Final Project: Robotics

Lucas Rizzolo

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1 Introduction

Using algebraic geometry, we can create methods to analyze the movements of robots. We'll look at three different questions posed by the want to analyze robotic movements:

- How do we create a mathematical description of these robots?
- Given a bunch of configurations of robotic pieces, can we know what the end position is?
- Given the end position of the robot, can we find all of the possible configurations that put the robot in that position?

The first problem asks the question of the geometry of the robots, second question asks the problem of Forward Kinetics, and the third asks the problem of Backward Kinetics.

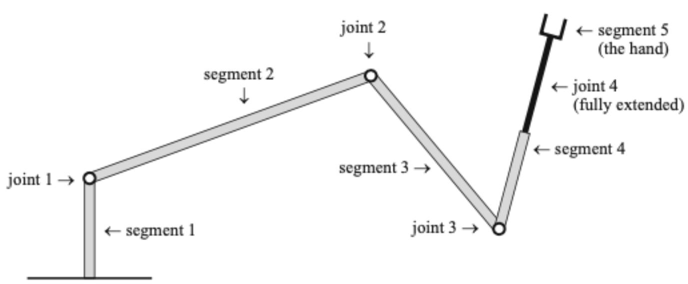


Figure 1: A basic robot with three revolute joints and one prismatic joint

2 The Geometry of Robots

In order to create a mathematical description of robots, we must first make some assumptions. The assumptions will define what pieces of the robot we can and cannot analyze.

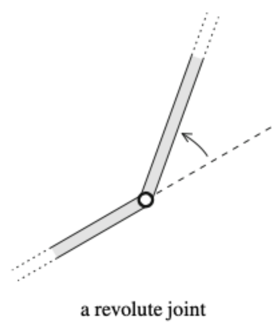
2.1 The Robot

The first assumption that we will make is that the robot is made up of rigid segments, connected by joints. To make things simple, we'll assume that they're connected in series, as in Figure 1. One end of the connected segments will be firmly set in place, and the other end will have a hand of sorts. This hand doesn't have to be a hand, but some sort of useful tool for the actual application.

We assume that the segments of the robot are rigid. Therefore, the only way the robot could move would be the joints. We'll define a few different kinds of joints:

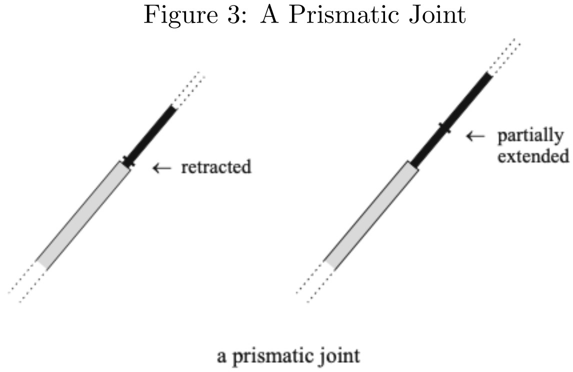
Definition 1 (Revolute Joints)**.** A **planar revolute joint** allows the rotation of one segment relative the another segment. This can be seen in Figure 2

Figure 2: A Revolute Planar Joint



Definition 2 (Prismatic joints)**.** A **prismatic joint** allows one segment to translate along an axis. This can be seen in Figure 3.

Figure 3: A Prismatic Joint



These basic joints permit two basic sets of movements for the robot: translation and rotation. We'll encounter two more complicated joints in the examples:

Definition 3 (Ball Joints)**.** A **ball joint** allows one segment to rotate freely about a three dimensional axis. This can be seen in Figure 4.

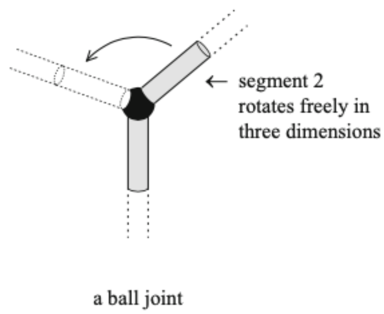


Figure 4: A Ball Joint

Definition 4 (Helical Joints)**.** A **helical joint** allows one segment to extend out of another segment, while rotating about a central axis. This can be seen in Figure 5.

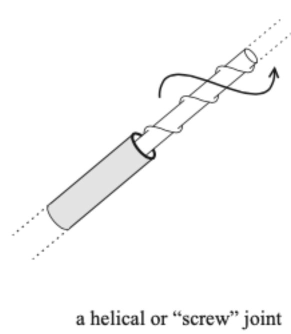


Figure 5: A Helical Joint

2.2 Describing the Robot Settings

For the purposes of clarity, we'll give a specific naming order to joints and segments. The fixed joint at one of the robot will be called Segment 1. The next segment will be Segment 2. Connecting these two joints will be Joint 1. The pattern continues, so that Joint i connects Segment i and Segment $i+1$.

For any two segments i and $i+1$ connected by a revolute joint, we measure the angle θ between the two in order to describe the position. In fact, we can parametrize a full circle of possible positions for the second segment, called S^i.

In a similar way, the setting of a prismatic joint can be defined by giving the distance that the joint is extended out. All of the possible settings of the i^{th} prismatic joint is called I^i. We can combine all of the possible joints settings in the robot with the following definition:

Definition 5 (Joint Space)**.** In a robot with r revolute joints and p prismatic joints, the **joint space** $\mathcal{J}$, or all possible configurations of the robot, can be parametrized by the Cartesian Product

$$\mathcal{J} = S^1 \times ...S^r \times I_1 \times ... \times I_p.$$

Finally, we need to be able to describe the position of the final "hand" segment. This robot is a planar, so we can set the fixed segment to the origin and set the position of the hand to be given by a single point (a, b). The hand can be rotated so we need to define some unit vector $\mathbf{u}$ to specify the orientation. This is seen in Figure 6

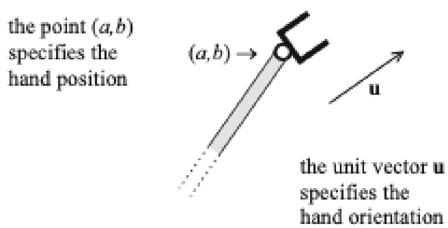


Figure 6: The Orientation and Position of the Hand

All of the possible unit vectors can be parametrized by a circle V, and all of the possible points (a, b) exist in some region $U \subseteq \mathbb{R}^2$. Therefore, all of the possible hand positions and orientations can be given by the following defintion

Definition 6 (Configuration Space)**.** The **configuration space** or **operational space** $\mathcal{C}$ is given by

$$\mathcal{C} = U \times V$$

3 Forward Kinetics

Forward Kinetics asks the following definition: Can we give an explicit definition of the maps from joint space to configuration space, in terms of the joint settings and configurations on our robot? We can formalize this with the definitions of joint space and configuration space.

Question 7. *Given a robot with configuration space* $\mathcal{C}$ *and joint space* $\mathcal{J}$*, we define a mapping* f *between the spaces as*

$$f \;:\; \mathcal{J} \to \mathcal{C}.$$

Can we give an explicit description or formula for f *in terms of the joint settings and dimensions of the segments of the robot "arm"?*

To answer this question, we'll consider all robots that are planar in $\mathbb{R}^2$. We'll establish a set of coordinates (x_1, y_1) at Joint 1 around the anchored segment at one end of the robot. Then, at each Joint i we establish a new coordinate system (x_{i+1}, y_{i+1}). For each $i \geq 2$, the (x_i, y_i) coordinates of i are $(l_i, 0)$ where l_i is the length of segment i. This can be seen in Figure 7

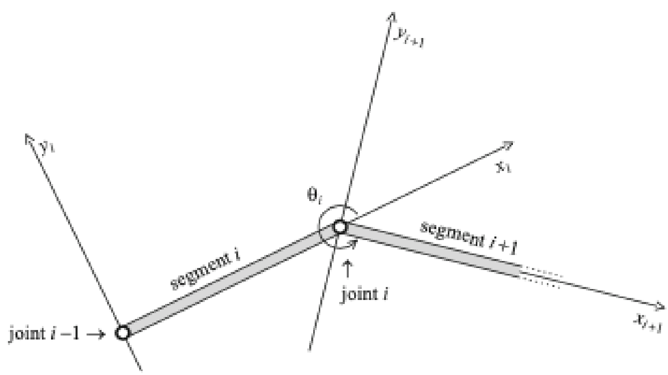


Figure 7: Multiple Axis

We need to be able to quickly change between coordinate systems. To get the location of point $q \in (x_{i+1}, y_{i+1})$ in $((x_i, y_i)$, this is accomplished by doing some matrix math.

Theorem 8 (Changing coordinates)**.** *Assume that we have some point $q = (a_{i+1}, b_{i+1})$ defined in coordinate system (x_{i+1}, y_{i+1}), and the length of segment i is defined as l_i. To get the point in (x_i, y_i), we can do the following matrix multiplication:*

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} cos\theta_i & -sin\theta_i \\ sin\theta_i & cos\theta_i \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} + \begin{pmatrix} l_i \\ 0 \end{pmatrix} \tag{1}$$

A quicker version of this is:

$$\begin{pmatrix} a_i \\ b_i 1 \end{pmatrix} = \begin{pmatrix} cos\theta_i & -sin\theta_i & l_i \\ sin\theta_i & cons\theta_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} = A_i \cdot \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ 1 \end{pmatrix} \tag{2}$$

This matrix multiplication accomplishes both a rotation to the new angle for the axis, and a translation to the end of the segment.

Let's apply the question of Forward Kinetics to a specific example.

Example 9 (The first example)**.** Given the planar robot in Figure 8, define a forumla for the mapping $f\ : \mathcal{J} \to \mathcal{C}$.

We will first calculate the matrices for calculating changing the points between coordinates for the three revolute joints. We'll use the simplified version

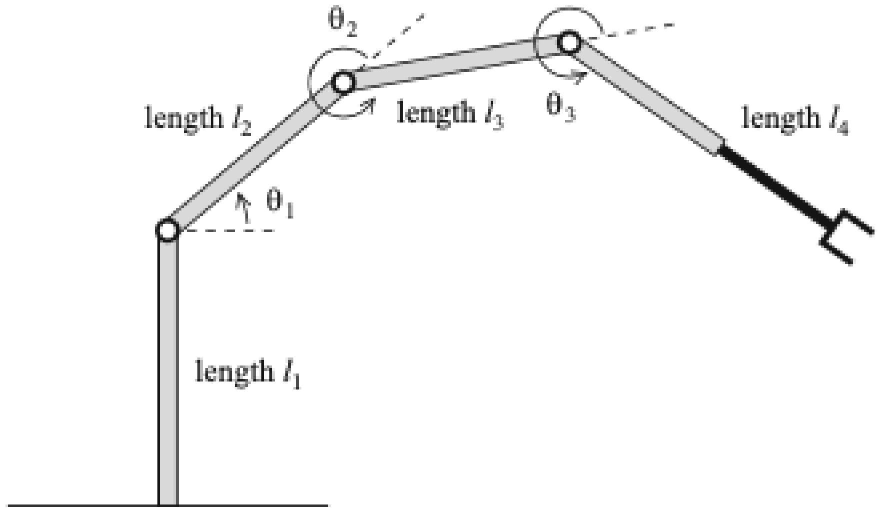


Figure 8: A general plane robot with three revolute joints and one prismatic joint

from Theorem 8. Due to the nature of matrix multiplication, we can take each coefficient matrix A_i and multiply them together as such:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = A_1 A_2 A_3 \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} cos\theta_1 & -sin\theta_1 & 0 \\ sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} cos\theta_2 & -sin\theta_2 & l_2 \\ sin\theta_2 & cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} cos\theta_3 & -sin\theta_3 & l_3 \\ sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using some trignometric addition formulas and multiplying them together, we can get:

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} cos(\theta_1+\theta_2+\theta_3) & -sin(\theta_1+\theta_2+\theta_3) & l_3cos(\theta_1+\theta_2)+l_2cos\theta_1 \\ sin(\theta_1+\theta_2+\theta_3) & cos(\theta_1+\theta_2+\theta_3) & l_3sin(\theta_1+\theta_2)+l_2cos\theta_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix}$$

If the hand was attached to the final segment instead of the prismatic joint, the head position in (x_4, y_4) are $(0, 0)$. Then, this would simplify down to

$$\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = \begin{pmatrix} l_3 cos(\theta_1 + \theta_2) + l_2 cos\theta_1 \\ l_3 sin(\theta_1 + \theta_2) + l_2 cos\theta_1 \\ 1 \end{pmatrix}$$

This gives us a 1×3 matrix that we can use to describe the mapping f. We know that the final angle of the hand will be the summation of the previous angles before it, so in the case without the prismatic joint, the function in terms of the revolute angles before are:

$$f(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} l_3 cos(\theta_1 + \theta_2) + l_2 cos\theta_1 \\ l_3 sin(\theta_1 + \theta_2) + l_2 cos\theta_1 \\ 1 \end{pmatrix}$$

The lengths of the segments are constant, so they are not a required input for the mapping.

Now, we know that we have a prismatic joint at the end. Let's say that the length is given by l_4, and l_4 can extend on values in the range $[m_1, m_2]$. Following the steps from before, we'd get a mapping:

$$g(\theta_1, \theta_2, \theta_2, l_4) = \begin{pmatrix} l_4 cos(\theta_1 + \theta_2 + \theta_3) + l_3 cos(\theta_1 + \theta_2) + l_2 cos\theta_1) \\ l_4 sin(\theta_1 + \theta_2 + \theta_3) + l_3 sin(\theta_1 + \theta_2) + l_2 sin\theta_1) \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}$$

Where l_3 and l_2 are constant and $l_4 \in [m_1, m_2]$. This gives us the x coordinate of the hand, the y coordinate of the hand, and the θ rotation of the hand.

It would make sense to describe the the matrix in terms of polynomial mappings, in order to describe this in terms of an algebraic variety. Let's make the parametrization

$$\begin{aligned} c_i &= cos\theta_i \\ s_i &= sin\theta_i \end{aligned}$$

subject to the constraints

$$c_i^2 + s_i^2 - 1 = 0$$

for $i = 1, 2, 3$. Using trignometric addition formulas, we can separate the sections into:

$$cos(\theta_1 + \theta_2) = cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 = c_1 c_2 + s_1 s_2$$

and

$$sin(\theta_1 + \theta_2) = sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2 = s_1 c_2 + c_1 s_2$$

and similarly for the other parts. Therefore the (x_1, y_1) hand positions are now

$$\begin{pmatrix} l_4(c_1(c_2c_3 - s_2s_3) - s_1(c_2s_3 + c_3s_2)) + l_3(c_1c_2 + s_1s_2) + l_2c_1 \\ l_4(s_1(c_2c_3 - s_2s_3) + c_1(c_2s_3 + c_32_2)) + l_3(s_1c_2 + c_1s_2) + l_2s_1 \end{pmatrix}$$

Now, we have $\mathcal{J}$as the subset $V \times [m_1, m_2]$ of the variety $V \times \mathbb{R}$, where $V = \mathbf{V}(x_1^2 + y_1^2 - 1, x_2^2 + y_2^2 - 1, x_3^2 + y_3^2 - 1)$.

4 Backward Kinetics

Given a specific configuration of the robot, can we determine all of the possible joint settings and configurations such that they map directly to that configuration? We can formalize this with the definitions of joint space and configuration space

Question 10. *Given a robot with configuration space $\mathcal{C}$ and joint space $\mathcal{J}$, we define a mapping f between the spaces as*

$$f \, : \, \mathcal{J} \to \mathcal{C}.$$

Given $c \in \mathcal{C}$, can we determine one or all the $j \in \mathcal{J}$ such that $f(j) = c$?

To answer this question, we'll use a Gröbner basis! Let's take the robot that we used for the previous example, but remove the prismatic joint at the hand. This is the theoretical that we considered, where the hand had position $(0, 0)$ in (x_4, y_4) coordinates. Using the methods of the previous section, we can get a system of polynomial equations:

$$\begin{aligned} a &= l_3(c_1c_2 - s_1s_2) + l_2c_1, \\ b &= l_3(c_1s_2 + c_2s_1) + l_2s_1, \\ 0 &= c_1^2 + s_1^2 - 1, \\ 0 &= c_2^2 + 2_2^2 - 1 \end{aligned}$$

for c_1, s_1, c_2, s_2. To solve these, let's make a grevlex Gröbner basis with

$$c_1 > s_1 > c_2 > s_2.$$

We get:

$$\begin{aligned} & c_1 - \frac{2bl_2l_3}{2l_2(a^2+b^2)}s_2 - \frac{a(a^2+b^2+l_2^2-l_3^2)}{2l_2(a^2+b^2)}, \\ & s_1 + \frac{2al_2l_3}{2l_2(a^2+b^2)}s_2 + \frac{b(a^2+b^2+l_2^2-l_3^2)}{2l_2(a^2+b^2)}, \\ & c_2 - \frac{a^2+b^2-l_2^2-l_3^2}{2l_2l_3}, \\ & s_2^2 + \frac{(a^2+b^2)^2 - 2(a^2+b^2)(l_2^2+l_3^2) + (l_2^2-l_3^2)^2}{4l_2^3l_3^2} \end{aligned}$$

This is a reduced Gröbner basis for the ideal I generated by the system of polynomials in the ring $\mathbb{R}(a, b, l_2, l_3)[c_1, s_1, c_2, s_2]$. To find specific real numbers for a, b, l_2, l_3, we need to substitute values for c_1, c_2, s_1, s_2. This creates an ideal $I_s \subseteq \mathbb{R}[c_1, s_1, c_2, s_2]$, which corresponds to a specific position of a robot with specific segment lengths. This replacement of variables is called the *specialization* of a Gröbner basis. For any values that do not vanish the denominators ($_2 \neq 0, l_3 \neq 0$, and $a^2 + b^2 \neq 0$), we will still get a Gröbner basis.

From the book, there should be two observations that follow:

- Any zero s_2 of the last polynomial can be extended uniquely to a full solution of the system
- The set of solutions of the set of polynomials is a finite set for this choice of a, b, l_2, l_3.

The last polynomial is quadratic in s_2 so there can be at most two real solutions. To make calculations simple and see which (a, b) values give real solutions, we'll specialize to the case $l_2 = l_3 = 1$. Substituting will get us:

$$\begin{aligned}
&c_1 - \frac{2b}{2(a^2+b^2)} s_2 - \frac{a}{2}, \\
&s_1 + \frac{2a}{2(a^2+b^2)} s_2 + \frac{b}{2}, \\
&c_2 - \frac{a^2+b^2}{2}, \\
&s_2^2 + \frac{(a^2+b^2)(a^2+b^2-4)}{4}
\end{aligned}$$

We need an (a, b) where $a^2 + b^2 \neq 0$. Solving the last equation of the basis, we get:

$$s_2 = \pm \frac{1}{2} \sqrt{(a^2+b^2)(4-(a^2+b^2))}$$

Therefore, the solutions are real iff. $a^2 + b^2 \leq 4$ and when $a^2 + b^2 = 4$. Given s_2 we can use the Gröbner basis to find the other solutions.

So, to summarize there are:

- infinitely many distinct settings of joint 1 when $a^2 + b^2 = 0$
- two distinct settings of joint 1 when $a^2 + b^2 < 4$
- one setting of joint 1 when $a^2 + b^2 = 4$
- no possible settings of joint 1 when $a^2 + b^2 > 4$

The case where $a^2 + b^2 = 0, 4$ is known as a kinematic singularity.

5 Proposition 3

Proposition 11 (Prop 3.). *Let $f \ : \mathcal{J} \to \mathcal{C}$ be the configuration mapping for a planar robot with $n \geq 3$ revolute joints. then there exist kinematic singularities $j \in \mathcal{J}$.*