Answer FOUR questions

1. Figure 1.1 shows a simplified model of a mechanical balancing system between two surfaces. Ignore gravity and any other external forces, such as air friction, and assume all forces are applied to the same point; i.e., the system dynamics are translational, not rotational.

(a) Write the differential equation that describes the balancing system, which is excited only by force $f(t)$.

[5 marks]

(b) Find the transfer function from the applied force $f(t)$ to the displacement, $y(t)$, of the mass. That is, find $Y(s)/F(s)$, with $F(s)$ and $Y(s)$ the Laplace transforms of $f(t)$ and $y(t)$, respectively. Draw the block diagram of the corresponding closed-loop system using only gain and integrator components and assuming $F(s)$ to be the input (set point) of the system and $Y(s)$ the output. How many feedback loops does the block diagram have?

[10 marks]

(c) (i) Is the system of Figure 1.1 stable for all possible values of its physical constants, M, K, B_1, B_2? Justify your answer analytically.

(ii) For the values of the physical constants that you find the system to be stable, what relationship between the physical constants must hold so that the system converges to the steady state without oscillation?

(iii) Further, what is the condition such that the system converges without oscillation *and* with the minimum time requirement?

[10 marks]

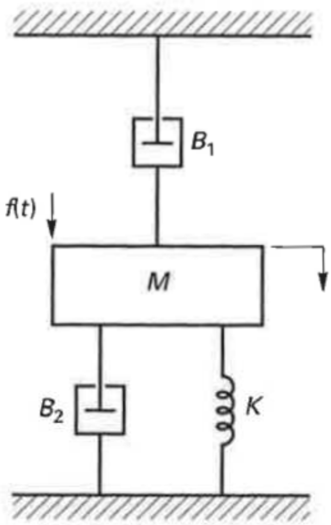

Figure 1.1. Mechanical balancing system comprising one mass, one spring and two viscous dampers, with physical constants: M, K, B_1, B_2, respectively.

2. (a) The normalised sensitivity of a system with transfer function $G_c(s)$ to a parameter p is:

$$E = \frac{\partial G_c(s)}{\partial p} \times \frac{p}{G_c(s)}$$

Consider $G_c(s)$ to be the closed-loop transfer function of a unity negative feedback system with gain K and transfer function $G_o(s)$ in the forward path. Show that the sensitivity of the closed-loop transfer function to the parameter $G_o(s)$ is:

$$E = \frac{1}{1 + KG_o(s)}$$

Note, therefore, that the forward path gain needs to be large to reduce the sensitivity of the closed loop to uncertainties in the properties of G_o, such as its gain or time constants.

[5 marks]

(b) Figure 2.1 shows the simplified block diagram for an aircraft automatic landing system, in which the output is the direction where the plane is heading. Two disturbances are present. One represents disturbance from the wind and the other represents noise from the radar monitoring the direction of the plane.

(i) Write down the closed loop transfer function $G_W(s) = \frac{X(s)}{W(s)}$ for the effect of the wind disturbance on the control loop output, and the closed loop transfer function $G_R(s) = \frac{X(s)}{R(s)}$ representing the effect of radar noise on the control loop output.

(ii) Show that the sensitivities of these transfer functions to variations in $G_o(s)$ are $E_W = -\frac{KG_o(s)}{1+KG_o(s)}$ (sensitivity of the wind transfer function) and $E_R = \frac{1}{1+KG_o(s)}$ (sensitivity of the radar noise transfer function).

[10 marks]

(c) Given E_W and E_R of part (b)(ii):

(i) Comment on the impact of the gain in these sensitivities, given that the performance of the automatic landing system will be enhanced if the above sensitivities are of small magnitude.

(ii) In conjunction with adjustment of K, the forward path transfer function, $G_o(s)$, could include a controller designed to shape the frequency response. Why might a phase lead controller, whose gain increases at high frequencies that can match the operating frequencies of the radar noise, be beneficial?

[10 marks]

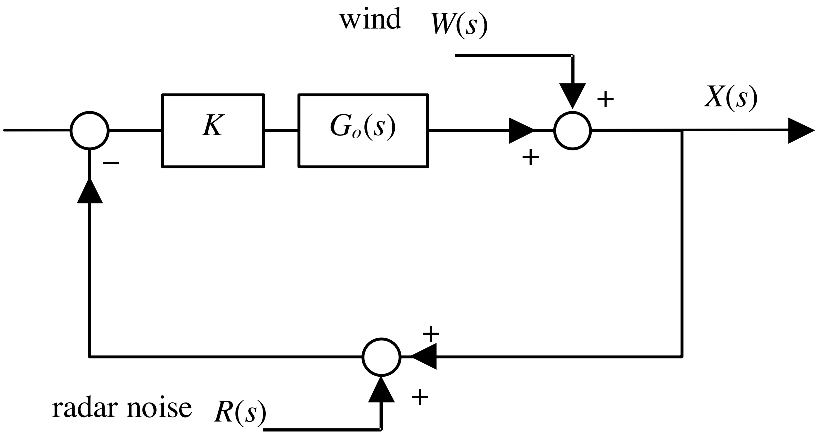

Figure 2.1. Simplified model of aircraft landing system.

3. (a) Starting from the Laplace expression of the PI (proportional plus integral) controller output, $C(s) = K\frac{1+\tau_i s}{\tau_i s}E(s)$, in the forward path of a closed-loop system, with $E(s)$ the error signal between the unity feedback and the set point, show that its time-domain output can be given by:

$$c(t) = K \times e(t) + \frac{K}{\tau_i}\int_{\tau=0}^{\tau=t} e(\tau)d\tau$$

[5 marks]

(b) (i) Consider the case where the error signal of a first-order system is $e(t) = e^{-t/3}$. If $K = 1$ and $\tau_i = 0.1$, show that the controller output reaches a steady value of 30.

(ii) Part (i) showed that the steady state controller output was not zero even though, in the steady state, the controller input, $e(t)$, becomes zero. Explain why this behaviour of a PI controller is of value in the design of high-precision control systems that aim to reach the set point precisely. Can a P-only (proportional) controller achieve zero steady-state error for a first-order system?

[10 marks]

(c) Design a digital control algorithm to implement the PI controller in a digital computer in terms of samples of the control loop set point, $u(t)$, and measured output variable of the closed-loop system, $x(t)$.

[10 marks]

4. (a) State how the class of a control system may be determined from an inspection of the system block diagram, if the transfer functions of its components are known. In what way is a class 1 system better than a class 0 system?

[5 marks]

(b) The positions of three cars on a road are denoted by x_1, x_2 and x_3 and their velocities are $dx_1/dt, dx_2/dt$ and dx_3/dt. A mathematical model for the traffic flow assumes a driver's acceleration is proportional to the distance from the car in front and to the velocity of his or her own car. There is no overtaking and the acceleration of the front car (number 1) is zero. Hence:

$$\frac{d^2x_1}{dt^2} = 0$$
$$\frac{d^2x_2}{dt^2} = k_2\left[(x_1 - x_2) - \tau_2\frac{dx_2}{dt}\right]$$
$$\frac{d^2x_3}{dt^2} = k_3\left[(x_2 - x_3) - \tau_3\frac{dx_3}{dt}\right]$$

Note that the steady-state case of zero acceleration has all the cars covering the distance at the same constant velocity.

Define a system state vector $\boldsymbol{z} = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{pmatrix}^T$ where:

$$\begin{aligned} z_1 &= x_1 & \qquad z_4 &= dx_1/dt \\ z_2 &= x_2 & z_5 &= dx_2/dt \\ z_3 &= x_3 & z_6 &= dx_3/dt \end{aligned}$$

and find the matrix $\boldsymbol{A}$ in the state-space representation $\dfrac{dz}{dt} = \boldsymbol{A}\,\boldsymbol{z}$.

[10 marks]

(c) The poles of the system of part (b) are the values of s that satisfy the state space characteristic equation $\det\left(s\boldsymbol{I} - \boldsymbol{A}\right) = 0$. By calculating this determinant, it can be shown that the characteristic equation of the system of part (b) is:

$$s^2\left(s^2 + k_2\tau_2 s + k_2\right)\left(s^2 + k_3\tau_3 s + k_3\right) = 0$$

(i) The factor s^2 in the characteristic equation gives a double pole at $s = 0$, which means that the traffic dynamics contain ramp-like and step-like behaviour in the time domain. What behaviour of the traffic is captured by the ramp and step signals?

(ii) Beyond the double pole at $s = 0$, determine the conditions that ensure the system is stable.

(iii) Assuming the system has parameters that ensure stability, due to the double pole at $s = 0$, neither the positions nor the velocities are zero in the steady state. Suggest what behaviour of the traffic might be captured by the other poles of the system, corresponding to exponentially decaying signals. Only qualitative, not quantitative, analysis is required.

[10 marks]

5. (a) Explain what is meant by a pole in a transfer function. Describe what features of the output of a system are determined by the poles when the input to the system is a unit step.

[5 marks]

(b) What is the influence of the system zeros on the output of a system in response to a step input? Use the example below to illustrate your answer by allowing the position of the zero to vary.

$$G(s) = \frac{s - s_a}{(s+1)(s+2)}$$

[10 marks]

(c) Figure 5.1 shows pole-zero maps for four transfer functions and four step responses. Copy the table below into your answer book and complete the table to indicate which step response plot (W, X, Y and Z) belongs with which pole zero map (A, B, C and D).
For each step response plot describe *all* the features relevant to your conclusion.

[10 marks]

Pole zero map	**A**	**B**	**C**	**D**
Step response plot				

QUESTION 5 CONTINUES

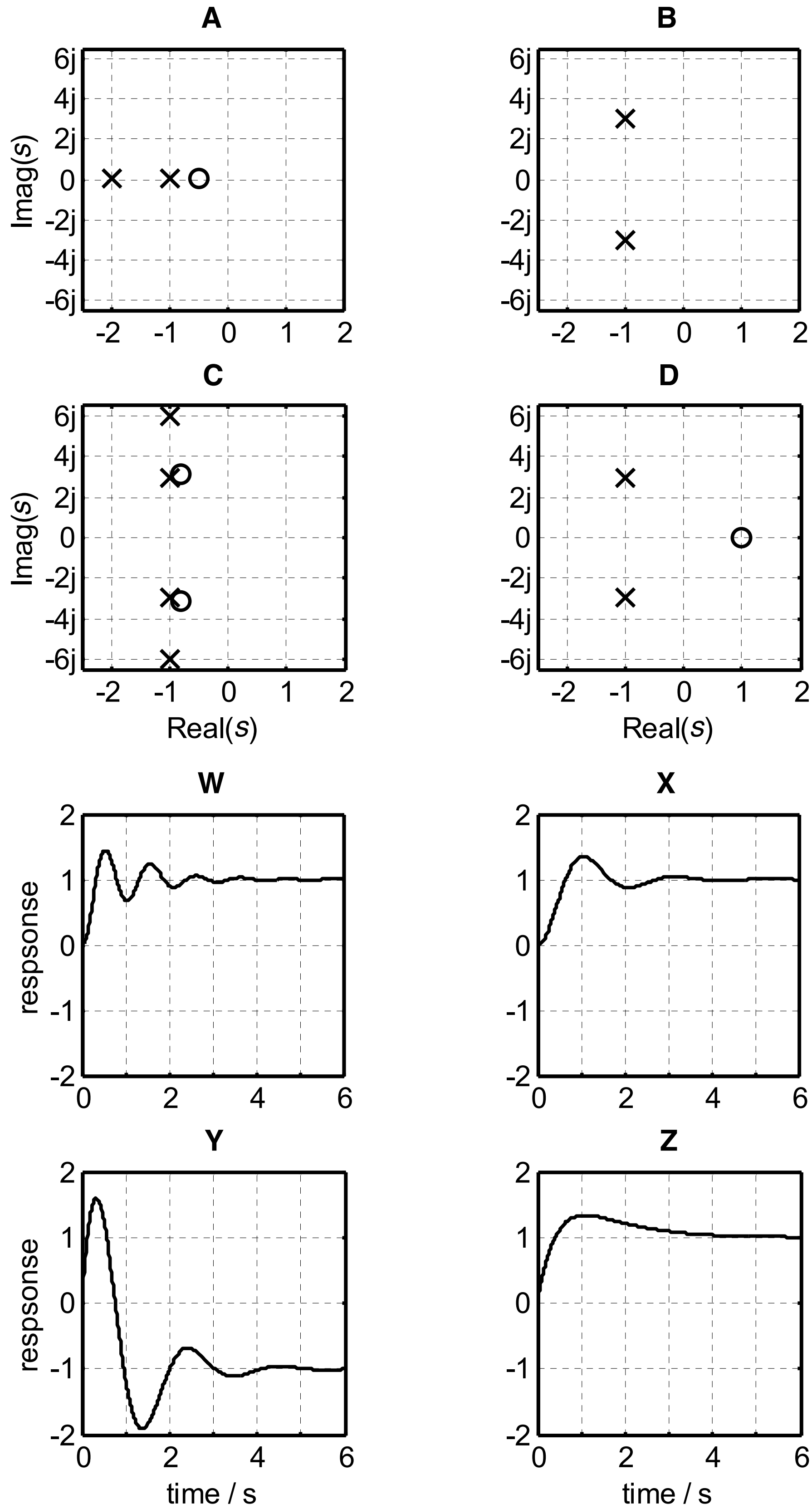

Figure 5.1. Top four plots(A, B, C, D): Pole-zero maps for four transfer functions; bottom four plots (W, X,Y, Z): four step responses.

END OF QUESTION 5

6. (a) By making use of Euler's theorem to decompose $\cos(t)$ and $\sin(t)$ into complex exponentials, or otherwise, prove that the Laplace transforms of $\cos(t)$ and $\sin(t)$ $(t \geq 0)$ are, respectively:

$$\frac{s}{s^2+1} \text{ and } \frac{1}{s^2+1}$$

[5 marks]

(b) (i) Explain what is wrong with the following reasoning:

"The Laplace transform of $\cos(t)$ is $\frac{s}{s^2+1}$. Since $\frac{d}{dt}\cos(t) = -\sin(t)$ it follows that the Laplace transform of $-\sin(t)$ is $s \times \frac{s}{s^2+1} = \frac{s^2}{s^2+1}$"

(ii) Use a partial fraction expansion to determine the time domain signal $y(t)$ whose Laplace transform is:

$$Y(s) = \frac{3(s+1)}{s(s-1)(s+2)^2}$$

Based on the partial fraction expansion, what is the value of $y(t)$ as $t \to \infty$?

[10 marks]

(c) (i) Determine the steady state value for $y(t)$ by application of the final value theorem to the expression for $Y(s)$ in part (b)(ii) above. Explain why the final value theorem gives a misleading result in this case.

(ii) Prove that the Laplace transform of a time delay is e^{-sT_d}, where T_d is the delay.

[10 marks]

END OF PAPER