Where does the eigenvalue λ_1 come from?

A calculus question

Find as many functions ψ which satisfy

$$-\psi''(x) = \lambda\psi(x) \quad \text{and} \quad \psi(x) = \psi(x+1).$$

[*Hint:* Try the sines or cosines!]

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$$\begin{aligned} \psi(x) &= \sin(\alpha x) \\ \psi'(x) &= \alpha\cos(\alpha x) \\ \psi''(x) &= -\alpha^2\sin(\alpha x) = -\alpha^2\psi(x) \end{aligned}$$

We also want

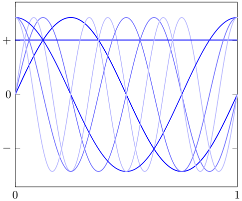
$$\sin(\alpha x) = \sin(\alpha(x+1))$$

Since sin is 2π-periodic, we can choose $\alpha = 2\pi k$ for any integer k.

$$\lambda = \alpha^2 = (2\pi k)^2.$$

Same story with $\psi(x) = \cos(2\pi kx)$.

Laplacian eigenvalues and eigenfunctions



$$-\psi''(x) = \lambda\psi(x) \quad \text{and} \quad \psi(x) = \psi(x+1).$$

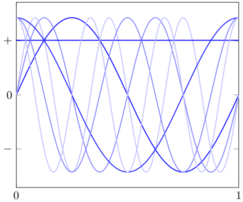
Infinite number of solutions.
We index the solutions according to increasing order in the eigenvalue λ:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \uparrow +\infty$$

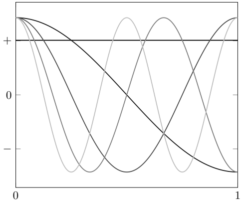
Here, $\lambda_1 = (2\pi)^2$, and $\psi_1(x)$ can be either $\sin(2\pi x)$ or $\cos(2\pi x)$.

The Laplace eigenvalue problem in 1D

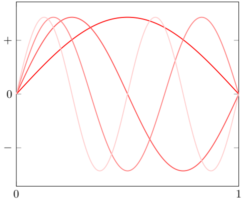
$-\psi''(x) = \lambda\psi(x)$ along with boundary conditions at the endpoints $\{0, 1\}$.



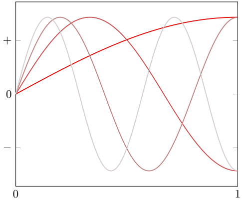
(a) Periodic b.c.



(b) Closed b.c.



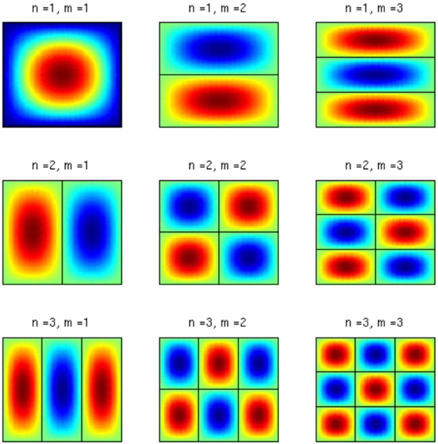
(c) Open b.c.



(d) Open at $\{0\}$, Closed at $\{1\}$

2D

$$-\Delta\psi(x,y) := -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) = \lambda\psi(x,y)$$



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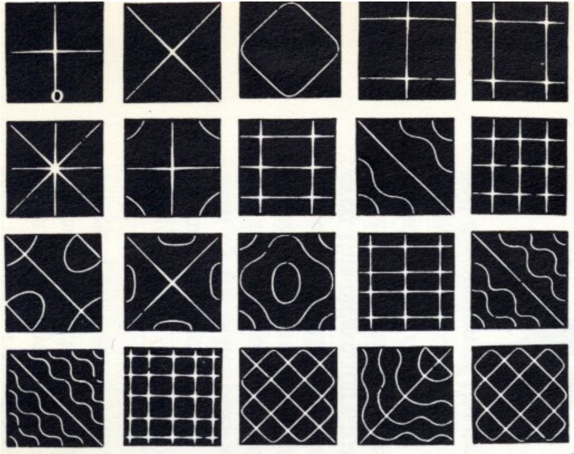

Physical demo of vibrations on a square drumhead (Chladni plates):
https://www.youtube.com/watch?v=1yaqUI4b974

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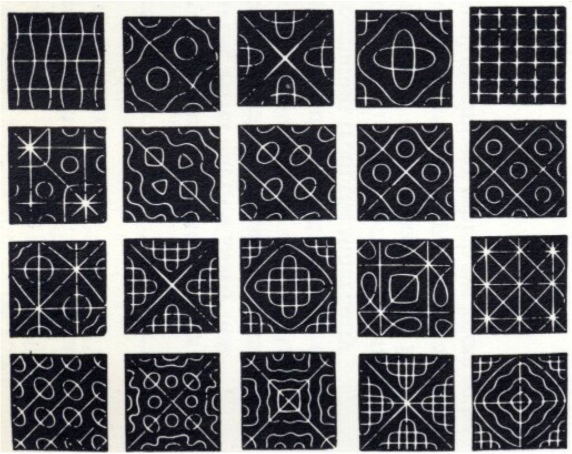

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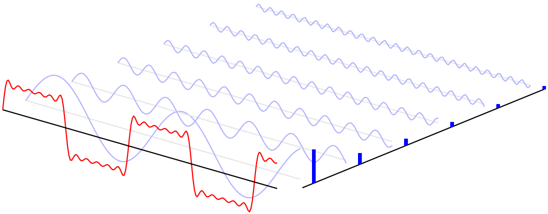
Fourier series

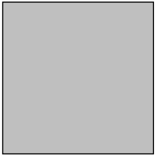


Any reasonably nice function f on the drumhead can be expressed as a linear combination of the eigenfunctions ψ_j, *a.k.a.* a Fourier series.

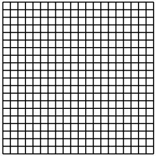
1D drumhead $$f(x) = \sum_{k=0}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx)$$

General drumheads $$f(x) = \sum_{j=1}^{\infty} c_j \psi_j(x)$$

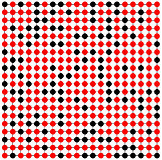
Back to the cutoff problem for card shuffles



Drumhead



N-by-N grid



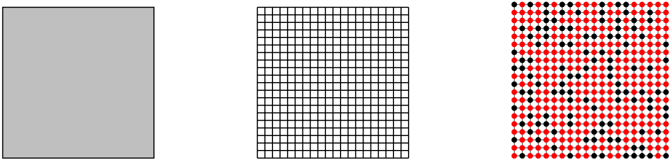
Cards on the grid

- For every positive integer N, on the N-by-N grid, place ϱ fraction of **black** cards and $1 - \varrho$ fraction of **red** cards.
- Shuffle = Choose an edge uniformly at random, and swap the positions of the two cards across that edge.

Question

What functions $f(x)$ to use in order to solve the cutoff problem?

Back to the cutoff problem for card shuffles



Drumhead　　　N-by-N grid　　　Cards on the grid

Answer: $\eta_t^N(x) = \begin{cases} 1, & \text{if a } \textbf{black} \text{ card is present at vertex } x \text{ at time } t, \\ 0, & \text{if a } \textbf{red} \text{ card is present at vertex } x \text{ at time } t. \end{cases}$

But we need to "smooth" it before applying the Fourier series expansion.

"Smoothing" is achieved by taking averages: multiply $\eta_{tN^2}^N(x)$ by a smooth function $F(x)$, sum over all the grid vertices x, then divide by the number of vertices $|V_N|$.

$$\frac{1}{|V_N|} \sum_{x \in V_N} \eta_{tN^2}^N(x) F(x)$$

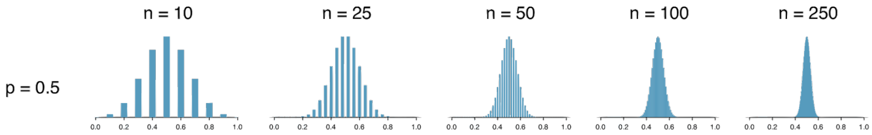
The law of averages emerges in the limit $N \to \infty$

The law of averages & the central limit theorem

Toss a fair coin (probability of heads $p = 0.5$) n times, count the # of heads turned up.

Frequency of heads in n tosses $\quad \hat{p}_n = \dfrac{\#\text{ of heads in } n \text{ tosses}}{n}$

Histogram for the frequency of heads

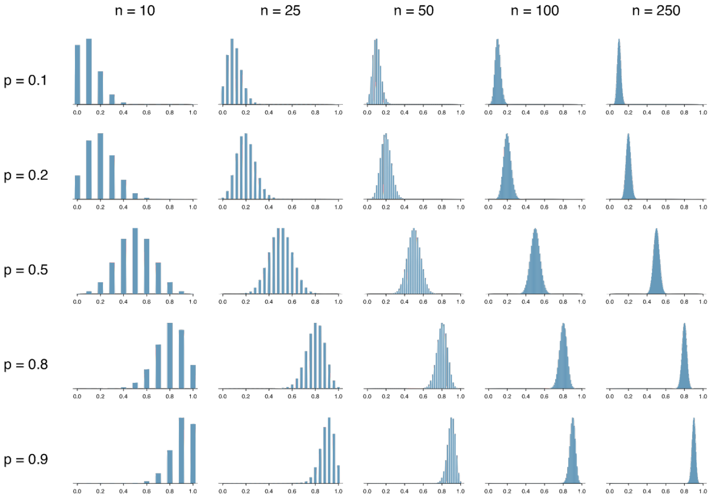


When n is large, $\hat{p}_n$ is well approximated by the normal distribution ("bell curve")

$$\mathcal{N}\left(\text{mean} = 0.5,\ \text{st. dev.} = \sqrt{\frac{0.5 \cdot 0.5}{n}}\right)$$

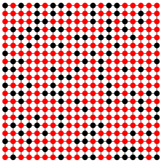
The law of averages & the central limit theorem

Toss a coin with probability of heads p, n times, find the frequency of heads.



When n is large, $\hat{p}_n \sim \mathcal{N}\left(\text{mean} = p,\ \text{st. dev.} = \sqrt{\frac{p(1-p)}{n}}\right).$

Limit theorems for card shuffling



- Let F be a smooth function on the drumhead.
- Law of averages

$$\frac{1}{|V_N|}\sum_{x\in V_N}\eta^N_{tN^2}(x)F(x) \xrightarrow[N\to\infty]{} \int_{\text{drumhead}} \rho(t,x)F(x)\,dx$$

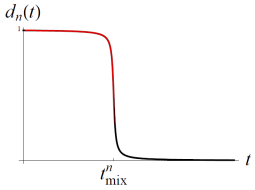
  where the **mean** space-time-varying **density** $\rho(t,x)$ is the solution to a **heat** (diffusion) **equation** on the drumhead.
- “Central limit theorem”

$$\frac{1}{\sqrt{|V_N|}}\sum_{x\in V_N}(\eta^N_{tN^2}(x)-\rho^N_{tN^2}(x))F(x) \text{ “}\xrightarrow[N\to\infty]{}\text{” } \mathcal{N}\left(\text{mean}=0,\ \text{st. dev.}=\sqrt{V(t,F)}\right).$$

- The observables on the left-hand side can be expanded in Fourier series.

Lower bound: $\lim_{N\to\infty} d_N(\kappa t_N) = 1$ for every $\kappa < 1$

$$t_N = \frac{N^2 \log |V_N|}{2\lambda_1}$$



Wilson's method ('04)

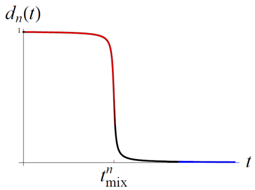
Construct the following "instrument" inspired by the law of averages, using the smooth function $F(x) = \psi_1^N(x)$, the first eigenfunction corresponding to eigenvalue λ_1^N:

$$\frac{1}{|V_N|} \sum_{x \in V_N} \eta^N_{tN^2}(x) \psi_1^N(x)$$

Computing its variance, and taking the limit $N \to \infty$, proves the mixing time lower bound.

Upper bound: $\lim_{N\to\infty} d_N(\kappa t_N) = 0$ for every $\kappa > 1$

$$t_N = \frac{N^2 \log|V_N|}{2\lambda_1}$$



- **Easy**: Adapting Wilson's argument one can show $\lim_{N\to\infty} d_N(\kappa t_N) = 0$ for every $\kappa > 2$ ("pre-cutoff").
- The sharp upper bound is very difficult to prove! Only proven in 1D; the proof relies strongly on the partial order of $\mathbb{Z}$. [Lacoin '16 (2x), Nam–Nestoridi '19]
  $\rightarrow$ Impossible to generalize to higher dimensions.
- **Breakthrough** (May 2020): We devised a new instrument, called the "cutoff martingale," which was inspired by the "CLT" observable.

"CLT" observable $\displaystyle \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} (\eta^N_{tN^2}(x) - \rho^N_{tN^2}(x))\psi^N_1(x)$

Upper bound: $\lim_{N\to\infty} d_N(\kappa t_N) = 0$ for every $\kappa > 1$

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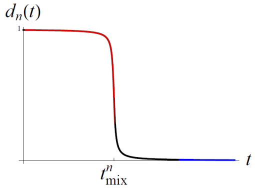

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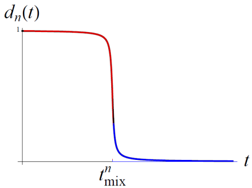

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Then "magic" happens, so long as the family of graphs converges geometrically and spectrally to a "nice" drumhead. $\to$ Works in any dimension.

Etymology of the cutoff martingale

Figure: Martingales[1]

- **Mar·tin·gale** (*noun*, French, 18th century): "any of several systems of betting in which a player increases the stake usually by doubling each time a bet is lost." Merriam-Webster
- **Martingale** (probability theory, 1930s): "a sequence of random variables for which, at a particular time, the conditional expectation of the next value in the sequence, regardless of all prior values, is equal to the present value." Wikipedia
- "**Cutoff martingale**" (2020): A mean-zero martingale which can be used to prove the sharp upper bound for our cutoff problems!

[1] Wikimedia: (Left) Montanabw / CC BY-SA; (Right) Ralf Roletschek / CC BY-SA.

Cutoffs can be proved on many graphs (road networks)

Discrete interval $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$

Cutoff time $\boxed{t_N = \dfrac{N^2 \log N}{2\lambda_1}}$.

Boundary condition: Open, Closed, Periodic.

Boundary condition	λ_1	*#1 proof*
	π^2	[Lacoin '16]
	$(2\pi)^2$	[Lacoin '16, '17]
	$(\pi/2)^2$	[Gantert–Nestoridi–Schmid '20]
	π^2	**[C.–Jara–M. '20+]**

Cutoffs can be proved on many graphs (road networks)

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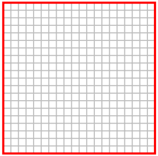
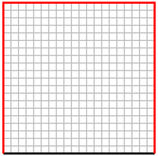
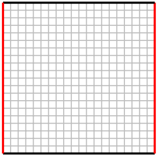


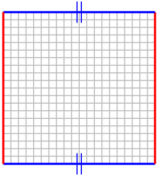
(a) $\lambda_1 = \pi^2 + \pi^2$



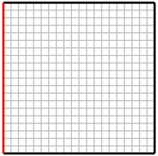
(b) $\lambda_1 = \pi^2 + (\pi/2)^2$



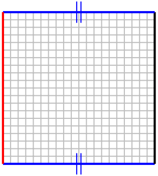
(c) $\lambda_1 = \pi^2 + 0$



(d) $\lambda_1 = \pi^2 + 0$



(e) $\lambda_1 = (\pi/2)^2 + 0$



(f) $\lambda_1 = (\pi/2)^2 + 0$

Cutoffs can be proved on many graphs (road networks)

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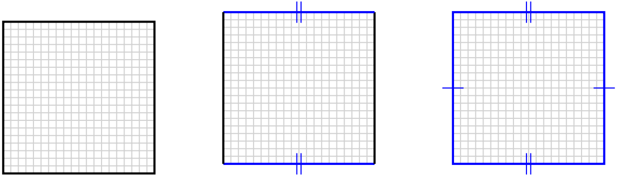
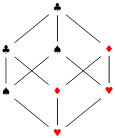


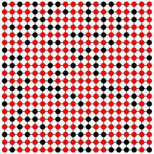
(g) $\lambda_1 = \pi^2 + 0$ (h) $\lambda_1 = \pi^2 + 0$ (i) $\lambda_1 = (2\pi)^2 + 0$

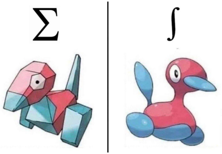
and more ...

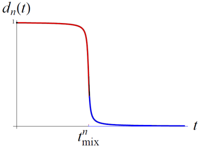
- Cutoffs hold on the d-dimensional grid for any integer d.
- Also hold on non-grid examples, such as a self-similar fractal network.
- In at least two shuffling protocols for which only pre-cutoff was established [Durrett '03, Saloff-Coste '04], we can prove cutoff.

This is our last slide









Preprints to appear soon on `arXiv.org`

- Cutoffs for road traffic with open (entry-exit) mechanisms: [C.–Jara–M. '20+]
- Cutoffs for card shuffling and of road traffic on a closed system: [C.–M. '20+]

Thank you! Obrigado!

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