UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**COMPGI13**
ASSESSMENT PATTERN	:	**COMPGI13B**
MODULE NAME	:	**Advanced Topics in Machine Learning**
DATE	:	**05-May-10**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

TURN OVER

Advanced Topics in Machine Learning, GI13, 2009/10

Answer ONE question from PART A and ONE question from PART B. Use separate answer books for each PART.

Marks for each part of each question are indicated in square brackets

Calculators are NOT permitted

Part A

1. a. Define a Gaussian process. You do not need to write down the density for a multivariate Gaussian random variable, but can use $\mathcal{N}(?,?)$ to refer to it.

[5 marks]

b. Assume you want to find the best place to fish in a lake, that is the place where the expected number of fish you catch in a fixed time is highest.

- Follow a Bayesian route and model the setting with a Gaussian process. Suggest a suitable covariance function.

[5 marks]

- The number of fish you get at a location is to some extent random: take account of this fact by defining an observation model.

[5 marks]

- Give the equations of the posterior mean after having tried locations $x_1,\ldots,x_n$.

[5 marks]

- A fishing company wants to optimize the amount of fish it harvests, that means they want to explore promising locations, while exploiting spots that are known to be good. Give a simple algorithm that uses the posterior mean $m_{\text{post}}(x)$ and the posterior covariance $k_{\text{post}}(x,y)$ to pick points for fishing. Motivate your algorithm.

[10 marks]

c. Use the Borell inequality to bound the probability that a Brownian motion X_t on $[0,T]$ crosses a level $\ell > \mathbb{U}$, where $\mathbb{U}$ denotes an upper bound on $\mathrm{E}\sup_{t\in[0,T]} X_t$ (Assume that $\mathbb{U}$ is given, i.e. you do not need to calculate an upper bound.)

Borell inequality: Let X_t be a centred Gaussian process. Let $||X|| = \sup_{t\in[0,1]} X_t$. Then

$$\mathbf{P}[|\,||X|| - \mathbf{E}||X||\,| > \lambda] \leq 2\exp\left(-\frac{1}{2}\frac{\lambda^2}{\sigma_T^2}\right),$$

where $\sigma_T = \sup_{t\in[0,T]} \sqrt{\mathbf{E}X_t^2}$.

[8 marks]

d. Give an upper bound on the expected supremum of a Brownian motion on $[0,T]$. Use

$$\mathbf{E}\sup_{t\in T} X_t \leq 14 \sup_{t\in T} \sum_{n\geq 0} 2^{n/2}\Delta(A_n(t)),$$

where the number of sets for n is at most N_n given by

$$N_0 = 1 \quad \text{and} \quad N_n = 2^{2^n},$$

and $\Delta(A)$ is the diameter of a set A measured with the distance

$$d(s,t) = \sqrt{k(s,s) - 2k(s,t) + k(t,t)}$$

based on the covariance function $k(s,t)$ of the Brownian motion at times s and t.

Remark: You are allowed to use $\int_{n=1}^{\infty} 2^{n/2}\Delta(A_n(t))dn$ as an approximation to $\sum_{n\geq 1} 2^{n/2}\Delta(A_n(t))$.

Hint: Make use of the integration by substitution method: $\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(x)dx$, where $g(t) = x$.

[12 marks]

[Total: 50 marks]

2. a. Explain the simple multi-armed bandit (MAB) problem with finite time horizon T and $K > 1$ arms, indicating what is known to the player and what is hidden. Explain why solutions must balance exploration with exploitation.

[8 marks]

b. Consider modelling the arm response probabilities with a Beta distribution:

$$P_{\alpha_0,\beta_0}(p) \propto p^{\alpha_0-1}(1-p)^{\beta_0-1}$$

with normalising constant

$$B(\alpha_0,\beta_0) = \int_0^1 p^{\alpha_0-1}(1-p)^{\beta_0-1}dp.$$

The distribution has mean $\alpha_0/(\alpha_0+\beta_0)$ and variance

$$\frac{\alpha_0\beta_0}{(\alpha_0+\beta_0)^2(\alpha_0+\beta_0+1)}.$$

Consider an MAB problem with $\{0,1\}$ reward, two arms and finite time horizon $T = 25$. Suppose that at the final trial the first arm has been played 15 times with 2 rewards and the other arm 9 times with 1 reward. What is the posterior distribution for each of the arms?

[7 marks]

c. Which arm would you propose to play at the next and final trial and why?

[7 marks]

d. Describe the Upper Confidence Bound (UCB) strategy indicating how it enables a trade-off between exploration and exploitation.

[10 marks]

e. Consider the situation described in part 2.b). Describe a UCB strategy making use of the Bayesian model updates.

[10 marks]

f. Describe the method of selecting an arm that relies on sampling from the posterior distribution obtained with the Bayesian model updates. Interpret the probability, in this method, of an arm being selected.

[8 marks]

[Total: 50 marks]

Part B

3. This question pertains to kernel methods and regularisation. Consider the following optimisation problem

$$\min_{w \in \mathbb{R}^d} \left\{ \sum_{i=1}^{m} (w^\mathsf{T} x_i - y_i)^2 + \gamma \|w\|^2 \right\}, \tag{1}$$

where $\|\cdot\|$ denotes the L_2 norm, $\gamma > 0$ is a fixed real number, and and $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^d \times \mathbb{R}$ are given datapoints.

a. Prove that the solution $\hat{w}$ of (1) has the form $\hat{w} = \sum_{i=1}^{n} c_i x_i$, where $c_1, \ldots, c_n$ are some real parameters.

[14 marks]

b. Let G be the $m \times m$ matrix with entries $G_{ij} = x_i^\mathsf{T} x_j$, for $i, j = 1, \ldots, m$. Derive the formula which expresses the vector c of coefficients $c = (c_1, \ldots, c_m)$ in terms of the matrix G and vector $y = (y_1, \ldots, y_m)$. Argue that, for every $x \in \mathbb{R}^d$ the computation of the quantity $\hat{w}^\mathsf{T} x$ is a function of the vector c and the vector $(x^\mathsf{T} x_1, \ldots, x^\mathsf{T} x_m)$.

[12 marks]

c. Show that the function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, defined as

$$K(x, t) := x^\mathsf{T} t \qquad \text{for all } x, t \in \mathbb{R}^d$$

is a symmetric positive semidefinite kernel.

[12 marks]

d. Consider the kernel $K(\mathbf{x}, \mathbf{t}) := (1 + \mathbf{x}'\mathbf{t})^2$, for $x, t \in \mathbb{R}^2$. Find a feature map associated to this kernel.

[12 marks]

[total 50 marks]

4. This question pertains to convex functions and convex optimisation.

a.

Consider the minimal ℓ_1-norm interpolation problem $\min\{\|w\|_1 : w \in \mathbb{R}^d,\ w^\mathsf{T} x_i = y_i,\ i = 1, \ldots, m\}$, where $w \in \mathbb{R}^d$ is a vector of parameters we minimize over, $\|w\| = \sum_{i=1}^{n} |w_i|$, and $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^d \times \mathbb{R}$ are given datapoints which can be interpolated. Show that this problem is a linear programming problem.

[12 marks]

b. Consider the problem in the previous question when $d = 2$ and $m = 1$. Give an instance (that is, provide the value (x_1, y_1) of the interpolating datapoint) for which the optimization problem has a unique solution.

[12 marks]

c. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(w) = (w_1 + w_2 - 1)^2 + \rho(|w_1| + |w_2|) \qquad \text{for every } w_1, w_2 \in \mathbb{R},$$

where ρ is a fixed positive constant. Show that the function f is convex and the optimisation problem $\min\{f(w) : w \in \mathbb{R}^2\}$ is a quadratic programming problem.

[12 marks]

d. Discuss the set of solutions to the above problem as a function of ρ.

[14 marks]

[total 50 marks]

END OF PAPER