Sandpiles in a Fractal Labyrinth

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Cornell Math SPUR/REU Smorgasbord Seminar
June 27, 2018

The abelian sandpile model [Bak–Tang–Wiesenfeld '87, Dhar, Majumdar, ...]

Place sand grains ("chips") on vertices of a graph (say, on the square lattice).
If # of chips at vertex $x \geq$ the degree of x, emit one chip to each neighbor of x. This is called a toppling at x.
Continue till the # of chips $<$ degree on every vertex. Then we say the configuration is stable.

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 4 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 0 & 4 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 6 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 2 & 1 \\ \hline 0 & 1 & 1 & 0 \\ \hline \end{array}$$

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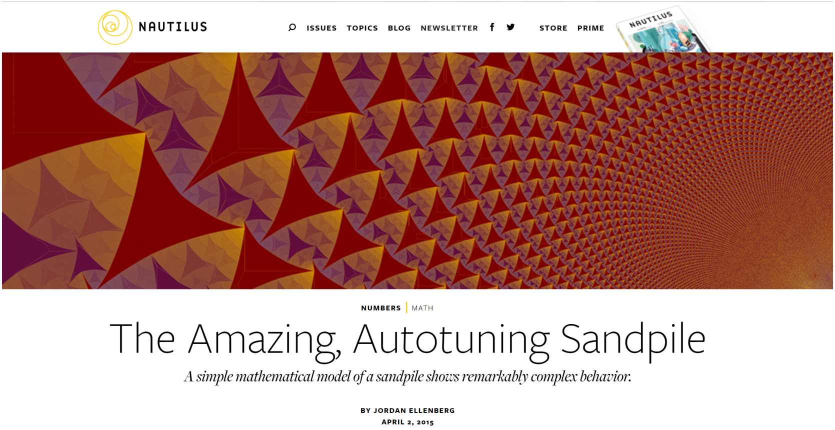
Why **abelian**? Because the final stable configuration does not depend on the order of topplings.

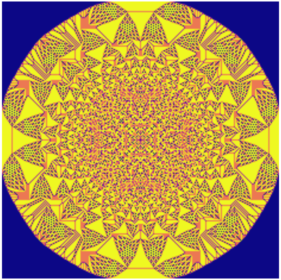
Some excellent references on sandpiles



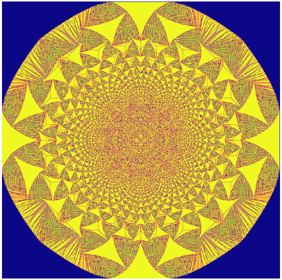
`http://nautil.us/issue/23/dominoes/the-amazing-autotuning-sandpile`

- Lionel Levine has many surveys and pictures: `http://pi.math.cornell.edu/~levine`.
- Wesley Pegden's pictures: `http://www.math.cmu.edu/~wes/sand.html`

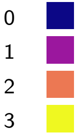
Sandpile cluster on $\mathbb{Z}^2$: Fractals in a sandpile



10^5 chips

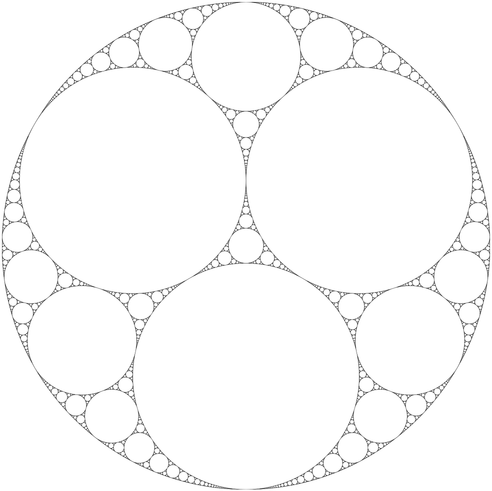


10^6 chips



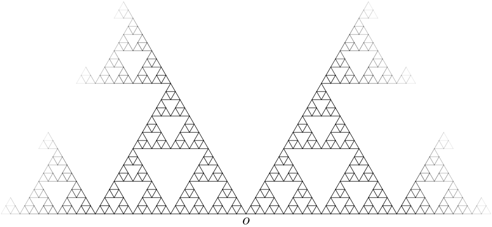
- Scaling limit of the patterns exists. [Pegden–Smart '11]
- Apollonian gaskets in the pattern. [Numerically observed since 90s, proved by Levine–Pegden–Smart '12, '14. Latter is published in *Ann. Math.* '17]
- Limiting cluster shape appears to be polygonal as opposed to an Euclidean ball, unlike other "internal" aggregation models on $\mathbb{Z}^d$.
  For an overview, see Levine–Peres, "Laplacian growth, sandpiles, and scaling limits." *Bull. Amer. Math. Soc.* (2017).

Apollonian gasket



By Time3000 [GFDL or CC BY-SA 4.0-3.0-2.5-2.0-1.0], from Wikimedia Commons
`https://commons.wikimedia.org/wiki/File:Apollonian_gasket.svg`

Sierpinski gasket (SG)



Problem: Study the sandpile model on SG!

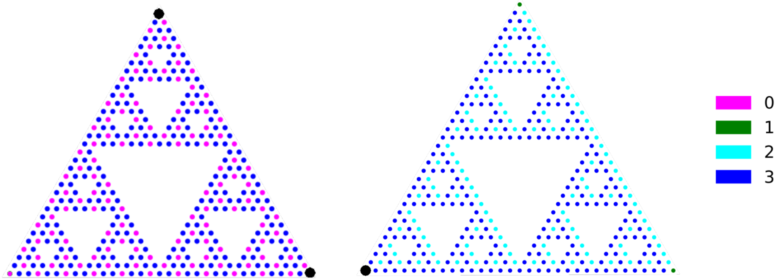
Random addition of chips & toppling (not described here)

- 90s: Daerden–Priezzhev–Vanderzande: Simulations and some exact results about the sandpile Markov chain on SG.
- 2012: Matter computed sandpile height distributions on the SG "cell graph" (or Hanoi-tower graph). [Matter PhD Thesis 2012, U. Geneve]

Deterministic single-source sandpile: Drop m chips at o and topple.

- 2013 & 14 Cornell math REU: Numerical study of sandpile patterns and growth. Made several key conjectures. [Fairchild–Haim–Setra–Strichartz–Westura '16, arXiv:1602.03424]
- March–May 2017: Jonah Kudler-Flam (Colgate '17, now UChicago PhD candidate) obtained extensive numerical results and proved a spherical shape theorem. AutomataSG

Smoking Gun #1: Self-similar sandpile configurations

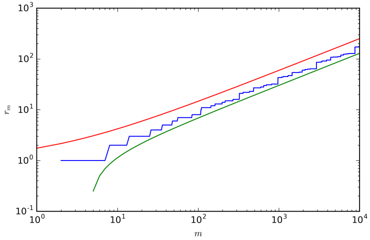


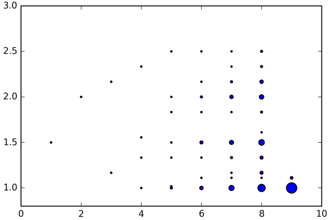
e_5 M_5

- The first configuration (e_n) was shown numerically (and conjectured) to be the **identity element of the sandpile group** (a finite abelian group) for the level-n SG (with 2 sinks) by [Fairchild–Haim–Setra–Strichartz–Westura '16]
- The second configuration (M_n) turns out to be the identity element of the sandpile group for the level-n SG (with 1 sink).

Smoking Gun #2: Patterns in sandpile growth

Jonah's experimental data, May '17

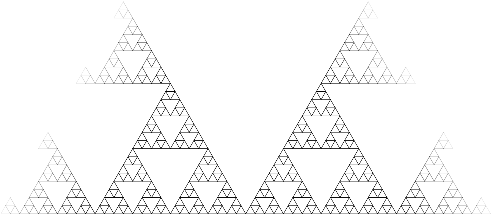




Radial jump data (right figure)

- Read from left to right, from bottom to top. Each dot represents a radial jump. The bigger the dot, the larger the radial increment.
- What do the scales mean?
- Jumps appear periodically. (But why no jumps between 2.5 and 3.0?)
- Picture is INCOMPLETE (limited by computing power).

Sandpile growth on SG



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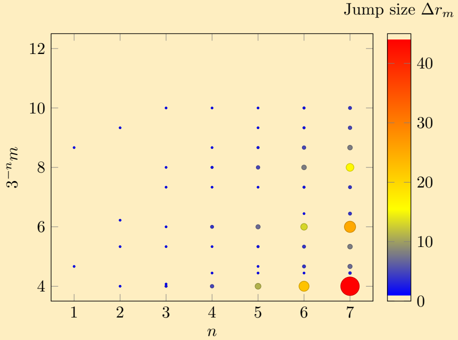
- March–May 2017: Jonah Kudler-Flam (Colgate '17, now UChicago PhD candidate) obtained extensive numerical results and proved a spherical shape theorem. `AutomataSG`
- **2018:** C. discovered an exact renormalization scheme to prove all the aforementioned "smoking guns." **Sandpile growth on *SG* is an exactly solvable problem!**

These latest results will appear in
C.–Kudler-Flam, "Laplacian growth and sandpiles on the Sierpinski gasket: limit shape universality and exact solutions," `arXiv:1807.xxxxx`.

Radial jumps on *SG* follow a periodic structure

Theorem

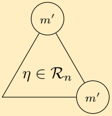


- m = Initial # of chips at the origin; n = graph approximation level of *SG*.
- Major jumps at $4 \cdot 3^n, \ 6 \cdot 3^n, \ 8 \cdot 3^n, \ 10 \cdot 3^n$. Period $= 2 \cdot 3^n$.
- Intermediate jumps within each period, EXCEPT between $10 \cdot 3^n$ and $12 \cdot 3^n = 4 \cdot 3^{n+1}$.

The fundamental diagram

Proposition. For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ = \left(\quad \right)^\circ \subseteq G_{n+1}.$$



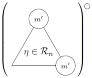
It follows that $r_m = 2^n + r_{m'-2}$.

Example: $n = 3$. Record values of m at which m' jumps.

$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m
4	**108**	**15**	**78**	**2**
$4\frac{2}{27}$	110	16	78	1
$4\frac{4}{9}$	120	19	82	
$4\frac{2}{3}$	126	20	86	
$5\frac{1}{3}$	144	28	88	1
6	162	42	78	1
$6\frac{2}{27}$	164	43	78	
$6\frac{4}{9}$	174	46	82	
$6\frac{2}{3}$	180	47	86	
$7\frac{1}{3}$	198	55	88	1

$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m
8	216	69	78	1
$8\frac{2}{27}$	218	70	78	
$8\frac{4}{9}$	228	73	82	
$8\frac{2}{3}$	234	74	86	
$9\frac{1}{3}$	252	82	88	
10	270	96	78	1
$10\frac{2}{27}$	272	97	78	
$10\frac{4}{9}$	282	100	82	
$10\frac{2}{3}$	288	101	86	
$11\frac{1}{3}$	306	109	88	

$\frac{m}{3^n}$	m	m'	$m-2m'$	Δr_m
	2	1	0	1
	8	4	0	1
		n = 1		
4	**12**	**3**	**6**	
$4\frac{2}{3}$	14	4	6	1
6	18	6	6	
$6\frac{2}{3}$	20	7	6	
8	24	9	6	
$8\frac{2}{3}$	26	10	6	1
10	30	12	6	
$10\frac{2}{3}$	32	13	6	
		n = 2		
4	**36**	**6**	**24**	**1**
$4\frac{2}{9}$	38	7	24	
$4\frac{2}{3}$	42	8	26	
$5\frac{1}{3}$	48	10	28	1
6	54	15	24	
$6\frac{2}{9}$	56	16	24	1
$6\frac{2}{3}$	60	17	26	
$7\frac{1}{3}$	66	19	28	
8	72	24	24	
$8\frac{2}{9}$	74	25	24	
$8\frac{2}{3}$	78	26	26	
$9\frac{1}{3}$	84	28	28	1
10	90	33	24	
$10\frac{2}{9}$	92	34	24	
$10\frac{2}{3}$	96	35	26	
$11\frac{1}{3}$	102	37	28	
		n = 3		
4	**108**	**15**	**78**	**2**
$4\frac{2}{27}$	110	16	78	1
$4\frac{4}{9}$	120	19	82	
$4\frac{2}{3}$	126	20	86	
$5\frac{1}{3}$	144	28	88	1
6	162	42	78	1
$6\frac{2}{27}$	164	43	78	
$6\frac{4}{9}$	174	46	82	
$6\frac{2}{3}$	180	47	86	
$7\frac{1}{3}$	198	55	88	1
8	216	69	78	1
$8\frac{2}{27}$	218	70	78	
$8\frac{4}{9}$	228	73	82	
$8\frac{2}{3}$	234	74	86	
$9\frac{1}{3}$	252	82	88	
10	270	96	78	1
$10\frac{2}{27}$	272	97	78	
$10\frac{4}{9}$	282	100	82	
$10\frac{2}{3}$	288	101	86	
$11\frac{1}{3}$	306	109	88	
		n = 4		
4	**324**	**42**	**240**	**5**
$4\frac{2}{81}$	326	43	240	
$4\frac{4}{9}$	360	55	250	1
$4\frac{2}{3}$	378	56	266	
$5\frac{1}{3}$	432	82	268	1
6	486	123	240	4
$6\frac{2}{81}$	488	124	240	
$6\frac{4}{9}$	522	136	250	
$6\frac{2}{3}$	540	137	266	
$7\frac{1}{3}$	594	163	268	1
8	648	204	240	2
$8\frac{2}{81}$	649	205	240	
$8\frac{4}{9}$	684	217	250	
$8\frac{2}{3}$	702	218	266	1
$9\frac{1}{3}$	756	244	268	
10	810	285	240	1
$10\frac{2}{81}$	812	286	240	
$10\frac{4}{9}$	846	298	250	
$10\frac{2}{3}$	864	299	266	
$11\frac{1}{3}$	918	325	268	
		n = 5		
4	**972**	**123**	**726**	**11**
$4\frac{2}{3^5}$	974	124	726	
$4\frac{4}{9}$	1080	163	754	1
$4\frac{2}{3}$	1134	164	806	1
$5\frac{1}{3}$	1296	244	808	2
6	1458	366	726	7
$6\frac{2}{3^5}$	1460	367	726	
$6\frac{4}{9}$	1566	406	754	
$6\frac{2}{3}$	1620	407	806	
$7\frac{1}{3}$	1782	487	808	1
8	1944	609	726	5
$8\frac{2}{3^5}$	1946	610	726	
$8\frac{4}{9}$	2052	649	754	
$8\frac{2}{3}$	2106	650	806	2
$9\frac{1}{3}$	2268	730	808	1
10	2430	852	726	1
$10\frac{2}{3^5}$	2432	853	726	
$10\frac{4}{9}$	2538	892	754	
$10\frac{2}{3}$	2592	893	806	
$11\frac{1}{3}$	2754	943	808	
		n = 6		
4	**2916**	**366**	**2184**	**22**
$4\frac{2}{3^6}$	2918	367	2184	
$4\frac{4}{9}$	3240	487	2266	1
$4\frac{2}{3}$	3402	488	2426	4
$5\frac{1}{3}$	3888	730	2428	4
6	4374	1095	2184	13
$6\frac{2}{3^6}$	4376	1096	2184	
$6\frac{4}{9}$	4698	1216	2266	1
$6\frac{2}{3}$	4860	1217	2426	
$7\frac{1}{3}$	5346	1459	2428	2
8	5832	1824	2184	8
$8\frac{2}{3^6}$	5834	1825	2184	
$8\frac{4}{9}$	6156	1945	2266	
$8\frac{2}{3}$	6318	1946	2426	5
$9\frac{1}{3}$	6804	2188	2428	2
10	7290	2553	2184	2
$10\frac{2}{3^6}$	7292	2554	2184	
$10\frac{4}{9}$	7614	2674	2266	
$10\frac{2}{3}$	7766	2675	2426	
$11\frac{1}{3}$	8262	2917	2428	
		n = 7		
4	**8748**	**1095**	**6558**	**44**
$4\frac{2}{3^7}$	8750	1096	6558	
$4\frac{4}{9}$	9720	1459	6802	3
$4\frac{2}{3}$	10206	1460	7286	7
$5\frac{1}{3}$	11664	2188	7288	8
6	13122	3282	6558	25

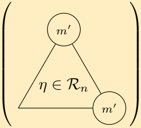
Legend: $(m\mathbb{1}_o)^\circ =$ 

; $\#\{\text{chips in } \eta\} = m - 2m'$.

Exact solution to the sandpile growth problem on SG

Theorem

For $n \geq 3$ and $m \in [4 \cdot 3^n, 4 \cdot 3^{n+1})$, $\quad (m\mathbb{1}_o)^\circ =$



$^\circ \subseteq G_{n+1}$*, where $m \mapsto m'$ is a piecewise constant càdlàg function which has jumps indicated in the following table.*

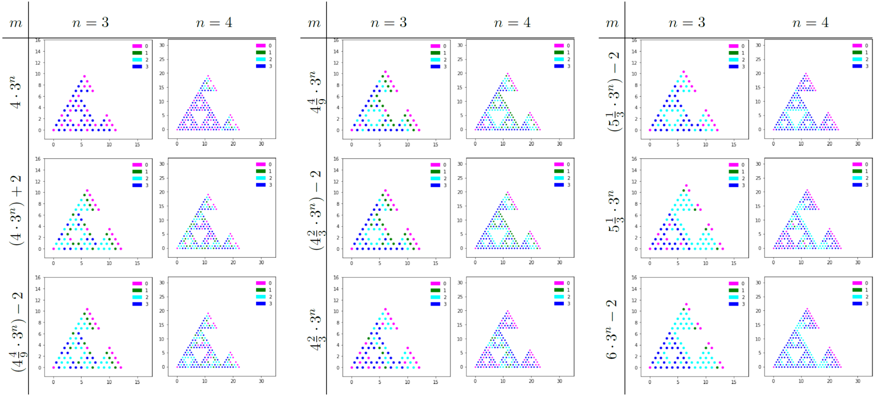
m	m'
$(4 + 2p) \cdot 3^n$	$b_n + p \cdot 3^n$
$(4 + 2p) \cdot 3^n + 2$	$(b_n + 1) + p \cdot 3^n$
$(4\frac{4}{9} + 2p) \cdot 3^n$	$2 \cdot 3^{n-1} + 1 + p \cdot 3^n$
$(4\frac{2}{3} + 2p) \cdot 3^n$	$2 \cdot 3^{n-1} + 2 + p \cdot 3^n$
$(5\frac{1}{3} + 2p) \cdot 3^n$	$3^n + 1 + p \cdot 3^n$

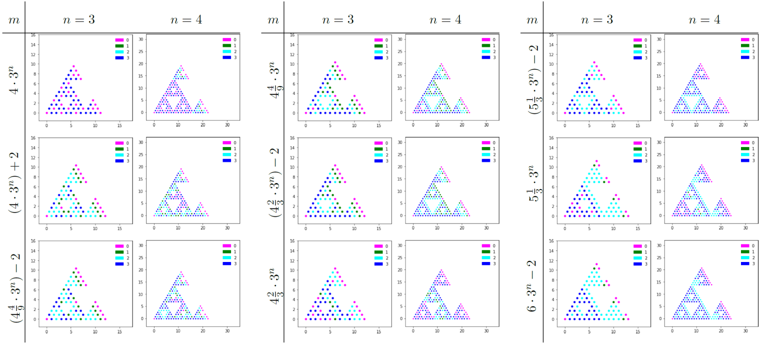
where $p \in \{0, 1, 2, 3\}$, and $b_n = |V(G_{n-1})| = \frac{3}{2}(3^{n-1} + 1)$.

Proof. Diagrammatic analysis of "sandpile tile renormalization" at, and prior to, each jump in m'.

“Periodic table” of sandpiles on SG

The configuration restricted to G_n is a gluing of 3 smaller, well-defined sandpile tiles.
[Exceptions: $(4 \cdot 3^n) + 2$, $(4\frac{4}{9} \cdot 3^n) - 2$.]





For the rest of this talk I will explain why:

1. The cluster is always an exact ball (in the graph metric).
2. The identity element of the sandpile group is as claimed.
3. $(2\cdot 3^n)$-periodicity of the sandpile patterns.
4. The cluster "explodes" in that the radius increments by >1 at periodic intervals.

   Remark. Explosion does NOT appear on $\mathbb{Z}^d$, trees, or even some pcf fractals.
   [*e.g.* On *SG*(3) sandpile cluster radius always increments by 1: computation by Ilias Stitou (Colgate '19).]

Abelian sandpiles: theory

- $G = (V \cup \{s\}, E)$: locally finite connected undirected graph with a distinguished vertex s ("sink"). (In practice s may also be a set of vertices identified as sink.) Chips that fall into the sink disappear.
- $\eta : V \to \mathbb{N}_0$ denotes a sandpile configuration on G.
- If we topple a configuration η at the vertex $x \in V$, this generates a new configuration $\eta' = \eta - \Delta'_G(x, \cdot)$, where

$$\Delta'_G(x,y) = \begin{cases} \deg(x), & \text{if } x = y \in V, \\ -\mathrm{N}_{xy}, & \text{if } x \neq y,\ x, y \in V. \end{cases}$$

  is the **graph Laplacian** on G, and N_{xy} is the number of edges connecting x and y.
- η is **stable** if $\eta(x) < \deg(x)$ for all $x \in V$.
- Ω_G: the set of all stable configurations on G.
- Given a configuration η, let $\eta^\circ \in \Omega_G$ be the stabilization of η.
- Define the binary operation $\oplus : \Omega_G \times \Omega_G \to \Omega_G$ by

$$\eta \oplus \xi = (\eta + \xi)^\circ$$

Sandpile group and its identity element

- We define a Markov chain on Ω_G with transitions
$$\eta \to (\eta + \mathbb{1}_x)^\circ \quad \text{with probability } p(x),$$
where $p(x) > 0$ for all $x \in V$ and $\sum_{x \in V} p(x) = 1$. Using the standard Markov chain language, we say that $\eta \in \Omega_G$ is **recurrent** if starting from η, the Markov chain returns to η with probability 1.
- $\mathcal{R}_G$: the set of recurrent configurations on G.
- **Theorem.** $(\mathcal{R}_G, \oplus)$ forms an abelian group, called the **sandpile group** (or the **critical group** in the combinatorics literature).
- **Theorem.** $(\mathcal{R}_G, \oplus)$ is isomorphic to the quotient group $\mathbb{Z}^V / \mathbb{Z}^V \Delta'_G$.
- There are many papers on identifying the sandpile group of a given graph (by computing the Smith normal form of the graph Laplacian). It is also of interest (and nontrivial!) to find the **identity element** e of the sandpile group.

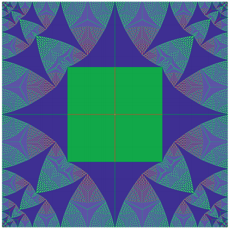


Figure: Identity element of the sandpile group for $([-n, n] \cap \mathbb{Z})^2$, with the entire boundary identified as sink.

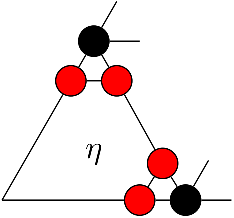
Dhar's "Multiplication by identity test"

Lemma. Let $\eta \in \mathcal{R}_G$. Then $\left(\eta + \sum_{y \in V} \mathsf{N}_{sy} \mathbb{1}_y\right)^\circ = \eta$, and each vertex topples exactly once upon stabilization.

Proof. For each $y \in V$,

$$\mathsf{N}_{sy} - \sum_{x \in V} \Delta'_G(x,y) = \mathsf{N}_{sy} - \left(\sum_{\substack{x \in V \\ x \neq y}} (-\mathsf{N}_{xy}) + \deg(y) \right) = 0.$$

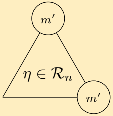
This implies that on a recurrent configuration η, after we add N_{sy} chips to each vertex $y \in V$, and then topple once at every vertex, the same configuration η is returned.



$\partial G_n =$ the two non-origin boundary vertices of G_n.

Proposition. For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ = \left(\quad \right)^\circ \subseteq G_{n+1}$$

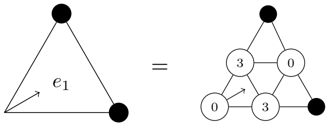


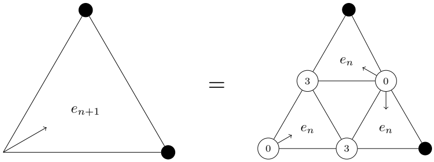
It follows that $r_m = 2^n + r_{m'-2}$.

Proof. First topple and stabilize at every vertex in $G_n \setminus \partial G_n$, and pause any excess chips on ∂G_n. This produces a recurrent config $\eta \in \mathcal{R}_n$, while leaving m' chips at each sink.
Then topple all vertices in G_n simultaneously. Upon toppling each vertex in ∂G_n loses 2 chips to generate the tail. Continue until 2 (resp. 3) chips remain at each vertex in ∂G_n, if m' is even (resp. odd).

To see that the cluster is an exact ball, we verify it for all $m < 12$, then use the above procedure to establish it for all m.

Application to SG #2: The identity element (with two sinks)

Let  , and for each $n \geq 1$, e_{n+1} is constructed by gluing three copies of e_n according to the rule



Here each arrow emanates from the origin vertex of G_n to indicate the orientation.

Proposition. e_n is the identity element of $(\mathcal{R}_n, \oplus)$.

Remark. The sequence (e_n) is nested: $e_{n+1}|_{G_n} = e_n$.

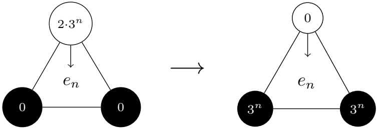
Proof that e_n is the identity element

Proposition. $e_n \oplus e_n = e_n$, and each sink receives $\frac{3}{2}(3^n - 1)$ chips upon stabilization.

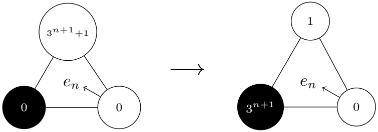
Proof. By induction on n.

- First, establish two types of toppling identities, one having two sink vertices and the other having one sink vertex.

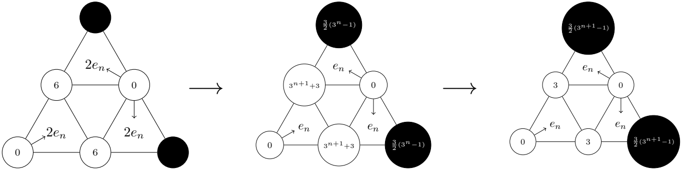
Two sinks at the bottom:



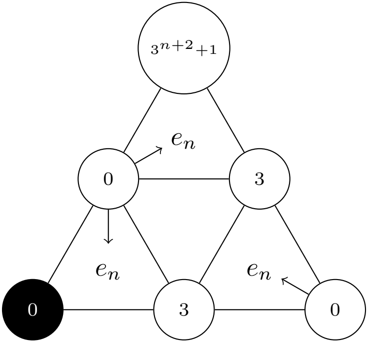
One sink at the bottom-left:



- Assuming the induction hypothesis, we then have

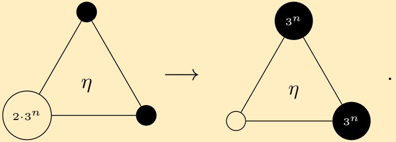


Induction step in the proof of toppling with one sink

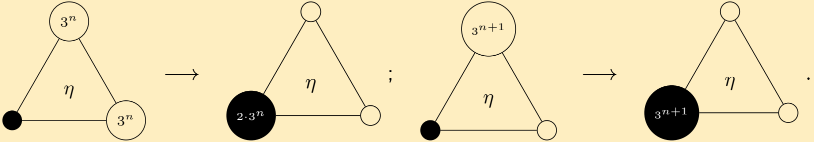


Toppling identities, periodicity

Proposition. For every $\eta \in \mathcal{R}_n^{(s)}$,

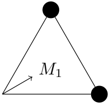


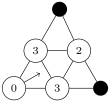
For every $\eta \in \mathcal{R}_n^{(o)}$,



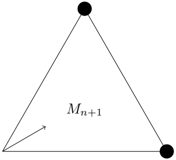
Proof. Already showed $(2 \cdot 3^n)\mathbb{1}_o \oplus e_n = e_n$. By the abelian property, $(2 \cdot 3^n)\mathbb{1}_o \oplus \eta = ((2 \cdot 3^n)\mathbb{1}_o \oplus e_n) \oplus \eta = e_n \oplus \eta = \eta$. The rest are similar.

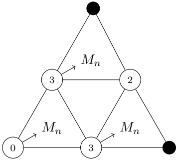
This explains the $(2 \cdot 3^n)$-periodicity in the sandpile growth (and the patterns restricted to G_n).

Let 

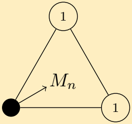
$=$ 

, and for each $n \geq 1$, M_{n+1} is constructed by gluing three copies of M_n according to the rule



$=$


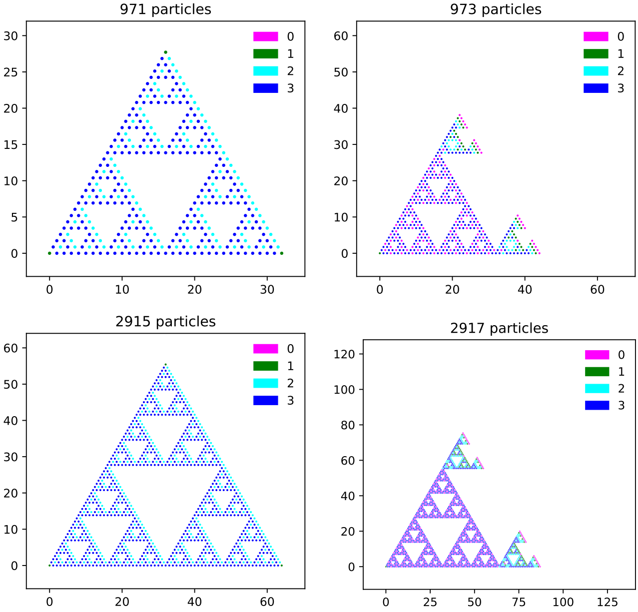
Proposition.



is the identity element w/ one sink.

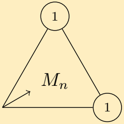
Proof. Same idea as the proof of the id element with two sinks.

The main explosions at $4 \cdot 3^n$: transition from M_n to e_n

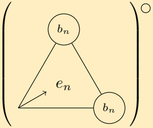


The main explosions at $4 \cdot 3^n$: transition from M_n to e_n

Proposition.

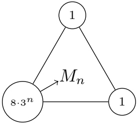
$((4 \cdot 3^n - 2)\mathbb{1}_o)^\circ =$ 

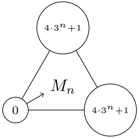
;

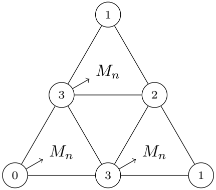
$((4 \cdot 3^n)\mathbb{1}_o)^\circ =$ 

, where $b_n = \frac{3}{2}\left(3^{n-1} + 1\right)$.

Proof of the induction step.

$(4 \cdot 3^{n+1} - 2)\mathbb{1}_o \rightarrow$ 

$\rightarrow$ 

$\rightarrow$ 

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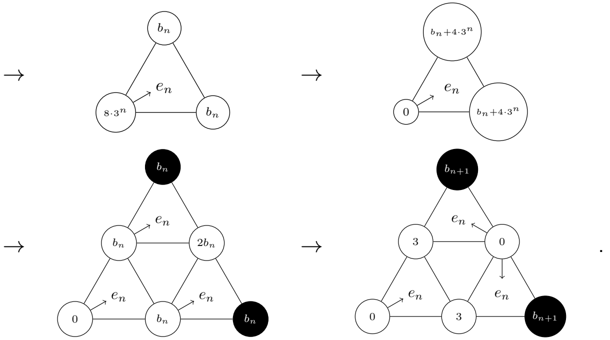
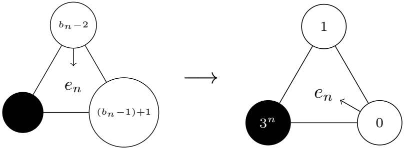
Proof (cont.).

$(4 \cdot 3^{n+1})\mathbb{1}_o \to$



The last step depends on the following "reflection lemma," using the axial symmetry of SG:



Also note $b_{n+1} = b_n + 3^n$.

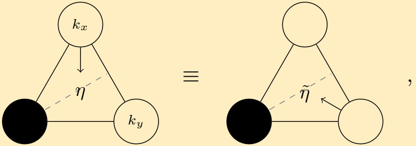
An axial reflection lemma (SG with one sink)

$e_n^{(o)}$ is the id element of $\mathcal{R}_n^{(o)}$; $\partial G_n = \{x, y\}$.

Lemma. Let $\eta \in \mathcal{R}_n^{(o)}$ be such that $\eta = e_n^{(o)} \oplus \alpha \mathbb{1}_x \oplus \beta \mathbb{1}_y$ for some $\alpha, \beta \in \mathbb{N}_0$. Let $k_x, k_y \in \mathbb{N}_o$ solve the system of equations

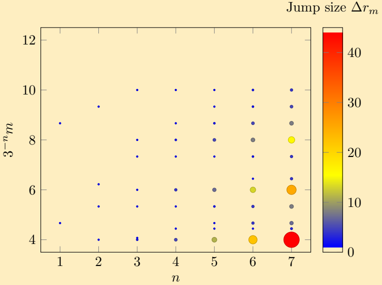
$$\begin{cases} \alpha + k_x &= \beta + p_0 \cdot 3^n + p_1 \cdot 3^{n+1} \\ \beta + k_y &= \alpha + p_0 \cdot 3^n + p_2 \cdot 3^{n+1} \end{cases}$$

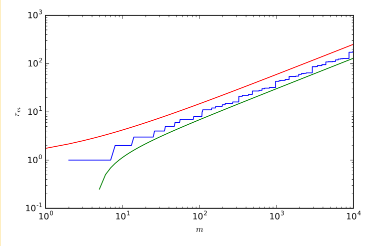
for some $p_0, p_1, p_2 \in \mathbb{Z}$ (which come from the toppling identities).
Then



where $\tilde{\eta} = e_n^{(o)} \oplus \beta \mathbb{1}_x \oplus \alpha \mathbb{1}_y$ is the reflection of η across the axis of symmetry.

Theorem (C.–Kudler-Flam '17∼'18)

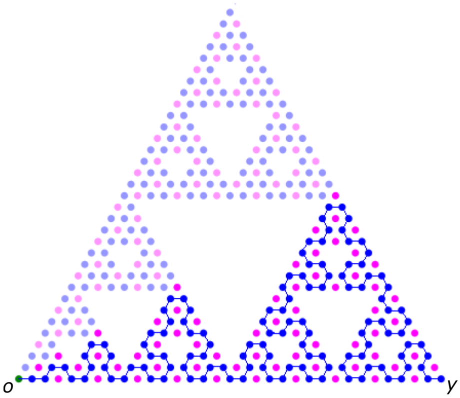




$$\begin{tikzcd}
 & 6\frac{2}{3} \arrow[d] & & 8\frac{4}{9} \arrow[r] & 7\frac{1}{3} \arrow[d] & 8 \arrow[l] & \\
6\frac{4}{9} \arrow[r] & 4\frac{2}{3} \arrow[r] & 6 \arrow[r] & 4\frac{4}{9} \arrow[r] & 5\frac{1}{3} \arrow[r] & 8\frac{2}{3} \arrow[u] & 10 \arrow[loop right] \\
 & & & 4 \arrow[u] & & 9\frac{1}{3} \arrow[u] &
\end{tikzcd}$$

$$a \to b: \quad r_{a\cdot 3^n} = 2^n + r_{b\cdot 3^{n-1}} \quad (n \geq 3)$$
$$a \to b: \quad r_{a\cdot 3^n} = 2^n + r_{b\cdot 3^{n-2}} \quad (n \geq 4)$$

Radial asymptotics: Power law (with exponent $\log_3 2$) modulated by log-periodic oscillation (with explicit Fourier series representation) → proved using the renewal theorem.



Thank you!