UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**ELEC3030**
ASSESSMENT PATTERN	:	**ELEC3030A**
MODULE NAME	:	**Numerical Methods**
DATE	:	**28-May-10**
TIME	:	**10:00**
TIME ALLOWED	:	**3 Hours 0 Minutes**

2009/10-ELEC3030A-001-EXAM-21

 TURN OVER

Answer FOUR questions

1. (a) Define the term *machine precision* of a floating point system, `eps`, and derive an expression for a system of base 10 and mantissa length t.

[5 marks]

(b) Find the normalised error bounds for the following two algorithms to calculate $y = (x-1)^3$, assuming that x is represented exactly in the floating point system.

(i) $y = (x-1)(x-1)(x-1)$ and

(ii) $y = x^3 - 3x^2 + 3x - 1$

Which of these two algorithms is more stable?

For simplicity, you may consider the errors in all the operations to be the same value $(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \ldots\ etc)$ but remember they can have different signs.

Neglect higher order error terms.

[12 marks]

(c) For a floating point system of base 10 with a mantissa length 4 and exponents between –9 and 9, determine the value of the machine precision `eps` and find the number of numbers in the intervals [0, 1] and [1, 2] which are exactly represented by this system. What is the smallest number ($\neq 0$) that can be represented exactly?

[8 marks]

2. (a) Use the fixed point iteration method to find a root of the function: $f(x) = 2x^2 - x - 6$. Start the iterations with $x_0 = 1.5$, do at least four iterations and use four significant figures.
Make sure your chosen procedure satisfies the convergence test before starting the iterations.

[8 marks]

(b) Use the Newton-Raphson method to find a root of the function: $f(x) = \sin(e^x)$. Start the iterations with the value $x_0 = 1$.
Use the following table in your answer.

iteration	x_n	$f(x)$	$f'(x)$	x_{n+1}
1	1			
2				
3				
4				

Please reproduce this table in your answer book

[10 marks]

(c) Find the solution to the following system of equations using the Gauss-Seidel method. Start the iterations with the vector: $(1, 1, 1)^T$ perform at least four iterations and use at least four significant figures.

$$\begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \\ 7 \end{bmatrix}$$

[7 marks]

3. (a) Use Lagrange interpolation to find a polynomial of second order that fits the data set:

x	y
1.0	4.05
2.5	2.25
4.0	1.35

[8 marks]

(b) Use Newton interpolation to find the polynomial of order 2 that passes through the same points specified in part (a).
Then, find the polynomial of order 3 that passes through the same points and also through the additional point: (5.0, 5.25).

[12 marks]

(c) To represent a data set of n points by a continuous function, one has the choice of interpolation with a single $(n-1)$-order polynomial, a spline interpolation and least squares fitting of a chosen curve. Comment on the characteristics, advantages and disadvantages of each of these options.

[5 marks]

4. (a) Using the Taylor expansions at the points $f(a+h)$ and $f(a-h)$ show that the following is a suitable formula to approximate the second derivative of $f(x)$ and show that the error is $O(h^2)$.

$$f''(a) \approx \frac{f(a-h) - 2f(a) + f(a+h)}{h^2}$$

Using this expression find a formula to approximate the 2-dimensional Laplacian operator:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

evaluated at a point (a, b) and using equal discretisation steps h in both directions.

[8 marks]

(b) Fig. 1 shows part of the cross-section of an infinitely long square coaxial structure.

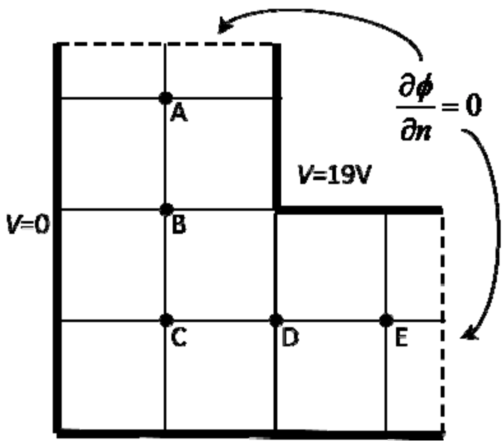

Fig. 1 *A quarter of the cross-section of a square coaxial line*

A voltage difference of V = 19 V is applied between the two conductors and the Neumann boundary condition is applied on the sections indicated by dotted lines. Using the mesh indicated in the figure, with spacing h in both dimensions, calculate using the finite differences method <u>and taking advantage of the symmetry of the structure</u>:

(i) The electric potential ϕ at the indicated points (A – E). [10 marks]

(ii) The electric field intensity at the points A, B and C. [7 marks]

<u>Note</u>: The electric potential ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$. The electric field is given by: $\vec{E} = -\nabla \phi = -\left(\frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} \right)$.

5. (a) Using the MacLaurin expansion (Taylor series centred at x=0) of $f(x)=\dfrac{\sin x}{x}$ truncated at order 4: $t(x)=1-\dfrac{x^2}{6}+\dfrac{x^4}{120}$, find the Padé approximant: $R_2^2(x)=\dfrac{p_2(x)}{q_2(x)}=\dfrac{a_2x^2+a_1x+a_0}{b_2x^2+b_1x+b_0}$ to the function $f(x)$, with the choice $b_0=1$.

Note: The general expression for the derivative of order i of the product: $g(x)=t(x)q(x)$ is given by: $g^{(i)}(x)=\sum_{j=0}^{i}\dfrac{i!}{j!(i-j)!}t^{(j)}(x)q^{(i-j)}(x)$, which evaluated at $x=0$ gives: $a_i=\sum_{j=0}^{i}c_{i-j}b_j$.

[10 marks]

(b) Calculate the integral: $\int_{-1}^{1}(2-x^2)\cos\dfrac{\pi x}{2}dx$

(i) using the trapezoid rule with 8 subintervals in [–1, 1].
(ii) using the Simpson's quadrature with 4 subintervals in [–1, 1].

Note: Use the expressions for a single interval repeatedly or derive the corresponding forms for multiple intervals. The Simpson's quadrature expression for one interval is:

$$\int_a^b f(x)dx\approx\frac{h}{3}(f(a)+4f(c)+f(b)),\quad \text{with}\quad c=\frac{a+b}{2}\quad\text{and}\quad h=\frac{b-a}{2}$$

Also, Note that the function is even and then, $\int_{-1}^{1}f(x)dx=2\int_0^1 f(x)dx$.

[8 marks]

(c) Use Gauss quadrature with 4 points to calculate the integral $\int_{-1}^{1}(2-x^2)\cos\dfrac{\pi x}{2}dx$

n	Nodes x_i^n	Weights w_i^n
1	0.0	2.0
2	$\pm\sqrt{3}/3$ =±0.577350269189	1.0
3	0	$8/9$ = 0.888888888889
	$\pm\sqrt{15}/5$ =±0.774596669241	$5/9$ = 0.555555555556
4	$\pm\sqrt{525-70\sqrt{30}}/35$=±0.339981043585	$(18+\sqrt{30})/36$ = 0.652145154863
	$\pm\sqrt{525+70\sqrt{30}}/35$=±0.861136311594	$(18-\sqrt{30})/36$ = 0.347854845137

[7 marks]

6. (a) Consider the differential equation: $\frac{\partial}{\partial x}\varepsilon(x)\frac{\partial \phi}{\partial x} = 0$ on a domain [a, b] with yet unspecified boundary conditions.
Starting from the weighted residuals form: $\langle r(x), w_i(x) \rangle = 0, \quad i = 1, \cdots, N$, formulate the solution of this equation using the Galerkin method, and find the expression for the elements of the resultant matrix problem. ($r(x)$ is the residual $r = \mathscr{L}\phi - s$, and $w_i(x)$ are the weighting functions).
Note: Your result will be a homogeneous matrix equation (with 0 in the right hand side). Do not worry about this because that will change once boundary conditions are imposed.
[10 marks]

(b) An appropriate variational expression for the differential equation specified in part (a) is:

$$\mathscr{J} = \int_a^b \varepsilon(x)\left(\frac{\partial \phi}{\partial x}\right)^2 dx$$

Formulate the solution of this equation using the variational method and find the expression for the elements of the resultant matrix problem.
Note: The same observation made above applies here.
[10 marks]

(c) Consider the following matrix problem for a finite element solution of the generalised Laplace equation:

$$\mathbf{A\Phi} = 0\,; \quad \mathbf{A} = \{a_{ij}\} \quad \text{with:}$$

(i) $$a_{ij} = \int_a^b b_i \left(\frac{\partial}{\partial x} \varepsilon \frac{\partial b_j}{\partial x} \right) dx$$

(ii) $$a_{ij} = \int_a^b \varepsilon \frac{\partial b_i}{\partial x} \frac{\partial b_j}{\partial x} dx$$

Comment on which of these two forms is more adequate for a finite element implementation. Show that the two forms are equivalent, converting one into the other using integration by parts. Consider that on the boundaries a Neumann condition applies
[5 marks]

END OF PAPER