

Hydrodynamic limit of the boundary-driven exclusion process on the Sierpinski gasket

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arXiv:1606.01677, [1702.03376](#), 1705.10290, [181x.yyyyy](#) (2x)

- **Goal:** Rigorous derivation of macroscopic fluid equations (e.g. Navier-Stokes) from microscopic systems (e.g. a volume of interacting water molecules).
- **Technical ingredients: Coarse-graining, ergodicity.** Both can be hard to verify.
- **A tractable approach:** Build *stochastic* interactions into the microscopic model, thereby ensuring ergodicity in the first place.
 - ▶ Examples of models: Exclusion process, zero-range process, stochastic Ising models, etc.
- This **hydrodynamic program**, started in the 80s by Kipnis, Olla, Varadhan, etc., and continues to this day, involves many areas of mathematics.
 - ▶ Stochastic processes (random walks, jump processes, diffusion processes), analysis of nonlinear (S)PDEs, potential theory, discrete harmonic analysis.

It also addresses, and is inspired from, questions from (non-)equilibrium statistical physics: e.g. density flow and fluctuations in an out-of-equilibrium physical system.

- Throughout the talk, I will mention new results connecting **interacting particle systems** to classical ideas from **random walks on graphs**, esp. the role of the **effective resistance**.



Boltzmann



Gibbs



Dirichlet



Thomson



Kirchhoff



Octopus?!

The boundary-driven exclusion process in 1D

- Identical particles performing random walks on $V_N := (0, N) \cap \mathbb{Z}$.
- In the bulk, $(0, N) \cap \mathbb{Z}$, particles undergo **exclusion dynamics**: No two random walkers can occupy the same vertex.
- At each boundary vertex $y \in \{0, N\}$ (“reservoir”), particles can be injected into or extracted from the bulk at respective rate $\lambda_+(y)$ and $\lambda_-(y)$.
- Denote by $\{\eta_t^N : t \geq 0\}$ the corresponding Markov chain on $\{0, 1\}^{V_N}$.
- **Goal**: Study the scaling limit of functionals of $\{\eta_{N^2 t}^N : t \in [0, T]\}$ under the diffusive time scale N^2 and space scale N .

The boundary-driven exclusion process in 1D

- The boundary rates $\lambda_{\pm}(y)$, $y \in \{0, N\}$ control the steady-state density:
 - ▶ On the boundary: When $y \in \{0, N\}$, $\bar{\rho}(y) = \frac{\lambda_+(y)}{\lambda_+(y) + \lambda_-(y)}$.
 - ▶ In the bulk: When $x \in (0, N)$, $\bar{\rho}(x)$ is the harmonic extension of $\bar{\rho}$ from $\{0, N\}$ to $\{0, 1, \dots, N\}$.
- By default, ergodicity is guaranteed: (limit) process has a unique invariant measure whose one-site marginal at x is $\bar{\rho}(x)$.
- If $\lambda_{\pm}(y)$ for all $y \in \{0, N\}$, $\bar{\rho}$ is constant everywhere: *equilibrium*
- Else: the one-site marginal $\bar{\rho}$ interpolates between a "hot" reservoir and a "cold" one: *out of equilibrium*

The boundary-driven exclusion process in 1D

- Empirical density measure on $[0, 1]$: $\pi_t^N = \sum_{x \in V_N} \eta_{tN^2}^N(x) \delta_{x/N}$.
- **LLN** for the symmetric exclusion process: Assume $\pi_0^N \xrightarrow[N \rightarrow \infty]{} \rho_0 dx$ weakly. Then
[Eyink–Lebowitz–Spohn '90] $\pi_t^N \xrightarrow[N \rightarrow \infty]{} \rho(t, x) dx$ where ρ is the unique sol'n of the heat eqn

$$\begin{cases} \partial_t \rho = \frac{1}{2} \Delta \rho & \text{on } [0, T] \times (0, 1) \\ \rho(0, \cdot) = \rho_0 & \text{on } [0, 1] \\ \rho(\cdot, \cdot) = \bar{\rho} & \text{on } [0, T] \times \{0, 1\} \end{cases}$$

- Fluctuations about the LLN are captured via a **large deviations principle**:

[Bertini–De Sole–Gabrielli–Jona-Lasinio–Landim '03, '07]

For each closed set \mathcal{C} and each open set \mathcal{O} of the Skorokhod space $D([0, T], \mathcal{M}_+)$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log Q^N[\mathcal{C}] \leq - \inf_{\pi \in \mathcal{C}} I_0(\pi), \quad \liminf_{N \rightarrow \infty} \frac{1}{N} \log Q^N[\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O}} I_0(\pi).$$

$$I_0(\pi) < \infty \Leftrightarrow \pi \in \mathcal{M}_{+,1}; \text{ then } \exists H \text{ s.t. } I_0(\pi) = \frac{1}{2} \int_0^T \int_{[0,1]} \rho(t, x) [1 - \rho(t, x)] |\nabla H_t|^2 dx dt.$$

Here $H \in C^{1,2}([0, T] \times [0, 1])$ is the **weak asymmetry** introduced into the exclusion process in the bulk.



$$\text{with rate } \frac{1}{2} e^{N[H_t(y) - H_t(x)]} \approx \frac{1}{2} (1 + N[H_t(y) - H_t(x)])$$

The boundary-driven exclusion process in 1D

- Empirical density measure on $[0, 1]$: $\pi_t^N = \sum_{x \in V_N} \eta_{tN^2}^N(x) \delta_{x/N}$.
- **LLN** for the **weakly asymmetric** exclusion process: Assume $\pi_0^N \xrightarrow[N \rightarrow \infty]{} \rho_0 dx$ weakly. Then [Eyink–Lebowitz–Spohn '91] $\pi_t^N \xrightarrow[N \rightarrow \infty]{} \rho(t, x) dx$ where ρ is the unique sol'n of the **semilinear heat eqn**

$$\begin{cases} \partial_t \rho = \frac{1}{2} \Delta \rho + \nabla \cdot (\rho(1 - \rho) \nabla H_t) & \text{on } [0, T] \times (0, 1) \\ \rho(0, \cdot) = \rho_0 & \text{on } [0, 1] \\ \rho(\cdot, \cdot) = \bar{\rho} & \text{on } [0, T] \times \{0, 1\} \end{cases}$$

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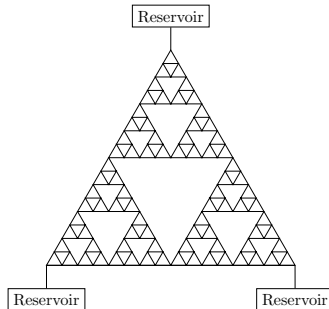
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with rate $\frac{1}{2} e^{N[H_t(y) - H_t(x)]} \approx \frac{1}{2} (1 + N[H_t(y) - H_t(x)])$

Generalization from 1D to higher dimensions

- Finite box $([0, M] \cap \mathbb{Z})^d$, $d \geq 2$: Analysis of fluctuations is more difficult than the $d = 1$ case, few results.
- State space with a **finite boundary set**: if it comes with 2 disjoint boundary components, then analysis is similar to the 1D case. [Akkermans–Bodineau–Derrida–Shpielberg '13; Bertini–De Sole–Gabrielli–Jona-Lasinio–Landim, *Macroscopic Fluctuation Theory* '15]
Interesting when there are ≥ 3 disjoint boundary components (hot, lukewarm, cold).
- For the analysis to succeed, we need good knowledge of scaling limits of random walks to diffusion on the candidate state space.
- This leads to our candidate: the Sierpinski gasket



Abstract setup: Exclusion process on a weighted graph

Let $G = (V, E)$ be a connected graph endowed with conductances $\mathbf{c} = (c_{xy})_{xy \in E}$.

The **symmetric exclusion process** on (G, \mathbf{c}) is a Markov chain on $\{0, 1\}^V$ with generator

$$(\mathcal{L}_{(G, \mathbf{c})}^{\text{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0, 1\}^V \rightarrow \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

Properties:

- 1 Total particle number is conserved in the process.
- 2 Each product Bernoulli measure ν_α , $\alpha \in [0, 1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure for this process.

Dirichlet energy: $\mathcal{E}_{(G, \mathbf{c}), \nu_\alpha}^{\text{EX}}(f) = \frac{1}{2} \sum_{zw \in E} c_{zw} \int_{\{0, 1\}^V} [(\nabla_{zw} f)(\eta)]^2 d\nu_\alpha(\eta).$

Weakly asymmetric exclusion process: Let $H : [0, T] \times V \rightarrow \mathbb{R}$ and $H_t = H(t, \cdot)$. Generator

$$(\mathcal{L}_{(G, \mathbf{c}), H}^{\text{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} \psi_{xy}(H_t, \eta) (\nabla_{xy} f)(\eta). \quad f : \{0, 1\}^V \rightarrow \mathbb{R},$$

where $\psi_{xy}(H, \eta) = \eta(x)[1 - \eta(y)]e^{H(y) - H(x)} + \eta(y)[1 - \eta(x)]e^{H(x) - H(y)}$

Abstract setup: Boundary-driven exclusion process

Declare a subset ∂V of V to be the boundary set. Assume WLOG that $c_{aa'} = 0$ for all $a, a' \in \partial V$.

Add a **birth-and-death chain** to each $a \in \partial V$.

At rate $\lambda_+(a)$, $\eta(a) = 0 \rightarrow \eta(a) = 1$ (birth); At rate $\lambda_-(a)$, $\eta(a) = 1 \rightarrow \eta(a) = 0$ (death).

Formally,

$$(\mathcal{L}_{\partial V}^b f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1 - \eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0, 1\}^V \rightarrow \mathbb{R},$$

where

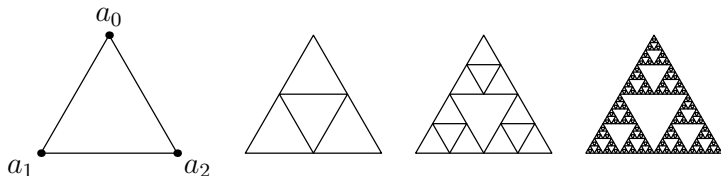
$$\eta^a(z) = \begin{cases} 1 - \eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

The **boundary-driven weakly asymmetric** exclusion process has generator

$$\mathcal{L}_{(G, c), H}^{\text{bEX}} = \mathcal{L}_{(G, c), H}^{\text{EX}} + \mathcal{L}_{\partial V}^b.$$

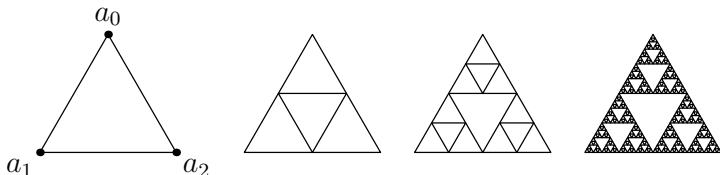
Remark. Due to the **boundary effect**, the invariant measure is no longer product Bernoulli. Nevertheless its one-site marginals $\bar{\rho}$ are known, being the harmonic extension of $\bar{\rho}$ from ∂V to V .

(Single-particle) diffusion on SG



- SG_N : Level- N Sierpinski gasket graph.
- m_N : self-similar probability measure on SG_N , assigns weight to vertex x which is proportional to $\deg(x)$.
- m_N converges weakly to m , the standard self-similar probability measure (with Hausdorff dimension $\log_2 3$), on the limit fractal K .
- $(X_t^N)_{t \geq 0}$: symmetric random walk process on SG_N .
- [Goldstein '87, Kusuoka '87, Barlow–Perkins '88]: [Probability on fractals](#)
 $X_{5^N t}^N \xrightarrow{N \rightarrow \infty} B_t$, called a Brownian motion on SG.

(Single-particle) diffusion on SG



- [Kigami '89+]: [Analysis on fractals](#)

Write down the Dirichlet energy on SG_N , renormalized by $(5/3)^N$:

$$\mathcal{E}_N(f) = \left(\frac{5}{3}\right)^N \sum_{x \sim y} [f(x) - f(y)]^2 \quad (f : K \rightarrow \mathbb{R})$$

Then $\{\mathcal{E}_N\}_N$ is a *monotone increasing* sequence, and hence has a limit \mathcal{E} .

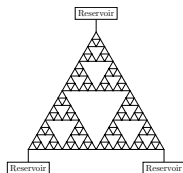
- Let \mathcal{F} be the [domain of finite energy](#) \mathcal{E} . $(\mathcal{E}, \mathcal{F})$ is a [strongly local regular Dirichlet form](#).
- **Operator convergence:** If Δ_N denotes the graph Laplacian on SG_N , then can prove the pointwise formula $\Delta = \lim_{N \rightarrow \infty} 5^N \Delta_N$, where Δ is the generator of B_t .
- *Key Remark.* The Euclidean formula “ $\mathcal{E}(f) = \int_K |\nabla f|^2 dm$ ” does NOT exist literally. Rather it should be understood in terms of [energy measure](#): $\mathcal{E}(f) = \int_K d\Gamma(f, f)$; note that $d\Gamma(f, f) \perp dm$ [Kusuoka '89]. **This complicates the analysis of the scaling limit of the exclusion process on SG.**

A general analytic framework: Resistance spaces

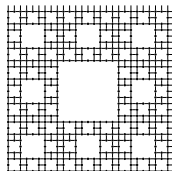
- The lack of translational invariance of the state space also poses a technical obstacle: need another property to generate **local ergodicity**.
- Knowing that the state space is bounded in the **electrical resistance** metric turns out to be a good one. (\Leftrightarrow diffusion is **strongly recurrent**, in the sense of Barlow, Delmotte, Telcs, ...)

Examples of resistance spaces

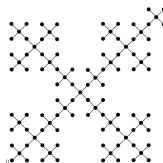
- Classical Dirichlet form $\int_{\Omega} |\nabla f|^2 dx$ on $L^2(\Omega, dx)$ is a resistance form $\Leftrightarrow \Omega$ has Euc dim 1.
- Other examples:



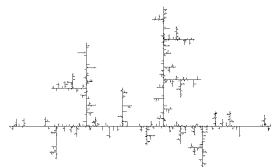
Sierpinski gasket



Sierpinski carpet



Vicsek tree



Random dendrite [by David Croydon]

A general analytic framework: Resistance spaces

- The lack of translational invariance of the state space also poses a technical obstacle: need another property to generate **local ergodicity**.
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Definition. [Kigami, *Analysis on Fractals* '01, also '04]

Let K be a nonempty set. A **resistance form** $(\mathcal{E}, \mathcal{F})$ on K is a pair such that

- 1 \mathcal{F} is a vector space of \mathbb{R} -valued functions on K containing the constants, and \mathcal{E} is a nonnegative definite symmetric quadratic form on \mathcal{F} satisfying

$$\mathcal{E}(u, u) = 0 \Leftrightarrow u \text{ is constant.}$$

- 2 $\mathcal{F}/\{\text{constants}\}$ is a Hilbert space with norm $\mathcal{E}(u, u)^{1/2}$.
- 3 Given a finite subset $V \subset K$ and a function $v : V \rightarrow \mathbb{R}$, there is $u \in \mathcal{F}$ s.t. $u|_V = v$.
- 4 For $x, y \in K$, the effective resistance

$$R_{\text{eff}}(x, y) := \sup \left\{ \frac{[u(x) - u(y)]^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \mathcal{E}(u, u) > 0 \right\} < \infty.$$

- 5 (Markovian property) If $u \in \mathcal{F}$, then $\bar{u} := 0 \vee (u \wedge 1) \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$.

- The lack of translational invariance of the state space also poses a technical obstacle: need another property to generate **local ergodicity**.
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Definition (Resistance form).

4 For $x, y \in K$,

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Remarks.

- The resistance space (K, R_{eff}) is a metric space. Can always assumed to be complete.
- From the definition of the effective resistance it follows that $|u(x) - u(y)|^2 \leq R_{\text{eff}}(x, y)\mathcal{E}(u, u)$, which then implies the Sobolev embedding $\mathcal{F} \subset C(K)$.
- **Analytic consequences:** leads to new functional inequalities for the exclusion process; local ergodicity; \exists and ! of the hydrodynamic PDE.

Hydrodynamic limit theorems on SG [C.–Hinz–Teplyaev, arXiv:181x.yyyyy]

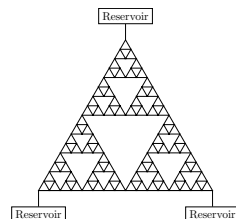
Boundary-driven WASEP: generator $5^N \mathcal{L}_{SG_N, H}^{\text{bEX}}$, law Q_H^N .

Take any $H \in C([0, T], G(\mathcal{F}_0)) \cap C^1((0, T), G(\mathcal{F}_0))$.

- \mathcal{F}_0 : domain of the Dirichlet Laplacian.
- G : (image of the) Dirichlet Green's operator.

Empirical density measure $\pi_t^N := \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \delta_x$

\mathcal{M}_+ : Space of nonnegative Borel measures on K .



Theorem (Density LLN for the boundary-driven WASEP)

Suppose $\pi_0^N \rightarrow \rho_0 dm$ weakly. Then w.p.1, $\pi_t^N \rightarrow \rho(t, x) dm(x)$ in the Skorokhod topology on $D([0, T], \mathcal{M}_+)$, where $\rho(t, x)$ is the unique weak solution of the **semilinear heat equation**

$$\begin{cases} \partial_t \rho_t = \Delta \rho_t - \partial^* (\chi(\rho_t) \partial H_t) & \text{on } (0, T) \times K \setminus V_0, \\ \rho(0, \cdot) = \rho_0 & \text{on } K \setminus V_0, \\ \rho(t, \cdot) = \bar{\rho} & \text{on } (0, T) \times V_0. \end{cases}$$

Here $\chi(\alpha) = \alpha(1 - \alpha)$, and ∂ (resp. the adjoint $-\partial^*$) is the abstract gradient (resp. divergence) operator induced by the resistance form $(\mathcal{E}, \mathcal{F})$ on SG .

Remark. Due to the energy singularity, \exists and ! of the PDE sol'n is proved using the monotone operator method of J.-L. Lions + input from resistance form theory. $\rho \in L^2(0, T, \mathcal{F})$.

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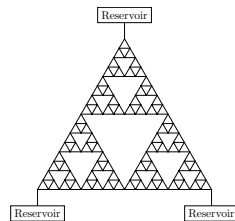
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Theorem (Density LDP: rate $|V_N| \sim \frac{3}{2} 3^N$ with good rate fcn l_0)

For each closed set \mathcal{C} and each open set \mathcal{O} of the Skorokhod space $D([0, T], \mathcal{M}_+)$, endowed with the Skorokhod topology of *weak convergence of measures with bounded energy*,

$$\limsup_{N \rightarrow \infty} \frac{1}{|V_N|} \log Q^N[\mathcal{C}] \leq - \inf_{\pi \in \mathcal{C}} l_0(\pi), \quad \liminf_{N \rightarrow \infty} \frac{1}{|V_N|} \log Q^N[\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O}} l_0(\pi).$$

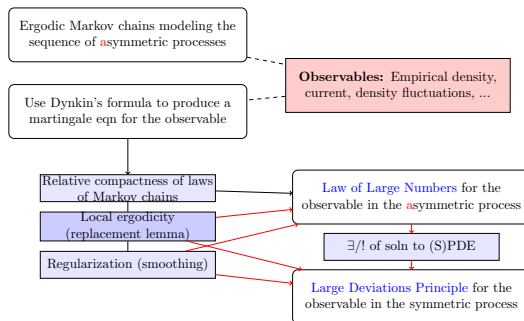
Let $\mathcal{M}_{+,1} = \{\mu \in \mathcal{M}_+ \mid \mu(dx) = \rho(x) m(dx), \ 0 \leq \rho \leq 1 \text{ } m\text{-a.e.}\}$ and

$$D_{+,1,\varepsilon}[0, T] := \{\pi \in D([0, T], \mathcal{M}_{+,1}) \mid \pi(t, dx) = \rho(t, x) m(dx), \ \rho \in L^2(0, T, \mathcal{F})\}.$$

$$l_0(\pi) < \infty \Leftrightarrow \pi \in D_{+,1,\varepsilon}[0, T]; \text{ then } \exists H \text{ s.t. } l_0(\pi) = \frac{1}{2} \int_0^T \int_K \chi(\rho(t, x)) \, d\Gamma(H_t, H_t) \, dt.$$

Technical Remark. The topology we use guarantees that $D_{+,1,\varepsilon}[0, T]$ is closed.

Proof: The entropy method [Guo–Papanicolaou–Varadhan '88, Kipnis–Olla–Varadhan '89]



- This **hydrodynamic program** has been carried out on \mathbb{Z}^d since the late 80's, described in the book [Kipnis–Landim '99].
- **Challenges:** Extend the program to non-translationally-invariant (and possibly energy singular) spaces.
- *Alternative method:* H.-T. Yau's **relative entropy method**, was used to prove LLN of the symmetric zero-range process on SG in [Jara '09].
- **Our contribution:** Established the entropy method on SG, which is more robust (enables the proof of LDP) and leads to further consequences.

Microscopic model

The **empirical density** measure on SG_N :

$$\pi_t^N := \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \delta_x$$

Let Q_H^N be the law of the Markov process generated by $5^N \mathcal{L}_{N,H}^{\text{bEX}}$.

Use **Dynkin's formula** to find that under Q_H^N , for all “nice” test functions F ,

$$\begin{aligned} \langle \pi_t^N, F \rangle &= \langle \pi_0^N, F \rangle + \int_0^t \langle \pi_s^N, \Delta_N F \rangle ds + \int_0^t \sum_{a \in V_0} \eta_s^N(a) (\partial_N^\perp F)(a) ds \\ &\quad + \frac{2}{3} \int_0^t \frac{5^N}{3^N} \sum_{xy \in E_N} \chi(\eta_s^N, xy) [H_s(x) - H_s(y)] [F_s(x) - F_s(y)] ds + \underbrace{M_t^{N,F}}_{\text{martingale}}. \end{aligned}$$

The martingale term $M_t^{N,F}$ has quadratic variation vanishing as $N \rightarrow \infty$.

-
- $(\Delta_N F)(x) = \frac{5^N}{3^N} \sum_{y \sim x} [F(y) - F(x)]$: (renormalized) Laplacian.
 - $(\partial_N^\perp F)(a) = \frac{2}{3} \frac{5^N}{3^N} \sum_{y \sim a} [F(y) - F(a)]$: (renormalized) normal derivative at $a \in V_0$.
 - $\chi(\eta, xy) = \eta(x)[1 - \eta(y)] + \eta(y)[1 - \eta(x)]$ is the conductivity of the exclusion process.

From the microscopic model to the macroscopic limit

$$\begin{aligned} \langle \pi_t^N, F \rangle &= \langle \pi_0^N, F \rangle + \int_0^t \langle \pi_s^N, \Delta_N F \rangle ds + \int_0^t \sum_{a \in V_0} \eta_s^N(a) (\partial_N^\perp F)(a) ds \\ &\quad + \frac{2}{3} \int_0^t \frac{5^N}{3^N} \sum_{xy \in E_N} \chi(\eta_s^N, xy) [H_s(x) - H_s(y)][F_s(x) - F_s(y)] ds + \underbrace{M_t^{N,F}}_{\text{martingale}}. \end{aligned}$$

Goal: Show that $\{Q_H^N\}_N$ is relatively compact, and the limit point Q_H^* concentrates on a.c. trajectories $\pi(t, dx) = \rho(t, x) m(dx)$ with $\rho \in L^2(0, T, \mathcal{F})$ and

$$\begin{aligned} \langle \pi_t, F \rangle &= \langle \pi_0, F \rangle + \int_0^t \langle \pi_s, \Delta F \rangle ds \\ &\quad + \int_0^t \sum_{a \in V_0} \bar{\rho}(a) (\partial^\perp F)(a) ds + \int_0^t \int_K \chi(\rho_s) d\Gamma(H_s, F) ds. \end{aligned}$$

- $\chi(\rho) := \rho(1 - \rho)$: conductivity in the exclusion process.
- $d\Gamma(H, F)$ is the **mutual energy measure** of $H, F \in \mathcal{F}$, analog of “ $(\nabla H \cdot \nabla F) dx$.” This is part of a broader theory of first-order calculus on Dirichlet spaces [Cipriani–Sauvageot '03, Hinz–Röckner–Teplyaev '13].

From the microscopic model to the macroscopic limit

$$\begin{aligned} \langle \pi_t^N, F \rangle &= \langle \pi_0^N, F \rangle + \int_0^t \langle \pi_s^N, \Delta_N F \rangle ds + \int_0^t \sum_{a \in V_0} \eta_s^N(a) (\partial_N^\perp F)(a) ds \\ &\quad + \frac{2}{3} \int_0^t \frac{5^N}{3^N} \sum_{xy \in E_N} \chi(\eta_s^N, xy) [H_s(x) - H_s(y)][F_s(x) - F_s(y)] ds + \underbrace{M_t^{N,F}}_{\text{martingale}}. \end{aligned}$$

Goal: Show that $\{Q_H^N\}_N$ is relatively compact, and the limit point Q_H^* concentrates on a.c. trajectories $\pi(t, dx) = \rho(t, x) m(dx)$ with $\rho \in L^2(0, T, \mathcal{F})$ and

$$\begin{aligned} \langle \pi_t, F \rangle &= \langle \pi_0, F \rangle + \int_0^t \langle \pi_s, \Delta F \rangle ds \\ &\quad + \int_0^t \sum_{a \in V_0} \bar{\rho}(a) (\partial^\perp F)(a) ds + \int_0^t \int_K \chi(\rho_s) d\Gamma(H_s, F) ds. \end{aligned}$$

Observe this is nothing but the weak formulation of the semilinear heat equation

$$\begin{cases} \partial_t \rho_t = \Delta \rho_t - \partial^* (\chi(\rho_t) \partial H_t) & \text{on } (0, T) \times K \setminus V_0, \\ \rho(0, \cdot) = \rho_0 & \text{on } K \setminus V_0, \\ \rho(t, \cdot) = \bar{\rho} & \text{on } (0, T) \times V_0. \end{cases}$$

From the microscopic model to the macroscopic limit

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Key issues: Terms on the RHS are NOT ALL in terms of the empirical density π^N . Need to make the following replacements:

- **Conductivity term:** Replace $\frac{1}{2} \chi(\eta_s^N, \cdot)$ by $\chi(A_{\vee B(\cdot, r_{\epsilon N})}[\eta_s^N])$ and then by $\chi(\rho_s)$.
- **Boundary term:** Replace $\eta_s^N(a)$ by $\bar{\rho}(a)$.

Replacement of the nonlinear conductivity term via local ergodicity

Basic idea: Replace functionals of η_t^N by coarse-grained functionals of the empirical density π_t^N , with negligible cost in the scaling limit.

- Call $\phi : V(\Gamma) \times \{0, 1\}^{V(\Gamma)} \rightarrow \mathbb{R}$ is a **local function bundle** if $\exists r \in (0, \infty)$ such that $\phi(x, \cdot)$ depends only on $\{\eta(z) : z \in B(x, r)\}$.
 - ▶ Examples of local function bundles: $\eta(x)$; $\sum_{y \sim x} \eta(x)\eta(y)$.
- Given ϕ and x , define $\Phi_x(\alpha) = \int \phi(x, \eta) d\nu_\alpha(\eta)$. Let

$$U_{N,\epsilon}(x, \eta) := \underbrace{\phi(x, \eta)}_{\text{microscopic}} - \underbrace{\Phi_x(\text{Av}_{B(x, r_{\epsilon N})}[\eta])}_{\text{macro avg}}.$$

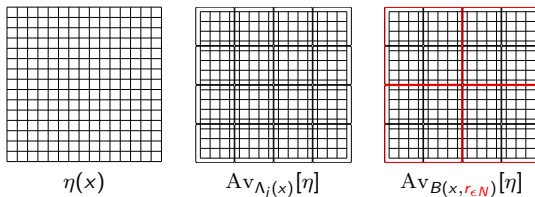
Local ergodicity (a.k.a. local equilibrium, replacement lemma)

For each $T > 0$ and each $\delta > 0$,

$$\limsup_{\epsilon \downarrow 0} \limsup_{N \rightarrow \infty} \sup_{x \in V_N} \frac{1}{|V_N|} \log Q_H^N \left\{ \left| \int_0^T U_{N,\epsilon}(x, \eta_t^N) dt \right| > \delta \right\} = -\infty.$$

- The local ergodicity theorem was proved in the more general setting of resistance spaces (viz. strongly recurrent weighted graphs) in [C. '17]. A “low-dimensional ($d < 2$)” result.
- All assumptions derive from potential theory of random walks on graphs. Nothing is assumed about the spatial symmetries of the underlying state space.

Technical estimates: 1-block and 2-blocks estimates



Strategy [GPV '88, KOV '89]: implement a two-scale coarse-graining procedure.

$$U_{N,\epsilon}(x, \eta) := \underbrace{\left[\phi(x, \eta) - \Phi_x \left(\text{Av}_{\Lambda_j(x)}[\eta] \right) \right]}_{\substack{U_{N,j}^{(1)} \quad \text{1-block}}} + \underbrace{\left[\Phi_x \left(\text{Av}_{\Lambda_j(x)}[\eta] \right) - \Phi_x \left(\text{Av}_{B(x, r_{\epsilon N})}[\eta] \right) \right]}_{\substack{U_{N,j,\epsilon}^{(2)} \quad \text{2-blocks}}}$$

- j sets the microscopic scale.
- $\epsilon \in [0, 1]$ sets the macroscopic aspect ratio.
- Ordering of limits: $N \rightarrow \infty$, then $\epsilon \downarrow 0$, then $j \rightarrow \infty$.

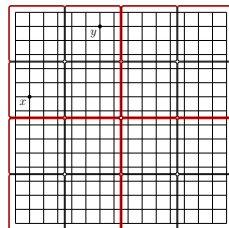
Separately show that $U_{N,j}^{(1)}$ and $U_{N,j,\epsilon}^{(2)}$ vanishes in the said limit with probability superexponentially close to 1 \rightarrow Requires the Feynman-Kac formula, a spectral gap estimate, and a local limit thm (equivalence of ensembles).

2-blocks estimate: A closer look

For the 2BE to be effective, the energy cost of moving between points x and y in any two micro blocks inside a macro block,

$$\int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 d\nu_\alpha(\eta),$$

should scale diffusively.



Problem: Due to exclusion, when moving a particle from x to y one also has to move over many “obstacles”!

- \mathbb{Z}^d : Just pick a shortest path with distance L , and carry out a sequence of nearest-neighbor “spin swaps,” and calculate the associated energy cost. \rightarrow **Cauchy-Schwarz + translation/rotation invariance of the**

exclusion process Dirichlet form. [GPV '88, KOV '89] $\rightarrow \int [(\nabla_{xy} f)(\eta)]^2 d\nu_\alpha(\eta) \lesssim L^2 \mathcal{E}^{\text{EX}}(f)$

- This argument was also used by [Diaconis–Saloff-Coste '93] to obtain eigenvalue bounds in the exclusion process on a finite graph.
- HOWEVER this method does NOT give the required $(5/3)^N$ diffusive scaling for SG. [Jara '09] already noticed this issue.
- **Solution:** Prove a functional inequality which uses the effective resistance distance instead of the

shortest-path distance. $\int [(\nabla_{xy} f)(\eta)]^2 d\nu_\alpha(\eta) \lesssim R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f)$

The crux of 2BE: the **moving particle lemma (MPL)** [C., ECP '17]

- $(G = (V, E), \mathbf{c} = (c_{zw})_{zw \in E})$: **finite** connected weighted graph.

Theorem (MPL for the symmetric exclusion process)

For all $f : \{0, 1\}^V \rightarrow \mathbb{R}$,

$$\underbrace{\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{zw} f)(\eta)]^2 d\nu_\alpha(\eta)}_{=2\mathcal{E}^{\text{EX}}(f)} \geq [R_{\text{eff}}(x, y)]^{-1} \underbrace{\int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 d\nu_\alpha(\eta)}_{\text{Cost of swapping configs } x \leftrightarrow y}.$$

Proof. Idea connected to the spectral gap conjecture of Aldous '92: $\lambda_2^{\text{EX}}(G) = \lambda_2^{\text{RW}}(G)$, which was resolved positively by [Caputo–Liggett–Richthammer '10] via their proof of the *octopus inequality* for the interchange process. I used OI, electric network reduction, and projection onto the exclusion process to obtain the MPL.

Harkens to...

Theorem (Dirichlet/Thomson 1867). For all $f : V \rightarrow \mathbb{R}$,

$$\underbrace{\sum_{zw \in E} c_{zw} [f(z) - f(w)]^2}_{=\mathcal{E}(f)} \geq [R_{\text{eff}}(x, y)]^{-1} [f(x) - f(y)]^2.$$

Equality is attained $\Leftrightarrow f$ is harmonic on $V \setminus \{x, y\}$: Ohm's Law: $V = IR$, Power = $IV = R^{-1}V^2$

The improved MPL has:

- enabled me to establish the local 2BE on resistance spaces. [C. '17]

There appear to be more benefits:

- A corresponding **MPL** in terms of resistance also holds for the **zero-range process** on a finite connected weighted graph.
- The MPL can be used to generalize the (2nd-order) **Boltzmann-Gibbs principle** from 1D to all resistance spaces; in particular, shortens the proof in the 1D case given in e.g. [Gonçalves-Jara-Simon '17].

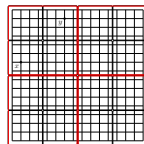
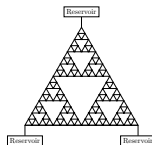
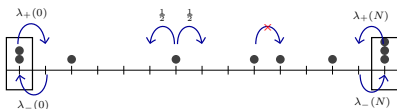
These functional analytic tools will be used to address related scaling limit questions, such as that of the **density fluctuation field** in the (W)ASEP under the equilibrium density ρ :

$$\mathcal{Y}_t^N = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} (\eta_t^N(x) - \rho) \delta_x.$$

A crossover between two scaling regimes

- **Gaussian (Edwards-Wilkinson) regime**: Ornstein-Uhlenbeck process.
 - ▶ For the ZRP on SG: [Jara '06, arXiv preprint].
- **Critical (“KPZ”) scaling** should lead to a version of stochastic Burgers' equation. Making sense of the nonlinearity via regularization is an outstanding open problem!
 - ▶ For a purely PDE approach to the viscous Burgers' eqn on SG: [Hinz-Meinert '17].

Het einde (voor nu)



Dank je!