*Answer FOUR questions*

1. (a) Consider two numbers $a$ and $b$ that are already in floating point representation (they are *machine numbers*) and consider the following two algorithms to calculate $y$:

i) $y = a^2 - b^2$

ii) $y = (a + b)(a - b)$

Determine with an error analysis which of the two algorithms is more stable. What ranges of values of $a$ and $b$ are likely to cause problems and to which algorithm?

[10 marks]

(b) For a floating point system specified by (10, $t$, $L$, $U$) with $t$ = 4, $L$ = –5 and $U$ = 5, what is the value of the *machine precision*? Using the same system, evaluate $y$ for $a$ = 37.6 and $b$ = 37.5 using both algorithms in part (a). How does the normalised error $\Delta y$ compare with the machine precision? Do your calculations confirm your results for part (a)?

[8 marks]

(c) For a large value of $x$ the values $a$ = cosh$x$ and $b$ = sinh$x$ are known. Two different algorithms can be used to calculate the value of $e^{-x}$:

i) $e^{-x} = \cosh x - \sinh x = a - b$ or

ii) $e^{-x} = \dfrac{1}{\cosh x + \sinh x} = \dfrac{1}{a+b}$, since $\cosh^2 x - \sinh^2 x = 1$

Knowing also that for large values of $x$, $\cosh x \approx \sinh x \approx e^x$, so $a$ and $b$ are large and close in value, which of the two algorithms will give the most accurate results? Note that $a$ and $b$ are not necessarily machine numbers.

<u>Hint</u>: In the evaluation of the error corresponding to the denominator of ii) above, consider the approximation: $|\varepsilon_1| = |\varepsilon_2| = |\varepsilon_3|$.

Also, use the relation: $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$

[7 marks]

2. (a) Use the bisection method to find a root of the function $f(x) = x^2 - 5.25x + 5$.
Use the starting values $x_1 = 1$ and $x_2 = 1.8$ and complete the table calculating 4 iterations.

| *iteration* | $x_1$ | $x_2$ | $c$ | $f(x_1)$ | $f(x_2)$ | $f(c)$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1.8 | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |

*please reproduce this table in your answer book* [8 marks]

(b) Describe the fixed point iteration method to find a root of a function $f(x)$. Indicate what the condition for convergence is.
Use the fixed point iteration to find the root close to 1 of the function: $f(x) = x^2 - 5.25x + 5$.
Start with the choice: $x_0 = 1$ and use at least 4 iterations.

[8 marks]

(c) Use the Newton-Raphson method to find a root of the function: $f(x) = x^2 - 5.25x + 5$.
Use the following table in your answer.

| *iteration* | $x_n$ | $f(x)$ | $f'(x)$ | $\rightarrow x_{n+1}$ |
|---|---|---|---|---|
| 1 | 1 | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

*please reproduce this table in your answer book* [9 marks]

3. (a) Consider a set of data points $(x_i, y_i)$ with $i = 1, \ldots, n$. Write down the expression for the least squares error functional and derive the expressions to determine a straight line that approximates the data point.

[8 marks]

(b) Use Lagrange interpolation to find a second order polynomial to interpolate the data points: $(-2, 9)$, $(0, -5)$ and $(2, -3)$.

[8 marks]

(c) Considering the 4 points: $x_0$, $x_1 = x_0 - h$, $x_2 = x_0 - 2h$, and $x_3 = x_0 - 3h$ and the Taylor expansion of a function $f(x)$ at $x_1$, $x_2$ and $x_3$, find a four-point backward difference expression to approximate the first derivative of $f(x)$. What is the order of approximation?

[9 marks]

4. (a) Define the following terms: *vector norm, matrix norm*, and *condition number of a matrix* **A**. For a system of equations **Ax** = **y**, show that if δ**x** and δ**y** are perturbations of the vectors **x** and **y**, respectively, they are related by:

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \mathrm{cond}(\mathbf{A})\frac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|}$$

[6 marks]

(b) Show that in the steepest descent method applied to the solution of the linear system of equations represented by **Ax** = **y**, with a symmetric matrix **A**, the value of $\alpha$ that minimizes the error functional $h^2$ along the line ($\mathbf{x}_i + \alpha\,\mathbf{p}_i$) is given by:

$$\alpha = \frac{\mathbf{p}_i^T(\mathbf{y}-\mathbf{A}\mathbf{x}_i)}{\mathbf{p}_i^T\mathbf{A}\mathbf{p}_i}$$

where $\mathbf{p}_i$ is the unit vector in the direction of $-\nabla h^2$ evaluated at $\mathbf{x}_\mathbf{i}$ and $\mathbf{x}_\mathbf{i}$ is the current point.

Hint: Remember that the error functional $h^2$ is defined as $h^2 = \mathbf{r}^T\mathbf{A}^{-1}\mathbf{r}$ where **r** is the residual $\mathbf{r} = \mathbf{y} - \mathbf{A}\mathbf{x}$, and that it can be simplified to: $h^2 = \mathbf{x}^T\mathbf{A}\mathbf{x} - 2\mathbf{x}^T\mathbf{y}$.

[7 marks]

(c) Consider the system of equations **Ax** = **y** and its solution using the steepest descent method:

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

Use the expression for $\alpha$ given in part (b) to find the iteration vector $\mathbf{x}_1$ from the starting vector $\mathbf{x}_0 = (1, 1, 1)^T$. In doing so, show that $h^2$ can be written as:

$$h^2(x_1, x_2, x_3) = 5x_1^2 + 3x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 - 4x_1 + 4x_2 + 4x_3$$

[12 marks]

5. (a) Use Gauss quadrature with 4 points to calculate the integral: $\int_{-1}^{1} e^{x} \sin \pi x \, dx$

| $n$ | Nodes $x_i^n$ | Weghts $w_i^n$ |
|---|---|---|
| 1 | 0.0 | 2.0 |
| 2 | $\pm\sqrt{3}/3 = \pm 0.577350269189$ | 1.0 |
| 3 | 0 | $8/9 = 0.888888888889$ |
| | $\pm\sqrt{15}/5 = \pm 0.774596669241$ | $5/9 = 0.555555555556$ |
| 4 | $\pm\sqrt{525-70\sqrt{30}}/35 = \pm 0.339981043585$ | $(18+\sqrt{30})/36 = 0.652145154863$ |
| | $\pm\sqrt{525+70\sqrt{30}}/35 = \pm 0.861136311594$ | $(18-\sqrt{30})/36 = 0.347854845137$ |

[8 marks]

(b) From the expression of the Simpson's quadrature (Simpson's rule) for one interval, derive the expression to calculate an integral using the Simpson quadrature with $n$ subintervals. Use this method to calculate the following integral with 2 subintervals.

$$\int_{-1}^{1} e^{x} \sin \pi x \, dx$$

[8 marks]

(c) Consider the solution of the following differential equation using finite differences:

$$\frac{d^2 f}{dx^2} + c\frac{df}{dx} + e\,f = g(x) \quad \text{for } a \le x \le b.$$

with $f(a) = 0$ and $\left.\dfrac{df}{dx}\right|_{x=b} = 0$.

Use the discretisation $x_i = a + i\Delta x,\ \ i = 0, \cdots, N$ and $(\Delta x = (b-a)/N)$. Use central differences for both derivatives. Determine the form of the discretised equation for an internal point $x_i$, and for $x_1$ and $x_N$. Show the form of the resultant matrix equation and the form of the corresponding matrix.

[9 marks]

6. (a) Describe the successive displacement or Gauss-Seidel method to solve a linear system of equations. Illustrate by calculating the first 4 iterations for the solution of the following system:

$$\begin{bmatrix} 5 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

Start the iterations with $\mathbf{x}_0 = (1, 1, 1)^T$.

[10 marks]

(b) Describe in detail the power method to find the dominant eigenvector of a matrix and explain how it works. Use the power method with the following matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Start the iterations with the vector $\mathbf{x}_0 = (1, 1, 1)^T$. Run at least 4 iterations and normalise the iteration vector using $\|\mathbf{x}\|_\infty$.

Use the Rayleigh quotient to find the corresponding eigenvector.

[9 marks]

(c) Describe the shifted inverse iteration method and explain in detail the steps necessary to find the eigenvalue of a matrix **A** that is closest to 5.0.

Knowing that the power method converges to the eigenvector corresponding to the largest eigenvalue of a matrix, show how the shifted iteration works.

[6 marks]



**END OF PAPER**