Spectral decimation on self-similar fractals: from singularly continuous spectrum to the Hofstadter butterfly

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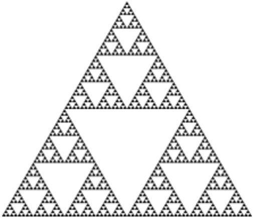
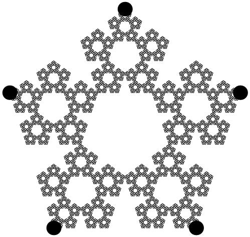
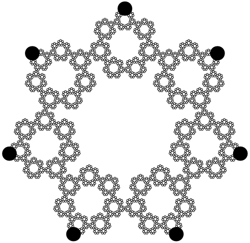
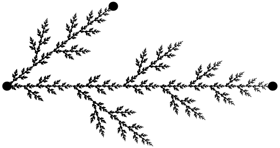
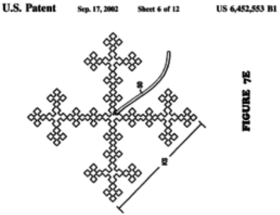
Motivation: Analysis on nonsmooth domains

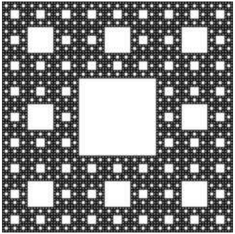




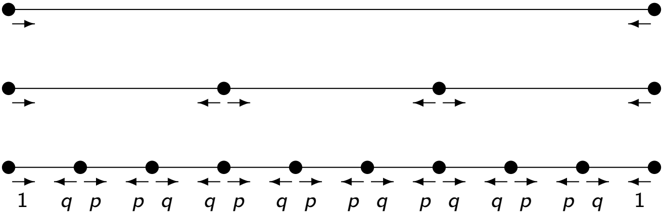


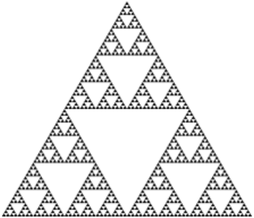


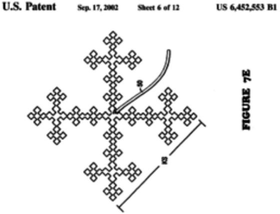




Some fractals are nicer than others

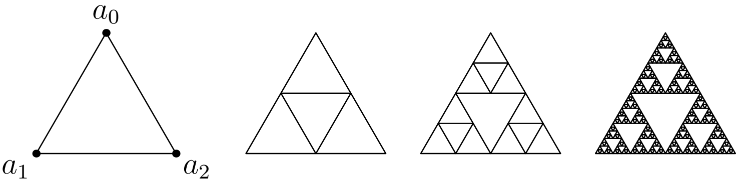






Each of these fractals is obtained from a nested sequence of graphs which has *nice, symmetric* replacement rules.

Spectral decimation (= spectral similarity)



Rammal-Toulouse ‘84, Bellissard ‘88, Fukushima-Shima ‘92, Shima ‘96, etc.
A recursive algorithm for identifying the Laplacian spectrum on highly symmetric, finitely ramified self-similar fractals.

Spectral decimation

Definition (Malozemov-Teplyaev '03)

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces. We say that an operator H on $\mathcal{H}$ is spectrally similar to H_0 on $\mathcal{H}_0$ with functions φ_0 and φ_1 if there exists a partial isometry $U : \mathcal{H}_0 \to \mathcal{H}$ (that is, $UU^* = I$) such that

$$U(H-z)^{-1}U^* = (\varphi_0(z)H_0 - \varphi_1(z))^{-1} =: \frac{1}{\varphi_0(z)}(H_0 - R(z))^{-1}$$

for any $z \in \mathbb{C}$ for which the two sides make sense.

A common class of examples: $\mathcal{H}_0$ subspace of $\mathcal{H}$, U^* is an ortho. projection from $\mathcal{H}$ to $\mathcal{H}_0$. Write $H - z$ in block matrix form w.r.t. $\mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$:

$$H - z = \begin{pmatrix} I_0 - z & \overline{X} \\ X & Q - z \end{pmatrix}.$$

Then $U(H-z)^{-1}U^*$ is the inverse of the Schur complement $S(z)$ w.r.t. to the lower-right block of $H - Z$: $S(z) = (I_0 - z) - \overline{X}(Q-z)^{-1}X$.

Issue: There may exist a set of z for which either $Q - z$ is not invertible, or $\varphi_0(z) = 0$.

Spectral decimation: the main theorem

Spectrum $\sigma(\Delta) = \{z \in \mathbb{C} : \Delta - z \text{ does not have a bounded inverse}\}$.

Definition

The exceptional set for spectral decimation is

$$\mathfrak{E}(H, H_0) \stackrel{def}{=} \{z \in \mathbb{C} : z \in \sigma(Q) \text{ or } \varphi_0(z) = 0\}.$$

Theorem (Malozemov-Teplyaev '03)

Suppose H is spectrally similar to H_0. Then for any $z \notin \mathfrak{E}(H, H_0)$:

- $R(z) \in \sigma(H_0) \iff z \in \sigma(H)$.
- *$R(z)$ is an eigenvalue of H_0 iff z is an eigenvalue of H. Moreover there is a one-to-one map between the two eigenspaces.*

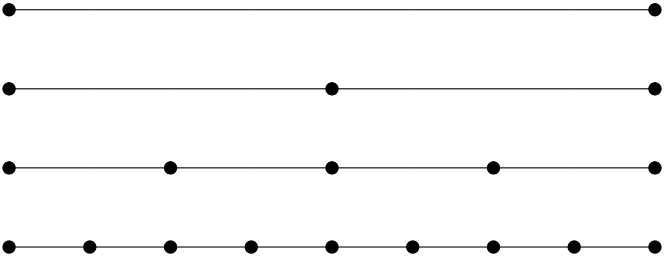
Consequence: For an operator H on a self-similar Hilbert space $\mathcal{H}$,

$$\mathcal{J}(R) \subset \sigma(H) \subset \mathcal{J}(R) \cup D,$$

where

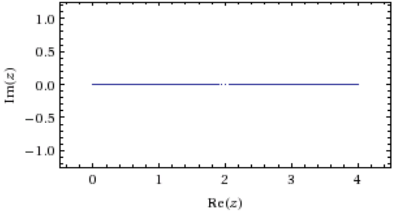
- $\mathcal{J}(R)$ is the **Julia set** of R ($=$ the complement in $\mathbb{C} \cup \{\infty\}$ of the domain in which $\{R^{\circ n}\}_n$ converges uniformly on compact subsets).
- D derives from the exceptional set $\mathfrak{E}$.

Example: $\mathbb{Z}_+$



Let Δ be the graph Laplacian on $\mathbb{Z}_+$ (with Neumann boundary condition at 0), realized as the limit of graph Laplacians on $[0, 2^n] \cap \mathbb{Z}_+$.
If $z \neq 2$ and $R(z) = z(4 - z)$, then

- $R(z) \in \sigma(-\Delta) \iff z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = \mathcal{J}(R)$.
- $\mathcal{J}(R)$ is the full interval $[0, 4]$.



Generalizing the interval: The *pq*-model

A one-parameter model of 1D fractals parametrized by $p \in (0,1)$. Set $q = 1-p$.

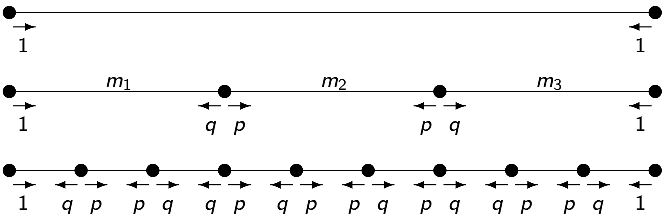
A triadic interval construction, "next easiest" fractal beyond the dyadic interval.

$$(\Delta_p f)(x) = \sum_y \mathfrak{p}(x,y) f(y) - f(x)$$

where $\mathfrak{p}(x,y) \in \{1, p, q\}$ depends on the arrow given below.
Assign probability weights to the three segments:

$$m_1 = m_3 = \frac{q}{1+q}, \quad m_2 = \frac{p}{1+q}$$

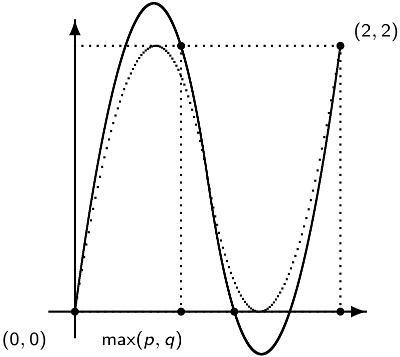
Then iterate. Let π be the resulting self-similar probability measure.



Spectral decimation for the pq-model

The spectral decimation polynomial is $R(z) = \frac{z(z^2-3z+(2+pq))}{pq}$.

$$\sigma(-\Delta_n) = \{0,2\} \cup \bigcup_{m=0}^{n-1} R^{-m}\{1 \pm q\}$$



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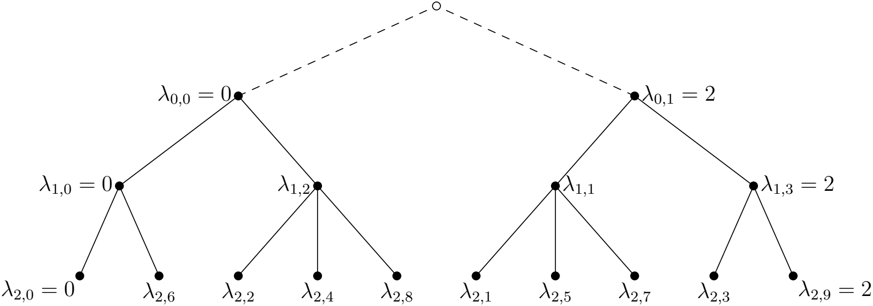
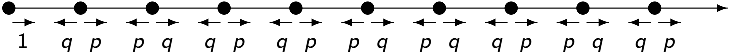


The pq-model on $\mathbb{Z}_+$

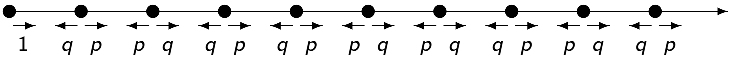


- Δ_p is not self-adjoint w.r.t. $\ell^2(\mathbb{Z}_+)$, but is self-adjoint w.r.t. the discretization of the aforementioned self-similar measure π.
- Let $\Delta_p^+ = D^* \Delta_p D$, where

$$D : \ell^2(\mathbb{Z}_+) \to \ell^2(3\mathbb{Z}_+), \quad (Df)(x) = f(3x).$$

Then Δ_p is spectrally similar to Δ_p^+. Moreover, Δ_p and Δ_p^+ are isometrically equivalent (in $L^2(\mathbb{Z}_+)$ or in $L^2(\mathbb{Z}_+, \pi)$).

The pq-model on $\mathbb{Z}_+$



Spectrum $\sigma(H) = \{z \in \mathbb{C} : H - z$ does not have a bounded inverse$\}$.
Facts from functional analysis:

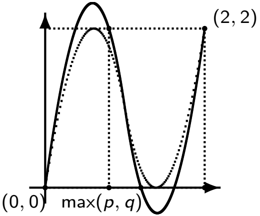
- $\sigma(H)$ is a nonempty compact subset of $\mathbb{C}$.
- $\sigma(H)$ equals the disjoint union $\sigma_{\mathrm{pp}}(H) \cup \sigma_{\mathrm{ac}}(H) \cup \sigma_{\mathrm{sc}}(H)$.
  pure point spectrum $\cup$ absolutely continuous spectrum $\cup$ singularly continuous spectrum

Theorem (C.-Teplyaev, *J. Math. Phys.* '16)

If $p \neq \frac{1}{2}$, the Laplacian Δ_p, regarded as an operator on either $\ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{Z}_+, \pi)$, has purely singularly continuous spectrum. The spectrum is the Julia set of the polynomial $R(z) = \frac{z(z^2-3z+(2+pq))}{pq}$, which is a topological Cantor set of Lebesgue measure zero.

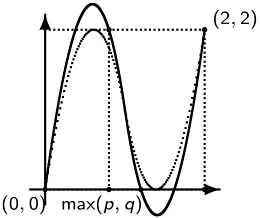
- One of the simplest realizations of purely singularly continuous spectrum. The mechanism appears to be simpler than those of quasi-periodic or aperiodic Schrodinger operators. (*cf.* Simon, Jitomirskaya, Avila, Damanik, Gorodetski, etc.)
- See also Grigorchuk-Lenz-Nagnibeda '14, '16 on spectra of Schreier graphs.

Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)



1. Spectral decimation: Δ_p is spectrally similar to Δ_p^+, and they are isometrically equivalent. After taking into account the exceptional set, $R(z) \in \sigma(\Delta_p) \iff z \in \sigma(\Delta_p)$. Notably, the repelling fixed points of R, $\{0, 1, 2\}$, lie in $\sigma(\Delta_p)$.
2. By 1, $\overline{\bigcup_{n=0}^{\infty} R^{\circ -n}(0)} \subset \sigma(\Delta_p)$. Meanwhile, since $0 \in \mathcal{J}(R)$, $\overline{\bigcup_{n=0}^{\infty} R^{\circ -n}(0)} = \mathcal{J}(R)$. So $\mathcal{J}(R) \subset \sigma(\Delta_p)$.
3. If $z \in \sigma(\Delta_p)$, then by 1, $R^{\circ n}(z) \in \sigma(\Delta_p)$ for each $n \in \mathbb{N}$. On the one hand, $\sigma(\Delta_p)$ is compact. On the other hand, the only attracting fixed point of R is ∞, so the Fatou set $(\mathcal{J}(R))^c$ contains the basin of attraction of ∞, whence non-compact. Infer that $z \notin (\mathcal{J}(R))^c$. So $\sigma(\Delta_p) \subset \mathcal{J}(R)$.

Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)



4. Thus $\sigma(\Delta_p) = \mathcal{J}(R)$. When $p \neq \frac{1}{2}$, $\mathcal{J}(R)$ is a disconnected Cantor set.
So $\sigma_{\rm ac}(\Delta_p) = \emptyset$.
5. Find the formal eigenfunctions corresponding to the fixed points of R, and show that none of them are in $\ell^2(\mathbb{Z}_+)$ or in $L^2(\mathbb{Z}_+, \pi)$. Thus none of the fixed points lie in $\sigma_{\rm pp}(\Delta_p)$. By spectral decimation, none of the pre-iterates of the fixed points under R are in $\sigma_{\rm pp}(\Delta_p)$.
So $\sigma_{\rm pp}(\Delta_p) = \emptyset$.
6. Conclude that $\sigma(\Delta_p) = \sigma_{\rm sc}(\Delta_p)$.

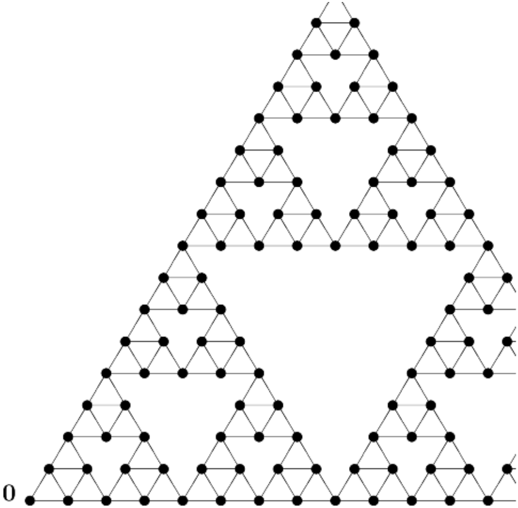
The Sierpinski gasket lattice (SGL)

Let Δ be the graph Laplacian on SGL.
If $z \notin \{2, 5, 6\}$ and $R(z) = z(5 - z)$, then

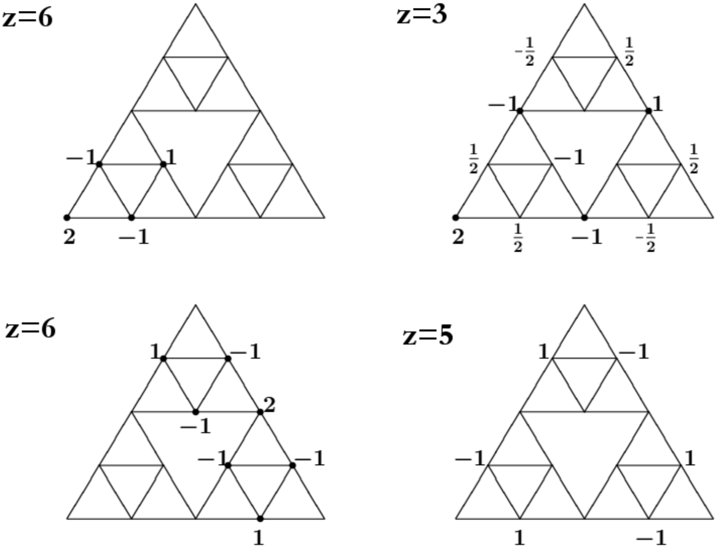
- $R(z) \in \sigma(-\Delta) \iff z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = \mathcal{J}_R \cup \mathcal{D}$, where $\mathcal{J}_R$ is the Julia set of $R(z)$ and $\mathcal{D} := \{6\} \cup \left(\bigcup_{m=0}^{\infty} R^{-m}\{3\}\right)$.
- $\mathcal{J}_R$ is a disconnected Cantor set.

Thm. (Teplyaev '98, Quint '09)
On SGL, $\sigma(\Delta) = \sigma_{\mathrm{pp}}(\Delta)$.
Eigenfunctions with finite support are complete.

$\rightarrow$ Localization due to geometry.



Localized eigenfunctions on SGL



A few related questions

1. Spectral decimation meets Kirchhoff's matrix-tree theorem:

On a finite connected graph G on n vertices,

$$\#\{\text{spanning trees on } G\} = \det(-\Delta_G[j]) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i,$$

where $\Delta_G[j]$ is the minor of Δ_G with the jth row and the jth column removed, and $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ are the eigenvalues of $-\Delta_G$.

- Counting spanning trees on fractals: Chang–Chen '05, Teufl–Wagner '06, Anema–Tsougkas '16 (uses spectral decimation).
- Randomly sampled **U**niform **S**panning **T**rees on *SG*: Shinoda–Teufl–Wagner '14.
- Regularized $\log\det\Delta$ formulas for various self-similar graphs: C.–Teplyaev–Tsougkas '18 (uses spectral decimation).

A few related questions

2. Wave propagation on fractals

$$u_{tt} = \Delta u, \quad \text{solution is } u(t,x) = \sum_{j=1}^{\infty} c_j \cos(t\sqrt{\lambda_j}) e_j(x) \quad \text{where } -\Delta e_j = \lambda_j e_j$$

- For the *pq*-model applied to a compact subinterval of $\mathbb{R}$, we obtain a good space-time approximation of the solution to the wave equation in Andrews–Bonik–C.–Martin–Teplyaev '17. Animation
- Gives a concrete example of "infinite speed of wave propagation" on fractals.

3. Anderson localization of $H = -\Delta + V_\omega$ on fractal lattices?
   - Molchanov: On finitely ramified lattices, $\sigma_{\mathrm{ac}}(H) = \emptyset$ (by the Simon-Wolff method).
   - (Still many unanswered questions ...)

Line bundle Laplacian (*a.k.a.* magnetic Schrödinger operator) on *SG*

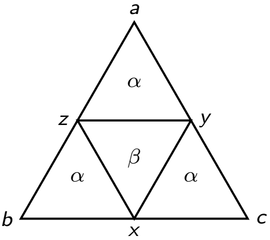
Let θ be a 1-form on the edge set of a connected graph: $\theta(x,y) = -\theta(y,x)$ for all $x \sim y$. The **line bundle Laplacian** on the vertex set is given by

$$(\Delta_\theta f)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} \left(f(x) - e^{i\theta(x,y)} f(y) \right), \quad f : V \to \mathbb{R}.$$

Line bundle = Vector bundle with a $U(1)$ connection = $\{e^{i\theta(x,y)}\}_{(x,y)\in E}$.

We are interested in a choice of θ which corresponds to "constant magnetic field" through the *SG* lattice:

Flux around a cycle, $\sum_{\mathbf{e}\in\text{cycle}} \theta(\mathbf{e})$, is $2\pi\alpha$ if traversed along an upward triangle, and $2\pi\beta$ if along a downward triangle. For consistency with self-similarity, $\alpha = \beta$.



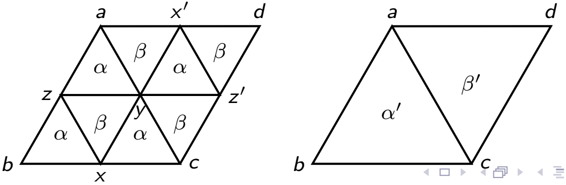
Spectral decimation of the line bundle Laplacian on *SG*

- Analyzed in several physics papers in the 80s, most notably Ghez–Wang–Rammal–Pannetier–Bellissard '88.
- Ruoyu Guo (Colgate '19) is working on a careful analysis of spectral decimation of Δ_θ as part of his senior honors thesis.

Proposition (GWRPB '88, C.–Guo '18+)

Let $(\Delta_\theta^{(n)}, \alpha)$ denote the line bundle Laplacian with constant flux $2\pi\alpha$ on the nth-level SG. Then there is spectral decimation from $(\Delta_\theta^{(n+1)}, \alpha)$ to $(\Delta_\theta^{(n)}, 4\alpha)$ with spectral decimation function $R(z,\alpha) = 1 + \frac{A(z,\alpha)}{2|\Psi(z,\alpha)|}$, where

$$\begin{aligned} A(z,\alpha) =& -64z^4 + 256z^3 - 356z^2 - [6\cos(2\pi\alpha) - 200]z \\ &+ 6\cos(2\pi\alpha) + \cos(4\pi\alpha) - 37, \\ \Psi(z,\alpha) =& 8z^2 - (2e^{2\pi i(3\alpha)} + 4e^{2\pi i\alpha} + 16)z + 2e^{2\pi i(3\alpha)} + \frac{3}{2}e^{4\pi i\alpha} + 4e^{2\pi i\alpha} + \frac{15}{2}. \end{aligned}$$



Spectrum of the line bundle Laplacian

An approximate spectrum: Initialize with points (λ_0, α_0) in $[0,2] \times [0,1]$. Let $(\lambda_n, \alpha_n) = R^{\circ n}(\lambda_0, \alpha_0)$. We expect the spectrum to *roughly* coincide with the Julia set of R. To get a picture of the filled Julia set, we keep points (λ_0, α_0) for which $|R^{\circ n}(\lambda_0, \alpha_0)| \leq C$ for all n.

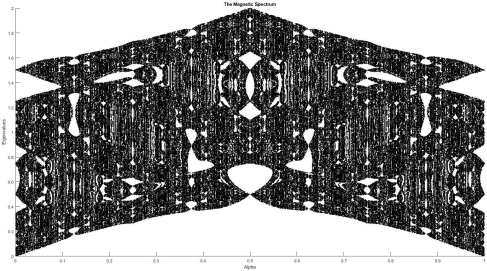
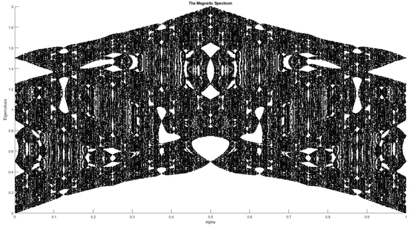
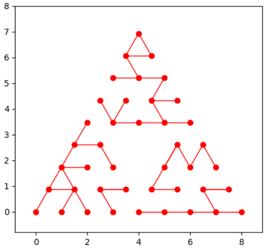


Figure: The "Hofstadter butterfly" on the Sierpinski gasket

Results to come (in 2019)





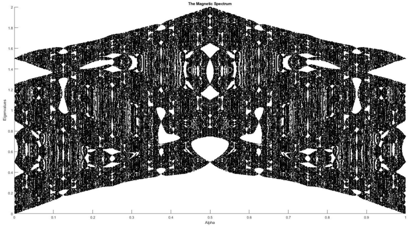
1. Make the connection between $\sigma((\Delta_\theta, \alpha))$ and $\mathcal{J}(R)$ exact.
2. Establish properties of the spectrum, e.g.: Is it true that for *every* $\alpha \in [0,1]$, $\sigma(\Delta_\theta, \alpha) = \sigma_{\rm pp}(\Delta_\theta, \alpha)$? Also, the bottom (and top) of the spectrum $\lambda_{\rm min}(\alpha)$ ($\lambda_{\rm max}(\alpha)$) seems to be continuous in α. Is this true?
3. The line-bundle version of Kirchhoff's matrix-tree theorem [Forman '93, Kenyon '10]

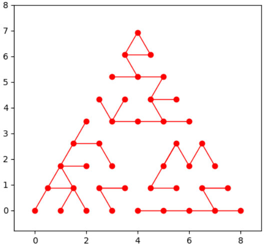
$$\det \Delta_\theta = \sum_{\text{CRSFs}} \prod_{\text{cycles}} 2\left(1 - \cos\left(\sum_{\mathbf{e}\in\text{cycle}} \theta(\mathbf{e})\right)\right),$$

where the sum runs over all **c**ycle-**r**ooted **s**panning **f**orests on the graph.

Leads to probabilistic analysis of random spatial processes on fractals.

Results to come (in 2019)





Thank you!