Mixing Times of Card Shuffling and Road Traffic

Rodrigo Marinho and Joe P. Chen

IST, University of Lisbon

Colgate University

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Card shuffling: some history

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How many times do you have to shuffle the deck like that to mix it up?

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- A deck will be considered mixed whenever all permutations are equiprobable.

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- $n = 52 \Rightarrow t^{52}_{\text{mix}} = 7$.

Riffle shuffle

The Annals of Applied Probability
1992, Vol. 2, No. 2, 294–313

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

BY DAVE BAYER[1] AND PERSI DIACONIS[2]

Columbia University and Harvard University

Riffle shuffle

2. A card trick. Rising sequences, the basic invariant of riffle shuffling, were discovered by magicians Williams and Jordan at the beginning of this

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR 297

century. A rising sequence is a maximal subset of an arrangement of cards,

Overhand shuffle

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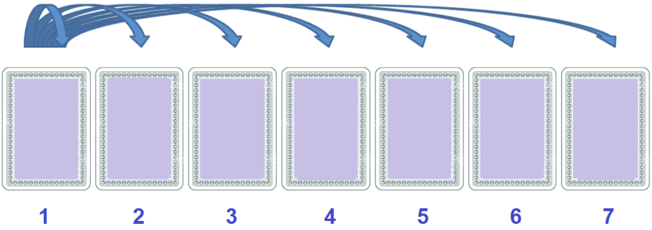
- $n = 52 \Rightarrow t^{52}_{\text{mix}} \approx 10000$.

Top-to-random shuffle



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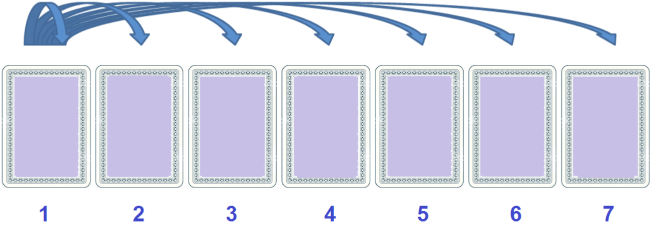

- Aldous and Diaconis: $t_{\text{mix}}^{n}(\varepsilon) = n \log n$.

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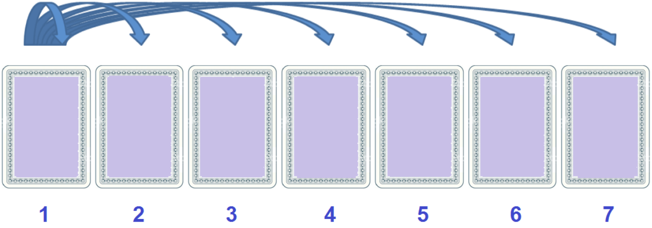
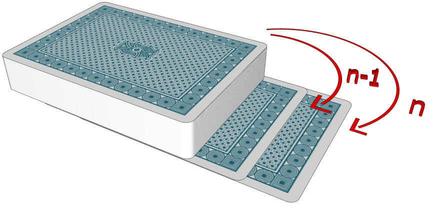


- Aldous and Diaconis: $t^n_{\text{mix}}(\varepsilon) = n \log n$.
- $n = 52 \Rightarrow t^{52}_{\text{mix}} \approx 200$.

Rudvalis shuffle



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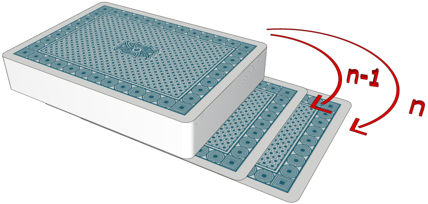

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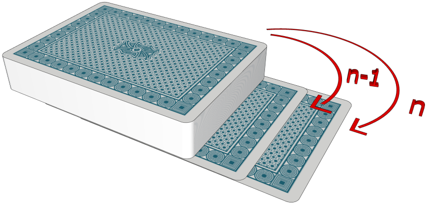

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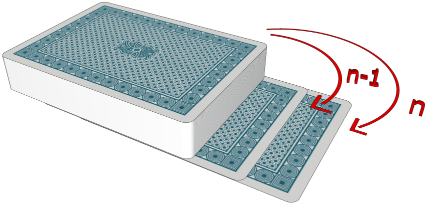

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- $n = 52 \Rightarrow t^{52}_{\text{mix}} \approx 14000$.

Random transposition shuffle

Example

1	2	3	4	5	6	7
1	5	3	4	2	6	7

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Random adjacent transposition shuffle

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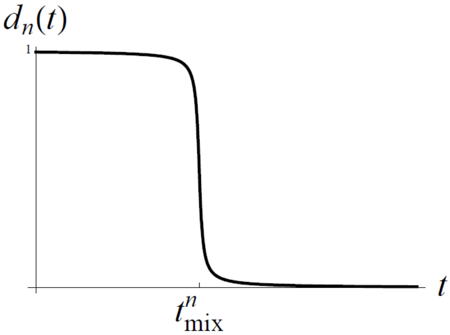
- $t^n_{\text{mix}}(\varepsilon) = \frac{1}{\pi^2} n^3 \log n$.

Cutoff phenomenon



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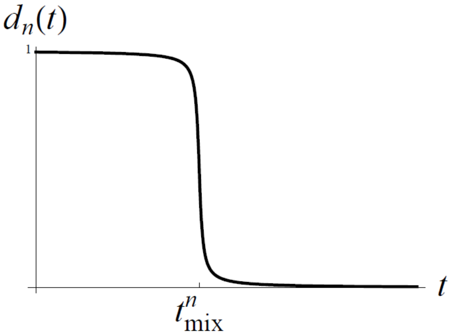

$$\lim_{n\to\infty} d_n(\kappa\, t^n_{mix}) = \begin{cases} 1, & \text{if } \kappa < 1; \\ 0, & \text{if } \kappa > 1. \end{cases}$$

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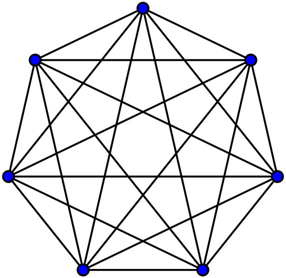

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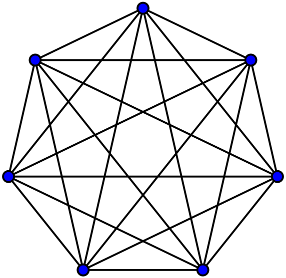
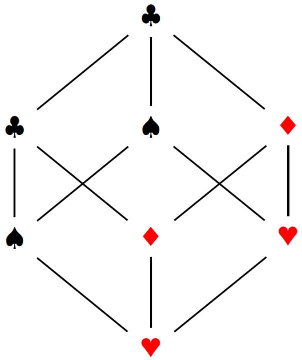


- At each vertex of G we put a card and at each edge of G we attach an exponential clock of rate 1.

Shuffling cards on graphs



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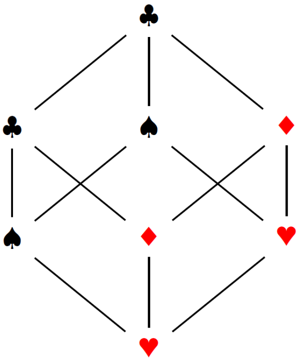

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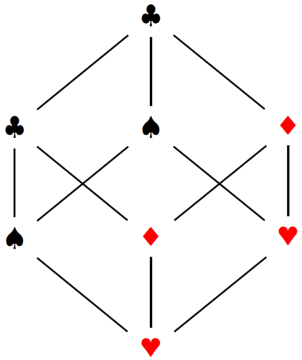
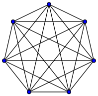


- Flip edges, exchanging the cards on the two incident vertices.
- Interchange process $\{\sigma_t : t \geq 0\}$.

Shuffling cards on graphs

- When G is the complete graph $K_n \Rightarrow$ Random transposition shuffle;



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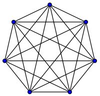
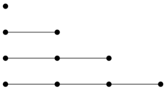


- When G is the graph $P_n \Rightarrow$ Adjacent transposition shuffle;



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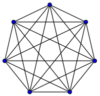

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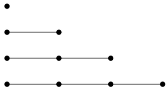
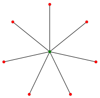


- The cutoff has been proven only for: K_n, P_n and star graphs ($n \log n$)



until now...

Shuffling cards on graphs

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$$\frac{1}{2\,\lambda_1^n}\,\log|V_n|,$$

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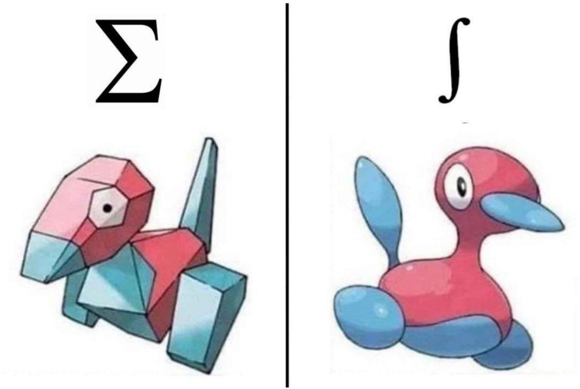
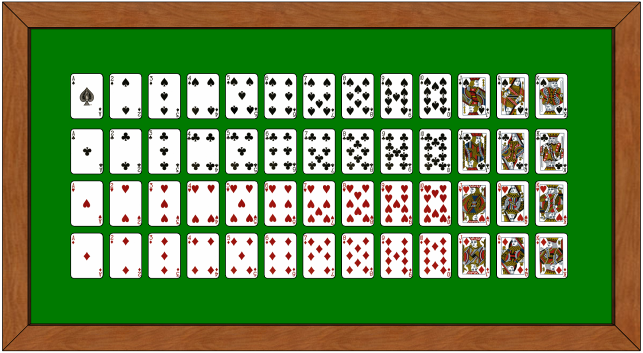
- We call λ_1^n by *spectral gap*. In spectral graph theory it is called *algebraic connectivity*.

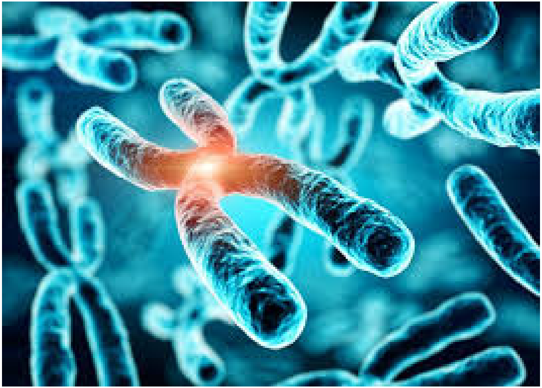
This is more or less what we are assuming on G_n



Example: adjacent transposition shuffle on a grid



Example: L-adjacent transposition shuffle



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$$\eta(x) = \begin{cases} 1, & \text{if the card at vertex x is black;} \\ 0, & \text{if the card at vertex x is red.} \end{cases}$$

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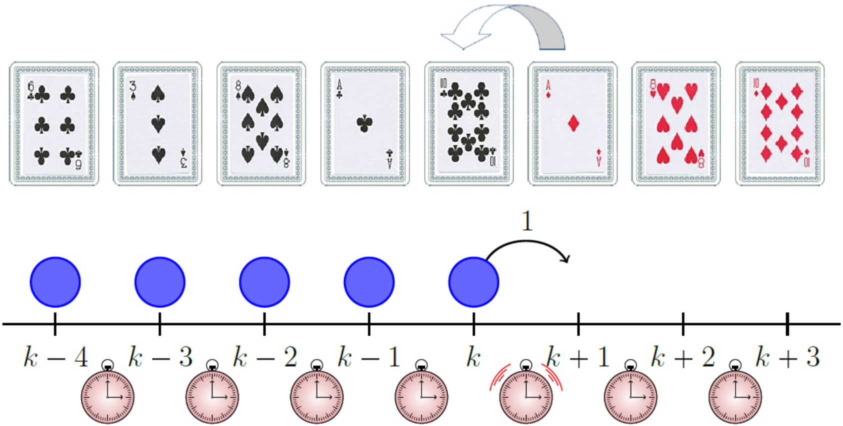

The Projection

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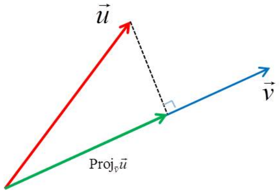

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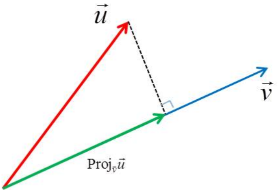

Thus, $d_{n,\ell}(t) \leq d_n(t)$ and

$$t_{\text{mix}}^{n,\ell}(\varepsilon) \leq t_{\text{mix}}^{n}(\varepsilon).$$