UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**ELEC3030**
ASSESSMENT PATTERN	:	**ELEC3030A**
MODULE NAME	:	**Numerical Methods**
DATE	:	**27-Apr-12**
TIME	:	**10:00**
TIME ALLOWED	:	**3 Hours 0 Minutes**

2011/12-ELEC3030A-001-EXAM-49

TURN OVER

Answer FOUR questions

1. (a) Explain the floating point representation of numbers and define the term *machine precision*.
[5 marks]

(b) For a floating point system of base 10 defined by the length of the mantissa, $t = 4$, and the lower and upper limits of the exponent, $L = -5$, $U = 5$, find:

(i) The smallest and the largest positive numbers
(ii) The total number of numbers that can be represented exactly in the system (or machine numbers).
(iii) The number of machine numbers in the intervals [0, 1], [1, 2] and [1000, 1001].
(iv) The machine precision of this system.

[8 marks]

(c) Determine the error bounds for the two following algorithms to calculate the same quantity assuming that x is already a machine number:

(i) $y = (x+2)(x-1)$

(ii) $y = x^2 + x - 2$

Determine with an error analysis the stability characteristics of both algorithms. What values of x are likely to produce problems and with which algorithm?

[12 marks]

2. (a) Use the Newton-Raphson method to find a root of the function: $f(x) = x^2 - 5\sin(x) + 2.5$.
Use the following table in your answer.

iteration	x_n	$f(x)$	$f'(x)$	$\rightarrow x_{n+1}$
1	0.0			
2				
3				
4				

Please reproduce this table in your answer book [9 marks]

(b) The function $f(x) = x^2 - 5\sin(x) + 2.5$ has two roots in the interval [0, 2]. Use the bisection method to find the largest root. Choose the search interval accordingly.

iteration	x_1	x_2	c	$f(x_1)$	$f(x_2)$	$f(c)$
1						
2						
3						
4						

Please reproduce this table in your answer book [8 marks]

(c) Describe the fixed point iteration method to find a root of a function $f(x)$. Indicate what the condition for convergence is.
Use the fixed point iteration to find the largest root in the interval [0, 2] of the function: $f(x) = x^2 - 5\sin(x) + 2.5$. Start with the choice: $x_0 = 1.2$ and use at least 4 iterations.

iteration	x_n	$g(x)$	$f(x)$
1	1.2		
2			
3			
4			

Please reproduce this table in your answer book [8 marks]

3. (a) The function: $f(x)=\dfrac{1}{2x}\log\left(\dfrac{1+x}{1-x}\right)$ has a singularity at $x = 1$ and its MacLaurin series becomes inaccurate to approximate this function in the vicinity of 1. The corresponding MacLaurin series is: $t(x)=\sum_{n=0}^{\infty}\dfrac{1}{2n+1}x^{2n}$. Use the first 3 terms (a polynomial of order 4), to construct a Padé approximant $R_2^2(x)=\dfrac{p_2(x)}{q_2(x)}=\dfrac{a_2x^2+a_1x+a_0}{b_2x^2+b_1x+b_0}$ to the function $f(x)$, with the choice b_0=1.

Note: The general expression for the derivative of order i of the product: $g(x)=t(x)q(x)$ is given by: $g^{(i)}(x)=\sum_{j=0}^{i}\dfrac{i!}{j!(i-j)!}t^{(j)}(x)q^{(i-j)}(x)$, which evaluated at $x=0$ gives: $a_i=\sum_{j=0}^{i}c_{i-j}b_j$. [10 marks]

(b) The list of values of x_i (i = 1, 2,…, 5) in the following table corresponds to a sequence of data values that is expected to converge to 2.4. Use the Aitken's δ method $\left(\overline{x}_n=x_n-\dfrac{(x_n-x_{n-1})^2}{x_n-2x_{n-1}+x_{n-2}}\right)$ to complete the table with the extrapolated values and calculate all the errors (in %).

i	x_i	error %	extrapolated	error %
1	2.3203			
2	2.3600			
3	2.3797			
4	2.3896			
5	2.3947			

Please reproduce this table in your answer book [5 marks]

(c) Use Newton interpolation to find the second order polynomial that interpolates the data: $x = 1, 2, 3$ and $y = 1.5, 1.0, 2.5$.
Then, find the third order polynomial that fits the same data plus the additional point (x=4, y=0). Complete the following table and give both polynomials in the standard form.

x_i	y_i	Dy_i	D^2y_i	D^3y_i
1	1.5			
2	1.0			
3	2.5			
4	0			

Please reproduce this table in your answer book [10 marks]

4. (a) Find the solution to the following system of equations $\mathbf{Ax} = \mathbf{b}$ using the Gauss-Seidel or successive displacement method:

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$$

Start the iterations with the vector $\mathbf{x}_0 = [1, 1, 1]^{\mathrm{T}}$. Calculate at least four iterations writing down clearly the successive iteration vectors.

[9 marks]

(b) To solve a linear system of equations $\mathbf{Ax} = \mathbf{b}$ with a symmetric matrix $\mathbf{A}$ using the steepest descent method, an error functional h^2, that can be written in a simplified form as: $h^2 = \mathbf{x}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{b}$ is minimised. Starting with a trial vector $\mathbf{x}_0$, the minimum is sought along the direction of the local gradient $-\nabla h^2$ evaluated at the current iteration point $\mathbf{x}_i$, giving $\mathbf{x}_{i+1} = \mathbf{x}_i + \alpha \mathbf{p}_i$ where $\mathbf{p}_i$ is the unit vector along $-\nabla h^2$. Show that the value of α that minimises h^2 along the line $\mathbf{x}_i + \alpha \mathbf{p}_i$ is given by:

$$\alpha = \frac{\mathbf{p}_i^{\mathrm{T}}(\mathbf{b} - \mathbf{Ax}_i)}{\mathbf{p}_i^{\mathrm{T}}\mathbf{Ap}_i}$$

[7 marks]

(c) Consider the solution of the system of equations in part (a) using the steepest descent method. Starting with the vector $\mathbf{x}_0 = [1, 1, 1]^{\mathrm{T}}$, find the iteration vector $\mathbf{x}_1$ using the expression for α given in part (b). Show also that the error functional h^2 for this system of equations can be written as:

$$h^2(\mathbf{x}) = 3x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_2x_3 - 2x_3^2 - 8x_1 + 4x_3$$

[9 marks]

5. (a) Use Gauss quadrature with 6 Gauss points to calculate the integral: $I = \int_0^1 e^{-x^2} \cos^2 x \, dx$.

Note that a change of variable is needed since the Gauss quadrature is defined over the interval [–1, 1].

Note: Gauss points and weights for order 6 are:

Nodes x_i^6	Weights w_i^6
±0.238619186	0.4679139
±0.661209386	0.3607616
±0.932469514	0.1713245

[8 marks]

(b) Use the Simpson quadrature with 4 subintervals to calculate the integral: $I = \int_0^1 e^{-x^2} \cos^2 x \, dx$

You can either divide the interval of integration into 4 subintervals and add the result for each or derive the expression for the Simpson quadrature using multiple subintervals.
Note: The expression for the Simpson quadrature for one interval is:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a)+4f(c)+f(b)\right] \text{ with } c=\frac{a+b}{2} \text{ and } h=\frac{b-a}{2}$$

[8 marks]

(c) Use finite differences to formulate the solution to the following differential equation over the interval [0, 1]:

$$\frac{d^2 f(x)}{dx^2} + x\frac{df(x)}{dx} + 2f(x) = \sin(\pi x)$$

with boundary conditions: $f(0) = f(1) = 0$.
Divide the interval into N subintervals, so the nodes are defined as: $x_n = nh, \; n = 0, 1, 2, \cdots N$ where $h = 1/N$. Note that the values are known for $n = 0$ and $n = N$, so there are only N–1 unknowns. Use central differences approximations for the first and second derivatives.
Write down the difference equation for a generic point x_n and the resultant matrix problem.
Write in full the matrix and right-hand-side vector corresponding to the choice $N = 5$.

[9 marks]

6. (a) The functional $\mathscr{J}(\phi) = \int_{\Omega} (\nabla \phi)^2 \, d\Omega$ is a variational expression for the Laplace equation $\nabla^2 \phi = 0$ defined over a domain Ω. Use the Rayleigh-Ritz method, approximating the function $\phi(x)$ as $\phi(x) = \sum_{i=1}^{N} c_i \, b_i(x)$, in terms of a set of basis functions $b_i(x)$, $i = 1, 2, \cdots, N$, where x is a point in the domain, to find the discretized form of the problem giving the general form of the matrix elements.

[10 marks]

(b) Consider the finite element mesh depicted in the Figure 6.1:

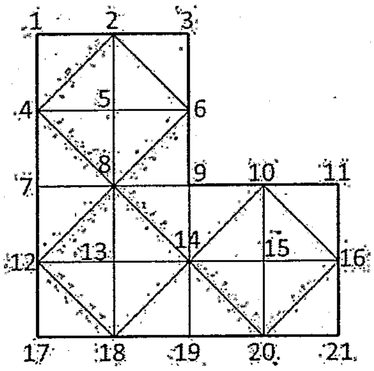

Figure 6.1

Copy the diagram below in your answer book and use it to indicate the sparsity pattern of the rows of the resultant matrix.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

For the matrix rows corresponding to nodes 1, 5, 9, 12 and 19 indicate with an "X" the positions with a non-zero entry. Leave blank the positions with zero. Use one row of boxes for each matrix row.

[10 marks]

(c) The Figure 6.2 below shows a second order triangular element. Write down the list of nodes (1 – 6) and their corresponding triangle area coordinates. Write the expression of the shape functions corresponding to the nodes 1, 3, 4 and 6 in terms of the area coordinates (ξ_1, ξ_2, ξ_3).

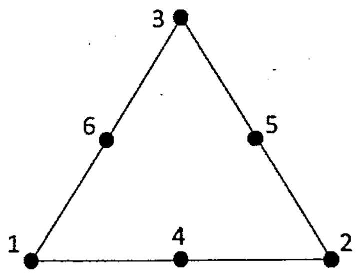

Figure 6.2

[5 marks]

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