

The abelian sandpile growth problem on the Sierpinski gasket is exactly solved

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Based on joint work with Jonah Kudler-Flam (UChicago)
[arXiv:1807.08748](https://arxiv.org/abs/1807.08748)

Limit shape results for Laplacian growth models (1992–2018)

\mathbb{Z}^d ($d \geq 2$) For all models: Launch $|B_o(n)|$ chips from o .

Model	<i>Shape theorem/conjecture</i>
IDLA	In/out-radius $\begin{cases} n \pm \mathcal{O}(\log n), & d = 2 \\ n \pm \mathcal{O}(\sqrt{\log n}), & d \geq 3 \end{cases}$ $\alpha, \beta, \gamma, \delta$
Rotor-router aggregation	In-radius $n - c \log n$, out-radius $n + c' \log n$ κ, ℓ
Divisible sandpiles	In-radius $n - c$, out-radius $n + c'$ κ
Abelian sandpiles	($d = 2$) Limit shape closer to a dodecagon than Euc ball κ Rigorous upper/lower estimates available (with a gap) κ, ℓ

^a Lawler–Bramson–Griffeath '92

β Lawler '95

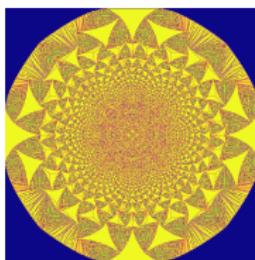
Asselah-Gaudilli  re '13 (2x)

^δ Jerison-Levine-Sheffield '13, '14

K Levine-Perez '09

Lovins, Perez '17

Eeviine Peres '10



Levine–Peres, “Laplacian growth, sandpiles, and scaling limits.” *Bull. Amer. Math. Soc.* (2017).

Limit shape results for Laplacian growth models (1992–2018)

Sierpinski gasket (*SG*) For all models: Launch from the corner vertex o .

Model	Initial chip #	Shape theorem/conjecture
IDLA	$ B_o(n) $	In/out-radius $n \pm \mathcal{O}(\sqrt{\log n})$ ^{1,2}
RRA	m	In-radius $n_m - 2$, out-radius n_m ²
DSM	m	In-radius $n_m - 1$, out-radius n_m ³
ASM	m	Receiving set is a ball $B_o(r_m)$ $r_m = m^{1/d_H} [\mathcal{G}(\log m) + o(1)]$ as $m \rightarrow \infty$ ² (\mathcal{G} is an explicit $(\log 3)$ -periodic function)

¹ C.–Huss–Sava–Huss–Teplyaev '17

² C.–Kudler–Flam '18

³ Huss–Sava–Huss '17

Theorem (Limit shape universality on *SG*)

On *SG*, clusters in all 4 single-source growth models fill balls in the graph metric.

First nontrivial state space (beyond \mathbb{Z}) where **limit shape universality** holds.

Limit shape results for Laplacian growth models (2018–?)

IDLA

Rotor-router

Abelian sandpiles

- Ecaterina Sava-Huss' talk: addressed the IDLA and divisible sandpile shape theorems.
[C.-Huss-Sava-Huss-Teplyaev '17, Huss-Sava-Huss '17]
- **This talk** Solving the **abelian sandpile growth problem** on SG . (Started with excellent numerical work by Jonah, ended with my proving every detail of his numerical findings.)
- For the rest of the limit shape universality story (e.g. rotor-router aggregation) which I can't get to, see [C.-Kudler-Flam '18]. Also consult local experts (Peres, Sava-Huss).

Conjecture. Limit shape universality holds on a nested fractal graph (Lindstrøm).

- Limit shapes for IDLA, DSM, and RRA are plausibly universal. (Estimate of harmonic functions, cut points, symmetries ...)
- Analysis of the abelian sandpile model is delicate ... (even on SG !)

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Place sand grains (“chips”) on vertices of a graph (say, on the square lattice).

If # of chips at vertex $x \geq$ the degree of x , emit one chip to each neighbor of x . This is called a **toppling** at x .

Continue till the # of chips < degree on every vertex. Then we say the configuration is **stable**.

0	0	0
0	4	0
0	0	0

→

0	1	0
1	0	1
0	1	0

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0	0	0
0	4	0
0	0	0

→

0	1	0
1	0	1
0	1	0

0	0	0	0
0	4	5	0
0	0	0	0

→

0	1	0	0
1	0	6	0
0	1	0	0

→

0	1	1	0
1	1	2	1
0	1	1	0

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0	0	0
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→

0	1	0
1	0	1
0	1	0

0	0	0	0
0	4	5	0
0	0	0	0

→

0	1	0	0
1	0	6	0
0	1	0	0

→

0	1	1	0
1	1	2	1
0	1	1	0

0	0	0	0
0	4	5	0
0	0	0	0

→

0	0	1	0
0	5	1	1
0	0	1	0

→

0	1	1	0
1	1	2	1
0	1	1	0

The abelian sandpile model [Bak–Tang–Wiesenfeld '87, Dhar, Majumdar, ...]

Place sand grains (“chips”) on vertices of a graph (say, on the square lattice).

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Continue till the # of chips < degree on every vertex. Then we say the configuration is **stable**.

0	0	0
0	4	0
0	0	0

→

0	1	0
1	0	1
0	1	0

0	0	0	0
0	4	5	0
0	0	0	0

→

0	1	0	0
1	0	6	0
0	1	0	0

→

0	1	1	0
1	1	2	1
0	1	1	0

0	0	0	0
0	4	5	0
0	0	0	0

→

0	0	1	0
0	5	1	1
0	0	1	0

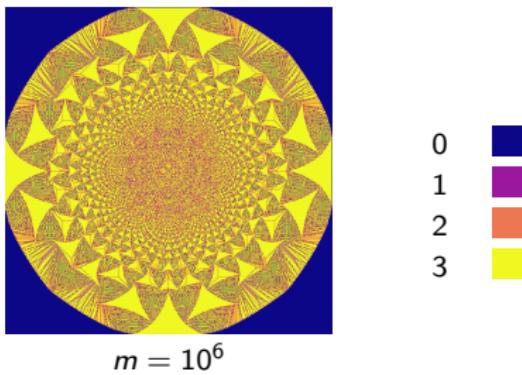
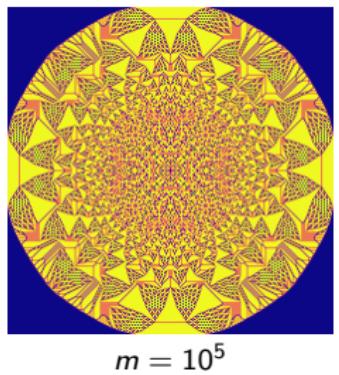
→

0	1	1	0
1	1	2	1
0	1	1	0

Why **abelian**? Because the final stable configuration does not depend on the order of topplings.

Sandpile growth on \mathbb{Z}^2 : Fractals in a sandpile

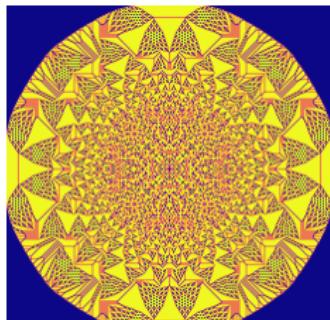
Lay m chips at the origin and stabilize. What can we say about the scaling limit of the sandpile clusters (length scaled by $m^{1/d}$)?



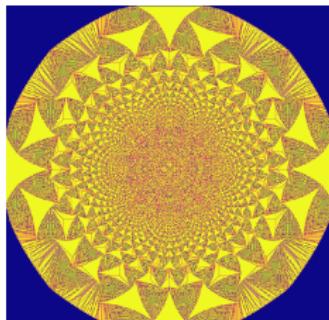
- Scaling limit of the patterns exists [in weak-* $L_\infty(\mathbb{R}^d)$]. [Pegden–Smart '11]
- **Apollonian gaskets in the pattern.** [Numerically observed since 90s, proved by Levine–Pegden–Smart '12, '14. Latter is published in *Ann. Math.* '17]

Sandpile growth on \mathbb{Z}^2 : Fractals in a sandpile

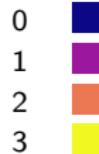
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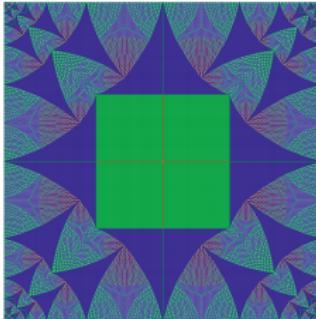
$m = 10^5$



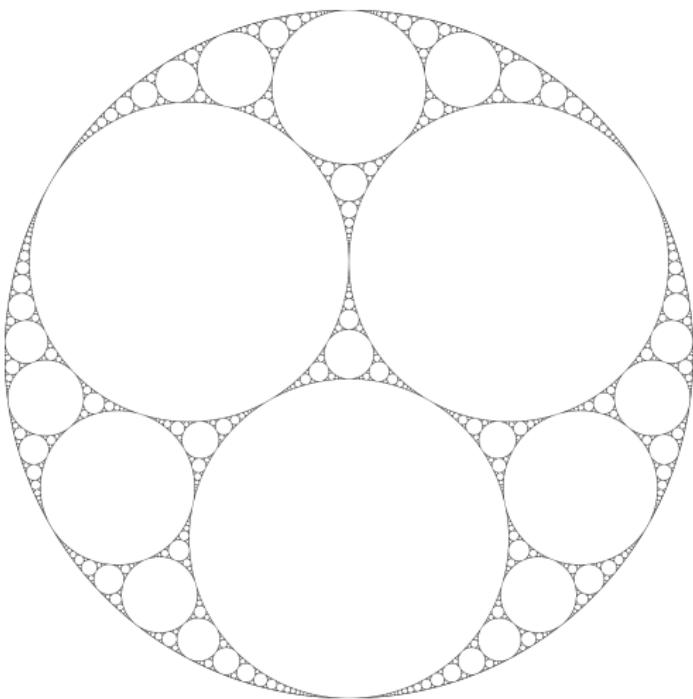
$m = 10^6$



- Fractal patterns also appear in the identity element of the sandpile group on a square lattice.

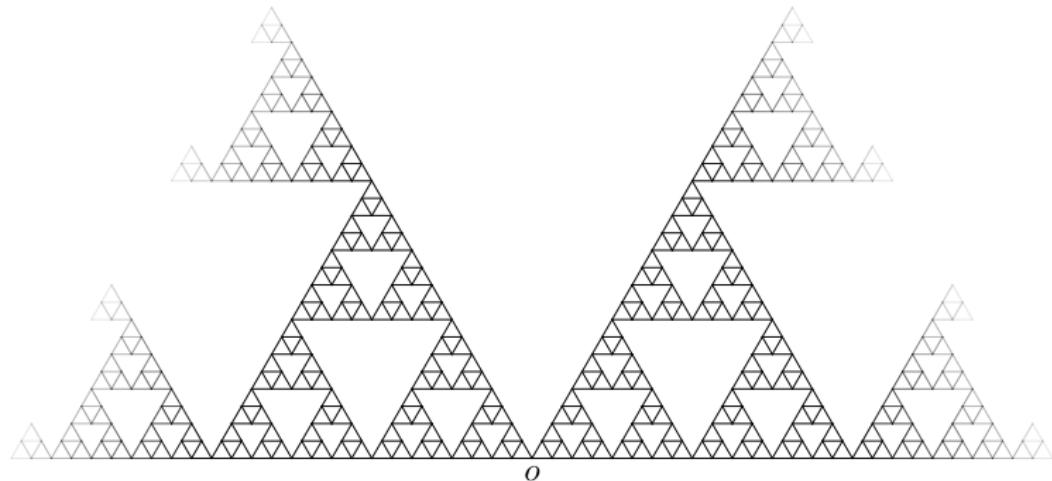


Apollonian gasket



By Time3000 [GFDL or CC BY-SA 4.0-3.0-2.5-2.0-1.0], from Wikimedia Commons
https://commons.wikimedia.org/wiki/File:Apollonian_gasket.svg

Sierpinski gasket (SG)



The abelian sandpile growth problem: Lay m chips at o and stabilize. Characterize the shape of, and patterns in, the stable sandpile cluster as a function of m .

Q. Is it possible to see fractal patterns in the sandpile on a fractal?

- It does not matter whether to solve it on the 1-sided SG or the 2-sided SG (symmetry).
- Key geometric inputs: Axial symmetries; cut points (junction points).

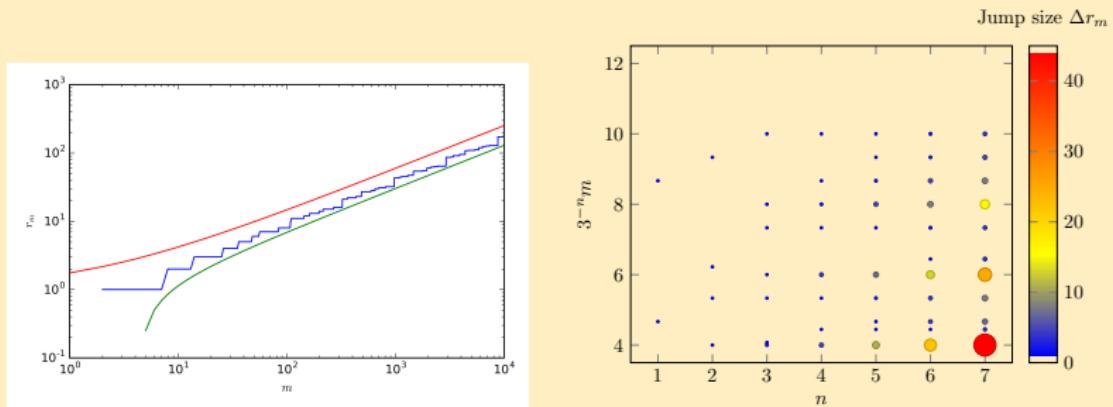
Key observations from simulations:

- Sandpile patterns exhibit periodicity. [Fairchild–Haim–Setra–Strichartz–Westura '16]
- The set of all vertices receiving at least 1 chip is ALWAYS a ball in the graph metric.
- Radial explosions occur at periodic values of m . (Not seen on \mathbb{Z}^d or trees!)

2018: C. can prove all the key observations.

Top-level results: radial growth & explosion

Theorem (Radial growth (blue) & radial jumps)



Major jumps at $4 \cdot 3^n$, $6 \cdot 3^n$, $8 \cdot 3^n$, $10 \cdot 3^n$. Period = $2 \cdot 3^n$.

Theorem (Radial asymptotics)

$r(x) = x^{1/d_H} [\mathcal{G}(\log x) + o(1)]$ as $x \rightarrow \infty$, where $d_H = \log_2 3$ is the Hausdorff dim of SG, and \mathcal{G} is an explicit non-constant log3-periodic function.

Proof. Sharp estimates of the remainder ($|r(3m) - 2r(m)| \leq 1$) + Renewal theorem.

The fundamental sandpile diagram on SG

Proposition. For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ = \left(\begin{array}{c} \text{circle } m' \\ \text{triangle } \eta \in \mathcal{R}_n \\ \text{circle } m' \end{array} \right)^\circ \subseteq G_{n+1}.$$

From this follows the fundamental equation $r_m = 2^n + r_{m'-2}$.

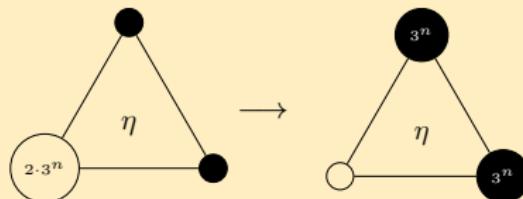
Key ideas:

- \mathcal{R}_n is the **sandpile group** of G_n (= class of **recurrent** sandpile configs on G_n) with two sinks ∂G_n . It is an abelian group under \oplus , **addition of chips followed by stabilization**.
- Stabilize $m\mathbb{1}_o$ in $G_n \setminus \partial G_n$ to obtain the config $\eta \in \mathcal{R}_n$, pausing the excess m' chips on ∂G_n (sink) → **using the abelian property**.
- Then topple the excess chips on ∂G_n (source). By **Dhar's multiplication by identity test** (a Laplacian identity), with each topple on ∂G_n , η is unchanged, while the # of chips on ∂G_n decrements in steps of 2.

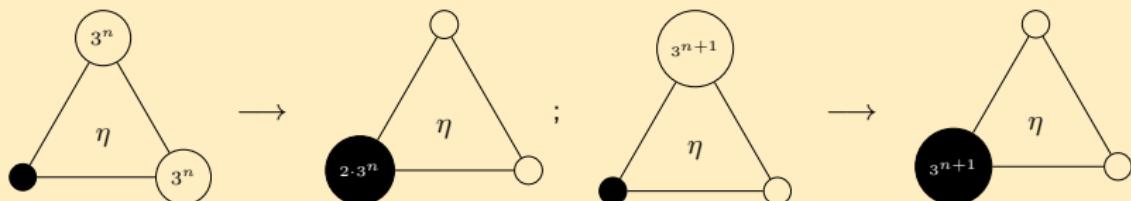
Using this Proposition we can inductively prove: cluster shape is a ball; periodicity; etc.

Toppling identities, periodicity

Proposition. For every $\eta \in \mathcal{R}_n^{(s)}$,



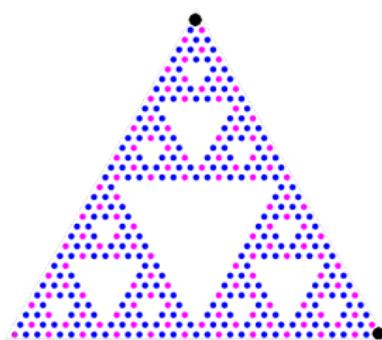
For every $\eta \in \mathcal{R}_n^{(o)}$,



This explains the **($2 \cdot 3^n$)-periodicity** in the sandpile growth (and the patterns restricted to G_n).

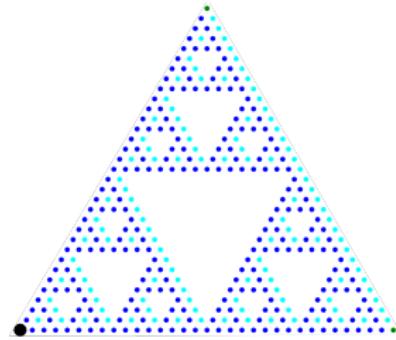
Identity element of the sandpile group of SG

2 sinks



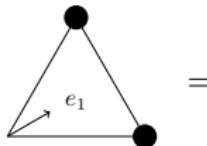
e_5

1 sink

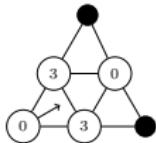


$$e_5^{(o)} := M_5 \oplus (\mathbf{1}_x + \mathbf{1}_y)$$

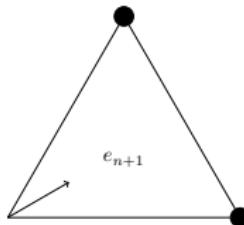
- 0
- 1
- 2
- 3



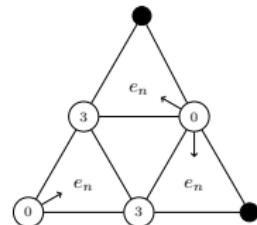
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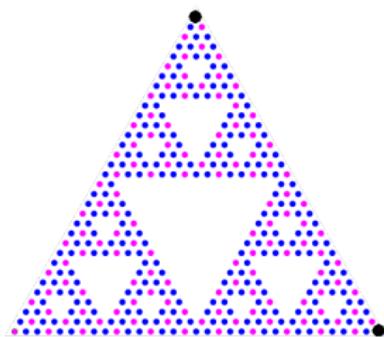


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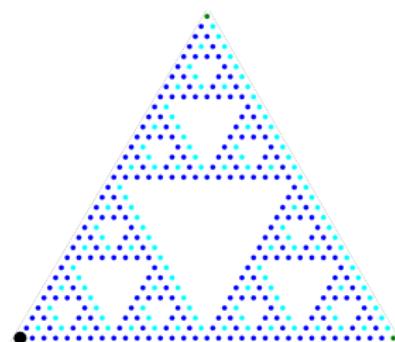
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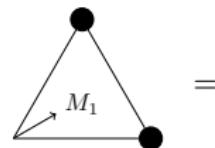


e_5

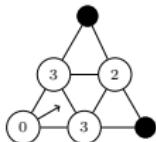
1 sink



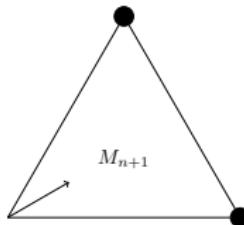
$$e_5^{(o)} := M_5 \oplus (\mathbf{1}_x + \mathbf{1}_y)$$



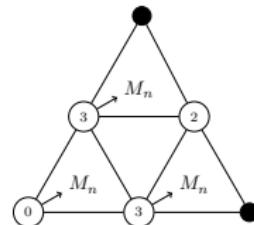
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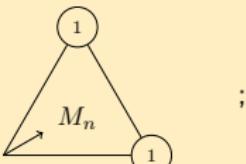


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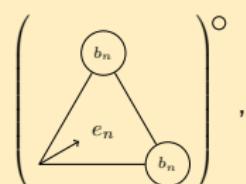


The main explosions at $4 \cdot 3^n$: transition from M_n to e_n

Proposition.

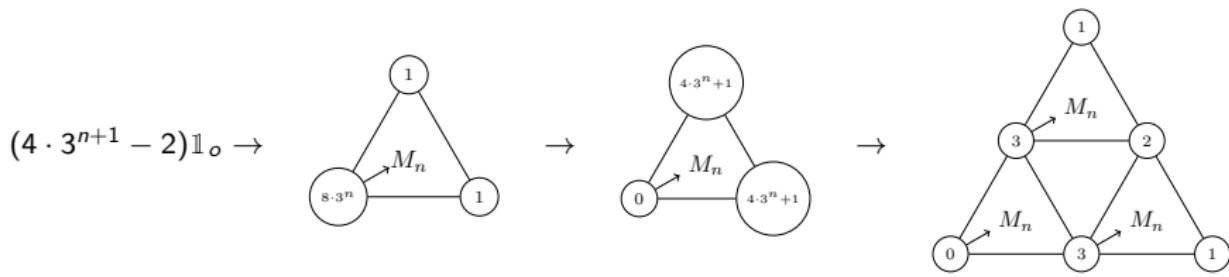
$$((4 \cdot 3^n - 2)\mathbb{1}_o)^\circ =$$


;

$$((4 \cdot 3^n)\mathbb{1}_o)^\circ =$$


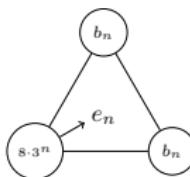
, where $b_n = \frac{3}{2} (3^{n-1} + 1)$.

Proof of the induction step.

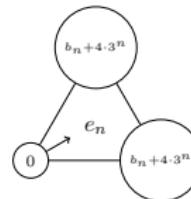


Proof (cont.).

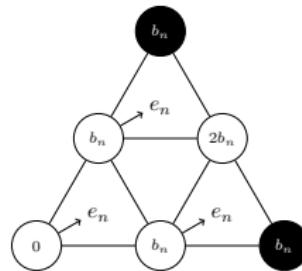
$$(4 \cdot 3^{n+1}) \mathbb{1}_o \rightarrow$$



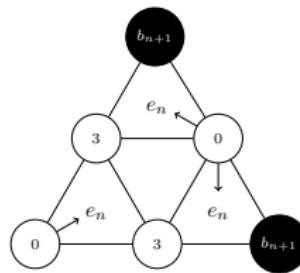
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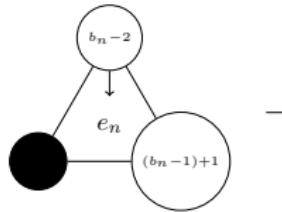


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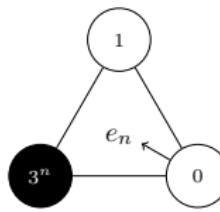


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The last step depends on the following “reflection lemma,” using the axial symmetry of SG :



\rightarrow



Also note $b_{n+1} = b_n + 3^n$.

An axial reflection lemma (SG with one sink)

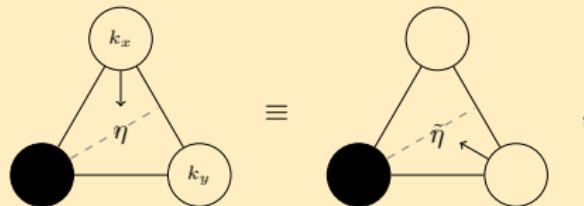
$e_n^{(o)}$ is the id element of $\mathcal{R}_n^{(o)}$; $\partial G_n = \{x, y\}$.

Lemma. Let $\eta \in \mathcal{R}_n^{(o)}$ be such that $\eta = e_n^{(o)} \oplus \alpha \mathbb{1}_x \oplus \beta \mathbb{1}_y$ for some $\alpha, \beta \in \mathbb{N}_0$. Let $k_x, k_y \in \mathbb{N}_0$ solve the system of equations

$$\begin{cases} \alpha + k_x &= \beta + p_0 \cdot 3^n + p_1 \cdot 3^{n+1} \\ \beta + k_y &= \alpha + p_0 \cdot 3^n + p_2 \cdot 3^{n+1} \end{cases}$$

for some $p_0, p_1, p_2 \in \mathbb{Z}$ (which come from the toppling identities).

Then



where $\tilde{\eta} = e_n^{(o)} \oplus \beta \mathbb{1}_x \oplus \alpha \mathbb{1}_y$ is the reflection of η across the axis of symmetry.

The fundamental sandpile diagram on SG

Proposition. For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m \mathbb{1}_o)^\circ = \left(\begin{array}{c} \text{circle } m' \\ \text{triangle } \eta \in \mathcal{R}_n \\ \text{circle } m' \end{array} \right)^\circ \subseteq G_{n+1}.$$

From this follows the fundamental equation $r_m = 2^n + r_{m'-2}$.

Example: $n = 3$. Record values of m at which m' jumps.

$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m	$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m
4	108	15	78	2	8	216	69	78	
$4\frac{2}{27}$	110	16	78	1	$8\frac{2}{27}$	218	70	78	
$4\frac{4}{9}$	120	19	82		$8\frac{4}{9}$	228	73	82	
$4\frac{2}{3}$	126	20	86		$8\frac{2}{3}$	234	74	86	
$5\frac{1}{3}$	144	28	88	1	$9\frac{1}{3}$	252	82	88	
6	162	42	78	1	10	270	96	78	1
$6\frac{2}{27}$	164	43	78		$10\frac{2}{27}$	272	97	78	
$6\frac{4}{9}$	174	46	82		$10\frac{4}{9}$	282	100	82	
$6\frac{2}{3}$	180	47	86		$10\frac{2}{3}$	288	101	86	
$7\frac{1}{3}$	198	55	88	1	$11\frac{1}{3}$	306	109	88	

$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m	$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m	$\frac{m}{3^n}$	m	m'	$m - 2m'$	Δr_m										
2	1	0	1		8	216	69	78	1	6 $\frac{2}{3}$	1620	407	806											
8	4	0	1		8 $\frac{2}{27}$	218	70	78		7 $\frac{1}{3}$	1782	487	808	1										
n = 1																								
4	12	3	6		8 $\frac{2}{9}$	234	74	86		8 $\frac{2}{27}$	1946	610	726											
4 $\frac{2}{9}$	14	4	6	1	10	270	96	78	1	8 $\frac{2}{9}$	2052	649	754											
6	18	6	6		10 $\frac{2}{27}$	272	97	78		8 $\frac{2}{3}$	2106	650	806	2										
6 $\frac{2}{27}$	20	7	6		10 $\frac{3}{9}$	282	100	82		9 $\frac{1}{3}$	2268	730	808	1										
8	24	9	6		10 $\frac{2}{81}$	288	101	86		10	2430	852	726	1										
8 $\frac{2}{81}$	26	10	6	1	11 $\frac{1}{3}$	306	109	88		10 $\frac{2}{27}$	2432	853	726											
10	30	12	6		n = 4																			
10 $\frac{2}{81}$	32	13	6		4	324	42	240	5	4	2916	366	2184	22										
n = 2																								
4	36	6	24	1	4 $\frac{2}{21}$	326	43	240		4 $\frac{2}{21}$	2918	367	2184											
4 $\frac{2}{9}$	38	7	24		4 $\frac{1}{9}$	360	55	250	1	4 $\frac{2}{27}$	3240	487	2266	1										
4 $\frac{2}{81}$	42	8	26		5 $\frac{1}{3}$	432	82	268	1	4 $\frac{2}{81}$	3402	488	2426	4										
5 $\frac{1}{3}$	48	10	28	1	6	486	123	240	4	5 $\frac{1}{3}$	3888	730	2428	4										
6	54	15	24		6 $\frac{2}{21}$	488	124	240		6	4374	1095	2184	13										
6 $\frac{2}{9}$	56	16	24	1	6 $\frac{1}{9}$	522	136	250		6 $\frac{2}{27}$	4376	1096	2184											
6 $\frac{2}{81}$	60	17	26		6 $\frac{4}{9}$	540	137	266		6 $\frac{4}{27}$	4698	1216	2266	1										
7 $\frac{1}{3}$	66	19	28		7 $\frac{1}{3}$	594	163	268	1	7 $\frac{1}{3}$	4860	1217	2426											
8	72	24	24		8	648	204	240	2	7 $\frac{1}{3}$	5346	1459	2428	2										
8 $\frac{2}{9}$	74	25	24		8 $\frac{2}{21}$	649	205	240		8	5832	1824	2184	8										
8 $\frac{2}{81}$	78	26	26		8 $\frac{1}{9}$	684	217	250																
9 $\frac{1}{3}$	84	28	28	1	8 $\frac{4}{21}$	702	218	266	1	8 $\frac{2}{27}$	5834	1825	2184											
10	90	33	24		9 $\frac{1}{3}$	756	244	268		8 $\frac{4}{81}$	6156	1945	2266											
10 $\frac{2}{9}$	92	34	24		10	810	285	240	1	8 $\frac{2}{27}$	6318	1946	2426	5										
10 $\frac{8}{81}$	96	35	26		10 $\frac{2}{81}$	812	286	240		9 $\frac{1}{3}$	6804	2188	2428	2										
11 $\frac{1}{3}$	102	37	28		10 $\frac{9}{81}$	846	298	250		10	7290	2553	2184	2										
n = 3																								
4	108	15	78	2	11 $\frac{1}{81}$	918	325	268		n = 5														
4 $\frac{2}{27}$	110	16	78	1	n = 7																			
4 $\frac{1}{9}$	120	19	82		4	972	123	726	11	4 $\frac{2}{3}$	8748	1095	6558	44										
4 $\frac{2}{81}$	126	20	86		4 $\frac{2}{3}$	1080	163	754	1	4 $\frac{2}{3}$	8750	1096	6558											
5 $\frac{1}{3}$	144	28	88	1	4 $\frac{2}{3}$	1134	164	806	1	4 $\frac{2}{3}$	9720	1459	6802	3										
6	162	42	78	1	4 $\frac{2}{3}$	1296	244	808	2	4 $\frac{2}{3}$	10206	1460	7286	7										
6 $\frac{2}{27}$	164	43	78		6	1458	366	726	7	5 $\frac{1}{3}$	11664	2188	7288	8										
6 $\frac{4}{9}$	174	46	82		6 $\frac{2}{3}$	1460	367	726		6	13122	3282	6558	25										
6 $\frac{2}{81}$	180	47	86		6 $\frac{2}{3}$	1566	406	754																
7 $\frac{1}{3}$	198	55	88	1	6 $\frac{1}{9}$																			

Legend: $(m\mathbb{I}_\theta)^\circ = \left(\begin{array}{c} \text{circle} \\ \text{triangle} \\ \eta \in \mathcal{R}_n \\ \text{circle} \end{array} \right)^\circ$; # {chips in η } = $m - 2m'$.

Detailed results: radial jumps

Theorem (Enumeration of radial jumps)

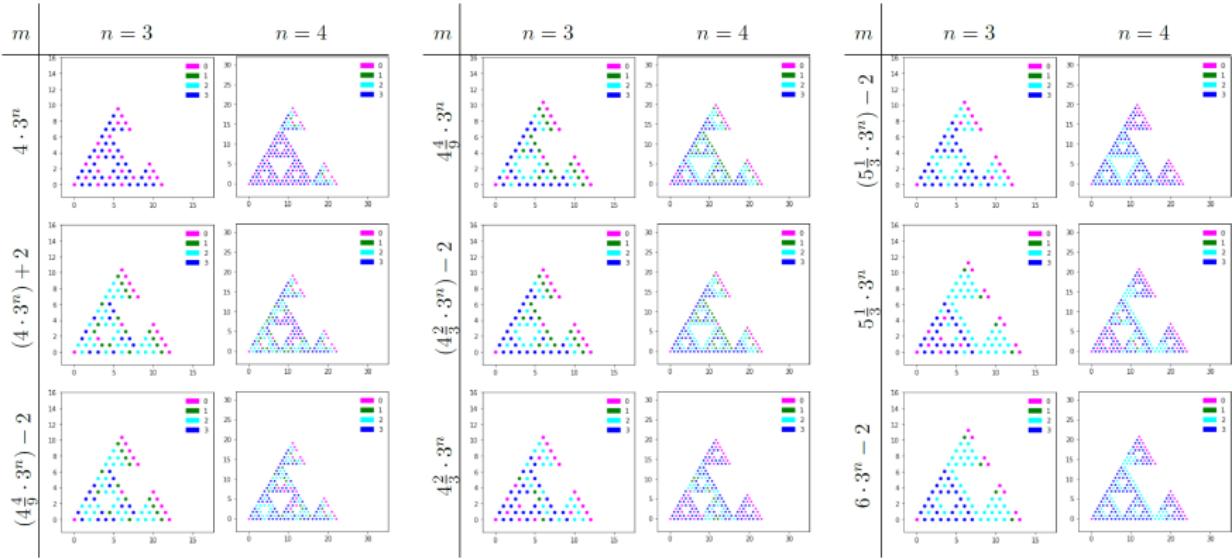
For $n \geq 3$ and $m \in [4 \cdot 3^n, 4 \cdot 3^{n+1})$, $(m \mathbb{1}_o)^\circ = \left(\begin{array}{c} m' \\ \eta \in \mathcal{R}_n \\ m' \end{array} \right)^\circ \subseteq G_{n+1}$, where $m \mapsto m'$ is a

piecewise constant right-continuous function which has jumps indicated in the following table.

m	m'
$(4 + 2p) \cdot 3^n$	$b_n + p \cdot 3^n$
$(4 + 2p) \cdot 3^n + 2$	$(b_n + 1) + p \cdot 3^n$
$(4 \frac{4}{9} + 2p) \cdot 3^n$	$2 \cdot 3^{n-1} + 1 + p \cdot 3^n$
$(4 \frac{2}{3} + 2p) \cdot 3^n$	$2 \cdot 3^{n-1} + 2 + p \cdot 3^n$
$(5 \frac{1}{3} + 2p) \cdot 3^n$	$3^n + 1 + p \cdot 3^n$

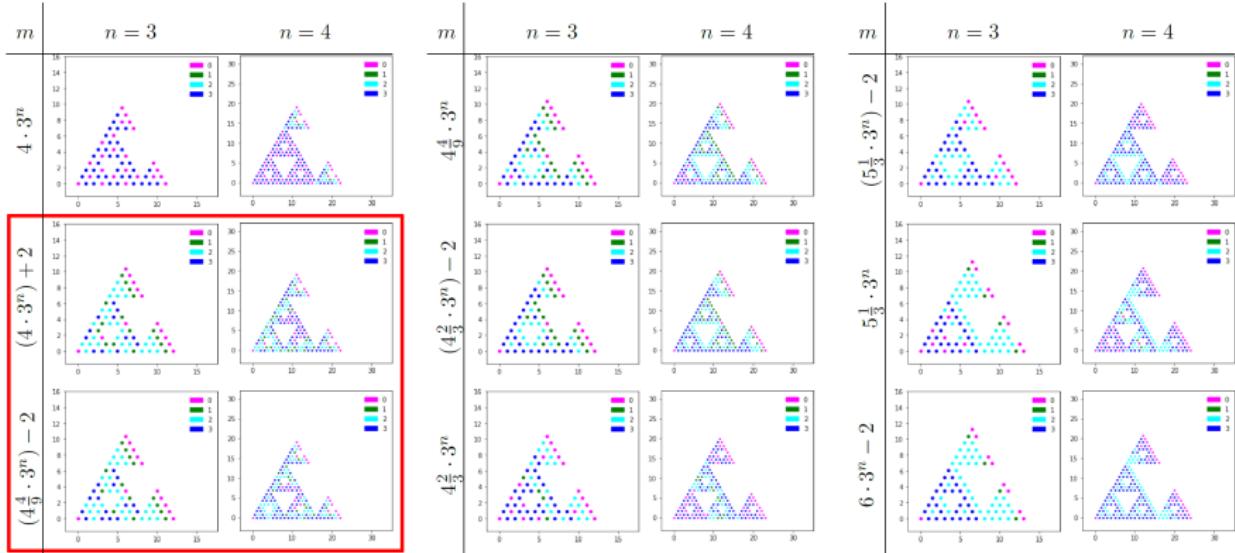
where $p \in \{0, 1, 2, 3\}$, and $b_n = |V(G_{n-1})| = \frac{3}{2}(3^{n-1} + 1)$.

Detailed results: **Fractals in the sandpile on a fractal**

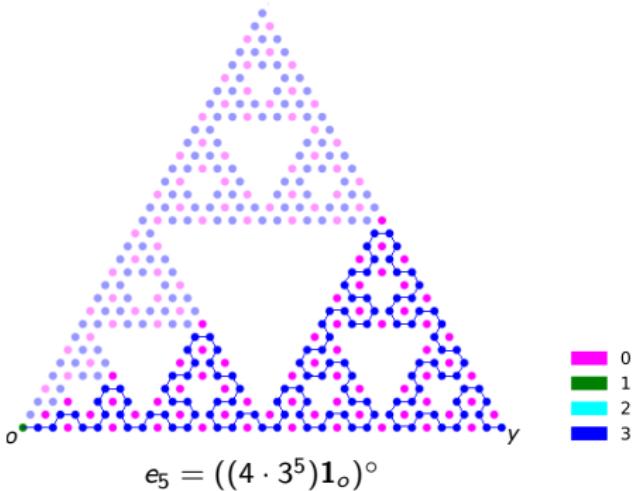


The proof of the radial jumps uses an inductive diagrammatic approach, *a.k.a.* **sandpile block renormalization**: Config restricted to G_n is the gluing of 3 well-defined sandpile tiles.

Detailed results: Fractals in the sandpile on a fractal

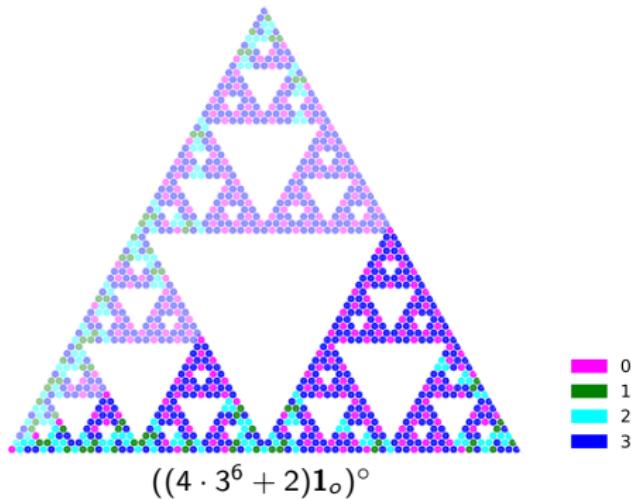


BUT there are **two exceptions** which cannot be explained by block renormalization.



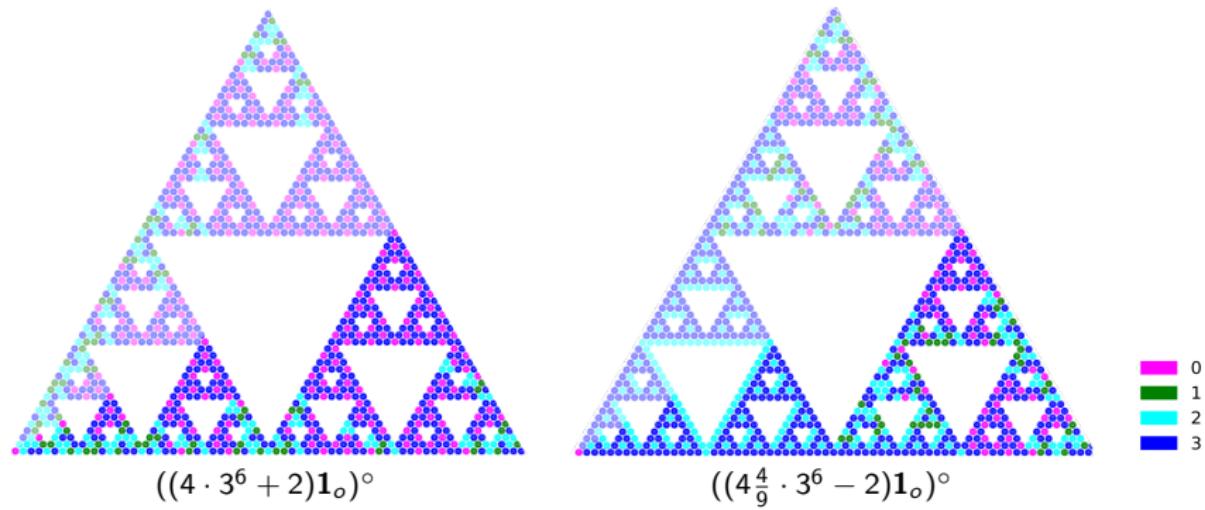
- The unique shortest blue path (Peano curve, in this case a.k.a. the Sierpinski arrowhead curve) of 3's connecting o to the sink y .
- What happens when 2 chips are added to o ?
- Triggers a chain reaction of topplings down the Peano curve, all the way to y ! This sends 1 extra chip to each sink vertex, which explains “ $4 \cdot 3^n + 2$ ” in the radial jump theorem.
- BUT ...

$e_n + 2$: Traps develop along the Peano curve



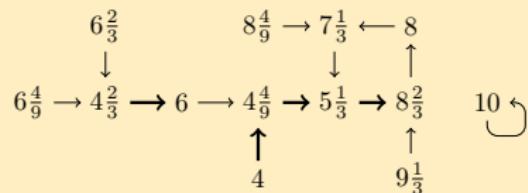
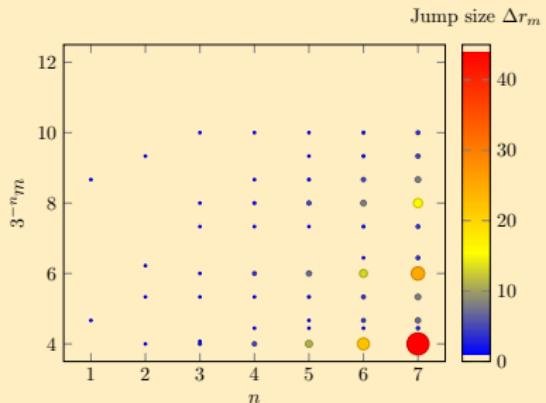
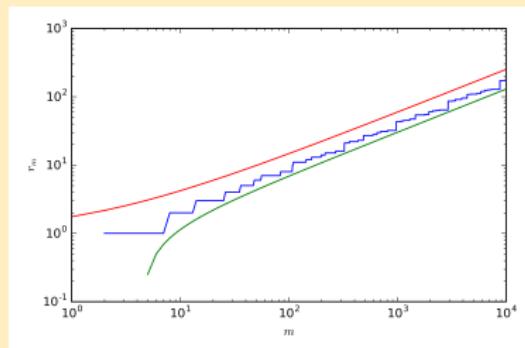
- Blotches of 1's and 2's ("traps") develop at well-defined locations, due to connections across vertices on the Peano curve.
- Best way to visualize this is to parametrize SG along the length of the Peano curve:

$"4\frac{4}{9} \cdot 3^n - 2"$: Inability to overcome the traps



Summary: Exact solution of abelian sandpile growth on SG

Theorem (Recursive radial growth formula)



$$a \rightarrow b: \quad r_{a \cdot 3^n} = 2^n + r_{b \cdot 3^{n-1}} \quad (n \geq 3)$$
$$a \rightarrow b: \quad r_{a \cdot 3^n} = 2^n + r_{b \cdot 3^{n-2}} \quad (n \geq 4)$$

Thank you!