(1+\dfrac{s}{n})^{-1} e^{z/n}$ where γ is the Euler-Mascheroni constant described by $\lim_{n\to\infty}(-\ln n + \sum_{k=1}^{n} \dfrac{1}{k})$