Quiz 7

(1) **Quiz 7: Complex roots of unity and linear group actions**

ESSAY | marked out of 1.0 | penalty 0.10 | HTML editor

Consider the (multiplicative) group of n-th complex roots of unity

$$U_n(\mathbb{C}) = \{z \in \mathbb{C} \mid z^n = 1\}$$

and let μ_n be the first complex root of unity. Notice that $U_n(\mathbb{C}) = \langle \mu_n \rangle$.

(a) (Complex roots of unity and Galois groups) Consider the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^i$. We can restrict this map to $U_n(\mathbb{C})$ and below you will study the properties of this "power map" on the group $U_n(\mathbb{C})$.

i. Let S be the set of group homomorphism $U_n(\mathbb{C}) \to U_n(\mathbb{C})$. Show that the map $z \mapsto z^i$ is in S. (So you need to show that the image is contained in $U_n(\mathbb{C})$ and that this is a group homomorphism). We will call α_i the map $\alpha_i : U_n(\mathbb{C}) \to U_n(\mathbb{C})$ given by $z \mapsto z^i$.

ii. Consider $\alpha_i, \alpha_j \in S$. Show that S is closed under composition of maps.

iii. Show that α_i is a group automorphism if and only if i and n are coprime.

iv. Consider the set G of automorphisms of $U_n(\mathbb{C})$, so G is the subset of the elements of S that have inverses under function composition. Show that G is a group by showing $G \cong \mathbb{Z}_n^*$.

v. Consider μ_8, the first complex 8th root of unity. Factor $t^8 - 1$ completely over $\mathbb{Q}$ and find the 8th cyclotomic polynomial.

vi. Describe the Galois group of $\mathbb{Q}(\mu_8) : \mathbb{Q}$.

vii. Prove/disprove. For any n the Galois group of $\mathbb{Q}(\mu_n) : \mathbb{Q}$ is cyclic.

(b) (Actions) In the following questions, for ease of notation, we are dropping the index n so $\mu = \mu_n$. Consider the action of the group $U_n(\mathbb{C}) = \langle \mu \rangle$ on the vector space $\mathbb{C}^2$ given by

$$\mu \cdot (x, y) = (\mu\, x, \overline{\mu}\, y)\,,$$

where the operation on the entries of the vector is multiplication of complex numbers.

i. Assuming that this is an action, find the vector $\mu^i\cdot(x,y)$.
ii. Find three vectors in the orbit of $(1,1)$.
iii. Assuming that this action is a linear action, so that the action of each group element is a linear transformation, find a matrix that represents the map given by the action of μ. So give a matrix for the linear map

$$(x,y)\mapsto(\mu\, x,\overline{\mu}\, y).$$

iv. If the matrix A represents the action of μ, which matrix would represent the action of μ^i?
v. Define the action of μ on the polynomial ring $\mathbb{C}[x,y]$ given by

$$\mu\cdot x=\mu\, x,\ \ \mu\cdot y=\overline{\mu}\, y$$

and extend the action to all polynomials by prescribing that $\mu\cdot p(x,y) = p(\mu\cdot x,\mu\cdot y) = p(\mu\, x,\overline{\mu}\, y)$. Study the orbit of $p(x,y)=xy$.
vi. Find at least one element in the stabilizer of $p(x,y)=xy$.

Information for graders:

Total of marks: 1