

# Random walks, electric networks, and coarse-graining in particle systems

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# Random walks and electric networks

Title of a famous monograph by Doyle & Snell.

<https://math.dartmouth.edu/~doyle/docs/walks/walks.pdf>

- Let  $G = (V, E)$  be a locally finite connected graph, and  $\mathbf{c} = \{c_{xy}\}_{xy \in E}$  be the set of positive weights (conductances) endowed on  $E$ .
- The (symmetric) random walk process on the **weighted graph** (=electric network)  $(G, \mathbf{c})$  is an irreducible Markov chain on  $V$  with transition probability

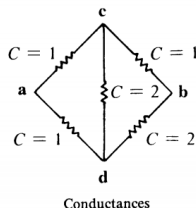
$$P(x, y) = \begin{cases} c_{xy}/c_x, & \text{if } xy \in E, \\ 0, & \text{otherwise.} \end{cases} \quad c_x := \sum_{z: xz \in E} c_{xz}.$$

- The RW process has  $\pi(\cdot) \propto c(\cdot)$  as reversible (invariant) measure, and the associated Dirichlet energy is

$$\mathcal{E}^{\text{RW}}(f) = \langle f, (\mathbf{I} - \mathbf{P})f \rangle_\pi = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2, \quad f: V \rightarrow \mathbb{R}.$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{2}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}$$

(The entries along each row must add up to 1.)



# Random walks and electric networks

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- Effective resistance** between  $A, B \subset V$ :

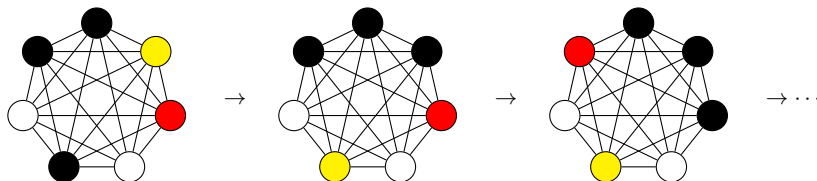
$$R_{\text{eff}}(A, B) = \sup \left\{ [\mathcal{E}^{\text{RW}}(f)]^{-1} \mid f: V \rightarrow \mathbb{R}, f|_A = 1, f|_B = 0 \right\}$$

In particular, if  $A = \{x\}$  and  $B = \{y\}$  we write  $R_{\text{eff}}(x, y)$ . By definition,

$$[f(x) - f(y)]^2 \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{RW}}(f), \quad f: V \rightarrow \mathbb{R}.$$

Also  $R_{\text{eff}}: V \times V \rightarrow \mathbb{R}_+$  is a metric on  $V$ .

# Interacting particle systems on an electric network



**Overarching question:** Can we study Markov processes involving MANY interacting “random walkers” on a weighted graph  $(G, c)$ ?

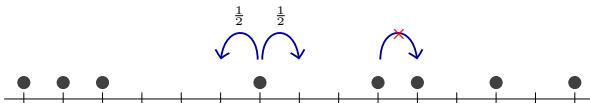
Mathematical development started with Spitzer (on the integer lattice).

Mathematically tractable models:

- 1 **Exclusion process** (state space  $\{0, 1\}^V$ ): Particles perform RWs subject to the exclusion constraint that **no two particles can occupy the same vertex at any time**.
- 2 **Zero-range process** (state space  $\mathbb{N}_0^V$ ): Particle at  $x$  jumps to neighboring  $y$  at rate depending on  $\mathbf{P}(x, y)$  [jump] and **the number of particles at  $x$  ONLY** [zero-range kinetics].

Both models are associated with a conserved quantity—the total # of particles (unless additional dynamics or “reservoirs” are attached).

# Particle system #1: Exclusion process



The **(symm.) exclusion process** on  $(G, c)$  is a Markov chain on  $\{0, 1\}^V$  with generator

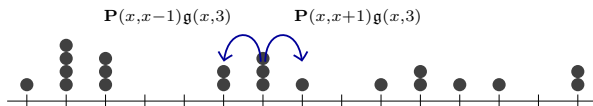
$$(\mathcal{L}^{\text{EX}} f)(\eta) = \sum_{xy \in E} c_{xy} (\nabla_{xy} f)(\eta). \quad f : \{0, 1\}^V \rightarrow \mathbb{R},$$

where  $(\nabla_{xy} f)(\eta) := f(\eta^{xy}) - f(\eta)$  and  $(\eta^{xy})(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$

- Each product Bernoulli measure  $\nu_\alpha$ ,  $\alpha \in [0, 1]$ , with marginal  $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$  for each  $x \in V$ , is an **invariant measure**.

- **Dirichlet energy:** 
$$\mathcal{E}^{\text{EX}}(f) = \frac{1}{2} \sum_{zw \in E} c_{zw} \int_{\{0,1\}^V} [(\nabla_{zw} f)(\eta)]^2 d\nu_\alpha(\eta).$$

## Particle system #2: Zero-range process



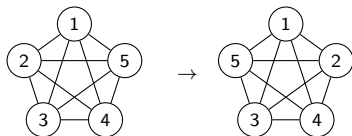
The **zero-range process** on  $(G, c)$  is a Markov chain on  $\mathbb{N}_0^V$  with generator

$$(\mathcal{L}^{\text{ZR}}f)(\xi) = \sum_{(x,y) \in V^2} \mathbf{P}(x,y)g(x,\xi(x)) [f(\xi + \mathbf{1}_y - \mathbf{1}_x) - f(\xi)], \quad f : \mathbb{N}_0^V \rightarrow \mathbb{R}.$$

where  $\mathbf{P}$  is an irreducible jump Markov matrix on  $V^2$ , and  $g : V \times \mathbb{N}_0 \rightarrow \mathbb{R}_+$  is the kinetic rate,  $g(x, 0) = 0$  always.

- **Invariant measure is a product one:**  $\mu(\xi) = \frac{1}{Z} \prod_{x \in V} \prod_{k=1}^{\xi(x)} \frac{\pi(x)}{g(x, k)}$ , where  $\pi$  is the invariant measure for  $\mathbf{P}$ .
- **Dirichlet energy:**  $\mathcal{E}^{\text{ZR}}(f) = \langle f, -\mathcal{L}^{\text{ZR}}f \rangle_\mu$ .

# Hierarchy of stochastic processes on a fixed graph

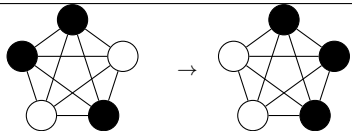


**Interchange process**  $f : \{\text{Permutations on } V\} \rightarrow \mathbb{R}$

$$\mathcal{E}^{\text{IP}}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 d\nu(\eta).$$

Reversible measure: uniform measure  $\nu$  on  $\{\text{Perms on } V\}$ .

↓ PROJECTION ↓

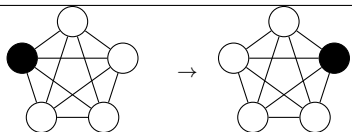


**Exclusion process**  $f : \{0, 1\}^V \rightarrow \mathbb{R}$

$$\mathcal{E}^{\text{EX}}(f) = \int \frac{1}{2} \sum_{zw \in E} c_{zw} [f(\eta^{zw}) - f(\eta)]^2 d\nu_\alpha(\eta).$$

Reversible measure: product Bernoulli measure  $\nu_\alpha$ ,  $\alpha \in [0, 1]$ ,  $\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha$  for all  $x \in V$ .

↓ PROJECTION ↓



**Random walk process**  $f : V \rightarrow \mathbb{R}$

$$\mathcal{E}^{\text{RW}}(f) = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2.$$

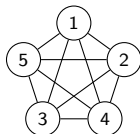
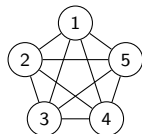
Reversible measure:  $c(\cdot) = \sum_{w \sim \cdot} c_{w \cdot}$ .

**Aldous' spectral gap conjecture '92:** Is  $\lambda_2^{\text{EX}}(G) = \lambda_2^{\text{RW}}(G)$ ?

Easy to see that  $\lambda_2^{\text{RW}}(G) \geq \lambda_2^{\text{EX}}(G)$ .

For the other direction, suffice to prove that  $\lambda_2^{\text{IP}}(G) \geq \lambda_2^{\text{RW}}(G)$ .

# Hierarchy of stochastic processes on a fixed graph

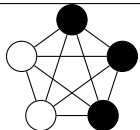
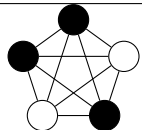


**Interchange process**  $f : \{\text{Permutations on } V\} \rightarrow \mathbb{R}$

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 d\nu(\eta) \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{IP}}(f).$$

**Moving particle lemma**

↓ PROJECTION ↓

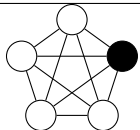
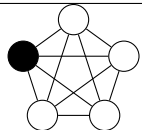


**Exclusion process**  $f : \{0, 1\}^V \rightarrow \mathbb{R}$

$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 d\nu_\alpha(\eta) \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f).$$

**Moving particle lemma**

↓ PROJECTION ↓



**Random walk process**  $f : V \rightarrow \mathbb{R}$

$$[f(x) - f(y)]^2 \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{RW}}(f).$$

**Dirichlet principle**

(Also a dual version involving flows: Thomson principle)

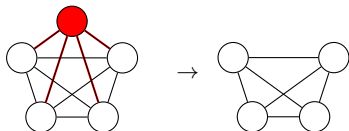
## Energy inequalities

Does the MPL follow trivially from the Dirichlet principle? NO!

However, a common idea is **electric network reduction** (Schur complementation in linear algebra).



# Network reduction: an exercise in Schur complements



**Idea:** Remove vertices (and edges attached to them) without changing the effective conductance between any of the non-removed vertices.

- Suppose we remove the vertex  $x \in V$  from  $(G, \mathbf{c})$ , as well as the edges attached to  $x$ . Call the reduced graph  $G_x = (V_x, E_x)$ . In the linear algebra language, we will reduce the Laplacian  $\mathbf{L}$  to a new Laplacian  $\mathbf{L}'$  (of one fewer dimension). This is attained by taking the **Schur complement** of the  $(x, x)$  block in  $\mathbf{L}$ :

$$\text{If } \mathbf{L} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{L}_{xx} \end{bmatrix}, \quad \text{then } \mathbf{L}' = \mathbf{X} - \mathbf{Y}(\mathbf{L}_{xx})^{-1}\mathbf{Z} = \mathbf{X} - \mathbf{YZ}. \quad (\text{Recall } \mathbf{L}_{xx} = -1.)$$

- In component form,  $\mathbf{L}'_{yz} = \mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}$  for  $y, z \in V_x$ . Since  $\mathbf{L}_{yz}^{(')} = -p_{yz}^{(')} = -\frac{c_{yz}}{c_y}$  whenever  $y \neq z$ , we see that the new conductances on  $E_x$  become

$$c'_{yz} = -c_y \mathbf{L}'_{yz} = -c_y (\mathbf{L}_{yz} - \mathbf{L}_{yx}\mathbf{L}_{xz}) = c_{yz} + \frac{c_{yx}c_{xz}}{c_x} =: c_{yz} + \tilde{c}_{yz}.$$

## Example 1: Series Law



Let  $c_{xy} = \alpha$  and  $c_{xz} = \beta$ .

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \textcolor{blue}{1} & 0 & -1 \\ 0 & \textcolor{blue}{1} & -1 \\ -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} & \textcolor{red}{1} \end{bmatrix}.$$

Let  $\mathbf{L}'$  be the Schur complement of the  $\textcolor{red}{1}$  block in  $\mathbf{L}$ :

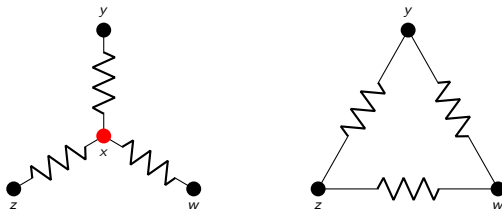
$$\mathbf{L}' = \begin{bmatrix} \textcolor{blue}{1} & 0 \\ 0 & \textcolor{blue}{1} \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} -\frac{\alpha}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & -\frac{\beta}{\alpha+\beta} \\ -\frac{\alpha}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

So  $\mathbf{L}'_{yz} = -\frac{\beta}{\alpha+\beta}$ . Since  $c_y = \alpha$ , we get  $c'_{yz} = -c_y \mathbf{L}'_{yz} = \frac{\alpha\beta}{\alpha+\beta}$ , i.e.,

$$R'_{yz} = \frac{1}{c'_{yz}} = \frac{1}{\alpha} + \frac{1}{\beta} = R_{xy} + R_{xz}.$$

(Resistors in series ADD!)

## Example 2: Y- $\Delta$ transform



Let  $c_{xy} = \alpha$ ,  $c_{xz} = \beta$ ,  $c_{xw} = \gamma$ , and  $\sigma = \alpha + \beta + \gamma$ .

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \alpha/\sigma & \beta/\sigma & \gamma/\sigma & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -\alpha/\sigma & -\beta/\sigma & -\gamma/\sigma & 1 \end{bmatrix}.$$

$$\mathbf{L}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} [-\alpha/\sigma \quad -\beta/\sigma \quad -\gamma/\sigma] = \frac{1}{\sigma} \begin{bmatrix} \beta + \gamma & -\beta & -\gamma \\ -\alpha & \alpha + \gamma & -\gamma \\ -\alpha & -\beta & \alpha + \beta \end{bmatrix}.$$

After a little more algebra we get

$$c'_{yz} = \frac{\alpha\beta}{\sigma}, \quad c'_{zw} = \frac{\beta\gamma}{\sigma}, \quad c'_{wy} = \frac{\gamma\alpha}{\sigma}.$$

(Anyone who studied electric circuits would find this familiar!)

# Proof of Dirichlet's principle via network reduction

$$\mathcal{E}(f) = \sum_{zw \in E} c_{zw} [f(z) - f(w)]^2.$$

In going from  $G$  to the reduced graph  $G_x$ , energy is

- **lost** due to the removal of edges attached to  $x$ : amount  $\sum_{y \in V_x} c_{xy} [f(x) - f(y)]^2$ .
- **gained** due to the increased conductance on the non-removed edges: amount  $\sum_{yz \in E_x} \tilde{c}_{yz} [f(y) - f(z)]^2$ .

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**Proposition** (“Octopus inequality” for electric network). For all  $f : V \rightarrow \mathbb{R}$ ,

$$\sum_{y \in V_x} c_{xy} [f(x) - f(y)]^2 \geq \sum_{yz \in E_x} \tilde{c}_{yz} [f(y) - f(z)]^2,$$

Energy lost from removed edges  $\geq$  Energy gained from increased conductances

where equality is attained iff  $(\mathbf{L}f)(x) = 0$ .

*Proof.* An exercise in high school algebra.

**Corollary.** The Dirichlet energy is *monotone non-increasing* upon successive network reductions.

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By carrying out network reduction one vertex at a time until two vertices  $z$  and  $y$  are left, we recover **Dirichlet's principle**:  $\mathcal{E}(f) \geq c_{\text{eff}}(z, y) [f(z) - f(y)]^2$ .

**Why the name “octopus”?** The tentacular nature of removing of a vertex and its edges may remind you of an octopus. [This name was christened by Pietro Caputo.]

# Octopus inequality & Aldous' spectral gap conjecture

Using the network reduction idea & delicately carrying out a series of Schur complementations, Caputo–Liggett–Richthammer *JAMS* '10 proved for the interchange process:

## Theorem (Octopus inequality, IP)

For all  $f : \mathcal{S}_{|V|} \rightarrow \mathbb{R}$ ,

$$\int \sum_{y \in V_x} c_{xy} [f(\eta^{xy}) - f(\eta)]^2 d\nu(\eta) \geq \int \sum_{yz \in E_x} \tilde{c}_{yz} [f(\eta^{yz}) - f(\eta)]^2 d\nu(\eta).$$

*Energy lost from removed edges  $\geq$  Energy gained from increased conductances*

This was the key inequality which resolved Aldous' '92 spectral gap conjecture:

$$(OI) \implies \lambda_2^{\text{IP}}(G) \geq \lambda_2^{\text{RW}}(G) \xRightarrow{+\text{proj.}} \lambda_2^{\text{IP}}(G) = \lambda_2^{\text{EX}}(G) = \lambda_2^{\text{RW}}(G).$$

- **MathSciNet review of CLR10, by L. Miclo:** “One leaves this beautiful paper with the dream that **maybe a simpler proof could be found.**”
- Since then there have been attempts at simplifying the CLR proof, but to little avail.
- Also it was unclear if the octopus has any applications beyond resolving the spectral gap conjecture...

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*Energy lost from removed edges  $\geq$  Energy gained from increased conductances*

## RECENT DEVELOPMENTS — Applications of the octopus:

- C. '17, Moving particle lemma, used to carry out coarse-graining in the exclusion process towards proving hydrodynamic limits.
- Alon–Kozma '18, Improved estimates of mixing times of interchange process, energy level ordering in the Heisenberg ferromagnetic model.  
arXiv:1811.10537: “The first to use the octopus lemma for something new was Chen.”
- (Related) Hermon–Salez '18: Analog of Aldous' spectral gap conjecture for the zero-range process, used to establish comparison theorems for two zero-range processes with the same kinetics on the same graph.

Bounding the energy cost of swapping two particles at  $x$  and  $y$  in an **interacting particle system** by the effective resistance between  $x$  and  $y$  w.r.t. the **random walk process**.

## Theorem (MPL, IP/EX)

$$\begin{aligned}\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 d\nu(\eta) &\leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{IP}}(f), \quad f : \mathcal{S}_{|V|} \rightarrow \mathbb{R}, \\ \frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 d\nu_{\alpha}(\eta) &\leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f), \quad f : \{0, 1\}^V \rightarrow \mathbb{R}.\end{aligned}$$

*Proof sketch.*

- (OI)  $\Leftrightarrow$  monotonicity of energy under 1-point network reductions. So reduce  $G$  successively until two vertices  $x, y$  are left, we get

$$\mathcal{E}^{\text{IP}}(f) \geq \cdots \geq \frac{1}{2} \int c_{\text{eff}}(x, y) [f(\eta^{xy}) - f(\eta)]^2 d\nu(\eta). \quad \text{MPL for IP}$$

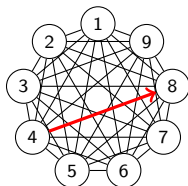
- To obtain the MPL for EX, use the projection of IP onto EX & disintegration of the uniform measure into orthonormal chambers with fixed particle number.

# Moving particle lemma for interchange/exclusion [C. ECP '17]

Bounding the energy cost of swapping two particles at  $x$  and  $y$  in an **interacting particle system** by the effective resistance between  $x$  and  $y$  w.r.t. the **random walk process**.

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$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 d\nu_{\alpha}(\eta) \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f), \quad f : \{0, 1\}^V \rightarrow \mathbb{R}.$$



Conventional approach is to pick a single path connecting  $x$  and  $y$  and obtain the energy cost. [Guo–Papanicolaou–Varadhan '88, Diaconis–Saloff-Coste '93].

Works just fine on finite integer lattices, but does NOT always give optimal cost on general weighted graphs.

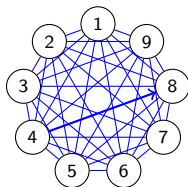


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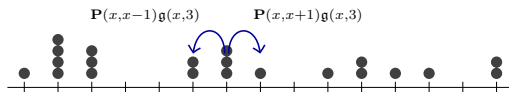
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$$\frac{1}{2} \int [f(\eta^{xy}) - f(\eta)]^2 d\nu_{\alpha}(\eta) \leq R_{\text{eff}}(x, y) \mathcal{E}^{\text{EX}}(f), \quad f : \{0, 1\}^V \rightarrow \mathbb{R}.$$



MPL bounds the energy cost by “optimizing electric flow over all paths connecting  $x$  and  $y$ .”

# Zero-range process $\leftrightarrow$ random walk process



$$(\mathcal{L}^{\text{ZR}} f)(\xi) = \sum_{(x,y) \in V^2} \mathbf{P}(x,y) \mathbf{g}(x, \xi(x)) [f(\xi + \mathbf{1}_y - \mathbf{1}_x) - f(\xi)], \quad \text{inv. meas. } \mu.$$

$$\text{Let } \Omega := \left\{ \xi \in \mathbb{N}_0^V : \sum_{x \in V} \xi(x) = m \right\} \text{ and } \hat{\Omega} := \left\{ \zeta \in \mathbb{N}_0^V : \sum_{x \in V} \zeta(x) = m-1 \right\}.$$

For each  $f : \Omega \rightarrow \mathbb{R}$  and  $\zeta \in \hat{\Omega}$ , define  $f_\zeta : V \rightarrow \mathbb{R}$  by  $f_\zeta(x) = f(\zeta + \mathbf{1}_x)$ .

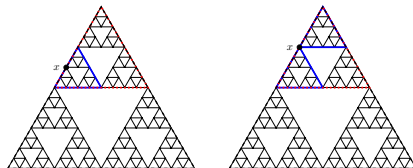
**Lemma.** For all  $f, g : \Omega \rightarrow \mathbb{R}$ ,  $\mathcal{E}_{(\mathbf{P}, \mathbf{g}, m)}^{\text{ZR}}(f, g) = \sum_{\zeta \in \hat{\Omega}} \mu(\zeta) \langle f_\zeta, (\mathbf{I} - \mathbf{P})g_\zeta \rangle_\pi$ . (Jump part decouples)

**Theorem** [Hermon-Salez '18]. For any two irred. jump matrices  $\mathbf{P}$  and  $\mathbf{Q}$ ,

$$\min_{\substack{f: \Omega \rightarrow \mathbb{R} \\ f \neq 0}} \left\{ \frac{\mathcal{E}_{(\mathbf{P}, \mathbf{g}, m)}^{\text{ZR}}(f, f)}{\mathcal{E}_{(\mathbf{Q}, \mathbf{g}, m)}^{\text{ZR}}(f, f)} \right\} = \min_{\substack{f: V \rightarrow \mathbb{R} \\ f \neq 0}} \left\{ \frac{\langle f, (\mathbf{I} - \mathbf{P})f \rangle_{\pi_{\mathbf{P}}}}{\langle f, (\mathbf{I} - \mathbf{Q})f \rangle_{\pi_{\mathbf{Q}}}} \right\}.$$

**Proposition** [C.]. If  $\mathbf{P}$  is associated to a symm. RW, then we have the **MPL**

$$\sum_{\zeta \in \hat{\Omega}} [f(\zeta + \mathbf{1}_y) - f(\zeta + \mathbf{1}_x)]^2 \mu(\zeta) \leq R_{\text{eff}}(x, y) \mathcal{E}_{(\mathbf{P}, \mathbf{g}, m)}^{\text{ZR}}(f, f).$$



For finite  $\Lambda \subset V$ , denote  $\text{Av}_\Lambda[\eta] := |\Lambda|^{-1} \sum_{z \in \Lambda} \eta(z)$ .

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator  $\mathcal{T}_N \mathcal{L}_N^{\text{EX}}$ ) on a sequence of graphs  $G_N = (V_N, E_N)$ , we need to prove that for every  $t > 0$ :

## Replacement lemma

$$\overline{\lim}_{\epsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t \left( \eta_s^N(x) - \text{Av}_{B(x, \epsilon N)}[\eta_s^N] \right) ds \right| \right] = 0, \quad x \in V_N.$$

where

- $\{\eta_t^N : t \geq 0\}$  is the exclusion process generated by  $\mathcal{T}_N \mathcal{L}_N^{\text{EX}}$ , where  $\mathcal{T}_N$  is the diffusive time acceleration factor.
- $\mu_N$  can be any measure on  $\{0, 1\}^{V_N}$ .
- $B(x, r)$  is a “ball” of radius  $r$  centered at  $x$  (in the graph metric).

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator  $\mathcal{T}_N \mathcal{L}_N^{\text{EX}}$ ) on a sequence of graphs  $G_N = (V_N, E_N)$ , we need to prove that for every  $t > 0$ :

## Replacement lemma

$$\lim_{\epsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t g(\eta_s^N) ds \right| \right] = 0, \text{ where } g(\eta) := \eta(x) - A_{V_{B(x, \epsilon N)}}[\eta], \quad x \in V_N.$$

The usual method to control additive functionals of the EX process is to employ the entropy inequality, Jensen's inequality, and the Feynman-Kac formula:

$$\mathbb{E}_{\mu_N} \left[ \left| \int_0^t g(\eta_s^N) ds \right| \right] \leq \frac{H(\mu_N | \nu_{\rho(\cdot)}^N)}{\kappa |V_N|} + \frac{1}{\kappa |V_N|} \sup_f \left\{ \int g(\eta) f(\eta) d\nu_{\rho(\cdot)}^N(\eta) - \frac{\mathcal{T}_N}{\kappa |V_N|} \langle \sqrt{f}, -\mathcal{L}_N \sqrt{f} \rangle_{\nu_{\rho(\cdot)}^N} \right\}$$

where

- $\rho(\cdot) \in \text{dom} \mathcal{E}$  is a (possibly non)constant reference density profile.
- $H(\mu | \nu) = \int \log \left( \frac{d\mu}{d\nu} \right) d\mu$  is the relative entropy of  $\mu$  w.r.t.  $\nu$ , assumed to be  $\mathcal{O}(|V_N|)$ .
- $\kappa > 0$ .
- The supremum is taken over all prob. densities  $f$  w.r.t. the product Bernoulli measure  $\nu_{\rho(\cdot)}^N$ .

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator  $\mathcal{T}_N \mathcal{L}_N^{\text{EX}}$ ) on a sequence of graphs  $G_N = (V_N, E_N)$ , we need to prove that for every  $t > 0$ :

## Replacement lemma

$$\overline{\lim}_{\epsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t g(\eta_s^N) ds \right| \right] = 0, \text{ where } g(\eta) := \eta(x) - \text{Av}_{B(x, \epsilon N)}[\eta], \quad x \in V_N.$$

Assume for this discussion that  $\rho(\cdot) = \rho$  constant. We wish to estimate

$$\int g(\eta) f(\eta) d\nu_\rho^N(\eta) - \frac{\mathcal{T}_N}{\kappa |V_N|} \langle \sqrt{f}, -\mathcal{L}_N \sqrt{f} \rangle_{\nu_\rho^N}$$

independent of  $f$  and the carré du champ

$$\mathcal{D}_N(\sqrt{f}, \nu_\rho^N) := \frac{1}{2} \int \sum_{zw \in E_N} c_{zw} \left( \sqrt{f(\eta^{zw})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta).$$

Using the Cauchy-Schwarz (Young) inequality and several elementary tricks, we get for any  $A > 0$ ,

$$\begin{aligned} \int g(\eta) f(\eta) d\nu_\rho^N(\eta) &\leq \frac{1}{2|B|} \sum_{z \in B} \left\{ \frac{A}{2} \int (\eta(z) - \eta(x))^2 \left( \sqrt{f(\eta^{zx})} + \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta) \right. \\ &\quad \left. + \frac{1}{2A} \int \left( \sqrt{f(\eta^{zx})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta) \right\}. \quad (B = B(x, \epsilon N)) \end{aligned}$$

# MPL & coarse-graining

In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator  $\mathcal{T}_N \mathcal{L}_N^{\text{EX}}$ ) on a sequence of graphs  $G_N = (V_N, E_N)$ , we need to prove that for every  $t > 0$ :

## Replacement lemma

$$\lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t g(\eta_s^N) ds \right| \right] = 0, \text{ where } g(\eta) := \eta(x) - A v_{B(x, \epsilon N)}[\eta], \quad x \in V_N.$$

This last term needs to be bounded by something times the carré du champ

$$\mathcal{D}_N(\sqrt{f}, \nu_\rho^N) := \frac{1}{2} \int \sum_{zw \in E_N} c_{zw} \left( \sqrt{f(\eta^{zw})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta).$$

Use the MPL:

$$\begin{aligned} \frac{1}{2|B|} \sum_{z \in B} \int \left( \sqrt{f(\eta^{zx})} - \sqrt{f(\eta)} \right)^2 d\nu_\rho^N(\eta) &\leq \frac{1}{|B|} \sum_{z \in B} R_{\text{eff}}(z, x) \mathcal{D}_N(\sqrt{f}, \nu_\rho^N) \\ &\leq \text{diam}_R(B) \mathcal{D}_N(\sqrt{f}, \nu_\rho^N), \end{aligned}$$

where  $\text{diam}_R(B)$  is the diameter of  $B$  in the resistance metric. ( $B = B(x, \epsilon N)$ )

Assuming that  $\frac{|V_N|}{\mathcal{T}_N} \text{diam}_R(B)$  is bounded for all  $N$ —this holds for **resistance spaces** in general— we can then choose  $A$  wisely to bound the **linear functional** from above by an expression which tends to 0 in the limit. This proves the replacement lemma.

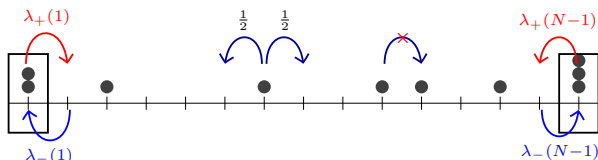
In the proof of the hydrodynamic limit for boundary-driven exclusion processes (w/ generator  $\mathcal{T}_N \mathcal{L}_N^{\text{EX}}$ ) on a sequence of graphs  $G_N = (V_N, E_N)$ , we need to prove that for every  $t > 0$ :

## Replacement lemma

$$\overline{\lim}_{\epsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t g(\eta_s^N) ds \right| \right] = 0, \text{ where } g(\eta) := \eta(x) - \text{Av}_{B(x, \epsilon N)}[\eta], \quad x \in V_N.$$

- AFAIK this is the first time such an argument works on a non-lattice weighted graph, where translational invariance is absent.
- Another instance where one needs to prove such a replacement lemma in the absence of translational invariance: Studying non-equilibrium density fluctuations on  $(\mathbb{Z}/N\mathbb{Z})^d$ . [Jara–Menezes '18](#) came up with their coarse-graining approach, called the “flow lemma,” which utilizes mass distribution on the lattice, and is reminiscent of the divisible sandpile problem [Levine–Peres '09].
- **Other usages of MPL:** Local 2-blocks estimate [C. '17]; 2nd-order Boltzmann-Gibbs principle for equilibrium density fluctuations [C. '19+].

# Adding reservoirs (Kawasaki dynamics) to the exclusion process



Designate a finite boundary set  $\partial V \subset V$ . For each  $a \in \partial V$ :

- At rate  $\lambda_+(a)$ ,  $\eta(a) = 0 \rightarrow \eta(a) = 1$  (birth).
- At rate  $\lambda_-(a)$ ,  $\eta(a) = 1 \rightarrow \eta(a) = 0$  (death).

Formally,  $(\mathcal{L}_{\partial V}^{\text{boun}} f)(\eta) = \sum_{a \in \partial V} [\lambda_+(a)(1 - \eta(a)) + \lambda_-(a)\eta(a)][f(\eta^a) - f(\eta)]$ ,  $f : \{0, 1\}^V \rightarrow \mathbb{R}$ , where

$$\eta^a(z) = \begin{cases} 1 - \eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

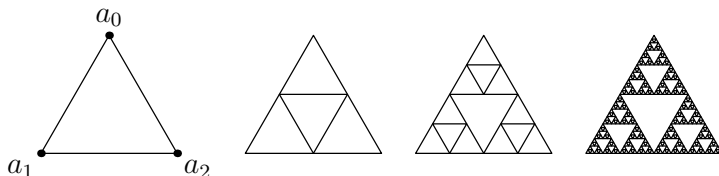
1D boundary-driven simple exclusion process: generator  $N^2 \left( \mathcal{L}_{\{1,2,\dots,N-1\}}^{\text{EX}} + \mathcal{L}_{\{1,N-1\}}^{\text{boun}} \right)$ .

Has been studied extensively for the past  $\sim 15$  years.

Hydrodynamic limits, fluctuations, large deviations, etc.

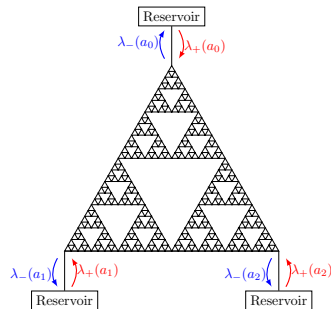


# Boundary-driven exclusion process on the Sierpinski gasket



- Construction of Brownian motion with invariant measure  $m$  (the standard self-similar measure) as scaling limit of RWs accelerated by  $\mathcal{T}_N = 5^N$ .  
[Goldstein '87, Kusuoka '88, Barlow-Perkins '88]
- A robust notion of calculus on SG which in some sense mimics (but in many other senses differs from) calculus in 1D: Laplacian, Dirichlet form, integration by parts, boundary-value problems, etc.  
[See the books by Kigami '01, Strichartz '06]
- A good model for rigorously studying (non)equilibrium stochastic dynamics with  $\geq 3$  boundary reservoirs.

# Boundary-driven exclusion process on the Sierpinski gasket



$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left( \mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

Parameter  $b > 0$  governs the inverse speed (relative to the bulk jump rate) at which the reservoir injects/extracts particles into/from the boundary vertices  $V_0$ .

Joint work with Patrícia Gonçalves (IST Lisboa), [arXiv:1904.xxxxx](#)

There is a **phase transition** in the scaling limit of the particle density depending on the value of  $b$ . The critical value of  $b$  is  $\frac{5}{3}$ .

Dirichlet ( $b < \frac{5}{3}$ ), Robin ( $b = \frac{5}{3}$ ), Neumann ( $b > \frac{5}{3}$ )

Assume that sequence of probability measures  $\{\mu_N\}_{N \geq 1}$  on  $\{0, 1\}^{V_N}$  is associated to a density profile  $\varrho : K \rightarrow [0, 1]$ : for any continuous function  $F : K \rightarrow \mathbb{R}$  and any  $\delta > 0$ ,

$$\lim_{N \rightarrow \infty} \mu_N \left\{ \eta \in \{0, 1\}^{V_N} : \left| \frac{1}{|V_N|} \sum_{x \in V_N} F(x) \eta(x) - \int_K F(x) \varrho(x) dm(x) \right| > \delta \right\} = 0.$$

Given the process  $\{\eta_t^N : t \geq 0\}$  generated by  $5^N \mathcal{L}_N^{\text{bEX}}$ , the **empirical density measure**  $\pi_t^N$  given by

$$\pi_t^N = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \delta_{\{x\}}$$

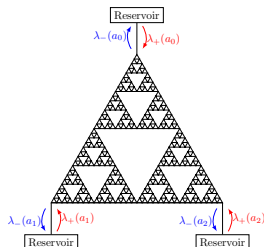
and for any test function  $F : K \rightarrow \mathbb{R}$ , we denote the integral of  $F$  wrt  $\pi_t^N$  by  $\pi_t^N(F)$  which equals

$$\pi_t^N(F) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x).$$

**Claim.** The sequence  $\{\pi_t^N\}_N$  converges in the Skorokhod topology on  $D([0, T], \mathcal{M}_+)$  to the unique measure  $\pi$ . with  $d\pi(x) = \rho(\cdot, x) dm(x)$ .

For any  $t \in [0, T]$ , any continuous  $F : K \rightarrow \mathbb{R}$  and any  $\delta > 0$ ,

$$\lim_{N \rightarrow \infty} \mu_N \left\{ \eta_t^N : \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t, x) dm(x) \right| > \delta \right\} = 0,$$



$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left( \mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

$$\lambda_{\Sigma}(a) = \lambda_{+}(a) + \lambda_{-}(a)$$

$$\bar{\rho}(a) = \frac{\lambda_{+}(a)}{\lambda_{\Sigma}(a)}$$

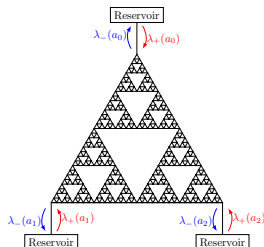
## Theorem (Density hydrodynamic limit)

For any  $t \in [0, T]$ , any continuous  $F : K \rightarrow \mathbb{R}$  and any  $\delta > 0$ ,

$$\lim_{N \rightarrow \infty} \mu_N \left\{ \eta_t^N : \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t, x) dm(x) \right| > \delta \right\} = 0,$$

where  $\rho$  is the unique weak solution of the heat equation  
with Dirichlet boundary condition if  $b < \frac{5}{3}$ :

$$\begin{cases} \partial_t \rho(t, x) = \frac{2}{3} \Delta \rho(t, x), & t \in [0, T], x \in K \setminus V_0, \\ \rho(t, a) = \bar{\rho}(a), & t \in (0, T], a \in V_0, \\ \rho(0, x) = \varrho(x), & x \in K. \end{cases}$$



$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left( \mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

$$\lambda_{\Sigma}(a) = \lambda_{+}(a) + \lambda_{-}(a)$$

$$\bar{\rho}(a) = \frac{\lambda_{+}(a)}{\lambda_{\Sigma}(a)}$$

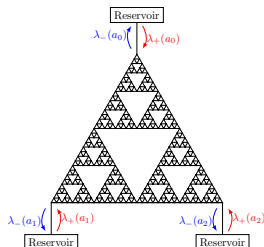
## Theorem (Density hydrodynamic limit)

For any  $t \in [0, T]$ , any continuous  $F : K \rightarrow \mathbb{R}$  and any  $\delta > 0$ ,

$$\lim_{N \rightarrow \infty} \mu_N \left\{ \eta_t^N : \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t, x) dm(x) \right| > \delta \right\} = 0,$$

where  $\rho$  is the unique weak solution of the heat equation with Neumann boundary condition if  $b > \frac{5}{3}$ :

$$\begin{cases} \partial_t \rho(t, x) = \frac{2}{3} \Delta \rho(t, x), & t \in [0, T], x \in K \setminus V_0, \\ (\partial^\perp \rho)(t, a) = 0, & t \in (0, T], a \in V_0, \\ \rho(0, x) = \varrho(x), & x \in K. \end{cases}$$



$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left( \mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

$$\lambda_{\Sigma}(a) = \lambda_{+}(a) + \lambda_{-}(a)$$

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## Theorem (Density hydrodynamic limit)

For any  $t \in [0, T]$ , any continuous  $F : K \rightarrow \mathbb{R}$  and any  $\delta > 0$ ,

$$\lim_{N \rightarrow \infty} \mu_N \left\{ \eta_t^N : \left| \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) F(x) - \int_K F(x) \rho(t, x) dm(x) \right| > \delta \right\} = 0,$$

where  $\rho$  is the unique weak solution of the heat equation with linear Robin boundary condition if  $b = \frac{5}{3}$ :

$$\begin{cases} \partial_t \rho(t, x) = \frac{2}{3} \Delta \rho(t, x), & t \in [0, T], x \in K \setminus V_0, \\ (\partial^\perp \rho)(t, a) = -\lambda_{\Sigma}(a)(\rho(t, a) - \bar{\rho}(a)), & t \in (0, T], a \in V_0, \\ \rho(0, x) = \varrho(x), & x \in K. \end{cases}$$

Analysis of Dynkin's martingale (which has QV tending to 0 as  $N \rightarrow \infty$ ):

$$M_t^N(F) := \pi_t^N(F_t) - \pi_0^N(F_0) - \int_0^t \pi_s^N \left( \left( \frac{2}{3} \Delta + \partial_s \right) F_s \right) ds \\ + \int_0^t \frac{3^N}{|V_N|} \sum_{a \in V_0} \left[ \eta_s^N(a) (\partial^\perp F_s)(a) + \frac{5^N}{3^N b^N} \lambda_\Sigma(a) (\eta_s^N(a) - \bar{\rho}(a)) F_s(a) \right] ds + o_N(1).$$

## ① Analysis on fractals

- ▶ Discrete Laplacian converges to the fractal Laplacian: Operator convergence, energy convergence.
- ▶ Integration by parts formula.
- ▶ Estimate of diameter of cells in the resistance metric.
- ▶ Sobolev embedding:  $\text{dom} \mathcal{E} \subset C(K)$ .

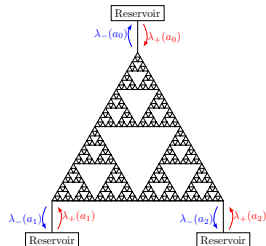
## ② Functional inequalities in the exclusion process (FOCUS of this talk)

- ▶ Moving particle lemma.
- ▶ Density replacement lemma **at each boundary point**: variational estimates of additive functionals via Feynman-Kac. Proof is sensitive to the regime of  $b$  (DRN).

## ③ Convergence of stochastic processes

- ▶ Prove tightness of the sequence  $\{\pi_t^N\}_N$  in  $D([0, T], \mathcal{M}_+)$  via Aldous' criterion.
- ▶ Characterization of the limit point: Almost surely, any limit point  $\pi$  is AC w.r.t. the standard self-similar measure  $m$ , and the density  $\rho \in L^2(0, T, \text{dom} \mathcal{E})$ .
- ▶ Uniqueness of the limit point: Show that  $\rho$  satisfies the weak formulation of the heat equation, and use Oleinik's method to prove uniqueness of the solution.

# A sneak preview of upcoming series of works, and **Thank you!**



$$5^N \mathcal{L}_N^{\text{bEX}} = 5^N \left( \mathcal{L}_N^{\text{EX}} + \frac{1}{b^N} \mathcal{L}_N^{\text{boun}} \right).$$

**Symmetric** exclusion process with **slowed** boundary on the Sierpinski gasket

Dirichlet ( $b < \frac{5}{3}$ ), Robin ( $b = \frac{5}{3}$ ), Neumann ( $b > \frac{5}{3}$ )

**Equilibrium**  $\Leftrightarrow \lambda_+(a) = \lambda_+$  and  $\lambda_-(a) = \lambda_-$  for all  $a \in V_0$ . (Otherwise, **nonequilibrium**.)

- (Non)equilibrium density hydrodynamic limit (DRN✓) [C.–Gonçalves, arXiv:1904.xxxxx]
- Ornstein-Uhlenbeck limit of equilibrium density fluctuations (DRN✓). [C.–Gonçalves, arXiv:1904.xxxxx]
- Large deviations principle for the (non)equilibrium density (D✓) [C.–Hinz '19]
- Hydrostatic limit, scaling limit of nonequilibrium density fluctuations (D in progress). [C.–Franceschini–Gonçalves–Menezes '19+]  $\rightarrow$  careful study of two-particle correlations
- **More in the pipeline:**
  - Motion of the tagged particle (a fractional BM on the gasket?).
  - Add (suitably rescaled) weak asymmetry to the jump rate, prove that the equilibrium density fluctuations converges (subsequentially) to a stochastic Burgers equation [C. '19+]