UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**ELEC3030**
ASSESSMENT PATTERN	:	**ELEC3030A**
MODULE NAME	:	**Numerical Methods**
DATE	:	**09-May-14**
TIME	:	**14:30**
TIME ALLOWED	:	**3 Hours 0 Minutes**

Answer FOUR questions

1. (a) When using floating point arithmetic, the quantity "*machine precision*" (eps) is used.
   (i) Give a definition and explain what this is.
   (ii) Demonstrate that for a decimal floating point system the machine precision can be calculated as:

$$eps = 5 \times 10^{-t}$$

   where t is the mantissa length.
   (iii) For a decimal system with a mantissa length of 4, calculate the value of eps for this system.
   (iv) Calculate the relative error bound for the product of two numbers, x_1 and x_2, which are not necessarily represented exactly as floating point numbers.
   (v) Now, calculate the relative error bound for the product of a set of n numbers, x_1 to x_n, which are not machine numbers (not exactly represented in the floating point system)

[10 marks]

(b) Consider the evaluation of the expression: $y = x^2 - 4x + 3$ where x is exactly represented as a floating point number in a decimal system with a mantissa length of 4.

   (i) Find the error bound for this calculation
   (ii) Calculate the actual error for this calculation within this floating point system if $x = 0.95$ and compare with the error bound calculated in (i). Note: Perform the calculations in the same order as the order of the terms on the right hand side of the expression above.
   (iii) Would there be any difference if the calculation was done using the terms on the right hand side in a different order? Which ordering would have been better in this case and why?

[15 marks]

2. (a) The function $f(x) = x^3 + 6\cos(x) + 2x$ has a maximum in the interval [0, 1]. Use the bisection method to find it. Use at least 4 decimal places in your calculations and write your answer using the following table.

iteration	x_1	c	x_2	$g(x_1)$	$g(c)$	$g(x_2)$
1						
2						
3						
4						
5						

Please reproduce this table in your answer book

Note that the question asks for the *maximum* of $f(x)$

[8 marks]

(b) Use the Newton-Raphson method to find the maximum of the same function specified in part (a) in the interval [0, 1], starting with the value $x_0 = 0$. Use at least 4 decimal places in your calculations and write your answer using the following table.

iteration	x_n	$g(x)$	$g'(x)$	$\rightarrow x_{n+1}$
1	0.0			
2				
3				
4				
5				

Please reproduce this table in your answer book [8 marks]

(c) Using Taylor expansions at the four points: $x_0 - 2h$, $x_0 - h$, $x_0 + h$ and $x_0 + 2h$, derive an expression for the first derivative of $f(x)$ at x_0 and use it to calculate the derivative of the function $f(x)$ given by the table:

Please reproduce this table in your answer book and use up to 5 decimals in your calculations.

x	$f(x)$	Derivative
0.0	0.00000	--
0.1	0.09983	--
0.2	0.19867	
0.3	0.29552	
0.4	0.38942	
0.5	0.47943	
0.6	0.56464	
0.7	0.64422	
0.8	0.71736	
0.9	0.78333	--
1.0	0.84147	--

[9 marks]

3. (a) The function $f(x) = \dfrac{e^{-x}}{1-x^2}$ can be approximated by the Taylor series $f(x) \approx 1 - x + \dfrac{3}{2}x^2 - \dfrac{7}{6}x^3 + \dfrac{37}{24}x^4 + \cdots$. Use the first 5 terms (as a truncated polynomial of order 4), to construct the Padé approximant $R_2^2(x) = \dfrac{p_2(x)}{q_2(x)} = \dfrac{a_2x^2 + a_1x + a_0}{b_2x^2 + b_1x + b_0}$ to the function $f(x)$, with the choice $b_0=1$.

Note: The general expression for the derivative of order i of the product: $g(x) = t(x)q(x)$ is given by: $g^{(i)}(x) = \sum_{j=0}^{i} \dfrac{i!}{j!(i-j)!} t^{(j)}(x) q^{(i-j)}(x)$, which evaluated at $x = 0$ gives: $a_i = \sum_{j=0}^{i} c_{i-j} b_j$.

[12 marks]

(b) Formulate the solution of the following differential equation using finite differences:

$$af''(x) + xf'(x) + bf(x) = s(x)$$

in the interval [0, 1] where the function $s(x)$ is a known function and $f(x)$ satisfies the boundary conditions: $f(0) = f(1) = 0$. Use 11 points equally spaced: $x_i = ih$ with $h = 0.1$ and write down the resultant matrix equation, specifying the values of matrix elements.
Use central difference expressions to approximate both derivatives.

[13 marks]

4. (a) Use the Gauss-Seidel method to solve the system of equations:

$$\begin{array}{rcrcrcr} 4x_1 & + & x_2 & - & 2x_3 & = & 3 \\ x_1 & + & 5x_2 & + & x_3 & = & 18 \\ -2x_1 & + & x_2 & + & 3x_3 & = & 7 \end{array}$$

Calculate at least 5 iterations starting with the guess vector $\mathbf{x}^{(0)} = [1, 1, 1]^T$. Show clearly the successive iteration vectors.

[9 marks]

(b) The LU decomposition of the matrix **A** below is given by the following **L** and **U** factors:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 5 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \mathbf{LU}, \text{ where } \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 10 \\ 0 & 0 & -14 \end{bmatrix}$$

Explain how this factorisation is used to solve the system of equations: $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ where $\mathbf{x} = (x_1, x_2, x_3)^T$ and **b** is the vector of the right hand side values.

Using $\mathbf{b} = (10, 2, 8)^T$, find the solution **x** showing all the calculations.

[8 marks]

(c) Simpson's quadrature method for numerical integration of a function in the interval $[a, b]$ uses the values at the points a, $(a+b)/2$ and b, with the weights $h/3$, $4h/3$ and $h/3$, respectively, where h is one half of the length of the interval Derive an expression for the calculation of the integral of a function over an interval subdivided into n subintervals and use it to calculate the following integral over the interval $[0, 1]$, using 4 subintervals.

$$I = \int_0^1 e^{-x^2} \cos(\pi x/2)\, dx$$

[8 marks]

5. (a) Describe the characteristics of the *power method* or direct iteration, and of the *inverse iteration* algorithms to find eigenvectors of a matrix. Specify in each case to which eigenvector these methods converge and how they are related when the methods are applied to the same matrix.

[6 marks]

(b) The inverse of the matrix **A** is the matrix **B**, both given below:

$$\mathbf{A} = \frac{1}{8}\begin{bmatrix} 5 & -3 & -1 \\ -3 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix} \qquad \mathbf{B} = \mathbf{A}^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Using the power method, find the eigenvector of **A** corresponding to the smallest eigenvalue and then use the Rayleigh quotient to calculate this eigenvalue. Start with the guess vector $(2, 2, 1)^T$ and perform at least 4 iterations. Normalise the trial vector in each iteration using the infinite norm (dividing by the largest value).

[10 marks]

(c) Using the data: $x = \{0, 1, 2, 3\}$, $y = \{2, 1, -2, -3\}$ find an interpolating polynomial with any method and estimate the value for which $y = 0$ using any root-finding method.

[9 marks]

6. (a) Explain in detail the method of Galerkin to solve a boundary value problem and use it to formulate the solution of the generalised Poisson equation: $\nabla \cdot \varepsilon \nabla \phi = s$ over a two-dimensional domain Ω, with an appropriate set of boundary conditions. The functions ε and s are known functions of (x, y).

   Specify the resultant matrix equation and give the expression for the matrix elements and for the elements of the right hand side vector.

   [10 marks]

   (b) An appropriate variational formulation to solve the equation. $\frac{\partial}{\partial x}\sigma(x)\frac{\partial y}{\partial x} = 0$ in the interval $[a, b]$ with the boundary conditions $y(a) = 0$ and $y(b) = 1$ is:

$$\mathcal{J} = \int_a^b \sigma(x)\left(\frac{\partial y}{\partial x}\right)^2 dx$$

   Formulate the solution of the above differential equation using the given variational form, showing all the steps and specifying all the details of the resultant matrix equation.

   If the solution were implemented using first order final elements using n nodes over the interval $[a, b]$ including both ends, explain how the boundary conditions can be implemented and show the resultant form of the matrix equation and the sparsity pattern of the matrix.

   [15 marks]

END OF PAPER