Single phase flow: governing equation

We write Navier-Stokes equations as

$$\begin{aligned} &\nabla \cdot \vec{u} = 0 \\ &\rho \left(\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{f} \end{aligned} \tag{1}$$

with $\vec{u} = \vec{u}\,(\vec{x}, t)$ the velocity field, $p = p(\vec{x}, t)$ the pressure field, ρ the density, μ the viscosity and $\vec{f}$ external volume forces.

Time discretization: predictor-corrector

The unknowns in (1) are $\vec{u}$ and p. A 2nd-order central scheme time discretization of equation (1), which contains both variables at new timestep, correspond to

$$\begin{aligned} &\frac{\vec{u}^{\,n+1} - \vec{u}^{\,n}}{\Delta t} = \\ &= \frac{1}{\rho} \left[-\nabla p^{n+1} - \vec{u}^{\,n+1/2} \cdot \nabla \vec{u}^{\,n+1/2} + \mu \nabla^2 \vec{u}^{\,n+1/2} + \rho \vec{f}^{\,n+1/2} \right] \end{aligned} \tag{2}$$

with $\vec{u}^{\,n} = \vec{u}\,(\vec{x}, n\Delta t)$. We replace the pressure p^{n+1} with the value at actual timestep p^n, so the updated velocity field does not satisfy mass conservation

$$\frac{\vec{u}^{\,*} - \vec{u}^{\,n}}{\Delta t} = -\frac{1}{\rho}\nabla p^n + \frac{3}{2}\vec{\mathcal{H}}^{\,n} - \frac{1}{2}\vec{\mathcal{H}}^{\,n-1} + \vec{f}^{\,n+1/2} \tag{3}$$

where $\mathcal{H}_i = -u_j \partial_j u_i + \nu \partial_j \partial_j u_i$ and we have extrapolated values at timestep $n + 1/2$ using values from timestep n and $n - 1$ (2nd order Adams-Bashfort scheme). We call this predicted velocity field u^*. To enforce mass conservation we introduce a function $\phi = \phi\,(\vec{x}, t)$ such that

$$\vec{u}^{\,n+1} = \vec{u}^{\,*} - \frac{\Delta t}{\rho} \nabla \phi^{\,n} \tag{4}$$

and ϕ must satisfy the condition

$$\nabla^2 \phi^{\,n} = \frac{\rho}{\Delta t} \nabla \cdot \vec{u}^{\,*} \tag{5}$$

The sum of equations (3) and (4) shows that (by comparison with (2))

$$p^{n+1} = p^n + \phi^n \tag{6}$$

hence, ϕ^n represent the pressure increment at timestep n which satisify mass conservation.

Space discretization

We use a staggered grid, as shown in figure 1.

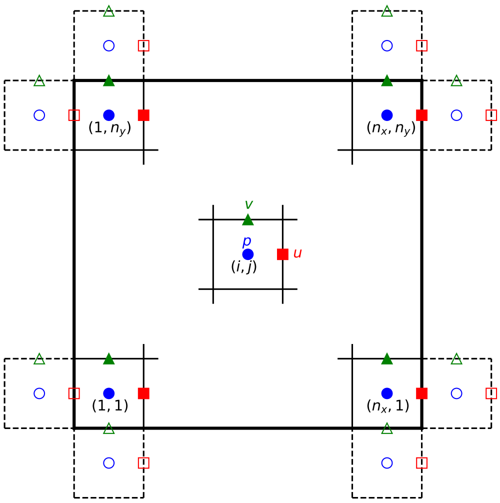

Figure 1: Sketch of the computational grid. Empty symbols represent ghost nodes.