Answer FOUR questions

1. (a) For a decimal floating point system with a mantissa length $t = 4$, and exponents between –6 and 6, calculate the value of the *machine precision* (*eps*).

[4 marks]

(b) What are the smallest and the largest numbers that can be represented exactly in the floating point system described in (a)?

[3 marks]

(c) Consider a real number a, which is large and positive and not necessarily represented exactly by the floating point system specified in part (a). What is the relative error bound for the following two algorithms to calculate $z = a - b$, where $b = \sqrt{a^2 - 1}$?

(i) $z = a - b$

(ii) $z = \dfrac{1}{a + b}$

If a = 58.4 what is the value of the relative error bound for both cases? What is then the absolute error bound in each case?
What would happen for even larger numbers?

<u>Hint:</u> In the evaluation of the error incurred calculating the denominator of the expression in (ii), consider that the individual errors are all equal ($\varepsilon_1 = \varepsilon_2 = \varepsilon_3$). Also, use the relation:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

[18 marks]

2. (a) Use the fixed point iteration to find a root of $f(x) = -0.2x^2 - 0.3x + 0.7$. Start with $x_0 = 0.5$, calculate at least four iterations and complete the following table:

n	*x*	*g*(*x*)	*f*(*x*)
0	0.5		
1			
2			
3			
4			

(Please copy this table in your answer sheet)

[12 marks]

(b) Find a root of the function $f(x) = -0.2x^2 - 0.3x + 0.7$ using the bisection method. Start with $x_1 = 0.5$ and $x_2 = 1.5$ and complete the following table:

x_1	x_2	c	$f(x_1)$	$f(x_2)$	$f(c)$
0.5	1.5		0.5	–0.2	

(Please copy this table in your answer sheet)

[13 marks]

3. (a) Use Newton interpolation to find the third order polynomial that fits the data:

$$x_i = 0, 1, 2, 3 \quad y_i = -1.2, 2.1, 1.8, 2.8 \,.$$

To find your answer fill the following table:

x_i	y_i			
0	–1.2			
1	2.1			
2	1.8			
3	2.8			

[9 marks]

(b) Using Lagrange interpolation find the polynomial of second order that fits the following data set: $x_i = \{-1.0, \quad 0.0, \quad 1.0\}$ $\qquad y_i = \{-1.2, \quad 1.3, \quad 2.4\}$

[8 marks]

(c) Derive the expressions to find the coefficients a and b of the straight line: $y(x) = a + bx$ that fits a set of data $\{x_i, \ y_i\}$ in the least squares sense and find the straight line that fits the data:

$$x_i = \{-1.0, \quad 0.0, \quad 1.0\} \qquad y_i = \{-1.0, \quad 1.0, \quad 2.4\} \,.$$

[8 marks]

4. (a) Use the successive displacement method (or Gauss-Seidel) to solve the following system of equations. Calculate at least four iterations and use at least four decimal places, starting with the trial values: (1, 1, 1).

$$\begin{bmatrix} 5 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

[8 marks]

(b) Use the power method or *direct iteration* to find the eigenvector corresponding to the largest eigenvalue of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 5 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

Calculate at least 4 iterations starting with the vector $\mathbf{x}_0 = [1,\ 1,\ 1]^T$ normalising the trial vector $\mathbf{x}_i$ after each iteration using $\tilde{\mathbf{x}}_i = \dfrac{\mathbf{x}_i}{\|\mathbf{x}_i\|_\infty}$ (Remember that $\|\mathbf{x}_i\|_\infty$ is the largest absolute value of the values of $\mathbf{x}_i$). Present your results using the table:

$\mathbf{x}_0$	$\tilde{\mathbf{x}}_0$	$\mathbf{x}_1$	$\tilde{\mathbf{x}}_1$	$\mathbf{x}_2$	$\tilde{\mathbf{x}}_2$	$\mathbf{x}_3$	$\tilde{\mathbf{x}}_3$	$\mathbf{x}_4$	$\tilde{\mathbf{x}}_4$
1	1								
1	1								
1	1								

Calculate the current estimate of the eigenvector after each iteration using the Rayleigh quotient.

[12 marks]

(c) Describe in detail the steps necessary to implement the *inverse iteration method* to find the eigenvector corresponding to the eigenvalue closest to zero of a matrix **B**. How can this method be modified to find the eigenvector corresponding to the eigenvalue closest to a given value σ?

[5 marks]

5. (a) Consider the four points: x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ and $x_3 = x_0 + 3h$. Use the Taylor expansions of a function $f(x)$ at points x_1, x_2 and x_3 around x_0 to find an expression for the forward difference approximation of the second derivative of $f(x)$ at x_0. Find also the order of the approximation.

[9 marks]

(b) Calculate the integral: $\int_0^1 e^{-x^2}\sqrt{1-x^2}\,dx$ using Simpson's quadrature with 4 subintervals.

Note: Use the expression for a single interval separately for each subinterval or derive the expression for multiple subintervals.
The expression for the Simpson's quadrature for one interval is:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a) + 4f(c) + f(b)\right] \quad \text{with} \quad c = \frac{a+b}{2} \quad \text{and} \quad h = \frac{b-a}{2}$$

[8 marks]

(c) Calculate the integral: $\int_0^1 e^{-x^2}\sqrt{1-x^2}\,dx$ using Gauss quadrature with 6 Gauss points.

Note: Gauss points and weights for order 6 are:

Nodes x_i^6	Weghts w_i^6
±0.238619186	0.4679139
±0.661209386	0.3607616
±0.932469514	0.1713245

Note also that Gauss quadrature is defined in the interval [–1, 1], so a change of variable is needed.

[8 marks]

6. (a) Describe in detail the procedure to use the Galerkin method to solve the differential equation $\nabla \cdot \sigma \nabla \phi = 0$, with some appropriate boundary conditions in the two dimensional domain Ω. Describe the form of the resultant matrix problem giving the generic expression for the matrix elements.

[8 marks]

(b) Use finite differences to formulate the solution of the following differential equation describing the vibration of a string of length L in a musical instrument:

$$\frac{\partial^2 y}{\partial x^2} + k^2 \frac{\partial^2 y}{\partial t^2} = 0 \, ,$$

where $y(x,t)$ is the lateral displacement of the string. The string is firmly attached at both ends so the boundary conditions are: $y(0,t) = y(L,t) = 0$.
The string is initially at rest and at time t = 0 the string is plucked at the centre ($x = L/2$) producing a unit impulse of displacement, so the initial conditions are: $y(x,t) = 0$ for $t < 0$ and $y(L/2, 0) = 1$.
Discretise the coordinate x and time t using: $x_i = i\Delta x$; $i = 0, \cdots, N$ and $t_m = m\Delta t$; $m = 0, 1, \cdots$. Use central difference approximations for both the x and time derivatives and write down the discretised equation corresponding to a general point x_i at time t_m. Write also the equations corresponding to the ends of the string at any time.
Rearrange the equations in the form of a time stepping process and write down the form of the resultant matrix equation, giving the generic form of the matrix elements.
Describe a procedure to solve the resultant time stepping matrix equation.

[17 marks]

END OF PAPER