1. Let $A_0,\ldots,A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1}+\overrightarrow{A_1A_2}+\cdots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_0}$.

2. In each of the following cases, decide if the indicated vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ can be represented with the vertices of a triangle:

1. $\mathbf{u}(7,3)$, $\mathbf{v}(-2,-8)$, $\mathbf{w}(-5,5)$.
2. $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.
3. $\|\mathbf{u}\|=7$, $\|\mathbf{v}\|=3$, $\|\mathbf{w}\|=11$.
4. $\mathbf{u}(1,0,1)$, $\mathbf{v}(0,1,0)$, $\mathbf{w}(2,2,2)$.

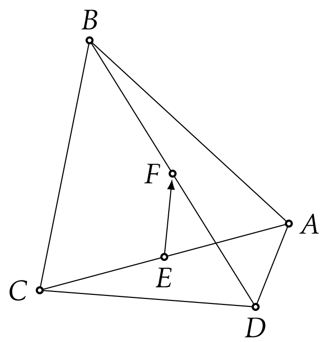
3. Let $ABCDEF$ be a regular hexagon centered at O.

1. Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of $\overrightarrow{OE}$ and $\overrightarrow{OF}$.
2. Show that $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=3\overrightarrow{AD}$.

4. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that $\overrightarrow{MN}+\overrightarrow{PQ}=0$. Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

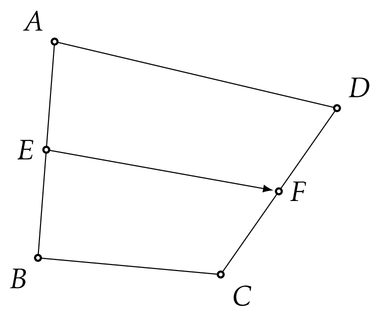
$$\overrightarrow{EF}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{CD})=\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{CB}).$$



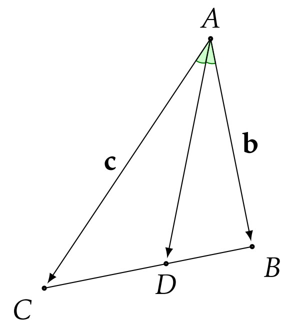
6. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF}=\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{BC}).$$

Deduce that the length of the midsegment in a trapezoid is the arithmetic mean of the lengths of the bases.

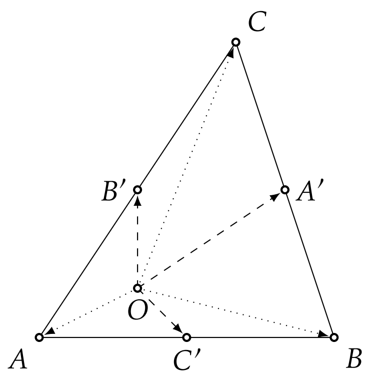


7. Let ABC be a triangle and let $D \in [BC]$ be such that AD is an angle bisector. Express $\overrightarrow{AD}$ in terms of $\mathbf{b} = \overrightarrow{AB}$ und $\mathbf{c} = \overrightarrow{AC}$.

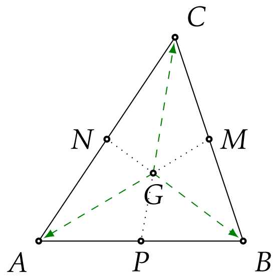


8. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$



9. Show that the medians in a triangle intersect in one point.



10. Let $ABCD$ be a tetrahedron. Determine the sums

1. $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$,
2. $\overrightarrow{AD}+\overrightarrow{BC}+\overrightarrow{DB}$,
3. $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{BC}+\overrightarrow{DA}$.

11. Let $ABCD$ be a tetrahedron. Show that $\overrightarrow{AD}+\overrightarrow{BC}=\overrightarrow{BD}+\overrightarrow{AC}$.

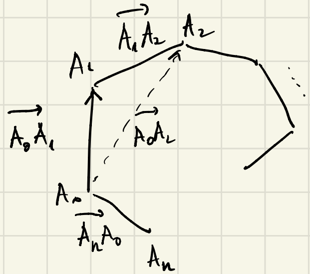
12. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC}+\overrightarrow{SD}=4\overrightarrow{SO}$$

where O is the center of the parallelogram.

13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

1. Let $A_0, \dots, A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \dots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$.



$$\underbrace{\underbrace{\underbrace{\underbrace{\overrightarrow{A_0A_1}+\overrightarrow{A_1A_2}}_{\overrightarrow{A_0A_2}}+\overrightarrow{A_2A_3}}_{\overrightarrow{A_0A_3}}+\dots+\overrightarrow{A_{n-1}A_n}}_{\overrightarrow{A_0A_n}}+\overrightarrow{A_nA_0}}_{\overrightarrow{A_0A_0}} = 0$$

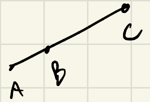
2. In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle:

1. $\mathbf{u}(7,3)$, $\mathbf{v}(-2,-8)$, $\mathbf{w}(-5,5)$.
2. $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.
3. $\|\mathbf{u}\| = 7$, $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 11$.
4. $\mathbf{u}(1,0,1)$, $\mathbf{v}(0,1,0)$, $\mathbf{w}(2,2,2)$.

1. $u + v + w = \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -8 \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ u, v, w can be represented on the sides of a polygon (with 3 sides, ie. Δ)

• is the triangle degenerate?



$\Updownarrow$

A, B, C collinear?

$\Updownarrow$

u, v linearly dependent which is not the case

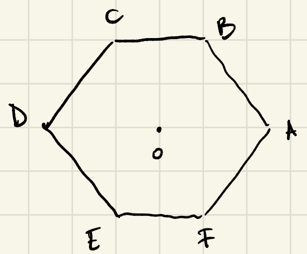
2. We have the same vectors as in 1 except v which is in the opposite direction so, the answer is yes.

3. $\|u\| + \|v\| = 10 < \|w\| = 11 \Rightarrow$ no, a triangle cannot have one side longer than the sum of the other sides

4. $\|u\| = \sqrt{2}$ $\quad \|v\| = 1$ $\quad \|w\| = 2\sqrt{3} \Rightarrow$ no, $\sqrt{2}+1 < 2\sqrt{3}$

3. Let $ABCDEF$ be a regular hexagon centered at O.

1. Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of $\overrightarrow{OE}$ and $\overrightarrow{OF}$.
2. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.



1.) $\overrightarrow{OA} = \overrightarrow{EF} = \overrightarrow{EO} + \overrightarrow{OF} = \overrightarrow{OF} - \overrightarrow{OE}$

$\overrightarrow{OB} = -\overrightarrow{OE}$

$\overrightarrow{OC} = -\overrightarrow{OF}$

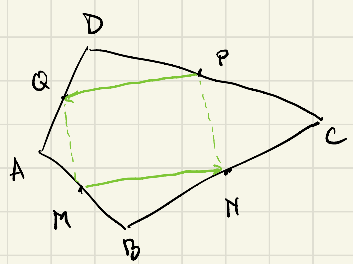
$\overrightarrow{OD} = \underset{\substack{\| \\ -\overrightarrow{EF} \\ \| \\ -\overrightarrow{OA}}}{\overrightarrow{FE}} = \overrightarrow{OE} - \overrightarrow{OF}$

2.) $$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = (\overrightarrow{AB} + \overrightarrow{AF}) + \overbrace{(\overrightarrow{AB} + \underset{\substack{\| \\ \overrightarrow{AO}}}{\overrightarrow{BC}})}^{\overrightarrow{AC}} + \overbrace{(\overrightarrow{AF} + \underset{\substack{\| \\ \overrightarrow{AO}}}{\overrightarrow{FE}})}^{\overrightarrow{AE}} + \underset{\substack{\| \\ 2\overrightarrow{AO}}}{\overrightarrow{AD}}$$

$$= 2\underbrace{(\overrightarrow{AB} + \overrightarrow{AF})}_{\overrightarrow{AO}} + 4\overrightarrow{AO}$$

$$= 6\overrightarrow{AO} = 3\overrightarrow{AD}$$

4. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that $\overrightarrow{MN} + \overrightarrow{PQ} = 0$. Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.



Fix a point $O \in E^2$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{2}(\cancel{\overrightarrow{OB}} + \overrightarrow{OC}) - \frac{1}{2}(\overrightarrow{OA} + \cancel{\overrightarrow{OB}})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \cancel{\overrightarrow{OD}}) - \frac{1}{2}(\cancel{\overrightarrow{OD}} + \overrightarrow{OC})$

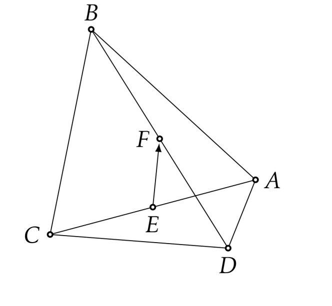
$+$

$\overrightarrow{MN} + \overrightarrow{PQ} = \ldots = 0$

$\Rightarrow \overrightarrow{MN} = \overrightarrow{QP} \Leftrightarrow MNPQ$ parallelogram

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB}).$$



Let O be any point in the plane E^2

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OB}) - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$$

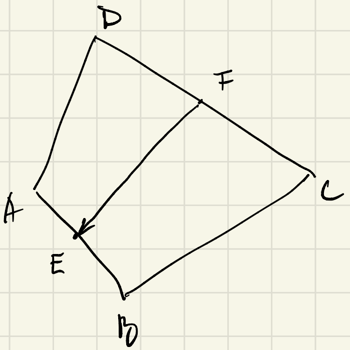
$$= \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OB} - \overrightarrow{OA} - \overrightarrow{OC}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD})$$

$$= \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB})$$

6. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC}).$$

Deduce that the length of the midsegment in a trapezoid is the arithmetic mean of the lengths of the bases.



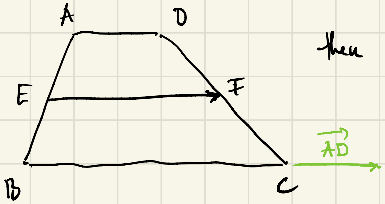
Let O be any point in E^2

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC}) - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

$$= \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{OB})$$

$$= \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})$$

If $ABCD$ is a trapezoid

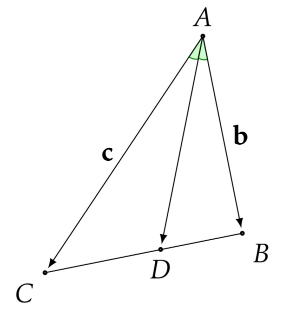


then $\overrightarrow{EF}, \overrightarrow{AD}, \overrightarrow{BC}$ are collinear.

$$|EF| = \|\overrightarrow{EF}\| = \left\|\frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})\right\| = \frac{1}{2}\|\overrightarrow{AD} + \overrightarrow{BC}\| = \frac{1}{2}(\|\overrightarrow{AD}\| + \|\overrightarrow{BC}\|) = \frac{|AD| + |BC|}{2}$$

Since $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are collinear and have the same orientation

7. Let ABC be a triangle and let $D \in [BC]$ be such that AD is an angle bisector. Express $\overrightarrow{AD}$ in terms of $\mathbf{b} = \overrightarrow{AB}$ und $\mathbf{c} = \overrightarrow{AC}$.



By the angle bisector theorem we have

$$\frac{|AC|}{|CD|} = \frac{|AB|}{|BD|} \qquad (*)$$

Now $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = c + \dfrac{|CD|}{|CB|}\,\overrightarrow{CB}$

$$= c + \frac{|CD|}{|CB|}(\overrightarrow{AB} - \overrightarrow{AC})$$

$$= c + \frac{|CD|}{|CB|}(b - c)$$

$$= \left(1 - \frac{|CD|}{|CB|}\right)c + \frac{|CD|}{|CB|}\,b$$

By $(*)$ we have $\dfrac{|BD|}{|CD|} = \dfrac{|AB|}{|AC|} \overset{+1}{\Rightarrow} \dfrac{\overbrace{|BD| + |CD|}^{|BC|}}{|CD|} = \dfrac{|AB| + |AC|}{|AC|}$

So $|CD| = \dfrac{|BC| \cdot |AC|}{|AB| + |AC|}$

$$\Rightarrow \quad \overrightarrow{AD} = \left(1 - \frac{|AC|}{|AB| + |AC|}\right)c + \frac{|AC|}{|AB| + |AC|}\,b$$

$$\overrightarrow{AD} = \frac{|AB|}{|AB| + |AC|}\,c + \frac{|AC|}{|AB| + |AC|}\,b$$

8. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

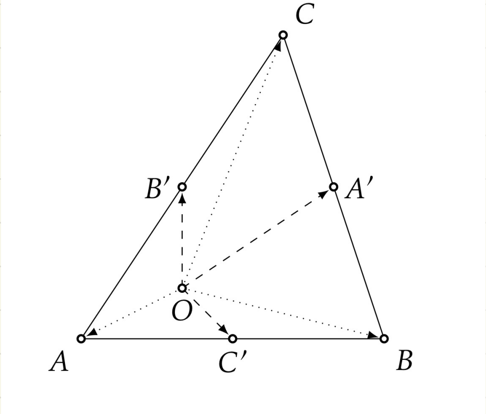
$$\overrightarrow{OA'} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)$$

$$\overrightarrow{OB'} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)$$

$$\overrightarrow{OC'} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$$

$+$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$



9. Show that the medians in a triangle intersect in one point.

Let P be the midpoint of AB

M —— " —— BC

N —— " —— CA

For any point $O \in \mathbb{E}^2$ we have

consider the vector

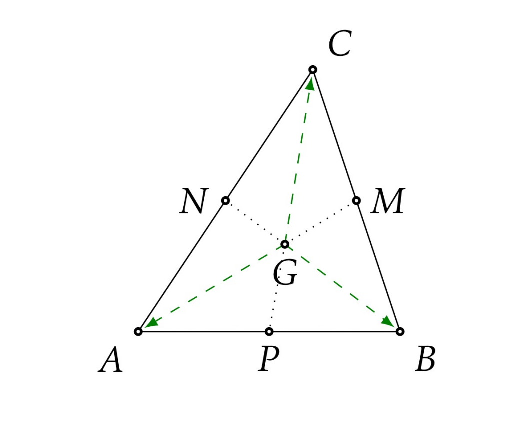
$$v = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

Then

$$v = \left(\overrightarrow{OA} + \overrightarrow{OB}\right) + \overrightarrow{OC} = 2\,\overrightarrow{OP} + \overrightarrow{OC} \qquad (1)$$

$$v = \left(\overrightarrow{OA} + \overrightarrow{OC}\right) + \overrightarrow{OB} = 2\,\overrightarrow{ON} + \overrightarrow{OB} \qquad (2)$$

$$v = \left(\overrightarrow{OB} + \overrightarrow{OC}\right) + \overrightarrow{OA} = 2\,\overrightarrow{OM} + \overrightarrow{OA} \qquad (3)$$



So, if we choose O on CP it follows from (1) that $v \parallel \overrightarrow{CP}$

Similar, if we choose O on BM —— $\parallel$ —— $v \parallel \vec{BM}$

—— $\parallel$ —— AM —— $\parallel$ —— $v \parallel \vec{AM}$

Now, if $O \in CP \cap BM \Rightarrow v \parallel \vec{CP}$ and $v \parallel \vec{BM}$

$\Rightarrow v = 0$ (else $\vec{CP} \parallel \vec{BM} \Leftrightarrow CP \parallel BM$ which is $\overset{\text{impossible}}{\smile}$)

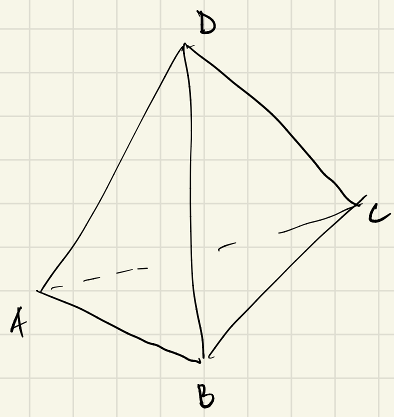
(3)
$\Rightarrow$ $0 = 2\vec{OM} + \vec{OA} \Rightarrow \vec{OM}, \vec{OA}$ are proportional $\Rightarrow O, M, A$ are collinear

So if $O \in CP \cap BM$ then $O \in AM$, i.e the intersection point of two medians lies on the third median

i.e all three medians intersect in one point

10. Let $ABCD$ be a tetrahedron. Determine the sums

1. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$,
2. $\overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DB}$,
3. $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA}$.



1. $\vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$

2. $\vec{AD} + \vec{BC} + \vec{DB} = \vec{AB} + \vec{BC} = \vec{AC}$

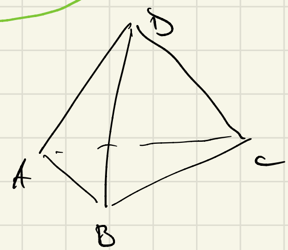
3. $\vec{AB} + \vec{CD} + \vec{BC} + \vec{DA} = \vec{AA} = 0$

① ② ③

11. Let $ABCD$ be a tetrahedron. Show that $\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}$.

$$\overrightarrow{AD} + \overrightarrow{BC} = (\overrightarrow{AB} + \overrightarrow{BD}) + (\overrightarrow{BA} + \overrightarrow{AC})$$

$$= \overrightarrow{BD} + \overrightarrow{AC} + \underbrace{\overrightarrow{AB} + \overrightarrow{BA}}_{=0}$$

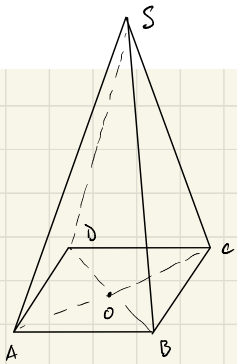


12. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.

$$\underbrace{\overrightarrow{SA} + \overrightarrow{SC}}_{= 2\overrightarrow{SO}} + \underbrace{\overrightarrow{SB} + \overrightarrow{SD}}_{= 2\overrightarrow{SO}} = 4\overrightarrow{SO}$$



13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

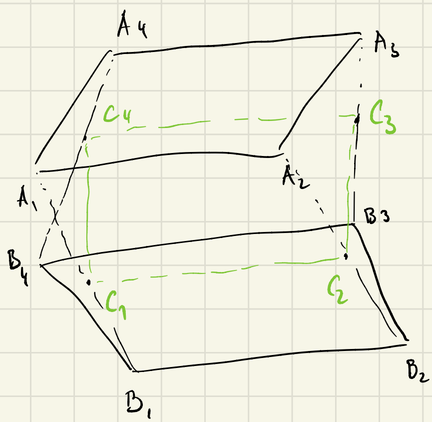
Let C_i be the midpoint of $[A_iB_i]$

$$\overrightarrow{C_1C_2} = \overrightarrow{OC_2} - \overrightarrow{OC_1} = \tfrac{1}{2}(\overrightarrow{OA_2} + \overrightarrow{OB_2} - \overrightarrow{OA_1} - \overrightarrow{OB_1})$$

$$= \tfrac{1}{2}(\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2})$$

Similar $\overrightarrow{C_3C_4} = \tfrac{1}{2}(\overrightarrow{A_3A_4} + \overrightarrow{B_3B_4})$

(see Problem 6)

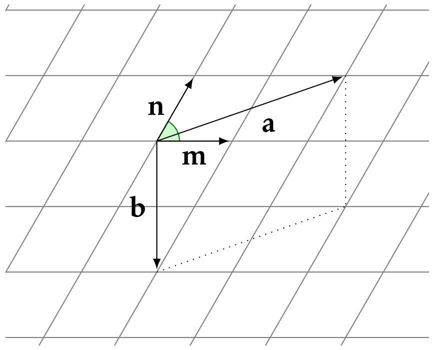


$$\Rightarrow \overrightarrow{C_1C_2} + \overrightarrow{C_3C_4} = \tfrac{1}{2}(\underbrace{\overrightarrow{A_1A_2} + \overrightarrow{A_3A_4}}_{=0} + \underbrace{\overrightarrow{B_1B_2} + \overrightarrow{B_3B_4}}_{=0}) = 0 \Rightarrow \overrightarrow{C_1C_2} = \overrightarrow{C_4C_3}$$

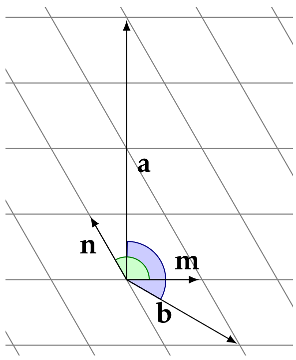
since $A_1A_2A_3A_4$ is a parallelogram

$\Rightarrow C_1C_2C_3C_4$ parallelogram.

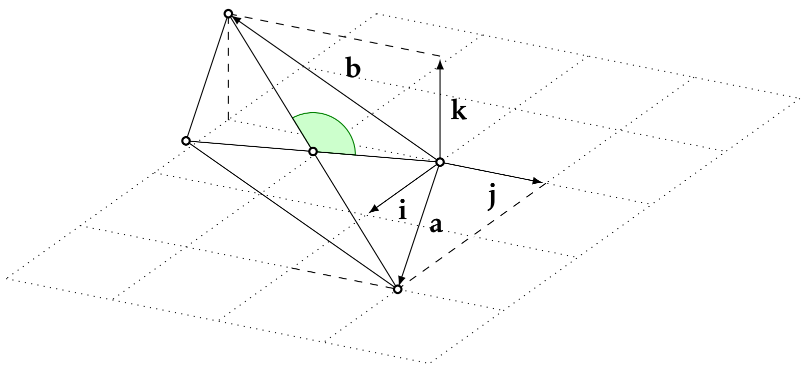
1. Let **m** and **n** be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$.



2. Let **m** and **n** be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.



3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by **a** and **b**.



4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

5. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC}$$

and deduce the law of cosines in a triangle.

6. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2.$$

7. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.

8. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,0)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

9. Show that the orthogonal reflection of a vector $\mathbf{b}$ parallel to $\mathbf{a}$ is

$$\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b} - 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \mathbf{b} - 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}).$$

Show that the orthogonal reflection of a vector $\mathbf{b}$ in the vector $\mathbf{a}$ is

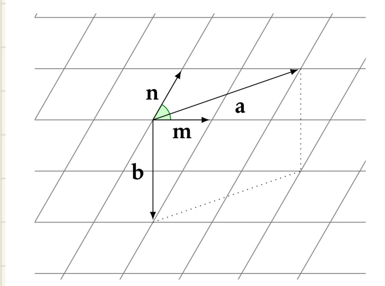
$$\mathrm{Ref}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = -\mathbf{b} + 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}).$$

10. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that

1. The set $\mathbf{v}^{\perp}$ is a vector subspace of $\mathbb{V}^n$.
2. There is a basis $\mathbf{v},\mathbf{v}_2,\mathbf{v}_3,\dots,\mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2,\dots,\mathbf{v}_{n-1}$ a basis of $\mathbf{v}^{\perp}$.

11. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w}\cdot\mathbf{v}$?

1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$.



- The length of the two diagonals are

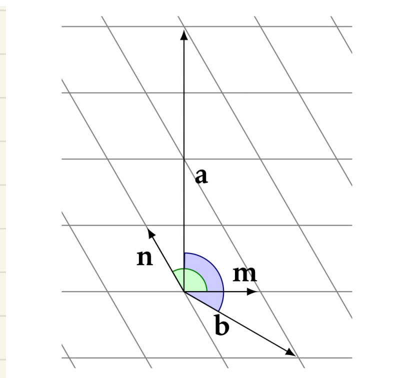
  $\|a+b\|$ and $\|a-b\|$

- $\|a+b\| = \|2m+n+m-2n\|$

$$\|a+b\|^2 = \|3m-n\|^2 = (3m-n)^2 = 9\underbrace{m^2}_{=1} - 6\underbrace{mn}_{=\underbrace{\|m\|}_{=1}\cdot\underbrace{\|n\|}_{=1}\cdot\underbrace{\cos\measuredangle(m,n)}_{=\cos 60°}} + \underbrace{n^2}_{=1} = 10 - 6\cdot\frac{1}{2} = 7 \Rightarrow \|a+b\| = \sqrt{7}$$

- $\|a-b\|^2 = \|2m+n-m+2n\| = (m+3n)^2 = m^2 + 6mn + 9n^2 = 10 + 6\cdot\frac{1}{2} = 13 \Rightarrow \|a-b\| = \sqrt{13}$

2. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.



$$\cos\measuredangle(a,b) = \frac{a\cdot b}{\|a\|\cdot\|b\|}$$

$$a\cdot b = (2m+4n)\cdot(m-n)$$

$$= 2m^2 - 2mn + 4nm - 4n^2$$

$$= 2m^2 + 2nm - 4n^2$$

$$= 2\|m\|^2 + 2\|n\|\cdot\|m\|\cdot\cos\measuredangle(n,m) - 4\|n\|^2$$

$$= -2 + 2\cdot\frac{-1}{2} = -3$$

$$\|a\|^2 = (2m+4n)^2 = 4m^2 + 16mn + 16n^2 = 20 + 16\cdot\frac{-1}{2} = 12 \quad \Rightarrow \|a\| = \sqrt{12}$$

$$\|b\|^2 = (m-n)^2 = m^2 - 2mn + n^2 = 2 - 2\frac{-1}{2} = 3 \quad \Rightarrow \|b\| = \sqrt{3}$$

$$\Rightarrow \cos\measuredangle(a,b) = \frac{-3}{2\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{2} \quad \Rightarrow \measuredangle(a,b) = 120°$$

3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

$$\cos \measuredangle(a+b, a-b) = \frac{(2,-1,1)\cdot(2,3,-1)}{\|(2,-1,1)\|\cdot\|(2,3,-1)\|} = \frac{4-3-1}{\sqrt{4+1+1}\,\sqrt{4+9+1}} = 0 \Rightarrow$$ the two diagonals are perpendicular to each other.

4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

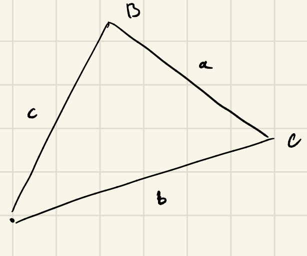
$$\cos \measuredangle(p,q) = \frac{p\cdot q}{\|p\|\cdot\|q\|} = \frac{3+2}{\sqrt{9+1}\,\sqrt{1+4+\lambda}} = \frac{5}{12} \iff \sqrt{10}\,\sqrt{5+\lambda} = 12 \iff 10(5+\lambda^2) = 144$$

So $\underbrace{\lambda^2 = 9.4}_{47/5} \Rightarrow \lambda = \sqrt{9.4}$

5. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC} \qquad (*)$$

and deduce the law of cosines in a triangle.



$$(*) \iff \overrightarrow{AB}^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{AC}^2 = \overrightarrow{BC}^2$$

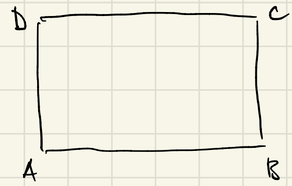
$$\iff (\underbrace{\overrightarrow{AB} - \overrightarrow{AC}}_{\overrightarrow{BC}})^2 = \overrightarrow{BC}^2$$

on the other hand $$(*) \iff c^2 + b^2 - a^2 = 2\,\underbrace{\|\overrightarrow{AB}\|}_{c}\cdot\underbrace{\|\overrightarrow{AC}\|}_{b}\,\underbrace{\cos\measuredangle(AB,AC)}_{\cos\hat{A}}$$

so $$a^2 = c^2 + b^2 - 2bc\cos\hat{A}$$

6. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC}=\overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{und} \quad \overrightarrow{OA}^2+\overrightarrow{OC}^2=\overrightarrow{OB}^2+\overrightarrow{OD}^2.$$



1) $\underbrace{\overrightarrow{OA}\cdot\overrightarrow{OC}} = \overrightarrow{OB}\cdot\overrightarrow{OD}$

$$(\overrightarrow{OB}+\overrightarrow{BA})(\overrightarrow{OD}+\overrightarrow{DC}) = \overrightarrow{OB}\cdot\overrightarrow{OD} + \overrightarrow{BA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{DC} + \overrightarrow{BA}\cdot\overrightarrow{DC}$$

$$= \overrightarrow{OB}\cdot\overrightarrow{OD} + \overrightarrow{BA}\cdot\overrightarrow{OC} + \overrightarrow{OB}\cdot\overrightarrow{DC}$$

$$- \overrightarrow{OB}\cdot\overrightarrow{BA} \qquad (\text{since } \overrightarrow{BA} = -\overrightarrow{DC})$$

$$= \overrightarrow{OB}\cdot\overrightarrow{OD} + \underbrace{\overrightarrow{BA}\,(\overset{\overrightarrow{CB}}{\overrightarrow{OC}-\overrightarrow{OD}})}_{=0} \quad (\text{since } \overrightarrow{BA}\perp\overrightarrow{CB}$$

2) $$\frac{\overrightarrow{OA}+\overrightarrow{OC}}{2} = \frac{\overrightarrow{OB}+\overrightarrow{OD}}{2}$$

$$\Rightarrow (\overrightarrow{OA}+\overrightarrow{OC})^2 = (\overrightarrow{OB}+\overrightarrow{OD})^2$$

$$\Rightarrow \overrightarrow{OA}^2 + 2\overrightarrow{OA}\overrightarrow{OC} + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \underbrace{2\overrightarrow{OB}\cdot\overrightarrow{OD}}_{2\overrightarrow{OA}\,\overrightarrow{OC}\ (\text{by part } 2)} + \overrightarrow{OD}^2$$

$$\Rightarrow \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2$$

7. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.

Let $\{e_1, \dots, e_n\}$ be a basis of V^n

Consider
$$v_1 = e_1 = v$$
$$v_2 = e_2 - \frac{v_1 \cdot e_2}{v_1 \cdot v_1} e_1$$
$$v_3 = e_3 - \frac{v_1 e_3}{v_1 \cdot v_1} v_1 - \frac{v_2 e_3}{v_2 \cdot v_2} v_2$$
$$\vdots$$
$$v_j = e_j - \sum_{i=1}^{j-1} \frac{v_i e_j}{v_i v_i} v_i$$
$$\vdots$$

Claim $\{v_1, \dots, v_n\}$ is a basis of V^n and $\{v_2, \dots v_n\}$ is a basis of $v^\perp$

- $\forall j$ $\{v_1, \dots, v_j\}$ is linearly independent

else $v_j =$ linear combination of $\{v_1 \dots v_{j-1}\}$

$\Leftrightarrow$ $v_j =$ linear combination of $\{e_1, \dots, e_{j-1}\}$

but $v_j = e_j +$ linear combination of $\{e_1, \dots, e_{j-1}\}$ $\Big\}$ contradiction

$\Rightarrow \{v_1, \dots, v_j\}$ is lin. indep. $\forall j \in \{1, \dots, n\}$

$\Rightarrow \{v_1, \dots, v_n\}$ is a basis of V^n

Claim $\{v_1, \dots, v_j\}$ is a set of mutually orthogonal vectors

i.e $\forall i<j$ $v_j \perp v_i$ we show this by induction

$$v_1 \cdot v_2 = v_1 \cdot \left(e_2 - \frac{v_1 e_2}{v_1 v_1} v_1\right) = v_1 \cdot e_2 - \frac{v_1 e_2}{v_1 v_1} \cdot v_1 v_1 = 0$$

assume statement holds for $\{v_1, \dots, v_{j-1}\}$

then, $\forall k<j$ we have

$$v_k \cdot v_j = v_k \cdot \left(e_j - \sum_{i=1}^{j-1} \frac{v_i e_j}{v_i v_i} v_i\right)$$

$$= v_k \cdot e_j - \frac{v_k e_j}{\cancel{v_k v_k}} \cancel{v_k v_k} - \sum_{\substack{i=1 \\ i\neq k}}^{j-1} \frac{v_i e_j}{v_i v_i} \cdot \underbrace{v_k \cdot v_i}_{= 0 \text{ by induction}}$$

$$= 0$$

$\Rightarrow$ the basis $\{v_1, \dots, v_n\}$ is orthogonal

$\Rightarrow$ the basis $\left\{\frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|}\right\}$ is orthonormal. (why ?)

8. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

$w_1 = v_1$

$$w_2 = v_2 - \frac{w_1 v_2}{w_1 w_1} w_1 = \begin{bmatrix}2\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix}}{1^2}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$w_3 = v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix} - 0\begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{\begin{bmatrix}2\\0\\0\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}}{4}\begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1/2\end{bmatrix}$$

So $w_1(0,1,0)$, $w_2(2,0,0)$ and $w_3(0,0,1/2)$ form an orthogonal basis

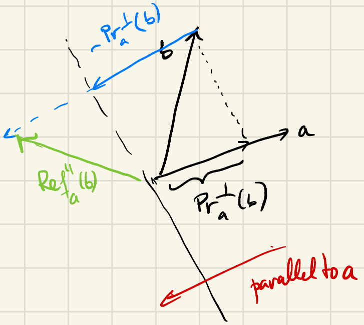
the corresponding orthonormal basis is $\frac{w_1}{\|w_1\|} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\frac{w_2}{\|w_2\|} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\frac{w_3}{\|w_3\|} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

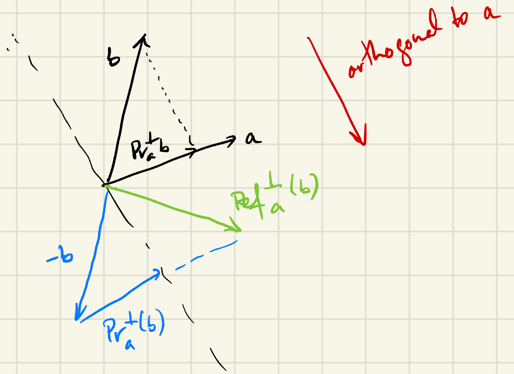
9. Show that the orthogonal reflection of a vector $\mathbf{b}$ parallel to $\mathbf{a}$ is

$$\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}) = \mathbf{b} - 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \mathbf{b} - 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}).$$

Show that the orthogonal reflection of a vector $\mathbf{b}$ in the vector $\mathbf{a}$ is

$$\mathrm{Ref}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathbf{b} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = -\mathbf{b} + 2\,\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}) = -\mathrm{Ref}^{\|}_{\mathbf{a}}(\mathbf{b}).$$





10. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that

1. The set $\mathbf{v}^{\perp}$ is a vector subspace of $\mathbb{V}^n$.

2. There is a basis $\mathbf{v}, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ a basis of $\mathbf{v}^{\perp}$.

1.) $v^{\perp} = \{ w \in \mathbb{V}^n : w \cdot v = 0 \}$

$\forall\, w, u \in v^{\perp}\ \forall\, \alpha, \beta \in \mathbb{R}\quad (\alpha w + \beta u)\cdot v = \alpha \overset{0}{\widetilde{w \cdot u}} + \beta \overset{0}{\widetilde{u \cdot v}} = 0$

$\Rightarrow \alpha w + \beta u \in v^{\perp}$ so $v^{\perp}$ is a vector subspace

2.) Follow the Gram-Schmidt process

Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{V}^n$

By Steiniz Theorem (Algebra, Lecture 6) we may assume that $e_1 = v$

Now consider

$$v_1 = e_1 = v$$

$$v_2 = e_2 - \frac{v_1 \cdot e_2}{v_1 \cdot v_1} e_1$$

$$v_3 = e_3 - \frac{v_1 e_3}{v_1 \cdot v_1} v_1 - \frac{v_2 e_3}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_j = e_j - \sum_{i=1}^{j-1} \frac{v_i e_j}{v_i v_i} v_i$$

by Exercise 7 $\{\overset{v}{\overset{\parallel}{v_1}}, \dots, v_n\}$ is a basis and $\overset{v}{\overset{\parallel}{v_1}} \perp v_k \ \forall k \in \{2, \dots, n\}$

So, since $v^\perp$ is a proper subspace of V^n it has dimension at most $n-1$

$\Rightarrow \{v_2, \dots, v_n\}$ is a basis of $v^\perp$

11. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v}$?

- That the map is linear follows from the properties of the scalar product v is fixed and

$$\phi(\alpha w + \beta u) = v \cdot (\alpha w + \beta u) = \alpha v \cdot w + \beta v \cdot u = \alpha \phi(w) + \beta \phi(u)$$

for any two vectors w, u and scalars $\alpha, \beta \in \mathbb{R}$

- if $v = v(v_1, \dots, v_n)$ with respect to the orthonormal basis $e_1, \dots, e_n$ then

$$\phi(e_1) = (v_1 e_1 + \dots + v_n e_n) \cdot e_1 = v_1 \cdot e_1^2 = v_1 \cdot \|e_1\|^2 = v_1$$

$$\vdots$$

$$\phi(e_j) = v_j \quad \Rightarrow \quad [\phi]_{e,1} = [v_1 \ v_2 \ \dots \ v_n]$$

- Nothing changes if we permute v and w since $w \cdot v = v \cdot w$.

12. $v = v(v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$

$v \perp a \iff v \cdot a = 0 \iff 4v_1 - 2v_2 - 3v_3 = 0$

$v \perp b \iff v \cdot b = 0 \iff v_2 + 3v_3 = 0 \Rightarrow v_2 = -3v_3$

$\Rightarrow 4v_1 + 3v_3 = 0$

$4v_1 = -3v_3 = v_2$

$\Downarrow$

$\left(\frac{1}{4}v_2, v_2, -\frac{1}{3}v_2\right)$

the vectors orthogonal to both a and b.

$\|v\| = 26 \iff v_1^2 + v_2^2 + v_3^2 = 676$

$\|v\| = 26 \Rightarrow \frac{1}{16}v_2^2 + v_2^2 + \frac{1}{9}v_2^2 = 26^2$

$\frac{9 + 144 + 16}{4^2 \cdot 3^2} v_2^2 = 26^2 \iff \frac{169}{4^2 \cdot 3^2} v_2^2 = 26^2$

$\iff v_2 = \pm \frac{12 \cdot 26}{13} = \pm 24$

So $v = (6, 24, -8)$ or $(-6, -24, 8)$

the angle with Ox is $\measuredangle(v, i)$ It is acute if $\cos\measuredangle(v,i) > 0$

$\Rightarrow v = (6, 24, -8)$

1. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

2. With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3, -1, -2)$ and $\mathbf{b}(1, 2, -1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6, 0, 1)$ and $\overrightarrow{AC} = (1.5, 2, 1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

4. Consider the vectors $\mathbf{a}(2, 3, -1)$ and $\mathbf{b}(1, -1, 3)$ with respect to an orthonormal basis.

1. Determine the vector subspace $\langle \mathbf{a}, \mathbf{b} \rangle^{\perp}$.
2. Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51$.

5. Consider the points $A(1, 2, 0)$, $B(3, 0, -3)$ and $C(5, 2, 6)$ with respect to an orthonormal coordinate system.

1. Determine the area of the triangle ABC.
2. Determine the distance from C to AB.

6. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

and deduce the law of sines in a triangle.

7. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2, -3, 1)$, $\mathbf{b}(-3, 1, 2)$ and $\mathbf{c}(1, 2, 3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

8. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

9. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi, \psi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = \mathbf{w} \times \mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v} \times \mathbf{u}$ where

1. $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$,
2. $\mathbf{w} = \mathbf{i} + \mathbf{k}$,
3. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$,
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

10. Prove the following identities:

1. the Jacobi identity,
2. the Lagrange identity,
3. the formula for the cross product of two cross products.

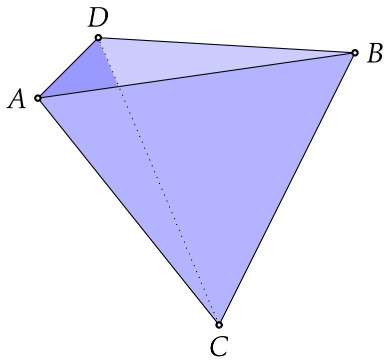
11. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}$, $\mathbf{b}=\mathbf{i}-\mathbf{k}$ and $\mathbf{c}=\mathbf{k}$. Determine if

1. $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a basis of $\mathbb{V}^3$,
2. if it is a basis, decide if it is left or right oriented.

12. The points $A(1,2-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

13. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $\mathbf{w}=\mathbf{k}$. Determine the matrix of the linear map $\phi:\mathbb{V}^3\to\mathbb{R}$ defined by $\phi(\mathbf{v})=[\mathbf{v},\mathbf{u},\mathbf{w}]$.

14. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.



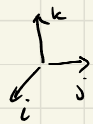
15. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D\in Oy$. Determine the coordinates of D.

16. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

1. the angles of $\mathbf{d}$ with $\mathbf{a}$ and with $\mathbf{b}$ are congruent,
2. $\mathbf{d}$ is orthogonal to $\mathbf{c}$,
3. $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{a},\mathbf{b},\mathbf{d})$ have the same orientation.

1. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

$$a \times b = (i + 2j - 2k) \times (7i + 4j + 6k)$$



$$= 7 \underbrace{i \times i}_{=0} + 4\, i \times j + 6\, i \times k +$$

$$+ 14\, j \times i + 8 \underbrace{j \times j}_{=0} + 12\, j \times k +$$

$$- 14\, k \times i - 8\, k \times j - 12 \underbrace{k \times k}_{=0}$$

$$= -10 \underbrace{i \times j}_{=k} + 20 \underbrace{i \times k}_{-j} + 20 \underbrace{j \times k}_{= i}$$

$$= 20i - 20j - 10k$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 7 & 4 & 6 \end{vmatrix} = 20i - 20j - 10k$$

2. With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 5i + j + 7k$$

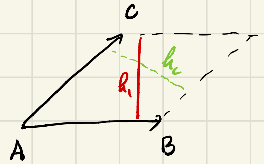
$(2a+b) \times b =$ you can calculate the components of $2a+b$ and a determinant, or

$$= 2a \times b + \underbrace{b \times b}_{=0} = 2a \times b = 10i + 2j + 14k.$$

$$(2a+b) \times (2a-b) = 4 \overbrace{a \times a}^{=0} - 2a \times b + 2 \underbrace{b \times a}_{-a \times b} - \overbrace{b \times b}^{=0} = -4\, a \times b = -20i - 4j - 28k$$

3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC} = (1.5, 2, 1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

Let a be the area of the parallelogram



$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = a = h_1 \cdot \|\overrightarrow{AB}\| = h_2 \cdot \|\overrightarrow{AC}\|$$

$$\Rightarrow h_1 = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{AB}\|} = \frac{\sqrt{673}}{2\sqrt{37}}$$

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \left\| \begin{vmatrix} i & j & k \\ 6 & 0 & 1 \\ 1.5 & 2 & 1 \end{vmatrix} \right\| = \| -2i - 4.5j + 12k \| = \sqrt{4 + \frac{81}{4} + 144}$$

$$= \frac{\sqrt{673}}{2}$$

$$\|\overrightarrow{AB}\| = \sqrt{37}$$

$$\|\overrightarrow{AC}\| = \sqrt{\frac{9}{4} + 4 + 1} = \frac{\sqrt{29}}{2}$$

$$\Rightarrow h_2 = \frac{\sqrt{673}}{\sqrt{29}}$$

4. Consider the vectors $\mathbf{a}(2,3,-1)$ and $\mathbf{b}(1,-1,3)$ with respect to an orthonormal basis.

1. Determine the vector subspace $\langle \mathbf{a}, \mathbf{b} \rangle^{\perp}$.
2. Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51$.

1. Method I take an arbitrary vector $v(x,y,z)$ and impose the conditions: $v \in \langle a,b \rangle^{\perp} \Leftrightarrow v \perp a$ and $v \perp b$

$$\Leftrightarrow v \cdot a = 0 \text{ and } v \cdot b = 0$$

$$\Leftrightarrow \begin{cases} 2x + 3y - z = 0 \\ x - y + 3z = 0 \end{cases}$$

this system allows us to express y and z in terms of x

then $v \in \langle a,b \rangle^{\perp} \Leftrightarrow v = v(x, y(x), z(x))$

Method II $\quad \langle a,b\rangle^{\perp} = \langle a\times b\rangle = \left\langle \begin{bmatrix} 8 \\ -7 \\ -5 \end{bmatrix} \right\rangle = \{(8t, -7t, -5t) : t\in\mathbb{R}\}$

$$a\times b = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 8i - 7j - 5k$$

2. $p \in \langle a,b\rangle^{\perp} \iff p = t\begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix}$ for some $t\in\mathbb{R}$

$p\cdot(2i-3j+4k) = 51 \iff (t8i - t7j - t5k)\cdot(2i-3j+4k) = 51$

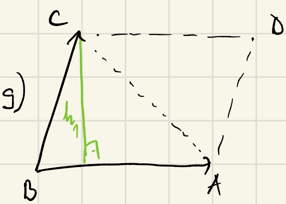
$\iff 16t + 21t - 20t = 51$

$\iff t = \frac{51}{17} = 3$

5. Consider the points $A(1,2,0)$, $B(3,0,-3)$ and $C(5,2,6)$ with respect to an orthonormal coordinate system.

1. Determine the area of the triangle ABC.
2. Determine the distance from C to AB.

$$\vec{BC}\times\vec{BA} = \begin{vmatrix} i & j & k \\ -2 & 2 & 3 \\ 2 & 2 & 9 \end{vmatrix}$$



$\vec{BC}(2,2,9)$

$\vec{BA}(-2,2,3)$

$= 12i - (-24)j + (-8)k$

$= 12i + 24j - 8k = 4(3i + 6j - 2k)$

$\Rightarrow$ area $\Delta ABC = \frac{\|\vec{BC}\times\vec{BA}\|}{2} = \frac{4\cdot\sqrt{49}}{2} = 14$

area $\Delta ABC = \frac{h_1\cdot|BA|}{2} = \frac{d(C,BA)\cdot\|\vec{BA}\|}{2}$

$\Rightarrow d(C,BA) = \frac{\|\vec{BC}\times\vec{BA}\|}{\|\vec{BA}\|}$

$\|\vec{BA}\| = \sqrt{4+4+9} = \sqrt{17}$

$\Rightarrow d(C,BA) = \frac{28}{\sqrt{17}}$

6. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}. \qquad (*)$$

and deduce the law of sines in a triangle.

- we have $u + v + w = 0$

$$\Rightarrow u = -v - w$$

$$\Rightarrow u\times v = (-v-w)\times v = -\underbrace{v\times v}_{=0} - w\times v = -w\times v = v\times w$$

$$\Rightarrow v = -u - w$$

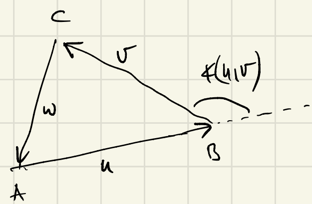
$$\Rightarrow u\times v = u\times(-u-w) = -u\times w = w\times u$$

- from $(*)$ we have $\|u\times v\| = \|v\times w\| = \|w\times u\|$

$$\Leftrightarrow \|u\|\cdot\|v\| \sin\measuredangle(u,v) = \|v\|\cdot\|w\|\cdot\sin\measuredangle(v,w) = \|w\|\cdot\|u\|\cdot\sin\measuredangle(w,u) \quad \Big| : \|u\|\cdot\|v\|\cdot\|w\|$$

$$\Leftrightarrow \frac{\sin\measuredangle(u,v)}{\|w\|} = \frac{\sin\measuredangle(v,w)}{\|u\|} = \frac{\sin\measuredangle(w,u)}{\|v\|}$$

$$\Leftrightarrow \frac{\sin\hat{B}}{\|w\|} = \frac{\sin\hat{C}}{\|u\|} = \frac{\sin\hat{A}}{\|v\|}$$



7. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

$$(a\times b)\times c = (a\cdot c)b - (b\cdot c)a = (2-6+3)b - (-3+2+6)a$$

$$= -\begin{bmatrix}-3\\1\\2\end{bmatrix} - 5\begin{bmatrix}2\\-3\\1\end{bmatrix} = \begin{bmatrix}-7\\14\\-3\end{bmatrix}$$

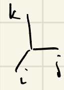
$$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c = -\begin{bmatrix}-3\\1\\2\end{bmatrix} + 7\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}10\\13\\19\end{bmatrix}$$

8. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

- The map is linear. During the lecture we showed Prop 2.5 from which it follows that

$$\phi(\alpha u + \beta w) = v \times (\alpha u + \beta w) = \alpha v \times u + \beta v \times w = \alpha \phi(u) + \beta \phi(w) \qquad \forall \alpha, \beta \in \mathbb{R} \quad \forall u, w \in \mathbb{V}^3$$

- let $v = v_1 i + v_2 j + v_3 k$ then $\phi(i) = (v_1 i + v_2 j + v_3 k) \times i$



$$= v_1 i \times i + v_2 j \times i + v_3 k \times i$$

$$= -v_2 k + v_3 j$$

similar $\phi(j) = v_1 k - v_3 i$

$\phi(k) = -v_1 j + v_2 i$

$\Rightarrow$ the matrix of ϕ with respect to the basis i, j, k is

$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

if we consider $\phi(w) = w \times v$ then, since $w \times v = -v \times w$,

the matrix of ϕ with respect to the basis i, j, k is

$$\begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & v_1 & 0 \end{bmatrix}$$

9. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi,\psi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = \mathbf{w}\times\mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v}\times\mathbf{u}$ where

1. $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$,
2. $\mathbf{w} = \mathbf{i} + \mathbf{k}$,
3. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$,
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

One can do a similar calculation as in the previous exercise

But since we already computed that matrix one could also just

use $\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$ for ϕ

so, for 1, the matrix of ϕ is $\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$

10. Prove the following identities:

1. the Jacobi identity,
2. the Lagrange identity,
3. the formula for the cross product of two cross products.

1.) $(a\times b)\times c + (b\times c)\times a + (c\times a)\times b = 0$?

$$(a\times b)\times c = \cancel{(a\cdot c) b} - \cancel{(b\cdot c)\cdot a}$$

$$(b\times c)\times a = \cancel{(b\cdot a) c} - \cancel{(c\cdot a)\cdot b}$$

$$(c\times a)\times b = \cancel{(c\cdot b) a} - \cancel{(a\cdot b)\cdot c} \qquad \oplus$$

$$(a\times b)\times c + (b\times c)\times a + (c\times a)\times b = 0$$

2.) $(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (b\cdot c)(a\cdot d)$

let $a=(a_1,a_2,a_3)$ $b=(b_1,b_2,b_3)$ $c=(c_1,c_2,c_3)$ $d=(d_1,d_2,d_3)$

then $a\times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2b_3-b_2a_3)+j(a_3b_1-a_1b_3)+k(a_1b_2-a_2b_1)$

and $c\times d = i(c_2d_3-d_2c_3)+j(c_3d_1-d_3c_1)+k(c_1d_2-d_1c_2)$

so

$$
\begin{aligned}
(a\times b)\cdot(c\times d) &= (a_2b_3-b_2a_3)(c_2d_3-d_2c_3)+(a_3b_1-b_3a_1)(c_3d_1-d_3c_1)+(a_1b_2-b_1a_2)(c_1d_2-d_1c_2) \\
&= a_2b_3c_2d_3 - a_2b_3d_2c_3 - b_2a_3c_2d_3 + b_2a_3d_2c_3 + \\
&\quad + a_3b_1c_3d_1 - a_3b_1d_3c_1 - b_3a_1c_3d_1 + b_3a_1d_3c_1 + \\
&\quad + a_1b_2c_1d_2 - a_1b_2d_1c_2 - b_1a_2c_1d_2 + b_1a_2d_1c_2
\end{aligned}
$$

$$
\begin{aligned}
(a\cdot c)(b\cdot d)-(b\cdot c)(a\cdot d) &= (a_1c_1+a_2c_2+a_3c_3)(b_1d_1+b_2d_2+b_3\cdot d_3) - \\
&\quad - (b_1c_1+b_2c_2+b_3c_3)(a_1d_1+a_2d_2+a_3d_3) \\
&= \cancel{a_1c_1b_1d_1} + a_1c_1b_2d_2 + a_1c_1b_3d_3 + \\
&\quad + a_2c_2b_1d_1 + \cancel{a_2c_2b_2d_2} + a_2c_2b_3d_3 \\
&\quad + a_3c_3b_1d_1 + a_3c_3b_2d_2 + \cancel{a_3c_3b_3d_3} \\
&\quad - \cancel{b_1c_1a_1d_1} - b_1c_1a_2d_2 - b_1c_1a_3d_3 \\
&\quad - b_2c_2a_1d_1 - \cancel{b_2c_2a_2d_2} - b_2c_2a_3d_3 \\
&\quad - b_3c_3a_1d_1 - b_3c_3a_2d_2 - \cancel{b_3c_3a_3d_3}
\end{aligned}
$$

one checks that these are equal

3) $(a\times b)\times(c\times d) = b\cdot[a,c,d] - a\cdot[b,c,d] = c\cdot[a,b,d] - d\cdot[a,b,c]$

one can do a calculation in coordinates as above

or

$$(a\times b)\times(c\times d) = \underbrace{(a\cdot(c\times d))}_{\substack{\parallel \\ (c\times d)\cdot a \\ \parallel \\ [c,d,a] \\ \parallel}} b - \underbrace{(b\cdot(c\times d))}_{\substack{\parallel \\ (c\times d)\cdot b \\ \parallel \\ [c,d,b] \\ \parallel}}\cdot a$$

$$= [a,c,d]\, b - [b,c,d]\, a$$

and

$$(a\times b)\times(c\times d) = [\underbrace{(a\times b)\cdot d}_{\parallel}]\cdot c - [\underbrace{(a\times b)\cdot c}_{\parallel}]\cdot d$$

$$= [a,b,d]\, c - [a,b,c]\, d$$

11. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}$, $\mathbf{b}=\mathbf{i}-\mathbf{k}$ and $\mathbf{c}=\mathbf{k}$. Determine if

1. $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a basis of $\mathbb{V}^3$,

2. if it is a basis, decide if it is left or right oriented.

$a=(1,1,0) \quad b=(1,0,-1) \quad c=(0,0,1)$

$$[a,b,c] = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$\neq 0 \Rightarrow a,b,c$ is a basis

$< 0 \Rightarrow a,b,c$ is left oriented

12. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

A, B, C, D coplanar $\iff \vec{AB}, \vec{AC}, \vec{AD}$ linearly dependent

$$\vec{AB} = \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}, \quad \vec{AC} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{AD} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\iff \underbrace{\begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix}}_{12-2-8-2=0} = 0 \quad \text{true}$$

13. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v}, \mathbf{u}, \mathbf{w}]$.

let $v = (v_1, v_2, v_3)$

$$\phi(v) = (v \times u) \cdot w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ -1 & 3 & 1 \end{vmatrix} \cdot k = \begin{vmatrix} v_2 & v_3 \\ 3 & 1 \end{vmatrix} \underbrace{i \cdot k}_{=0} - \begin{vmatrix} v_1 & v_3 \\ -1 & 1 \end{vmatrix} \underbrace{j \cdot k}_{=0} + \begin{vmatrix} v_1 & v_2 \\ -1 & 3 \end{vmatrix} k \cdot k$$

$$= 3v_1 + v_2$$

so $\phi(1,0,0) = 3$

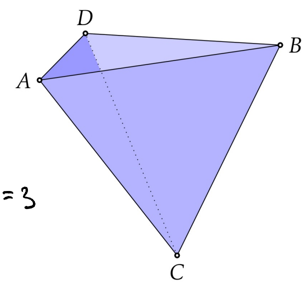
$\phi(0,1,0) = 1$

$\phi(0,0,1) = 0$

So the matrix is $[3 \quad 1 \quad 0]$

14. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.

$$\left| \frac{1}{6} [\vec{AB}, \vec{AC}, \vec{AD}] \right| = \left| \frac{1}{6} \begin{vmatrix} 3 & 6 & 3 \\ 1 & 3 & -2 \\ 2 & 2 & 2 \end{vmatrix} \right| = |-3| = 3$$



15. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D \in Oy$. Determine the coordinates of D.

$D \in Oy \Rightarrow D(0,\lambda,0)$

the volume of the tetrahedron is $V = \left| \frac{1}{6} \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & \lambda-1 & 1 \end{vmatrix} \right| = \frac{1}{6} \left| -2+8-8-4(\lambda-1) \right| = \frac{1}{6} \left| 2-4\lambda \right|$

so $5 = \frac{|1-2\lambda|}{3} \iff 15 = |1-2\lambda| \iff \begin{cases} \lambda = -7 \\ \lambda = 8 \end{cases}$

16. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

1. the angles of $\mathbf{d}$ with $\mathbf{a}$ and with $\mathbf{b}$ are congruent,
2. $\mathbf{d}$ is orthogonal to $\mathbf{c}$,
3. $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{a},\mathbf{b},\mathbf{d})$ have the same orientation.

Let $d = (d_1, d_2, d_3)$

$\measuredangle(d,a) = \measuredangle(d,b) \iff \cos\measuredangle(d,a) = \cos\measuredangle(d,b)$

$\iff \frac{a \cdot d}{\|a\| \cdot \cancel{\|d\|}} = \frac{b \cdot d}{\|a\| \cdot \cancel{\|d\|}}$

$\iff \frac{8d_1 + 4d_2 + d_3}{9} = \frac{2d_1 + 2d_2 + d_3}{3}$

$\iff 2d_1 - 2d_2 - 2d_3 = 0$

$\iff d_1 - d_2 - d_3 = 0$

$d \perp c \iff d \cdot c = 0 \iff d_1 + d_2 + d_3 = 0$

$\Big\} \Rightarrow \begin{matrix} d_1 = 0 \\ d_2 = -d_3 \end{matrix} \Big\} \Rightarrow d = (0,\lambda,-\lambda)$

$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 16+2+4-2-8-8 = 4 \Rightarrow a,b,c$ is right oriented

$$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 0 & \lambda & -\lambda \end{vmatrix} = -16\lambda + 2\lambda + 8\lambda - 8\lambda = -14\lambda$$ should be positive

$\Rightarrow$ all possible vectors are $d(0, \lambda, -\lambda)$ with $\lambda < 0$

All objects considered here are in the plane $\mathbb{E}^2$.

1. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,
2. ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,
3. ℓ contains the point $M(1,7)$ and is parallel to Oy,
4. ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2. For the lines ℓ in the previous exercise

1. give a Cartesian equation for ℓ,
2. describe all direction vectors for ℓ.

3. Determine a Cartesian equations for the line ℓ in the following cases:

1. ℓ has slope -5 and contains the point $A(1,-2)$,
2. ℓ has slope 8 and is at distance 2 from the origin,
3. ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,
4. ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

4. For the lines ℓ in the previous exercise

1. give parametric equations for ℓ,
2. describe all normal vectors for ℓ.

5. Consider a line ℓ. Show that

1. if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,
2. if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

6. Consider the points $A(1,1)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

7. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

8. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

1. the length of the altitude from A,
2. the line containing the altitude from A.

9. Determine the circumcenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

10. Determine the angle between the lines $\ell_1 : y = 2x + 1$ and $\ell_2 : y = -x + 2$.

11. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

12. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x + 4y - 20 = 0$. Determine the coordinates of A'.

13. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

1. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,
2. ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,
3. ℓ contains the point $M(1,7)$ and is parallel to Oy,
4. ℓ contains the points $M(2,4)$ and $N(2,-5)$.

1\. $\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad t \in \mathbb{R} \iff \begin{cases} x = 1 + 3t \\ y = 2 - t \end{cases} \quad t \in \mathbb{R}$

2\. $\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad s \in \mathbb{R} \iff \begin{cases} x = 4s \\ y = 5s \end{cases} \quad s \in \mathbb{R}$

3\. $\ell \parallel Oy \Rightarrow j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a direction vector for $\ell \Rightarrow \ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; \lambda \in \mathbb{R} \iff \ell : \begin{cases} x = 1 \\ y = 7 + \lambda \end{cases}$

4\. $\ell \ni M, N$, $M \neq N \Rightarrow \overrightarrow{MN}(0,-9)$ is a dir. vect for $\ell \Rightarrow \ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ -9 \end{bmatrix} \alpha \in \mathbb{R} \iff \ell : \begin{cases} x = 2 \\ y = 4 - 9\alpha \end{cases}$

2. For the lines ℓ in the previous exercise

1. give a Cartesian equation for ℓ,
2. describe all direction vectors for ℓ.

(1.1) $\ell : \dfrac{x-1}{3} = \dfrac{y-2}{-1} \; (=t) \iff \ell : -x - 3y + 1 + 6 = 0 \iff \ell : x + 3y - 7 = 0$

$a(3,-1)$ is a dir. vect. for $\ell \Rightarrow$ all dir. vectors for ℓ are

$\{ta : t \in \mathbb{R} \; t \neq 0\} = \{(3t, -t) : t \neq 0\}$

(1.2) $\ell : \dfrac{x}{4} = \dfrac{y}{5} \; (=s) \iff \ell : 5x - 4y = 0$, dir vectors for $\ell = \{(4\alpha, 5\alpha) : \alpha \in \mathbb{R} \; \alpha \neq 0\}$

(1.3) $\ell : \; x = 1$ $\quad$ dir. vect. for $\ell = \{\beta j : \beta \in \mathbb{R}, \beta \neq 0\} = \{(0,\beta) : \beta \in \mathbb{R} \; \beta \neq 0\}$

(1.4) $\ell : \; x = 2$ $\quad$ dir. vect. $\{(0,\gamma) : \gamma \in \mathbb{R} \; \gamma \neq 0\}$

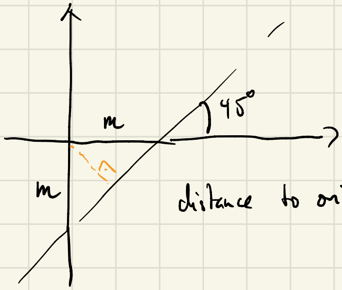
3. Determine a Cartesian equations for the line ℓ in the following cases:

1. ℓ has slope -5 and contains the point $A(1,-2)$,
2. ℓ has slope 1 and is at distance 2 from the origin,
3. ℓ contains the point $A(-2,3)$ and has an angle of 60° with the Ox-axis,
4. ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

1. $\ell: y = -5x + m \ni A(1,-2) \Rightarrow -2 = -5 + m \Rightarrow m = 3$

$\Rightarrow \ell: y = -5x + 3 \iff \ell: 5x + y - 3 = 0$

2.



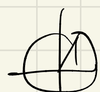
$\ell: \frac{x}{m} + \frac{y}{m} = 1$

distance to origin is $\frac{\sqrt{2}\cdot m}{2} = 2 \Rightarrow m = \frac{4}{\sqrt{2}}$

$\Rightarrow \ell: \sqrt{2}x + \sqrt{2}y - 4 = 0$

$\frac{\sqrt{3}}{2}$

$\frac{2}{2}$



3. $\ell: y - 3 = \underbrace{\tan 60^\circ}^{\sqrt{3}} (x+2) \iff \ell: \sqrt{3}x - y + 2\sqrt{3} - 3 = 0$

4.

n is a normal vector for ℓ

$\ell: 4(x-1) + 3(y-7) = 0$

$\iff \ell: 4x + 3y - 25 = 0$

4. For the lines ℓ in the previous exercise

1. give parametric equations for ℓ,
2. describe all normal vectors for ℓ.

(3.1) $\ell: \begin{cases} y = -5x + 3 \\ x = x \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \end{bmatrix} \quad t \in \mathbb{R}$

$n(5,1)$ is a normal vect. all others are $\alpha \cdot n$ $\alpha \in \mathbb{R}$, $\alpha \neq 0$

(3.2) $\ell: \begin{cases} x = \frac{4}{\sqrt{2}} - y \\ y = y \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \lambda \in \mathbb{R}$, $n(1,1)$ is a normal vect

$\Rightarrow$ all other normal vects are (β, β) $\beta \in \mathbb{R}$, $\beta \neq 0$

(3.3) $\ell: \begin{cases} x = x \\ y = \sqrt{3}x + 2\sqrt{3} - 3 \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sqrt{3}-3 \end{bmatrix} + s\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \quad s \in \mathbb{R}$

$n(\sqrt{3}, 1)$ is a normal vect for ℓ $\Rightarrow$ all other normal vectors for ℓ are $(\sqrt{3}t, t)$ $t \in \mathbb{R}$, $t \neq 0$

(3.4) $\ell: \begin{cases} x = \frac{25}{4} - \frac{3}{4}y \\ y = y \end{cases} \iff \ell: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25/4 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3/4 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$, $(4, 3)$ is a normal vector for ℓ

5. Consider a line ℓ. Show that

1. if $\mathbf{v}(v_1, v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2, -v_1)$ is a normal vector for ℓ,
2. if $\mathbf{n}(n_1, n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2, -n_1)$ is a direction vector for ℓ.

- If v is a dir. vect. for ℓ then n is a normal vector for ℓ if and only if

$v \perp n \iff v \cdot n = 0$ which is true in our case since

$$v(v_1, v_2) \cdot n(v_2, -v_1) = v_1 v_2 - v_2 v_1 = 0$$

- If n is a normal vector for ℓ then v is a direction vector for ℓ iff

$n \perp v \iff n \cdot v = 0$ again true if $v = v(n_2, -n_1)$

6. Consider the points $A(1, 2)$, $B(-2, 3)$ and $C(4, 7)$. Determine the medians of the triangle ABC.

$\Rightarrow$ give equations

Method I calculate midpoints of sides, then write equations for each median

Method II calculate centroid G,
$$G = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 7 \end{bmatrix}}{3} = \frac{\begin{bmatrix} 3 \\ 12 \end{bmatrix}}{3} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

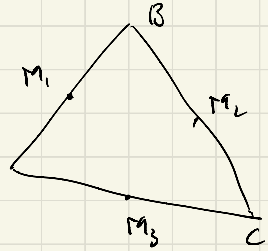
the medians are AG, BG, CG

$$AG: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\vec{AG}} \quad t \in \mathbb{R}$$

$$BG: \frac{x+2}{1+2} = \frac{y-3}{4-3}$$

CG: ....

7. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.



Method I

$AB \ni M_1$, $AB \parallel M_2M_3$

$\Downarrow$

$\overrightarrow{M_2M_3}(2,-5)$ is a direction vector for AB

$\Rightarrow$ $AB: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ or $AB: \dfrac{x-1}{2} = \dfrac{y-2}{-5}$

similar for BC and CD

Method II $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$

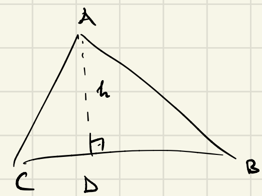
then $\begin{cases} 1 = \dfrac{x_A + x_B}{2} \\ 2 = \dfrac{y_A + y_B}{2} \\ 3 = \dfrac{x_B + x_C}{2} \\ 4 = \dfrac{y_B + y_C}{2} \\ 5 = \dfrac{x_C + x_A}{2} \\ -1 = \dfrac{y_C + y_A}{2} \end{cases}$

solving this system one obtains the coordinates of the vertices of the triangle ABC

so, one can write down the equations required

8. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

1. the length of the altitude from A,
2. the line containing the altitude from A.



1) h = length of the altitude

$\text{area}_{\Delta ABC} = \underbrace{\frac{1}{2}\|\overrightarrow{AC} \times \overrightarrow{AB}\|}_{\text{area of parallelogram spanned by } \overrightarrow{AC}, \overrightarrow{AB}} = \frac{1}{2} h \cdot \|\overrightarrow{CB}\|$

$\|\overrightarrow{AC} \times \overrightarrow{AB}\| = \left|\begin{vmatrix} 1 & 3 & 1 \\ -4 & 3 & 1 \\ 2 & 9 & 1 \end{vmatrix}\right| = |3 - 36 + 6 - 6 + 12 - 9| = 30$ and $\|\overrightarrow{CB}\| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$

since $A, B, C \in Oxy$

$\Rightarrow h = \dfrac{30}{6\sqrt{2}} = \dfrac{5}{\sqrt{2}}$

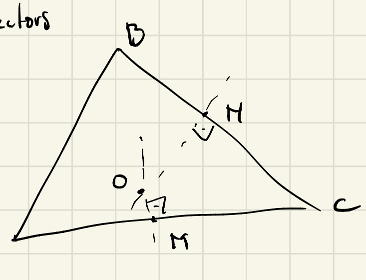
2.) $\ell \perp CB \Rightarrow \vec{CB}$ is a normal vector, $\vec{CB}(-6,6) \Rightarrow n(-1,1)$ is a normal vector

$\ell \ni A \Rightarrow \ell: \ -(x-1)+(y-3)=0$

Rem. one can intersect ℓ with BC to obtain D then $h=|AD|$

9. Determine the $\underbrace{\text{circumcenter}}$ of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

O = intersection of perpendicular bisectors



$O = MO \cap NO$ where M is the midpoint of AC
N —— 〃 —— BC

$M=\begin{bmatrix}3\\4\end{bmatrix}$, $N=\begin{bmatrix}4\\2\end{bmatrix}$

$MO \perp AC \Rightarrow \vec{AC}(4,4)$ is a normal vector for MO, and so is $(1,1)$

$NO \perp BC \Rightarrow \vec{BC}(2,8)$ —— 〃 —— NO, and so is $(1,4)$

$\Rightarrow MO: \ (x-3)+(y-4)=0 \iff MO: \ x+y-7=0$

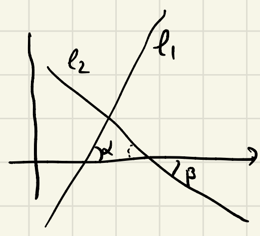
$NO: \ (x-4)+4(y-2)=0 \iff NO: \ x+4y-12=0$

$$\Rightarrow MO\cap NO: \begin{cases} x+y-7=0 \\ x+4y-12=0 \end{cases} \ominus$$

$$3y-5=0 \Rightarrow y=\frac{5}{3} \Rightarrow x=7-\frac{5}{3}=\frac{16}{3} \Rightarrow O\left(\frac{16}{3},\frac{5}{3}\right)$$

10. Determine the angle between the lines $\ell_1: y=2x+1$ and $\ell_2: y=-x+2$.

method I $\tan\alpha=2$, $\tan\beta=-1$



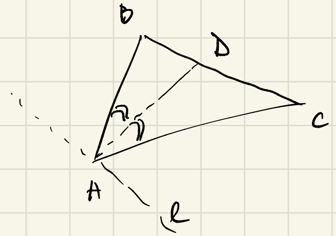
$\Rightarrow$ one angle is $\alpha+\beta \Rightarrow \tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{2-1}{1+2}=\dfrac{1}{3}$

$\simeq 18.43^\circ$

method II $\left.\begin{array}{l} n_1(2,-1) \text{ normal vector for } \ell_1 \\ n_2(1,1) \text{ —〃—} \end{array}\right\} \Rightarrow \cos\sphericalangle(n_1,n_2)=\dfrac{2-1}{\sqrt{5}\sqrt{2}}=\dfrac{1}{\sqrt{10}} \simeq 71.57^\circ$

11. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

a direction vector for the $\underset{\text{interior}}{\text{angle}}$ bisector is



$$v = \frac{\vec{AB}}{\|\vec{AB}\|} + \frac{\vec{AC}}{\|\vec{AC}\|} = \frac{1}{2\sqrt{4+9}}\begin{bmatrix}4\\6\end{bmatrix} + \frac{1}{\sqrt{9+4}}\begin{bmatrix}-3\\-2\end{bmatrix}$$

$$\vec{AB}(4,6),\ \vec{AC}(-3,-2)$$

$$\Rightarrow v = \frac{1}{\sqrt{13}}\begin{bmatrix}-1\\1\end{bmatrix} \quad \Rightarrow \tilde{v} = \begin{bmatrix}-1\\1\end{bmatrix} \text{ is also a direction vector for } AD$$

$$\Rightarrow AD: \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} + t\begin{bmatrix}-1\\1\end{bmatrix} \quad \text{or} \quad AD: \frac{x-1}{-1} = \frac{y+2}{1} \iff AD: x+y+1=0$$

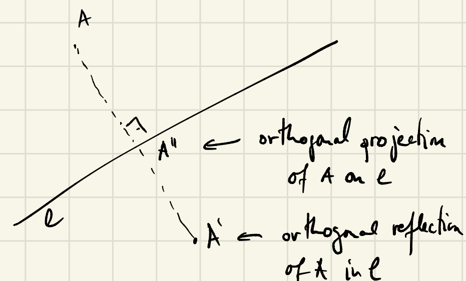
the exterior angle bisector is orthogonal to AD so v is a normal vector for this line

$$\Rightarrow \ell: -1(x-1) + 1(y+2) = 0 \iff \ell: -x+y+3=0$$

12. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x+4y-20=0$. Determine the coordinates of A'.

method I

$(3,4)$ is a normal vector for ℓ



$\Rightarrow (3,4)$ is a direction vector for AA'

$\Rightarrow (-4,3)$ is a normal vector for AA'

$$\Rightarrow AA': -4(x-10) + 3(y-10) = 0 \iff AA': -4x+3y+10=0$$

$$\Rightarrow A'': \begin{cases} -4x+3y+10=0 \\ 3x+4y-20=0 \end{cases}$$

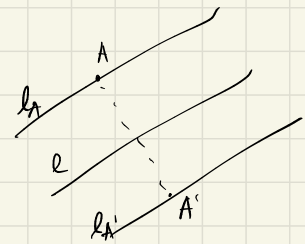
$$\underline{\quad} \oplus$$

$$-x+7y-10=0 \Rightarrow x = 7y-10 \quad \Big\} \Rightarrow 21y-30+4y-20=0 \Rightarrow 25y=50 \Rightarrow y=2 \Rightarrow x=4$$

$$\text{So } A'' = \begin{bmatrix}2\\4\end{bmatrix} \Rightarrow \vec{AA''} = \begin{bmatrix}-8\\-6\end{bmatrix} \Rightarrow \vec{AA'} = \begin{bmatrix}-16\\-12\end{bmatrix} \Rightarrow A' = A + \vec{AA'} = \begin{bmatrix}-6\\-2\end{bmatrix}$$

method II $\quad d(A,l) = d(A',l) \qquad A' = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\frac{50}{\sqrt{9+16}} = \frac{|3x+4y-20|}{\sqrt{9+16}} \iff |3x+4y-20| = 50$$



$l_A: \; 3x+4y=70$ $\qquad -3x-4y+20=50$

$l_{A'}: \; 3x+4y=-30$

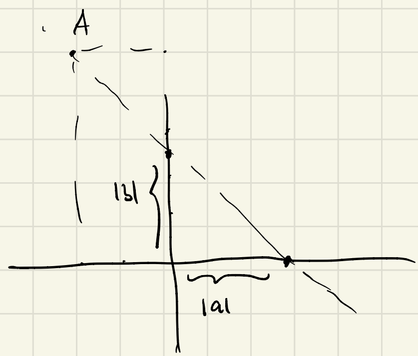
as above $\quad AA': \; -4x+3y+10=0$

$$\Rightarrow A': \begin{cases} -4x+3y+10=0 \\ 3x+4y=-30 \end{cases}$$ solving we obtain the coordinates of A'

13. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

Consider l with the eq.

$$l: \frac{x}{a} + \frac{y}{b} = 1$$



$\Rightarrow \; l \cap Ox = (a,0) \qquad l \cap Oy = (0,b)$

$\Rightarrow$ the segments are of length $|a|$ and $|b|$

Congruent segment $\Rightarrow |a| = |b|$ so $a = \pm b$

I $\quad a=b$ then $l: \frac{x}{a} + \frac{y}{a} = 1$

$l \ni A \Rightarrow \frac{-2}{a} + \frac{5}{a} = 1 \iff a=3$ so $l: \frac{x}{3} + \frac{y}{3} = 1$

$x+y=3$

II $\quad a=-b$ then $l: \frac{x}{a} - \frac{y}{a} = 1$

$l \ni A \Rightarrow \frac{-2}{a} - \frac{5}{a} = 1 \iff a=-7$ so $l: \frac{x}{-7} - \frac{y}{-7} = 1$

$x-y=-7$

Planes in the Euclidean space $\mathbb{E}^3$.

1. Determine parametric equations for the plane π in the following cases:

1. π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1)$ and $\mathbf{a}_2(0,3,1)$,
2. π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,
3. π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,
4. π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,
5. π contains the point $A(1,5,7)$ and the coordinate axis Ox.

2. Determine Cartesian equations for the plane π in the following cases:

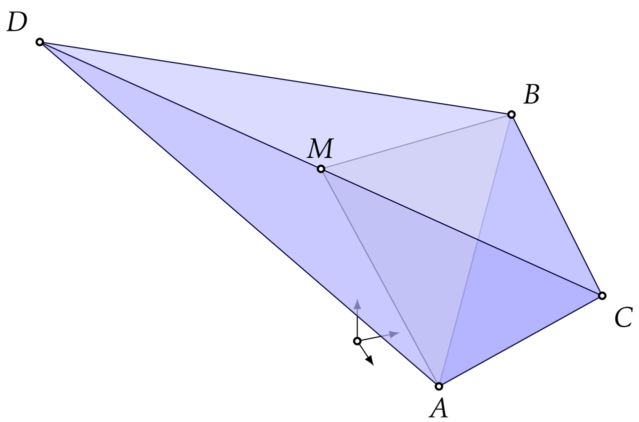
1. $\pi : x = 2+3u-4v,\ y = 4-v,\ z = 2+3u$;
2. $\pi : x = u+v,\ y = u-v,\ z = 5+6u-4v$.

3. Determine parametric equations for the plane π in the following cases:

1. $3x-6y+z=0$;
2. $2x-y-z-3=0$;

4. Determine an equation for each plane passing through $A(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

5. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.



6. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

7. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

8. Determine the distance between the planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

Lines in the Euclidean space $\mathbb{E}^3$.

9. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,
2. ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,
3. ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

10. Give Cartesian equations for the lines ℓ in the previous exercise.

11. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

12. Determine the relative positions of the lines $x=-3t, y=2+3t, z=1$ and $x=1+5s, y=1+13s, z=1+10s$.

13. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

14. Determine the parameter m for which the line $x=-1+3t, y=2+mt, z=-3-2t$ doesn't intersect the plane $x+3y+3z-2=0$.

15. Determine the values a and d for which the line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

16. Determine the values a and c for which the line $3x-2y+z+3=0 \cap 4x-3y+4z+1=0$ is perpendicular to the plane $ax+8y+cz+2=0$.

17. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.

18. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$.

19. Determine the orthogonal projection of the point $A(1,3,5)$ on the line $2x+y+z-1=0 \cap 3x+y+2z-3=0$.

1. Determine parametric equations for the plane π in the following cases:

1. π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1\overset{(3,-1,1)}{(\cancel{3,-1})}$ and $\mathbf{a}_2(0,3,1)$,
2. π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,
3. π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,
4. π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,
5. π contains the point $A(1,5,7)$ and the coordinate axis Ox.

1.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad t,s \in \mathbb{R}$

2.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \lambda,\mu \in \mathbb{R}$

3.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} + \mu\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad u,v \in \mathbb{R}$

4.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + t\overset{\overrightarrow{M_2M_1}}{\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}} + s\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \quad t,s \in \mathbb{R}$

5.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + \mu\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda,\mu \in \mathbb{R}$

2. Determine Cartesian equations for the plane π in the following cases:

1. $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u;$
2. $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v.$

1. $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + u\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + v\begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} \iff \begin{aligned} x &= 2 + 3u - 4v \\ y &= 4 - v \\ z &= 2 + 3u \end{aligned}$

$\pi: \begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0 \iff \pi: \cancel{3}(x-2) - \overset{4}{\cancel{12}}(y-4) - \cancel{3}(z-2) = 0$

$x - 2 - 4y + 16 - z + 2 = 0$

$\iff \pi: x - 4y - z + 16 = 0$

2. $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + v \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$

$$\pi: \begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0 \iff \pi: x(-4+6) - y(-4-6) + (z-5)(-2) = 0$$

$$\iff \pi: 2x + 10y - 2z + 10 = 0$$

3. Determine parametric equations for the plane π in the following cases:

1. $3x - 6y + z = 0$;
2. $2x - y - z - 3 = 0$;

1. $3x - 6y + z = 0 \iff \begin{cases} z = -3x + 6y \\ x = x \\ y = y \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$ $x, y \in \mathbb{R}$

2. $2x - y - z - 3 = 0 \iff \begin{cases} y = 2x - z - 3 \\ x = x \\ z = z \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$x, z \in \mathbb{R}$

4. Determine an equation for each plane passing through $P(3, 5, -7)$ and intersecting the coordinate axes in congruent segments.

$\pi: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\pi \cap Ox = (a, 0, 0) = A$$
$$\pi \cap Oy = (0, b, 0) = B$$
$$\pi \cap Oz = (0, 0, c) = C$$

"congruent segments" $\iff |OA| = |OB| = |OC| \iff |a| = |b| = |c| = m.$

$\iff a = \pm m, \ b = \pm m, \ c = \pm m$

I $\quad a = m, \ b = m, \ c = m$

$$\pi: \frac{x}{m} + \frac{y}{m} + \frac{z}{m} = 1 \iff \pi: x + y + z = m$$

$\Pi \ni P(3,5,-7) \Leftrightarrow 3+5-7=m$ So $m=1$

so $\Pi : x+y+z=1$

$\underline{\text{II}}$ $a=m$, $b=m$, $c=-m$

$\Pi : x+y-z=m \ni P(3,5-7) \Rightarrow 3+5+7=m$

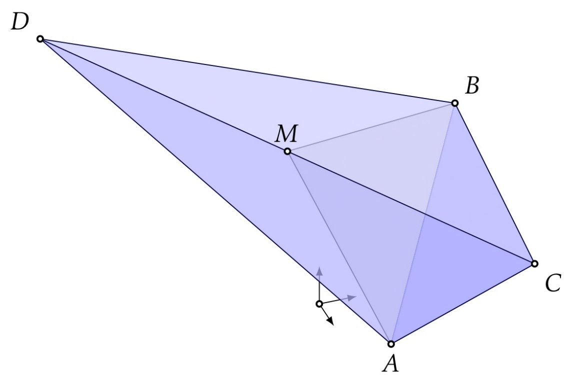
so $\Pi : x+y-z=15$

$\underline{\text{II}}$ $a=m$, $b=-m$, $c=m$

...

all the other cases are treated similarly

5. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.



$M = \left(\frac{6+0}{2}, \frac{3-7}{2}, \frac{4+8}{2}\right) = (3,-2,6)$ $\quad \vec{AB} = (-1,2,5)$ $\quad \vec{AM} = (1,-3,-6)$

$$ABM : \begin{vmatrix} x-2 & y-1 & z \\ -1 & 2 & 5 \\ 1 & -3 & -6 \end{vmatrix} = 0 \Leftrightarrow ABM : (x-2)\overbrace{(-12+15)}^{3} - (y-1)\overbrace{(6-5)}^{1} + z\overbrace{(3-2)}^{1} = 0$$

$$\Leftrightarrow ABM : 3x - y + z - 5 = 0$$

6. Show that a parallelepiped with faces in the planes $\underbrace{2x+y-2z+6=0}_{\Pi_1}$, $\underbrace{2x-2y+z-8=0}_{\Pi_2}$ and $\underbrace{x+2y+2z+1=0}_{\Pi_3}$ is rectangular.

$\Downarrow$

if the angles between these planes are right angles

$n_1(2,1,-2)$ is a normal vector for Π_1

$n_2(2,-2,1)$ —— $\shortparallel$ —— Π_2

$n_3(1,2,2)$ —— $\shortparallel$ —— Π_3

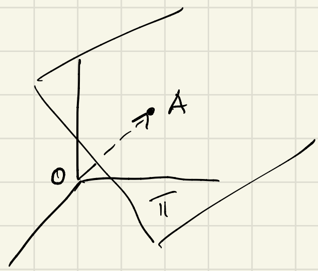
$\Pi_1 \perp \Pi_2 \iff n_1 \perp n_2 \iff n_1 \cdot n_2 = 0$, and $n_1 \cdot n_2 = 4-2-2=0$ so $\Pi_1 \perp \Pi_2$

$n_1 \cdot n_3 = 2+2-4=0 \Rightarrow \Pi_1 \perp \Pi_3$

$n_2 \cdot n_3 = 2-4+2=0 \Rightarrow \Pi_2 \perp \Pi_3$

7. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

A = orthogonal projection of O on Π



$\Rightarrow OA \perp \Pi$

$\Rightarrow \vec{OA} \perp \Pi$

$\Rightarrow \vec{OA}$ is a normal vector for Π

$\Rightarrow \Pi: 1\cdot(x-1)+(-1)(y-(-1))+3(z-3)=0$ (since $A \in \Pi$)

so $\Pi: x-y+3z-11=0$

8. Determine the distance between the planes $\overbrace{x-2y-2z+7=0}^{\Pi_1}$ and $\overbrace{2x-4y-4z+17=0}^{\Pi_2}$.

$n_1(1,-2,-2)$ is a normal vector for Π_1

$n_2(2,-4,-4)$ is a normal vector of Π_2

$2n_1 = n_2$, they are proportional

$\Rightarrow$ Π_1 and Π_2 have the same normal vectors $\Rightarrow$ $\Pi_1 \parallel \Pi_2$

(Another way to see this is by noticing that $\operatorname{rank}\begin{bmatrix} 1 & -2 & -2 \\ 2 & -4 & -4 \end{bmatrix} = 1$)

So $\Pi_1 \parallel \Pi_2$

$$\Rightarrow d(\Pi_1, \Pi_2) = d(\Pi_1, P) \quad \forall\, P \in \Pi_2$$

choose $P = \left(-\frac{17}{2}, 0, 0\right) \in \Pi_2$

then
$$d(\Pi_1, P) = \frac{\left|-\frac{17}{2}+7\right|}{\sqrt{1+4+4}} = \frac{\frac{3}{2}}{\sqrt{9}} = \frac{1}{2}$$

9. Determine parametric equations for the line ℓ in the following cases:

1. ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,
2. ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,
3. ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

1. $$\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \quad t \in \mathbb{R}$$

2. $$\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad s \in \mathbb{R}$$

3. $$\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{\overrightarrow{M_1M_2}} \quad \lambda \in \mathbb{R}$$

10. Give Cartesian equations for the lines ℓ in the previous exercise.

1. $$\ell: \frac{x-2}{3} = \frac{y}{-2} = \frac{z-3}{-2} \iff \ell: \begin{cases} -2x+4 = 3y \\ y = z-3 \end{cases} \iff \ell: \begin{cases} 2x+3y-4=0 \\ y-z+3=0 \end{cases}$$

2. $$\ell: \begin{cases} x = 1 \\ y = 2 \\ z = 3+s \end{cases} \iff \ell: \begin{cases} x = 1 \\ y = 2 \end{cases}$$

3. $$\left.\ell: \begin{cases} x = 1+3\lambda \\ y = 2+2\lambda \\ z = 3+\lambda \quad \Rightarrow \lambda = z-3 \end{cases}\right\} \Rightarrow \ell \begin{cases} x = 1+3z-9 \\ y = 2+2z-6 \end{cases} \iff \ell: \begin{cases} x-3z+8=0 \\ y-2z+4=0 \end{cases}$$

11. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

method I

$$\ell : \begin{cases} x+y+2z-3=0 \\ x-y+z-1=0 \end{cases} \quad \begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 3 & -4 \\ 0 & -2 & -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x = -3z+4 \\ -2y = z-2 \end{cases} \Rightarrow \begin{cases} x = -\frac{3}{2}z+2 \\ y = -\frac{1}{2}z+1 \\ z = z \end{cases} \Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$z \in \mathbb{R}$

method II

$n_1\,(1,1,2)$ is a normal vector for $\Pi_1 : x+y+2z-3=0$

$n_2\,(1,-1,1)$ —— $\|$ —— $\Pi_2 : x-y+z-1=0$

$\Rightarrow n_1 \times n_2$ is a direction vector for ℓ $\quad n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 3i + j - 2z$

choose a point P in $\ell \in \Pi_1 \cap \Pi_1$, i.e a point P such that the coordinates of P satisfy the equations of Π_1 and Π_2

for example $P = (2,1,0)$

then

$$\ell : \begin{bmatrix} x \\ y \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{P} + \lambda \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \lambda \in \mathbb{R}$$

12. Determine the relative positions of the lines $x=-3t, y=2+3t, z=1$ and $x=1+5s, y=1+13s, z=1+10s$.

$$l_1: \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{P_1} + t \underbrace{\begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}}_{v_1 \text{ a direction vector for } l_1} \quad t \in \mathbb{R} \qquad l_2: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{P_2} + s \underbrace{\begin{bmatrix} 5 \\ 13 \\ 10 \end{bmatrix}}_{v_2 \text{ a direction vector for } l_2} \quad s \in \mathbb{R}$$

v_1 and v_2 are linearly independent (they are not proportional)

$\Rightarrow$ l_1 is not parallel to l_2

$\Rightarrow$ either l_1 and l_2 intersect or they are skew

$l_1 \cap l_2 \neq \emptyset \iff$ the two lines are coplanar

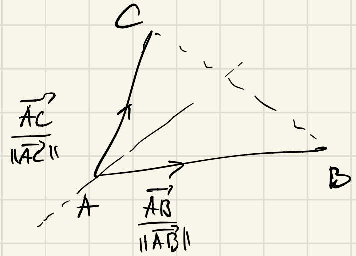
$\iff$ $\overrightarrow{P_1P_2}, v_1, v_3$ are linearly dependent

$$\iff \underbrace{\begin{vmatrix} 1 & -1 & 0 \\ -3 & 3 & 0 \\ 5 & 13 & 10 \end{vmatrix}}_{= \, 10 \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} = 0} = 0$$

so, yes the two lines intersect.

13. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

a direction vector for the angle bisector ℓ is $\dfrac{\vec{AB}}{\|\vec{AB}\|} + \dfrac{\vec{AC}}{\|\vec{AC}\|}$



$$\vec{AB} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \frac{\vec{AB}}{\|\vec{AB}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \frac{\vec{AC}}{\|\vec{AB}\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + \lambda \begin{bmatrix} 1 + 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

14. Determine the parameter m for which the line $x = -1 + 3t, y = 2 + mt, z = -3 - 2t$ doesn't intersect the plane $x + 3y + 3z - 2 = 0$.

$$\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + t \underbrace{\begin{bmatrix} 3 \\ m \\ -2 \end{bmatrix}}_{\substack{\parallel \\ v \text{ a dir. vector} \\ \text{for } \ell}}$$

$\pi: x + 3y + 3z - 2 = 0$

$\hookrightarrow n(1,3,3)$ is a normal vector for π

$$\ell \cap \pi = \emptyset \iff \ell \parallel \pi \iff v \perp n \iff v \cdot n = 0$$

$$v \cdot n = 3 + 3m - 6 = 0 \iff m = 1$$

15. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax + y - 2z + d = 0$.

$$l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}}_{P} + t \underbrace{\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}}_{v}$$

$\Pi: ax + y + 2z + d = 0$

$\hookrightarrow n(a,1,2)$ is a nomal vector for Π

$l \subseteq \Pi \iff l \parallel \Pi$ and any point in l, for example $P = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ belongs to Π

$l \parallel \Pi \iff n \cdot v = 0 \iff 3a + 2 - 4 = 0 \iff a = \frac{2}{3}$

$P \in \Pi \iff \frac{2}{3} \cdot 2 - 1 + 6 + d = 0 \iff d = \frac{-19}{3}$

16. Determine the values a and c for which the line $3x - 2y + z + 3 = 0 \cap 4x - 3y + 4z + 1 = 0$ is perpendicular to the plane $ax + 8y + cz + 2 = 0$.

$$l: \begin{cases} 3x - 2y + z + 3 = 0 & \leftarrow \text{eq. of plane } \Pi_1 \text{ with normal. vect } n_1(3,-2,1) \\ 4x - 3y + 4z + 1 = 0 & \leftarrow \text{—"— } \Pi_2 \text{ —"— } n_2(4,-3,4) \end{cases}$$

$\Rightarrow$ a direction vector for l is $n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 4 & -3 & 4 \end{vmatrix} = -5i - 8j - k$

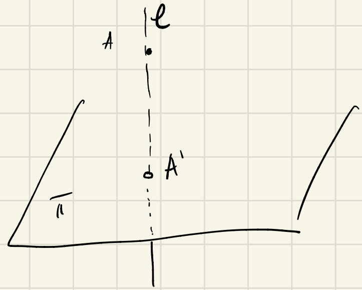
$\Pi: ax + 8y + cz + 2 = 0 \Rightarrow n(a,8,c)$ is a nomal vector for Π

$l \perp \Pi \iff v \parallel n \iff \frac{a}{-5} = \frac{8}{-8} = \frac{c}{-1} \Rightarrow a = 5$ and $c = 1$

$\Rightarrow \Pi: 5x + 8y + z + 2 = 0$

17. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $\underbrace{x+4y-3z+7=0}_{\pi}$.

is the intersection of
the line ℓ, passing through A and
orthogonal to π, with the plane π



$$Pr^{\perp}_{\pi}(A) = A'$$

$\ell \perp \pi \Rightarrow$ the normal vectors of π are direction vectors of ℓ

from the eq. of π we see that $n(1,4,-3)$ is a normal vector for ℓ

$$\Rightarrow \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}}_{A} + t \underbrace{\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}}_{n} \quad t \in \mathbb{R} \quad \Leftrightarrow \quad \begin{cases} x = 2+t \\ y = 11+4t \\ z = -5-3t \end{cases}$$

$$\ell \cap \pi : \quad (2+t) + 4(11+4t) - 3(-5-3t) + 7 = 0$$

$$2 + t + 44 + 16t + 15 + 9t + 7 = 0$$

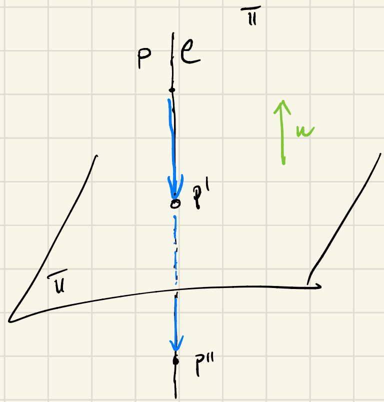
$$26t = -68 \Leftrightarrow t = -\frac{34}{13}$$

$$\Rightarrow \ell \cap \pi = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix} - \frac{34}{13} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -8 \\ 7 \\ 37 \end{bmatrix}$$

18. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $\underbrace{2x-3y+z-4=0}_{\pi}$.

of the point P is the point P'' such that

$Pr^{\perp}_{\pi}(P) = P'$ is the midpoint of the segment $[PP'']$



we calculate P' first.

$P' = l \cap \pi$ where $l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

↳ normal vector of π is direction vector for l

so $l: \begin{cases} x = 6+2t \\ y = -5-3t \\ z = 5+t \end{cases}$

$\Rightarrow l \cap \pi: \; 2(6+2t) - 3(-5-3t) + (5+t) - 4 = 0$

$12 + 4t + 15 + 9t + 5 + t - 4 = 0$

$14t = -28 \Rightarrow t = -2 \quad \Rightarrow \quad l \cap \pi = \begin{bmatrix} 6 \\ -5 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Now we can obtain $P'' = Ref^{\perp}_{\pi}(P)$ as $P'' = P + 2\overrightarrow{PP'}$

or by using the fact that P' is the midpoint of PP''

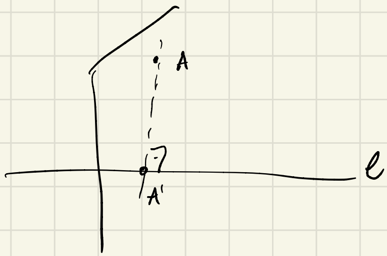
$\Rightarrow P'' = (x'', y'', z'') \quad \begin{cases} 2 = \dfrac{x''+6}{2} \\ 1 = \dfrac{y''-5}{2} \\ 3 = \dfrac{z''+5}{2} \end{cases} \Rightarrow P'' = \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \dots$

19. Determine the orthogonal projection of the point $A(1,3,5)$ on the line $2x+y+z-1=0 \cap 3x+y+2z-3=0$.

$\downarrow$ denote this line by ℓ

Let A' be the orthogonal proj of A on ℓ

$A' = \ell \cap \Pi$ where Π is the plane containing A and orthogonal to ℓ



Changing reference frames.

1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 0.1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

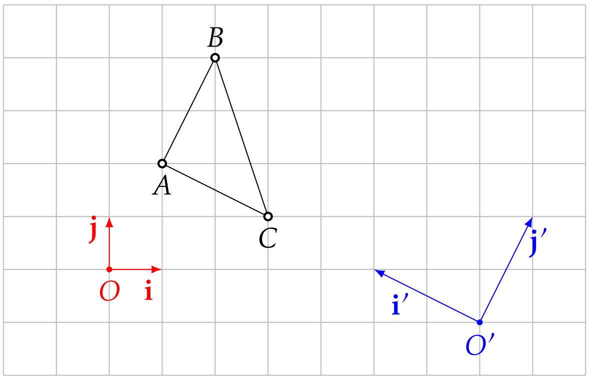


Figure 0.1: Coordinate systems 2D.

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

3. Consider the tetrahedron $ABCD$ (see Fig. 0.2) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

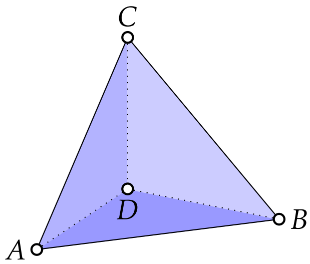


Figure 0.2: Tetrahedron

4. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 0.3) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

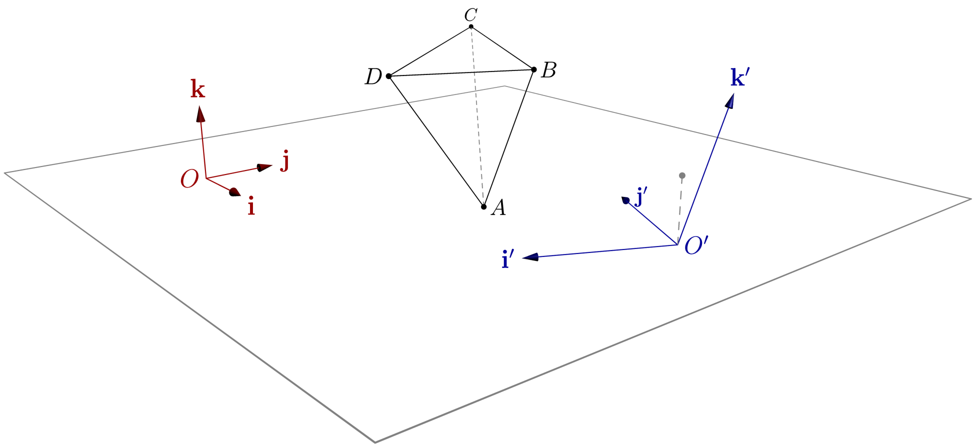


Figure 0.3: Coordinate systems 3D.

5. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

Projections and reflections on/in hyperplanes.

6. Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

7. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

8. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

9. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

10. Give Cartesian equations for the line passing through the point $M(1,0,7)$, parallel to the plane $\pi : 3x - y + 2z - 15 = 0$ and intersecting the line

$$\ell : \frac{x-1}{4} = \frac{y-3}{2} = \frac{z}{1}.$$

11. In $\mathbb{E}^3$, show that the orthogonal reflection $\mathrm{Ref}^{\perp}_{\pi}(x)$ in the plane $\pi : \langle n, x \rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2} n$.

12. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$

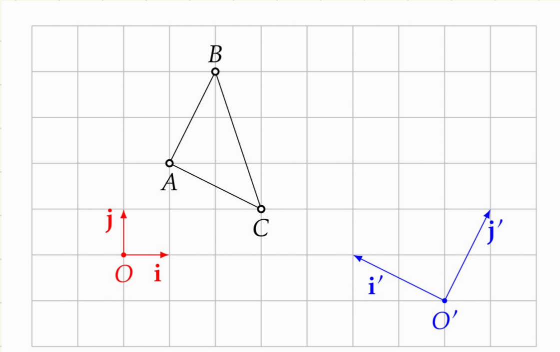
1. We consider two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ (see Fig. 1) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.



• Let $v = (i, j)$ and $w = (i', j')$

• The base change matrix from K' to K is the matrix $M_{v,w}(Id)$

$$M_{K,K'} = M_{v,w} = \begin{array}{c} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \\ \;[i']_K \;\; [j']_K \end{array}$$

are given

• In order to change coordinates, we have $[A]_{\mathcal{K}'} = \mathrm{M}_{\mathcal{K}',\mathcal{K}}([A]_{\mathcal{K}} - [O']_{\mathcal{K}}) = \mathrm{M}_{\mathcal{K}',\mathcal{K}}[A]_{\mathcal{K}} + [O]_{\mathcal{K}'}$

• $M_{K',K} = M_{K,K'}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \cdot \frac{1}{-5}$

So $[A]_{K'} = \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \frac{1}{-5} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} \frac{1}{-5} \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -15 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$[B]_{K'} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$[C]_{K'} = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

• In order to change coordinates backwards we use $[A]_K = M_{K,K'}([A]_{K'} - [O]_{K'}) = M_{K,K'}[A]_{K'} + [O']_K$

Notice that $[A]_K = M_{K,K'}(M_{K',K}[A]_K + [O]_{K'}) + [O']_K$
$= [A]_K + M_{K,K'}[O]_{K'} + [O']_K = [A]_K$

2. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

- In K we have $\overrightarrow{AB} = \begin{bmatrix}1\\2\end{bmatrix} \Rightarrow AB_K : \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} + t\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{=[v]_K}$ $\quad t \in \mathbb{R} \qquad (*)$

In K': $AB_{K'}: \begin{bmatrix}x'\\y'\end{bmatrix} = M_{K',K}\left(\begin{bmatrix}1\\2\end{bmatrix} + t\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}7\\-1\end{bmatrix}\right)$

$$= \frac{1}{5}\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}\left(\begin{bmatrix}t\\2t\end{bmatrix} + \begin{bmatrix}-6\\3\end{bmatrix}\right)$$

$$= \frac{1}{5}\left(\begin{bmatrix}0\\5t\end{bmatrix} + \begin{bmatrix}15\\0\end{bmatrix}\right) = \begin{bmatrix}0\\t\end{bmatrix} + \begin{bmatrix}3\\0\end{bmatrix}$$

$$AB_{K'}: \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} + t\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{=[v]_{K'}}$$

Notice that if $\ell: \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x_A\\y_A\end{bmatrix} + t\begin{bmatrix}v_x\\v_y\end{bmatrix}$ in K then

in K' we have $\ell: \begin{bmatrix}x'\\y'\end{bmatrix} = M_{K',K}\left(\begin{bmatrix}x_A\\y_A\end{bmatrix} + t\begin{bmatrix}v_x\\v_y\end{bmatrix} - [O']_K\right)$

$$= \underbrace{M_{K',K}\left(\begin{bmatrix}x_A\\y_A\end{bmatrix} - [O']_K\right)}_{=[A]_{K'}} + t\underbrace{M_{K',K}\begin{bmatrix}v_x\\v_y\end{bmatrix}}_{=[v]_{K'}}$$

- $(*) \Rightarrow \begin{cases} x = 1+t \\ y = 2+2t\end{cases} \Rightarrow t = x-1 = \frac{y-2}{2} \Rightarrow \ell: 2x - \cancel{2} - y + \cancel{2} = 0$

$\ell: 2x - y = 0$

So, in K we have $\ell: \begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$

$\Rightarrow$ in K' we have $\ell: \begin{bmatrix}2 & -1\end{bmatrix}\left(M_{K,K'}\begin{bmatrix}x'\\y'\end{bmatrix} + [O']_K\right)$

$$\Leftrightarrow \begin{bmatrix}2 & -1\end{bmatrix}\left(\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix} + \begin{bmatrix}7\\-1\end{bmatrix}\right) = 0$$

$$\Leftrightarrow \begin{bmatrix}-5, & 0\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix} + 15 = 0$$

$$\Leftrightarrow -5x' + 15 = 0$$

$$\ell_{K'}: x' - 3 = 0$$

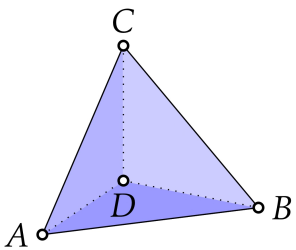
AC and BC are treated similarly

3. Consider the tetrahedron $ABCD$ (see Fig. 3) and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

1. the coordinates of the vertices of the tetrahedron in the three coordinate systems,
2. the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,
3. the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.



a.) $[A]_{K_A} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $[B]_{K_A} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $[C]_{K_A} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $D_{K_A} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$[A]_{K'_A} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $[B]_{K'_A} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $[C]_{K'_A} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $[D]_{K'_A} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$[A]_{K_B} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $[B]_{K_B} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $[C]_{K_B} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $[D]_{K_B} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

b.) Let $M = M_{K'_A, K_A}$ $\quad M[A]_{K_A} = [A]_{K'_A}$

$M\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\quad M\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad M\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

basis vectors in K'_A relative to K_A

$$M\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \Leftrightarrow M = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

c.) $\overrightarrow{BA} = -\overrightarrow{AB}$ $\quad [\overrightarrow{BA}]_{K_A} = \begin{bmatrix}-1\\0\\0\end{bmatrix}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$ $\quad [\overrightarrow{BC}]_{K_A} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$ $\quad [\overrightarrow{BD}]_{K_A} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

So $M_{K_A,K_B} = \begin{bmatrix}-1&-1&-1\\0&1&0\\0&0&1\end{bmatrix}$ is the base change matrix from K_B to K_A

4. We consider the coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ (see Fig. 4) where

$$[O']_{\mathcal{K}} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}, \quad [\mathbf{i}']_{\mathcal{K}} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad [\mathbf{j}']_{\mathcal{K}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{k}']_{\mathcal{K}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

$$M_{KK'} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow M_{K'K} = M_{KK'}^{-1} = \begin{bmatrix} 2 & 4 & 0 \\ 4 & -2 & 0 \\ -2 & 1 & -5 \end{bmatrix}^T \cdot \frac{1}{-10} = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\det M_{KK'} = -2 - 8$$

$$\Rightarrow [A]_{K'} = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[B]_{K'} = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

similarly for $[C]_{K'}$ and $[D]_{K'}$

Notice that $[A]_K = M_{K,K'}\left(M_{K'K}[A]_K + [O]_{K'}\right) + [O']_K = [A]_K + M_{K,K'}[O]_{K'} + [O']_K = [A]_K$

$$M_{KK'}[O]_{K'} = [O']_K$$

5. With the assumptions in the previous exercise, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

line AB in K
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{A}{\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}} + t \underset{\overrightarrow{AB}}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} \qquad [O']_K$$

$\Rightarrow$ line AB in K'
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M_{K'K}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}\right)$$
$$= M_{K'K}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}\right) + t\, M_{K'K}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
$$= \underset{[A]_{K'}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} + t\, M_{K'K}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \ldots\ldots$$

If the plane ACD in K is $ax + by + cz + d = 0 \iff [a\ b\ c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -d$

Then the plane ACD in K' is $[a\ b\ c]\left(M_{K,K'}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \underset{}{\overset{[O']_K}{\begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}}}\right) = -d$

6. Consider $\mathbf{v}(2,1,1) \in \mathbb{V}^3$ and $Q(2,2,2) \in \mathbb{E}^3$.

1. Give the matrix form for the parallel projection on the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.
2. Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ along the line $Q + \langle \mathbf{v} \rangle$.

- Let $\ell = Q + \langle v \rangle$

Consider a point $P(x_0, y_0, z_0) \in \mathbb{A}^3(\mathbb{R})$

the line containing P and parallel to ℓ is $\ell_p : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 + 2t \\ y_0 + t \\ z_0 + t \end{bmatrix}$

$\ell_p \cap \pi : \quad z_0 + t = 0 \iff t = -z_0$ so $\ell \cap \pi = \begin{bmatrix} x_0 - 2z_0 \\ y_0 - z_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

So, $\quad Pr_{\pi,\ell} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- For the reflection let $P'(x_1, y_1, z_1)$ be the reflection of P in π along ℓ

Then $\quad \frac{1}{2}\left(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) = \begin{bmatrix} x_0 - 2z_0 \\ y_0 - z_0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 - 4z_0 \\ y_0 - 2z_0 \\ -z_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

Rem clearly, one can also apply the general formulas that we deduced
$\hookrightarrow$ do this and check that you get the same answer.

7. Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

- We show this for projections in hyperplanes (the other cases are similar)
- the vector forms don't change:

$$\mathrm{Pr}_{H,\mathbf{v}}(P) = P - \frac{\varphi(P)}{\mathrm{lin}\,\varphi(\mathbf{v})}\mathbf{v}.$$

but H is in this case a plane so it has an equation of the form

$H: ax+by+cz+d=0$

and $v = v(v_x, v_y, v_z)$

- For the matrix form we have

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

which in our case becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{av_x + bv_y + cv_z}\begin{pmatrix} v_x a & v_x b & v_x c \\ v_y a & v_y b & v_y c \\ v_z a & v_z b & v_z c \end{pmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x + bv_y + cv_z}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$= \left[\frac{1}{av_x + bv_y + cv_z}\begin{bmatrix} bv_y + cv_z & v_x b & v_x c \\ v_y a & av_x + cv_z & v_y c \\ v_z a & v_z b & av_x + bv_y \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x + bv_y + cv_z}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

8. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi : ax + by + c = 0, \quad \ell : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

As in the previous exercise we consider $\mathrm{Pr}_{\pi,\ell}$ since $\mathrm{Ref}_{\pi,\ell}$ is similar

$$\mathrm{Pr}_{\pi,\ell}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{av_1+bv_2}\begin{bmatrix} bv_2 & cv_2 \\ bv_1 & cv_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \frac{d}{av_1+bv_1}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

9. Let H be a hyperplane and let $\mathbf{v}$ be a vector. Use the deduced compact matrix forms to show that

1. $\mathrm{Pr}_{H,\mathbf{v}} \circ \mathrm{Pr}_{H,\mathbf{v}} = \mathrm{Pr}_{H,\mathbf{v}}$ and
2. $\mathrm{Ref}_{H,\mathbf{v}} \circ \mathrm{Ref}_{H,\mathbf{v}} = \mathrm{Id}$.

Rem from the definition of these maps it should be clear that the indicated relations are true

1. $$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$

$$\text{So } \left[\mathrm{Pr}_{H,v} \circ \mathrm{Pr}_{H,v}(P)\right]_K = \left[\mathrm{Pr}_{H,v}\left(\mathrm{Pr}_{H,v}([P]_K)\right)\right]$$

$$= \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\mathrm{Pr}_{H,v}([P]_K) - \frac{a_{n+1}}{v^t\cdot a}v$$

$$= \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\left[\left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)[P]_K - \frac{a_{n+1}}{v^t\cdot a}v\right] - \frac{a_{n+1}}{v^t\cdot a}v$$

$$(*) = \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)^2[P]_K - \left(\mathrm{Id}_n - \frac{v\cdot a^t}{v^t\cdot a}\right)\frac{a_{n+1}}{v^t\cdot a}v - \frac{a_{n+1}}{v^t\cdot a}v$$

$$- \frac{a_{n+1}}{(v^t\cdot a)^2}\left(2v^t\cdot a\cdot \mathrm{Id}_n - v\cdot a^t\right)\cdot v$$

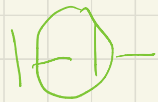
$$\underbrace{2v^t\cdot a\cdot v - \underbrace{v\cdot a^t\cdot v}_{= v^t\cdot a\cdot v}}_{v^t\cdot a\cdot v}$$

$$= -\frac{a_{n+1}}{v^t\cdot a}v$$

so

$$(*) = \underbrace{\left(Id_n - \frac{v \cdot a^t}{v^t \cdot a} \right)^2}_{} [p]_k - \frac{a_{n+1}}{v^t \cdot a} v$$

$v^t a \cdot v \cdot a^t$



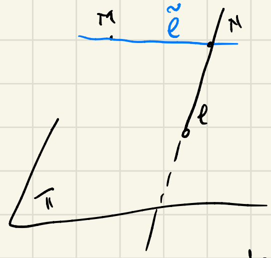
$$\underbrace{Id_n - 2\frac{v \cdot a^t}{v^t \cdot a} + \underbrace{\frac{v \cdot a^t}{v^t \cdot a} \cdot \frac{v \cdot a^t}{v^t \cdot a}}_{= \frac{\cancel{(v^t a)} (v \cdot a^t)}{\cancel{(v^t a)} \cdot (v^t \cdot a)}}}_{= Id_n - \frac{v \cdot a^t}{v^t \cdot a}}$$

so $(*) = \left(Id - \frac{v \cdot a^t}{v^t \cdot a} \right) [p]_k - \frac{a_{n+1}}{v^t a} v = [Pr_{H,v}(p)]_k$ $\square$

for 2. the calculation is similar

10. Give Cartesian equations for the line $\tilde{\ell}$ passing through the point $M(1,0,7)$, parallel to the plane $\pi: 3x-y+2z-15=0$ and intersecting the line

$$\ell: \frac{x-1}{4}=\frac{y-3}{2}=\frac{z}{1}.$$



$\boxed{\text{Method 1}}$ Determine the point $N=\ell\cap\tilde{\ell}$ using a projection
$N=\mathrm{Pr}_{\ell,\tilde{\pi}}(M)$

$$\mathrm{Pr}_{\ell,\tilde{\pi}}(p)=p'=q-\frac{\varphi(q)}{(\mathrm{lin}\,\varphi)(v)}v \quad \text{where}$$

- q is a point on ℓ
- v is a direction vector of ℓ
- $\varphi^{-1}(0)$ is the plane parallel to π and containing p

in our case $\ell: \begin{pmatrix}x\\y\\z\end{pmatrix}=\overset{q}{\begin{pmatrix}1\\3\\0\end{pmatrix}}+t\overset{v}{\begin{pmatrix}4\\2\\1\end{pmatrix}}$ and, if $\tilde{\pi}\parallel\pi$ and $\tilde{\pi}\ni p$ then

$\mathrm{lin}\,\varphi(x,y,z)=\mathrm{lin}\,\psi(x,y,z)$

$\tilde{\pi}: \underbrace{3x-y+2z+D}_{\psi(x,y,z)}=0$

D is determined by the condition $\varphi(p)=0 \Rightarrow \psi(x,y,z)=(\mathrm{lin}\,\varphi)(x,y,z)-(\mathrm{lin}\,\varphi)(p)$

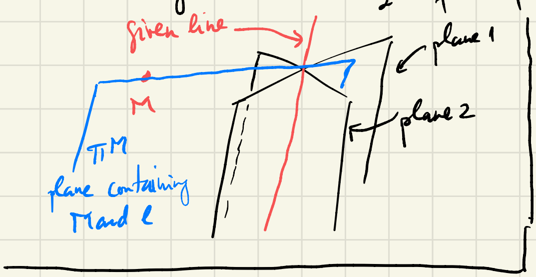
$\Rightarrow$ We obtain $N=\mathrm{Pr}_{\ell,\tilde{\pi}}(M)=\begin{pmatrix}-\frac{14}{3}\\ \frac{1}{6}\\ -\frac{17}{2}\end{pmatrix}$ $\Rightarrow$ we have two points (M and N) on $\tilde{\ell}$, so we can write down eq. for $\tilde{\ell}$

$\boxed{\text{Method 2}}$ Determine $\tilde{\ell}$ using the pencil of planes passing through the line ℓ

$\ell: \frac{x-1}{4}=\frac{y-3}{2}=z \Leftrightarrow \ell: \begin{cases}\frac{x-1}{4}=z\\ \frac{y-3}{2}=z\end{cases} \Leftrightarrow \begin{cases}x-4z-1=0\\ y-2z-3=0\end{cases}$ and any other plane

$\alpha^2+\beta^2>0$, $\alpha,\beta\in\mathbb{R}$

containing ℓ has an eq. of the form $\pi_{\alpha,\beta}: \alpha(x-4z-1)+\beta(y-2z-3)=0$



- If we determine π^M then the line $\tilde{\ell}$ is $\pi^M\cap\tilde{\pi}$ (where $\tilde{\pi}$ is as before the plane parallel to π and containing M)
- $\pi^M=\pi_{\alpha,\beta}$ for some $\alpha,\beta\in\mathbb{R}$

$M\in\pi_{\alpha,\beta} \Rightarrow \alpha(1-28-1)+\beta(-14-3)=0 \Rightarrow \alpha=-\frac{17}{28}\beta \Rightarrow \pi^M=\pi_{-\frac{17}{28}\beta,\beta}: -17x+28y+12z-67=0$

$M\in\tilde{\pi} \Rightarrow M=\begin{pmatrix}1\\0\\7\end{pmatrix}$ satisfies $3x-y+2z+D=0 \Rightarrow D=-17$

$$\Rightarrow \tilde{\ell}: \begin{cases}-17x+28y+12z-67=0\\ 3x-y+2z-17=0\end{cases}$$

11. Consider the Euclidean space $\mathbb{E}^3$. Show that the orthogonal reflection $\mathrm{Ref}^{\perp}_{\pi}(x)$ in the plane $\pi : \langle n, x\rangle = p$ is given by

$$\mathrm{Ref}_{\pi}(x) = Ax + b$$

where $A = \left(I - 2\frac{nn^t}{\|n\|^2}\right)$ and $b = \frac{2p}{\|n\|^2}n$.

if $n = n(n_1, n_2, n_3)$ and $x = (x_1, x_2, x_3)$

then $\langle n \cdot x\rangle = p \iff n_1 x_1 + n_2 x_2 + n_3 \cdot x_3 - p = 0$

Use the compact matrix form and notice that

$\langle n, n\rangle = n^t \cdot n = \|n\|^2$

12. Give the matrix form for the orthogonal reflections in the planes

$$\pi_1 : 3x - 4z = -1 \quad \text{and} \quad \pi_2 : 10x - 2y + 3z = 4 \quad \text{respectively.}$$

$\boxed{\text{for } \pi_1}$

- $\pi_1 = \varphi^{-1}(0)$ where $\varphi(x,y,z) = 3x - 4z + 1$
- the direction of the reflection is given by the normal vector to π_1 $\quad n = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$

$$\Rightarrow \mathrm{Ref}^{\perp}_{\pi_1}(p) = p + 2\mu n \text{ with } \mu = -\frac{\varphi(p)}{(\mathrm{lin}\,\varphi)(n)} = \frac{3x - 4z + 1}{9 + 16} \quad \text{if } p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \mathrm{Ref}^{\perp}_{\pi_1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{6x - 8z + 2}{25}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \frac{1}{25}\begin{pmatrix} 25x - 18x + 24z - 6 \\ 25y \\ 25z + 24x - 32z + 8 \end{pmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 7x & & 24z \\ & 25y & \\ 24x & & -7z \end{bmatrix} + \frac{2}{25}\begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 7 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{2}{25}\begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

Check:

- this needs to be the same as replacing in the generic matrix form which we deduced for dimension 3
- the points in π_1 do not change if they are reflected with $\mathrm{Ref}^{\perp}_{\pi}$. Check this on an affine basis of π_1

$\boxed{\text{for } \pi_2}$ the calculations are similar