Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Nonequilibrium fluctuations
in the boundary-driven exclusion process
on a resistance space

Joe P. Chen

Department of Mathematics
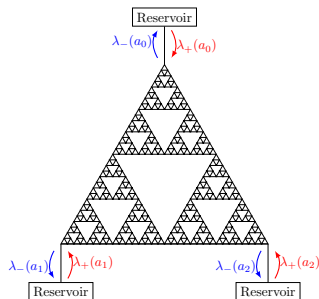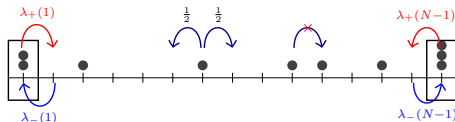Colgate University
Hamilton, NY, USA

The 41st Stochastic Processes & their Applications Conference
Session "Fluctuations of Interacting Particle Systems"
Northwestern University
Evanston, IL, USA
July 8, 2019

COLGATE UNIVERSITY

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Overview of results

Scaling limits of empirical density in the boundary-driven SEP on the Sierpinski gasket

- **LLN & eqFluct:** Joint work with Patrícia Gonçalves (IST Lisboa), `arXiv:1904.08789`.
- LDP: Joint work with Michael Hinz (Bielefeld) (2019+).
- **NoneqFluct & hydrostatics:** Joint w/ Chiara Franceschini, Patrícia Gonçalves, and Otávio Menezes (all IST Lisboa) (2019+).

Functional inequalities and local averging tools (C.)

- **Moving particle lemma:** *ECP* '17, `arXiv:1606.01577`.
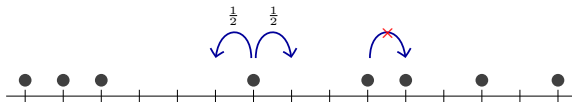- Local ergodicity (1-block & 2-blocks estimates): `arXiv:1705.10290`

Outline

Motivation: Generalizing the analysis of the exclusion process from 1D to higher dimensions

Boundary-driven exclusion process on the Sierpinski gasket

New tools & ideas for resistance spaces

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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Exclusion process

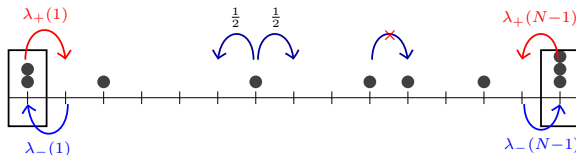The **(symm.) exclusion process** on (G, c) is a Markov chain on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}} f)(\eta) = \sum_{xy \in E} c_{xy}(\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) := \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

- Each product Bernoulli measure ν_α, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure.
- Dirichlet energy: $\mathcal{E}^{\mathrm{EX}}(f) = \dfrac{1}{2} \sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta).$

Adding reservoirs (Glauber dynamics) to the exclusion process

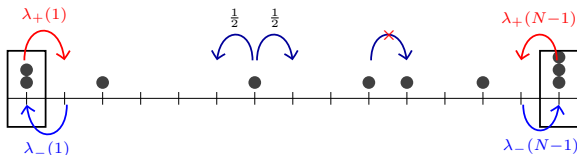Designate a finite boundary set $\partial V \subset V$. For each $a \in \partial V$:

- At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth).
- At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).

Formally,

$$(\mathcal{L}_{\partial V}^{\mathrm{boun}} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1 - \eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R}, \text{ where}$$
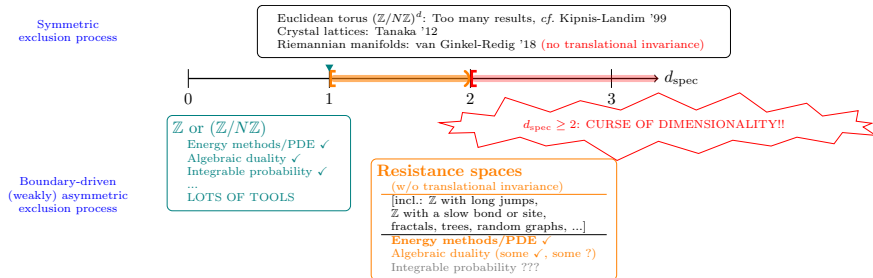
$$\eta^a(z) = \begin{cases} 1 - \eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

Adding reservoirs (Glauber dynamics) to the exclusion process

- **1D boundary-driven simple exclusion process**: generator $N^2 \left(\mathcal{L}^{\mathrm{EX}}_{\{1,2,\cdots,N-1\}} + \mathcal{L}^{\mathrm{boun}}_{\{1,N-1\}} \right)$.

- Has been studied extensively for the past $\sim$ 15 years:
  Hydrodynamic limits, fluctuations, large deviations, etc.
  Bertini–DeSole–Gabrielli–Landim–Jona-Lasinio '03, '07; Landim–Milanes–Olla '08;
  Franco–Gonçalves–Neumann '13, '17; Baldasso–Menezes–Neumann–Souza '17;
  Gonçalves–Jara–Menezes–Neumann '18+; ...

- **Difficulties**: # of particles is no longer conserved; the invariant measure is in general not explicit.

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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Extending the analysis to higher dims & with > 2 reservoirs?

- **Today's message:** On state spaces with spectral dimension $d_{\mathrm{spec}} \in [1, 2)$ (diffusion is strongly recurrent), we have a path towards proving scaling limits of SSEP/WASEP w/o requiring translational invariance.

- **Open question:** Prove scaling limits of boundary-driven SSEP/WASEP on state spaces with $d_{\mathrm{spec}} \geq 2$ (diffusion is NOT strongly recurrent).

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Exclusion process on *SG*: Main results
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Resistance spaces [Kigami '03]

Let K be a nonempty set. A **resistance form** $(\mathcal{E}, \mathcal{F})$ on K is a pair such that

1 $\mathcal{F}$ is a vector space of $\mathbb{R}$-valued functions on K containing the constants, and $\mathcal{E}$ is a nonnegative definite symmetric quadratic form on $\mathcal{F}$ satisfying

$$\mathcal{E}(u, u) = 0 \iff u \text{ is constant.}$$

2 $\mathcal{F}/\{\text{constants}\}$ is a Hilbert space with norm $\mathcal{E}(u, u)^{1/2}$.

3 Given a finite subset $V \subset K$ and a function $v : V \to \mathbb{R}$, there is $u \in \mathcal{F}$ s.t. $u|_V = v$.

4 For $x, y \in K$, the **effective resistance**

$$R_{\text{eff}}(x, y) := \sup \left\{ \frac{[u(x) - u(y)]^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \ \mathcal{E}(u, u) > 0 \right\} < \infty.$$

5 (Markovian property) If $u \in \mathcal{F}$, then $\bar{u} := 0 \vee (u \wedge 1) \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$.

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Exclusion process on *SG*: Main results
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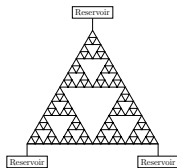# Resistance spaces [Kigami '03]

Point-to-point effective resistance is finite

$$R_{\text{eff}}(x, y) := \sup \left\{ \frac{[u(x) - u(y)]^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \ \mathcal{E}(u, u) > 0 \right\} < \infty.$$
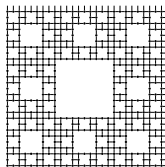
Examples of resistance spaces

- Classical Dirichlet form $\int_{\Omega} |\nabla u|^2 \, dx$ on $L^2(\Omega, dx)$ is a resistance form $\Leftrightarrow \Omega$ has Euc dim 1.
- α-stable process on $\mathbb{R}$ with $\alpha \in (1, 2]$:

$$\mathcal{E}^{(\alpha)}(u) = \int_{\mathbb{R}^2} \frac{[u(x) - u(y)]^2}{|x - y|^{1+\alpha}} \, dy \, dx.$$
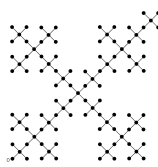
- Diffusion on (some) fractals, trees, random graphs:
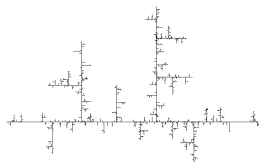

Sierpinski gasket

Sierpinski carpet
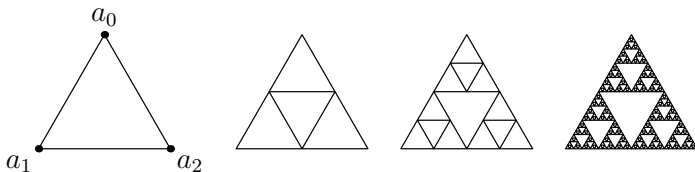

Vicsek tree

Random dendrite [by David Croydon]

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Boundary-driven exclusion process on the Sierpinski gasket

- Construction of **Brownian motion** with invariant measure m (the standard self-similar measure) as scaling limit of RWs accelerated by $\mathcal{T}_N = 5^N$.

  [Goldstein '87, Kusuoka '88, Barlow-Perkins '88]

- A **robust notion of calculus** on *SG* which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.

  [Kigami, *Analysis on Fractals* '01; Strichartz, *Differential Equations on Fractals* '06]

- A good model for rigorously studying (non)equilibrium stochastic dynamics with ≥ 3 boundary reservoirs.

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Exclusion process on *SG*: Main results
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Analysis on fractals (à la Kigami–Strichartz)
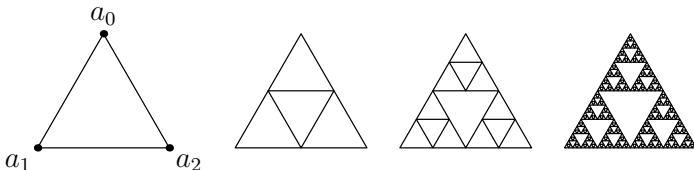
- Define the discrete renormalized Dirichlet energy on G_N:

$$\mathcal{E}_N(f) = \frac{5^N}{3^N} \frac{1}{2} \sum_{\substack{x,y \in V_N \\ x \sim y}} [f(x) - f(y)]^2, \quad f : K \to \mathbb{R}.$$

  Fact. $\{\mathcal{E}_N(f)\}_N$ is monotone nondecreasing, so it either converges to a finite quantity or diverges to $+\infty$.

  Define $\mathcal{F} := \{f : \lim_{N\to\infty} \mathcal{E}_N(f) < +\infty\}$, and for each $f \in \mathcal{F}$, we denote the limit energy by $\mathcal{E}(f)$.

- Analogy to the 1D interval:

$$\left(\int_{[0,1]} |\nabla f|^2 \, dx, \ H_1([0,1]) \right) \quad \text{vs.} \quad \left(\mathcal{E}(f) = \int_K \text{``}|\nabla f|^2\text{''} \, dm, \ \mathcal{F} \right)$$

  Sobolev embedding: $H_1([0,1]) \subset C([0,1])$, $\mathcal{F} \subset C(K)$.

- **Caveat.** The "$|\nabla f|^2$" does NOT exist literally.

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Exclusion process on SG: Main results
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Analysis on fractals (à la Kigami–Strichartz)

- Define the discrete renormalized Dirichlet energy on $G_N = (V_N, E_N)$:

$$\mathcal{E}_N(f) = \frac{5^N}{3^N} \frac{1}{2} \sum_{\substack{x,y \in V_N \\ x \sim y}} [f(x) - f(y)]^2, \quad f : K \to \mathbb{R}.$$

  Fact. $\{\mathcal{E}_N(f)\}_N$ is monotone nondecreasing, so it either converges to a finite quantity or diverges to $+\infty$.

  Define $\mathcal{F} := \{f : \lim_{N \to \infty} \mathcal{E}_N(f) < +\infty\}$, and for each $f \in \mathcal{F}$, we denote the limit energy by $\mathcal{E}(f)$.

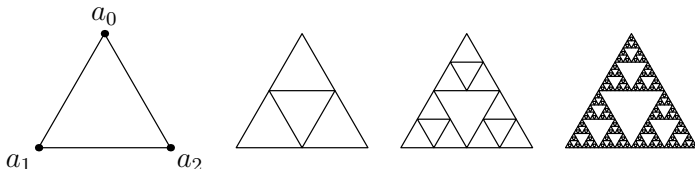- Analogy to the 1D interval:
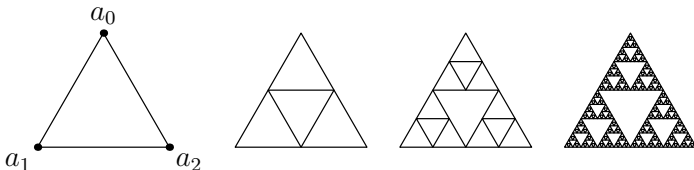
$$\left(\int_{[0,1]} |\nabla f|^2 \, dx, \ H_1([0,1]) \right) \quad \text{vs.} \quad \left(\mathcal{E}(f) = \int_K d\Gamma(f), \ \mathcal{F} \right)$$

  Sobolev embedding: $H_1([0,1]) \subset C([0,1])$, $\mathcal{F} \subset C(K)$.

- **Caveat.** For nonconstant $f \in \mathcal{F}$, $d\Gamma(f) \perp dm$. This is a source of great technical difficulty in the analysis of RW/IPS on fractals.

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Analysis on fractals (à la Kigami–Strichartz)

- **Laplacian**: the following two formulations coincide.
  - Weak formulation: Say $u = -\Delta f \in C(K)$ if $\mathcal{E}(v, f) = \int_K vu\, dm$ for all $v \in \mathcal{F}_0 := \{\phi \in \mathcal{F} : \phi|_{V_0} = 0\}$.
  - Pointwise formulation ($x \in V_N \setminus V_0$): $(\Delta f)(x) := \lim_{N \to \infty} \frac{3}{2} 5^N \sum_{\substack{y \in V_N \\ y \sim x}} [f(y) - f(x)]$.

  Denote by $\mathrm{dom}\Delta$ the operator domain of the Laplacian.

For each $f \in \mathrm{dom}\Delta$ we can further give:

- (Outward) Normal derivative at the boundary ($a \in V_0$): $(\partial^\perp f)(a) = \lim_{N \to \infty} \frac{5^N}{3^N} \sum_{\substack{y \in V_N \\ y \sim a}} [f(a) - f(y)]$.

- Integration by parts formula:
$$\mathcal{E}(f, g) = \int_K (-\Delta f)g\, dm + \sum_{a \in V_0} (\partial^\perp f)(a)g(a) \qquad (f \in \mathrm{dom}\Delta,\ g \in \mathcal{F})$$

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Exclusion process on the Sierpinski gasket with slowed boundary
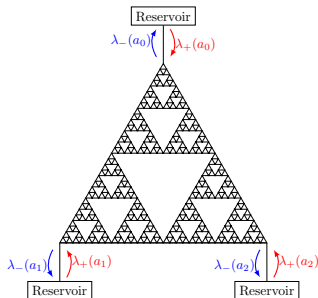
$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left(\mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

Parameter $b > 0$ governs the inverse speed at which the reservoir injects/extracts particles into/from the boundary vertices V_0.

Main result in a nutshell

A phase transition in the scaling limit of the particle density with respect to $b > 0$, reflected by the different boundary conditions.

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

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Hydrodynamic limit: a LLN result

Assume that sequence of probability measures $\{\mu_N\}_{N \geq 1}$ on $\{0, 1\}^{V_N}$ is associated to a density profile $\varrho : K \to [0, 1]$:
$\forall F \in C(K), \ \forall \delta > 0,$

$$\lim_{N \to \infty} \mu_N \left\{ \eta \in \{0, 1\}^{V_N} \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} F(x)\eta(x) - \int_K F(x)\varrho(x) \, dm(x) \right| > \delta \right\} = 0.$$

Given the process $\{\eta_t^N : t \geq 0\}$ generated by $5^N \mathcal{L}_N^{\mathrm{bEX}}$, the **empirical density measure** (and its pairing with test functions $F : K \to \mathbb{R}$):

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \mathbb{1}_{\{x\}} \qquad \left(\pi_t^N(F) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x). \right)$$

Claim. $\{\pi_\cdot^N\}_N$ converges in the Skorokhod topology on $D([0, T], \mathcal{M}_+)$ to the unique measure $\pi_\cdot$ with $d\pi_\cdot(x) = \rho(\cdot, x) \, dm(x)$.
$\forall t \in [0, T], \ \forall F \in C(K), \ \forall \delta > 0,$

$$\lim_{N \to \infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t, x) \, dm(x) \right| > \delta \right\} = 0,$$

Hydrodynamic limit: a LLN result
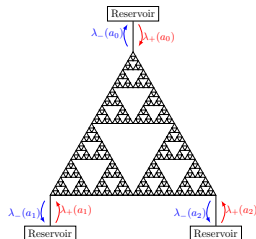
$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$

Theorem (Density hydrodynamic limit (C.–Gonçalves '19))

For any $t \in [0, T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N \to \infty} \mu_N \left\{ \eta^N_. \; : \; \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta^N_t(x) F(x) - \int_K F(x) \rho(t, x) \, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with Dirichlet boundary condition if $b < \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t, x) = \frac{2}{3} \Delta \rho(t, x), & t \in [0, T], \; x \in K \setminus V_0, \\ \rho(t, a) = \bar{\rho}(a), & t \in (0, T], \; a \in V_0, \\ \rho(0, x) = \varrho(x), & x \in K. \end{cases}$$

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Hydrodynamic limit: a LLN result

$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$
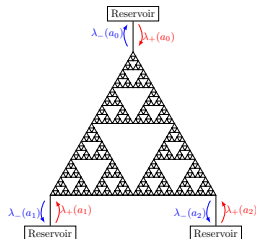
Theorem (Density hydrodynamic limit (C.–Gonçalves '19))

For any $t \in [0, T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N \to \infty} \mu_N \left\{ \eta^N \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t, x) \, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with Neumann boundary condition if $b > \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t, x) = \frac{2}{3} \Delta \rho(t, x), & t \in [0, T], \ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t, a) = 0, & t \in (0, T], \ a \in V_0, \\ \rho(0, x) = \varrho(x), & x \in K. \end{cases}$$

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Hydrodynamic limit: a LLN result

$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$

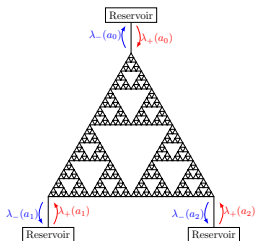## Theorem (Density hydrodynamic limit (C.–Gonçalves '19))

For any $t \in [0, T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N \to \infty} \mu_N \left\{ \eta^{\cdot} \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t, x) \, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with linear Robin boundary condition if $b = \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t, x) = \frac{2}{3} \Delta \rho(t, x), & t \in [0, T], \ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t, a) = -\lambda_\Sigma(a)(\rho(t, a) - \bar{\rho}(a)), & t \in (0, T], \ a \in V_0, \\ \rho(0, x) = \varrho(x), & x \in K. \end{cases}$$

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Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N \left(\left(\frac{2}{3}\Delta + \partial_s \right) F_s \right) ds$$

$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a))F_s(a) \right] ds + o_N(1).$$

[Ingredient #1] **Analysis on fractals**
This part will produce the weak formulation of the heat equation.

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Exclusion process on SG: Main results
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New tools & ideas for resistance spaces
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Summary
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Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N \left(\left(\frac{2}{3}\Delta + \partial_s \right) F_s \right) ds$$

$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar\rho(a))F_s(a) \right] ds + o_N(1).$$

[Ingredient #2] **Analysis of the boundary term**

- $b > 5/3$: The first term dominates, should converge to $\displaystyle\int_0^t \frac{2}{3} \sum_{a \in V_0} \rho_s(a)(\partial^\perp F_s)(a)\, ds$

- $b = 5/3$: Both terms contribute equally, should converge to
  $\displaystyle\int_0^t \frac{2}{3} \sum_{a \in V_0} \left[\rho_s(a)(\partial^\perp F_s)(a) + \lambda_\Sigma(a)(\rho_s(a) - \bar\rho(a))F_s(a) \right] ds$

- $b < 5/3$: Impose $\rho_t(a) = \bar\rho(a)$ for all $a \in V_0$, should converge to $\displaystyle\int_0^t \frac{2}{3} \sum_{a \in V_0} \bar\rho(a)(\partial^\perp F_s)(a)$

Require a series of **replacement lemmas**: not trivial on state spaces without translational invariance!
$\to$**Octopus inequality, moving particle lemma**

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Heuristics for hydrodynamics

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

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$$+ \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a)(\eta_s^N(a) - \bar{\rho}(a))F_s(a) \right] \, ds + o_N(1).$$

$$\Big\downarrow N \to \infty$$

$$0 = \pi_t(F_t) - \pi_0(F_0) - \int_0^t \pi_s \left(\left(\frac{2}{3}\Delta + \partial_s \right) F_s \right) \, ds + (\text{boundary term})$$

[Ingredient #3] **Convergence of stochastic processes**

- Show that $\{\pi_\cdot^N\}_N$ is tight in the Skorokhod topology on $D([0, T], \mathcal{M}_+)$ via Aldous' criterion.

- Prove that any limit point $\pi_\cdot$ is absolutely continuous w.r.t. the self-similar measure m, with $\pi_t(dx) = \rho(t, x) \, dm(x)$, and $\rho \in L^2(0, T, \mathcal{F})$.

- Finally, prove ! of the weak solution to the heat equation to conclude ! of the limit point.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Density fluctuation field: Heuristics

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
Equilibrium: the product Bernoulli measure ν_ρ^N with $\rho = \lambda_+/(\lambda_+ + \lambda_-)$ is stationary for the process.
Not true in the non-equilibrium setting.

Density fluctuation field (DFF)
$$\mathcal{Y}_t^N(F) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \underbrace{\left(\eta_t^N(x) - \mathbb{E}_{\mu_N}[\eta_t^N(x)] \right)}_{=: \bar{\eta}_t^N(x)} F(x)$$

The corresponding Dynkin's martingale is

$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F) \, ds + o_N(1)$$

$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma(a) F(a) \right] \, ds,$$

which has QV

$$\langle \mathcal{M}^N(F) \rangle_t = \int_0^t \frac{5^N}{|V_N|^2} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} (\eta_s^N(x) - \eta_s^N(y))^2 (F(x) - F(y))^2 \, ds$$

$$+ \int_0^t \sum_{a \in V_0} \frac{5^N}{b^N |V_N|^2} \{ \lambda_-(a) \eta_s^N(a) + \lambda_+(a)(1 - \eta_s^N(a)) \} F^2(a) \, ds.$$

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Density fluctuation field: Heuristics

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$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma(a) F(a) \right] \, ds,$$

which, as $N \to \infty$, has the QV of a space-time white noise (with boundary condition)

$$\frac{2}{3} \cdot 2 \int_0^t \int_K \chi(\rho_s) \, d\Gamma_b(F) \, ds, \quad \text{where } \chi(\alpha) = \alpha(1 - \alpha), \quad \mathcal{E}_b(F) = \mathcal{E}(F) + \sum_{a \in V_0} \lambda_\Sigma(a) F^2(a) \mathbf{1}_{\{b=5/3\}},$$

and $\Gamma_b(F)$ is the energy measure associated to $\mathcal{E}_b(F)$: $\mathcal{E}_b(F) = \int_K d\Gamma_b(F)$.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Density fluctuation field: Heuristics

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)
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$$\mathcal{Y}_t^N(F) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \underbrace{\left(\eta_t^N(x) - \mathbb{E}_{\mu_N}[\eta_t^N(x)] \right)}_{=:\bar{\eta}_t^N(x)} F(x)$$

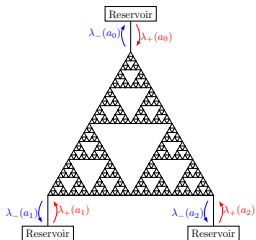The corresponding Dynkin's martingale is
$$\mathcal{M}_t^N(F) = \mathcal{Y}_t^N(F) - \mathcal{Y}_0^N(F) - \int_0^t \mathcal{Y}_s^N(\Delta_N F) \, ds + o_N(1)$$

$$+ \frac{3^N}{\sqrt{|V_N|}} \int_0^t \sum_{a \in V_0} \bar{\eta}_s^N(a) \left[(\partial_N^\perp F)(a) + \frac{5^N}{b^N 3^N} \lambda_\Sigma(a) F(a) \right] ds,$$

We then argue that the test function $F \in \mathrm{dom}\Delta_b$ be chosen appropriate to each boundary condition such that the boundary term vanishes as $N \to \infty$.

$$\mathrm{dom}\Delta_b := \begin{cases} \{F \in \mathrm{dom}\Delta : F|_{V_0} = 0\}, & \text{if } b < 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = -\lambda_\Sigma F|_{V_0}\}, & \text{if } b = 5/3, \\ \{F \in \mathrm{dom}\Delta : (\partial^\perp F)|_{V_0} = 0\}, & \text{if } b > 5/3. \end{cases}$$

For technical reasons we use a smaller test function space $\mathcal{S}_b := \{F \in \mathrm{dom}\Delta_b : \Delta_b F \in \mathrm{dom}\Delta_b\}$, which can be made into a Frechét space. Let $\mathcal{S}_b'$ be the topological dual of $\mathcal{S}_b$.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Scaling limit of density fluctuations: Equilibrium

$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Dirichlet $(b < \frac{5}{3})$, Robin $(b = \frac{5}{3})$, Neumann $(b > \frac{5}{3})$

Eq. $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_- \ \forall a \in V_0$.

Let $\mathbb{Q}_\rho^{N,b}$ be the probability measure on $D([0, T], \mathcal{S}_b')$ induced by the DFF $\mathcal{Y}_\cdot^N$ started from ν_ρ^N and boundary parameter b.

Theorem (EqCLT (C.–Gonçalves '19))

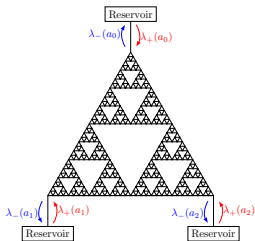*The sequence $\{\mathbb{Q}_\rho^{N,b}\}_N$ converges in distribution, as $N \to \infty$, to a unique solution of the* **Ornstein-Uhlenbeck equation** *with covariance*

$$\mathbb{E}[\mathcal{Y}_t(F)\mathcal{Y}_s(G)] = \chi(\rho) \int_K (\tilde{T}_t^b F)(\tilde{T}_s^b G) \, dm + \frac{2}{3} \cdot 2 \cdot \chi(\rho) \int_0^s \mathscr{E}_b \left(\tilde{T}_{t-r}^b F, \tilde{T}_{s-r}^b G \right) \, dr$$

for $0 \leq s \leq t \leq T$ and $F, G \in \mathcal{S}_b$.

$\left\{ \tilde{T}_t^b \right\}_{t>0}$ is the heat semigroup associated to $\frac{2}{3}\mathscr{E}_b$.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Scaling limit of density fluctuations: Non-equilibrium, Dirichlet case

$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \mathcal{L}_N^{\mathrm{boun}} \right).$$

Assumptions

1. $\{\mu_N\}_N$ is associated to a profile $\varrho : K \to [0,1]$.

2. $\displaystyle\sup_{x,y \in V_N} \left| \mathbb{E}_{\mu_N}[\bar{\eta}^N(x)\bar{\eta}^N(y)] \right| \lesssim |V_N|^{-1}$.

Let $\mathbb{Q}_{\mu_N}$ be the probability measure on $D([0,T], \mathcal{S}'_{\mathrm{Dir}})$ induced by the DFF $\mathcal{Y}^N_\cdot$ started from μ_N.
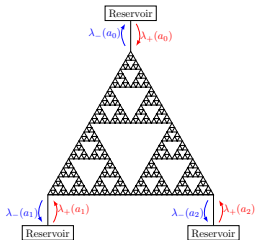
Theorem (NoneqFluct (C.–Franceschini–Gonçalves–Menezes '19+))

Under the above Assumptions, any limit point $\mathbb{Q}^*$ of $\{\mathbb{Q}_{\mu_N}\}_N$ concentrates on paths

$$\mathcal{Y}_t(F) = \mathcal{Y}_0(\tilde{\mathsf{T}}_t^{\mathrm{Dir}} F) + \mathcal{W}_t(F) \qquad \forall F \in \mathcal{S}_{\mathrm{Dir}},$$

where $\mathcal{Y}_0$ and $\mathcal{W}_t$ are uncorrelated mean-zero random fields, and $\mathcal{W}_t$ is Gaussian with variance
$\dfrac{2}{3} \cdot 2 \displaystyle\int_0^t \int_K \chi(\rho_s)\, d\Gamma(\tilde{T}_{t-s}^{\mathrm{Dir}} F)\, ds.$

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Scaling limit of density fluctuations: Non-equilibrium, Dirichlet case

$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \mathcal{L}_N^{\mathrm{boun}} \right).$$

Assumptions

1. $\{\mu_N\}_N$ is associated to a profile $\varrho : K \to [0, 1]$.

2. $\displaystyle\sup_{x,y \in V_N} \left| \mathbb{E}_{\mu_N}[\bar{\eta}^N(x)\bar{\eta}^N(y)] \right| \lesssim |V_N|^{-1}$.

3. $\mathcal{Y}_0^N \xrightarrow{d} \mathcal{Y}_0$ Gaussian.

Let $\mathbb{Q}_{\mu_N}$ be the probability measure on $D([0, T], \mathcal{S}'_{\mathrm{Dir}})$ induced by the DFF $\mathcal{Y}^N_{\cdot}$ started from μ_N.

Theorem (NoneqCLT (C.–Franceschini–Gonçalves–Menezes '19+)**)**

Under the above Assumptions, $\{\mathbb{Q}_{\mu_N}\}_N$ converges to a generalized O-U process *with covariance*

$$\mathbb{E}[\mathcal{Y}_t(F)\mathcal{Y}_s(G)] = \mathbb{E}\left[\mathcal{Y}_0(\tilde{T}_t^{\mathrm{Dir}} F)\mathcal{Y}_0(\tilde{T}_s^{\mathrm{Dir}} G) \right]$$

$$+ \frac{2}{3} \cdot 2 \int_0^s \int_K \chi(\rho_r) \, d\Gamma\left(\tilde{T}_{t-r}^{\mathrm{Dir}} F, \tilde{T}_{s-r}^{\mathrm{Dir}} G \right) \, dr$$

for $0 \leq s \leq t \leq T$ and $F, G \in \mathcal{S}_{\mathrm{Dir}}$.

Outline

Motivation: Generalizing the analysis of the exclusion process from 1D to higher dimensions

Boundary-driven exclusion process on the Sierpinski gasket

New tools & ideas for resistance spaces

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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New/old tools & ideas

Microscopics: Exclusion process on a non-lattice state space

NO translational invariance.

- How to carry out **local averaging** without using translation?
  Ans: Use the effective resistance for the random walk process, in conjunction with space-time scaling limits of random walks to a diffusion process (invariance principle).

- How to characterize **nonequilibrium correlations** $\phi(x, y) = \mathbb{E}[\bar{\eta}(x)\bar{\eta}(y)]$ in the exclusion process on a general graph?
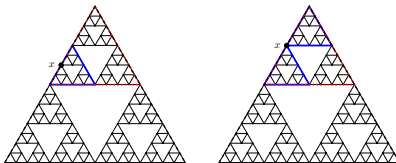 *Ans:* Identify ϕ as the solution to a discretized Poisson's equation on the product graph , and "invert the Laplacian."

Macroscopics: Analysis of (S)PDEs on fractals / metric measure spaces

NO explicit representation formulas, DELICATE notion of gradient ∇, but EXCELLENT notion of Laplacian Δ.

- Dirichlet forms for diffusion $\mathcal{E}(f, g) = \langle f, -\Delta g \rangle_m$, heat semigroup $\{T_t\}_{t>0}$
- Heat kernel bounds $p_t(x, y)$ (Nash ineq.), spectral asymptotics, Green's function $G(x, y)$.

Motivation
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Exclusion process on SG: Main results
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New tools & ideas for resistance spaces
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Summary
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Local averaging

For finite $\Lambda \subset V$, denote the **average density over** Λ by $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z \in \Lambda} \eta(z)$.

In the proof of the hydrodynamic limit for Markov processes, with generator $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$ on a sequence of graphs $G_N = (V_N, E_N)$, we use that for every $t > 0$:
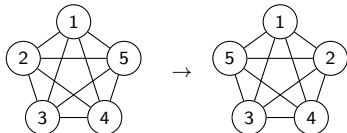
Replacement lemma

$$\overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{N \to \infty}} \, \mathbb{E}_{\mu_N} \left[\left| \int_0^t \left(\eta_s^N(x) - \mathrm{Av}_{B(x, \epsilon N)}[\eta_s^N] \right) ds \right| \right] = 0, \quad x \in V_N.$$

where

- $\{\eta_t^N : t \geq 0\}$ is the exclusion process generated by $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$, where $\mathcal{T}_N$ is the diffusive time acceleration factor.

- μ_N can be any measure on $\{0, 1\}^{V_N}$.

- $B(x, r)$ is a "ball" of radius r centered at x (in the graph metric).

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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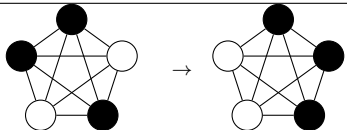○

Hierarchy of stochastic processes on a fixed graph

Interchange process $f : \{\text{Permutations on } V\} \to \mathbb{R}$
$$\mathcal{E}^{\mathrm{IP}}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw}[f(\eta^{zw}) - f(\eta)]^2 \, d\nu(\eta).$$
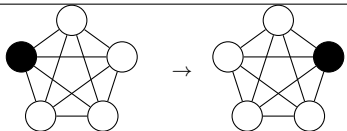Reversible measure: uniform measure ν on $\{\text{Perms on } V\}$.

↓ PROJECTION ↓

Exclusion process $f : \{0, 1\}^V \to \mathbb{R}$
$$\mathcal{E}^{\mathrm{EX}}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw}[f(\eta^{zw}) - f(\eta)]^2 \, d\nu_\alpha(\eta).$$
Reversible measure: product Bernoulli measure ν_α, $\alpha \in [0, 1]$,
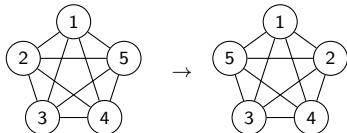$\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for all $x \in V$.

↓ PROJECTION ↓

Random walk process $f : V \to \mathbb{R}$
$$\mathcal{E}^{\mathrm{RW}}(f) = \sum_{zw \in E} c_{zw}[f(z) - f(w)]^2.$$

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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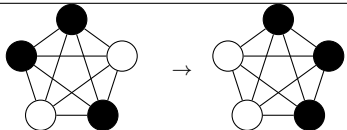
Hierarchy of stochastic processes on a fixed graph

Interchange process $f : \{\text{Permutations on } V\} \to \mathbb{R}$
$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{IP}}(f).$$
Moving particle lemma [C. *ECP* 2017]

↓ PROJECTION ↓

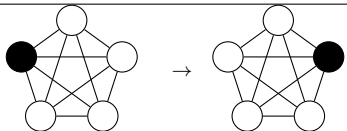**Exclusion process** $f : \{0, 1\}^V \to \mathbb{R}$
$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f).$$
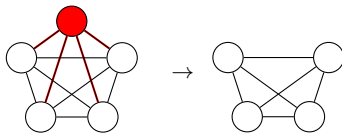**Moving particle lemma** [C. *ECP* 2017]

↓ PROJECTION ↓

Random walk process $f : V \to \mathbb{R}$
$$[f(x) - f(y)]^2 \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{RW}}(f).$$
Dirichlet principle [1867]

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Octopus inequality & Aldous' spectral gap conjecture

Using the network reduction idea & delicately carrying out a series of Schur complementations,
Caputo–Liggett–Richthammer JAMS '10 proved for the **interchange process**:

Theorem (Octopus inequality, IP (Caputo–Liggett–Richthammer *JAMS* '10))

For all $f : \mathcal{S}_{|V|} \to \mathbb{R}$,

$$\int \sum_{y \in V_x} c_{xy}[f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \geq \int \sum_{yz \in E_x} \tilde{c}_{yz}[f(\eta^{yz}) - f(\eta)]^2 \, d\nu(\eta).$$

Energy lost from removed edges ≥   Energy gained from increased conductances

---

This was the key inequality which resolved Aldous' '92 spectral gap conjecture:

$$\left\{ \begin{array}{l} \text{Projection argument gives } \lambda_2^{\mathrm{RW}}(G) \leq \lambda_2^{\mathrm{EX}}(G) \leq \lambda_2^{\mathrm{IP}}(G) \\ \text{(OI)} \implies \lambda_2^{\mathrm{IP}}(G) \geq \lambda_2^{\mathrm{RW}}(G) \end{array} \right\} \implies \lambda_2^{\mathrm{IP}}(G) = \lambda_2^{\mathrm{EX}}(G) = \lambda_2^{\mathrm{RW}}(G)$$

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Moving particle lemma for interchange/exclusion

Bounding the energy cost of swapping two particles at x and y in an interacting particle system by the **effective resistance** between x and y w.r.t. the random walk process.

Theorem (MPL, IP/EX (C. *ECP* '17))

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{IP}}(f), \quad f : \mathcal{S}_{|V|} \to \mathbb{R},$$

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f), \quad f : \{0, 1\}^V \to \mathbb{R}.$$

Proof.

- (OI) $\Leftrightarrow$ monotonicity of energy under 1-point network reductions. So reduce G successively until two vertices x, y are left, we get MPL for IP.

- A further projection argument yields the MPL for EX.

Moving particle lemma for interchange/exclusion

Bounding the energy cost of swapping two particles at x and y in an interacting particle system by the **effective resistance** between x and y w.r.t. the random walk process.

Theorem (MPL, IP/EX (C. *ECP* '17))

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{IP}}(f), \quad f : \mathcal{S}_{|V|} \to \mathbb{R},$$

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f), \quad f : \{0, 1\}^V \to \mathbb{R}.$$
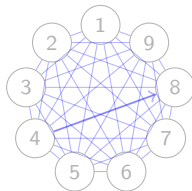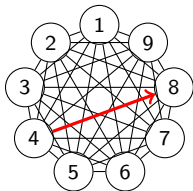
Conventional approach is to pick a shortest path connecting x and y, and telescope along the path to obtain the energy cost. [Guo–Papanicolaou–Varadhan '88, Diaconis–Saloff-Coste '93].
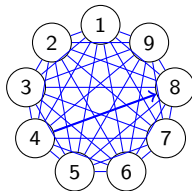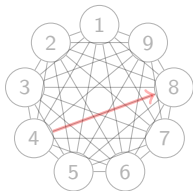
OK on finite integer lattices, but does NOT always give optimal cost on general weighted graphs.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Moving particle lemma for interchange/exclusion

Bounding the energy cost of swapping two particles at x and y in an interacting particle system by the **effective resistance** between x and y w.r.t. the random walk process.

Theorem (MPL, IP/EX (C. *ECP* '17))

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{IP}}(f), \quad f : \mathcal{S}_{|V|} \to \mathbb{R},$$

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MPL bounds the energy cost by "optimizing electric flow over all paths connecting x and y."

Motivation
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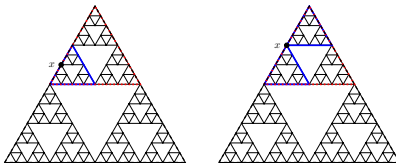
Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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MPL & local averaging

For finite $\Lambda \subset V$, denote the **average density over** Λ by $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z \in \Lambda} \eta(z)$.

In the proof of the hydrodynamic limit for Markov processes, with generator $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$ on a sequence of graphs $G_N = (V_N, E_N)$, we use that for every $t > 0$:

Replacement lemma

$$\overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{N \to \infty}} \, \mathbb{E}_{\mu_N} \left[\left| \int_0^t \left(\eta_s^N(x) - \mathrm{Av}_{B(x, \epsilon N)}[\eta_s^N] \right) ds \right| \right] = 0, \quad x \in V_N.$$

$$\eta(x) - \mathrm{Av}_B[\eta] = \frac{1}{|B|} \sum_{z \in B} \left(\eta(x) - \eta(z) \right).$$

Estimating this cost using the variational characterization of the largest eigenvalue requires telescoping or MPL. Works for resistance spaces; UNCLEAR if there is an analog of this for $d_{\mathrm{spec}} \geq 2$.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Two-point correlation functions, nonequilibrium

- μ_{ss}^N: unique invariant measure for $5^N \mathcal{L}_N^{\mathrm{bEX}}$, $b = 1$.

- Steady-state density: $\rho_{\mathrm{ss}}^N(x) = \mathbb{E}_{\mu_{\mathrm{ss}}^N}[\eta(x)]$.

- Steady-state correlation: $\phi_{\mathrm{ss}}^N(x, y) = \mathbb{E}_{\mu_{\mathrm{ss}}^N}[(\eta(x) - \rho_{\mathrm{ss}}^N(x))(\eta(y) - \rho_{\mathrm{ss}}^N(y))]$.

  Related to the local time for two particles in EX to stay adjacent to each other.

- In 1D, $\phi_{\mathrm{ss}}^N(x, y)$ is exactly a multiple of the Green's function for RW, $-\frac{1}{N-1}\mathsf{G}^N(x, y)$.

- How to find $\phi_{\mathrm{ss}}^N(x, y)$ on *SG*? Or on a general graph?

Poisson's eqn on the product graph

$$\begin{cases} \boldsymbol{\Delta}_N \phi_{\mathrm{ss}}^N(x, y) = \mathbb{1}_{\{x \sim y\}} 5^N \left(\rho_{\mathrm{ss}}^N(x) + \rho_{\mathrm{ss}}^N(y) - 2\rho_{\mathrm{ss}}^N(x)\rho_{\mathrm{ss}}^N(y) - 2\phi_{\mathrm{ss}}^N(x, y) \right), & x, y \in V_N \setminus V_0, \ x \neq y, \\ \boldsymbol{\Delta}_N \phi_{\mathrm{ss}}^N(x, x) = 2 \cdot 5^N \sum_{y \sim x} \left(\phi_{\mathrm{ss}}^N(x, y) - \chi \left(\rho_{\mathrm{ss}}^N(x) \right) \right), & x \in V_N \setminus V_0, \\ \left((\partial_N^\perp \phi_{\mathrm{ss}}^N)(x, \cdot) \right)(a) = \left((\partial_N^\perp \phi_{\mathrm{ss}}^N)(\cdot, x) \right)(a) = -\frac{5^N}{3^N} \lambda_\Sigma(a) \phi_{\mathrm{ss}}^N(x, a), & a \in V_0. \end{cases}$$

Source term is nonzero only if x and y are adjacent.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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"Invert the Laplacian" to solve for the correlation (in terms of the Green's function G^N)

$$\phi_{ss}^N(x, y) = -\frac{5^N}{|V_N|^2} \sum_{x' \in V_N} \sum_{y' \sim x'} G^N(x, x') G^N(y, y')(\rho_{ss}^N(x') - \rho_{ss}^N(y'))^2$$

$$+ \frac{1}{|V_N|} G^N(x, y) \left(\chi(\rho_{ss}^N(x)) + \chi(\rho_{ss}^N(y)) \right) - \frac{2}{|V_N|^2} \sum_{a \in V_0} \lambda_\Sigma(a) G^N(x, a) G^N(y, a) \chi(\rho_{ss}^N(a))$$

$$- \frac{5^N}{|V_N|^2} \sum_{x' \in V_N} \sum_{y' \sim x'} \phi_{ss}^N(x', y') \left[G^N(x, x') - G^N(x, y') \right] \left[G^N(y, x') - G^N(y, y') \right].$$

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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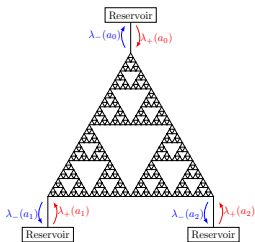
After some estimates we get

Lemma

There exists a positive constant $C = C(\rho_{\mathrm{ss}})$ such that for all N and $x, y \in V_N$,

$$|\phi_{\mathrm{ss}}^N(x, y)| \le \frac{C}{|V_N|}\max\left\{\mathsf{G}^N(x, y), \sup_{(x', y') \in V_N^2 : x' \sim y'} \mathsf{G}^N(x, x')\mathsf{G}^N(y, y')\right\}.$$

Correlation scales as (inverse volume)×(Green's function for RW).
This Lemma (and its time-dependent version) is needed to establish tightness/convergence of the density fluctuation field in non-equilibrium.

Motivation
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Exclusion process on *SG*: Main results
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New tools & ideas for resistance spaces
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Summary
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Summary, and **Thank you!**

$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left(\mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

Symmetric exclusion process with **slowed** boundary on the Sierpinski gasket

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)

- (Non)equilibrium density hydrodynamic limit (DRN✓) [C.–Gonçalves '19]
- Ornstein-Uhlenbeck limit of equilibrium density fluctuations (DRN✓). [C.–Gonçalves '19]
- Large deviations principle for the (non)equilibrium density (D✓) [C.–Hinz '19+]
- Hydrostatic limit, scaling limit of nonequilibrium density fluctuations (D✓RN?). [C.–Franceschini–Gonçalves–Menezes '19+]

Future directions

- Generalization to *any* resistance space (with a good theory of boundary-value problems).
- Incorporate asymmetry in the exclusion jump rates → microscopic derivation of stochastic Burgers' equation on resistance spaces.