Random walks, electric networks, and coarse-graining in particle systems

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Random walks and electric networks

Title of a famous monograph by Doyle & Snell.
https://math.dartmouth.edu/~doyle/docs/walks/walks.pdf

- Let $G = (V, E)$ be a locally finite connected graph, and $\mathbf{c} = \{c_{xy}\}_{xy \in E}$ be the set of positive weights (conductances) endowed on E.
- The (symmetric) random walk process on the **weighted graph** (=**electric network**) $(G, \mathbf{c})$ is an irreducible Markov chain on V with transition probability

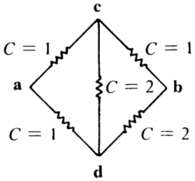
$$\mathbf{P}(x,y) = \begin{cases} c_{xy}/c_x, & \text{if } xy \in E, \\ 0, & \text{otherwise.} \end{cases} \qquad c_x := \sum_{z : xz \in E} c_{xz}.$$

- The RW process has $\pi(\cdot) \propto c(\cdot)$ as reversible (invariant) measure, and the associated Dirichlet energy is

$$\mathcal{E}^{\mathrm{RW}}(f) = \langle f, (\mathbf{I} - \mathbf{P}) f \rangle_\pi = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2, \quad f : V \to \mathbb{R}.$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{2}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}$$

(The entries along each row must add up to 1.)



Conductances

Random walks and electric networks

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- The RW process has $\pi(\cdot) \propto c(\cdot)$ as reversible (invariant) measure, and the associated Dirichlet energy is

$$\mathcal{E}^{\rm RW}(f) = \langle f, (\mathbf{I} - \mathbf{P}) f \rangle_\pi = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2, \quad f : V \to \mathbb{R}.$$

- **Effective resistance** between $A, B \subset V$:

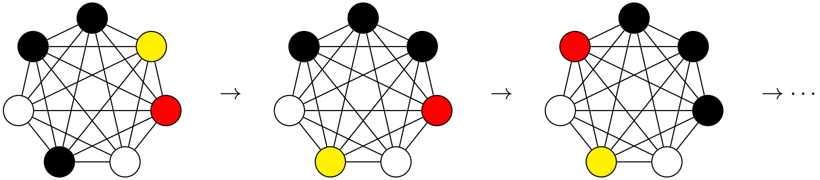
$$R_{\rm eff}(A,B) = \sup \left\{ [\mathcal{E}^{\rm RW}(f)]^{-1} \,\middle|\, f : V \to \mathbb{R},\ f|_A = 1,\ f|_B = 0 \right\}$$

  In particular, if $A = \{x\}$ and $B = \{y\}$ we write $R_{\rm eff}(x,y)$. By definition,

$$[f(x) - f(y)]^2 \leq R_{\rm eff}(x,y) \mathcal{E}^{\rm RW}(f), \quad f : V \to \mathbb{R}.$$

  Also $R_{\rm eff} : V \times V \to \mathbb{R}_+$ is a metric on V.

Interacting particle systems on an electric network



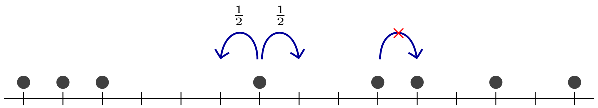
Overarching question: Can we study Markov processes involving MANY interacting "random walkers" on a weighted graph $(G, \mathbf{c})$?

Mathematical development started with Spitzer (on the integer lattice).
Mathematically tractable models:

1. **Exclusion process** (state space $\{0,1\}^V$): Particles perform RWs subject to the exclusion constraint that no two particles can occupy the same vertex at any time.
2. **Zero-range process** (state space $\mathbb{N}_0^V$): Particle at x jumps to neighboring y at rate depending on $\mathbf{P}(x,y)$ [jump] and the number of particles at x ONLY [zero-range kinetics].

Both models are associated with a conserved quantity—the total # of particles (unless additional dynamics or "reservoirs" are attached).

Particle system #1: Exclusion process



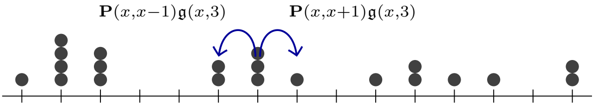
The **(symm.) exclusion process** on $(G, \mathbf{c})$ is a Markov chain on $\{0,1\}^V$ with generator

$$(\mathcal{L}^{\mathrm{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0,1\}^V \to \mathbb{R},$$

where $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$ and $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

- Each product Bernoulli measure ν_α, $\alpha \in [0,1]$, with marginal $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in V$, is an invariant measure.
- Dirichlet energy: $\boxed{\mathcal{E}^{\mathrm{EX}}(f) = \frac{1}{2} \sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{xy} f)(\eta)]^2 \, d\nu_\alpha(\eta)}$.

Particle system #2: Zero-range process



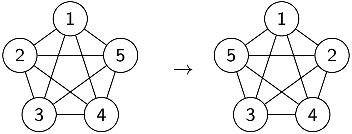
The **zero-range process** on $(G, \mathbf{c})$ is a Markov chain on $\mathbb{N}_0^V$ with generator

$$(\mathcal{L}^{\mathrm{ZR}} f)(\xi) = \sum_{(x,y)\in V^2} \mathbf{P}(x,y)\mathfrak{g}(x,\xi(x)) \left[f(\xi + \mathbf{1}_y - \mathbf{1}_x) - f(\xi) \right], \quad f : \mathbb{N}_0^V \to \mathbb{R}.$$

where $\mathbf{P}$ is an irreducible jump Markov matrix on V^2, and $\mathfrak{g} : V \times \mathbb{N}_0 \to \mathbb{R}_+$ is the kinetic rate, $\mathfrak{g}(x,0) = 0$ always.

- Invariant measure is a product one: $\mu(\xi) = \dfrac{1}{Z} \displaystyle\prod_{x\in V} \prod_{k=1}^{\xi(x)} \frac{\pi(x)}{\mathfrak{g}(x,k)}$, where π is the invariant measure for $\mathbf{P}$.
- Dirichlet energy: $\mathcal{E}^{\mathrm{ZR}}(f) = \langle f, -\mathcal{L}^{\mathrm{ZR}} f \rangle_\mu$.

Hierarchy of stochastic processes on a fixed graph

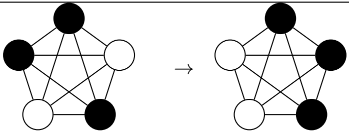


Interchange process $f : \{\text{Permutations on } V\} \to \mathbb{R}$

$$\mathcal{E}^{\rm IP}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 \, d\nu(\eta).$$

Reversible measure: uniform measure ν on $\{\text{Perms on } V\}$.

↓ PROJECTION ↓



Exclusion process $f : \{0,1\}^V \to \mathbb{R}$

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Reversible measure: product Bernoulli measure ν_α, $\alpha \in [0,1]$, $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for all $x \in V$.

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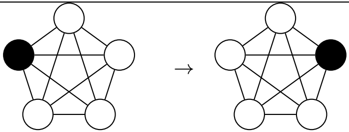


Random walk process $f : V \to \mathbb{R}$

$$\mathcal{E}^{\rm RW}(f) = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2.$$

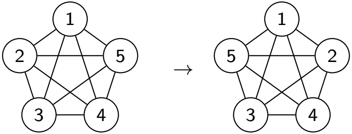
Reversible measure: $c(\cdot) = \sum_{w \sim \cdot} c_{w\cdot}$.

Aldous' spectral gap conjecture '92: Is $\lambda_2^{\rm EX}(G) = \lambda_2^{\rm RW}(G)$?

Easy to see that $\lambda_2^{\rm RW}(G) \geq \lambda_2^{\rm EX}(G)$.

For the other direction, suffice to prove that $\lambda_2^{\rm IP}(G) \geq \lambda_2^{\rm RW}(G)$.

Hierarchy of stochastic processes on a fixed graph

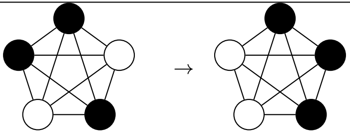


Interchange process $f : \{\text{Permutations on } V\} \to \mathbb{R}$

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \le R_{\text{eff}}(x,y)\mathcal{E}^{\text{IP}}(f).$$

Moving particle lemma

↓ PROJECTION ↓

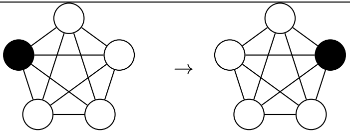


Exclusion process $f : \{0,1\}^V \to \mathbb{R}$

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2 \, d\nu_\alpha(\eta) \le R_{\text{eff}}(x,y)\mathcal{E}^{\text{EX}}(f).$$

Moving particle lemma

↓ PROJECTION ↓



Random walk process $f : V \to \mathbb{R}$

$$[f(x) - f(y)]^2 \le R_{\text{eff}}(x,y)\mathcal{E}^{\text{RW}}(f).$$

Dirichlet principle

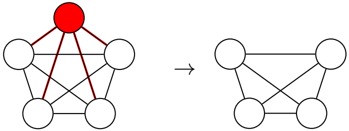
(Also a dual version involving flows: Thomson principle)

Energy inequalities

Does the MPL follow trivially from the Dirichlet principle? NO!

However, a common idea is **electric network reduction** (Schur complementation in linear algebra).

Network reduction: an exercise in Schur complements



Idea: Remove vertices (and edges attached to them) without changing the effective conductance between any of the non-removed vertices.

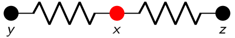
- Suppose we remove the vertex $x \in V$ from $(G, \mathbf{c})$, as well as the edges attached to x. Call the reduced graph $G_x = (V_x, E_x)$. In the linear algebra language, we will reduce the Laplacian $\mathbf{L}$ to a new Laplacian $\mathbf{L}'$ (of one fewer dimension). This is attained by taking the Schur complement of the (x, x) block in $\mathbf{L}$:

$$\text{If } \mathbf{L} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{L}_{xx} \end{bmatrix}, \quad \text{then } \mathbf{L}' = \mathbf{X} - \mathbf{Y}(\mathbf{L}_{xx})^{-1}\mathbf{Z} = \mathbf{X} - \mathbf{Y}\mathbf{Z}. \qquad (\text{Recall } \mathbf{L}_{xx} = -1.)$$

- In component form, $\mathbf{L}'_{yz} = \mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}$ for $y, z \in V_x$. Since $\mathbf{L}^{(\prime)}_{yz} = -p^{(\prime)}_{yz} = -\frac{c^{(\prime)}_{yz}}{c_y}$ whenever $y \neq z$, we see that the new conductances on E_x become

$$c'_{yz} = -c_y \mathbf{L}'_{yz} = -c_y(\mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}) = c_{yz} + \frac{c_{yx}c_{xz}}{c_x} =: c_{yz} + \tilde{c}_{yz}.$$

Example 1: Series Law



Let $c_{xy} = \alpha$ and $c_{xz} = \beta$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} & 1 \end{bmatrix}.$$

Let $\mathbf{L}'$ be the Schur complement of the 1 block in $\mathbf{L}$:

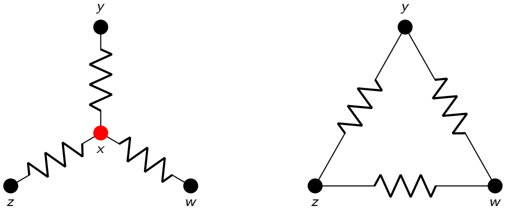
$$\mathbf{L}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \\ -\frac{\alpha}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

So $\mathbf{L}'_{yz} = -\frac{\beta}{\alpha+\beta}$. Since $c_y = \alpha$, we get $c'_{yz} = -c_y \mathbf{L}_{yz} = \dfrac{\alpha\beta}{\alpha+\beta}$, *i.e.*,

$$R'_{yz} = \frac{1}{c'_{yz}} = \frac{1}{\alpha} + \frac{1}{\beta} = R_{xy} + R_{xz}.$$

(Resistors in series ADD!)

Example 2: Y-Δ transform



Let $c_{xy} = \alpha$, $c_{xz} = \beta$, $c_{xw} = \gamma$, and $\sigma = \alpha+\beta+\gamma$.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \alpha/\sigma & \beta/\sigma & \gamma/\sigma & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma & 1 \end{bmatrix}.$$

$$\mathbf{L}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma \end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix} \beta+\gamma & -\beta & -\gamma \\ -\alpha & \alpha+\gamma & -\gamma \\ -\alpha & -\beta & \alpha+\beta \end{bmatrix}.$$

After a little more algebra we get

$$c'_{yz} = \frac{\alpha\beta}{\sigma}, \quad c'_{zw} = \frac{\beta\gamma}{\sigma}, \quad c'_{wy} = \frac{\gamma\alpha}{\sigma}.$$

(Anyone who studied electric circuits would find this familiar!)

Proof of Dirichlet's principle via network reduction

$$\mathcal{E}(f) = \sum_{zw\in E} c_{zw}[f(z)-f(w)]^2.$$

In going from G to the reduced graph G_x, energy is

- lost due to the removal of edges attached to x: amount $\sum_{y\in V_x} c_{xy}[f(x)-f(y)]^2$.
- gained due to the increased conductance on the non-removed edges: amount $\sum_{yz\in E_x} \tilde{c}_{yz}[f(y)-f(z)]^2$.

Proposition ("Octopus inequality" for electric network). For all $f : V \to \mathbb{R}$,

$$\sum_{y\in V_x} c_{xy}[f(x)-f(y)]^2 \geq \sum_{yz\in E_x} \tilde{c}_{yz}[f(y)-f(z)]^2,$$

Energy lost from removed edges $\geq$ Energy gained from increased conductances

where equality is attained iff $(\mathbf{L}f)(x) = 0$.

Proof. An exercise in high school algebra.

Corollary. The Dirichlet energy is *monotone non-increasing* upon successive network reductions.

By carrying out network reduction one vertex at a time until two vertices z and y are left, we recover Dirichlet's principle: $\mathcal{E}(f) \geq c_{\rm eff}(z,y)[f(z)-f(y)]^2$.

Why the name "octopus"? The tentacular nature of removing of a vertex and its edges may remind you of an octopus. [This name was christened by Pietro Caputo.]

Octopus inequality & Aldous' spectral gap conjecture

Using the network reduction idea & delicately carrying out a series of Schur complementations, Caputo–Liggett–Richthammer *JAMS* '10 proved for the interchange process:

Theorem (Octopus inequality, IP)

For all $f : \mathcal{S}_{|V|} \to \mathbb{R}$,

$$\int \sum_{y \in V_x} c_{xy} [f(\eta^{xy}) - f(\eta)]^2 \, d\nu(\eta) \geq \int \sum_{yz \in E_x} \tilde{c}_{yz} [f(\eta^{yz}) - f(\eta)]^2 \, d\nu(\eta).$$

Energy lost from removed edges $\geq$ *Energy gained from increased conductances*

This was the key inequality which resolved Aldous' '92 spectral gap conjecture:

$$\text{(OI)} \implies \lambda_2^{\mathrm{IP}}(G) \geq \lambda_2^{\mathrm{RW}}(G) \underset{+\text{proj.}}{\implies} \lambda_2^{\mathrm{IP}}(G) = \lambda_2^{\mathrm{EX}}(G) = \lambda_2^{\mathrm{RW}}(G).$$

- **MathSciNet review of CLR10, by L. Miclo:** "One leaves this beautiful paper with the dream that maybe a simpler proof could be found."
- Since then there have been attempts at simplifying the CLR proof, but to little avail.
- Also it was unclear if the octopus has any applications beyond resolving the spectral gap conjecture...

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RECENT DEVELOPMENTS — Applications of the octopus:

- C. '17, Moving particle lemma, used to carry out coarse-graining in the exclusion process towards proving hydrodynamic limits.
- Alon–Kozma '18, Improved estimates of mixing times of interchange process, energy level ordering in the Heisenberg ferromagnetic model.
  `arXiv:1811.10537`: "The first to use the octopus lemma for something new was Chen."
- (Related) Hermon–Salez '18: Analog of Aldous' spectral gap conjecture for the zero-range process, used to establish comparison theorems for two zero-range processes with the same kinetics on the same graph.

Moving particle lemma for interchange/exclusion [C. *ECP* '17]

Bounding the energy cost of swapping two particles at x and y in an interacting particle system by the effective resistance between x and y w.r.t. the random walk process.

Theorem (MPL, IP/EX)

$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2\, d\nu(\eta) \leq R_{\rm eff}(x,y)\mathcal{E}^{\rm IP}(f), \quad f : \mathcal{S}_{|V|} \to \mathbb{R},$$
$$\frac{1}{2}\int [f(\eta^{xy}) - f(\eta)]^2\, d\nu_\alpha(\eta) \leq R_{\rm eff}(x,y)\mathcal{E}^{\rm EX}(f), \quad f : \{0,1\}^V \to \mathbb{R}.$$

Proof sketch.

- (OI) $\Leftrightarrow$ monotonicity of energy under 1-point network reductions. So reduce G successively until two vertices x, y are left, we get

$$\mathcal{E}^{\rm IP}(f) \geq \cdots \geq \frac{1}{2}\int c_{\rm eff}(x,y)[f(\eta^{xy}) - f(\eta)]^2\, d\nu(\eta). \quad \text{MPL for IP}$$

- To obtain the MPL for EX, use the projection of IP onto EX & disintegration of the uniform measure into orthonormal chambers with fixed particle number.

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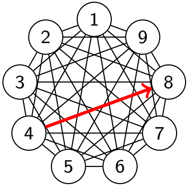

Conventional approach is to pick a single path connecting x and y and obtain the energy cost. [Guo–Papanicolaou–Varadhan '88, Diaconis–Saloff-Coste '93].

Works just fine on finite integer lattices, but does NOT always give optimal cost on general weighted graphs.

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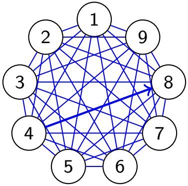
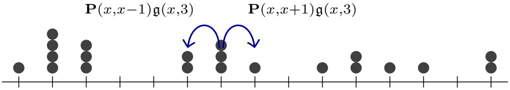


MPL bounds the energy cost by "optimizing electric flow over all paths connecting x and y."

Zero-range process $\leftrightarrow$ random walk process



$$(\mathcal{L}^{\mathrm{ZR}}f)(\xi) = \sum_{(x,y)\in V^2} \mathbf{P}(x,y)\mathfrak{g}(x,\xi(x))\left[f(\xi+\mathbf{1}_y-\mathbf{1}_x)-f(\xi)\right], \quad \text{inv. meas. } \mu.$$

Let $\Omega := \left\{\xi \in \mathbb{N}_0^V : \sum_{x\in V}\xi(x) = m\right\}$ and $\hat{\Omega} := \left\{\zeta \in \mathbb{N}_0^V : \sum_{x\in V}\zeta(x) = m-1\right\}$.

For each $f:\Omega\to\mathbb{R}$ and $\zeta\in\hat{\Omega}$, define $f_\zeta : V\to\mathbb{R}$ by $f_\zeta(x) = f(\zeta+\mathbf{1}_x)$.

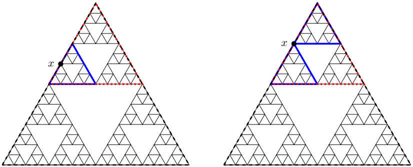
Lemma. For all $f,g:\Omega\to\mathbb{R}$, $\mathcal{E}^{\mathrm{ZR}}_{(\mathbf{P},\mathfrak{g},m)}(f,g) = \sum_{\zeta\in\hat{\Omega}}\mu(\zeta)\langle f_\zeta, (\mathbf{I}-\mathbf{P})g_\zeta\rangle_\pi.$ (Jump part decouples)

Theorem [Hermon–Salez '18]. For any two irred. jump matrices **P** and **Q**,

$$\min_{\substack{f:\Omega\to\mathbb{R}\\ f\neq 0}}\left\{\frac{\mathcal{E}^{\mathrm{ZR}}_{(\mathbf{P},\mathfrak{g},m)}(f,f)}{\mathcal{E}^{\mathrm{ZR}}_{(\mathbf{Q},\mathfrak{g},m)}(f,f)}\right\} = \min_{\substack{f:V\to\mathbb{R}\\ f\neq 0}}\left\{\frac{\langle f,(\mathbf{I}-\mathbf{P})f\rangle_{\pi_{\mathbf{P}}}}{\langle f,(\mathbf{I}-\mathbf{Q})f\rangle_{\pi_{\mathbf{Q}}}}\right\}.$$

Proposition [C.]. If **P** is associated to a symm. RW, then we have the **MPL**

$$\sum_{\zeta\in\hat{\Omega}}[f(\zeta+\mathbf{1}_y)-f(\zeta+\mathbf{1}_x)]^2\mu(\zeta) \le R_{\mathrm{eff}}(x,y)\,\mathcal{E}^{\mathrm{ZR}}_{(\mathbf{P},\mathfrak{g},m)}(f,f).$$



For finite $\Lambda \subset V$, denote $\mathrm{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z\in\Lambda} \eta(z)$.
In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$) on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

Replacement lemma

$$\varlimsup_{\epsilon\downarrow 0} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t \left(\eta_s^N(x) - \mathrm{Av}_{B(x,\epsilon N)}[\eta_s^N]\right) ds\right|\right] = 0, \quad x \in V_N.$$

where

- $\{\eta_t^N : t \geq 0\}$ is the exclusion process generated by $\mathcal{T}_N \mathcal{L}_N^{\mathrm{EX}}$, where $\mathcal{T}_N$ is the diffusive time acceleration factor.
- μ_N can be any measure on $\{0,1\}^{V_N}$.
- $B(x,r)$ is a "ball" of radius r centered at x (in the graph metric).

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator $\mathcal{T}_N\mathcal{L}_N^{\rm EX}$) on a sequence of graphs $G_N=(V_N,E_N)$, we need to prove that for every $t>0$:

Replacement lemma

$$\overline{\lim_{\epsilon\downarrow 0}}\,\overline{\lim_{N\to\infty}}\,\mathbb{E}_{\mu_N}\left[\left|\int_0^t g(\eta_s^N)\,ds\right|\right]=0,\ \text{where } g(\eta):=\eta(x)-{\rm Av}_{B(x,\epsilon N)}[\eta],\ x\in V_N.$$

The usual method to control additive functionals of the EX process is to employ the entropy inequality, Jensen's inequality, and the Feynman-Kac formula:

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t g(\eta_s^N)\,ds\right|\right]\leq\frac{H(\mu_N|\nu^N_{\rho(\cdot)})}{\kappa|V_N|}+\frac{1}{\kappa|V_N|}\sup_f\left\{\int g(\eta)f(\eta)d\nu^N_{\rho(\cdot)}(\eta)-\frac{\mathcal{T}_N}{\kappa|V_N|}\langle\sqrt{f},-\mathcal{L}_N\sqrt{f}\rangle_{\nu^N_{\rho(\cdot)}}\right\}$$

where

- $\rho(\cdot)\in{\rm dom}\mathcal{E}$ is a (possibly non)constant reference density profile.
- $H(\mu|\nu)=\int\log\left(\frac{d\mu}{d\nu}\right)d\mu$ is the relative entropy of μ w.r.t. ν, assumed to be $\mathcal{O}(|V_N|)$.
- $\kappa>0$.
- The supremum is taken over all prob. densities f w.r.t. the product Bernoulli measure $\nu^N_{\rho(\cdot)}$.

MPL & coarse-graining

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator $\mathcal{T}_N \mathcal{L}_N^{\rm EX}$) on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

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Assume for this discussion that $\rho(\cdot) = \rho$ constant. We wish to estimate

$$\int g(\eta) f(\eta) d\nu_\rho^N(\eta) - \frac{\mathcal{T}_N}{\kappa |V_N|}\langle \sqrt{f}, -\mathcal{L}_N \sqrt{f}\rangle_{\nu_\rho^N}$$

independent of f and the carré du champ

$$\mathcal{D}_N(\sqrt{f}, \nu_\rho^N) := \frac{1}{2}\int \sum_{zw \in E_N} c_{zw}\left(\sqrt{f(\eta^{zw})} - \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta).$$

Using the Cauchy-Schwarz (Young) inequality and several elementary tricks, we get for any $A > 0$,

$$\int g(\eta) f(\eta)\, d\nu_\rho^N(\eta) \le \frac{1}{2|B|}\sum_{z\in B}\left\{\frac{A}{2}\int (\eta(z) - \eta(x))^2 \left(\sqrt{f(\eta^{zx})} + \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta) + \frac{1}{2A}\int \left(\sqrt{f(\eta^{zx})} - \sqrt{f(\eta)}\right)^2 d\nu_\rho^N(\eta)\right\}. \quad (B = B(x, \epsilon N))$$

MPL & coarse-graining

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator $\mathcal{T}_N \mathcal{L}_N^{\rm EX}$) on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

Replacement lemma

$$\varlimsup_{\epsilon\downarrow 0} \varlimsup_{N\to\infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t g(\eta_s^N)\, ds \right| \right] = 0, \text{ where } g(\eta) := \eta(x) - {\rm Av}_{B(x,\epsilon N)}[\eta], \; x \in V_N.$$

This last term needs to be bounded by something times the carré du champ

$$\mathcal{D}_N(\sqrt{f}, \nu_\rho^N) := \frac{1}{2} \int \sum_{zw\in E_N} c_{zw} \left(\sqrt{f(\eta^{zw})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta).$$

Use the MPL:

$$\begin{aligned} \frac{1}{2|B|} \sum_{z\in B} \int \left(\sqrt{f(\eta^{zx})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta) &\leq \frac{1}{|B|} \sum_{z\in B} R_{\rm eff}(z,x) \mathcal{D}_N(\sqrt{f}, \nu_\rho^N) \\ &\leq {\rm diam}_R(B) \mathcal{D}_N(\sqrt{f}, \nu_\rho^N), \end{aligned}$$

where ${\rm diam}_R(B)$ is the diameter of B in the resistance metric. $(B = B(x, \epsilon N))$

Assuming that $\dfrac{|V_N|}{\mathcal{T}_N} {\rm diam}_R(B)$ is bounded for all N—this holds for **resistance spaces** in general— we can then choose A wisely to bound the variational functional from above by an expression which tends to 0 in the limit. This proves the replacement lemma.

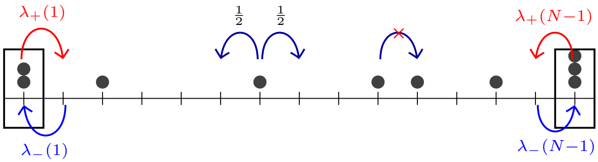
MPL & coarse-graining

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator $\mathcal{T}_N \mathcal{L}_N^{\rm EX}$) on a sequence of graphs $G_N = (V_N, E_N)$, we need to prove that for every $t > 0$:

Replacement lemma

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{N \to \infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t g(\eta_s^N)\, ds\right|\right] = 0, \text{ where } g(\eta) := \eta(x) - \mathrm{Av}_{B(x,\epsilon N)}[\eta], \;\; x \in V_N.$$

- AFAIK this is the first time such an argument works on a non-lattice weighted graph, where translational invariance is absent.
- Another instance where one needs to prove such a replacement lemma in the absence of translational invariance: Studying non-equilibrium density fluctuations on $(\mathbb{Z}/N\mathbb{Z})^d$. Jara–Menezes '18 came up with their coarse-graining approach, called the "flow lemma," which utilizes mass distribution on the lattice, and is reminiscent of the divisible sandpile problem [Levine–Peres '09].
- **Other usages of MPL:** Local 2-blocks estimate [C. '17]; 2nd-order Boltzmann-Gibbs principle for equilibrium density fluctuations [C. '19+].



Designate a finite boundary set $\partial V \subset V$. For each $a \in \partial V$:

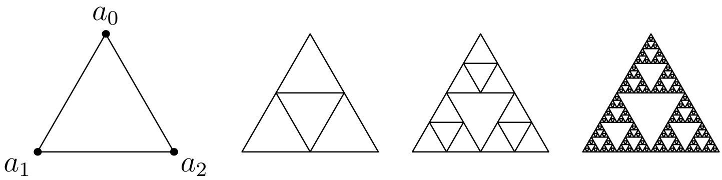
- At rate $\lambda_+(a)$, $\eta(a) = 0 \to \eta(a) = 1$ (birth).
- At rate $\lambda_-(a)$, $\eta(a) = 1 \to \eta(a) = 0$ (death).

Formally, $(\mathcal{L}^{\text{boun}}_{\partial V} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1-\eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)], \quad f : \{0,1\}^V \to \mathbb{R}$, where

$$\eta^a(z) = \begin{cases} 1-\eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

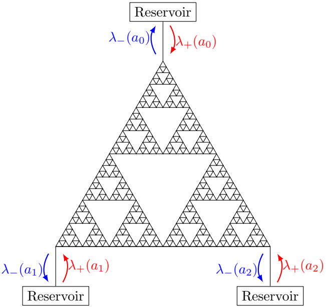
1D boundary-driven simple exclusion process: generator $N^2 \left(\mathcal{L}^{\text{EX}}_{\{1,2,\cdots,N-1\}} + \mathcal{L}^{\text{boun}}_{\{1,N-1\}}\right)$.
Has been studied extensively for the past ~ 15 years.
Hydrodynamic limits, fluctuations, large deviations, etc.

Boundary-driven exclusion process on the Sierpinski gasket



- Construction of Brownian motion with invariant measure m (the standard self-similar measure) as scaling limit of RWs accelerated by $\mathcal{T}_N = 5^N$.
  [Goldstein '87, Kusuoka '88, Barlow-Perkins '88]
- A robust notion of calculus on SG which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.
  [See the books by Kigami '01, Strichartz '06]
- A good model for rigorously studying (non)equilibrium stochastic dynamics with ≥ 3 boundary reservoirs.

Boundary-driven exclusion process on the Sierpinski gasket



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Parameter $b > 0$ governs the inverse speed (relative to the bulk jump rate) at which the reservoir injects/extracts particles into/from the boundary vertices V_0.

Joint work with Patrícia Gonçalves (IST Lisboa), `arXiv:1904.xxxxx`

There is a **phase transition** in the scaling limit of the particle density depending on the value of b. The critical value of b is $\frac{5}{3}$.

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Hydrodynamic limit with slowed boundary [C.–Gonçalves, `arXiv:1904.xxxxx`]

Assume that sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on $\{0,1\}^{V_N}$ is associated to a density profile $\varrho : K \to [0,1]$: for any continuous function $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta \in \{0,1\}^{V_N} \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} F(x)\eta(x) - \int_K F(x)\varrho(x)\, dm(x) \right| > \delta \right\} = 0.$$

Given the process $\{\eta_t^N : t \geq 0\}$ generated by $5^N \mathcal{L}_N^{\mathrm{bEX}}$, the **empirical density measure** π_t^N given by

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) \delta_{\{x\}}$$

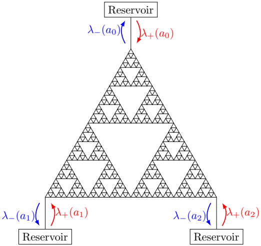
and for any test function $F : K \to \mathbb{R}$, we denote the integral of F wrt π_t^N by $\pi_t^N(F)$ which equals

$$\pi_t^N(F) = \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x).$$

Claim. The sequence $\{\pi_\cdot^N\}_N$ converges in the Skorokhod topology on $D([0,T], \mathcal{M}_+)$ to the unique measure $\pi_\cdot$ with $d\pi_\cdot(x) = \rho(\cdot, x)\, dm(x)$.
For any $t \in [0,T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

Hydrodynamic limit with slowed boundary [C.–Gonçalves, arXiv:1904.xxxxx]



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

$$\bar{\rho}(a) = \frac{\lambda_+(a)}{\lambda_\Sigma(a)}$$

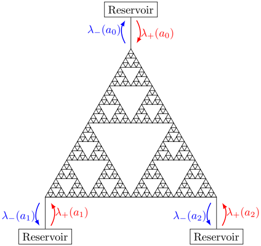
Theorem (Density hydrodynamic limit)

For any $t \in [0, T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N \ : \ \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t,x)\, dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with Dirichlet boundary condition if $b < \frac{5}{3}$:

$$\begin{cases} \partial_t \rho(t,x) = \frac{2}{3} \Delta \rho(t,x), & t \in [0,T],\ x \in K \setminus V_0, \\ \rho(t,a) = \bar{\rho}(a), & t \in (0,T],\ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{cases}$$



$$5^N \mathcal{L}_N^{\rm bEX} = 5^N \left(\mathcal{L}_N^{\rm EX} + \frac{1}{b^N} \mathcal{L}_N^{\rm boun} \right).$$

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Theorem (Density hydrodynamic limit)

For any $t \in [0,T]$, any continuous $F : K \to \mathbb{R}$ and any $\delta > 0$,

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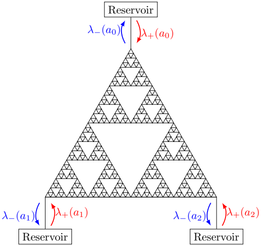
where ρ is the unique weak solution of the heat equation
with Neumann boundary condition if $b > \frac{5}{3}$:

$$\left\{ \begin{array}{ll} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t\in[0,T],~ x \in K \setminus V_0, \\ (\partial^\perp \rho)(t,a) = 0, & t \in (0,T],~ a \in V_0, \\ \rho(0,x) = \varrho(x), & x \in K. \end{array} \right.$$



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$$\lambda_\Sigma(a) = \lambda_+(a) + \lambda_-(a)$$

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Theorem (Density hydrodynamic limit)

For any $t \in [0,T]$, any continuous $F: K \to \mathbb{R}$ and any $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left\{ \eta_\cdot^N ~:~ \left| \frac{1}{|V_N|} \sum_{x\in V_N} \eta_t^N(x) F(x) - \int_K F(x)\rho(t,x)\,dm(x) \right| > \delta \right\} = 0,$$

where ρ is the unique weak solution of the heat equation
with linear Robin boundary condition if $b = \frac{5}{3}$:

$$\left\{ \begin{array}{ll} \partial_t \rho(t,x) = \frac{2}{3}\Delta\rho(t,x), & t\in[0,T],~ x \in K\setminus V_0, \\ (\partial^\perp \rho)(t,a) = -\lambda_\Sigma(a)(\rho(t,a) - \bar\rho(a)), & t \in (0,T],~ a\in V_0, \\ \rho(0,x) = \varrho(x), & x\in K. \end{array} \right.$$

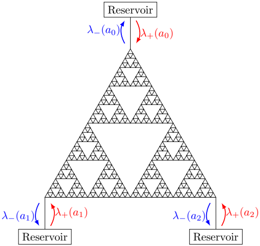
Proof ingredients

Analysis of Dynkin's martingale (which has QV tending to 0 as $N \to \infty$):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N\left(\left(\frac{2}{3}\Delta + \partial_s\right)F_s\right)\,ds$$
$$+ \int_0^t \frac{3^N}{|V_N|}\sum_{a\in V_0}\left[\eta_s^N(a)(\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N}\lambda_\Sigma(a)(\eta_s^N(a)-\bar{\rho}(a))F_s(a)\right]\,ds + o_N(1).$$

1. Analysis on fractals
   - Discrete Laplacian converges to the fractal Laplacian: Operator convergence, energy convergence.
   - Integration by parts formula.
   - Estimate of diameter of cells in the resistance metric.
   - Sobolev embedding: $\mathrm{dom}\mathcal{E} \subset C(K)$.
2. **Functional inequalities in the exclusion process** (FOCUS of this talk)
   - Moving particle lemma.
   - Density replacement lemma at each boundary point: variational estimates of additive functionals via Feynman-Kac. Proof is sensitive to the regime of b (DRN).
3. Convergence of stochastic processes
   - Prove tightness of the sequence $\{\pi_\cdot^N\}_N$ in $D([0,T],\mathcal{M}_+)$ via Aldous' criterion.
   - Characterization of the limit point: Almost surely, any limit point π is AC w.r.t. the standard self-similar measure m, and the density $\rho \in L^2(0,T,\mathrm{dom}\mathcal{E})$.
   - Uniqueness of the limit point: Show that ρ satisfies the weak formulation of the heat equation, and use Oleinik's method to prove uniqueness of the solution.

A sneak preview of upcoming series of works, and **Thank you!**



$$5^N \mathcal{L}_N^{\mathrm{bEX}} = 5^N \left(\mathcal{L}_N^{\mathrm{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\mathrm{boun}} \right).$$

Symmetric exclusion process with **slowed** boundary on the Sierpinski gasket

Dirichlet ($b < \frac{5}{3}$), Robin ($b = \frac{5}{3}$), Neumann ($b > \frac{5}{3}$)

Equilibrium $\Leftrightarrow \lambda_+(a) = \lambda_+$ and $\lambda_-(a) = \lambda_-$ for all $a \in V_0$. (Otherwise, **nonequilibrium**.)

- (Non)equilibrium density hydrodynamic limit (DRN✓) [C.–Gonçalves, `arXiv:1904.xxxxx`]
- Ornstein-Uhlenbeck limit of equilibrium density fluctuations (DRN✓). [C.–Gonçalves, `arXiv:1904.xxxxx`]
- Large deviations principle for the (non)equilibrium density (D✓) [C.–Hinz '19]
- Hydrostatic limit, scaling limit of nonequilibrium density fluctuations (D in progress). [C.–Franceschini–Gonçalves–Menezes '19+] $\to$ *careful study of two-particle correlations*
- **More in the pipeline:**

  Motion of the tagged particle (a fractional BM on the gasket?).

  Add (suitably rescaled) weak asymmetry to the jump rate, prove that the equilibrium density fluctuations converges (subsequentially) to a stochastic Burgers equation [C. '19+]