Single phase flow: governing equation

We write incompressible Navier-Stokes equations as

$$\begin{cases} \nabla \cdot \vec{u} = 0 & \text{(1.a)} \\ \rho \left(\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{f} & \text{(1.b)} \end{cases}$$

with $\vec{u} = \vec{u}\,(\vec{x}, t)$ the velocity field, $p = p(\vec{x}, t)$ the pressure field, ρ the density, μ the viscosity and $\vec{f} = \vec{f}\,(\vec{x}, t)$ external volume forces.

Time discretization: predictor-corrector

The unknowns in (1) are $\vec{u}$ and p. A 2nd-order central scheme time discretization of equation (1.b), which contains both variables at new timestep, correspond to

$$\begin{aligned} &\frac{\vec{u}^{\,n+1} - \vec{u}^{\,n}}{\Delta t} = \\ &= \frac{1}{\rho} \left[-\nabla p^{n+1} - \vec{u}^{\,n+1/2} \cdot \nabla \vec{u}^{\,n+1/2} + \mu \nabla^2 \vec{u}^{\,n+1/2} + \rho \vec{f}^{\,n+1/2} \right] \end{aligned} \tag{2}$$

with $\vec{u}^{\,n} = \vec{u}\,(\vec{x}, n\Delta t)$. We replace the pressure p^{n+1} with the value at actual timestep p^n, so the updated velocity field does not satisfy mass conservation

$$\frac{\vec{u}^{\,*} - \vec{u}^{\,n}}{\Delta t} = -\frac{1}{\rho} \nabla p^n + \frac{3}{2} \vec{\mathcal{H}}^{\,n} - \frac{1}{2} \vec{\mathcal{H}}^{\,n-1} + \vec{f}^{\,n+1/2} \tag{3}$$

where $\mathcal{H}_i = -u_j \partial_j u_i + \nu \partial_j \partial_j u_i$ and we have extrapolated values at timestep $n + 1/2$ using values from timestep n and $n - 1$ (2nd order Adams-Bashfort scheme). We call this predicted velocity field u^*. To enforce mass conservation we introduce a function $\phi = \phi\,(\vec{x}, t)$ such that

$$\vec{u}^{\,n+1} = \vec{u}^{\,*} - \frac{\Delta t}{\rho} \nabla \phi^{\,n} \tag{4}$$

and ϕ must satisfy the condition

$$\nabla^2 \phi^{\,n} = \frac{\rho}{\Delta t} \nabla \cdot \vec{u}^{\,*} \tag{5}$$

The sum of equations (3) and (4) shows that (by comparison with (2))

$$p^{n+1} = p^n + \phi^n \tag{6}$$

hence, ϕ^n represent the pressure increment at timestep n which satisify mass conservation.

Space discretization

We use a staggered grid, as shown in figure 1. We replace space coordinates as $\vec{x} = (x, y) = ((i + s_x)\Delta x, (j + s_y)\Delta y)$, where $i = 1, 2, \ldots, N_x$, $j = 1, 2, \ldots, N_y$ are integers, Δx and Δy are the grid spacing in x and y direction (*i.e.* $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$); s_x and s_y are coefficient to get the value at cell center of cell face. Pressure is defined at cell center ($s_x = -0.5$, $s_y = -0.5$), while velocity on cell faces: for horizontal velocity component $s_x = 0$, $s_y = -0.5$, while for vertical velocity component $s_x = -0.5$, $s_y = 0$.

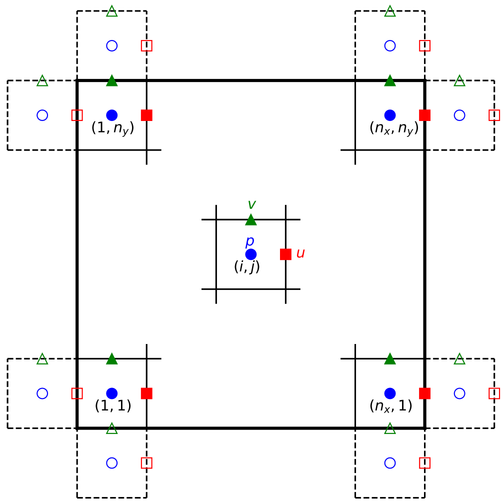

Figure 1: Sketch of the computational grid. Empty symbols represent ghost nodes.

By using the staggered grid the pressure gradient term in the x component of equation (1.b), for example, can be expressed as

$$\partial_t u + \cdots \approx -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i,j}}{\Delta x} + \ldots$$

and correspond to a 2nd order central scheme approximation of the pressure gradient on the x-face of cell (i, j) (red square in figure 1). The diffusion term, in the same equation can be approximated with 2nd order accuracy as

$$\nabla^2 u\big|_{i,j} \approx \nu \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}.$$

For incompressible flows we can write

$$\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} = u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} =$$

$$= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\nabla\cdot\vec{u}=0} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

A 2nd order approximation of the advection term in x direction can be written as

$$\left.\frac{\partial uu}{\partial x}\right|_{i,j} \approx \frac{(uu)_{i+1/2,j} - (uu)_{i-1/2,j}}{\Delta x} =$$

$$= \left[\frac{(u_{i+1,j} + u_{i,j})^2}{4} - \frac{(u_{i,j} + u_{i-1,j})^2}{4}\right]\frac{1}{\Delta x}$$

$$\left.\frac{\partial uv}{\partial y}\right|_{i,j} \approx \frac{uv_{i,j+1/2} - uv_{i,j-1/2}}{\Delta y} =$$

$$\left[\frac{u_{i,j+1} + u_{i,j}}{2}\frac{v_{i+1,j} + v_{i,j}}{2} - \frac{u_{i,j} + u_{i,j-1}}{2}\frac{v_{i+1,j-1} + v_{i,j-1}}{2}\right]\frac{1}{\Delta y}$$

Poisson Solver