CUTOFFS FOR EXCLUSION AND INTERCHANGE PROCESSES ON FINITE GRAPHS

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ABSTRACT. We prove a general theorem on cutoffs for symmetric exclusion and interchange processes on finite graphs $G_N = (V_N, E_N)$, under the assumption that either the graphs converge geometrically and spectrally to a compact metric measure space, or they are isomorphic to discrete Boolean hypercubes. Specifically, cutoffs occur at times $t_N = (2\gamma_1^N)^{-1} \log |V_N|$, where γ_1^N is the spectral gap of the symmetric random walk process on G_N. Under the former assumption, our theorem is applicable to the said processes on graphs such as: the d-dimensional discrete grids and tori for any integer dimension d; the L-th powers of cycles for fixed L, *a.k.a.* the L-adjacent transposition shuffle; and self-similar fractal graphs and products thereof.

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1. INTRODUCTION

There are many ways to shuffle a deck of N cards. Fixing a choice of the shuffling method, we may ask how many shuffles it takes to well mix the deck. Mathematically, we want to find the *ε-mixing time* of the shuffling Markov chain, which is the first time that the total variation distance $d_N(t)$ between the probability distribution of the deck at time t and the uniform measure on the symmetric group on N elements is smaller than $\varepsilon \in (0, 1)$.

The purpose of this article is prove, for a family of (old and new) shuffling methods—and via projection, a family of symmetric exclusion processes—that the total variation distance $d_N(t)$ converges abruptly from 1 to 0 at well-defined time scales $\{t_N\}_{N \in \mathbb{N}}$, as $N \to \infty$. This is known as the *cutoff phenomenon*.

1.1. **Synopsis of the cutoff problem.** One of the first card shuffles where cutoff is proven is the *random transposition shuffle*, described as follows: Choose a card uniformly at random with the right hand, and another card, also uniformly at random, with the left hand (it is okay to choose the same card with both hands); then transpose the two cards. Diaconis and Shahshahani [DS81] proved that for any $\kappa < 1$, the sequence $d_N \left(\frac{\kappa}{2} N \log N \right)$ converges to 1 as $N \to \infty$, and later Matthews [Mat88] proved that this sequence converges to 0 for any $\kappa > 1$. Combining both results proves cutoff for the random transposition shuffle.

A variation of the previous card shuffle is the *adjacent transposition shuffle*. Here the only allowed transpositions are the ones between adjacent cards, that is: cards at positions $k, k+1$, for $k \in \{1, \ldots, N-1\}$. Although the model has been introduced as a discrete-time Markov chain, it is more convenient to consider its continuous-time version: Assign to each pair $(k, k+1)$ of positions in the deck an i.i.d. rate-1 exponential clock, and when a clock rings at that pair, transpose the cards at the pair's positions. Lacoin [Lac16b] proved that the continuous-time adjacent transposition shuffle on N cards presents cutoff at times $\frac{1}{2\pi^2} N^2 \log N$,

obtaining an upper bound which matched a lower bound given previously by Wilson [Wil04]. In fact, *Wilson's method* is known to be robust for proving sharp lower bounds on the mixing times of several Markov chains. His idea is to construct an observable of the process that involves the eigenfunction of the process generator with the largest non-zero eigenvalue, so that it mixes as fast as the process itself. Then by a standard probabilistic argument, the problem boils down to estimating the first two moments of the observable with respect to the distribution at time t and the invariant measure, respectively, of the process.

Observe that the two aforementioned card shuffles are particular cases of the **interchange process** on a finite connected graph $G = (V, E)$. At each vertex of V we put a card, and at each edge of E we attach an i.i.d. rate-1 exponential clock. Whenever a clock rings, we flip the associated edge, transposing the two cards adjacent to the edge. Under this description, the random transposition and the adjacent transposition shuffles are nothing but the interchange process on, respectively, the complete graph on N vertices ($\mathcal{K}_N$) and the path on N vertices ($\mathcal{P}_N$).

Via a simple projection, the interchange process can be used to obtain a related process, the **exclusion process**, on G. If one colors some cards in black and the rest in red, one can see the black cards interact as indistinguishable particles.

Projections cannot increase the total variation distance to equilibrium. Therefore, the mixing time of the exclusion process on G is bounded above by the mixing time of the interchange process on G. Intuitively, if there are as many black as red cards, both processes should not mix too differently and so, it should be enough to obtain a lower bound on the mixing time of the exclusion process and an upper bound on the mixing time of the interchange process. That is what Lacoin does in [Lac16b]. He obtains an upper bound that matches the lower bound obtained by Wilson. Curiously, Wilson's bound on the mixing time of the simple exclusion process on a general sequence of graphs $\{G_N\}_{N \in \mathbb{N}}$ is usually given by

$$(1.1) \qquad\qquad t_N = \frac{\log |V_N|}{2\gamma_1^N},$$

where $-\gamma_1^N$ is the largest non-zero eigenvalue of the generator of the simple random walk on G_N.

Lacoin also proved cutoff at times t_N (1.1) for the symmetric exclusion process on the circle ($\mathcal{C}_N$) [Lac16, Lac17]. His approach is very robust, and almost everything in his proof can be adapted to the symmetric exclusion process on a higher-dimensional lattice, *except* for the construction of a monotone grand coupling for the process.

So far the only graphs for which cutoffs have been proved for the symmetric exclusion or interchange processes are the star graphs [Dia88, Example 4.1], complete graphs $\mathcal{K}_N$ and the 1D graphs—$\mathcal{P}_N$ and $\mathcal{C}_N$ mentioned above. For higher-dimensional graphs, it is believed that the lower bound obtained via Wilson's method is optimal, but there has been no complete proof of a matching upper bound. Accomplishing the latter has proven to be a very difficult task.

1.2. **Our contribution to the cutoff problem.**

In this work we establish **sharp upper bounds** on the mixing times for exclusion and interchange processes on finite graphs $G_N = (V_N, E_N)$, assuming that these graphs *either* converge geometrically and spectrally to a compact metric measure space, *or* are isomorphic to discrete (Boolean) hypercubes. In conjunction with the lower bounds obtained via Wilson's method, we then establish cutoffs for these processes at times t_N (1.1).

Our examples cover symmetric exclusion and interchange processes on: the discrete d-dimensional grid or torus, for every integer dimension d; the powers of cycles, *a.k.a.* the L-adjacent transposition shuffle [Dur03]; self-similar fractal graphs, such as (products of) Sierpinski gaskets; and the discrete hypercubes [Wil04]. To put our result in perspective, we mention the excellent survey of Saloff-Coste [SC04] on total variation pre-cutoffs and cutoffs in various random walk processes on the symmetric group S_n. As of this paper, the first three chains in his "pre-cutoffs" table, [SC04, Table 3]—the adjacent transposition, the L-adjacent transposition, and the nearest neighbors transposition on a square grid—are proven to exhibit cutoffs.

Our result also reinforces the main result of Oliveira [Oli13], which asserts that there exists an universal constant $c > 0$ such that the mixing time of the symmetric exclusion process on a graph $G_N = (V_N, E_N)$ is $c \log |V_N|$ times the $\frac{1}{4}$-mixing time of the symmetric random walk process on G_N. Oliveira also conjectured that the mixing time of the simple exclusion process should be at most the mixing time of the process where particles perform independent random walks. This conjecture was proven by Hermon and Pymar [HP18] for

d-regular graphs. Since our proof method can be applied to the independent random walks model, Oliveira's conjecture holds for the graphs considered in this paper.

Our main technical contribution is the construction and analysis of the **cutoff martingale** in the exclusion processes, which enables us to prove the sharp upper bound (see §5 for details). For full disclosure, this method is simultaneously implemented in [CJM20] to prove cutoffs for exclusion processes with open boundary, again in any dimension. While there are similarities between [CJM20] and this paper, the difference in the boundary condition (open vs. closed) triggers different assumptions on the spectral convergence; a different coupling of the microscopic processes; and a different analytic role played by the lowest nonzero Laplacian eigenvalue and its corresponding eigenfunction.

Organization of the paper. In §2 we introduce the models, the notation, and the assumptions on the geometric and spectral convergence of the graphs under consideration. In §3 we state our main cutoff Theorem 1, and discuss examples of graphs to which our theorem applies. The proof of Theorem 1 is given in the subsequent two sections. §4 proves lower bounds on the mixing times of the exclusion processes on graphs satisfying our assumptions, using Wilson's method [Wil04]. §5, which is our new contribution, establishes the matching upper bounds on the mixing times via the cutoff martingale method. We close the paper with an open problem in §6.

Asymptotic notation used in the paper. Given two real sequences $\{f_N ; N \in \mathbb{N}\}$ and $\{g_N ; N \in \mathbb{N}\}$, we will use the notation

- $f_N = \mathcal{O}(g_N)$ (or $f_N \lesssim g_N$ when it suits better) if and only if there exists a positive constant C, independent of N, such that $f_N \leq C g_N$ for all sufficiently large N.
- $f_N = \Theta(g_N)$ if and only if there exist positive constants $C_1 < C_2$, independent of N, such that $C_1 g_N \leq f_N \leq C_2 g_N$ for all sufficiently large N.

2. NOTATION AND ASSUMPTIONS

2.1. **Interchange process.** Let $G_N = (V_N, E_N)$ be a finite undirected graph. At each vertex of V_N we put a card and at each edge of E_N we attach an exponential clock of rate 1. Whenever a clock rings, we flip that edge and transpose the two cards incident to it. This card shuffle on G_N, known as the *interchange process* on G_N, is the continuous time Markov process $\{\sigma_t^N ; t \geq 0\}$ whose state space is the symmetric group $S_{|V_N|}$ of the permutations in a $|V_N|$-element set and whose generator is given by

$$\mathcal{L}_{\mathrm{int}}^N f(\sigma) = \sum_{xy \in E_N} (f(\sigma^{xy}) - f(\sigma)),$$

for every function $f : S_{|V_N|} \to \mathbb{R}$, where σ^{xy} is the permutation obtained from σ after transposing the cards at vertices x and y. Since G_N is a finite graph, this Markov process is irreducible, and the uniform measure U_{int}^N on $S_{|V_N|}$ is its unique invariant measure. We write ν_t^N for the distribution of $\{\sigma_t^N ; t \geq 0\}$ at time t.

Recall that the total variation distance between two probability measures μ and ν on a finite state space Ω is defined as

$$\|\mu - \nu\|_{TV} = \max_{A \subset \Omega} (\mu(A) - \nu(A)).$$

The distance to equilibrium of the process $\{\sigma_t^N ; t \geq 0\}$ is defined as

$$D_N(t) = \|\nu_t^N - U_{\mathrm{int}}^N\|_{TV}.$$

The ε-mixing time of the process $\{\sigma_t^N ; t \geq 0\}$ is defined as

$$T_{\mathrm{mix}}^N(\varepsilon) = \inf \{t \geq 0 : D_N(t) \leq \varepsilon\}.$$

2.2. **Exclusion process.** Now we will define the *exclusion process* on the graph G_N by coloring the cards of the interchange process on G_N. Indeed, let us color $k = k_N$ cards in black and $|V_N| - k$ cards in red. We will assume that $\rho_{\mathrm{ss}}^N := k_N/|V_N| \to \rho$ for some $\rho \in (0,1)$ as $N \to \infty$. Define $\eta_t^N : V_N \to \{0,1\}$ as the

function that assigns value 1 to sites with black cards and value 0 to sites with red cards. We can see η_t^N as the result of a projection $P : S_{|V_N|} \to \{0,1\}^{V_N}$ of the interchange process which can be defined as

$$(2.1) \qquad \eta(x) = P(\sigma)(x) = \begin{cases} 1, & \text{if the card at } x \text{ is black;} \\ 0, & \text{if the card at } x \text{ is red.} \end{cases}$$

The process $\{\eta_t^N ; t \geq 0\}$ obtained as the image of $\{\sigma_t^N ; t \geq 0\}$ under P is known as the exclusion process on G_N. As a helpful visual, black cards represent particles in the exclusion process, while red cards represent vacancies. The corresponding generator is

$$(2.2) \qquad \mathcal{L}_{\text{exc}}^N f(\eta) = \sum_{xy \in E_N} \left(f(\eta^{xy}) - f(\eta) \right).$$

for every function $f : \{0,1\}^{V_N} \to \mathbb{R}$, where η^{xy} is the configuration obtained from η after exchanging the occupations of the vertices x and y, that is,

$$\eta^{xy}(z) = \begin{cases} \eta(y), & \text{if } z = x; \\ \eta(x), & \text{if } z = y; \\ \eta(z), & \text{if } z \notin \{x, y\}. \end{cases}$$

We will write $\mathbb{P}_{\mu_N}$ for the law of $\{\eta_t^N ; t \geq 0\}$ with initial measure μ_N, and $\mu_t^N = \mathbb{P}_{\mu_N}(\eta_t^N \in \cdot)$ for its marginal at time $t \geq 0$.

Moreover, we will write U_{exc}^N for its invariant measure, which is the uniform measure in the configuration space

$$(2.3) \qquad \Omega_{N,k} = \left\{ \eta \in \{0,1\}^{V_N} ; \sum_{x \in V_N} \eta(x) = k \right\}.$$

The total variation distance to equilibrium of the exclusion process $\{\eta_t^N ; t \geq 0\}$ is defined as

$$d_{N,k}(t) = \sup_{\mu_N} \left\| \mu_t^N - U_{\text{exc}}^N \right\|_{TV},$$

where the supremum is taken over all possible initial distributions μ_N. The ε-mixing time of the chain $\{\eta_t^N ; t \geq 0\}$ is defined as

$$t_{\text{mix}}^{N,k}(\varepsilon) = \inf \{t \geq 0 : d_{N,k}(t) \leq \varepsilon\}.$$

2.3. **Geometric and analytic assumptions.** In this subsection we lay out the assumptions needed to prove our main result. From the analytic point of view, it sometimes helps to declare a subset ∂V_N of the vertex set V_N as the "boundary" of G_N. So in what follows we assume one of the following scenarios:

- Either that $\partial V_N = \emptyset$ for all $N \in \mathbb{N}$;
- or that there exists a sequence of nonempty subsets ∂V_N of V_N indexed by N.

In the former scenario, any condition stated below that involves ∂V_N can be ignored.

Assumption 1 (Geometric convergence). Let $\{(G_N, \partial V_N)\}_N$ be a sequence of connected, bounded-degree graphs, possibly with boundaries; in particular the degree bound is assumed to be uniform in N. We say that the sequence of graphs $\{(G_N, \partial V_N)\}_N$ **converges geometrically** to a compact metric measure space $(K, d, \mathfrak{m})$ with boundary ∂K and boundary measure $\mathfrak{s}$ if:

(1) For every $N \in \mathbb{N}$, $V_N \subseteq K$ and $\partial V_N \subseteq \partial K$.

Moreover, as $N \to \infty$:

(2) $|\partial V_N|/|V_N| \to 0$.
(3) $\mathfrak{m}_N := \frac{1}{|V_N|} \sum_{x \in V_N} \delta_x$ converges weakly to $\mathfrak{m}$.
(4) $\mathfrak{s}_N := \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \delta_a$ converges weakly to $\mathfrak{s}$.

Above δ_x is the Dirac measure at x. Without loss of generality, we assume that $\mathfrak{m}$ and $\mathfrak{s}$ have full support on K and ∂K, respectively.

2.3.1. *Laplacian, Dirichlet form, and the spectral problem.* To define our next objects, we shall introduce a sequence of positive numbers $\{\mathcal{T}_N\}_N$, which stand for the time scales to speed up the microscopic processes $\{\eta_t^N ; t \geq 0\}$. Typically $\mathcal{T}_N$ monotonically increases to $+\infty$ as $N \to \infty$, but it is possible to consider $\mathcal{T}_N = 1$ for all N (*cf.* the hypercube discussed in §3.5).

Consider the exclusion process generated by $\mathcal{T}_N \mathcal{L}_{\mathrm{Exc}}^N$, where $\mathcal{L}_{\mathrm{Exc}}^N$ was defined in (2.2). This generator induces a *Laplacian* Δ_N on functions $f : V_N \to \mathbb{R}$ given by

$$(2.4) \qquad (\Delta_N f)(x) = \mathcal{T}_N \sum_{\substack{y \in V_N \\ y \sim x}} (f(y) - f(x)).$$

We leave the reader to check that $\mathcal{T}_N \mathcal{L}_{\mathrm{Exc}}^N \eta(x) = (\Delta_N \eta)(x)$ for all $x \in V_N$, and that Δ_N generates the symmetric simple random walk process on G_N.

Meanwhile, if we adopt the scenario of graphs with boundaries, we can define the **outward normal derivative** of f at $a \in \partial V_N$ as

$$(2.5) \qquad (\partial_N^\perp f)(a) = -\frac{\mathcal{T}_N}{|V_N|/|\partial V_N|} \sum_{\substack{y \in V_N \\ y \sim a}} (f(y) - f(a)).$$

Note the difference in the scaling between (2.4) and (2.5).

Next we introduce the **Dirichlet form** on G_N. For any $f, g : V_N \to \mathbb{R}$, define

$$(2.6) \qquad \mathcal{E}_N(f, g) = \frac{1}{2} \frac{\mathcal{T}_N}{|V_N|} \sum_{x \in V_N} \sum_{\substack{y \in V_N \\ y \sim x}} [f(x) - f(y)][g(x) - g(y)].$$

Using summation by parts and (2.5), we can rewrite (2.6) as

$$\mathcal{E}_N(f, g) = \frac{1}{|V_N|} \sum_{x \in V_N} f(x)(-\Delta_N g)(x)$$

$$= \frac{1}{|V_N|} \sum_{x \in V_N \setminus \partial V_N} f(x)(-\Delta_N g)(x) + \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} f(a)(\partial_N^\perp g)(a).$$

The last expression is suggestive of the integration by parts formula

$$(2.7) \qquad \mathcal{E}_N(f, g) = \int_{V_N \setminus \partial V_N} f(x)(-\Delta_N g)(x)\, d\mathfrak{m}_N(x) + \int_{\partial V_N} f(a)(\partial_N^\perp g)(a)\, d\mathfrak{s}_N(a).$$

As usual we adopt the shorthand $\mathcal{E}_N(f) := \mathcal{E}_N(f, f)$. It is direct to verify that $-\Delta_N$ is a nonnegative self-adjoint operator on $L^2(K, \mathfrak{m}_N)$.

Last but not least, we discuss the spectral problem. A function $\psi : V_N \to \mathbb{R}$ is an eigenfunction of $-\Delta_N$ if there exists an eigenvalue $\lambda \in \mathbb{C}$ such that

$$(2.8) \qquad \begin{cases} -\Delta_N \psi = \lambda \psi & \text{on } V_N \setminus \partial V_N, \\ \lambda \psi(a) = \frac{|V_N|}{|\partial V_N|}(\partial_N^\perp \psi)(a), & a \in \partial V_N. \end{cases}$$

To explain the boundary condition (2.8), we first use the eigenvalue equation and (2.7) to obtain

$$(2.9) \qquad \mathcal{E}_N(f, \psi) = \frac{1}{|V_N|} \sum_{x \in V_N \setminus \partial V_N} f(x)(\lambda \psi(x)) + \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} f(a)(\partial_N^\perp \psi)(a)$$

for any $f : V_N \to \mathbb{R}$. On the other hand, by definition we have

$$(2.10) \qquad \mathcal{E}_N(f, \psi) = \frac{1}{|V_N|} \sum_{x \in V_N} f(x)(\lambda \psi(x)).$$

Matching the RHS of (2.9) and (2.10) and then setting $f = \mathbb{1}_a$ for each $a \in \partial V_N$ yields (2.8).

Since $-\Delta_N$ is a nonnegative self-adjoint operator on $L^2(\mathfrak{m}_N)$, the eigenvalue problem (2.8) has $|V_N|$ solutions, each of which is associated with a $\mathbb{R}_+$-valued eigenvalue λ_i^N. We choose the index i such that $\{\lambda_i^N\}_{i=0}^{|V_N|-1}$ forms an increasing sequence.

By the Perron-Frobenius theorem, $\lambda_0^N = 0$ is the unique lowest eigenvalue. In the case where the eigenvalue λ_i^N, $i \geq 1$, is repeated with multiplicity k, the corresponding eigenspace is k-dimensional, and we apply Gram-Schmidt to generate k orthonormal eigenfunctions. In this way we can construct an orthonormal basis $\{\psi_i^N\}_{i=0}^{|V_N|-1}$ for $L^2(\mathfrak{m}_N)$.

Let us note the elementary result that for any $i \geq 1$, the integral of ψ_i^N equals 0. This follows from the orthogonality of $\{\psi_i^N\}_{i=0}^{|V_N|-1}$ and that $\psi_0^N \equiv 1$:

$$
(2.11) \qquad \sum_{x \in V_N} \psi_i^N(x) = |V_N| \int_{V_N} \psi_0^N \, \psi_i^N \, d\mathfrak{m}_N = 0.
$$

To state our next assumption, we shall extend each ψ_i^N from V_N to K via continuous interpolation, and call the extension ψ_i^N still.

Assumption 2 (Spectral convergence). The collection of discrete Laplacian eigensolutions

$$
\left\{ (\lambda_i^N, \psi_i^N) : 1 \leq i \leq |V_N| \right\}_{N \in \mathbb{N}}
$$

defined above satisfies the following conditions:

(1) There exist $\{\lambda_i\}_{i=1}^\infty \subseteq \mathbb{R}_+$ such that $\lim\limits_{N \to \infty} \lambda_i^N = \lambda_i$ for each $i \in \mathbb{N}$, and in particular, $\lambda_1 > 0$.
(2) There exist $\{\psi_i\}_{i=1}^\infty \subseteq C(K)$ such that $\psi_i^N \to \psi_i$ in $C(K)$ for each $i \in \mathbb{N}$.
(3) Each eigenfunction in the sequence $\{\psi_i\}_{i=1}^\infty$ from Item (2) satisfies $\sup\limits_{N} \sup\limits_{a \in \partial V_N} \left| (\partial_N^\perp \psi_i)(a) \right| < \infty$.

When implementing Assumption 1-(2) and Assumption 2 into the boundary condition in (2.8), we see that this gives rise to Neumann boundary condition on the eigenfunction ψ_i^N in the limit $N \to \infty$.

Remark 2.1 (Eigenfunctions corresponding to a multiple eigenvalue). If λ_i is a simple eigenvalue (*i.e.*, has multiplicity 1), then the statements in Items (2) and (3) of Assumption 2 are clear. If λ_i is a multiple eigenvalue, then strictly speaking Item (2) should be replaced by the convergence of the corresponding eigenspaces. That said, there is no loss of generality in assuming that an orthonormal basis has been chosen for each eigenspace such that Item (2) holds.

Remark 2.2 (Tuning the boundary interchange rates). Under Assumptions 1 and 2, we can adjust the rates of the interchange process (and in turn, the exclusion process obtained by projection) between $a \in \partial V_N$ and an adjacent vertex by a $\Theta(1)$ multiplicative factor, uniformly for all $a \in \partial V_N$ and its neighboring vertices, without affecting the spectral data in the limit $N \to \infty$. This is because the normal derivative at the boundary decays to 0 at $\Theta(1)$ times the original rate as $N \to \infty$.

3. Main result and examples

We now state our main result. Set the following figure of merit

$$
t_N := \frac{\mathcal{T}_N}{2\lambda_1^N} \log |V_N|.
$$

Recall that $D_N(t)$ (resp. $d_{N,k}(t)$) denotes the total variance distance to equilibrium of the interchange process (resp. the exclusion process with k particles), and $T_{\mathrm{mix}}^N(\varepsilon)$ (resp. $t_{\mathrm{mix}}^{N,k}(\varepsilon)$) the ε-mixing times.

Theorem 1. *Suppose Assumptions 1 and 2 hold. Let $\{k_N\}_{N \in \mathbb{N}}$ be any sequence of nonnegative integers such that $\lim_{N \to \infty} k_N/|V_N| = \rho \in (0,1)$. Then*

$$
\lim_{N \to \infty} d_{N,k_N}(\kappa \, t_N) = \lim_{N \to \infty} D_N(\kappa \, t_N) = \begin{cases} 1, & \text{if } \kappa < 1, \\ 0, & \text{if } \kappa > 1; \end{cases}
$$

equivalently, for any $\varepsilon \in (0,1)$,

$$
\lim_{N \to \infty} \frac{t_{\mathrm{mix}}^{N,k_N}(\varepsilon)}{t_N} = \lim_{N \to \infty} \frac{T_{\mathrm{mix}}^N(\varepsilon)}{t_N} = 1.
$$

Theorem 1 asserts that the distances to equilibrium of the exclusion and interchange processes converges abruptly from 1 to 0 at times t_N, a phenomenon known as *cutoff*.

Recall that the exclusion process is obtained from projecting the interchange process. Since projections cannot increase distances, for any $t \geq 0$ we have

(3.1) $\qquad d_{N,k}(t) \leq D_N(t)$ and $t_{\mathrm{mix}}^{N,k}(\varepsilon) \leq T_{\mathrm{mix}}^N(\varepsilon)$ for any $k \in \{0, 1, \ldots, |V_N|\}$ and $\varepsilon \in (0, 1)$.

Therefore, in order to prove Theorem 1 it is sufficient to obtain a lower bound on the mixing time of the exclusion process on G_N and a matching upper bound on the mixing time of the interchange process on G_N. We present the proof in §4 and §5: the lower bound uses *Wilson's method* [Wil04], while the upper bound uses the *cutoff martingale method*. In both parts of the proof, we require that the fractions of black cards, $k_N/|V_N|$ (whence the fractions of red cards, $1 - k_N/|V_N|$) converge as $N \to \infty$ to a number which is bounded away from 0 and 1.

In the rest of the section, we discuss some explicit examples to which Theorem 1 applies.

3.1. **The d-dimensional grid.** Let $\{e_i \, ; \, i \in \{1, \ldots, d\}\}$ stand for the canonical basis of $\mathbb{R}^d$. Let G_N be the d-dimensional grid of side length N, where $V_N = \{0, \ldots, N\}^d$, and for each pair of vertices $x, y \in V_N$, declare $xy \in E_N$ if and only if $x = y \pm e_i$ for some $i \in \{1, \ldots, d\}$. We set $\partial V_N := \{(x_1, \ldots, x_d) : x_i \in \{0, N\}$ for at least one $i\}$. Then it is clear that $\{N^{-1} G_N\}_{N \in \mathbb{N}}$ forms a discrete approximation of the unit cube $[0, 1]^d$ (with boundary $\partial([0, 1]^d)$), thereby verifying Assumption 1.

Next, we choose the diffusive time scaling $\mathcal{T}_N = N^2$, and observe that in the presence of the boundary,

$$\mathcal{E}_N(f, g) \xrightarrow[N \to \infty]{} \mathcal{E}(f, g) := \int_{(0,1)^d} f(-\Delta g) \, dx + \int_{\partial([0,1]^d)} f(\partial^\perp g) \, ds$$

for all $f, g \in C^2([0, 1]^d)$ with $\partial^\perp g = 0$ on $\partial([0, 1]^d)$. By the usual integration by parts formula we have that

$$\mathcal{E}(f, g) = \int_{[0,1]^d} \nabla f \cdot \nabla g \, dx$$

where $f, g \in C^2([0, 1]^d)$ with $\partial^\perp f = \partial^\perp g = 0$ on $\partial([0, 1]^d)$. The eigenfunctions are the products of the 1D Neumann eigenfunctions: adopting the multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{N}_0^d$, we have that $\boldsymbol{\psi}_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^d \psi_{k_i}(x_i)$, where

$$\psi_k(x) = \begin{cases} 1, & \text{if } k = 0, \\ \sqrt{2} \cos(k\pi x), & \text{if } k \in \mathbb{N}. \end{cases}$$

The corresponding eigenvalue is $\boldsymbol{\lambda_k} = \pi^2 \sum_{i=1}^d (k_i)^2$. It is direct to verify Assumption 2; in particular, the lowest nonzero eigenvalue is $\lambda_1 = \pi^2$ with corresponding eigenspace being the linear span of $\{\cos(\pi x_1), \cos(\pi x_2), \ldots, \cos(\pi x_d)\}$.

Therefore, the sequences of exclusion and interchange processes on G_N present cutoffs at times $t_N = \dfrac{d}{2\pi^2} N^2 \log N$.

Remark 3.1. In the $d = 1$ case, the cutoff result coincides with [Lac16b, Theorem 2.4]. In the $d = 2$ case, the process is also known as the *nearest neighbor transposition shuffle on a square grid* [DSC93, Wil04], for which only "pre-cutoff" has been established. Hence we provide the first proof that this process exhibits cutoff. Finally, if we replace the square grid by a rectangular $aN \times bN$ grid (Figure 1), then we can use scaling and the aforementioned argument to show that both the interchange and the exclusion processes exhibit cutoffs at times $(\max(a, b))^2 \frac{N^2 \log N}{\pi^2}$.

3.2. **The discrete d-dimensional torus $\mathbb{T}_N^d$.** Let G_N be the d-dimensional torus $\mathbb{T}_N^d$, that is: $V_N = (\mathbb{Z}/N\mathbb{Z})^d$, and for each pair of vertices $x, y \in V_N$, we declare that $xy \in E_N$ if and only if there exists $j \in \{1, \ldots, d\}$ such that $x_j = y_j \pm 1 \pmod{N}$ and $x_i = y_i$ for all $i \neq j$. Then $\{N^{-1} \mathbb{T}_N^d\}_{N \in \mathbb{N}}$ forms a discrete approximation of the d-dimensional unit torus $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, which verifies Assumption 1. Again taking the diffusive time scaling $\mathcal{T}_N = N^2$, one may verify that

$$\mathcal{E}_N(f, g) \xrightarrow[N \to \infty]{} \mathcal{E}(f, g) := \int_{\mathbb{T}^d} \nabla f \cdot \nabla g \, dx$$
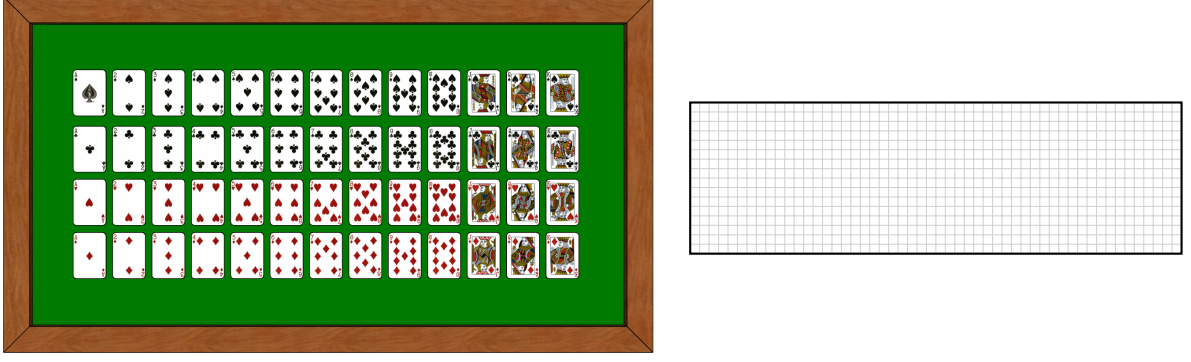
FIGURE 1. (Left) Nearest neighbors transposition shuffle on a 13×4 grid: cards are vertically or horizontally interchanged each time that a clock rings; (Right) The discretization of a rectangle with width-to-height ratio $13 : 4$. According to Theorem 1, we can exhibit cutoff at times $t_N = \dfrac{N^2 \log N}{(\pi/13)^2}$ for the interchange processes on the family of graphs $\{0, 1, \ldots, 13N\} \times \{0, 1, \ldots, 4N\}$, $N \in \mathbb{N}$.

(A) $\lambda_1 = \pi^2 + 0$ (B) $\lambda_1 = \pi^2 + 0$ (C) $\lambda_1 = (2\pi)^2 + 0$
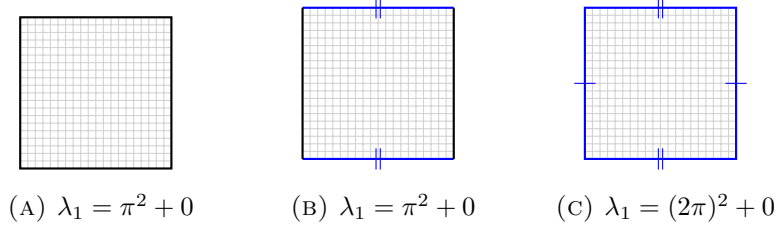
FIGURE 2. A summary of results on cutoffs for exclusion and interchange processes on the discrete unit square. The boundary condition can be one of the following: closed (reflecting) or periodic. In each case where λ_1 is given, cutoff is established: for every $\varepsilon \in (0, 1)$,

$$\lim_{N \to \infty} \frac{\lambda_1 t_{\mathrm{mix}}^N(\varepsilon)}{N^2 \log N} = 1.$$

for all $f, g \in C^2(\mathbb{T}^d)$. The eigenfunctions are $\psi_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^d \psi_{k_i}(x_i)$, where

$$\psi_k(x) \begin{cases} = 1, & \text{if } k = 0, \\ \in \{\sqrt{2}\sin(2k\pi x), \sqrt{2}\cos(2k\pi x)\}, & \text{if } k \in \mathbb{N}. \end{cases}$$

The corresponding eigenvalue is $\boldsymbol{\lambda}_{\mathbf{k}} = \pi^2 \sum_{i=1}^d (2k_i)^2$. Checking Assumption 2 is now routine. We highlight the first nonzero eigenvalue $\lambda_1 = (2\pi)^2$ whose corresponding eigenspace is the linear span of

$$\{\sin(2\pi x_1), \ldots, \sin(2\pi x_d), \cos(2\pi x_1), \ldots, \cos(2\pi x_d)\}.$$

As a result, the sequences of exclusion and interchange processes on $\mathbb{T}_N^d$ present cutoffs at times

$$t_N = \frac{d}{8\pi^2} N^2 \log N.$$

Remark 3.2. In the $d = 1$ case, the result coincides [Lac16, Theorem 2.4] and [Lac17, Theorem 1.1] for the simple exclusion process. For $d \geq 2$ we give the first proof of cutoff on the torus. It is possible to pick, for each coordinate direction, either Neumann or periodic boundary condition, and prove cutoff on the discrete unit cube with that boundary condition. (On the other hand, if one uses Dirichlet boundary condition somewhere, then the analysis and the cutoff theorem differ from the present work; see [CJM20] and Figures 1 and 2 therein.) Figure 2 provides a visual summary in the case $d = 2$.
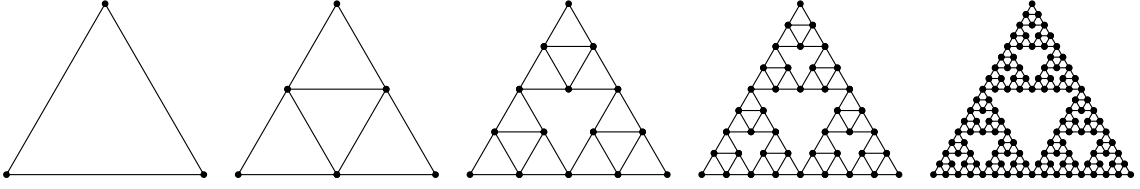
FIGURE 3. The level-N approximating graph $\mathcal{G}_N$ of the Sierpinski gasket, for $N = 0, 1, 2, 3, 4$ (from left to right). For every $N \in \mathbb{N}$, the boundary set ∂V_N is the set of 3 corner vertices of the outer triangle.

3.3. **The L-adjacent transposition shuffle.** Let G_N be the L-th power of a cycle of length N, $\mathcal{C}_N^L$, for fixed L, namely: $V_N = \mathbb{Z}_N$ and for each pair of vertices $x, y \in V_N$, we declare $xy \in E_N$ if and only if $x = y \pm \ell \pmod N$ for some $\ell \in \{1, 2, \ldots, L\}$. The interchange process on this choice of G_N was introduced in [Dur03] as the L-adjacent transposition shuffle. Observe that $\{N^{-1}\mathbb{Z}_N\}_{N \in \mathbb{N}}$ forms a discrete approximation of the unit torus $\mathbb{T}$, so Assumption 1 is immediate. To verify Assumption 2, let us note that by taking $\mathcal{T}_N = N^2$, and using the fact that $\mathcal{C}_N^L$ is a circulant graph, we can use [LY93, Corollary 5] to find the eigensolutions of $-\Delta_N$ are indexed by $k \in \{0, 1, \ldots, N-1\}$,

$$\lambda_k^N = 4N^2 \sum_{\ell=1}^{L} \sin^2\left(\frac{\pi k \ell}{N}\right); \quad \psi_k^N(x) = \sqrt{2} \sum_{\ell=1}^{L} \sin(2\pi k \ell x) \text{ for } x \in N^{-1}\mathbb{Z}_N.$$

Thus Items (1) and (2) hold. In particular

$$\lambda_1^N = 4N^2 \sum_{\ell=1}^{L} \sin^2\left(\frac{\pi \ell}{N}\right) = 4\pi^2 \left(\sum_{\ell=1}^{L} \ell^2\right)(1 + \mathcal{O}(N^{-2})) \xrightarrow[N \to \infty]{} 4\pi^2 \left(\sum_{\ell=1}^{L} \ell^2\right) =: \lambda_1.$$

Thus the sequences of exclusion and interchange processes on G_N present cutoffs at times

$$t_N = \frac{N^2 \log N}{8\pi^2 \left(\sum_{\ell=1}^{L} \ell^2\right)}.$$

Previously Durrett [Dur03] has proven the sharp lower bound and an upper bound of the same order.

3.4. **Self-similar fractal graphs.** For background and details on analysis on fractals, the reader is referred to the monographs [Bar98, Kig01, Str06]. We will use the Sierpinski gasket as the working example. Figure 3 shows the graph approximations of the gasket: for each N, the level-N graph has $|V_N| = \frac{3}{2}(3^N + 1)$, and we take ∂V_N to be the 3 corner vertices of the outer triangle. It is a routine argument that the normalized counting measure on V_N converges weakly, as $N \to \infty$, to the standard self-similar measure on the limit fractal K. Thus Assumption 1 is verified. As for Assumption 2, we note that the diffusive time scale is $\mathcal{T}_N = 5^N$. This choice ensures that for every $f : K \to \mathbb{R}$, the sequence $\{\mathcal{E}_N(f)\}_{N \in \mathbb{N}}$ is monotone increasing, and hence allows us to define $\mathcal{E}(f) := \lim_{N \to \infty} \mathcal{E}_N(f)$ for all f such that $\mathcal{E}(f) < \infty$. Indeed this is a consequence of the construction of a *resistance form* on K, described in [Kig01, Chapter 2], which implies that for each $i \in \mathbb{N}$, $\lambda_i^N \to \lambda_i$ with $\lambda_1 > 0$ (*i.e.,* Perron-Frobenius holds), and $\psi_i^N \to \psi_i$ in $C(K)$. We can then use the summation by part formula (2.7) to justify the existence of the normal derivative $(\partial_N^\perp \psi_i^N)(a)$, $a \in \partial V_N$, and show that the limit $\lim_{N \to \infty} (\partial_N^\perp \psi_i^N)(a) = 0$ for all $i \in \mathbb{N}$. Thus Assumption 2 is verified.

As a result, the sequences of exclusion and interchange processes on the Sierpinski gasket G_N present cutoffs at times

$$(3.2) \qquad t_N = \frac{(\log 3)N 5^N}{2\lambda_1^{SG}}.$$

The value of λ_1^{SG} is obtained via a recursive procedure called *spectral decimation*: $\lambda_1^{SG} = \dfrac{3}{2} \lim_{k \to \infty} 5^k \phi^{\circ k}(6)$, where $\phi(t) := \frac{5 - \sqrt{25 - 4t}}{2}$. The interested reader is referred to [Str06, Sections 3.2 and 3.3] for details.

Generalizations of the cutoff result to the "double SG" (gluing two copies of SG at the boundary vertices [Str12]); the d-dimensional Sierpinski simplex; the Vicsek tree; and other post-critically finite fractals are
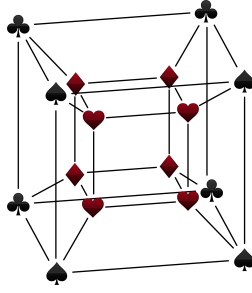
FIGURE 4. The 4-dimensional hypercube, also known as the *tesseract.*

more or less immediate. Moreover, we also can deal with products of fractals. For example, on the Cartesian product of d ($d \geq 2$) copies of the Sierpinski gasket, $\bigotimes_{i=1}^{d} G_N$, cutoffs occur at times

$$t_N = \frac{d(\log 3) N 5^N}{2\lambda_1^{SG}}.$$

3.5. **The N-dimensional hypercube.** Let G_N be the N-dimensional (Boolean) hypercube, that is: $V_N = \mathbb{Z}_2^N$, and for each pair of vertices $x, y \in V_N$, we declare $xy \in E_N$ if and only if $x = y \pm e_i$ for some $i \in \{1, \ldots, N\}$. (Figure 4 shows G_4.) While we cannot regard $\{G_N\}_N$ as some discretization of a compact space, and therefore Assumption 1 does not apply, we can nonetheless discuss the spectrum. Fix $\mathcal{T}_N = 1$ for all N. Then the eigenfunction of $-\Delta_N$ can be indexed by $\mathbf{k} = (k_1, k_2, \ldots, k_N) \in \{0, 1\}^N$, and easily seen to be $\psi_{\mathbf{k}}^N(\mathbf{x}) = \prod_{i=1}^{N} \psi_{k_i}(x_i)$, $\mathbf{x} \in \mathbb{Z}_2^N$, where

$$\psi_k(x) = \begin{cases} 1, & \text{if } k = 0, \\ \cos(\pi x), & \text{if } k = 1. \end{cases}$$

The corresponding eigenvalue is $\boldsymbol{\lambda}_{\mathbf{k}}^N = 2(\sum_{i=1}^{N} k_i)$. In particular, the spectral gap $\lambda_1^N = 2$ for all N, and the corresponding eigenspace is the linear span of $\{\cos(\pi x_1), \cos(\pi x_2), \ldots, \cos(\pi x_N)\}$. Thus we will prove that the sequences of exclusion and interchange processes on G_N present cutoffs at times

$$t_N = \frac{\log 2}{4} N.$$

The proofs are nearly identical to the ones where Assumptions 1 and 2 hold, except for a minor argument involving the support of the positive part of ψ_1^N (*cf.* Remark 4.3).

4. LOWER BOUND ON THE MIXING TIME OF THE EXCLUSION PROCESS

In this section we obtain a lower bound on the mixing time of the exclusion process on graphs G_N that either satisfy Assumptions 1 and 2, or are isomorphic to the hypercube (§3.5). We use Wilson's method [Wil04]: the key is to find a good observable of the process which allows us to prove the optimal lower bound.

It is natural to look at empirical density measure

$$\pi_t^N := \frac{1}{|V_N|} \sum_{x \in V_N} \eta_{t\mathcal{T}_N}^N(x) \delta_{\{x\}}.$$

associated with the process $\{\eta_t^N \, ; \, t \geq 0\}$ generated by $\mathcal{T}_N \mathcal{L}_{\text{exc}}^N$. The reason is because in several well-known settings, one can prove a law of large numbers, or a *hydrodynamic limit*, for $\{\pi_t^N \, ; \, t \geq 0\}_N$. Specifically, assuming that the sequence of initial measures $\{\pi_0^N\}_N$ is associated to a density profile, then it can be proven that at all later times $t > 0$, π_t^N converges (in the Skorokhod topology on measure-valued trajectories) to the measure $\rho(t, x) \, d\mathfrak{m}(x)$ where ρ is the unique weak solution of a hydrodynamic equation. In particular the total mass $\int_K \rho(t, \cdot) \, d\mathfrak{m}$ is invariant for all times $t \geq 0$. Since our exclusion process has symmetric rates, the hydrodynamic equation is the heat equation.

In the proof of the hydrodynamic limit, we pair a sufficiently smooth test function $f : V_N \to \mathbb{R}$ with the empirical measure π_t^N:

$$\pi_t^N(f) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_{t\mathcal{T}_N}^N(x) f(x). \tag{4.1}$$

In Wilson's method, we fix f to be a (possibly non-unique) eigenfunction ψ_1^N corresponding to the lowest nonzero eigenvalue λ_1^N of $-\Delta_N$. In fact we shall choose $\{\psi_1^N\}_N$ such that $\psi_1^N \to \psi_1$ in $C(K)$, per Assumption 2-(2). Applying the generator $\mathcal{T}_N \mathcal{L}_{\text{exc}}^N$ to the function $\pi_s^N(\psi_1^N)$ and summing by parts, we obtain

$$\mathcal{T}_N \mathcal{L}_{\text{exc}}^N \pi_s^N(\psi_1^N) = \pi_s^N(\Delta_N \psi_1^N) = -\lambda_1^N \pi_s^N(\psi_1^N),$$

which shows that $\pi_s^N(\psi_1^N)$ is an eigenfunction of $-\mathcal{T}_N \mathcal{L}_{\text{exc}}^N$ with eigenvalue λ_1^N.

Remark 4.1. This computation shows that the spectral gap of $-\mathcal{L}_{\text{exc}}^N$ is no more than the spectral gap of $-\mathcal{T}_N^{-1} \Delta_N$. It turns out that on any finite connected graph with symmetric weights (conductances), the spectral gaps of the three operators $-\mathcal{T}_N^{-1} \Delta_N$, $-\mathcal{L}_{\text{exc}}^N$ and $-\mathcal{L}_{\text{int}}^N$ are identical. This was known famously as Aldous' spectral gap conjecture in the early 90s, which was proven 20 years later by Caputo, Liggett, and Richthammer [CLR10].

Now observe that for any $s > 0$

$$\partial_s \pi_s^N(\psi_1^N) e^{\lambda_1^N s} = \lambda_1^N \pi_s^N(\psi_1^N) e^{\lambda_1^N s}.$$

Hence by Dynkin's formula (see [KL99, Appendix 1, Lemma 5.1]) applied to the function $\pi_t^N(\psi_1^N) e^{\lambda_1^N t}$, we obtain

$$\pi_t^N(\psi_1^N) e^{\lambda_1^N t} = \pi_0^N(\psi_1^N) + \mathcal{M}_t^N, \tag{4.2}$$

where $\{\mathcal{M}_t^N \, ; \, t \geq 0\}$ is a mean-zero martingale with respect to the natural filtration generated by $\{\pi_t^N(\psi_1^N) \, ; \, t \geq 0\}$, and its quadratic variation is

$$\langle \mathcal{M}_t^N(\psi_1^N) \rangle_t = \frac{\mathcal{T}_N}{|V_N|^2} \int_0^t \sum_{xy \in E_N} e^{2\lambda_1^N s} \left(\eta_{s\mathcal{T}_N}^N(x) - \eta_{s\mathcal{T}_N}^N(y) \right)^2 \left(\psi_1^N(x) - \psi_1^N(y) \right)^2 ds \tag{4.3}$$

$$\leq \frac{\mathcal{T}_N}{|V_N|^2} \int_0^t \sum_{xy \in E_N} e^{2\lambda_1^N s} \left(\psi_1^N(x) - \psi_1^N(y) \right)^2 ds$$

$$= \frac{1}{|V_N|} \left(e^{2\lambda_1^N t} - 1 \right) \frac{1}{\lambda_1^N} \, \mathcal{E}_N \left(\psi_1^N, \psi_1^N \right)$$

$$= \frac{1}{|V_N|} \left(e^{2\lambda_1^N t} - 1 \right) \| \psi_1^N \|_{L^2(\mathfrak{m}_N)}^2 = |V_N|^{-1} (e^{2\lambda_1^N t} - 1). \tag{4.4}$$

Next we state [LP17, Proposition 7.9] which holds on a finite state space Ω. In what follows, given a random variable $X : \Omega \to \mathbb{R}$ and a probability measure μ on Ω, we denote the variance of X with respect to μ by $V_\mu(X)$.

Proposition 4.2. *Let μ and ν be two probability measures on the finite state space Ω, and $X : \Omega \to \mathbb{R}$ be a random variable. If*

$$\left| \int X \, d\mu - \int X \, d\nu \right| \geq A \sqrt{\max\{ V_\mu(X), V_\nu(X) \}},$$

for some $A > 0$, then

$$\| \mu - \nu \|_{TV} \geq 1 - \frac{8}{A^2}.$$

Our goal is to find a lower bound for $\| \mu_t^N - U_{\text{exc}}^N \|_{TV}$, so according to Proposition 4.2, it suffices to estimate the expectation and the variance of the random variable $\pi_t^N(\psi_1^N)$ with respect to the measures $\mathbb{P}_{\mu_N}$ and U_{exc}^N.

Let us start with the estimates with respect to U_{exc}^N. By (2.11), we have

$$(4.5) \qquad \int \pi_t^N(\psi_1^N) \, dU_{\text{exc}}^N = \frac{1}{|V_N|} \sum_{x \in V_N} \left(\int \eta(x) \, dU_{\text{exc}}^N \right) \psi_1^N(x) = \rho_{\text{ss}}^N \int \psi_1^N \, d\mathfrak{m}_N = 0.$$

Furthermore, for any $x, y \in V_N$ with $x \neq y$, we have

$$\int \eta(x)\eta(y) \, dU_{\text{exc}}^N - \int \eta(x) \, dU_{\text{exc}}^N \int \eta(y) \, dU_{\text{exc}}^N = -\frac{1}{|V_N|-1} \frac{k_N}{|V_N|} \left(1 - \frac{k_N}{|V_N|} \right) = -\frac{\rho_{\text{ss}}^N}{|V_N|-1} \left(1 - \rho_{\text{ss}}^N \right),$$

where $\rho_{\text{ss}}^N := k_N/|V_N|$ is the (constant) steady-state density of the process. So by (4.5) we have

$$V_{U_{\text{exc}}^N}(\pi_t^N(\psi_1^N)) = \int \left(\pi^N(\psi_1^N) \right)^2 dU_{\text{exc}}^N$$

$$(4.6) \qquad\qquad = \int \left(\frac{1}{|V_N|} \sum_{x \in V_N} \eta(x)\psi_1^N(x) \right)^2 dU_{\text{exc}}^N = \mathcal{O}(|V_N|^{-1}).$$

Next we estimate with respect to $\mathbb{P}_{\mu_N}$, which will involve the support of the positive part of ψ_1^N. Using (4.2) and the fact that $\mathcal{M}_t^N$ is a mean-zero martingale, we have

$$(4.7) \qquad\qquad \mathbb{E}_{\mu_N}[\pi_t^N(\psi_1^N)] = \pi_0^N(\psi_1^N) \, e^{-\lambda_1^N t}.$$

Let us find an initial distribution μ_N from which the process starts and then mixes slowly enough. Indeed, let $u \in K$ be such that $\psi_1(u) > 0$. Since ψ_1 is continuous by virtue of Assumption 2-(2), there exists an open ball $B(u, r) := \{v \in K \, ; \, d(u, v) < r\}$ (d is the metric on K) such that $\psi_1(v) > 0$ for any $v \in B(u, r)$. For each sufficiently large N, this induces a nonempty set $B_+^N := \{x \in V_N : \psi_1^N(x) > 0\}$. Moreover, for any small enough $\delta > 0$ we have $\lim_{N \to \infty} |B_+^N|/|V_N| > \delta$. Therefore, by a projection argument, if $\delta < \rho_{\text{ss}}^N$, and we can prove a lower bound on the mixing time of the process with density δ, then we also have the same lower bound on the mixing time of the process with density ρ_{ss}^N. Thus, let η_0^N be a configuration in Ω_{N,k_N} (see (2.3)) for which $\eta_0^N(x) = \mathbb{1}_{\{x \in B_+^N\}}$, and let μ_N be the Dirac measure concentrated at η_0^N. Then

$$(4.8) \qquad\qquad \pi_0^N(\psi_1^N) = \frac{1}{|V_N|} \sum_{x \in B_+^N} \psi_1^N(x) = \Theta(1).$$

From the point of view of cards, if one puts cards of the same color together, then it should take some time until the black cards start to mix with the red cards.

Remark 4.3. In the case of the discrete hypercube (§3.5) the last paragraph is replaced by the following: Since ψ_1^N takes values in $\{-1, 1\}$ and defines a 2-coloring on V_N, one can choose $B_+^N \subset \{x \in V_N \, ; \, x_1 = 0\}$ with $|B_+^N| \leq k_N$.

We are now ready to estimate the mean and the variance of $\pi_t^N(\psi_1^N)$ with respect to $\mathbb{P}_{\mu_N}$. On the one hand, by (4.7) and (4.8),

$$(4.9) \qquad\qquad \mathbb{E}_{\mu_N}[\pi_t^N(\psi_1^N)] = \Theta \left(e^{-\lambda_1^N t} \right).$$

On the other hand, by (4.2) and (4.3) we have

$$(4.10) \qquad V_{\mathbb{P}_{\mu_N}}(\pi_t^N(\psi_1^N)) = e^{-2\lambda_1^N t} \mathbb{E}_{\mu_N}[\langle M_t^N(\psi_1^N) \rangle_t] \lesssim \frac{1}{|V_N|} (1 - e^{-2\lambda_1^N t}) \operatorname{Var}(\psi_1^N) \leq \frac{1}{|V_N|}.$$

Combining the estimates (4.5), (4.6), (4.9) and (4.10), then using Proposition 4.2, we conclude that there exists a positive constant c such that for all N,

$$\|\mu_t^N - U_{\text{exc}}^N\|_{TV} \geq 1 - \frac{c \, e^{2\lambda_1^N t/\mathcal{T}_N}}{|V_N|}.$$

It follows that $\lim_{N \to \infty} d_{N,k_N}(\kappa t_N) = 1$ for all $\kappa < 1$.

5. UPPER BOUND ON THE MIXING TIME OF THE INTERCHANGE PROCESS

In this section we obtain a matching upper bound on the mixing time of the interchange process on graphs G_N that either satisfy Assumptions 1 and 2, or are isomorphic to a discrete hypercube (§3.5). The proof is by way of the *cutoff martingale method*.

The method proceeds in three steps. First, one constructs a coupling between two copies $\{\sigma_t^{1,N} ; t \geq 0\}$, $\{\sigma_t^{2,N} ; t \geq 0\}$ of the interchange process $\{\sigma_t^N ; t \geq 0\}$, such that they evolve independently until they merge at an almost-surely finite random time T_N, the coupling time. Denote the joint law of the coupled processes by $\mathbb{P}$, and the corresponding expectation by $\mathbb{E}$, in what follows. By [LP17, Corollary 5.5], this implies the bound

$$(5.1) \qquad D_N(t) \leq \max_{\sigma,\tau} \mathbb{P}(T_N > t \mid \sigma_0^{1,N} = \sigma , \sigma_0^{2,N} = \tau).$$

Second, one identifies a mean-zero observable of the processes—the cutoff martingale—which has magnitude $\mathcal{O}(1)$ at the coupling time T_N almost surely. Finally, one proves that for any $\kappa \geq 1$, the variance of the cutoff martingale at times κt_N tends to ∞ as $N \to \infty$. Combining the last two items, we deduce that $\mathbb{P}(T_N > \kappa t_N) \to 0$ as $N \to \infty$ for any $\kappa > 1$. And then in conjunction with (5.1), we arrive at the desired conclusion that $D_N(\kappa t_N) \to 0$ as $N \to \infty$ for any $\kappa > 1$.

5.1. **Coupling.** We define a coupling between two independent copies $\{\sigma_t^{1,N} ; t \geq 0\}$ and $\{\sigma_t^{2,N} ; t \geq 0\}$ of the process $\{\sigma_t^N ; t \geq 0\}$ starting from respective permutations σ and τ, as follows. Recall that the process with generator $\mathcal{L}_{\text{int}}^N$ presents the following dynamics: we associate with each edge $xy \in E_N$ an independent rate-1 Poisson process $\{\mathcal{T}_k^{xy} ; k \in \mathbb{N}_0\}$. Namely, for each $xy \in E_N$, $\mathcal{T}_0^{xy} = 0$ and $\{(\mathcal{T}_k^{xy} - \mathcal{T}_{k-1}^{xy}) ; k \in \mathbb{N} \cup \{0\}\}$ is a field of i.i.d. exponential random variables with mean 1. The process $\{\mathcal{T}_k^{xy} ; k \in \mathbb{N} \cup \{0\}$ and $xy \in E_N\}$ is known as *Harris process* or *the clock process*, and it is important to observe that Harris construction makes the probability that two clocks rings simultaneously equal to 0. Whenever the clock rings at xy, *i.e.*, at times $t = \mathcal{T}_k^{xy}$, $k \in \mathbb{N}$, we flip the edge $xy \in E_N$. Our coupling is defined with two independent clock processes, one for each copy $\{\sigma_t^{i,N} ; t \geq 0\}$, $i \in \{1,2\}$. Therefore, it is immediate that this constructs two independent copies of the same Markov process generated by $\mathcal{L}_{\text{int}}^N$, and that these two processes merge at some random time $T_N = \inf\{t \geq 0 ; \sigma_t^{1,N} = \sigma_t^{2,N}\}$. After the two processes merge, *i.e.*, for times $t \geq T_N$, both processes are updated using the same clock process. So $\sigma_t^{1,N} = \sigma_t^{2,N}$ for any $t \geq T_N$.

5.2. **Projection to exclusion and hydrodynamic evolution.** For each $i \in \{1,2\}$, let $\{\eta_t^{i,N} : t \geq 0\}$ be the process obtained via projection (2.1) from $\{\sigma_t^{i,N} : t \geq 0\}$. Then let $\rho_t^{i,N}(x) = \mathbb{E}[\eta_t^{i,N}(x)]$ for each $x \in V_N$. Since the interchange process takes place on the symmetric group $S_{|V_N|}$, there is no loss of generality in taking $\sigma_0^{1,N}$ as the identity permutation, and $\sigma_0^{2,N}$ a permutation that maximizes inequality (5.1). A choice on the initial 2-coloring of the cards at $\sigma_0^{1,N}$ generates the two initial states of the exclusion process projections $\{\eta_t^{i,N} ; t \geq 0\}$, $i \in \{1,2\}$. Recall from our assumptions on k_N that we require a nonzero fraction of black and red cards; this is important for estimating the magnitude of (5.10) below.

Applying Kolmogorov's equation $\partial_t \mathbb{E}_{\mu_N^i}[\eta_t^{i,N}(x)] = \mathbb{E}_{\mu_N^i}[\mathcal{T}_N \mathcal{L}_{\text{exc}}^N \eta_t^{i,N}(x)]$, we find that $\rho^{i,N}$ is the unique solution of the discrete heat equation

$$\begin{cases} \partial_t \rho_{t\mathcal{T}_N}^{i,N}(x) = \Delta_N \rho_{t\mathcal{T}_N}^{i,N}(x), & x \in V_N, \\ \rho_0^{i,N}(x) = \eta_0^{i,N}(x), & x \in V_N. \end{cases}$$

Using the spectral decomposition we find that

$$(5.2) \qquad \rho_t^{i,N}(x) = \rho_{\text{ss}}^N + r_t^{i,N}(x),$$

where

$$r_t^{i,N}(x) = \sum_{j=1}^{|V_N|-1} \mathsf{c}_j^{i,N} \, e^{-\lambda_j^N t/\mathcal{T}_N} \, \psi_j^N(x),$$

and

$$\mathsf{c}_j^{i,N} = \int_{V_N} \eta_0^{i,N} \, \psi_j^N \, d\mathfrak{m}_N$$

is the jth Fourier coefficient of $\eta_0^{i,N}$ in the eigenbasis $\{\psi_j^N\}_j$.

5.3. The cutoff martingale and its estimate.

While the empirical measure was good enough to obtain the lower bound on the mixing time, it is not strong enough to deduce a matching upper bound. We need to identify another observable with which a *central limit theorem* can be stated and (possibly) proved. Thus, for $t \geq 0$ we define the observable

$$(5.3) \qquad h_t^N := \sum_{x \in V_N} (\eta_t^{1,N}(x) - \eta_t^{2,N}(x))\psi_1^N(x)$$

and its recentered version

$$(5.4) \qquad \begin{aligned} H_t^N :&= \sum_{x \in V_N} \left((\eta_t^{1,N}(x) - \rho_t^{1,N}(x)) - (\eta_t^{2,N}(x) - \rho_t^{2,N}(x))\right)\psi_1^N(x) \\ &= h_t^N - \sum_{x \in V_N} (\rho_t^{1,N}(x) - \rho_t^{2,N}(x))\psi_1^N(x). \end{aligned}$$

Up to an overall scaling with some power of $|V_N|$, the observable H_t^N is the difference between the *density fluctuation fields* of the coupled processes $\{\eta_t^{i,N} \,;\, t \geq 0\}$, $i \in \{1,2\}$, paired with a lowest eigenfunction ψ_1^N. Plugging (5.2) into (5.4), and using the orthonormality of $\{\psi_j^N\}_j$, we find that

$$H_t^N = h_t^N - \left(\mathsf{c}_1^{1,N} - \mathsf{c}_1^{2,N}\right)|V_N|e^{-\lambda_1^N t/\mathcal{T}_N}.$$

In order to see the putative cutoff we need to change H_t^N in two ways. First, speed up the process by t_N. Second, divide H_t^N by $|V_N|e^{-\lambda_1^N t/\mathcal{T}_N}$ so as to retain $\mathsf{c}_1^{1,N} - \mathsf{c}_1^{2,N}$, a $\mathcal{O}(1)$–magnitude quantity. Altogether this means that we define, for $\kappa \geq 0$,

$$(5.5) \qquad \chi_\kappa^N := |V_N|^{\frac{\kappa}{2}-1} H_{\kappa t_N}^N = |V_N|^{\frac{\kappa}{2}-1} h_{\kappa t_N}^N - \left(\mathsf{c}_1^{1,N} - \mathsf{c}_1^{2,N}\right).$$

By applying Dynkin's formula (*cf.* the proof of Proposition 5.1 below) one can check that $\{\chi_\kappa^N \,;\, \kappa > 0\}$ is a mean-zero martingale with respect to the natural filtration generated by the coupled processes. This is the **cutoff martingale** alluded to in the introduction.

Finally we can explain how the cutoff martingale can be used to establish the sharp upper bound on the mixing time. Assume the coupling time $T_N = \kappa t_N$ for some fixed $\kappa > 0$. On the one hand, we have by definition of the coupling and (5.3) that, $\mathbb{P}$-a.s., $h_t^N = 0$ for any $t \geq T_N$. It follows that

$$\left(\chi_\kappa^N\right)^2 = \left(\mathsf{c}_1^{1,N} - \mathsf{c}_1^{2,N}\right)^2 = \mathcal{O}(1) \quad \mathbb{P}\text{-a.s.}$$

On the other hand, if $\kappa \geq 1$ then the above equality is contradicted by Proposition 5.1 below. We therefore deduce that $\lim_{N \to \infty} \mathbb{P}(T_N > \kappa t_N) = 0$ for any $\kappa > 1$. In conjunction with (5.1) we then conclude that $\lim_{N \to \infty} D_N(\kappa t_N) = 0$ for any $\kappa > 1$, yielding the matching upper bound.

Thus it remains to prove

Proposition 5.1. *For any $\kappa \geq 1$, $\lim_{N \to \infty} \mathbb{E}\left[(\chi_\kappa^N)^2\right] = +\infty$.*

Proof. For $i \in \{1,2\}$ define

$$\chi_\kappa^{i,N} := |V_N|^{\kappa/2-1} \sum_{x \in V_N} (\eta_{\kappa t_N}^{i,N}(x) - \rho_{\kappa t_N}^{i,N}(x))\psi_1^N(x),$$

so clearly $\chi_\kappa^N = \chi_\kappa^{1,N} - \chi_\kappa^{2,N}$. Firstly, we will show that for any $i \in \{1,2\}$, $\chi_\kappa^{i,N}$ is a mean-zero martingale with variance

$$(5.6) \qquad \mathbb{E}\left[(\chi_\kappa^{i,N})^2\right] = t_N|V_N|^{\kappa-2} \int_0^\kappa \sum_{x \in V_N} \sum_{y \sim x} \mathbb{E}\left[\left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y)\right)^2\right] \left(\psi_1^N(x) - \psi_1^N(y)\right)^2 ds.$$

Indeed, observe that

$$(5.7) \qquad t_N \mathcal{L}_{\mathrm{exc}}^N \chi_\kappa^{i,N} = -\frac{1}{2}|V_N|^{\kappa/2-1}\log|V_N| \sum_{x \in V_N} \eta_{\kappa t_N}^{i,N}(x)\psi_1^N(x) = -\partial_\kappa \chi_\kappa^{i,N}.$$

So by Dynkin's formula,

$$\chi_\kappa^{i,N} - \chi_0^{i,N} - \int_0^\kappa \left(t_N \mathcal{L}_{\text{exc}}^N + \partial_s \right) \chi_s^{i,N} \, ds = \chi_\kappa^{i,N}$$

is a mean-zero martingale whose quadratic variation equals

$$t_N |V_N|^{\kappa-2} \int_0^\kappa \sum_{x \in V_N} \sum_{y \sim x} \left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y) \right)^2 \left(\psi_1^N(x) - \psi_1^N(y) \right)^2 \, ds.$$

The claim (5.6) follows.

Now we estimate the variance of χ_κ^N. Since

$$\begin{aligned}
\mathbb{E}\left[(\chi_\kappa^N)^2 \right] &= \mathbb{E}\left[\left(\chi_\kappa^{1,N} - \chi_\kappa^{2,N} \right)^2 \right] \\
&= \mathbb{E}\left[\left(\chi_\kappa^{1,N} \right)^2 - 2 \left(\chi_\kappa^{1,N} \right) \left(\chi_\kappa^{2,N} \right) + \left(\chi_\kappa^{2,N} \right)^2 \right] \\
&= \mathbb{E}\left[\left(\chi_\kappa^{1,N} \right)^2 \right] + \mathbb{E}\left[\left(\chi_\kappa^{2,N} \right)^2 \right] - 2\mathbb{E}\left[\left(\chi_\kappa^{1,N} \right) \left(\chi_\kappa^{2,N} \right) \right]
\end{aligned}$$

and $\chi_\kappa^{1,N}$ and $\chi_\kappa^{2,N}$ are independent martingales prior to the coupling time, the right-hand side of the last display equals

$$\mathbb{E}\left[\left(\chi_\kappa^{1,N} \right)^2 \right] + \mathbb{E}\left[\left(\chi_\kappa^{2,N} \right)^2 \right].$$

By (5.6), it is sufficient to prove that for $\kappa \geq 1$,

$$(5.8) \qquad \lim_{N \to \infty} t_N |V_N|^{\kappa-2} \int_0^\kappa \sum_{i=1}^2 \sum_{x \in V_N} \sum_{y \sim x} \mathbb{E}\left[\left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y) \right)^2 \right] \left(\psi_1^N(x) - \psi_1^N(y) \right)^2 \, ds = +\infty.$$

To this end let us define, for each $i \in \{1, 2\}$, the two-point correlation function

$$\varphi_t^{i,N}(x, y) = \mathbb{E}[(\eta_t^{i,N}(x) - \rho_t^{i,N}(x))(\eta_t^{i,N}(y) - \rho_t^{i,N}(y))].$$

Observe that we can express $\mathbb{E}\left[\left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y) \right)^2 \right]$ as

$$(5.9) \qquad \begin{aligned}
&2\rho_{\text{ss}}^N(1 - \rho_{\text{ss}}^N) - 2\varphi_{st_N}^{i,N}(x, y) \\
&+ \left(1 - 2\rho_{\text{ss}}^N - r_{st_N}^{i,N}(x) \right) r_{st_N}^{i,N}(x) + \left(1 - 2\rho_{\text{ss}}^N - r_{st_N}^{i,N}(y) \right) r_{st_N}^{i,N}(y) + \left(r_{st_N}^{i,N}(x) - r_{st_N}^{i,N}(y) \right)^2 \\
&= 2\rho_{\text{ss}}^N(1 - \rho_{\text{ss}}^N) - 2\varphi_{st_N}^{i,N}(x, y) + (1 - 2\rho_{\text{ss}}^N) \left(r_{st_N}^{i,N}(x) + r_{st_N}^{i,N}(y) \right) - 2r_{st_N}^{i,N}(x) r_{st_N}^{i,N}(y).
\end{aligned}$$

Since we assumed that $\{\eta_t^{i,N} ; t \geq 0\}$ starts from a deterministic configuration, by [And88] the correlation function $\varphi_t^{i,N}(x, y)$ is non-positive for every $t > 0$ and every $x \neq y$. Thus (5.9) is bounded below by

$$(5.10) \qquad 2\rho_{\text{ss}}^N(1 - \rho_{\text{ss}}^N) + (1 - 2\rho_{\text{ss}}^N) \left(r_{st_N}^{i,N}(x) + r_{st_N}^{i,N}(y) \right) - 2r_{st_N}^{i,N}(x) r_{st_N}^{i,N}(y).$$

We claim that under our assumption on k_N, the first term $2\rho_{\text{ss}}^N(1 - \rho_{\text{ss}}^N)$ is the dominant term.

Indeed, since $\lim_{N \to \infty} \rho_{\text{ss}}^N = \rho \in (0, 1)$ and t_N is of order $\Theta(|\mathcal{T}_N| \log |V_N|)$, the last two terms in (5.10) converge to 0 as $N \to \infty$, uniformly for all $x, y \in V_N$ with $x \sim y$. Meanwhile $\lim_{N \to \infty} 2\rho_{\text{ss}}^N(1 - \rho_{\text{ss}}^N) = 2\rho(1 - \rho)$, which is strictly positive. Thus for any $\epsilon > 0$,

$$\mathbb{E}\left[\left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y) \right)^2 \right] > 2\rho(1 - \rho) - \epsilon$$

for all sufficiently large N; and in particular

$$\mathbb{E}\left[\left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y) \right)^2 \right] > \rho(1 - \rho)$$

for all large N. Hence the expression

$$t_N|V_N|^{\kappa-2} \int_0^\kappa \sum_{i=1}^2 \sum_{x \in V_N} \sum_{y \sim x} \mathbb{E}\left[\left(\eta_{st_N}^{i,N}(x) - \eta_{st_N}^{i,N}(y)\right)^2\right] \left(\psi_1^N(x) - \psi_1^N(y)\right)^2 \, ds$$

is bounded below by

$$2\rho(1-\rho)\kappa|V_N|^{\kappa-1}\log|V_N|\left(\frac{1}{\lambda_1^N}\mathcal{E}_N(\psi_1^N, \psi_1^N)\right)$$
$$= 2\rho(1-\rho)\kappa|V_N|^{\kappa-1}\log|V_N| \cdot \|\psi_1^N\|_{L^2(\mathfrak{m}_N)}^2$$
$$= 2\rho(1-\rho)\kappa|V_N|^{\kappa-1}\log|V_N|.$$

This implies (5.8) for $\kappa \geq 1$. $\hfill\square$

6. An open problem

For the discrete Boolean hypercubes G_N, although Assumption 1 does not apply, we can still succeed in proving the sharp convergence to equilibrium of the interchange processes on these graphs. The reason is because $\{G_N\}_N$ is a sequence of transitive graphs whose $(\star)$ **spectral gaps** γ_1^N **are uniformly bounded away from zero**. Inspired by [LP17, Chapter 26.2, Question 5], we ask: should the family of interchange processes on transitive graphs satisfying $(\star)$ present cutoffs at times t_N given in (1.1)?

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