# Nonequilibrium fluctuations in the boundary-driven exclusion process on a resistance space

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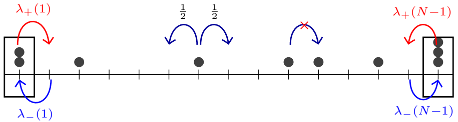
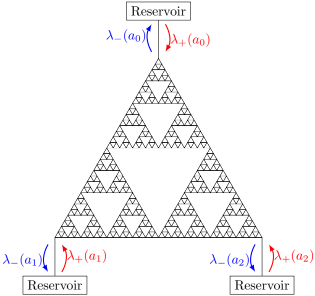
# Overview of results




Scaling limits of empirical density in the boundary-driven SEP on the Sierpinski gasket

- LLN & eqFluct: Joint work with Patrícia Gonçalves (IST Lisboa), `arXiv:1904.08789`.
- LDP: Joint work with Michael Hinz (Bielefeld) (2019+).
- NoneqFluct & hydrostatics: Joint w/ Chiara Franceschini, Patrícia Gonçalves, and Otávio Menezes (all IST Lisboa) (2019+).

Functional inequalities and local averging tools (C.)

- Moving particle lemma: *ECP* '17, `arXiv:1606.01577`.
- Local ergodicity (1-block & 2-blocks estimates): `arXiv:1705.10290`.

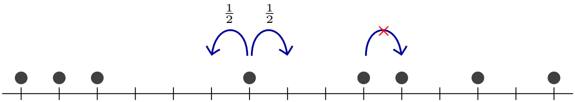
# Outline

Motivation: Generalizing the analysis of the exclusion process from 1D to higher dimensions

Boundary-driven exclusion process on the Sierpinski gasket

New tools & ideas for resistance spaces

# Exclusion process




The **(symm.) exclusion process** on $(G, \mathbf{c})$ is a Markov chain on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

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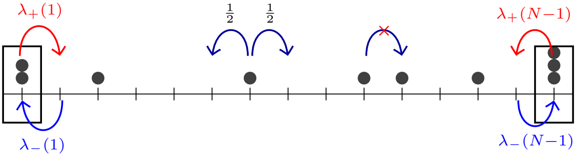
- Each product Bernoulli measure $\nu_\alpha$, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure.
- Dirichlet energy: $\mathcal{E}^{\mathrm{EX}}(f) = \dfrac{1}{2} \displaystyle\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta).$

# Adding reservoirs (Glauber dynamics) to the exclusion process



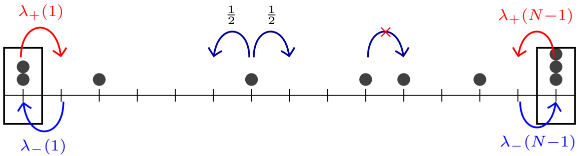

Designate a finite boundary set $\partial V \subset V$. For each $a \in \partial V$:

- At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth).
- At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).

Formally,
$(\mathcal{L}^{\rm boun}_{\partial V} f)(\eta) = \sum_{a\in\partial V} [\lambda_+(a)(1-\eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R}$, where

$$\eta^a(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

# Adding reservoirs (Glauber dynamics) to the exclusion process




- 1D boundary-driven simple exclusion process: generator $N^2\left(\mathcal{L}^{\rm EX}_{\{1,2,\cdots,N-1\}} + \mathcal{L}^{\rm boun}_{\{1,N-1\}}\right)$.
- Has been studied extensively for the past $\sim$ 15 years:
  Hydrodynamic limits, fluctuations, large deviations, etc.
  Bertini–DeSole–Gabrielli–Landim–Jona-Lasinio '03, '07; Landim–Milanes–Olla '08; Franco–Gonçalves–Neumann '13, '17; Baldasso–Menezes–Neumann–Souza '17; Gonçalves–Jara–Menezes–Neumann '18+; ...
- Difficulties: # of particles is no longer conserved; the invariant measure is in general not explicit.

# Extending the analysis to higher dims & with $> 2$ reservoirs?

Symmetric exclusion process

Euclidean torus $(\mathbb{Z}/N\mathbb{Z})^d$: Too many results, *cf.* Kipnis-Landim '99
Crystal lattices: Tanaka '12
Riemannian manifolds: van Ginkel-Redig '18 (no translational invariance)


$d_{\rm spec} \geq 2$: CURSE OF DIMENSIONALITY!!

Boundary-driven (weakly) asymmetric exclusion process

$\mathbb{Z}$ or $(\mathbb{Z}/N\mathbb{Z})$
Energy methods/PDE ✓
Algebraic duality ✓
Integrable probability ✓
...
LOTS OF TOOLS

**Resistance spaces**
(w/o translational invariance)
[incl.: $\mathbb{Z}$ with long jumps,
$\mathbb{Z}$ with a slow bond or site,
fractals, trees, random graphs, ...]
**Energy methods/PDE ✓**
Algebraic duality (some ✓, some ?)
Integrable probability ???

- **Today's message:** On state spaces with spectral dimension $d_{\rm spec} \in [1,2)$ (diffusion is strongly recurrent), we have a path towards proving scaling limits of SSEP/WASEP w/o requiring translational invariance.
- **Open question:** Prove scaling limits of boundary-driven SSEP/WASEP on state spaces with $d_{\rm spec} \geq 2$ (diffusion is NOT strongly recurrent).

# Resistance spaces [Kigami '03]

Let $K$ be a nonempty set. A **resistance form** $(\mathcal{E}, \mathcal{F})$ on $K$ is a pair such that

1. $\mathcal{F}$ is a vector space of $\mathbb{R}$-valued functions on $K$ containing the constants, and $\mathcal{E}$ is a nonnegative definite symmetric quadratic form on $\mathcal{F}$ satisfying
$$\mathcal{E}(u,u) = 0 \quad \Leftrightarrow \quad u \text{ is constant.}$$
2. $\mathcal{F}/\{\text{constants}\}$ is a Hilbert space with norm $\mathcal{E}(u,u)^{1/2}$.
3. Given a finite subset $V \subset K$ and a function $v : V \to \mathbb{R}$, there is $u \in \mathcal{F}$ s.t. $u|_V = v$.
4. For $x, y \in K$, the **effective resistance**
$$R_{\rm eff}(x,y) := \sup\left\{ \frac{[u(x)-u(y)]^2}{\mathcal{E}(u,u)} : u \in \mathcal{F},\ \mathcal{E}(u,u) > 0 \right\} < \infty.$$
5. (Markovian property) If $u \in \mathcal{F}$, then $\bar{u} := 0 \vee (u \wedge 1) \in \mathcal{F}$ and $\mathcal{E}(\bar{u},\bar{u}) \leq \mathcal{E}(u,u)$.

# Resistance spaces [Kigami '03]

## Point-to-point effective resistance is finite

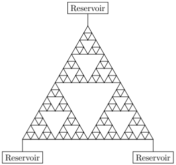
$$R_{\rm eff}(x,y) := \sup\left\{\frac{[u(x)-u(y)]^2}{\mathcal{E}(u,u)} : u \in \mathcal{F},\ \mathcal{E}(u,u)>0\right\} < \infty.$$

## Examples of resistance spaces

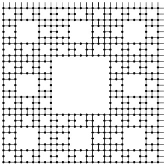
- Classical Dirichlet form $\int_\Omega |\nabla u|^2\,dx$ on $L^2(\Omega, dx)$ is a resistance form $\Leftrightarrow$ $\Omega$ has Euc dim 1.
- $\alpha$-stable process on $\mathbb{R}$ with $\alpha \in (1,2]$:

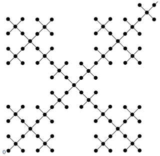
$$\mathcal{E}^{(\alpha)}(u) = \int_{\mathbb{R}^2} \frac{[u(x)-u(y)]^2}{|x-y|^{1+\alpha}}\,dy\,dx.$$

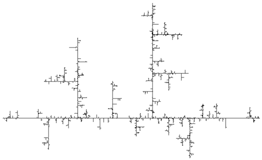
- Diffusion on (some) fractals, trees, random graphs:



Sierpinski gasket


Sierpinski carpet


Vicsek tree


Random dendrite [by David Croydon]

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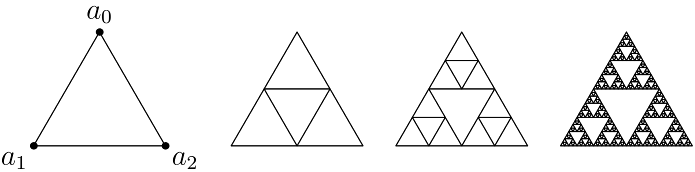
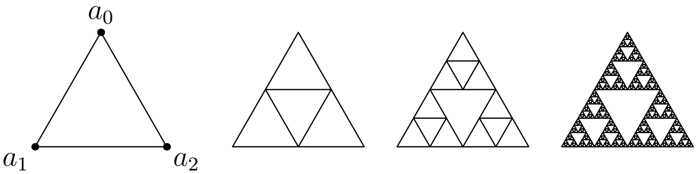
# Boundary-driven exclusion process on the Sierpinski gasket




- Construction of **Brownian motion** with invariant measure $m$ (the standard self-similar measure) as scaling limit of RWs accelerated by $\mathcal{T}_N = 5^N$.
  [Goldstein '87, Kusuoka '88, Barlow-Perkins '88]
- A **robust notion of calculus** on $SG$ which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.
  [Kigami, *Analysis on Fractals* '01; Strichartz, *Differential Equations on Fractals* '06]
- A good model for rigorously studying (non)equilibrium stochastic dynamics with $\geq 3$ boundary reservoirs.

# Analysis on fractals (à la Kigami–Strichartz)




- Define the discrete renormalized Dirichlet energy on $G_N$:

$$\mathcal{E}_N(f) = \frac{5^N}{3^N}\frac{1}{2}\sum_{\substack{x,y\in V_N \\ x\sim y}} [f(x)-f(y)]^2, \quad f: K \to \mathbb{R}.$$

**Fact.** $\{\mathcal{E}_N(f)\}_N$ is monotone nondecreasing, so it either converges to a finite quantity or diverges to $+\infty$.

Define $\mathcal{F} := \{f : \lim_{N\to\infty} \mathcal{E}_N(f) < +\infty\}$, and for each $f \in \mathcal{F}$, we denote the limit energy by $\mathcal{E}(f)$.

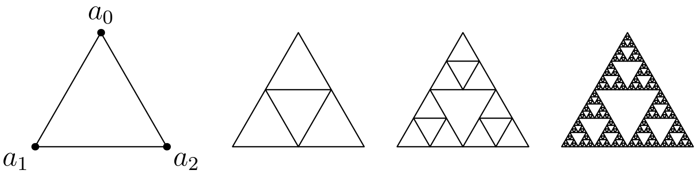
- Analogy to the 1D interval:

$$\left(\int_{[0,1]} |\nabla f|^2\, dx,\ H_1([0,1])\right) \quad \text{vs.} \quad \left(\mathcal{E}(f) = \int_K \text{“}|\nabla f|^2\text{”}\, dm,\ \mathcal{F}\right)$$

Sobolev embedding: $H_1([0,1]) \subset C([0,1])$, $\mathcal{F} \subset C(K)$.

- **Caveat.** The “$|\nabla f|^2$” does NOT exist literally.

# Analysis on fractals (à la Kigami–Strichartz)




- Define the discrete renormalized Dirichlet energy on $G_N = (V_N, E_N)$:

$$\mathcal{E}_N(f) = \frac{5^N}{3^N} \frac{1}{2} \sum_{\substack{x,y \in V_N \\ x \sim y}} [f(x) - f(y)]^2, \quad f : K \to \mathbb{R}.$$

  **Fact.** $\{\mathcal{E}_N(f)\}_N$ is monotone nondecreasing, so it either converges to a finite quantity or diverges to $+\infty$.
  Define $\mathcal{F} := \{f : \lim_{N\to\infty} \mathcal{E}_N(f) < +\infty\}$, and for each $f \in \mathcal{F}$, we denote the limit energy by $\mathcal{E}(f)$.

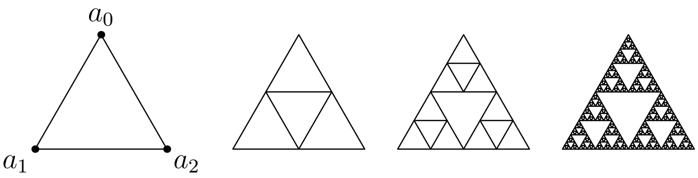
- Analogy to the 1D interval:

$$\left( \int_{[0,1]} |\nabla f|^2 \, dx, \ H_1([0,1]) \right) \quad \text{vs.} \quad \left( \mathcal{E}(f) = \int_K d\Gamma(f), \ \mathcal{F} \right)$$

  Sobolev embedding: $H_1([0,1]) \subset C([0,1])$, $\mathcal{F} \subset C(K)$.

- **Caveat.** For nonconstant $f \in \mathcal{F}$, $d\Gamma(f) \perp dm$. This is a source of great technical difficulty in the analysis of RW/IPS on fractals.

# Analysis on fractals (à la Kigami–Strichartz)




- **Laplacian**: the following two formulations coincide.
  - Weak formulation: Say $u = -\Delta f \in C(K)$ if $\mathcal{E}(v, f) = \int_K vu\, dm$ for all $v \in \mathcal{F}_0 := \{\phi \in \mathcal{F} : \phi|_{V_0} = 0\}$.
  - Pointwise formulation ($x \in V_N \setminus V_0$): $(\Delta f)(x) := \lim_{N\to\infty} \frac{3}{2} 5^N \sum_{\substack{y\in V_N \\ y\sim x}} [f(y) - f(x)]$.

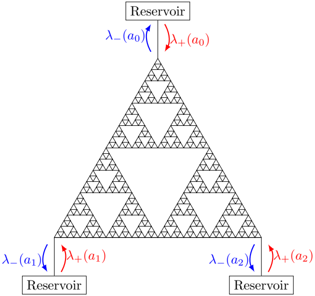
  Denote by $\mathrm{dom}\Delta$ the operator domain of the Laplacian.

For each $f \in \mathrm{dom}\Delta$ we can further give:

- (Outward) Normal derivative at the boundary ($a \in V_0$): $(\partial^\perp f)(a) = \lim_{N\to\infty} \frac{5^N}{3^N} \sum_{\substack{y\in V_N \\ y\sim a}} [f(a) - f(y)]$.
- Integration by parts formula:

$$\mathcal{E}(f, g) = \int_K (-\Delta f) g\, dm + \sum_{a\in V_0} (\partial^\perp f)(a) g(a) \qquad (f \in \mathrm{dom}\Delta,\ g \in \mathcal{F})$$

# Exclusion process on the Sierpinski gasket with slowed boundary




$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left( \mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Parameter $b > 0$ governs the inverse speed at which the reservoir injects/extracts particles into/from the boundary vertices $V_0$.

## Main result in a nutshell

A phase transition in the scaling limit of the particle density with respect to $b > 0$, reflected by the different boundary conditions.

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

# Hydrodynamic limit: a LLN result

Assume that sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on $\{0,1\}^{V_N}$ is associated to a density profile $\varrho : K \to [0,1]$:
$\forall F \in C(K)$, $\forall \delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \{0,1\}^{V_N} \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} F(x)\eta(x) - \int_K F(x)\varrho(x)\, dm(x) \right| > \delta \right\} = 0.$$

Given the process $\{\eta_t^N : t \geq 0\}$ generated by $5^N \mathcal{L}_N^{\rm bEX}$, the **empirical density measure** (and its pairing with test functions $F : K \to \mathbb{R}$):

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) \mathbb{1}_{\{x\}} \qquad \left( \pi_t^N(F) = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x). \right)$$

**Claim.** $\{\pi^N_\cdot\}_N$ converges in the Skorokhod topology on $D([0,T],\mathcal{M}_+)$ to the unique measure $\pi_\cdot$ with $d\pi_\cdot(x) = \rho(\cdot,x)\, dm(x)$.
$\forall t \in [0,T]$, $\forall F \in C(K)$, $\forall \delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta^N_\cdot \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

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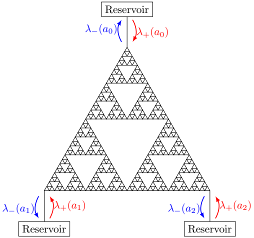


$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left( \mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$

## Theorem (Density hydrodynamic limit (C.–Gonçalves '19))

*For any $t \in [0,T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,*

$$\lim_{N\to\infty} \mu_N \left\{ \eta^N_\cdot \;:\; \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta^N_t(x) F(x) - \int_K F(x) \rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

where $\rho$ is the unique weak solution of the heat equation
with Dirichlet boundary condition if $b < \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ \rho(t,a) = \bar{\rho}(a), & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

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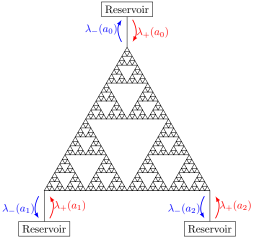


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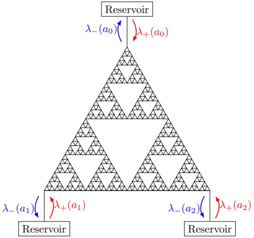
where $\rho$ is the unique weak solution of the heat equation
with Neumann boundary condition if $b > \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t,a) = 0, & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

# Hydrodynamic limit: a LLN result

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where $\rho$ is the unique weak solution of the heat equation
with linear Robin boundary condition if $b = \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t,a) = -\lambda_\Sigma(a)(\rho(t,a) - \bar\rho(a)), & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$

# Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N\left(\left(\frac{2}{3}\Delta + \partial_s\right) F_s\right) ds$$
$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a)) F_s(a)\right] ds + o_N(1).$$

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[Ingredient #1] **Analysis on fractals**

This part will produce the weak formulation of the heat equation.

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$$+ \int_0^t \frac{3^N}{|V_N|}\sum_{a\in V_0}\left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a) - \bar\rho(a))F_s(a)\right]\,ds + o_N(1).$$

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[Ingredient #2] **Analysis of the boundary term**

- $b > 5/3$: The first term dominates, should converge to $\int_0^t \frac{2}{3}\sum_{a\in V_0}\rho_s(a)(\partial^\perp F_s)(a)\,ds$
- $b = 5/3$: Both terms contribute equally, should converge to $\int_0^t \frac{2}{3}\sum_{a\in V_0}\left[\rho_s(a)(\partial^\perp F_s)(a) + \lambda_\Sigma(a)(\rho_s(a) - \bar\rho(a))F_s(a)\right]\,ds$
- $b < 5/3$: Impose $\rho_t(a) = \bar\rho(a)$ for all $a \in V_0$, should converge to $\int_0^t \frac{2}{3}\sum_{a\in V_0}\bar\rho(a)(\partial^\perp F_s)(a)$

Require a series of **replacement lemmas**: not trivial on state spaces without translational invariance!
→**Octopus inequality, moving particle lemma**

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$$+ \int_0^t \frac{3^N}{|V_N|}\sum_{a \in V_0}\left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a))F_s(a)\right]\,ds + o_N(1).$$

$$\Big\downarrow N \to \infty$$

$$0 = \pi_t(F_t) - \pi_0(F_0) - \int_0^t \pi_s\left(\left(\frac{2}{3}\Delta + \partial_s\right)F_s\right)\,ds + (\text{boundary term})$$

---

[Ingredient #3] **Convergence of stochastic processes**

- Show that $\{\pi_\cdot^N\}_N$ is tight in the Skorokhod topology on $D([0,T],\mathcal{M}_+)$ via Aldous' criterion.
- Prove that any limit point $\pi_\cdot$ is absolutely continuous w.r.t. the self-similar measure $m$, with $\pi_t(dx) = \rho(t,x)\,dm(x)$, and $\rho \in L^2(0,T,\mathcal{F})$.
- Finally, prove ! of the weak solution to the heat equation to conclude ! of the limit point.

# Density fluctuation field: Heuristics

**Equilibrium** $\Leftrightarrow$ $\lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
**Equilibrium**: the product Bernoulli measure $\nu_\rho^N$ with $\rho = \lambda_+/(\lambda_+ + \lambda_-)$ is stationary for the process. Not true in the non-equilibrium setting.

**Density fluctuation field (DFF)**

$$\mathcal{Y}_t^N(F) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \underbrace{\left(\eta_t^N(x) - \mathbb{E}_{\mu_N}[\eta_t^N(x)]\right)}_{=:\bar{\eta}_t^N(x)} F(x)$$

The corresponding Dynkin's martingale is

$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F)\, ds + o_N(1)$$
$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[ (\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma(a) F(a) \right] ds,$$

which has QV

$$\langle \mathcal{M}^N(F) \rangle_t = \int_0^t \frac{5^N}{|V_N|^2} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} (\eta_s^N(x) - \eta_s^N(y))^2 (F(x) - F(y))^2 ds$$
$$+ \int_0^t \sum_{a \in V_0} \frac{5^N}{b^N |V_N|^2} \{\lambda_-(a)\eta_s^N(a) + \lambda_+(a)(1 - \eta_s^N(a))\} F^2(a) ds.$$

# Density fluctuation field: Heuristics

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which, as $N \to \infty$, has the QV of a space-time white noise (with boundary condition)

$$\frac{2}{3} \cdot 2 \int_0^t \int_K \chi(\rho_s)\, d\Gamma_b(F)\, ds, \quad \text{where } \chi(\alpha) = \alpha(1-\alpha), \quad \mathscr{E}_b(F) = \mathcal{E}(F) + \sum_{a \in V_0} \lambda_\Sigma(a) F^2(a) \mathbf{1}_{\{b=5/3\}},$$

and $\Gamma_b(F)$ is the energy measure associated to $\mathscr{E}_b(F)$: $\mathscr{E}_b(F) = \int_K d\Gamma_b(F)$.

# Density fluctuation field: Heuristics

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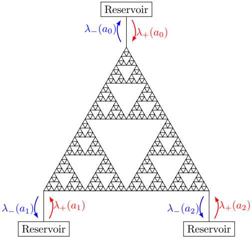
We then argue that the test function $F \in \mathrm{dom}\Delta_b$ be chosen appropriate to each boundary condition such that the boundary term vanishes as $N \to \infty$.

$$\mathrm{dom}\Delta_b := \begin{cases} \{F \in \mathrm{dom}\Delta : F|_{V_0} = 0\}, & \text{if } b < 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = -\lambda_\Sigma F|_{V_0}\}, & \text{if } b = 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = 0\}, & \text{if } b > 5/3. \end{cases}$$

For technical reasons we use a smaller test function space $\mathcal{S}_b := \{F \in \mathrm{dom}\Delta_b : \Delta_b F \in \mathrm{dom}\Delta_b\}$, which can be made into a Frechét space. Let $\mathcal{S}_b'$ be the topological dual of $\mathcal{S}_b$.

# Scaling limit of density fluctuations: Equilibrium




$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left( \mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

**Eq.** $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ $\forall a \in V_0$.

Let $\mathbb{Q}_\rho^{N,b}$ be the probability measure on $D([0,T], \mathcal{S}_b')$ induced by the DFF $\mathcal{Y}_\cdot^N$ started from $\nu_\rho^N$ and boundary parameter $b$.

## Theorem (EqCLT (C.–Gonçalves '19))

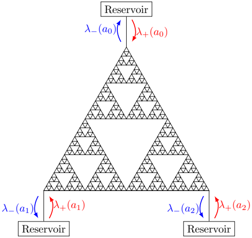
*The sequence $\{\mathbb{Q}_\rho^{N,b}\}_N$ converges in distribution, as $N \to \infty$, to a unique solution of the Ornstein-Uhlenbeck equation with covariance*

$$\mathbb{E}[\mathcal{Y}_t(F)\mathcal{Y}_s(G)] = \chi(\rho) \int_K (\tilde{T}_t^b F)(\tilde{T}_s^b G)\, dm + \frac{2}{3} \cdot 2 \cdot \chi(\rho) \int_0^s \mathscr{E}_b \left( \tilde{T}_{t-r}^b F, \tilde{T}_{s-r}^b G \right) dr$$

*for $0 \le s \le t \le T$ and $F, G \in \mathcal{S}_b$.*

$\left\{ \tilde{T}_t^b \right\}_{t>0}$ is the heat semigroup associated to $\frac{2}{3}\mathscr{E}_b$.

# Scaling limit of density fluctuations: Non-equilibrium, Dirichlet case




$$5^N \mathcal{L}_N^{\rm bEX} = 5^N \left(\mathcal{L}_N^{\rm EX} + \mathcal{L}_N^{\rm boun}\right).$$

**Assumptions**

1. $\{\mu_N\}_N$ is associated to a profile $\varrho : K \to [0,1]$.
2. $\displaystyle\sup_{x,y\in V_N} \left|\mathbb{E}_{\mu_N}[\bar\eta^N(x)\bar\eta^N(y)]\right| \lesssim |V_N|^{-1}$.

Let $\mathbb{Q}_{\mu_N}$ be the probability measure on $D([0,T],\mathcal{S}'_{\rm Dir})$ induced by the DFF $\mathcal{Y}^N_\cdot$ started from $\mu_N$.

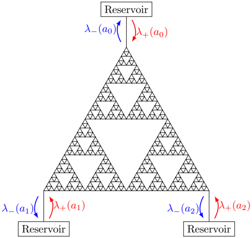
## Theorem (NoneqFluct (C.–Franceschini–Gonçalves–Menezes '19+))

*Under the above Assumptions, any limit point $\mathbb{Q}^*$ of $\{\mathbb{Q}_{\mu_N}\}_N$ concentrates on paths*

$$\mathcal{Y}_t(F) = \mathcal{Y}_0(\tilde{\mathsf{T}}_t^{\rm Dir} F) + \mathcal{W}_t(F) \qquad \forall F \in \mathcal{S}_{\rm Dir},$$

*where $\mathcal{Y}_0$ and $\mathcal{W}_t$ are uncorrelated mean-zero random fields, and $\mathcal{W}_t$ is Gaussian with variance* $\displaystyle\frac{2}{3}\cdot 2\int_0^t \int_K \chi(\rho_s)\, d\Gamma(\tilde{T}^{\rm Dir}_{t-s}F)\, ds.$

# Scaling limit of density fluctuations: Non-equilibrium, Dirichlet case




$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left( \mathcal{L}_N^{\mathrm{EX}} + \mathcal{L}_N^{\mathrm{boun}} \right).$$

Assumptions

1. $\{\mu_N\}_N$ is associated to a profile $\varrho : K \to [0,1]$.
2. $\sup_{x,y \in V_N} \left| \mathbb{E}_{\mu_N}[\bar{\eta}^N(x)\bar{\eta}^N(y)] \right| \lesssim |V_N|^{-1}$.
3. $\mathcal{Y}_0^N \xrightarrow{d} \mathcal{Y}_0$ Gaussian.

Let $\mathbb{Q}_{\mu_N}$ be the probability measure on $D([0,T], \mathcal{S}'_{\mathrm{Dir}})$ induced by the DFF $\mathcal{Y}^N_\cdot$ started from $\mu_N$.

**Theorem (NoneqCLT** (C.–Franceschini–Gonçalves–Menezes '19+)**)**

*Under the above Assumptions, $\{\mathbb{Q}_{\mu_N}\}_N$ converges to a generalized O-U process with covariance*

$$\mathbb{E}[\mathcal{Y}_t(F)\mathcal{Y}_s(G)] = \mathbb{E}\left[\mathcal{Y}_0(\tilde{T}_t^{\mathrm{Dir}} F)\mathcal{Y}_0(\tilde{T}_s^{\mathrm{Dir}} G)\right] + \frac{2}{3}\cdot 2\int_0^s \int_K \chi(\rho_r)\, d\Gamma\left(\tilde{\mathrm{T}}_{t-r}^{\mathrm{Dir}} F, \tilde{\mathrm{T}}_{s-r}^{\mathrm{Dir}} G\right) dr$$

*for $0 \le s \le t \le T$ and $F, G \in \mathcal{S}_{\mathrm{Dir}}$.*

# Outline

# New/old tools & ideas

## **Microscopics**: Exclusion process on a non-lattice state space

NO translational invariance.

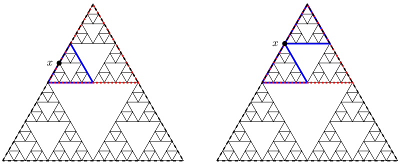
- How to carry out **local averaging** without using translation?
  *Ans:* Use the **effective resistance** for the random walk process, in conjunction with space-time scaling limits of random walks to a diffusion process ( **invariance principle** ).
- How to characterize **nonequilibrium correlations** $\phi(x,y) = \mathbb{E}[\bar{\eta}(x)\bar{\eta}(y)]$ in the exclusion process on a general graph?
  *Ans:* Identify $\phi$ as the solution to a discretized **Poisson's equation on the product graph** , and "invert the Laplacian."

## **Macroscopics**: Analysis of (S)PDEs on fractals / metric measure spaces

NO explicit representation formulas, DELICATE notion of gradient $\nabla$, but EXCELLENT notion of Laplacian $\Delta$.

- Dirichlet forms for diffusion $\mathcal{E}(f,g) = \langle f, -\Delta g\rangle_m$, heat semigroup $\{T_t\}_{t>0}$
- Heat kernel bounds $p_t(x,y)$ (Nash ineq.), spectral asymptotics, Green's function $G(x,y)$.

# Local averaging




For finite $\Lambda \subset V$, denote the **average density over** $\Lambda$ by $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z\in\Lambda} \eta(z)$.
In the proof of the hydrodynamic limit for Markov processes, with generator $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$ on a sequence of graphs $G_N = (V_N, E_N)$, we use that for every $t > 0$:

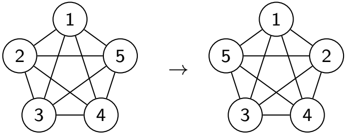
## Replacement lemma

$$\overline{\lim_{\epsilon\downarrow 0}}\ \overline{\lim_{N\to\infty}}\ \mathbb{E}_{\mu_N}\left[\left|\int_0^t \left(\eta_s^N(x) - \mathrm{Av}_{B(x,\epsilon N)}[\eta_s^N]\right) ds\right|\right] = 0, \quad x \in V_N.$$

where

- $\{\eta_t^N : t \geq 0\}$ is the exclusion process generated by $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$, where $\mathcal{T}_N$ is the diffusive time acceleration factor.
- $\mu_N$ can be any measure on $\{0,1\}^{V_N}$.
- $B(x,r)$ is a "ball" of radius $r$ centered at $x$ (in the graph metric).

# Hierarchy of stochastic processes on a fixed graph

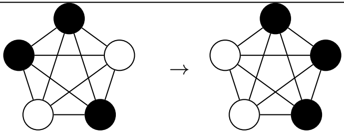



**Interchange process** $f : \{\text{Permutations on } V\} \to \mathbb{R}$

$$\mathcal{E}^{\rm IP}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 \, d\nu(\eta).$$

Reversible measure: uniform measure $\nu$ on $\{\text{Perms on } V\}$.

↓ PROJECTION ↓

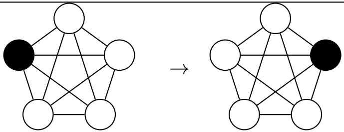



**Exclusion process** $f : \{0,1\}^V \to \mathbb{R}$

$$\mathcal{E}^{\rm EX}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 \, d\nu_\alpha(\eta).$$

Reversible measure: product Bernoulli measure $\nu_\alpha$, $\alpha \in [0,1]$, $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for all $x \in V$.

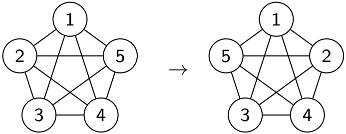
↓ PROJECTION ↓




**Random walk process** $f : V \to \mathbb{R}$

$$\mathcal{E}^{\rm RW}(f) = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2.$$

# Hierarchy of stochastic processes on a fixed graph

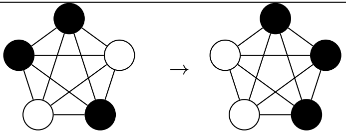



**Interchange process** $f : \{\text{Permutations on } V\} \to \mathbb{R}$

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \leq R_{\rm eff}(x,y)\mathcal{E}^{\rm IP}(f).$$

**Moving particle lemma** [C. *ECP* 2017]

↓ PROJECTION ↓

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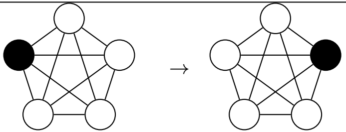
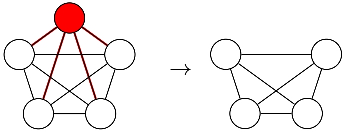



**Random walk process** $f : V \to \mathbb{R}$

$$[f(x) - f(y)]^2 \leq R_{\rm eff}(x,y)\mathcal{E}^{\rm RW}(f).$$

**Dirichlet principle** [1867]

# Octopus inequality & Aldous' spectral gap conjecture



Using the network reduction idea & delicately carrying out a series of Schur complementations, **Caputo–Liggett–Richthammer JAMS '10** proved for the **interchange process**:

## Theorem (Octopus inequality, IP (Caputo–Liggett–Richthammer *JAMS* '10))

*For all* $f : \mathcal{S}_{|V|} \to \mathbb{R}$,

$$\int \sum_{y \in V_x} c_{xy}[f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \geq \int \sum_{yz \in E_x} \tilde{c}_{yz}[f(\eta^{yz}) - f(\eta)]^2 \, d\nu(\eta).$$

*Energy lost from removed edges* $\geq$ *Energy gained from increased conductances*

This was the key inequality which resolved Aldous' '92 spectral gap conjecture:

$$\left\{ \begin{array}{l} \text{Projection argument gives } \lambda_2^{\mathrm{RW}}(G) \leq \lambda_2^{\mathrm{EX}}(G) \leq \lambda_2^{\mathrm{IP}}(G) \\ \text{(OI)} \implies \lambda_2^{\mathrm{IP}}(G) \geq \lambda_2^{\mathrm{RW}}(G) \end{array} \right\} \implies \lambda_2^{\mathrm{IP}}(G) = \lambda_2^{\mathrm{EX}}(G) = \lambda_2^{\mathrm{RW}}(G)$$

# Moving particle lemma for interchange/exclusion

Bounding the energy cost of swapping two particles at $x$ and $y$ in an interacting particle system by the **effective resistance** between $x$ and $y$ w.r.t. the random walk process.

## Theorem (MPL, IP/EX (C. *ECP* '17))

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\rm eff}(x,y)\mathcal{E}^{\rm IP}(f), \quad f : \mathcal{S}_{|V|} \to \mathbb{R},$$
$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\rm eff}(x,y)\mathcal{E}^{\rm EX}(f), \quad f : \{0,1\}^V \to \mathbb{R}.$$

*Proof.*

- (OI) $\Leftrightarrow$ monotonicity of energy under 1-point network reductions. So reduce $G$ successively until two vertices $x, y$ are left, we get MPL for IP.
- A further projection argument yields the MPL for EX.

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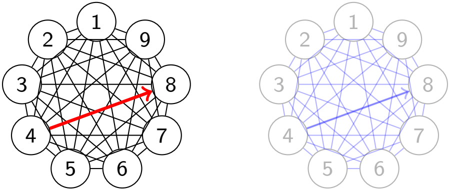


Conventional approach is to pick a shortest path connecting $x$ and $y$, and telescope along the path to obtain the energy cost. [Guo–Papanicolaou–Varadhan '88, Diaconis–Saloff-Coste '93].

OK on finite integer lattices, but does NOT always give optimal cost on general weighted graphs.

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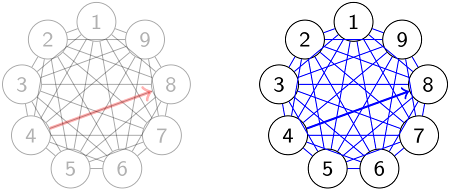
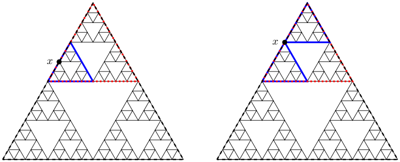



MPL bounds the energy cost by "optimizing electric flow over all paths connecting $x$ and $y$."

# MPL & local averaging

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$$\eta(x) - \mathrm{Av}_B[\eta] = \frac{1}{|B|}\sum_{z\in B}\left(\eta(x)-\eta(z)\right).$$

Estimating this cost using the variational characterization of the largest eigenvalue requires telescoping or MPL. Works for resistance spaces; UNCLEAR if there is an analog of this for $d_{\mathrm{spec}} \geq 2$.

# Two-particle correlation functions, nonequilibrium

- $\mu_{\rm ss}^N$: unique invariant measure for $5^N \mathcal{L}_N^{\rm bEX}$, $b=1$.
- Steady-state density: $\rho_{\rm ss}^N(x) = \mathbb{E}_{\mu_{\rm ss}^N}[\eta(x)]$.
- Steady-state correlation: $\phi_{\rm ss}^N(x,y) = \mathbb{E}_{\mu_{\rm ss}^N}[(\eta(x)-\rho_{\rm ss}^N(x))(\eta(y)-\rho_{\rm ss}^N(y))]$.

  Related to the local time for two particles in EX to stay adjacent to each other.
- In 1D, $\phi_{\rm ss}^N(x,y)$ is exactly a multiple of the Green's function for RW, $-\frac{1}{N-1}G^N(x,y)$.
- How to find $\phi_{\rm ss}^N(x,y)$ on *SG*? Or on a general graph?

## Poisson's eqn on the product graph

$$\begin{cases} \boldsymbol{\Delta}_N \phi_{\rm ss}^N(x,y) = \mathbb{1}_{\{x\sim y\}} 5^N \left(\rho_{\rm ss}^N(x) + \rho_{\rm ss}^N(y) - 2\rho_{\rm ss}^N(x)\rho_{\rm ss}^N(y) - 2\phi_{\rm ss}^N(x,y)\right), & x,y \in V_N \setminus V_0,\ x \neq y, \\ \boldsymbol{\Delta}_N \phi_{\rm ss}^N(x,x) = 2\cdot 5^N \sum_{y\sim x} \left(\phi_{\rm ss}^N(x,y) - \chi\left(\rho_{\rm ss}^N(x)\right)\right), & x \in V_N \setminus V_0, \\ \left((\partial_N^\perp \phi_{\rm ss}^N)(x,\cdot)\right)(a) = \left((\partial_N^\perp \phi_{\rm ss}^N)(\cdot,x)\right)(a) = -\frac{5^N}{3^N}\lambda_\Sigma(a)\phi_{\rm ss}^N(x,a), & a \in V_0. \end{cases}$$

Source term is nonzero only if $x$ and $y$ are adjacent.

# Two-particle correlation functions, nonequilibrium

- $\mu^N_{\rm ss}$: unique invariant measure for $5^N \mathcal{L}^{\rm bEX}_N$, $b=1$.
- Steady-state density: $\rho^N_{\rm ss}(x) = \mathbb{E}_{\mu^N_{\rm ss}}[\eta(x)]$.
- Steady-state correlation: $\phi^N_{\rm ss}(x,y) = \mathbb{E}_{\mu^N_{\rm ss}}[(\eta(x)-\rho^N_{\rm ss}(x))(\eta(y)-\rho^N_{\rm ss}(y))]$.

  Related to the local time for two particles in EX to stay adjacent to each other.
- In 1D, $\phi^N_{\rm ss}(x,y)$ is exactly a multiple of the Green's function for RW, $-\frac{1}{N-1}{\rm G}^N(x,y)$.
- How to find $\phi^N_{\rm ss}(x,y)$ on $SG$? Or on a general graph?

## "Invert the Laplacian" to solve for the correlation (in terms of the Green's function ${\rm G}^N$)

$$
\begin{aligned}
\phi^N_{\rm ss}(x,y) = &-\frac{5^N}{|V_N|^2}\sum_{x'\in V_N}\sum_{y'\sim x'} {\rm G}^N(x,x'){\rm G}^N(y,y')(\rho^N_{\rm ss}(x')-\rho^N_{\rm ss}(y'))^2 \\
&+\frac{1}{|V_N|}{\rm G}^N(x,y)\left(\chi(\rho^N_{\rm ss}(x))+\chi(\rho^N_{\rm ss}(y))\right) - \frac{2}{|V_N|^2}\sum_{a\in V_0}\lambda_\Sigma(a){\rm G}^N(x,a){\rm G}^N(y,a)\chi(\rho^N_{\rm ss}(a)) \\
&-\frac{5^N}{|V_N|^2}\sum_{x'\in V_N}\sum_{y'\sim x'}\phi^N_{\rm ss}(x',y')\left[{\rm G}^N(x,x')-{\rm G}^N(x,y')\right]\left[{\rm G}^N(y,x')-{\rm G}^N(y,y')\right].
\end{aligned}
$$

# Two-particle correlation functions, nonequilibrium

- $\mu_{\rm ss}^N$: unique invariant measure for $5^N \mathcal{L}_N^{\rm bEX}$, $b=1$.
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  Related to the local time for two particles in EX to stay adjacent to each other.
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After some estimates we get

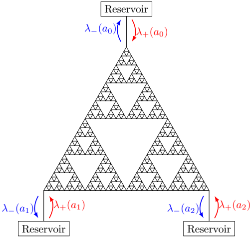
## Lemma

*There exists a positive constant $C = C(\rho_{\rm ss})$ such that for all $N$ and $x, y \in V_N$,*

$$|\phi_{\rm ss}^N(x,y)| \leq \frac{C}{|V_N|} \max\left\{ \mathsf{G}^N(x,y), \sup_{(x',y')\in V_N^2 : x'\sim y'} \mathsf{G}^N(x,x')\mathsf{G}^N(y,y') \right\}.$$

Correlation scales as (inverse volume)$\times$(Green's function for RW).
This Lemma (and its time-dependent version) is needed to establish tightness/convergence of the density fluctuation field in non-equilibrium.

# Summary, and **Thank you!**




$$5^N \mathcal{L}_N^{\rm bEX} = 5^N \left(\mathcal{L}_N^{\rm EX} + \frac{1}{b^N}\mathcal{L}_N^{\rm boun}\right).$$

**Symmetric** exclusion process with **slowed** boundary on the Sierpinski gasket

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

**Equilibrium** $\Leftrightarrow$ $\lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)

- (Non)equilibrium density hydrodynamic limit (DRN✓) [C.–Gonçalves '19]
- Ornstein-Uhlenbeck limit of equilibrium density fluctuations (DRN✓). [C.–Gonçalves '19]
- Large deviations principle for the (non)equilibrium density (D✓) [C.–Hinz '19+]
- Hydrostatic limit, scaling limit of nonequilibrium density fluctuations (D✓RN?). [C.–Franceschini–Gonçalves–Menezes '19+]

## Future directions

- Generalization to *any* resistance space (with a good theory of boundary-value problems).
- Incorporate asymmetry in the exclusion jump rates $\to$ microscopic derivation of stochastic Burgers' equation on resistance spaces.