Proving Stirling's Formula

We want to prove that $\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi - \int_0^\infty \frac{P_1(t)}{z+t}\,dt$

The residues of Γ at $z=-n$ are $\frac{(-1)^n}{n!}$, so we will consider a set U which is the complex plane with the negative real axis removed.

We first need the Euler Summation Formula:

$$(1) \quad \sum_{k=0}^{n} f(k) = \int_0^n f(t)\,dt + \frac{1}{2}\left(f(n)+f(0)\right) + \int_0^n P_1(t) f'(t)\,dt$$

We next let $P_2(t) = \frac{1}{2}(t^2-t)$, and for z in U we have $\int_0^\infty \frac{P_1(t)}{z+t}\,dt = \int_0^\infty \frac{P_2(t)}{(z+t)^2}\,dt$

This is easily shown using integration by parts, and a differentiation theorem.

Then it is easily shown that $\lim_{y\to\infty} \int_0^\infty \frac{P_1(t)}{iy+t}\,dt = 0 \qquad (2)$

We then apply (1) to $f(t) = \log(z+t)$ and $g(t) = \log(1+t)$ and subtract:

$$\log \frac{z(z+1)\cdots(z+n)}{n!\,(n+1)} = z\log(z+n) + n\log(z+n) - z\log z - (z+n) + z + \frac{1}{2}\left(\log(z+n) + \log z\right) - (n+1)\log(n+1) + (n+1) - 1$$
$$-\frac{1}{2}\left(\log(n+1)\right) + \left[\text{terms with integrals of } P_1(t)\right]$$

Then using: $\log(n+1) = \log n + \log\left(1+\frac{1}{n}\right)$ and $z\log(z+n) = z\log n + z\log\left(1+\frac{z}{n}\right)$, and for large n

$$\log\left(1+\frac{z}{n}\right) \approx \frac{z}{n} \qquad \log\left(1+\frac{1}{n}\right) \approx \frac{1}{n}$$

We reduce to: and take $\lim_{n\to\infty}$

$$\log \frac{n!\,n^z}{z(z+1)\cdots(z+n)} = \Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + 1 + \int_0^\infty \frac{P_1(t)}{1+t}\,dt - \int_0^\infty \frac{P_1(t)}{z+t}\,dt \qquad (3)$$

Valid for all z in U by analyticity of terms.

We then use the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ to get $|\Gamma(iy)| = \left(\frac{2\pi}{y(e^{\pi y} - e^{-\pi y})}\right)^{1/2} \qquad (4)$

Then rearrange (3): $1 + \int_0^\infty \frac{P_1(t)}{1+t}\,dt = \mathrm{Re}\left\{\log\Gamma(iy) - \left(iy-\frac{1}{2}\right)\log(iy) + iy + \int_0^\infty \frac{P_1(t)}{iy+t}\,dt\right\}$

Using (2) and (4) we can see that $1 + \int_0^\infty \frac{P_1(t)}{1+t}\,dt = \frac{1}{2}\log 2\pi$

Thus we've proven the formula. //