The abelian sandpile growth problem on the Sierpinski gasket is exactly solved

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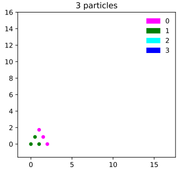


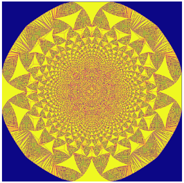
Based on joint work with Jonah Kudler-Flam (UChicago)
arXiv:1807.08748

Limit shape results for Laplacian growth models (1992–2018)

$\mathbb{Z}^d$ $(d \geq 2)$ For all models: Launch $|B_o(n)|$ chips from o.

Model	*Shape theorem/conjecture*
IDLA	In/out-radius $\begin{cases} n \pm \mathcal{O}(\log n), & d = 2 \\ n \pm \mathcal{O}(\sqrt{\log n}), & d \geq 3 \end{cases}$ [α,β,γ,δ]
Rotor-router aggregation	In-radius $n - c\log n$, out-radius $n + c' \log n$ [κ,ℓ]
Divisible sandpiles	In-radius $n - c$, out-radius $n + c'$ [κ]
Abelian sandpiles	$(d = 2)$ Limit shape closer to a dodecagon than Euc ball [κ] Rigorous upper/lower estimates available (with a gap) [κ,ι]

[α] Lawler–Bramson–Griffeath '92
[β] Lawler '95
[γ] Asselah–Gaudillière '13 (2x)
[δ] Jerison–Levine–Sheffield '13, '14
[κ] Levine–Peres '09
[ℓ] Levine–Peres '17
[ι] Fey–Levine–Peres '10



Levine–Peres, "Laplacian growth, sandpiles, and scaling limits." *Bull. Amer. Math. Soc.* (2017).

Limit shape results for Laplacian growth models (1992–2018)

Sierpinski gasket (SG) For all models: Launch from the corner vertex o.

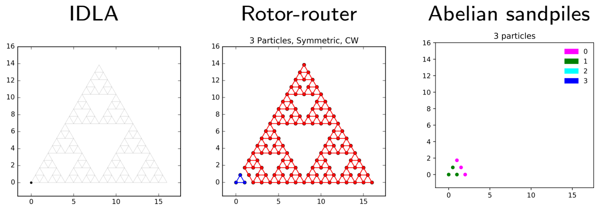
Model	*Initial chip #*	*Shape theorem/conjecture*
IDLA	$\lvert B_o(n)\rvert$	In/out-radius $n \pm \mathcal{O}(\sqrt{\log n})$ [1,2]
RRA	m	In-radius $n_m - 2$, out-radius n_m [2]
DSM	m	In-radius $n_m - 1$, out-radius n_m [3]
ASM	m	Receiving set is a ball $B_o(r_m)$ $r_m = m^{1/d_H}[\mathcal{G}(\log m) + o(1)]$ as $m \to \infty$ [2] ($\mathcal{G}$ is an explicit $(\log 3)$-periodic function)

[1] C.–Huss–Sava-Huss–Teplyaev '17
[2] C.–Kudler-Flam '18
[3] Huss–Sava-Huss '17

Theorem (Limit shape universality on SG)

On SG, clusters in all 4 single-source growth models fill balls in the graph metric.

First nontrivial state space (beyond $\mathbb{Z}$) where **limit shape universality** holds.



- Ecaterina Sava-Huss' talk: addressed the IDLA and divisible sandpile shape theorems. [C.–Huss–Sava-Huss–Teplyaev '17, Huss–Sava-Huss '17]
- **This talk** Solving the **abelian sandpile growth problem** on *SG*. (Started with excellent numerical work by Jonah, ended with my proving every detail of his numerical findings.)
- For the rest of the limit shape universality story (*e.g.* rotor-router aggregation) which I can't get to, see [C.–Kudler-Flam '18]. Also consult local experts (Peres, Sava-Huss).

Conjecture. Limit shape universality holds on a nested fractal graph (Lindstrøm).

- Limit shapes for IDLA, DSM, and RRA are plausibly universal. (Estimate of harmonic functions, cut points, symmetries ...)
- Analysis of the abelian sandpile model is delicate ... (even on *SG*!)

The abelian sandpile model [Bak–Tang–Wiesenfeld '87, Dhar, Majumdar, ...]

Place sand grains ("chips") on vertices of a graph (say, on the square lattice).
If # of chips at vertex $x \geq$ the degree of x, emit one chip to each neighbor of x. This is called a toppling at x.
Continue till the # of chips $<$ degree on every vertex. Then we say the configuration is stable.

0	0	0
0	4	0
0	0	0

$\longrightarrow$

0	1	0
1	0	1
0	1	0

0	0	0	0
0	4	5	0
0	0	0	0

$\longrightarrow$

0	1	0	0
1	0	6	0
0	1	0	0

$\longrightarrow$

0	1	1	0
1	1	2	1
0	1	1	0

0	0	0	0
0	4	5	0
0	0	0	0

$\longrightarrow$

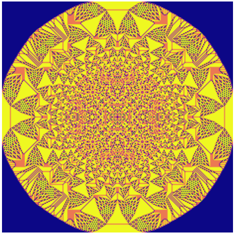
0	0	1	0
0	5	1	1
0	0	1	0

$\longrightarrow$

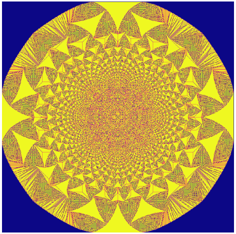
0	1	1	0
1	1	2	1
0	1	1	0

Why **abelian**? Because the final stable configuration does not depend on the order of topplings.

Lay m chips at the origin and stabilize. What can we say about the scaling limit of the sandpile clusters (length scaled by $m^{1/d}$)?

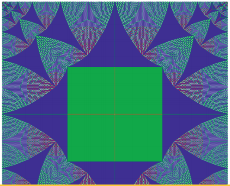


$m = 10^5$

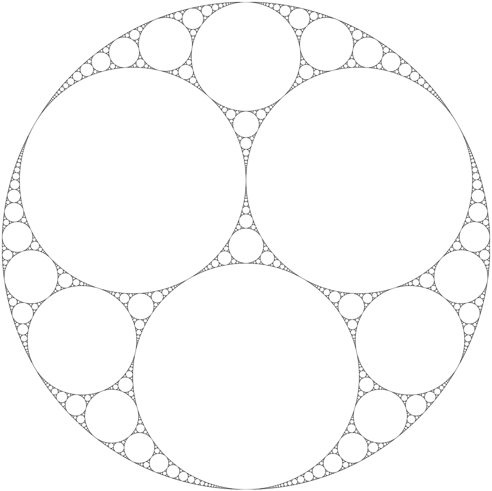


$m = 10^6$

- Scaling limit of the patterns exists [in weak-$*$ $L_\infty(\mathbb{R}^d)$]. [Pegden–Smart '11]
- Apollonian gaskets in the pattern. [Numerically observed since 90s, proved by Levine–Pegden–Smart '12, '14. Latter is published in *Ann. Math.* '17]
- Fractal patterns also appear in the identity element of the sandpile group on a square lattice.

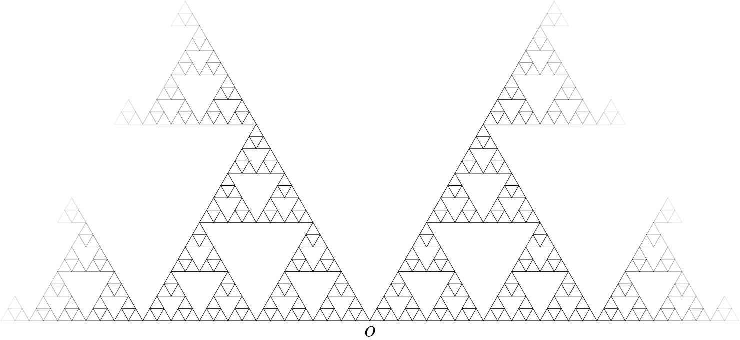


Apollonian gasket



By Time3000 [GFDL or CC BY-SA 4.0-3.0-2.5-2.0-1.0], from Wikimedia Commons
`https://commons.wikimedia.org/wiki/File:Apollonian_gasket.svg`

Sierpinski gasket (SG)

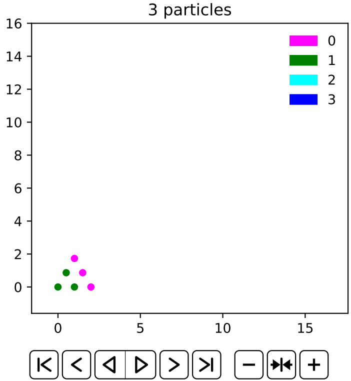


The abelian sandpile growth problem: Lay m chips at o and stabilize. Characterize the shape of, and patterns in, the stable sandpile cluster as a function of m.

Q. Is it possible to see fractal patterns in the sandpile on a fractal?

- It does not matter whether to solve it on the 1-sided SG or the 2-sided SG (symmetry).
- Key geometric inputs: Axial symmetries; cut points (junction points).

Sandpile growth on the 1-sided SG



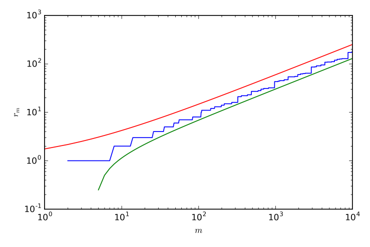
Key observations from simulations:

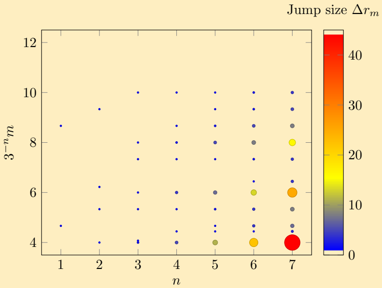
- Sandpile patterns exhibit periodicity. [Fairchild–Haim–Setra–Strichartz–Westura '16]
- The set of all vertices receiving at least 1 chip is ALWAYS a ball in the graph metric.
- Radial explosions occur at periodic values of m. (Not seen on $\mathbb{Z}^d$ or trees!)

2018: C. can prove all the key observations.

Top-level results: radial growth & explosion

Theorem (Radial growth (blue) & radial jumps)





Major jumps at $4 \cdot 3^n,\ 6 \cdot 3^n,\ 8 \cdot 3^n,\ 10 \cdot 3^n$. Period $= 2 \cdot 3^n$.

Theorem (Radial asymptotics)

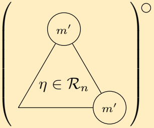
$r(x) = x^{1/d_H}[\mathcal{G}(\log x) + o(1)]$ as $x \to \infty$, where $d_H = \log_2 3$ is the Hausdorff dim of SG, and $\mathcal{G}$ is an explicit non-constant $\log 3$-periodic function.

Proof. Sharp estimates of the remainder $(|r(3m) - 2r(m)| \leq 1)$ + Renewal theorem.

The fundamental sandpile diagram on SG

Proposition. For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ =$$



$$\subseteq G_{n+1}.$$

From this follows the fundamental equation $r_m = 2^n + r_{m'-2}$.

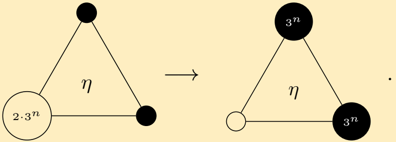
Key ideas:

- $\mathcal{R}_n$ is the **sandpile group** of G_n (= class of **recurrent** sandpile configs on G_n) with two sinks ∂G_n. It is an abelian group under $\oplus$, addition of chips followed by stabilization.
- Stabilize $m\mathbb{1}_o$ in $G_n \setminus \partial G_n$ to obtain the config $\eta \in \mathcal{R}_n$, pausing the excess m' chips on ∂G_n (sink) $\rightarrow$ using the abelian property.
- Then topple the excess chips on ∂G_n (source). By Dhar's multiplication by identity test (a Laplacian identity), with each topple on ∂G_n, η is unchanged, while the # of chips on ∂G_n decrements in steps of 2.

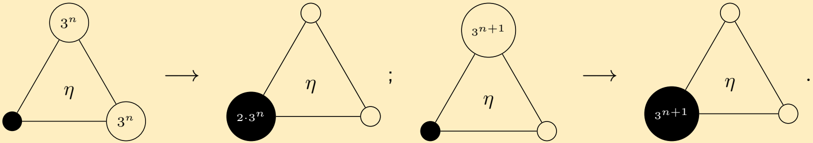
Using this Proposition we can inductively prove: cluster shape is a ball; periodicity; etc.

Toppling identities, periodicity

Proposition. For every $\eta \in \mathcal{R}_n^{(s)}$,



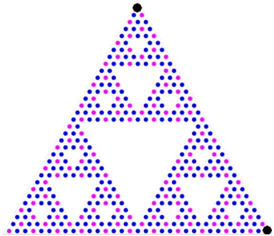
For every $\eta \in \mathcal{R}_n^{(o)}$,



This explains the $(2 \cdot 3^n)$-periodicity in the sandpile growth (and the patterns restricted to G_n).

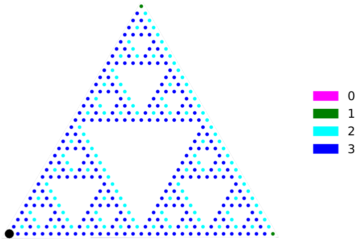
Identity element of the sandpile group of SG

2 sinks

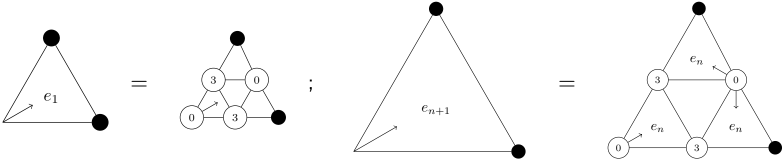


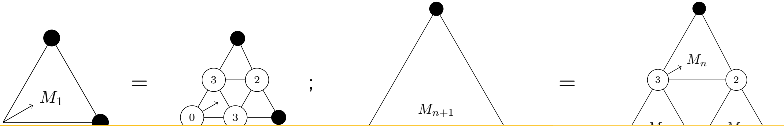
e_5

1 sink

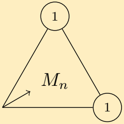


$e_5^{(o)} := M_5 \oplus (\mathbf{1}_x + \mathbf{1}_y)$

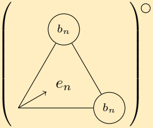




Proposition.

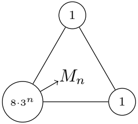
$((4 \cdot 3^n - 2)\mathbb{1}_o)^\circ =$ 

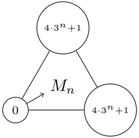
;

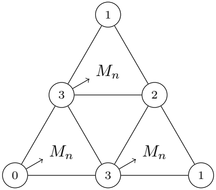
$((4 \cdot 3^n)\mathbb{1}_o)^\circ =$ 

, where $b_n = \frac{3}{2}\left(3^{n-1} + 1\right)$.

Proof of the induction step.

$(4 \cdot 3^{n+1} - 2)\mathbb{1}_o \to$ 

$\to$ 

$\to$ 

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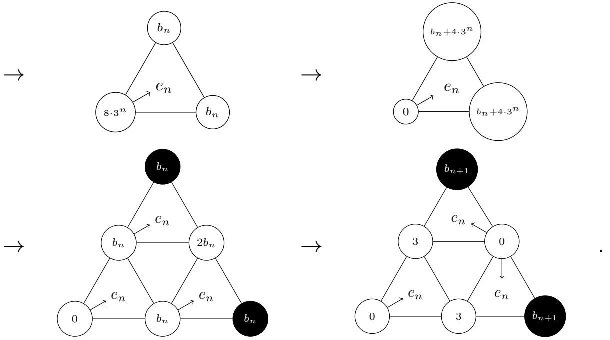
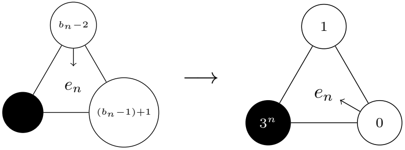
Proof (cont.).

$(4 \cdot 3^{n+1})\mathbb{1}_o \to$



The last step depends on the following "reflection lemma," using the axial symmetry of SG:



Also note $b_{n+1} = b_n + 3^n$.

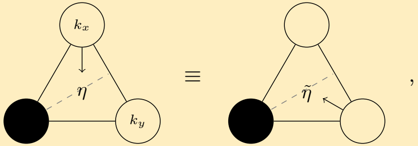
An axial reflection lemma (SG with one sink)

$e_n^{(o)}$ is the id element of $\mathcal{R}_n^{(o)}$; $\partial G_n = \{x, y\}$.

Lemma. Let $\eta \in \mathcal{R}_n^{(o)}$ be such that $\eta = e_n^{(o)} \oplus \alpha \mathbb{1}_x \oplus \beta \mathbb{1}_y$ for some $\alpha, \beta \in \mathbb{N}_0$. Let $k_x, k_y \in \mathbb{N}_o$ solve the system of equations

$$\begin{cases} \alpha + k_x &= \beta + p_0 \cdot 3^n + p_1 \cdot 3^{n+1} \\ \beta + k_y &= \alpha + p_0 \cdot 3^n + p_2 \cdot 3^{n+1} \end{cases}$$

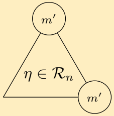
for some $p_0, p_1, p_2 \in \mathbb{Z}$ (which come from the toppling identities).
Then



where $\tilde{\eta} = e_n^{(o)} \oplus \beta \mathbb{1}_x \oplus \alpha \mathbb{1}_y$ is the reflection of η across the axis of symmetry.

Proposition. For each $m \geq 12$, there exists a unique $(n, m') \in \mathbb{N}^2$ such that

$$(m\mathbb{1}_o)^\circ = \Bigg(\qquad \Bigg)^\circ \subseteq G_{n+1}.$$



From this follows the fundamental equation $r_m = 2^n + r_{m'-2}$.

Example: $n = 3$. Record values of m at which m' jumps.

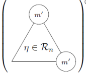
$\frac{m}{3^n}$	m	m'	$m-2m'$	Δr_m
4	**108**	**15**	**78**	**2**
$4\frac{2}{27}$	110	16	78	1
$4\frac{4}{9}$	120	19	82	
$4\frac{2}{3}$	126	20	86	
$5\frac{1}{3}$	144	28	88	1
6	162	42	78	1
$6\frac{2}{27}$	164	43	78	
$6\frac{4}{9}$	174	46	82	
$6\frac{2}{3}$	180	47	86	
$7\frac{1}{3}$	198	55	88	1

$\frac{m}{3^n}$	m	m'	$m-2m'$	Δr_m
8	216	69	78	1
$8\frac{2}{27}$	218	70	78	
$8\frac{4}{9}$	228	73	82	
$8\frac{2}{3}$	234	74	86	
$9\frac{1}{3}$	252	82	88	
10	270	96	78	1
$10\frac{2}{27}$	272	97	78	
$10\frac{4}{9}$	282	100	82	
$10\frac{2}{3}$	288	101	86	
$11\frac{1}{3}$	306	109	88	

$\frac{m}{3^n}$	m	m'	$m-2m'$	Δr_m
	2	1	0	1
	8	4	0	1
		n = 1		
4	**12**	**3**	**6**	
$4\frac{2}{3}$	14	4	6	1
6	18	6	6	
$6\frac{2}{3}$	20	7	6	
8	24	9	6	
$8\frac{2}{3}$	26	10	6	1
10	30	12	6	
$10\frac{2}{3}$	32	13	6	
		n = 2		
4	**36**	**6**	**24**	**1**
$4\frac{2}{9}$	38	7	24	
$4\frac{2}{3}$	42	8	26	
$5\frac{1}{3}$	48	10	28	1
6	54	15	24	
$6\frac{2}{9}$	56	16	24	1
$6\frac{2}{3}$	60	17	26	
$7\frac{1}{3}$	66	19	28	
8	72	24	24	
$8\frac{2}{9}$	74	25	24	
$8\frac{2}{3}$	78	26	26	
$9\frac{1}{3}$	84	28	28	1
10	90	33	24	
$10\frac{2}{9}$	92	34	24	
$10\frac{2}{3}$	96	35	26	
$11\frac{1}{3}$	102	37	28	
		n = 3		
4	**108**	**15**	**78**	**2**
$4\frac{2}{27}$	110	16	78	1
$4\frac{4}{9}$	120	19	82	
$4\frac{2}{3}$	126	20	86	
$5\frac{1}{3}$	144	28	88	1
6	162	42	78	1
$6\frac{2}{27}$	164	43	78	
$6\frac{4}{9}$	174	46	82	
$6\frac{2}{3}$	180	47	86	
$7\frac{1}{3}$	198	55	88	1

$\frac{m}{3^n}$	m	m'	$m-2m'$	Δr_m
8	216	69	78	1
$8\frac{2}{27}$	218	70	78	
$8\frac{4}{9}$	228	73	82	
$8\frac{2}{3}$	234	74	86	
$9\frac{1}{3}$	252	82	88	
10	270	96	78	1
$10\frac{2}{27}$	272	97	78	
$10\frac{4}{9}$	282	100	82	
$10\frac{2}{3}$	288	101	86	
$11\frac{1}{3}$	306	109	88	
		n = 4		
4	**324**	**42**	**240**	**5**
$4\frac{2}{81}$	326	43	240	
$4\frac{4}{9}$	360	55	250	1
$4\frac{2}{3}$	378	56	266	
$5\frac{1}{3}$	432	82	268	1
6	486	123	240	4
$6\frac{2}{81}$	488	124	240	
$6\frac{4}{9}$	522	136	250	
$6\frac{2}{3}$	540	137	266	
$7\frac{1}{3}$	594	163	268	1
8	648	204	240	2
$8\frac{2}{81}$	649	205	240	
$8\frac{4}{9}$	684	217	250	
$8\frac{2}{3}$	702	218	266	1
$9\frac{1}{3}$	756	244	268	
10	810	285	240	1
$10\frac{2}{81}$	812	286	240	
$10\frac{4}{9}$	846	298	250	
$10\frac{2}{3}$	864	299	266	
$11\frac{1}{3}$	918	325	268	
		n = 5		
4	**972**	**123**	**726**	**11**
$4\frac{2}{3^5}$	974	124	726	
$4\frac{4}{9}$	1080	163	754	1
$4\frac{2}{3}$	1134	164	806	1
$5\frac{1}{3}$	1296	244	808	2
6	1458	366	726	7
$6\frac{2}{3^5}$	1460	367	726	
$6\frac{4}{9}$	1566	406	754	

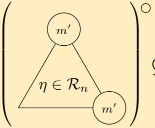
$\frac{m}{3^n}$	m	m'	$m-2m'$	Δr_m
$6\frac{2}{3}$	1620	407	806	
$7\frac{1}{3}$	1782	487	808	1
8	1944	609	726	5
$8\frac{2}{3^5}$	1946	610	726	
$8\frac{4}{9}$	2052	649	754	
$8\frac{2}{3}$	2106	650	806	2
$9\frac{1}{3}$	2268	730	808	1
10	2430	852	726	1
$10\frac{2}{3^5}$	2432	853	726	
$10\frac{4}{9}$	2538	892	754	
$10\frac{2}{3}$	2592	893	806	
$11\frac{1}{3}$	2754	943	808	
		n = 6		
4	**2916**	**366**	**2184**	**22**
$4\frac{2}{3^6}$	2918	367	2184	
$4\frac{4}{9}$	3240	487	2266	1
$4\frac{2}{3}$	3402	488	2426	4
$5\frac{1}{3}$	3888	730	2428	4
6	4374	1095	2184	13
$6\frac{2}{3^6}$	4376	1096	2184	
$6\frac{4}{9}$	4698	1216	2266	1
$6\frac{2}{3}$	4860	1217	2426	
$7\frac{1}{3}$	5346	1459	2428	2
8	5832	1824	2184	8
$8\frac{2}{3^6}$	5834	1825	2184	
$8\frac{4}{9}$	6156	1945	2266	
$8\frac{2}{3}$	6318	1946	2426	5
$9\frac{1}{3}$	6804	2188	2428	2
10	7290	2553	2184	2
$10\frac{2}{3^6}$	7292	2554	2184	
$10\frac{4}{9}$	7614	2674	2266	
$10\frac{2}{3}$	7766	2675	2426	
$11\frac{1}{3}$	8262	2917	2428	
		n = 7		
4	**8748**	**1095**	**6558**	**44**
$4\frac{2}{3^7}$	8750	1096	6558	
$4\frac{4}{9}$	9720	1459	6802	3
$4\frac{2}{3}$	10206	1460	7286	7
$5\frac{1}{3}$	11664	2188	7288	8
6	13122	3282	6558	25

Legend: $(m\mathbb{1}_o)^\circ = \Big($ 

$\Big)^\circ$; $\#\{\text{chips in } \eta\} = m - 2m'$.

Theorem (Enumeration of radial jumps)

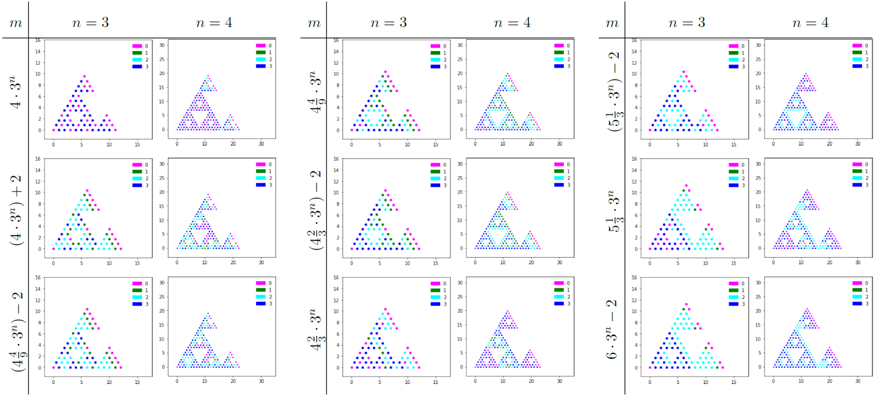
For $n \geq 3$ *and* $m \in [4 \cdot 3^n, 4 \cdot 3^{n+1})$, $\quad (m\mathbb{1}_o)^\circ =$



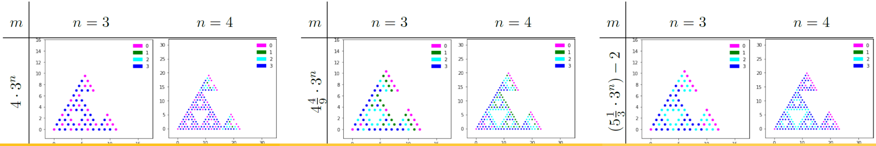
$\subseteq G_{n+1}$, *where* $m \mapsto m'$ *is a piecewise constant right-continuous function which has jumps indicated in the following table.*

m	m'
$(4 + 2p) \cdot 3^n$	$b_n + p \cdot 3^n$
$(4 + 2p) \cdot 3^n + 2$	$(b_n + 1) + p \cdot 3^n$
$(4\frac{4}{9} + 2p) \cdot 3^n$	$2 \cdot 3^{n-1} + 1 + p \cdot 3^n$
$(4\frac{2}{3} + 2p) \cdot 3^n$	$2 \cdot 3^{n-1} + 2 + p \cdot 3^n$
$(5\frac{1}{3} + 2p) \cdot 3^n$	$3^n + 1 + p \cdot 3^n$

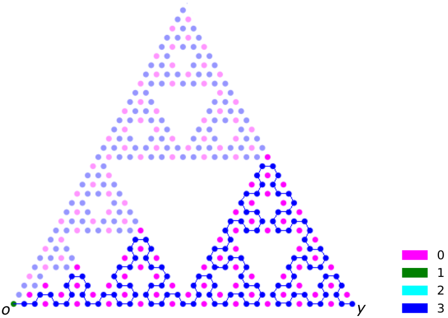
where $p \in \{\mathbf{0}, 1, 2, 3\}$, *and* $b_n = |V(G_{n-1})| = \frac{3}{2}(3^{n-1} + 1)$.



The proof of the radial jumps uses an inductive diagrammatic approach, *a.k.a.* **sandpile block renormalization**: Config restricted to G_n is the gluing of 3 well-defined sandpile tiles.



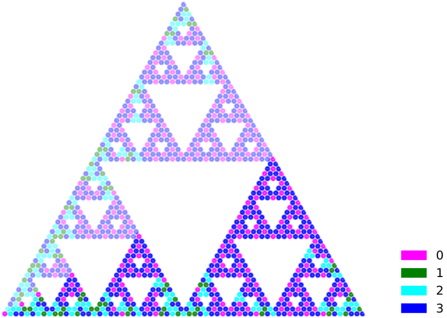
e_n: A fractal within a fractal!



$$e_5 = ((4 \cdot 3^5)\mathbf{1}_o)^\circ$$

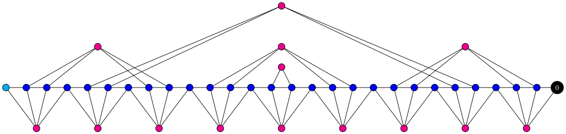
- The unique shortest blue path (Peano curve, in this case a.k.a. the Sierpinski arrowhead curve) of 3's connecting o to the sink y.
- What happens when 2 chips are added to o?
- Triggers a chain reaction of topplings down the Peano curve, all the way to y! This sends 1 extra chip to each sink vertex, which explains "$4 \cdot 3^n + 2$" in the radial jump theorem.
- BUT ...

$e_n + 2$: Traps develop along the Peano curve

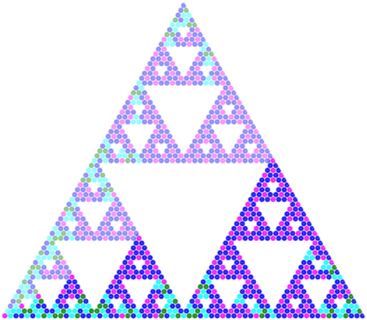


$((4 \cdot 3^6 + 2)\mathbf{1}_o)^\circ$

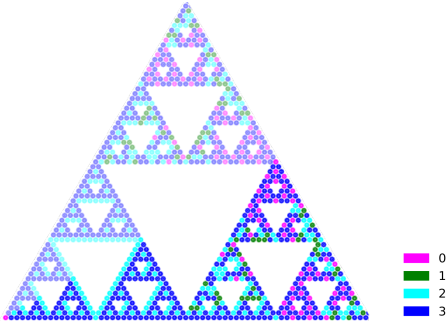
- Blotches of 1's and 2's ("traps") develop at well-defined locations, due to connections across vertices on the Peano curve.
- Best way to visualize this is to parametrize SG along the length of the Peano curve:



“$4\frac{4}{9} \cdot 3^n - 2$”: Inability to overcome the traps

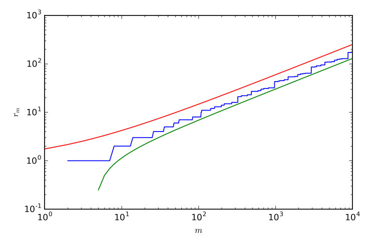


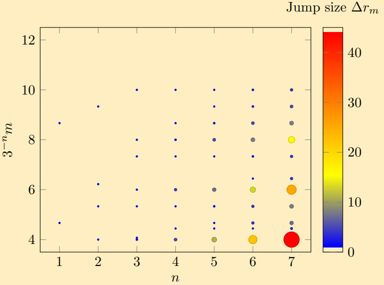
$((4 \cdot 3^6 + 2)\mathbf{1}_o)^\circ$



$((4\frac{4}{9} \cdot 3^6 - 2)\mathbf{1}_o)^\circ$

Theorem (Recursive radial growth formula)





$$
\begin{tikzcd}
 & 6\frac{2}{3} \arrow[d] & & 8\frac{4}{9} \arrow[r] & 7\frac{1}{3} \arrow[d] & 8 \arrow[l] & \\
6\frac{4}{9} \arrow[r] & 4\frac{2}{3} \arrow[r, thick] & 6 \arrow[r] & 4\frac{4}{9} \arrow[r, thick] & 5\frac{1}{3} \arrow[r, thick] & 8\frac{2}{3} \arrow[u] & 10 \arrow[loop right] \\
 & & & 4 \arrow[u, thick] & & 9\frac{1}{3} \arrow[u] &
\end{tikzcd}
$$

$$
\begin{aligned}
a \rightarrow b: &\quad r_{a\cdot 3^n} = 2^n + r_{b\cdot 3^{n-1}} \quad (n \geq 3) \\
a \longrightarrow b: &\quad r_{a\cdot 3^n} = 2^n + r_{b\cdot 3^{n-2}} \quad (n \geq 4)
\end{aligned}
$$

Thank you!