Hydrodynamic limit of the boundary-driven exclusion process on the Sierpinski gasket

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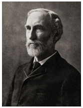
arXiv:1606.01677, 1702.03376, 1705.10290, 181x.yyyyy (2x)

Hydrodynamic limits

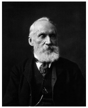
- **Goal:** Rigorous derivation of macroscopic fluid equations (*e.g.* Navier-Stokes) from microscopic systems (*e.g.* a volume of interacting water molecules).
- **Technical ingredients: Coarse-graining**, **ergodicity**. Both can be hard to verify.
- **A tractable approach:** Build *stochastic* interactions into the microscopic model, thereby ensuring ergodicity in the first place.
    - Examples of models: Exclusion process, zero-range process, stochastic Ising models, etc.
- This **hydrodynamic program**, started in the 80s by Kipnis, Olla, Varadhan, etc., and continues to this day, involves many areas of mathematics.
    - Stochastic processes (random walks, jump processes, diffusion processes), analysis of nonlinear (S)PDEs, potential theory, discrete harmonic analysis.

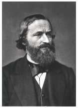
  It also addresses, and is inspired from, questions from (non-)equilibrium statistical physics: *e.g.* density flow and fluctuations in an out-of-equilibrium physical system.
- Throughout the talk, I will mention new results connecting **interacting particle systems** to classical ideas from **random walks on graphs**, esp. the role of the **effective resistance**.

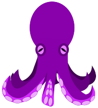

Boltzmann


Gibbs


Dirichlet

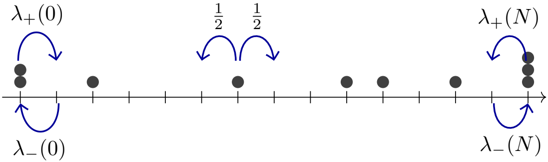

Thomson


Kirchhoff


Octopus?!

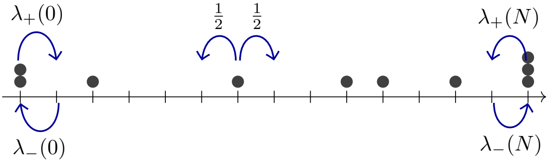
The boundary-driven exclusion process in 1D

- Identical particles performing random walks on $V_N := (0, N) \cap \mathbb{Z}$.
- In the bulk, $(0, N) \cap \mathbb{Z}$, particles undergo **exclusion dynamics**: No two random walkers can occupy the same vertex.
- At each boundary vertex $y \in \{0, N\}$ ("reservoir"), particles can be injected into or extracted from the bulk at respective rate $\lambda_+(y)$ and $\lambda_-(y)$.
- Denote by $\{\eta_t^N : t \geq 0\}$ the corresponding Markov chain on $\{0, 1\}^{V_N}$.
- **Goal:** Study the scaling limit of functionals of $\left\{\eta^N_{N^2 t} : t \in [0, T]\right\}$ under the diffusive time scale N^2 and space scale N.



The boundary-driven exclusion process in 1D

- The boundary rates $\lambda_\pm(y)$, $y \in \{0, N\}$ control the steady-state density:
  - On the boundary: When $y \in \{0, N\}$, $\bar{\rho}(y) = \dfrac{\lambda_+(y)}{\lambda_+(y) + \lambda_-(y)}$.
  - In the bulk: When $x \in (0, N)$, $\bar{\rho}(x)$ is the harmonic extension of $\bar{\rho}$ from $\{0, N\}$ to $\{0, 1, \cdots, N\}$.
- By default, ergodicity is guaranteed: (limit) process has a unique invariant measure whose one-site marginal at x is $\bar{\rho}(x)$.
- If $\lambda_\pm(y)$ for all $y \in \{0, N\}$, $\bar{\rho}$ is constant everywhere: *equilibrium*
- Else: the one-site marginal $\bar{\rho}$ interpolates between a "hot" reservoir and a "cold" one: *out of equilibrium*



The boundary-driven exclusion process in 1D

- Empirical density measure on $[0,1]$: $\pi_t^N = \sum_{x \in V_N} \eta^N_{tN^2}(x)\delta_{x/N}$.
- **LLN** for the symmetric exclusion process: Assume $\pi_0^N \xrightarrow[N\to\infty]{} \rho_0\, dx$ weakly. Then [Eyink–Lebowitz–Spohn '90] $\pi_t^N \xrightarrow[N\to\infty]{} \rho(t,x)dx$ where ρ is the unique sol'n of the heat eqn

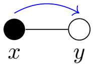
$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho & \text{on } [0,T]\times(0,1) \\ \rho(0,\cdot) = \rho_0 & \text{on } [0,1] \\ \rho(\cdot,\cdot) = \bar{\rho} & \text{on } [0,T]\times\{0,1\} \end{cases}$$

- Fluctuations about the LLN are captured via a **large deviations principle**: [Bertini–De Sole–Gabrielli–Jona-Lasinio–Landim '03, '07]
For each closed set $\mathcal{C}$ and each open set $\mathcal{O}$ of the Skorokhod space $D([0,T],\mathcal{M}_+)$,

$$\limsup_{N\to\infty} \frac{1}{N}\log Q^N[\mathcal{C}] \leq -\inf_{\pi\in\mathcal{C}} I_0(\pi), \quad \liminf_{N\to\infty}\frac{1}{N}\log Q^N[\mathcal{O}] \geq -\inf_{\pi\in\mathcal{O}} I_0(\pi).$$

$I_0(\pi) < \infty \Leftrightarrow \pi \in \mathcal{M}_{+,1}$; then $\exists H$ s.t. $I_0(\pi) = \frac{1}{2}\int_0^T \int_{[0,1]} \rho(t,x)[1-\rho(t,x)]|\nabla H_t|^2\, dx\, dt$.

Here $H \in C^{1,2}([0,T]\times[0,1])$ is the **weak asymmetry** introduced into the exclusion process in the bulk.



with rate $\frac{1}{2}e^{N[H_t(y)-H_t(x)]} \approx \frac{1}{2}(1+N[H_t(y)-H_t(x)])$

The boundary-driven exclusion process in 1D

- Empirical density measure on $[0,1]$: $\pi_t^N = \sum_{x \in V_N} \eta^N_{tN^2}(x)\delta_{x/N}$.
- **LLN** for the weakly asymmetric exclusion process: Assume $\pi_0^N \underset{N\to\infty}{\longrightarrow} \rho_0\, dx$ weakly. Then [Eyink–Lebowitz–Spohn '91] $\pi_t^N \underset{N\to\infty}{\longrightarrow} \rho(t,x)dx$ where ρ is the unique sol'n of the semilinear heat eqn

$$\begin{cases} \partial_t \rho = \frac{1}{2}\Delta\rho + \nabla\cdot(\rho(1-\rho)\nabla H_t) & \text{on } [0,T]\times(0,1) \\ \rho(0,\cdot) = \rho_0 & \text{on } [0,1] \\ \rho(\cdot,\cdot) = \bar{\rho} & \text{on } [0,T]\times\{0,1\} \end{cases}$$

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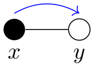
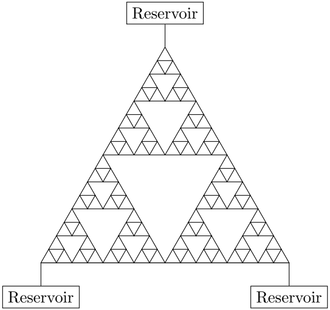


with rate $\frac{1}{2}e^{N[H_t(y)-H_t(x)]} \approx \frac{1}{2}(1+N[H_t(y)-H_t(x)])$

Generalization from 1D to higher dimensions

- Finite box $([0, N] \cap \mathbb{Z})^d$, $d \geq 2$: Analysis of fluctuations is more difficult than the $d = 1$ case, few results.
- State space with a **finite boundary set**: if it comes with 2 disjoint boundary components, then analysis is similar to the 1D case. [Akkermans–Bodineau–Derrida–Shpielberg '13; Bertini–De Sole–Gabrielli–Jona-Lasinio–Landim, *Macroscopic Fluctuation Theory* '15]
  Interesting when there are ≥ 3 disjoint boundary components (hot, lukewarm, cold).
- For the analysis to succeed, we need good knowledge of scaling limits of random walks to diffusion on the candidate state space.
- This leads to our candidate: the Sierpinski gasket



Abstract setup: Exclusion process on a weighted graph

Let $G = (V, E)$ be a connected graph endowed with conductances $\mathbf{c} = (c_{xy})_{xy \in E}$.

The symmetric exclusion process on $(G, \mathbf{c})$ is a Markov chain on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}}_{(G,\mathbf{c})} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

Properties:

1. Total particle number is conserved in the process.
2. Each product Bernoulli measure ν_α, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure for this process.

Dirichlet energy: $\mathcal{E}^{\mathrm{EX}}_{(G,\mathbf{c}),\nu_\alpha}(f) = \dfrac{1}{2} \displaystyle\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta).$

Weakly asymmetric exclusion process: Let $H : [0, T] \times V \to \mathbb{R}$ and $H_t = H(t, \cdot)$. Generator

$$(\mathcal{L}^{\mathrm{EX}}_{(G,\mathbf{c}),H} f)(\eta) = \sum_{xy \in E} c_{xy} \psi_{xy}(H_t, \eta) (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $\psi_{xy}(H, \eta) = \eta(x)[1 - \eta(y)] e^{H(y) - H(x)} + \eta(y)[1 - \eta(x)] e^{H(x) - H(y)}$.

Abstract setup: Boundary-driven exclusion process

Declare a subset ∂V of V to be the boundary set. Assume WLOG that $c_{aa'} = 0$ for all $a, a' \in \partial V$.

Add a **birth-and-death chain** to each $a \in \partial V$.
At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth); At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).
Formally,

$$(\mathcal{L}^{\mathrm{b}}_{\partial V} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1-\eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R},$$

where

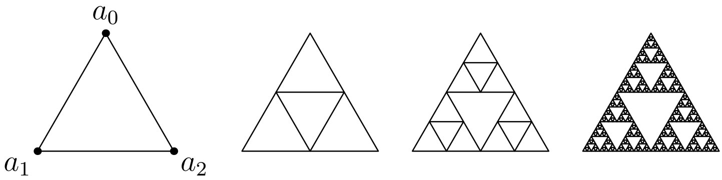
$$\eta^a(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

The boundary-driven weakly asymmetric exclusion process has generator

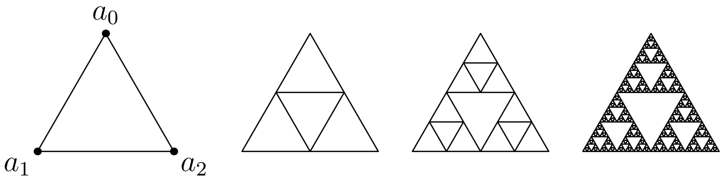
$$\mathcal{L}^{\mathrm{bEX}}_{(G,\mathbf{c}),H} = \mathcal{L}^{\mathrm{EX}}_{(G,\mathbf{c}),H} + \mathcal{L}^{\mathrm{b}}_{\partial V}.$$

Remark. Due to the boundary effect, the invariant measure is no longer product Bernoulli. Nevertheless its one-site marginals $\bar{\rho}$ are known, being the harmonic extension of $\bar{\rho}$ from ∂V to V.

(Single-particle) diffusion on SG



- SG_N: Level-N Sierpinski gasket graph.
- m_N: self-similar probability measure on SG_N, assigns weight to vertex x which is proportional to $\deg(x)$.
- m_N converges weakly to m, the standard self-similar probability measure (with Hausdorff dimension $\log_2 3$), on the limit fractal K.
- $(X^N_t)_{t\geq 0}$: symmetric random walk process on SG_N.
- [Goldstein '87, Kusuoka '87, Barlow–Perkins '88]: Probability on fractals
  $X^N_{5^N t} \underset{N\to\infty}{\longrightarrow} B_t$, called a Brownian motion on SG.



- [Kigami '89+]: Analysis on fractals
  Write down the Dirichlet energy on SG_N, renormalized by $(5/3)^N$:

$$\mathcal{E}_N(f) = \left(\frac{5}{3}\right)^N \sum_{x\sim y} [f(x) - f(y)]^2 \qquad (f : K \to \mathbb{R})$$

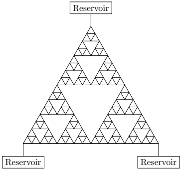
  Then $\{\mathcal{E}_N\}_N$ is a *monotone increasing* sequence, and hence has a limit $\mathcal{E}$.
- Let $\mathcal{F}$ be the domain of finite energy $\mathcal{E}$. $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form.
- **Operator convergence:** If Δ_N denotes the graph Laplacian on SG_N, then can prove the pointwise formula $\Delta = \lim_{N\to\infty} 5^N \Delta_N$, where Δ is the generator of B_t.
- *Key Remark.* The Euclidean formula "$\mathcal{E}(f) = \int_K |\nabla f|^2 \, dm$" does NOT exist literally. Rather it should be understood in terms of energy measure: $\mathcal{E}(f) = \int_K d\Gamma(f,f)$; note that $d\Gamma(f,f) \perp dm$ [Kusuoka '89]. **This complicates the analysis of the scaling limit of the exclusion process on *SG*.**

A general analytic framework: Resistance spaces

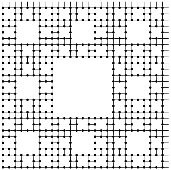
- The lack of translational invariance of the state space also poses a technical obstacle: need another property to generate **local ergodicity**.
- Knowing that the state space is bounded in the **electrical resistance** metric turns out to be a good one. ($\Leftrightarrow$ diffusion is **strongly recurrent**, in the sense of Barlow, Delmotte, Telcs, ...)

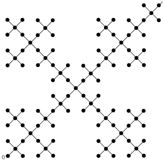
Examples of **resistance spaces**

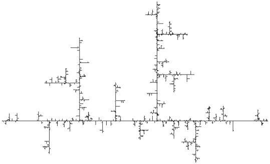
- Classical Dirichlet form $\int_\Omega |\nabla f|^2 \, dx$ on $L^2(\Omega, dx)$ is a resistance form $\Leftrightarrow$ Ω has Euc dim 1.
- Other examples:



Sierpinski gasket


Sierpinski carpet


Vicsek tree


Random dendrite [by David Croydon]

- The lack of translational invariance of the state space also poses a technical obstacle: need another property to generate **local ergodicity**.
- Knowing that the state space is bounded in the **electrical resistance** metric turns out to be a good one. ($\Leftrightarrow$ diffusion is **strongly recurrent**, in the sense of Barlow, Delmotte, Telcs, ...)

Definition. [Kigami, *Analysis on Fractals* '01, also '04]
Let K be a nonempty set. A **resistance form** $(\mathcal{E}, \mathcal{F})$ on K is a pair such that

1. $\mathcal{F}$ is a vector space of $\mathbb{R}$-valued functions on K containing the constants, and $\mathcal{E}$ is a nonnegative definite symmetric quadratic form on $\mathcal{F}$ satisfying

$$\mathcal{E}(u,u) = 0 \ \Leftrightarrow \ u \text{ is constant.}$$

2. $\mathcal{F}/\{\text{constants}\}$ is a Hilbert space with norm $\mathcal{E}(u,u)^{1/2}$.
3. Given a finite subset $V \subset K$ and a function $v : V \to \mathbb{R}$, there is $u \in \mathcal{F}$ s.t. $u|_V = v$.
4. For $x, y \in K$, the effective resistance

$$R_{\rm eff}(x,y) := \sup\left\{ \frac{[u(x)-u(y)]^2}{\mathcal{E}(u,u)} : u \in \mathcal{F},\ \mathcal{E}(u,u) > 0 \right\} < \infty.$$

5. (Markovian property) If $u \in \mathcal{F}$, then $\bar{u} := 0 \vee (u \wedge 1) \in \mathcal{F}$ and $\mathcal{E}(\bar{u},\bar{u}) \le \mathcal{E}(u,u)$.

- The lack of translational invariance of the state space also poses a technical obstacle: need another property to generate **local ergodicity**.
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Definition (Resistance form).

4. For $x, y \in K$,

$$R_{\rm eff}(x,y) := \sup\left\{\frac{[u(x)-u(y)]^2}{\mathcal{E}(u,u)} : u \in \mathcal{F},\ \mathcal{E}(u,u) > 0\right\} < \infty.$$

Remarks.

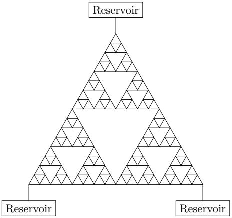
- The resistance space $(K, R_{\rm eff})$ is a metric space. Can always assumed to be complete.
- From the definition of the effective resistance it follows that $|u(x)-u(y)|^2 \le R_{\rm eff}(x,y)\mathcal{E}(u,u)$, which then implies the Sobolev embedding $\mathcal{F} \subset C(K)$.
- **Analytic consequences:** leads to new functional inequalities for the exclusion process; local ergodicity; $\exists$ and ! of the hydrodynamic PDE.

Boundary-driven WASEP: generator $5^N \mathcal{L}^{\mathrm{bEX}}_{SG_N,H}$, law Q^N_H.
Take any $H \in C([0,T], G(\mathcal{F}_0)) \cap C^1((0,T), G(\mathcal{F}_0))$.

- $\mathcal{F}_0$: domain of the Dirichlet Laplacian.
- G : (image of the) Dirichlet Green's operator.

Empirical density measure $\pi^N_t := \frac{1}{|V_N|} \sum_{x \in V_N} \eta^N_t(x) \delta_x$

$\mathcal{M}_+$: Space of nonnegative Borel measures on K.



Theorem (Density LLN for the boundary-driven WASEP)

Suppose $\pi^N_0 \to \rho_0\, dm$ weakly. Then w.p.1, $\pi^N_t \to \rho(t,x)\, dm(x)$ in the Skorokhod topology on $D([0,T], \mathcal{M}_+)$, where $\rho(t,x)$ is the unique weak solution of the semilinear heat equation

$$\begin{cases} \partial_t \rho_t = \Delta \rho_t - \partial^* \left(\chi(\rho_t) \partial H_t \right) & \text{on } (0,T) \times K \setminus V_0, \\ \rho(0,\cdot) = \rho_0 & \text{on } K \setminus V_0, \\ \rho(t,\cdot) = \bar{\rho} & \text{on } (0,T) \times V_0. \end{cases}$$

Here $\chi(\alpha) = \alpha(1-\alpha)$, and ∂ (resp. the adjoint $-\partial^*$) is the abstract gradient (resp. divergence) operator induced by the resistance form $(\mathcal{E}, \mathcal{F})$ on SG.

Remark. Due to the energy singularity, $\exists$ and ! of the PDE sol'n is proved using the monotone operator method of J.-L. Lions + input from resistance form theory. $\rho \in L^2(0,T,\mathcal{F})$.

Hydrodynamic limit theorems on SG [C.–Hinz–Teplyaev, arXiv:181x.yyyyy]

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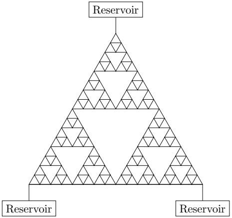


Theorem (Density LDP: rate $|V_N| \sim \frac{3}{2} 3^N$ with good rate fcn I_0)

For each closed set $\mathcal{C}$ and each open set $\mathcal{O}$ of the Skorokhod space $D([0,T], \mathcal{M}_+)$, endowed with the Skorokhod topology of weak convergence of measures with bounded energy,

$$\limsup_{N\to\infty} \frac{1}{|V_N|} \log Q^N[\mathcal{C}] \leq -\inf_{\pi \in \mathcal{C}} I_0(\pi), \quad \liminf_{N\to\infty} \frac{1}{|V_N|} \log Q^N[\mathcal{O}] \geq -\inf_{\pi \in \mathcal{O}} I_0(\pi).$$

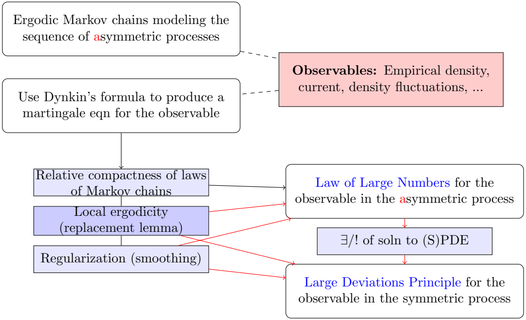
Let $\mathcal{M}_{+,1} = \{\mu \in \mathcal{M}_+ \mid \mu(dx) = \rho(x)\, m(dx),\ 0 \leq \rho \leq 1 \ m\text{-a.e.}\}$ and

$$D_{+,1,\mathcal{E}}[0,T] := \left\{\pi \in D([0,T], \mathcal{M}_{+,1}) \mid \pi(t,dx) = \rho(t,x)\, m(dx),\ \rho \in L^2(0,T,\mathcal{F})\right\}.$$

$I_0(\pi) < \infty \Leftrightarrow \pi \in D_{+,1,\mathcal{E}}[0,T]$; then $\exists H$ s.t. $\boxed{I_0(\pi) = \dfrac{1}{2} \displaystyle\int_0^T \int_K \chi(\rho(t,x))\, d\Gamma(H_t, H_t)\, dt}$.

Technical Remark. The topology we use guarantees that $D_{+,1,\mathcal{E}}[0,T]$ is closed.

Proof: The entropy method [Guo–Papanicolaou–Varadhan '88, Kipnis–Olla–Varadhan '89]



- This **hydrodynamic program** has been carried out on $\mathbb{Z}^d$ since the late 80's, described in the book [Kipnis–Landim '99].
- **Challenges:** Extend the program to non-translationally-invariant (and possibly energy singular) spaces.
- *Alternative method:* H.-T. Yau's relative entropy method, was used to prove LLN of the symmetric zero-range process on SG in [Jara '09].
- **Our contribution:** Established the entropy method on SG, which is more robust (enables the proof of LDP) and leads to further consequences.

Microscopic model

The **empirical density** measure on SG_N:

$$\pi_t^N := \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x)\delta_x$$

Let Q_H^N be the law of the Markov process generated by $5^N \mathcal{L}_{N,H}^{\mathrm{bEX}}$.

Use Dynkin's formula to find that under Q_H^N, for all "nice" test functions F,

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \int_0^t \langle \pi_s^N, \Delta_N F\rangle \, ds + \int_0^t \sum_{a \in V_0} \eta_s^N(a)(\partial_N^\perp F)(a)\, ds$$
$$+ \frac{2}{3} \int_0^t \frac{5^N}{3^N} \sum_{xy \in E_N} \chi(\eta_s^N, xy)[H_s(x) - H_s(y)][F_s(x) - F_s(y)]\, ds + \underbrace{M_t^{N,F}}_{\text{martingale}} .$$

The martingale term $M_t^{N,F}$ has quadratic variation vanishing as $N \to \infty$.

- $(\Delta_N F)(x) = \dfrac{5^N}{3^N} \displaystyle\sum_{y \sim x} [F(y) - F(x)]$: (renormalized) Laplacian.
- $(\partial_N^\perp F)(a) = \dfrac{2}{3}\dfrac{5^N}{3^N} \displaystyle\sum_{y \sim a} [F(y) - F(a)]$: (renormalized) normal derivative at $a \in V_0$.
- $\chi(\eta, xy) = \eta(x)[1 - \eta(y)] + \eta(y)[1 - \eta(x)]$ is the conductivity of the exclusion process.

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \int_0^t \langle \pi_s^N, \Delta_N F\rangle\, ds + \int_0^t \sum_{a\in V_0} \eta_s^N(a)\, (\partial_N^\perp F)(a)\, ds$$
$$+ \frac{2}{3}\int_0^t \frac{5^N}{3^N} \sum_{xy\in E_N} \chi(\eta_s^N, xy)\, [H_s(x) - H_s(y)][F_s(x) - F_s(y)]\, ds + \underbrace{M_t^{N,F}}_{\text{martingale}} .$$

Goal: Show that $\{Q_H^N\}_N$ is relatively compact, and the limit point Q_H^* concentrates on a.c. trajectories $\pi(t, dx) = \rho(t, x)\, m(dx)$ with $\rho \in L^2(0, T, \mathcal{F})$ and

$$\langle \pi_t, F\rangle = \langle \pi_0, F\rangle + \int_0^t \langle \pi_s, \Delta F\rangle\, ds$$
$$+ \int_0^t \sum_{a\in V_0} \bar{\rho}(a)\, (\partial^\perp F)(a)\, ds + \int_0^t \int_K \chi(\rho_s)\, d\Gamma(H_s, F)\, ds.$$

- $\chi(\rho) := \rho(1-\rho)$: conductivity in the exclusion process.
- $d\Gamma(H, F)$ is the **mutual energy measure** of $H, F \in \mathcal{F}$, analog of "$(\nabla H \cdot \nabla F)\, dx$." This is part of a broader theory of first-order calculus on Dirichlet spaces [Cipriani–Sauvageot '03, Hinz–Röckner–Teplyaev '13].

From the microscopic model to the macroscopic limit

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \int_0^t \langle \pi_s^N, \Delta_N F\rangle \, ds + \int_0^t \sum_{a\in V_0} \eta_s^N(a) \, (\partial_N^\perp F)(a) \, ds$$

$$+ \frac{2}{3}\int_0^t \frac{5^N}{3^N} \sum_{xy\in E_N} \chi(\eta_s^N, xy) \, [H_s(x) - H_s(y)][F_s(x) - F_s(y)] \, ds + \underbrace{M_t^{N,F}}_{\text{martingale}} .$$

Goal: Show that $\{Q_H^N\}_N$ is relatively compact, and the limit point Q_H^* concentrates on a.c. trajectories $\pi(t, dx) = \rho(t,x)\, m(dx)$ with $\rho \in L^2(0, T, \mathcal{F})$ and

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$$+ \int_0^t \sum_{a\in V_0} \bar{\rho}(a) \, (\partial^\perp F)(a)\, ds + \int_0^t \int_K \chi(\rho_s) \, d\Gamma(H_s, F)\, ds.$$

Observe this is nothing but the weak formulation of the semilinear heat equation

$$\begin{cases} \partial_t \rho_t = \Delta \rho_t - \partial^* \left(\chi(\rho_t)\partial H_t\right) & \text{on } (0,T)\times K\setminus V_0, \\ \rho(0,\cdot) = \rho_0 & \text{on } K\setminus V_0, \\ \rho(t,\cdot) = \bar{\rho} & \text{on } (0,T)\times V_0. \end{cases}$$

From the microscopic model to the macroscopic limit

$$\langle \pi_t^N, F\rangle = \langle \pi_0^N, F\rangle + \int_0^t \langle \pi_s^N, \Delta_N F\rangle\, ds + \int_0^t \sum_{a\in V_0} \eta_s^N(a)\, (\partial_N^\perp F)(a)\, ds$$

$$+ \frac{2}{3}\int_0^t \frac{5^N}{3^N} \sum_{xy\in E_N} \chi(\eta_s^N, xy)\, [H_s(x) - H_s(y)][F_s(x) - F_s(y)]\, ds + \underbrace{M_t^{N,F}}_{\text{martingale}}\, .$$

Goal: Show that $\{Q_H^N\}_N$ is relatively compact, and the limit point Q_H^* concentrates on a.c. trajectories $\pi(t, dx) = \rho(t,x)\, m(dx)$ with $\rho \in L^2(0, T, \mathcal{F})$ and

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$$+ \int_0^t \sum_{a\in V_0} \bar{\rho}(a)\, (\partial^\perp F)(a)\, ds + \int_0^t \int_K \boldsymbol{\chi}(\rho_s)\, d\Gamma(H_s, F)\, ds.$$

Key issues: Terms on the RHS are NOT ALL in terms of the empirical density π^N. Need to make the following replacements:

- Conductivity term: Replace $\frac{1}{2}\chi(\eta_s^N, \cdot)$ by $\boldsymbol{\chi}\left(\mathrm{Av}_{B(\cdot, r_\epsilon N)}[\eta_s^N]\right)$ and then by $\boldsymbol{\chi}(\rho_s)$.
- Boundary term: Replace $\eta_s^N(a)$ by $\bar{\rho}(a)$.

Replacement of the nonlinear conductivity term via local ergodicity

Basic idea: Replace functionals of η_t^N by coarse-grained functionals of the empirical density π_t^N, with negligible cost in the scaling limit.

- Call $\phi : V(\Gamma) \times \{0,1\}^{V(\Gamma)} \to \mathbb{R}$ is a **local function bundle** if $\exists r \in (0,\infty)$ such that $\phi(x,\cdot)$ depends only on $\{\eta(z) : z \in B(x,r)\}$.
  - Examples of local function bundles: $\eta(x)$; $\sum_{y\sim x} \eta(x)\eta(y)$.
- Given ϕ and x, define $\Phi_x(\alpha) = \int \phi(x,\eta)\, d\nu_\alpha(\eta)$. Let

$$U_{N,\epsilon}(x,\eta) := \underbrace{\phi(x,\eta)}_{\text{microscopic}} - \underbrace{\Phi_x\left(\mathrm{Av}_{B(x,r_\epsilon N)}[\eta]\right)}_{\text{macro avg}}.$$

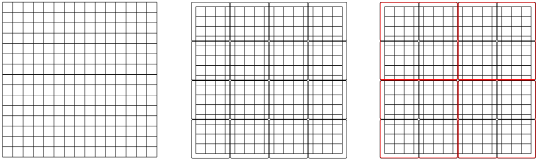
Local ergodicity (*a.k.a.* local equilibrium, replacement lemma)

For each $T>0$ and each $\delta>0$,

$$\limsup_{\epsilon\downarrow 0} \limsup_{N\to\infty} \sup_{x\in V_N} \frac{1}{|V_N|} \log Q_H^N \left\{ \left| \int_0^T U_{N,\epsilon}(x,\eta_t^N)\, dt \right| > \delta \right\} = -\infty.$$

- The local ergodicity theorem was proved in the more general setting of resistance spaces (viz. strongly recurrent weighted graphs) in [C. '17]. A "low-dimensional ($d<2$)" result.
- All assumptions derive from potential theory of random walks on graphs. Nothing is assumed about the spatial symmetries of the underlying state space.

Technical estimates: 1-block and 2-blocks estimates



$\eta(x)$ $\quad$ $\mathrm{Av}_{\Lambda_j(x)}[\eta]$ $\quad$ $\mathrm{Av}_{B(x,r_{\epsilon N})}[\eta]$

Strategy [GPV '88, KOV '89]: implement a two-scale coarse-graining procedure.

$$U_{N,\epsilon}(x,\eta) := \underbrace{\left[\phi(x,\eta) - \Phi_x\left(\mathrm{Av}_{\Lambda_j(x)}[\eta]\right)\right]}_{U^{(1)}_{N,j} \quad \text{1-block}} + \underbrace{\left[\Phi_x\left(\mathrm{Av}_{\Lambda_j(x)}[\eta]\right) - \Phi_x\left(\mathrm{Av}_{B(x,r_{\epsilon N})}[\eta]\right)\right]}_{U^{(2)}_{N,j,\epsilon} \quad \text{2-blocks}}$$

- j sets the microscopic scale.
- $\epsilon \in [0,1]$ sets the macroscopic aspect ratio.
- Ordering of limits: $N \to \infty$, then $\epsilon \downarrow 0$, then $j \to \infty$.

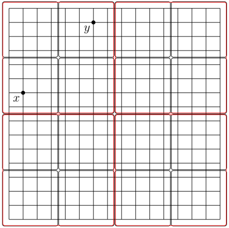
Separately show that $U^{(1)}_{N,j}$ and $U^{(2)}_{N,j,\epsilon}$ vanishes in the said limit with probability superexponentially close to 1 $\to$ Requires the Feynman-Kac formula, a spectral gap estimate, and a local limit thm (equivalence of ensembles).

2-blocks estimate: A closer look

For the 2BE to be effective, the energy cost of moving between points x and y in any two micro blocks inside a macro block,

$$\int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta),$$

should scale diffusively.



Problem: Due to exclusion, when moving a particle from x to y one also has to move over many "obstacles"!

- $\mathbb{Z}^d$: Just pick a shortest path with distance L, and carry out a sequence of nearest-neighbor "spin swaps," and calculate the associated energy cost. → Cauchy-Schwarz + translation/rotation invariance of the exclusion process Dirichlet form. [GPV '88, KOV '89] → $\int [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta) \lesssim L^2 \mathcal{E}^{\mathrm{EX}}(f)$
- This argument was also used by [Diaconis–Saloff-Coste '93] to obtain eigenvalue bounds in the exclusion process on a finite graph.
- HOWEVER this method does NOT give the required $(5/3)^N$ diffusive scaling for SG. [Jara '09] already noticed this issue.
- **Solution:** Prove a functional inequality which uses the effective resistance distance instead of the shortest-path distance. $\int [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta) \lesssim R_{\mathrm{eff}}(x,y) \mathcal{E}^{\mathrm{EX}}(f)$

The crux of 2BE: the **moving particle lemma (MPL)** [C., *ECP* '17]

- $(G = (V, E), \mathbf{c} = (c_{zw})_{zw \in E})$: **finite** connected weighted graph.

Theorem (MPL for the symmetric exclusion process)

For all $f : \{0,1\}^V \to \mathbb{R}$,

$$\underbrace{\sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{zw} f)(\eta)]^2 \, d\nu_\alpha(\eta)}_{=2\mathcal{E}^{\rm EX}(f)} \geq [R_{\rm eff}(x,y)]^{-1} \underbrace{\int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta)}_{\text{Cost of swapping configs } x \leftrightarrow y}.$$

Proof. Idea connected to the spectral gap conjecture of Aldous '92: $\lambda_2^{\rm EX}(G) = \lambda_2^{\rm RW}(G)$, which was resolved positively by [Caputo–Liggett–Richthammer '10] via their proof of the *octopus inequality* for the interchange process. I used OI, electric network reduction, and projection onto the exclusion process to obtain the MPL.

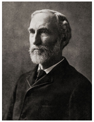
Harkens to...

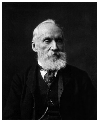
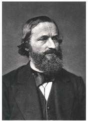
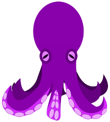
Theorem (Dirichlet/Thomson 1867). For all $f : V \to \mathbb{R}$,

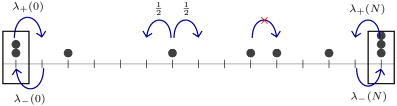
$$\underbrace{\sum_{zw \in E} c_{zw} [f(z) - f(w)]^2}_{=\mathcal{E}(f)} \geq [R_{\rm eff}(x,y)]^{-1} [f(x) - f(y)]^2.$$

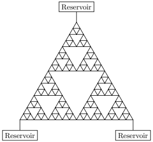
Equality is attained $\Leftrightarrow$ f is harmonic on $V \setminus \{x, y\}$: Ohm's Law: $V = IR$, Power$= IV = R^{-1} V^2$.

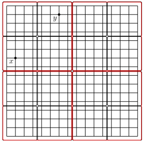
Het einde (voor nu)







Dank je!