UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**ELEC3030**
ASSESSMENT PATTERN	:	**ELEC3030A**
MODULE NAME	:	**Numerical Methods**
DATE	:	**08-May-13**
TIME	:	**10:00**
TIME ALLOWED	:	**3 Hours 0 Minutes**

TURN OVER

Answer FOUR questions

1. (a) Explain how a number is represented in a floating point format in a computing machine and in this context, define the terms *mantissa*, *base* and *exponent*.
Define the term *machine precision* and show that for a floating point system with base 10 and mantissa length t, its value can be given by:

$$eps = 5 \times 10^{-t}$$

Consider for this calculation that the machine precision is the upper bound for the relative error in representing any number in the system, that is: $\left|\frac{fl(x) - x}{x}\right| \leq eps$.

Evaluate the machine precision for a system with mantissa length 8, base 10 and exponent limits: [−15, 15] and compare its value with the smallest positive number that can be represented in that system. How many numbers in the interval [0, 1] can be represented with this system?

[12 marks]

(b) Find the error bounds in terms of the machine precision eps for the following two algorithms to calculate y when x is already a floating point number in the system:

(i) $y = (x-1)(x-3)$

(ii) $y = x^2 - 4x + 3$

Compare the stability characteristics of both algorithms. Which of them is more stable? Specify the conditions under which one of them can become inaccurate.

[13 marks]

2. (a) The function $f(x) = e^{-x} - x\sin(x)$ has a single root in the interval [0, 1]. Use the bisection method to find it. Use 4 decimal places in your calculations and write your answer using the following table.

iteration	x_1	x_2	c	$f(x_1)$	$f(x_2)$	$f(c)$
1						
2						
3						
4						
5						

please reproduce this table in your answer book

[8 marks]

(b) Use the Newton-Raphson method to find a root in the interval [0, 1] of the same function specified in part (a), starting with the value $x_0 = 0$. Use 4 decimal places in your calculations and write your answer using the following table.

iteration	x_n	$f(x)$	$f'(x)$	$\rightarrow x_{n+1}$
1	0.0			
2				
3				
4				
5				

please reproduce this table in your answer book [9 marks]

(c) The function $f(x) = \dfrac{\tan(x)}{x}$ can be approximated by the Taylor series:

$f(x) \approx 1 + \frac{1}{3}x^2 + \frac{2}{15}x^4 + \frac{17}{315}x^6 + \cdots$ for $|x| \leq \frac{\pi}{2}$. Use the first 3 terms (as a truncated polynomial of order 4 with $c_1 = c_3 = 0$), to construct a Padé approximant $R_2^2(x) = \dfrac{p_2(x)}{q_2(x)} = \dfrac{a_2x^2 + a_1x + a_0}{b_2x^2 + b_1x + b_0}$ to the function $f(x)$, with the choice b_0=1.

Hence, derive the R_2^2 Padé approximant to the function $f(x) = \tan(x)$.

Note: The general expression for the derivative of order i of the product: $g(x) = t(x)q(x)$ is given by: $g^{(i)}(x) = \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} t^{(j)}(x) q^{(i-j)}(x)$, which evaluated at $x = 0$ gives:

$a_i = \sum_{j=0}^{i} c_{i-j} b_j$. [8 marks]

3. (a) The sequence of values x_i (i = 1, 2,...,5) in the following table corresponds to successive iterations of a convergent algorithm. Use the Aitken's δ method to find an accelerated sequence.

i	x_i	extrapolated
1	0.6205	–
2	0.5936	–
3	0.5892	
4	0.5886	
5	0.5885	

Please reproduce this table in your answer book [5 marks]

Note: $\bar{x}_n = x_n - \dfrac{(x_n - x_{n-1})^2}{x_n - 2x_{n-1} + x_{n-2}}$

(b) Use Lagrange interpolation to find the second order polynomial that fits the data: $x = \{-1, 0, 1\}$ and $y = \{-0.7, -1, 2.7\}$.

[10 marks]

(c) Use Newton interpolation to find the second order polynomial that interpolates the data specified in part (b).
Then, find the third order polynomial that fits the same data plus the additional point (x=2, y=14.6). Complete the following table and write down the expression of both polynomials in the standard form.

x_i	y_i	Dy_i	D^2y_i	D^3y_i
−1.0	−0.7			
0.0	−1.0			
1.0	2.7			
2.0	14.6			

Please reproduce this table in your answer book [10 marks]

4. (a) Explain in detail the least squares method to fit a curve to a set of data values.
Derive the equations needed to find the coefficients of a straight line to fit a set of data and use them to find the straight line that fits the data:

x_i	y_i
0	0.3
1	1.6
2	3.2
3	4.6
4	6.2

[12 marks]

(b) Describe the methods of successive displacement (or Gauss-Seidel) and simultaneous displacement (or Jacobi) to find the solution to the system of equations expressed in matrix form as:

$$\begin{bmatrix} 5 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

Start the iterations with the vector $\mathbf{x}^{(0)} = [0, 0, 0]^{\mathrm{T}}$. Calculate at least five iterations writing down clearly the successive iteration vectors.

[13 marks]

5. (a) Calculate the integral: $I = \int_0^1 x^3 e^{-x}\, dx$ using:

(i) the trapezoid rule with 8 subintervals.
(ii) the Simpson's quadrature with 4 subintervals.

Note: Use the methods repeatedly in each subinterval or derive the corresponding expressions for multiple subintervals.
Remember that the Simpson quadrature for the interval $[a, b]$ uses 3 points: a, $(a+b)/2$ and b, with the weights $h/3$, $4h/3$ and $h/3$ respectively, with $h = (b-a)/2$.

[12 marks]

(b) Use Gauss quadrature with 6 Gauss points to calculate the integral: $I = \int_0^1 x^3 e^{-x}\, dx$. Note that the Gauss quadrature is defined in the interval $[-1, 1]$ so a change of variable is needed before integrating. Note also that the function to integrate is neither even nor odd.

Note: The Gauss points and weights for a quadrature of order 6 are:

Nodes x_i^6	Weights w_i^6
±0.238619186	0.4679139
±0.661209386	0.3607616
±0.932469514	0.1713245

[8 marks]

(c) Use the Taylor expansions for $f(a+h)$, $f(a-h)$, $f(a+2h)$ and $f(a-2h)$ to show that the following expression is a formula for the first derivative of $f(x)$ and that the error is $O(h^4)$.

$$f'(x) = \frac{f(a-2h) - 8f(a-h) + 8f(a+h) + f(a+2h)}{12h}$$

[5 marks]

6. (a) The figure shows the cross-section of two infinitely long conducting plates, where the perpendicular sides of the first conductor are of equal length and the second conductor forms an angle of 45° with the first. One of the electrodes is grounded and the other is kept at a voltage of 14 V.

Using the mesh shown, with spacing h, calculate using the finite differences method and taking advantage of the symmetry of the structure:

(i) The approximate electric potential at the points A, B and C.

(ii) The approximate field intensity (x and y components) at A, B and C.

(iii) The approximate surface charge density at the points P, Q and R.

Leave the results in terms of h and ε_0.

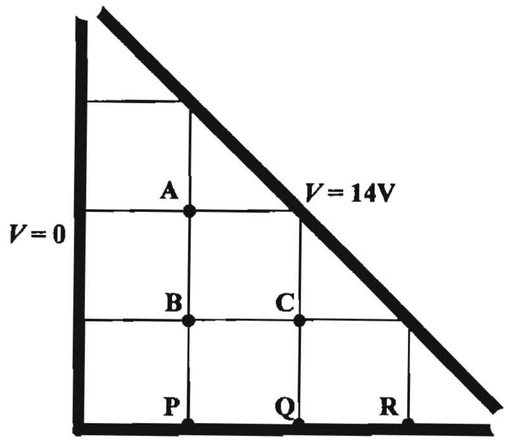

Fig. 6.1 Cross-section view

Note: The electric potential ϕ satisfies the Laplace equation: $\nabla^2\phi = 0$. The electric field is given by: $\vec{E} = -\nabla\phi$ and the surface charge density is given by $\rho_s = \varepsilon_0 \vec{E}\cdot\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the normal to the surface.

[15 marks]

(b) Describe in detail the *weighted residuals method* to solve a differential equation. In particular, explain the special form known as the *Galerkin method*. Mention other common variations of the general method.

Formulate the solution of the following differential equation using the Galerkin method:

$$\frac{d^2u}{dx^2} + 7\frac{du}{dx} + 3u = \sin x \quad \text{for } x \text{ in } [0, \pi]$$

Show that the application of this method leads to a matrix problem of the form $\mathbf{Au} = \mathbf{s}$, where the vector $\mathbf{u}$ contains the coefficients c_i of the expansion of $u(x)$ in terms of the chosen (unspecified) basis functions, say, $b_j(x)$ and the vector $\mathbf{s}$ contains the inner products of the right hand side function $\sin x$ with the basis functions.

Give in detail the form of the elements a_{ij} of the matrix $\mathbf{A}$.

If the boundary conditions satisfied by $u(x)$ are: $u(0) = u(\pi) = 0$, modify the expression of a_{ij} in order to have only first order derivatives.

Hint: Use integration by parts on the second order derivative term of a_{ij}.

[10 marks]

END OF PAPER