Answer FOUR questions

1. (a) Define the term *machine precision* of a floating point system and give an expression for the machine precision of a system specified by (10, t, L, U) where 10 is the base of the system, t is the length of the mantissa and L and U are the lower and upper limits of the exponent.

[3 marks]

(b) For the floating point system (10, 5, −5, 5) find

(i) the machine precision `eps`.

(ii) the smallest positive number that can be represented exactly in this system.

(iii) the largest positive number that can be represented exactly in this system.

(iv) the total number of floating point numbers in the system.

(v) the number of floating point numbers in the interval [0, 1].

(vi) the number of floating point numbers in the interval [1, 2].

(vii) the number of floating point numbers in the interval [10001, 10002].

[12 marks]

(c) Find the normalised error bounds ($\Delta y = \dfrac{y - fl(y)}{y}$) for the following two algorithms to calculate $(a+b)^2$ assuming that a and b are machine numbers (represented exactly in the floating point system):

(i) $y = (a+b)(a+b)$

(ii) $y = a^2 + b^2 + 2ab$

Which of the two algorithms is more stable? Explain and specify the conditions that can make one of them inaccurate. Illustrate this by evaluating the error bounds for the case: $a = 135$, $b = -134$.

[10 marks]

2. (a) The following values of x_i (i = 1,2,…5) correspond to a sequence of data values that is expected to converge to 1.0. Complete the table with the extrapolated values using Aitken's δ^2 method $\left(\bar{x}_n = x_n - \dfrac{(x_n - x_{n-1})^2}{x_n - 2x_{n-1} + x_{n-2}}\right)$ and calculate all the errors (in %).

i	x_i	error	extrapolated	error
1	1.3679		–	
2	1.2431		–	
3	1.1769			
4	1.1353			
5	1.1069			

Please reproduce this table in your answer book [5 marks]

(b) Use Lagrange interpolation to find the second order polynomial that interpolates the data points: (−1, 3), (0, −1) and (1, 1).

[8 marks]

(c) Use Newton interpolation to find the second order polynomial that interpolates the data points: (−1, 3), (0, −1) and (1, 1).
Then, find the third order polynomial that fits the same data plus the additional point (2, 0).
Complete the following table and give the standard form of both polynomials.

x_i	y_i	Dy_i	D^2y_i	D^3y_i
−1	3			
0	−1			
1	1			
2	0			

Please reproduce this table in your answer book [12 marks]

3. (a) With the help of a diagram explain and derive the expression for the Newton-Raphson procedure to find a root of a function $f(x)$.

[5 marks]

(b) Use the Newton-Raphson method to find the value of $\sqrt{3}$ by finding the positive zero of the function $f(x) = x^2 - 3$. Start with the value $x_0 = 1$. Calculate the values for the first 4 iterations completing the following table.

iteration	x_n	$f(x)$	$f'(x)$	$\rightarrow x_{n+1}$
1	1			
2				
3				
4				

Please reproduce this table in your answer book [12 marks]

(c) Use the bisection method to find the value of $\sqrt{3}$ by finding the positive zero of the function $f(x) = x^2 - 3$. Use the starting values x_1 = 1 and x_2 = 2 and complete the table calculating 4 iterations.

iteration	x_1	x_2	c	$f(c)$
1	1	2		
2				
3				
4				

Please reproduce this table in your answer book [8 marks]

4. (a) Give a detailed account of the least-squares method to fit a curve to a set of data values. Derive and write down the equations that determine the parameters to define a first order curve (straight line) that fits a set of data: (x_i, y_i), $i = 1, 2, \ldots, n$.

[7 marks]

(b) Explain what is meant by the term *spline interpolation*. What are *first order splines*, and *cubic splines*? Comment on the benefits or disadvantages of using splines to interpolate a set of n data points (n large) compared to using a single high order polynomial.

[5 marks]

(c) What is meant when it is said that a set of functions is *orthogonal* over a domain Ω? Explain the basic ideas of approximating a given function $f(x)$ by an expansion in terms of a set of (basis) functions. Derive expressions for the coefficients of this expansion and explain the advantage of using an orthogonal set of functions for this expansion.

[6 marks]

(d) Using the Taylor (MacLaurin) expansion of $f(x) = \cos x$ truncated to order 4: $t(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$, find the Padé approximant $R_2^2(x) = \dfrac{p_n(x)}{q_m(x)} = \dfrac{a_2 x^2 + a_1 x + a_0}{b_2 x^2 + b_1 x + 1}$ to the function $f(x)$, where $n = m = 2$.

Note: Calling $g(x) = t(x)\, q(x)$, the general expression for the i-derivative of $g(x)$ is: $g^{(i)}(x) = \sum_{j=0}^{i} \dfrac{i!}{j!(i-j)!} t^{(j)}(x) q^{(i-j)}(x)$, which evaluated at $x = 0$ gives: $a_i = \sum_{j=0}^{i} c_j b_{i-j}$, for $i = 0, \ldots, k$ ($k = m + n$).

[7 marks]

5. (a) Consider the points x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ and $x_3 = x_0 + 3h$. Using Taylor expansions of a function $f(x)$ at these points, truncated at order 3, find an expression for the second derivative of $f(x)$ at $x = x_0$. What is the order of approximation of this formula?

[8 marks]

(b) Consider the differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on the rectangular domain: $0 \le x \le 1$, $0 \le y \le 1$, with the boundary conditions:

$$
\begin{array}{lll}
u(x,\,0) = 1 & u(x,\,1) = 0 & 0 \le x \le 1 \\
\frac{\partial}{\partial x} u(0, y) = 0 & \frac{\partial}{\partial x} u(1, y) = 0 & 0 \le y \le 1
\end{array}
$$

Discretize the domain using a regular grid of size h = 1/3. Draw a diagram and mark and number the points of the domain where the solution is unknown.
Establish the form of the discretized equation for all these points.
Take account of the Newman boundary condition by introducing fictitious grid points, *e.g.* at positions $(-h, y_i)$ and approximating the derivative by:

$$\frac{\partial}{\partial x} u(0, y_i) = \frac{u(0, y_i) - u(-h, y_i)}{h}$$

One obtains a system of linear equations of the form **Az** = **b** with eight unknowns. Specify **A** and **b**.

[17 marks]

6. (a) Define the "*condition number*" of a square matrix **A** and discuss its relevance in solving a linear system of equations **A x = b**, where **x** and **b** are vectors.

[5 marks]

(b) Describe the simultaneous displacement or Jacobi method and the successive displacement or Gauss-Seidel method to solve a linear system of equations, indicating their differences. Illustrate by using the Jacobi method to calculate the first 3 iterations for the solution of the following system:

$$\begin{bmatrix} 4 & -1 & 1 \\ 1 & 6 & 2 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

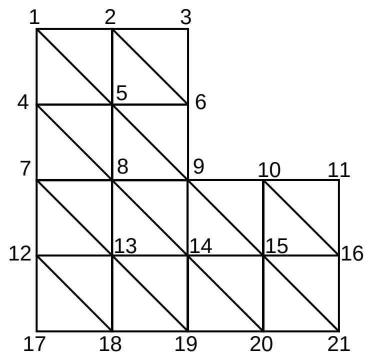
Start the iterations with $\mathbf{x} = (0, 0, 0)^T$.

[9 marks]

(c) Explain the fundamental aspects of the variational method and describe its use in a general procedure to solve a differential equation of the form $\mathcal{L}u = s$ using the finite element method.

[6 marks]

(d) The mesh shown in the figure is used in finite element calculations with first order nodal elements. If the nodes are numbered as in the figure, give the list of the non-zero entries in row 9 of the corresponding matrix. And what are the nonzero values in row 11?



[5 marks]

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