# Network Science Analytics
## Option Applied Math and M.Sc. in DSBA

Lecture 1B

Graph theory and linear recap; basic network properties

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# Acknowledgements


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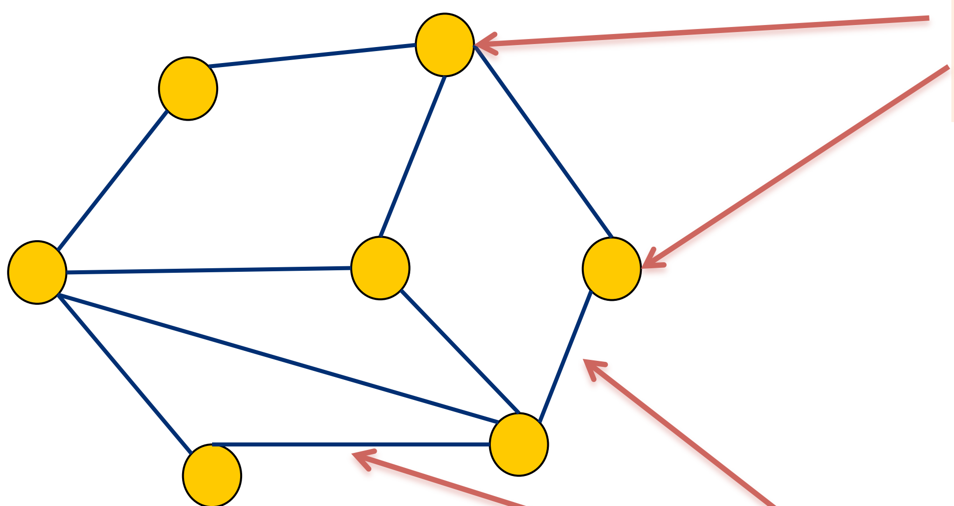
  - R. Zafarani, M. A. Abbasi, and H. Liu, Social Media Mining: An Introduction, Cambridge University Press, 2014. Free book and slides at http://socialmediamining.info/

Thank you!

# Basic graph-theoretic concepts and definitions

# Graphs and Networks

## Graphs: modeling dependencies



Nodes (or vertices)
(objects/entities)

Edges (or links)
(interconnections)

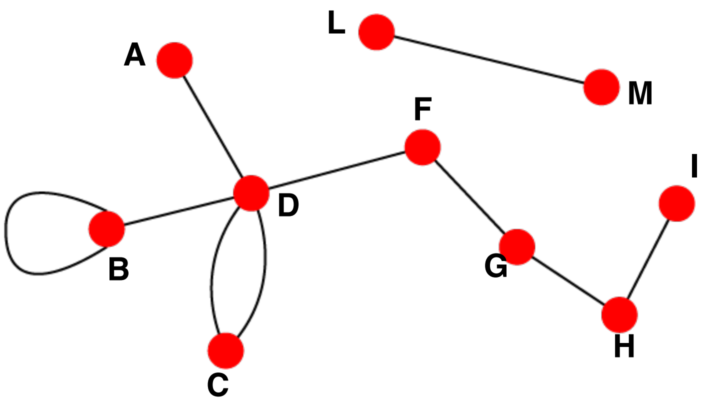
$G = (V, E)$
(network or graph)

$n = |V|$ is the number of nodes
$m = |E|$ is the number of edges

# Undirected vs. Directed Networks

## Undirected

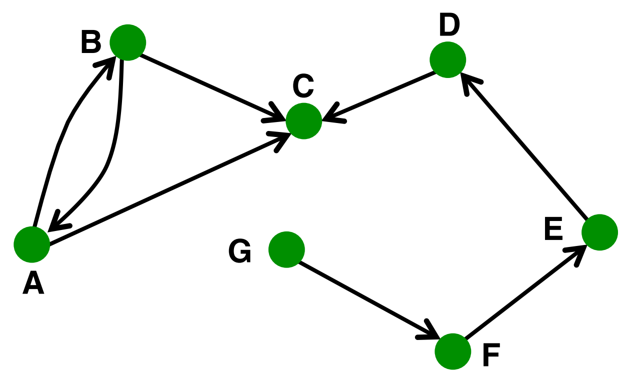
- **Links:** undirected (symmetrical, reciprocal)




- Examples
  - Collaborations
  - Friendship on Facebook

## Directed

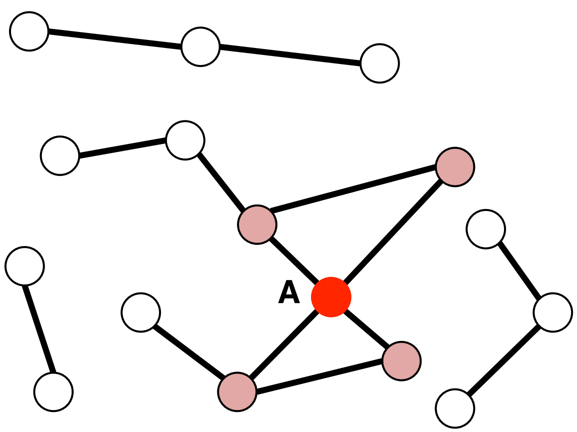
- **Links:** directed (arcs)




- Examples
  - Phone calls
  - Following on Twitter

# Node Degree

Undirected



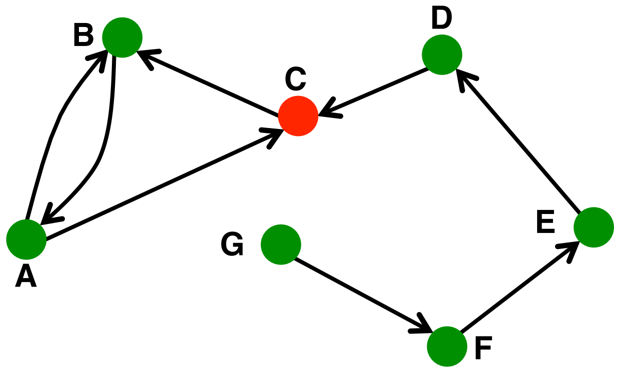

Node degree $k_i$: the number of edges adjacent to node $i$

$$k_A = 4$$

Average degree:

$$\bar{k} = \langle k \rangle = \frac{1}{n}\sum_{i=1}^{n} k_i = \frac{2|E|}{n}$$

Directed




In directed networks we define an in-degree and out-degree

The (total) degree of a node is the sum of in- and out-degrees

$$k_C^{in} = 2 \qquad k_C^{out} = 1 \qquad k_C = 3$$

Source: Node with $k^{in} = 0$

Sink: Node with $k^{out} = 0$

Average: $\overline{k^{in}} = \overline{k^{out}}$

# Properties of the Degree

- Property 1: The sum of the degrees in an undirected graph is twice the number of edges

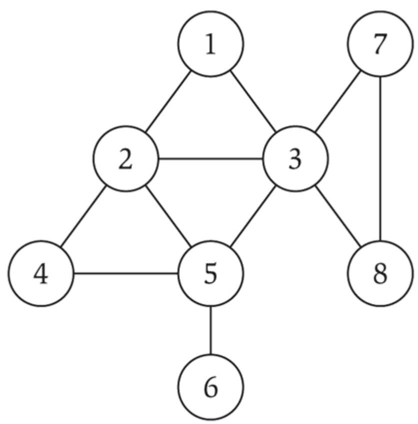
$$\sum_{i=1}^{n} k_i = 2\,|E|$$

- Property 2: In any directed graph, the sum of in-degrees is equal to the sum of out-degrees

$$\sum_{i=1}^{n} k_i^{out} = \sum_{j=1}^{n} k_j^{in}$$

# Graph Representation: Adjacency Matrix

- A graph can be represented by the adjacency matrix $A$
  - Matrix of size $n \times n$, where $n=|V|$ is the number of nodes
  - $A_{ij} > 0$, if $i$ and $j$ are connected
  - $A_{ij} = 0$, if $i$ and $j$ are not connected
  - In case of **unweighted** graphs, $A_{ij} = 1$, if $(i, j)$ is an edge of the graph
  - Space proportional to $n^2$




Undirected graph

Node indexing

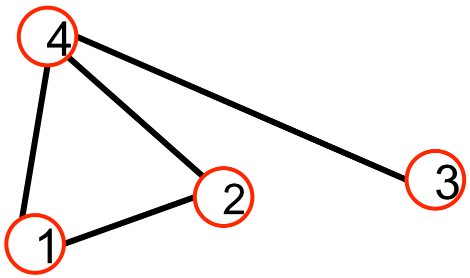
Node indexing

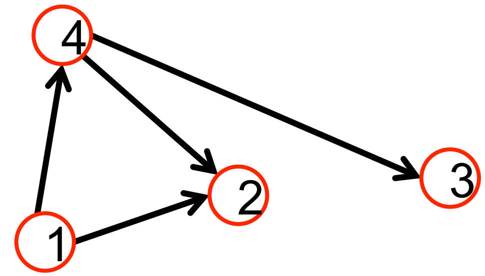
| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| **2** | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| **3** | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| **4** | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| **5** | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| **6** | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| **7** | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| **8** | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |

Adjacency matrix

More matrix representations in a while

# Adjacency Matrix




$A_{ij} = 1$ if there is a link from node *i* to node *j*

$A_{ij} = 0$ otherwise

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Symmetric matrix

Undirected graph

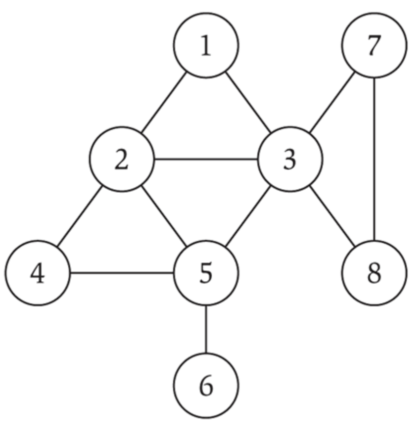
$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

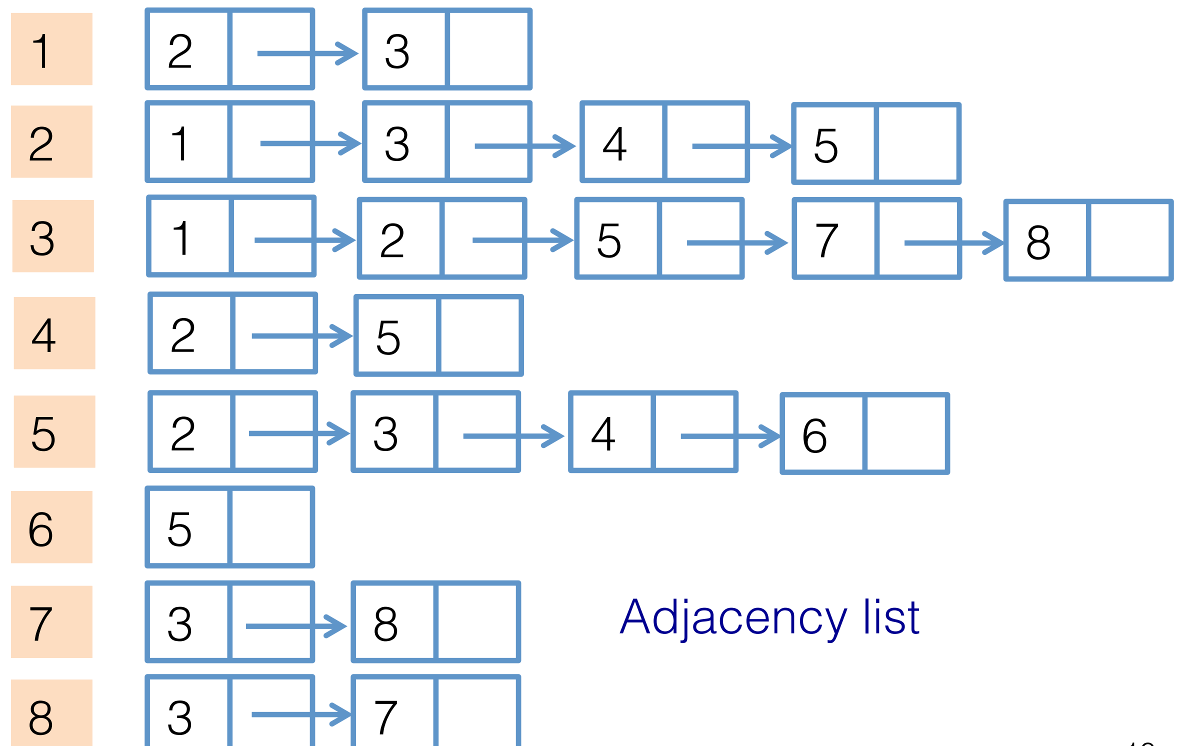
Nonsymmetric matrix

Directed graph

# Graph Representation: Adjacency List

- Adjacency lists
  - Representation of a graph with *n* nodes using an array of *n* lists of nodes
  - List *i* contains node *j* if there is an edge *(i, j)*
  - A weighted graph can be represented with a list of node/weight pairs
  - Space proportional to $\Theta(m+n)$
  - Checking if *(i, j)* is an edge takes $O(k_i)$ time

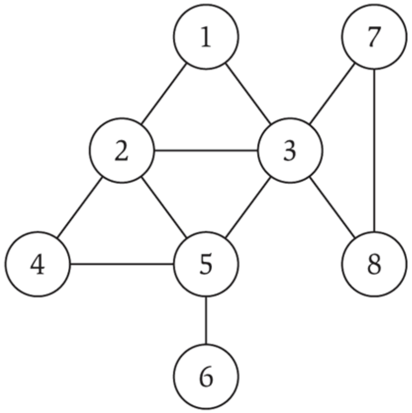



Adjacency list

# Graph Representation: Edge List

- Edge list
  - Very simple way to represent a graph
  - List of $|E|$ edges
  - An edge is represented as as pair of vertices (e.g., source, destination)
  - Space proportional to $\Theta(|E|)$
  - Checking if $(i, j)$ is an edge takes $O(|E|)$ time (if the edges appear in no particular order)

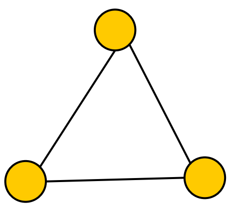



| | |
|---|---|
| 1 | 2 |
| 1 | 3 |
| 2 | 3 |
| 2 | 4 |
| 2 | 5 |
| 3 | 5 |
| 3 | 7 |
| 3 | 8 |
| 4 | 5 |
| 5 | 6 |
| 7 | 8 |

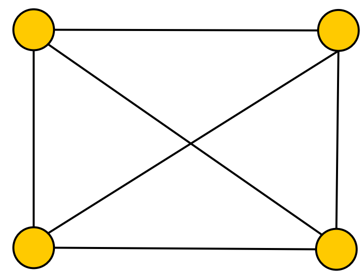
Edge list

# Complete Graph

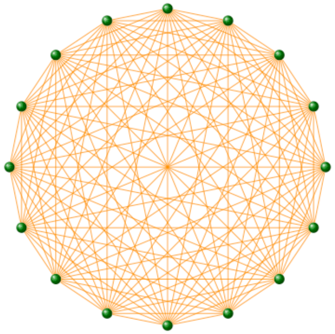
- Definition: A graph $G = (V, E)$ is called complete $K_n$ if every pair of nodes is connected by an edge



Complete graph with 3 nodes: triangle ($K_3$)



$K_4$



$K_{16}$

- Q: What is the number of edges in $K_n$ (complete graph with $n$ nodes)?

$$\binom{n}{2} = \frac{n!}{2!(n-1)!} = \frac{n(n-1)}{2}$$

# Real Networks are Sparse Graphs

Most real-world networks are sparse: $|E| << |E_{max}|$ (or avg. $k << |V|-1$)

| Network | # of nodes | Avg. degree |
|---|---|---|
| WWW (Stanford-Berkeley) | 319,717 | 9.65 |
| Social networks (LinkedIn) | 6,946,668 | 8.87 |
| Communication (MSN IM) | 242,720,596 | 11.1 |
| Coauthorships (DBLP) | 317,080 | 6.62 |
| Internet (AS-Skitter): | 1,719,037 | 14.91 |
| Roads (California) | 1,957,027 | 2.82 |
| Proteins (S. Cerevisiae) | 1,870 | 2.39 |

**Consequence:**
The adjacency matrix is filled with zeros!

**Graph density:**

$$D = \frac{2|E|}{|V|(|V|-1)}$$

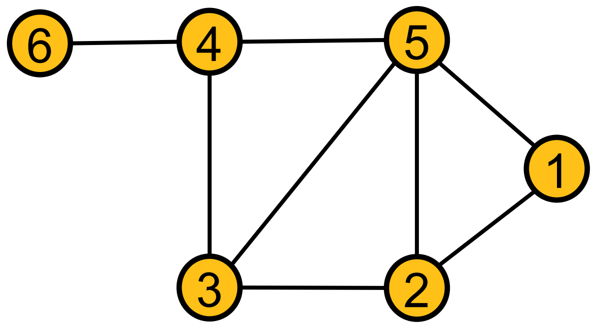
**Density of the graph:**

- WWW=$1.51\times10^{-5}$
- MSN IM = $2.27\times10^{-8}$

Source: Leskovec et al., Internet Mathematics, 2009

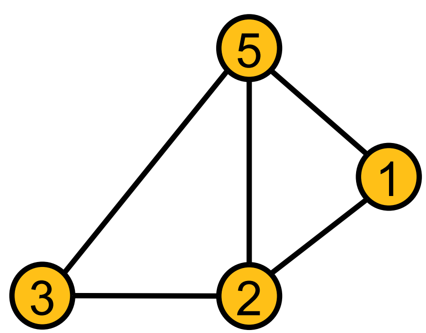
# Subgraphs

- Let *G = (V, E)* be a graph and let $S \subseteq V$ be any subset of its vertices



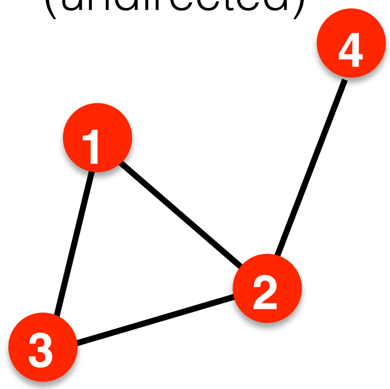

- Definition: The induced subgraph *G[S] = (S, E')* is the graph whose vertex set is *S* and its edge set consists of all of the edges in *E* that have both endpoints in *S*

S={1, 2, 3, 5}



# More Types of Graphs (1/2)

Unweighted
(undirected)



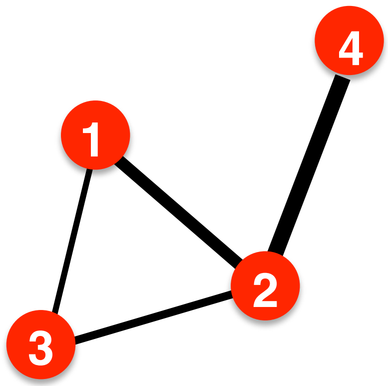

$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$|E| = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij} \qquad \bar{k} = \frac{2|E|}{n}$$

Examples: Friendship, Hyperlink

Weighted
(undirected)




$$A_{ij} = \begin{pmatrix} 0 & 2 & 0.5 & 0 \\ 2 & 0 & 1 & 4 \\ 0.5 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

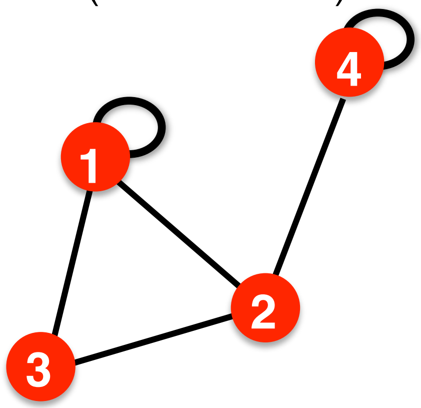
$$A_{ii} = 0 \qquad A_{ij} = A_{ji}$$

$$|E| = \frac{1}{2}\sum_{i,j=1}^{n} nonzero(A_{ij}) \qquad \bar{k} = \frac{2|E|}{n}$$

Examples: Collaboration, Internet, Roads

# More Types of Graphs (2/2)

Self-edges (self-loops)
(undirected)



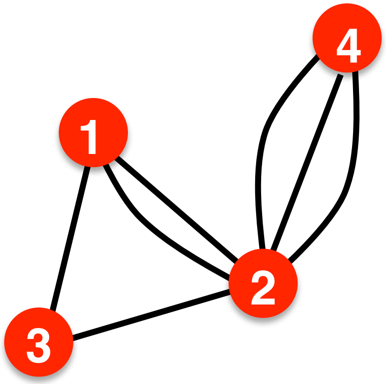

$$A_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$A_{ii} \neq 0$ $\quad A_{ij} = A_{ji}$

$$|E| = \frac{1}{2}\sum_{i,j=1, i\neq j}^{n} A_{ij} + \sum_{i=1}^{n} A_{ii}$$

Examples: Proteins, Hyperlinks

Multigraph
(undirected)




$$A_{ij} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

$A_{ii} = 0$ $\quad A_{ij} = A_{ji}$

$$|E| = \frac{1}{2}\sum_{i,j=1}^{n} nonzero(A_{ij}) \quad \bar{k} = \frac{2|E|}{n}$$

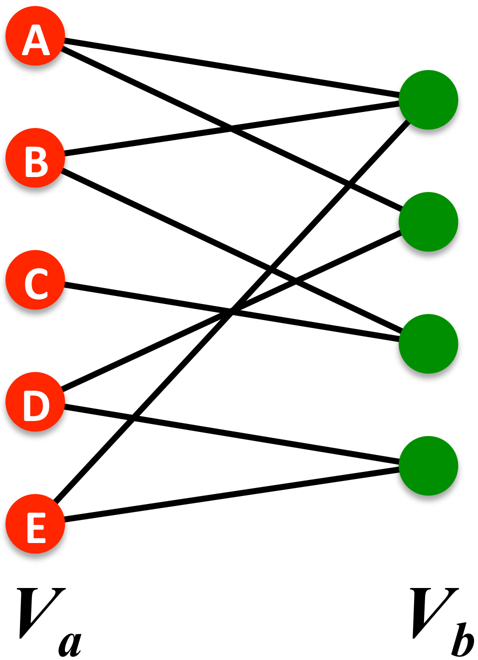
Examples: Communication, Collaboration

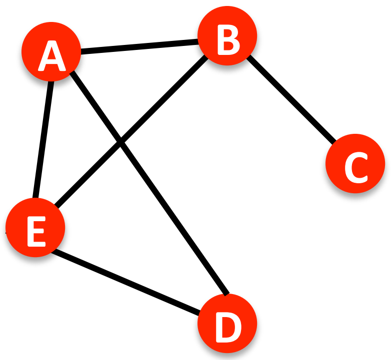
# Bipartite Graph

- Definition: A graph $G=(V,E)$ is called bipartite if the node set $V$ can be partitioned into two disjoint sets $V_a$, $V_b$ and every edge $(u,v)$ connects a node of $V_a$ to a node of $V_b$

- Examples
  - Authors-to-papers (they authored)
  - Actors-to-Movies (they appeared in)
  - Users-to-Movies (they rated)
  - Customers-to-Products (they purchased)

- Folded (or one-mode) networks
  - Author collaboration networks
  - Movie co-rating networks




Folded version of the graph above

# Network Representations

direction, edge weight, self-edges, acyclic, multigraph

WWW > directed multigraph with self-edges

Facebook friendships > undirected, unweighted

Citation networks > directed, unweighted, acyclic

Collaboration networks > undirected multigraph or weighted graph

Mobile phone calls > directed, (weighted?) multigraph

Twitter graph > directed, weighted (?)

Protein Interactions > undirected, unweighted with self-interactions

# Representation Matters!

Choice of the proper network representation of a given system determines our ability to use networks successfully

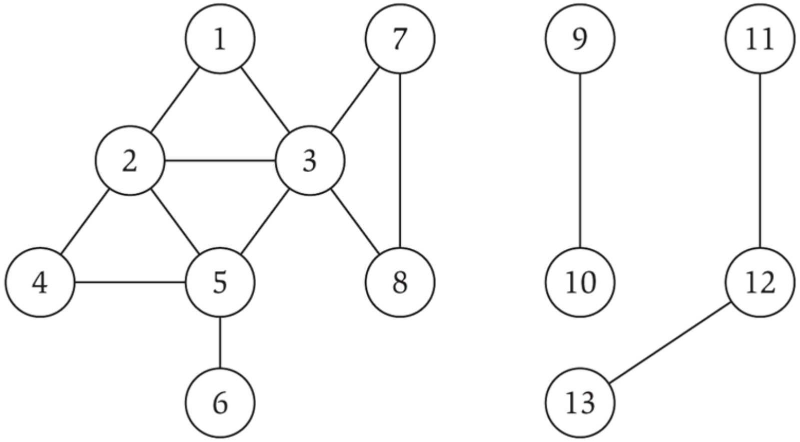
# Paths,
# graph connectivity and distance

# Paths and Connectivity in Graphs Trees

- Definition: A path in an undirected graph $G=(V,E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair $v_{i-1}, v_i$ is joined by an edge in $E$

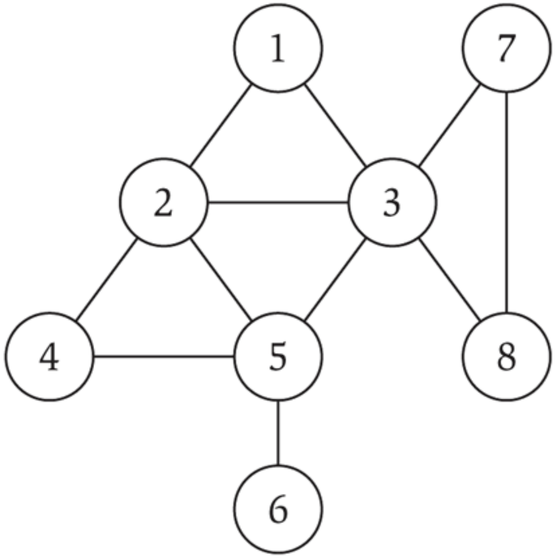
- Definition: An undirected graph is connected if for every pair of nodes $u$ and $v$, there is a path between $u$ and $v$

Is this graph connected?



# Cycles in Graphs

- Definition: A cycle is a path $v_1, v_2, \ldots, v_k$ in which $v_1 = v_k$, $k > 2$ and the first $k$-$1$ nodes are all distinct

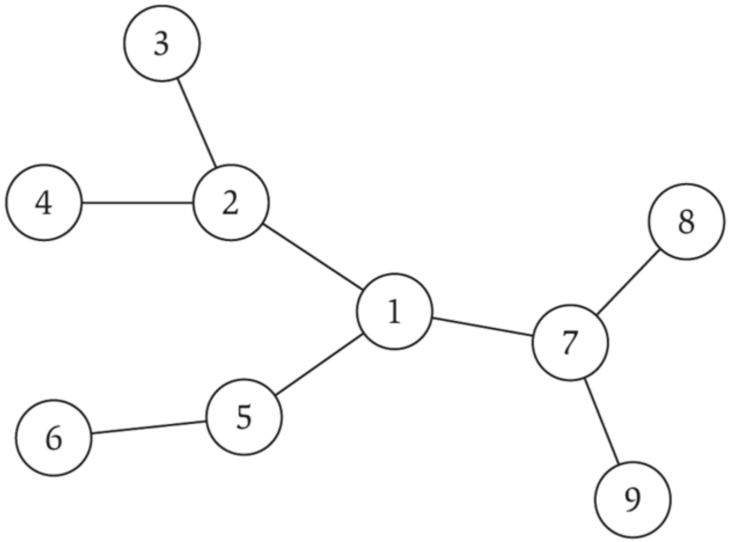



Cycle C = 1 – 2 – 4 – 5 – 3 – 1

# Trees

- Definition: An undirected graph is a tree if it is connected and does not contain a cycle

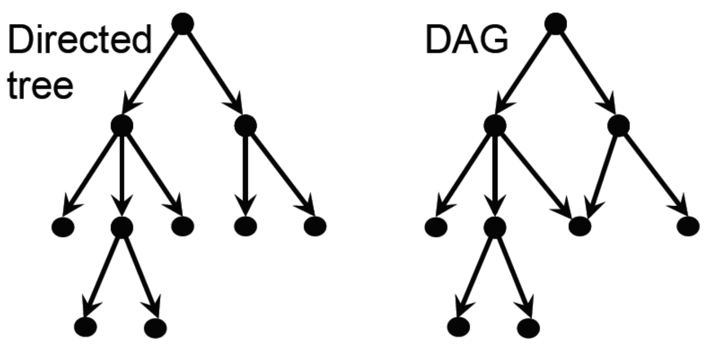
- Theorem: Let $G$ be an undirected graph with $n$ nodes. Then, any two of the following statements imply the third:
  - $G$ is connected
  - $G$ does not contain a cycle
  - $G$ has $n-1$ edges



# Directed Acyclic Graph

- Definition: A directed tree is a directed graph whose underlying undirected graph is a tree
  - Root is the only vertex with paths to all the other nodes
  - Vertex terminology: parent, children, ancestor, descendant, leaf

- A directed acyclic graph (DAG) is a directed graph with no directed cycles

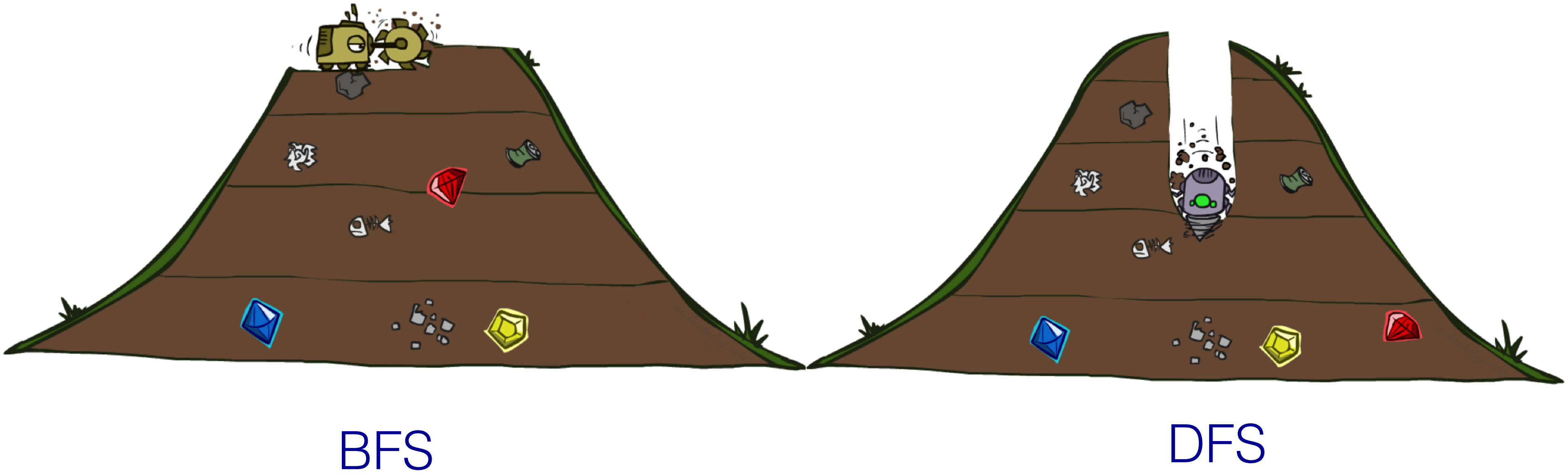


# Graph Traversal

- Graph traversal is the problem of visiting all the nodes in the graph
  - Also known as graph search

- Graph traversal algorithms
  - Breadth-first search (BFS)
  - Depth-first search (DFS)

# BFS vs. DFS




Source: A. Shehu, F. Li, GMU

# Breadth-First Search (BFS)

- Strategy for searching in graphs when search is limited to two operations:
  - Visit and inspect a node of the graph
  - Visit the neighborhood nodes of currently visited node

```
procedure BFS(G, v):
    create a queue Q
    add v onto Q
    mark v
    while Q is not empty:
        t ← Q.dequeue()
        if t is what we are looking for:
            return t
        for all edges e in G.adjacentEdges(t) do
            o ← G.adjacentVertex(t,e)
            if o is not marked:
                mark o
                add o in Q
```

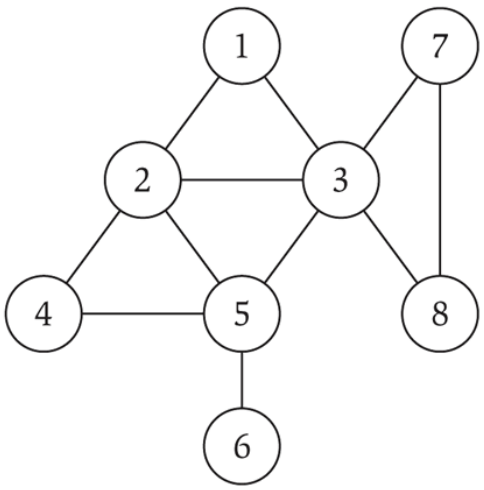
Input: Graph $G$ and a node $v$

Output: An assignment (labeling) of the nodes of the graph into layers (or the node closest to $v$ in $G$ satisfying some conditions)

Complexity: $\mathcal{O}(|V| + |E|)$

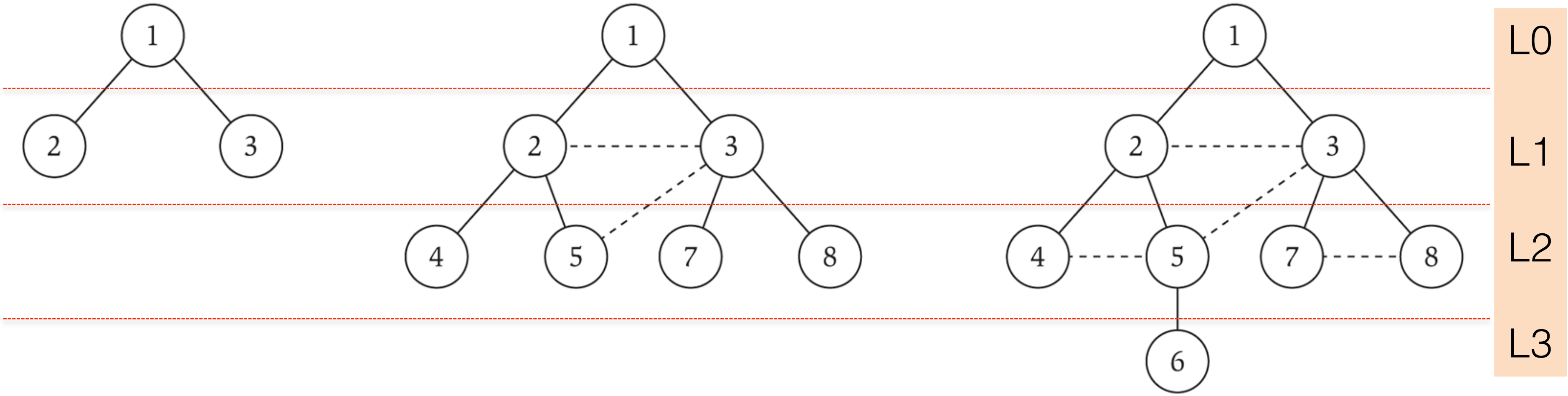
# Breadth-First Search (BFS)

## Example




Start BFS from node 1

Property: Let T be a BFS tree of *G* and let *(u, v)* be an edge of *G*. Then, the levels of *u* and *v* differ by at most 1



# Depth-First Search (DFS)

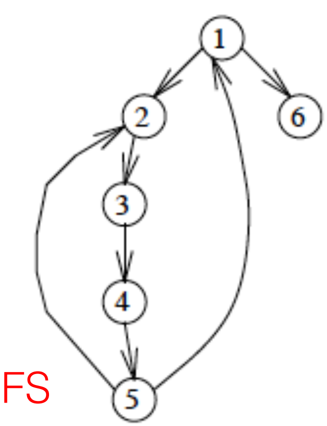
- Strategy for traversing graphs:
  - Visits the children nodes before visiting the sibling nodes
  - Traverses the depth of any particular path before exploring the breadth

```
procedure DFS(G,v):
    label v as explored
    for all edges e in G.adjacentEdges(v) do
        if edge e is unexplored then
            w ← G.adjacentVertex(v,e)
            if vertex w is unexplored then
                label e as a discovery edge
                recursively call DFS(G,w)
            else
                label e as a back edge
```

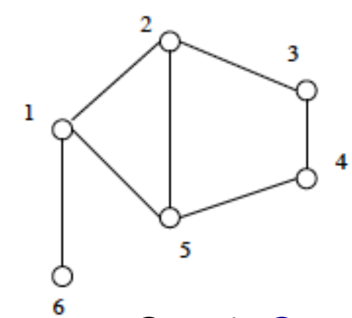
**Input:** Graph *G* and a node *v*

**Output:** A labeling of the edges of the graph as discovery and back edges

Complexity: $\mathcal{O}(|V| + |E|)$



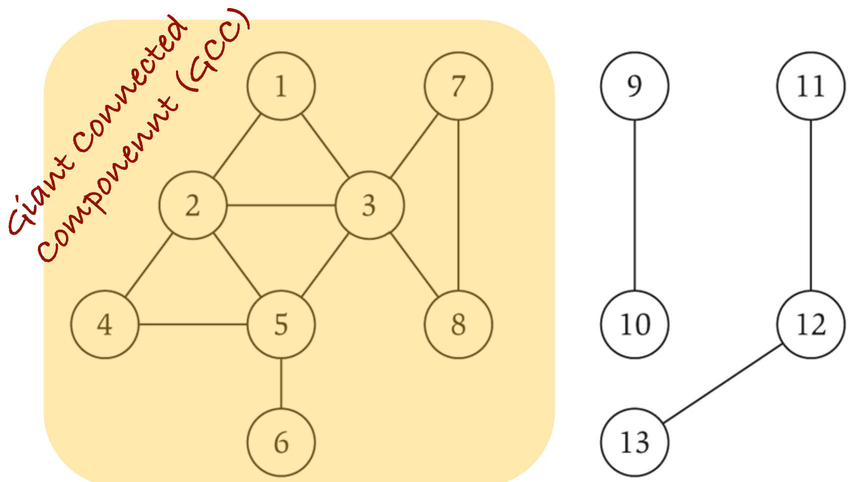

DFS




Graph G

# Connected Components

- A connected component is a maximal connected subgraph of a graph G (there is a path between any pair of nodes)
  - Maximal means adding another vertex will ruin connectivity




Connected component containing node 1: {1, 2, 3, 4, 5, 6, 7, 8}

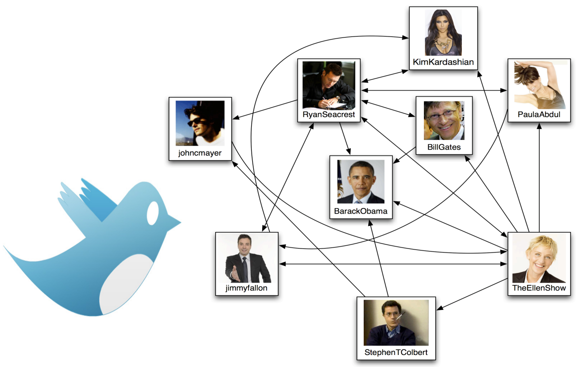
Graph with 3 connected components

Question: How can we compute the connected components of a graph?

A: Apply BFS

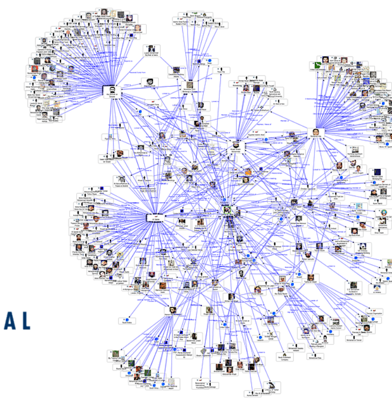
# Connectivity in Directed Graphs (1/2)

- A plethora of network data from several applications is from their nature directed

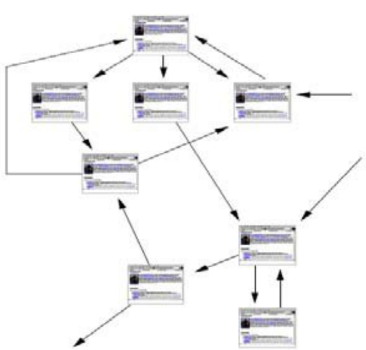



Twitter

[Image: http://sites.davidson.edu/mathmovement/]

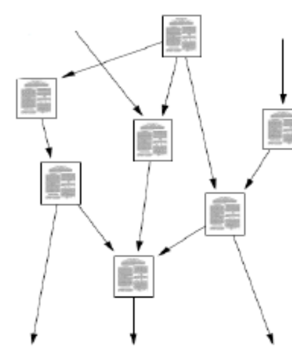


Online Social Networks



Web Graph

Wikipedia

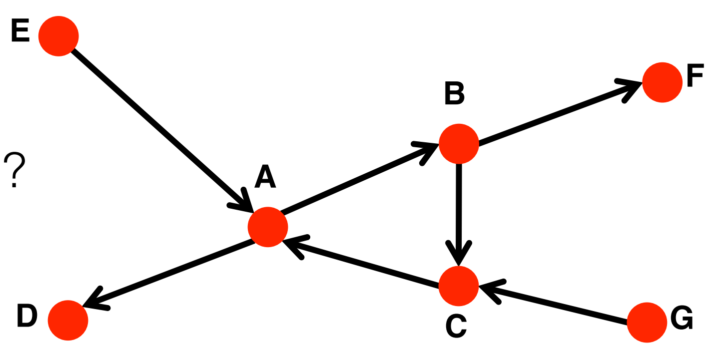


Citation Graph

# Connectivity in Directed Graphs (2/2)

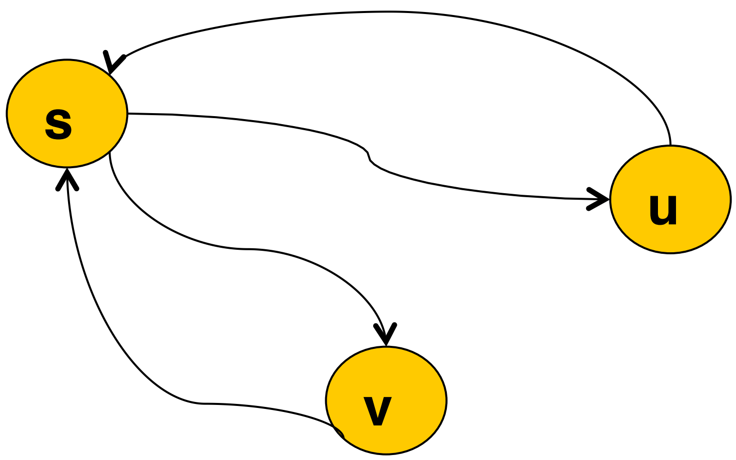
- Directed reachability: Given a node *s* in a directed graph *G*, find all nodes reachable from *s*
  - Web crawler: start from a webpage *s*, find all webpages linked from s, either directly or indirectly
- The notion of connectivity in directed graphs is replaced by the
  - Strong connectivity: a graph is called strongly connected if it is possible to reach any node taking into account the directionality of the edges
  - Weak connectivity: a graph is called weakly connected if it is possible to reach any node if we do not take into account the directionality of the edges

Q: Is this graph strongly connected?



# Strong Connectivity

- Lemma: Let s be any node. Graph *G* is strongly connected iff every node is reachable from *s*, and *s* is reachable from every node
- Proof:
  – --> Follows from definition
  – <-- Path from *u* to *v*: concatenate *u* --> *s* path with *s* --> *v* path
    Path from *v* to *u*: concatenate *v* --> *s* path with *s* --> *u* path



# Strong Connectivity: Algorithm

- **Theorem:** We can determine if $G$ is strongly connected in $O(m+n)$ (i.e., linear) time

- **Proof:**

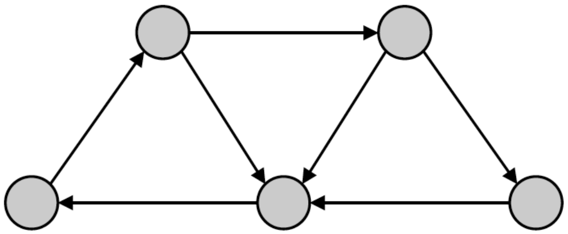
  Idea: use BFS

  – Pick any node $s$

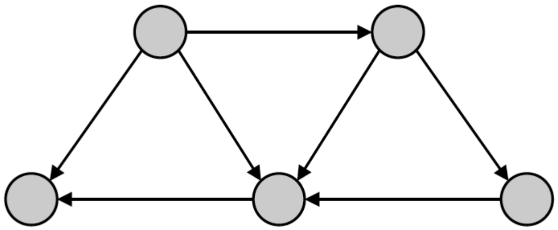
  – Run BFS from $s$ in $G$

  – Run BFS from $s$ in $G_{reverse}$ (reverse the orientation of the edges)

  – Return true iff all nodes reached in both BFS executions



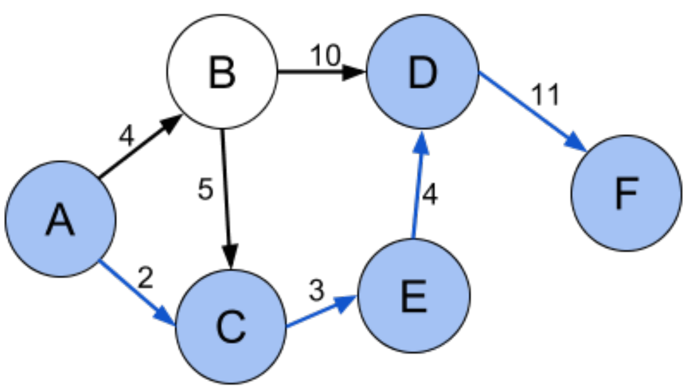
Strongly connected



Not strongly connected

# Shortest Paths

- Definition: find a path between two nodes in a graph, in such a way that the sum of the weights of its constituent edges is minimized
  - Many applications (e.g., road networks, community detection, communications)
- Variants
  - Single-source shortest path problem
  - Single-destination shortest path problem
  - All-pairs shortest path problem




Shortest path (A, C, E, D, F) between vertices A and F in the weighted directed graph

Various algorithms:
- Dijkstra
- Bellman-Ford
  (works with negative edge weights)

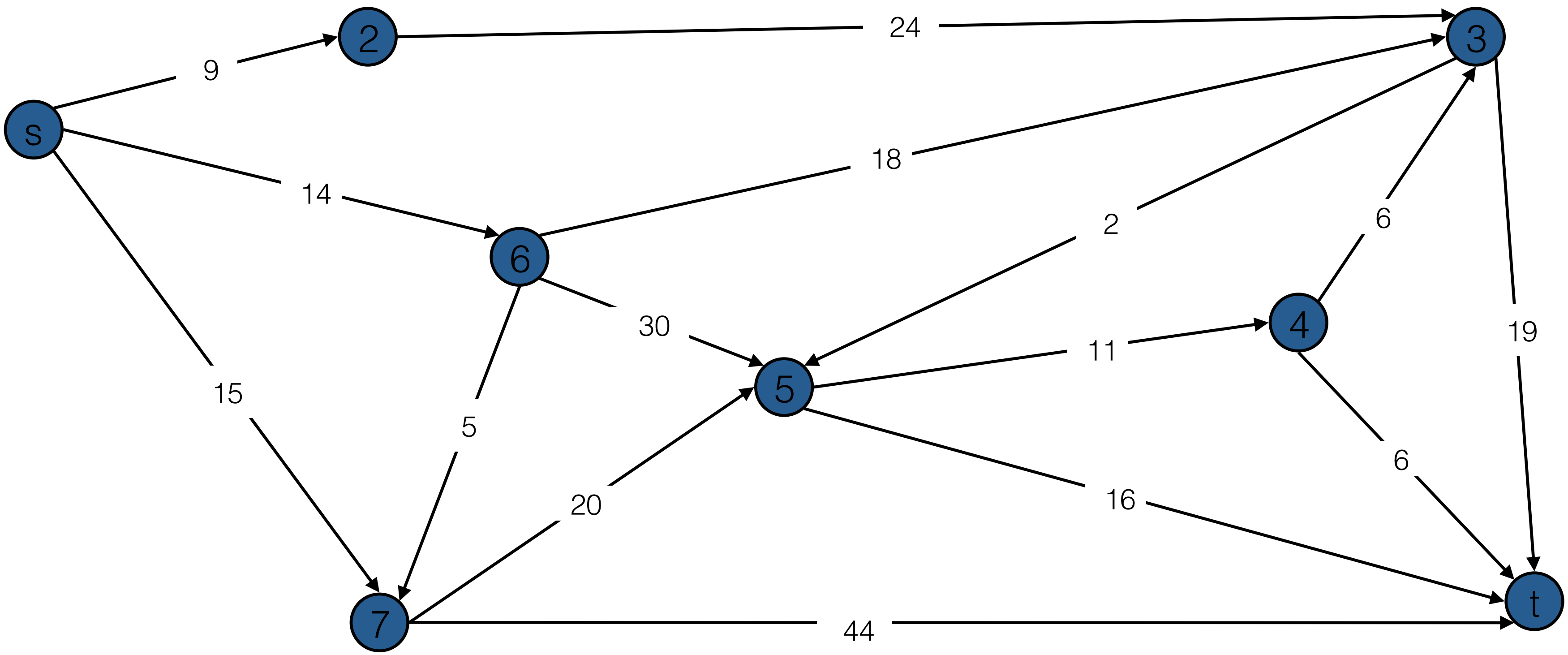
See: https://en.wikipedia.org/wiki/Shortest_path_problem

# Dijkstra's Algorithm

```
function Dijkstra(Graph, source):

        create vertex set Q

        for each vertex v in Graph:                  // Initialization
                dist[v] := infinity                  // initial distance from source to vertex v is set to infinite
                previous[v] := undefined             // Previous node in optimal path from source
                add v to Q                           // All nodes initially in Q (unvisited nodes)

        dist[source] := 0                            // Distance from source to source


        while Q is not empty:                        // Main loop
                u := node in Q with smallest dist[ ]
                remove u from Q

                for each neighbor v of u:            // Where v has not been removed yet from Q
                        alt := dist[u] + length(u, v)
                        if alt < dist[v]             // A shorter path to v has been found
                                dist[v] := alt
                                previous[v] := u

return  dist[], previous[ ]
```

# Dijkstra's Shortest Path Algorithm

- Find shortest path from *s* to *t*

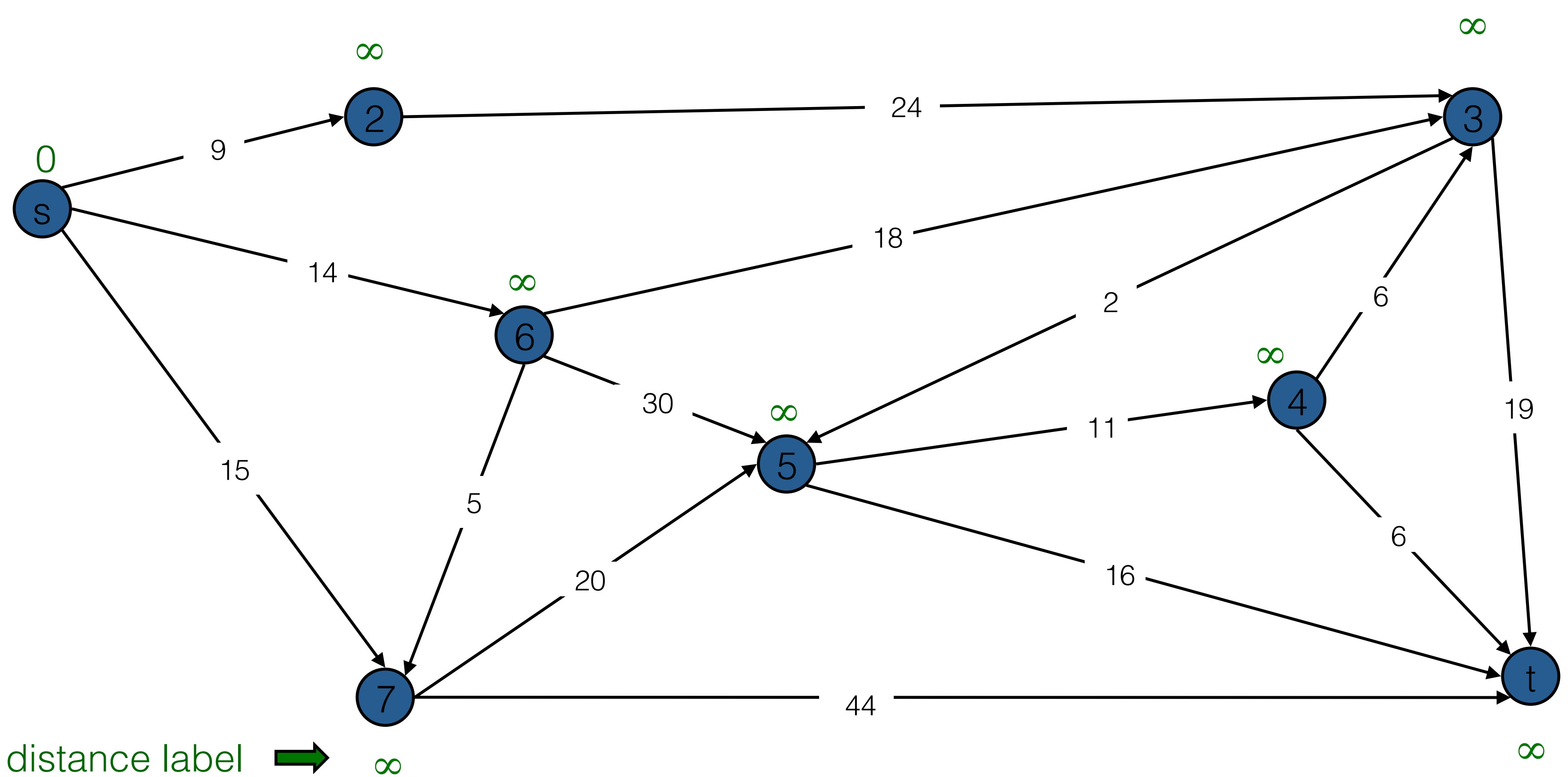



Source: L. Wang, City Univ. of Hong Kong

# Dijkstra's Shortest Path Algorithm

Visited = { }

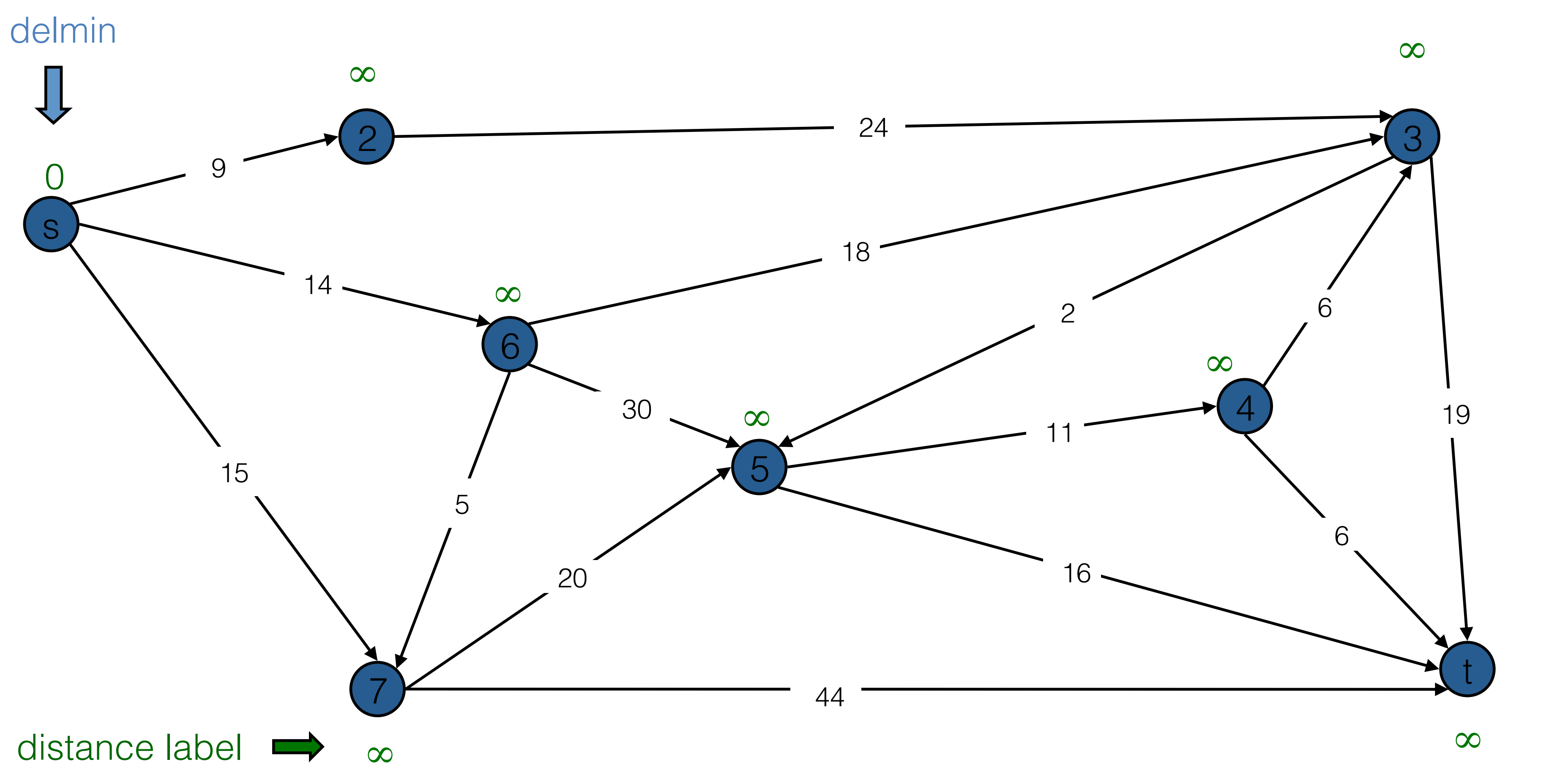
Unvisited = { s, 2, 3, 4, 5, 6, 7, t }



# Dijkstra's Shortest Path Algorithm

Visited = { }

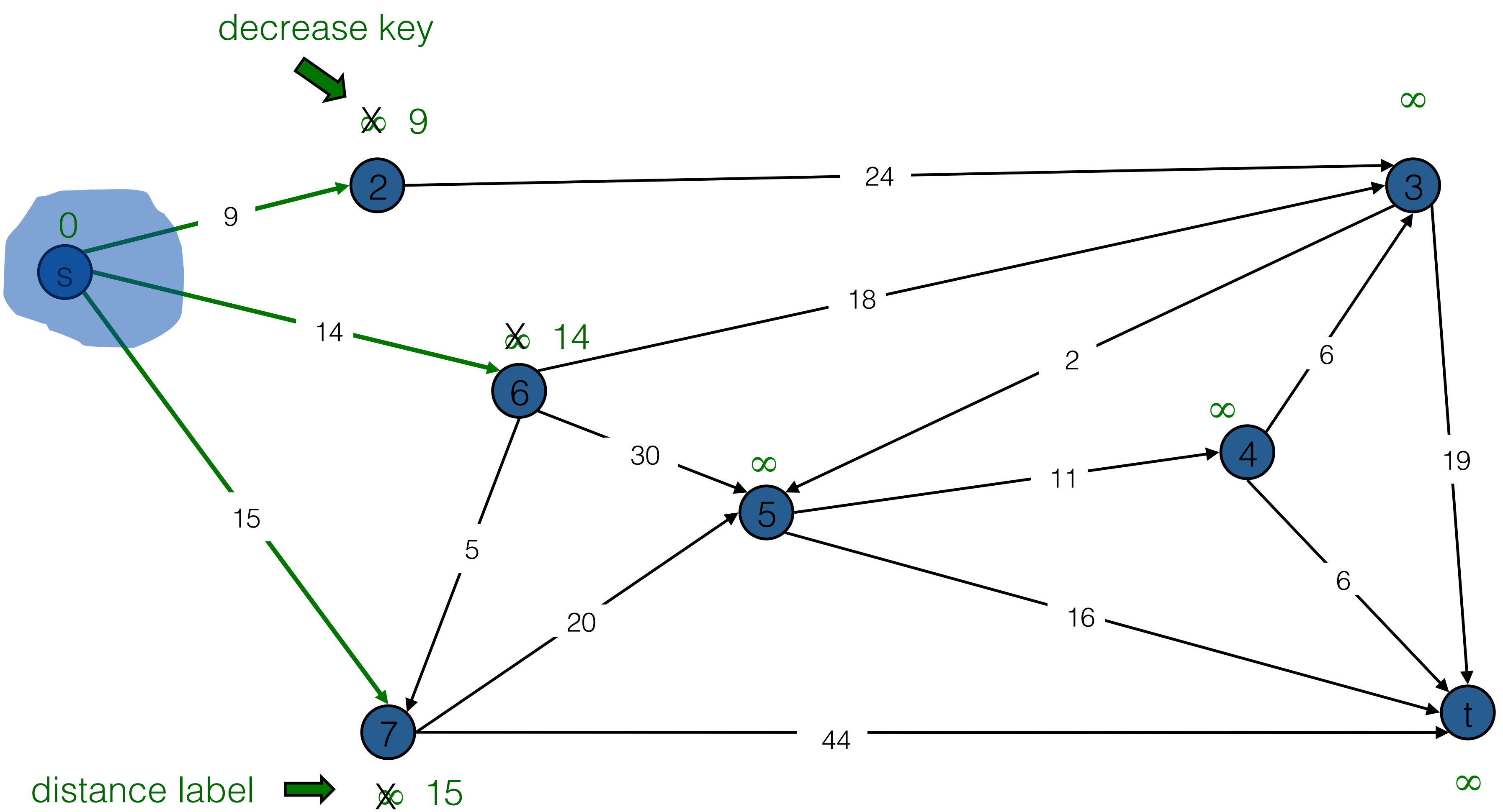
Unvisited = { s, 2, 3, 4, 5, 6, 7, t }



# Dijkstra's Shortest Path Algorithm

Visited = { s }

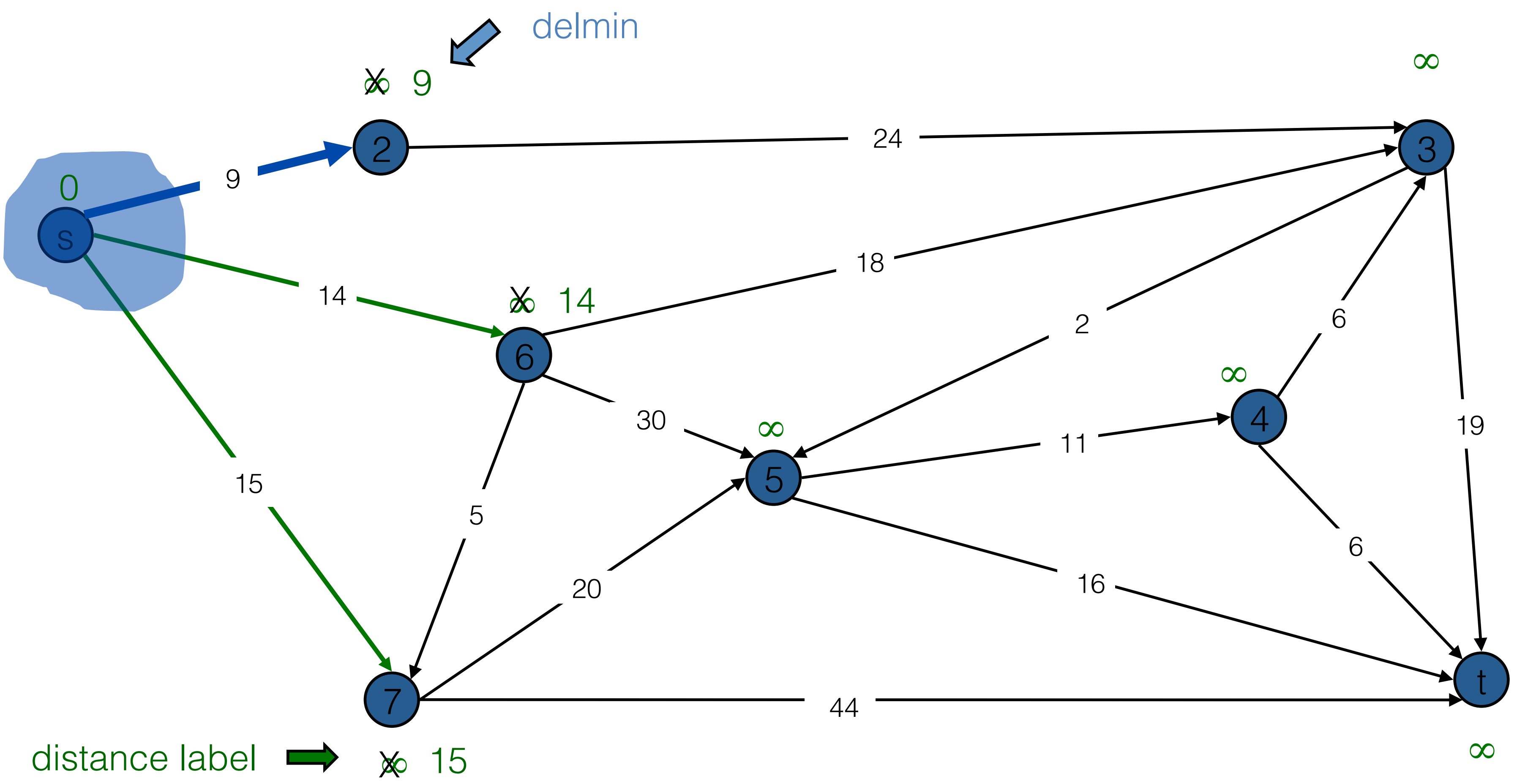
Unvisited = { 2, 3, 4, 5, 6, 7, t }



# Dijkstra's Shortest Path Algorithm

Visited = { s }

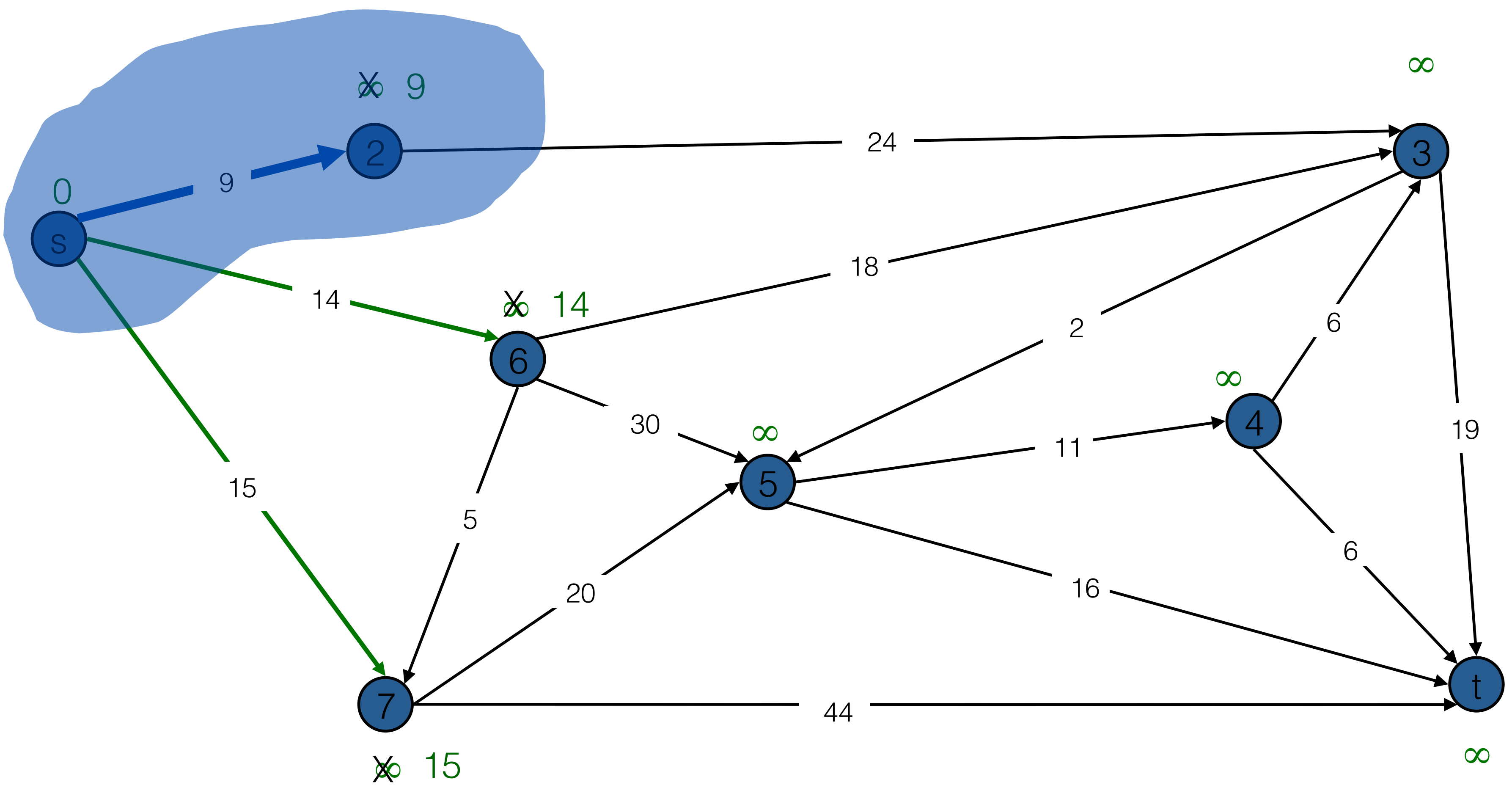
Unvisited = { 2, 3, 4, 5, 6, 7, t }



# Dijkstra's Shortest Path Algorithm

Visited = { s, 2 }

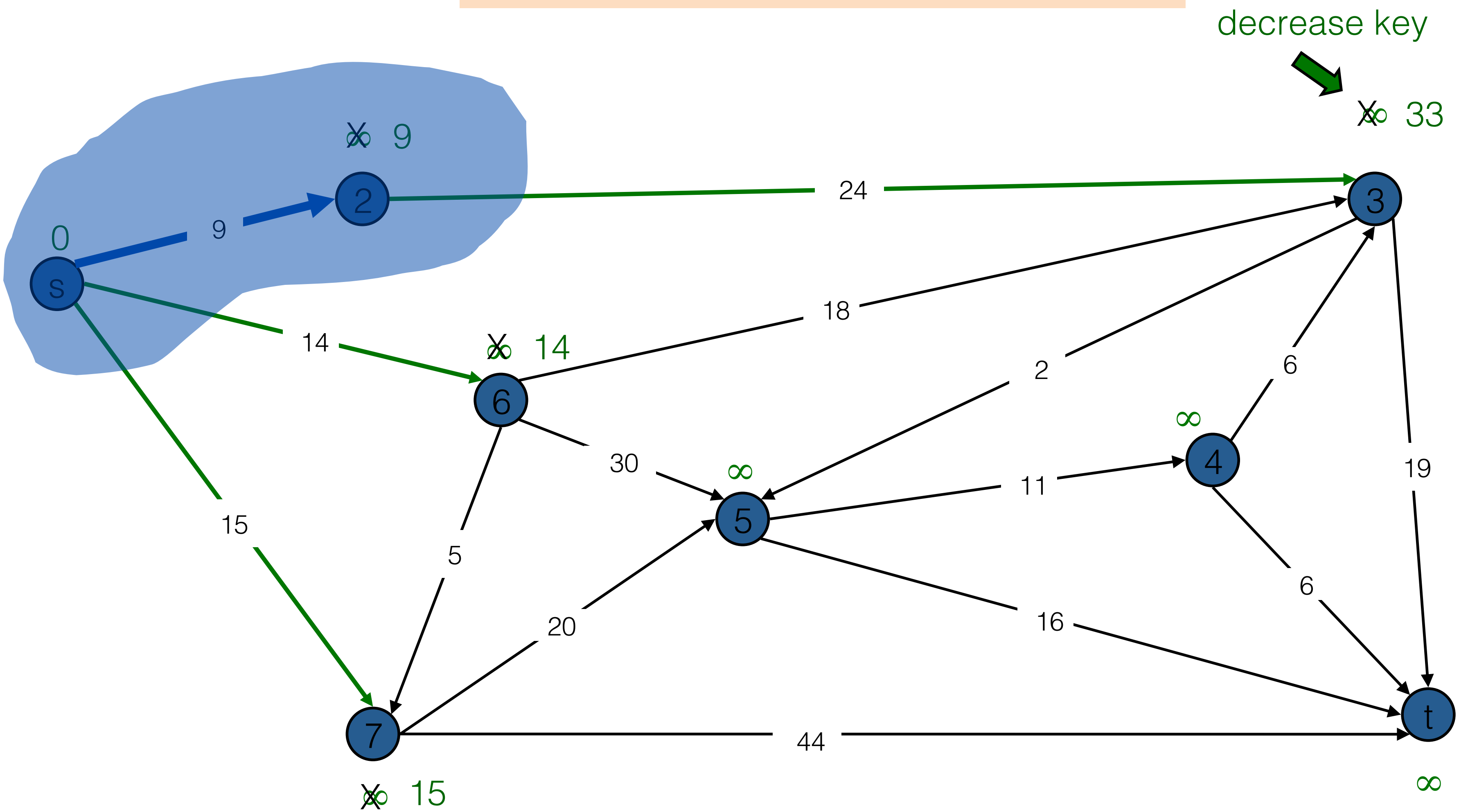
Unvisited = { 3, 4, 5, 6, 7, t }



# Dijkstra's Shortest Path Algorithm

Visited = { s, 2 }

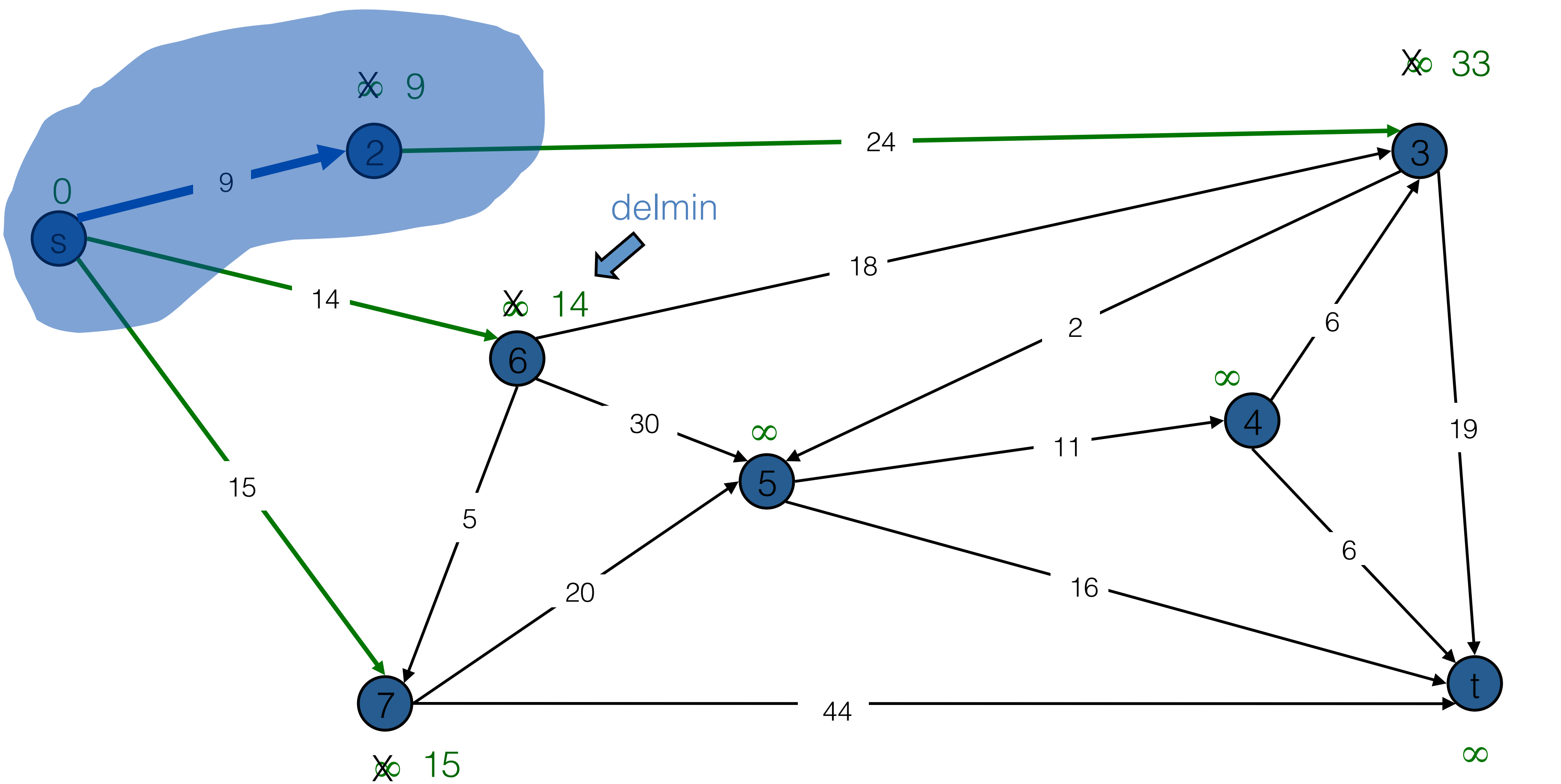
Unvisited = { 3, 4, 5, 6, 7, t }



# Dijkstra's Shortest Path Algorithm

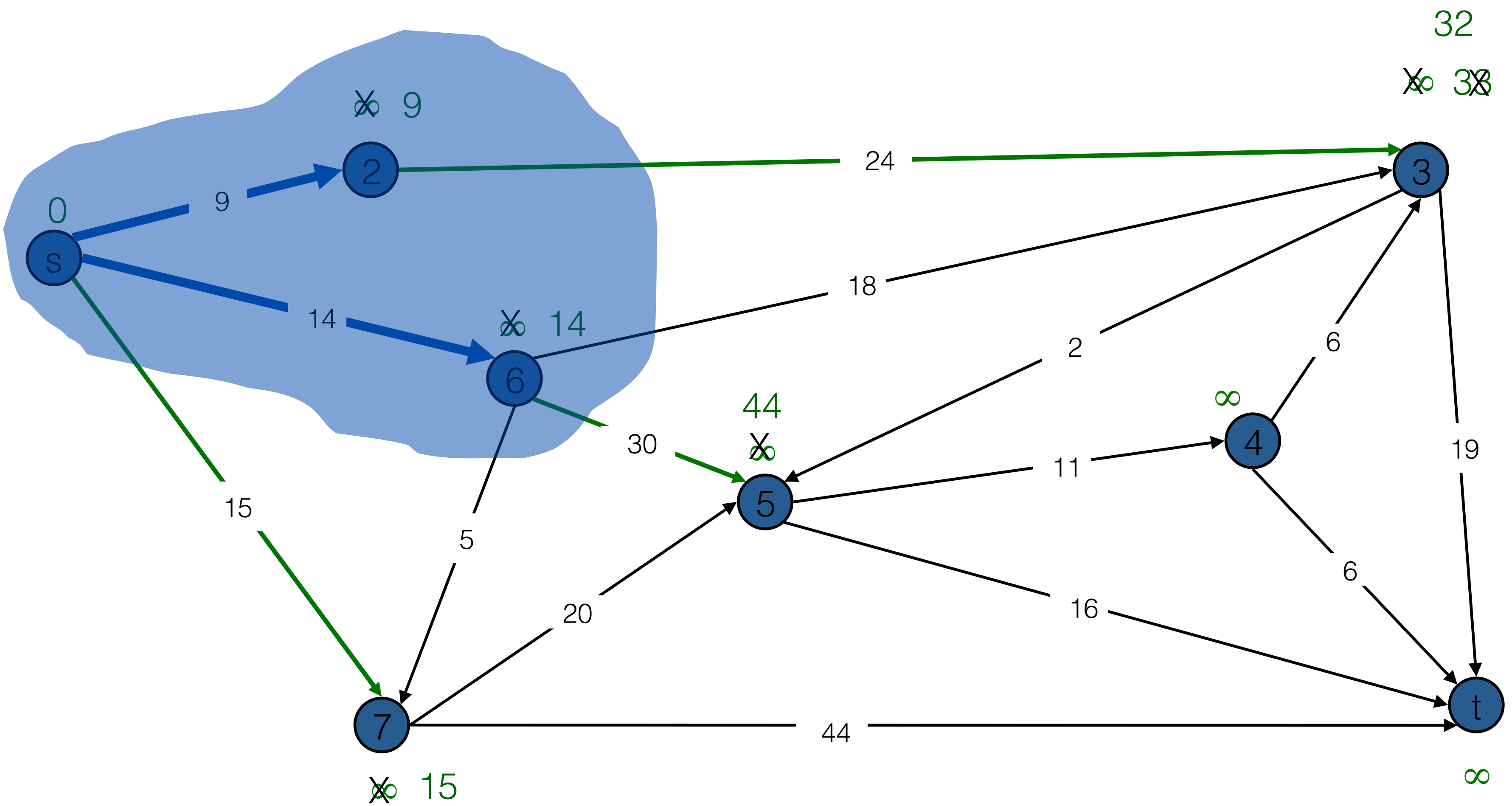
Visited = { s, 2 }

Unvisited = { 3, 4, 5, 6, 7, t }



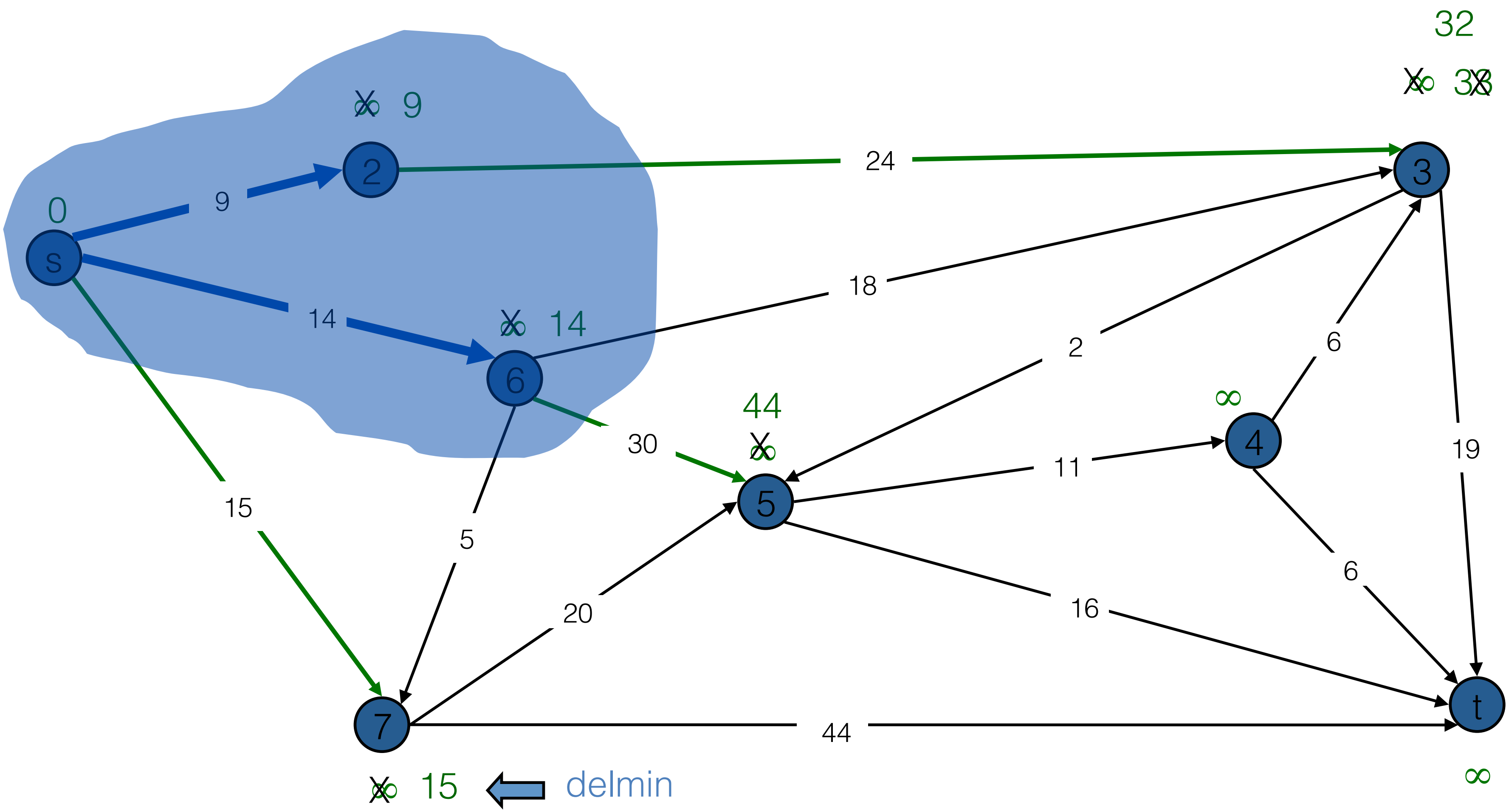
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 6 }
Unvisited = { 3, 4, 5, 7, t }



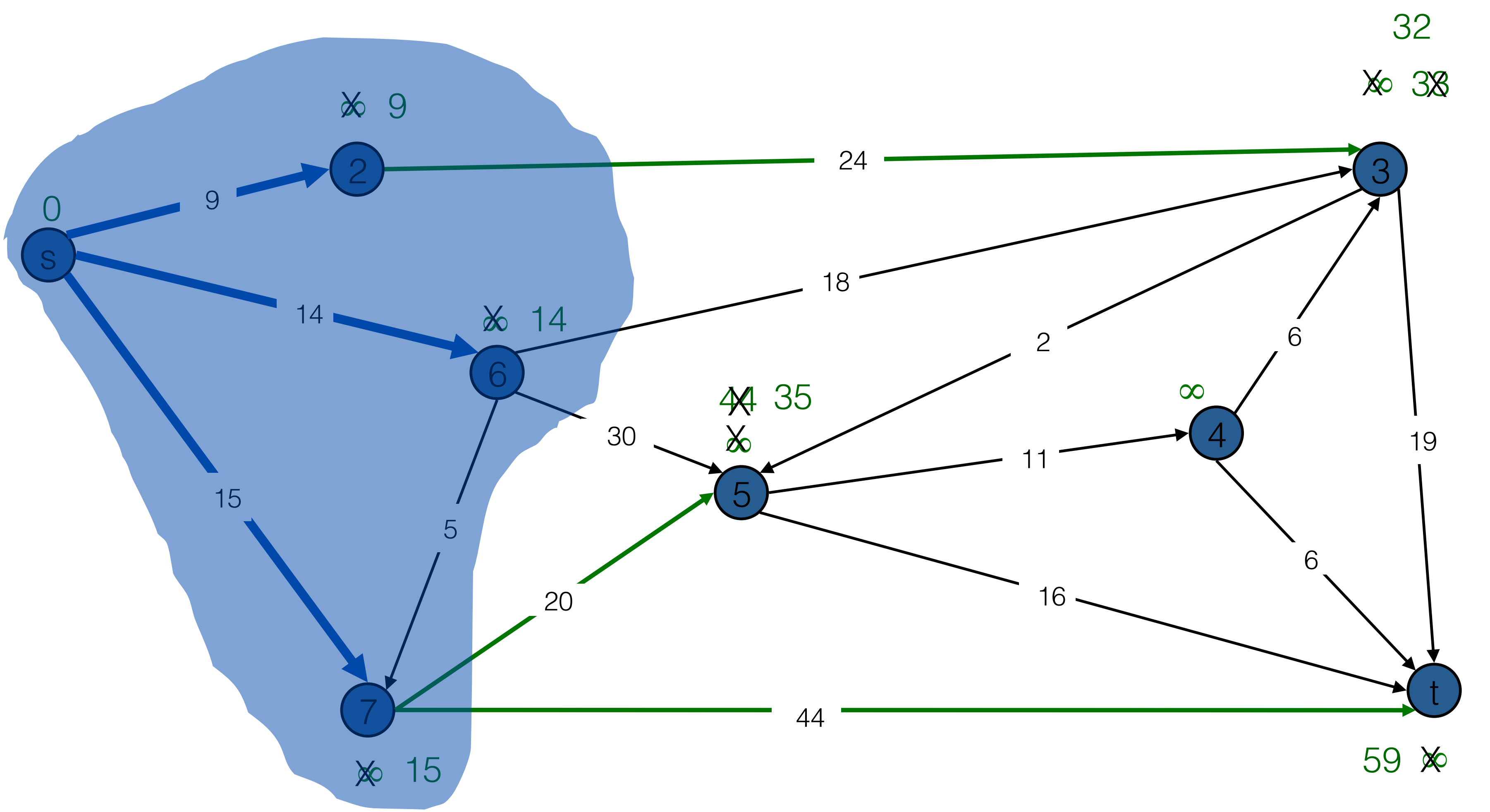
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 6 }
Unvisited = { 3, 4, 5, 7, t }



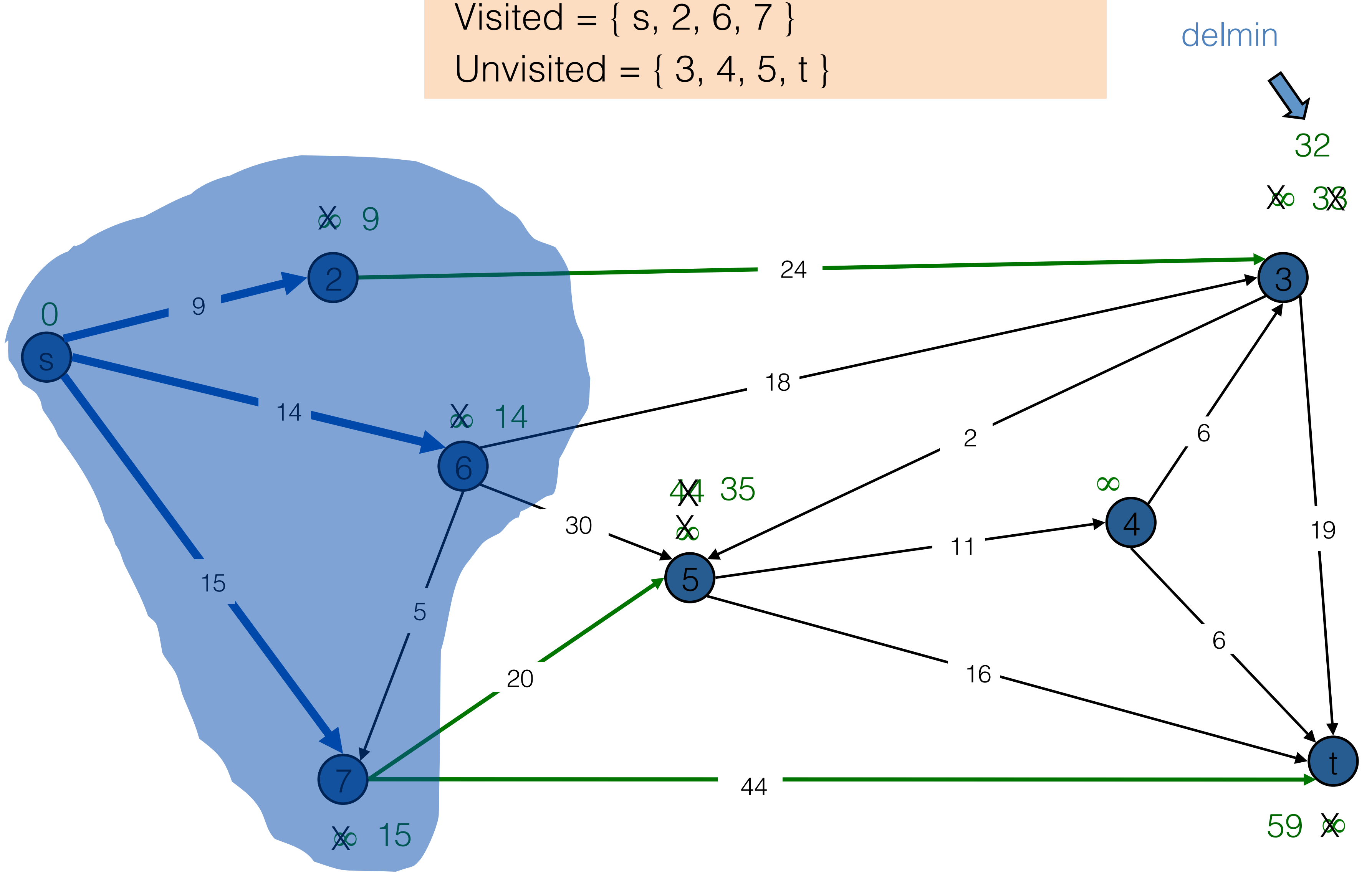
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 6, 7 }
Unvisited = { 3, 4, 5, t }



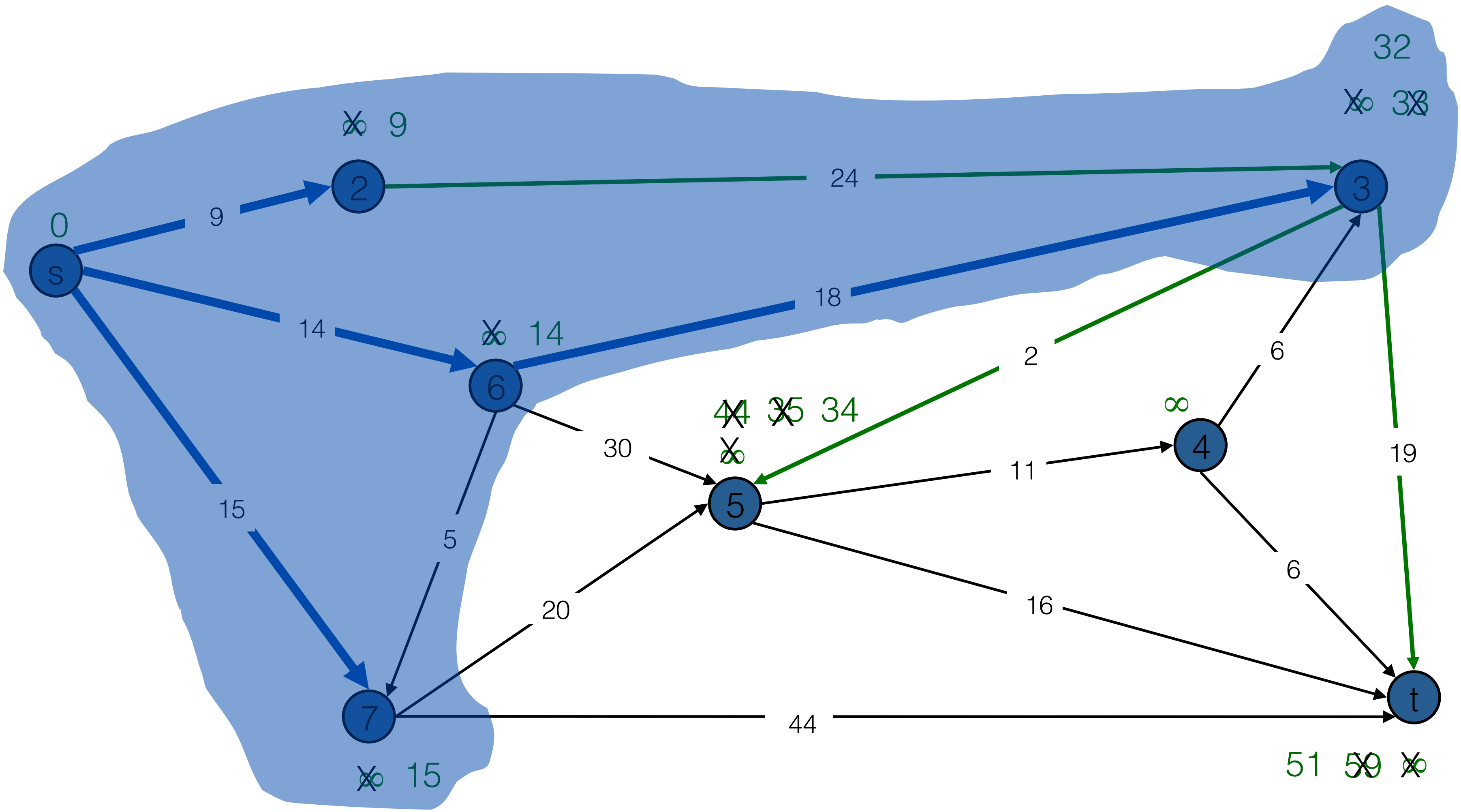
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 6, 7 }
Unvisited = { 3, 4, 5, t }



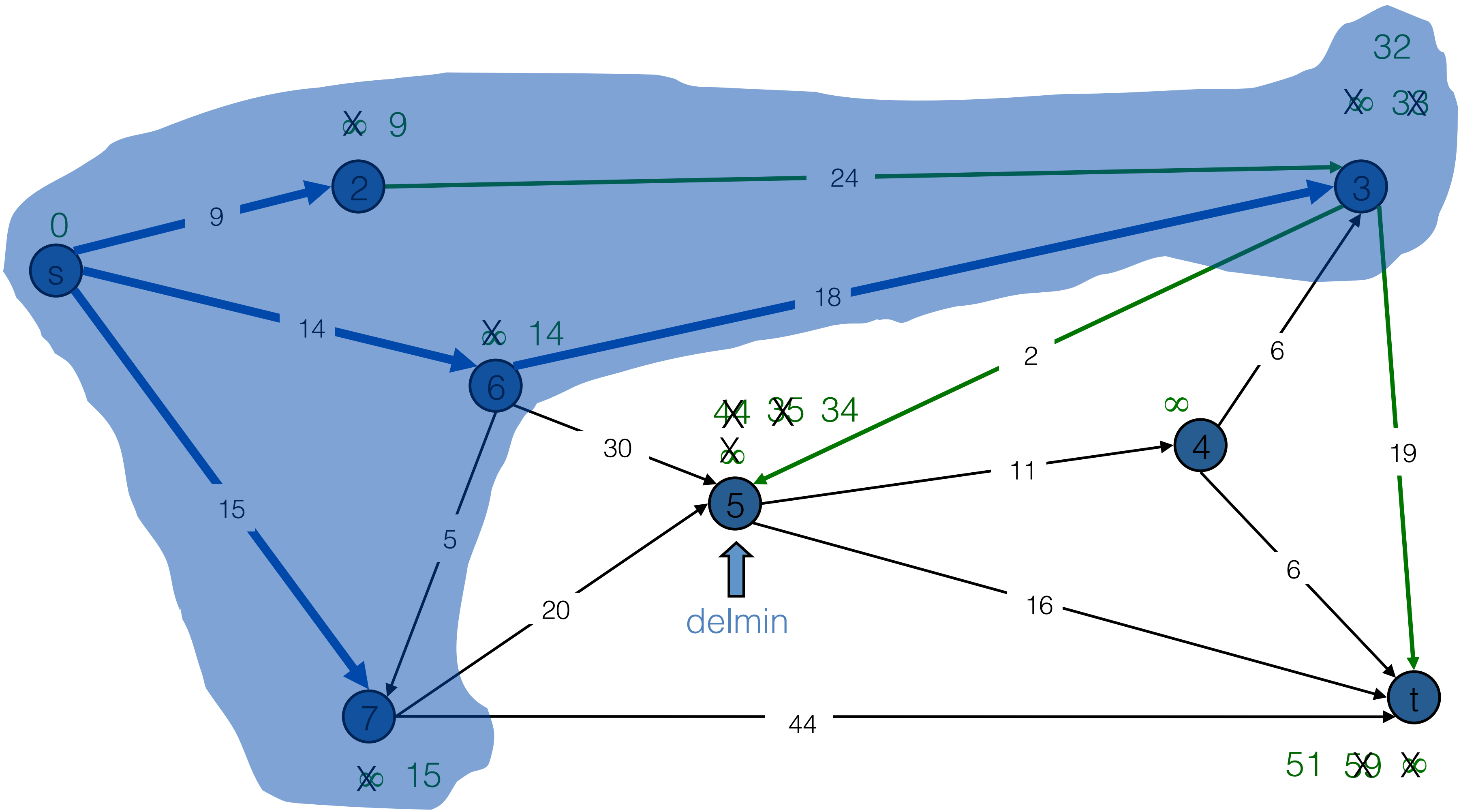
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 3, 6, 7 }
Unvisited = { 4, 5, t }



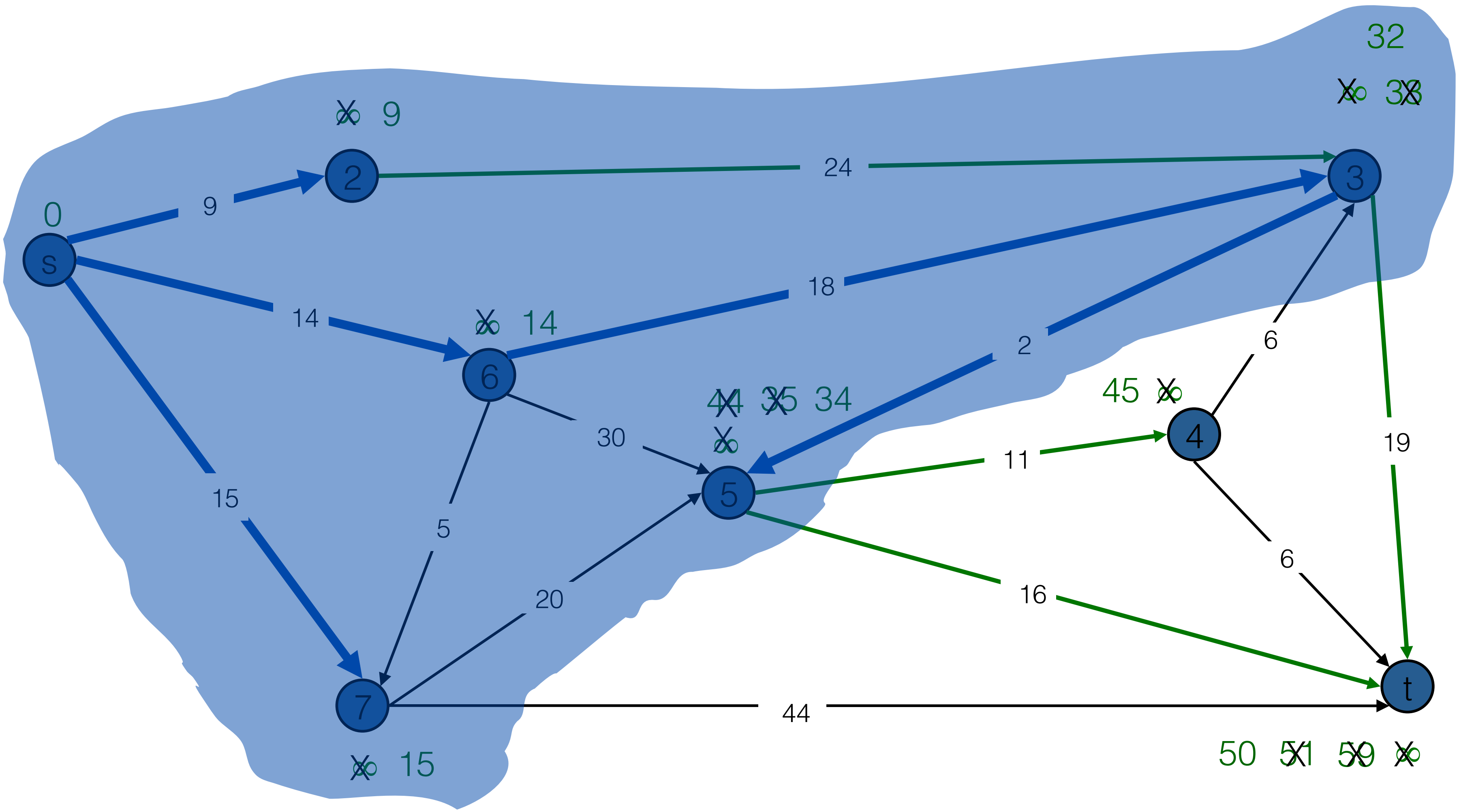
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 3, 6, 7 }
Unvisited = { 4, 5, t }



# Dijkstra's Shortest Path Algorithm

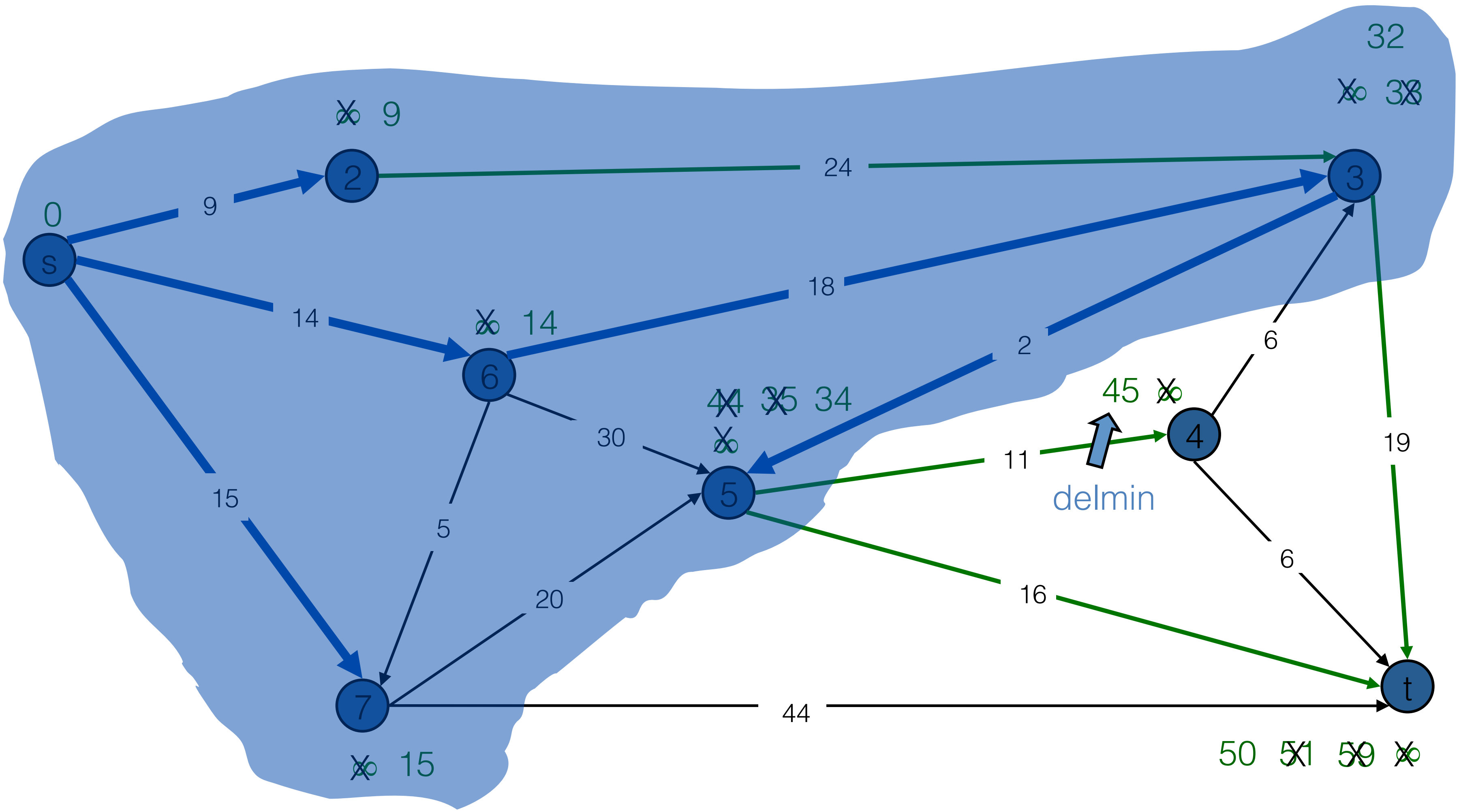
Visited = { s, 2, 3, 5, 6, 7 }
Unvisited = { 4, t }



# Dijkstra's Shortest Path Algorithm

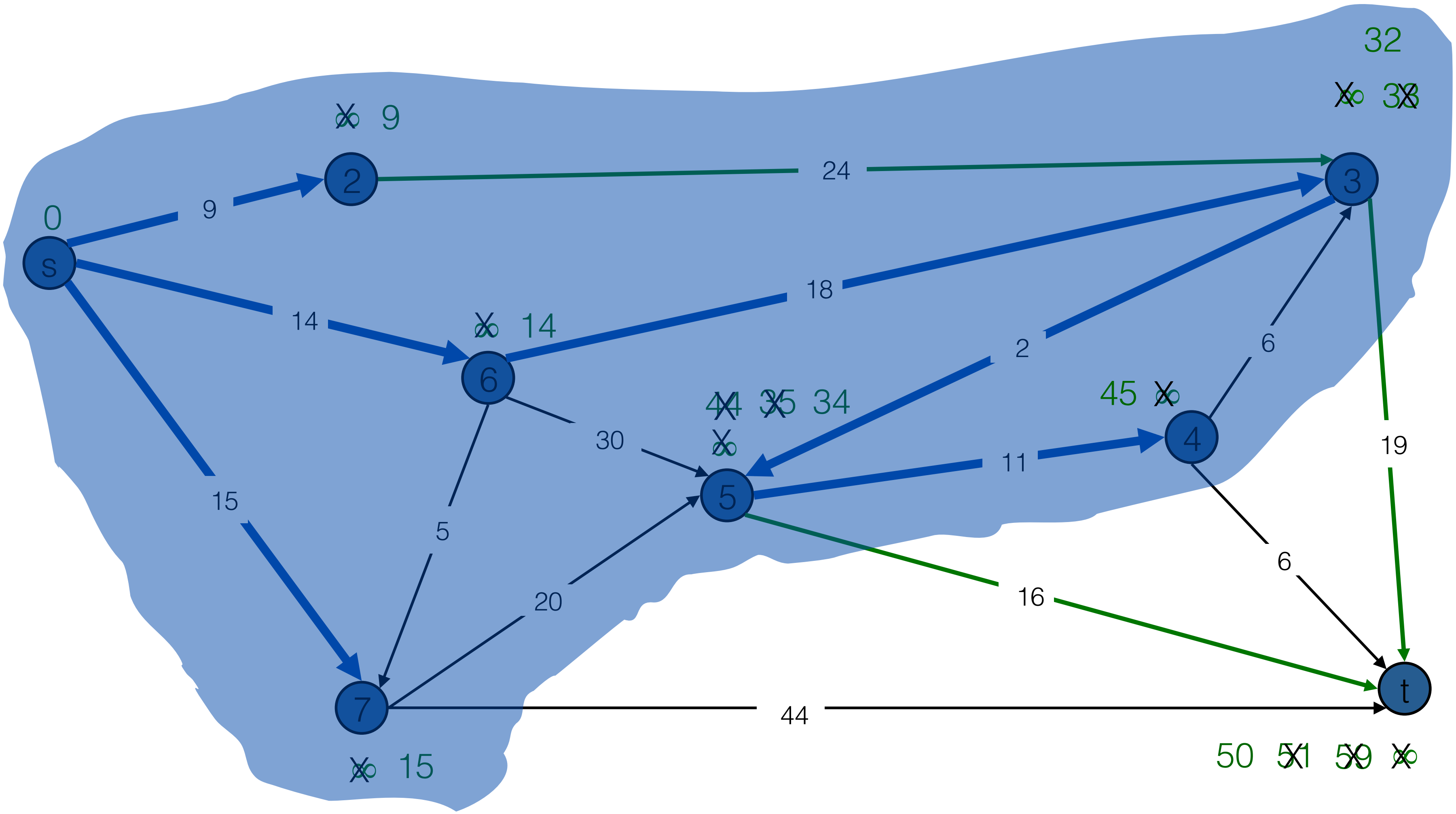
Visited = { s, 2, 3, 5, 6, 7 }

Unvisited = { 4, t }



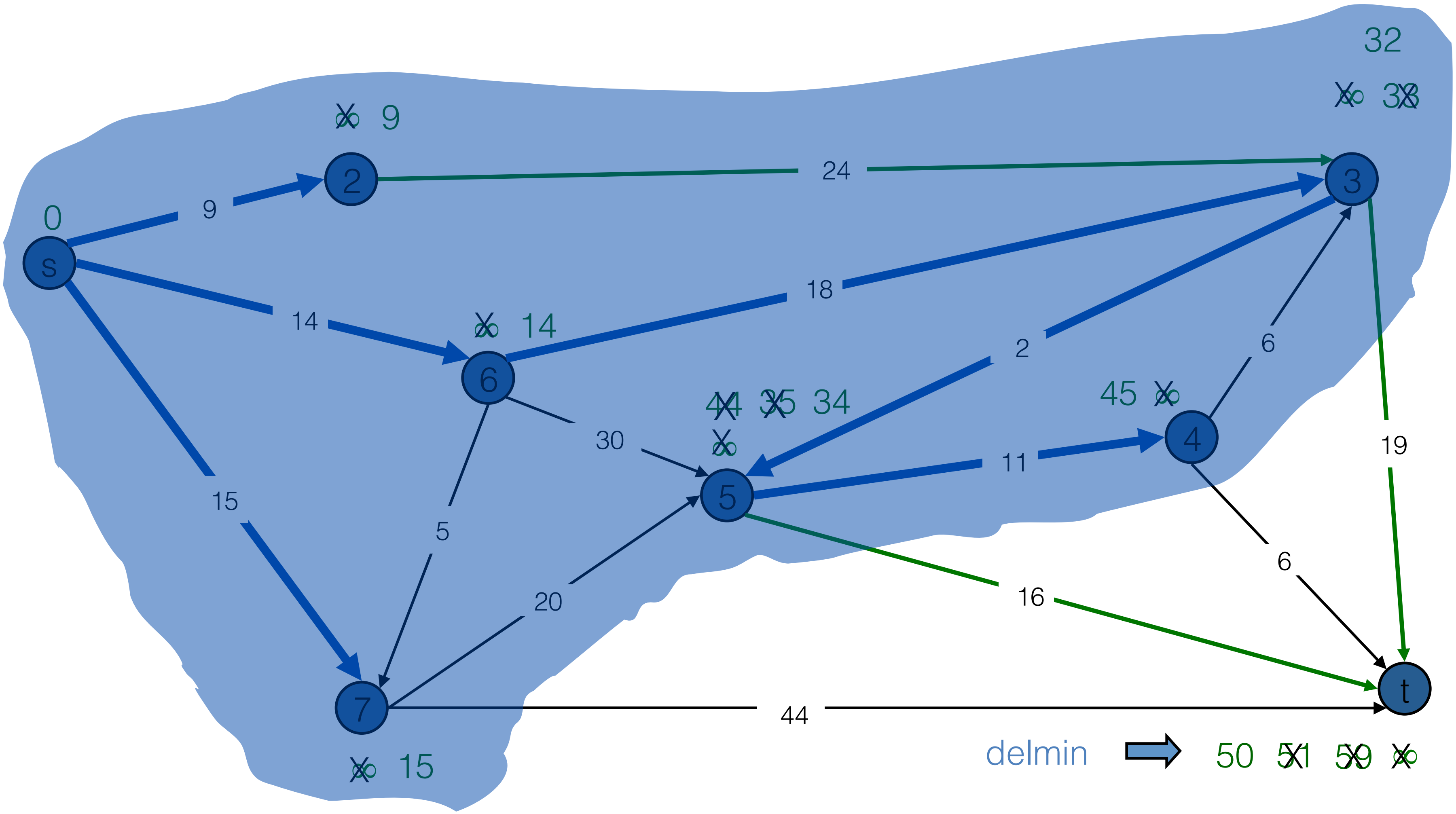
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 3, 4, 5, 6, 7 }
Unvisited = { t }



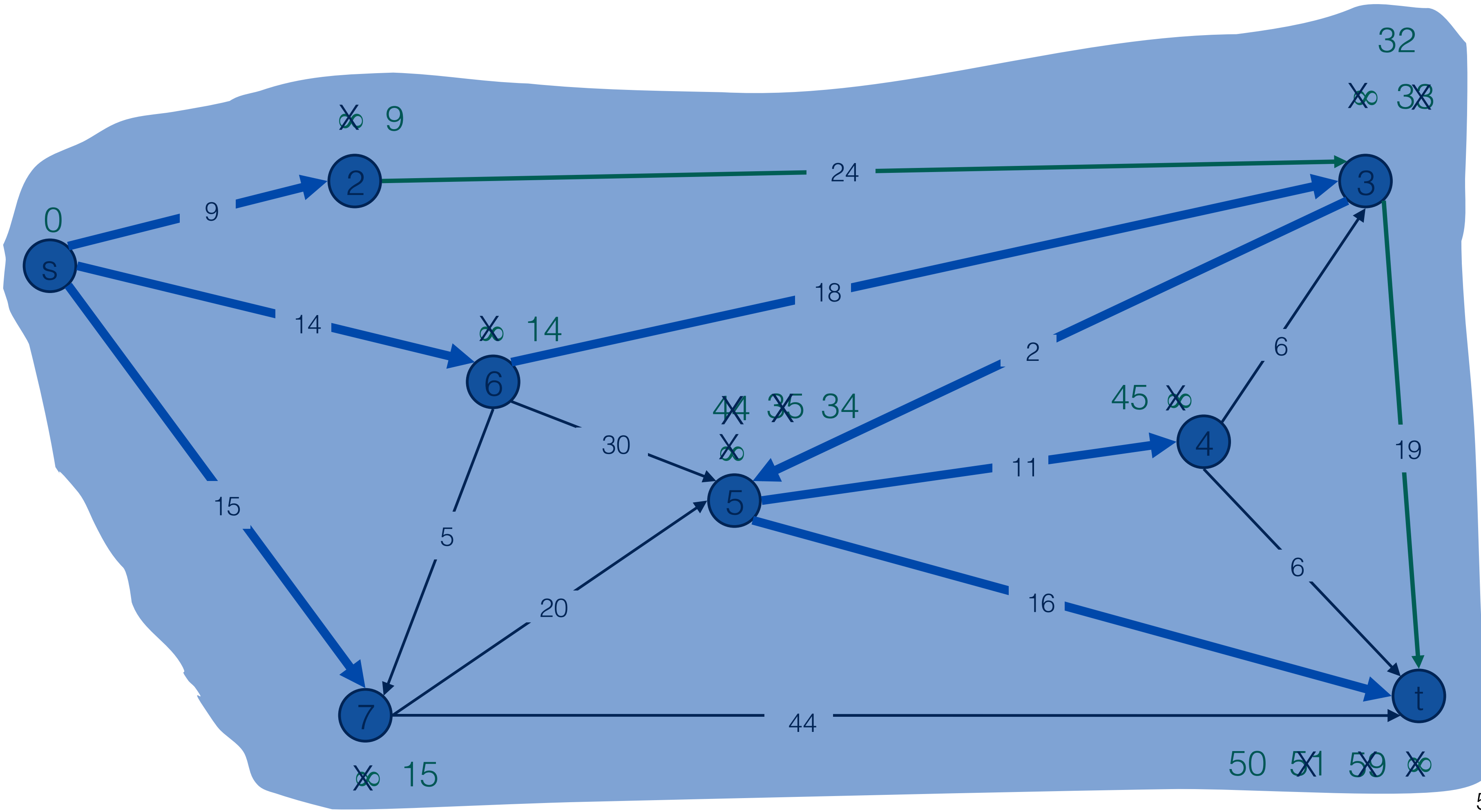
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 3, 4, 5, 6, 7 }
Unvisited = { t }



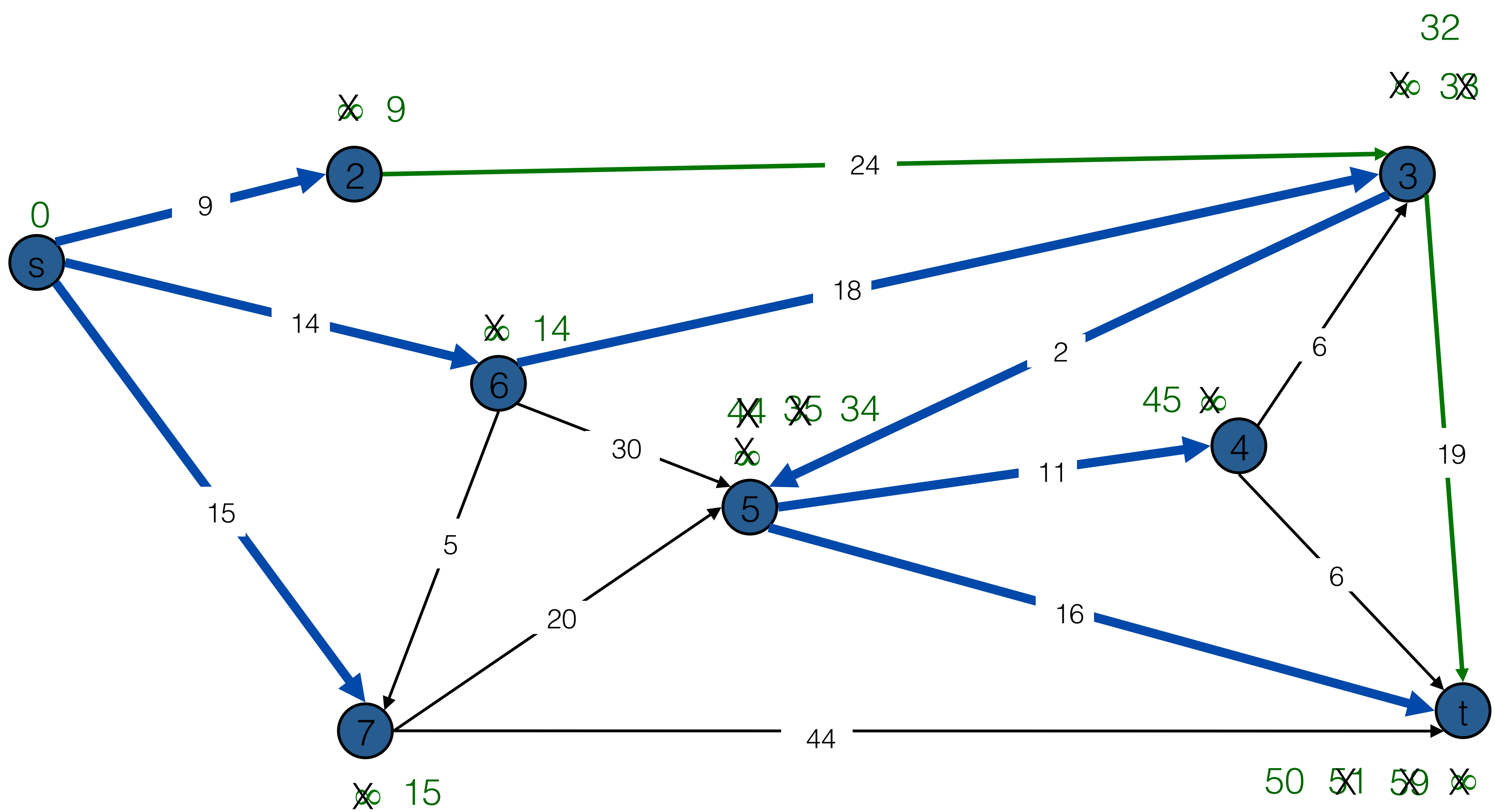
# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 3, 4, 5, 6, 7, t }
Unvisited = { }



# Dijkstra's Shortest Path Algorithm

Visited = { s, 2, 3, 4, 5, 6, 7, t }
Unvisited = { }



# Basics in Linear Algebra

# Eigenvalues and Eigenvectors

- Definition: Let $A$ be a symmetric $n \times n$ matrix. A vector $u$ is defined as eigenvector of $A$ if and only if $A u = \lambda u$, where $\lambda$ is a scalar called eigenvalue corresponding to u
  - In other words, $\lambda$ is an eigenvalue of $A$ iff the equation

    $$(A - \lambda I)\, u = 0$$

    has a non trivial solution

- Then, $A$ can be written as

$$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{U}^T$$

  - where the orthogonal matrix $U$ contains as columns the eigenvectors $u_1, \ldots, u_n$ that correspond to $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$
  - $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

# Singular Value Decomposition (SVD)

- Definition: The SVD of a real matrix $A \in R^{m \times n}$ is defined as

$$\mathbf{A} = \mathbf{U}\ \mathbf{\Sigma}\ \mathbf{V}^T$$

  - $U$ is the $m \times m$ orthogonal matrix that contains the left-singular vectors of $A$
  - $V$ is the $n \times n$ orthogonal matrix that contains the right-singular vector of $A$
  - $\Sigma$ is the $m \times n$ diagonal matrix with nonnegative entries comprised of the singular values $\sigma_i$ sorted from high to low
  - Note: for symmetric matrices, the singular values correspond to the absolute values of the eigenvalues

- Rank $r$ of a matrix $A$
  - Number of linearly independent rows or columns
  - $r \leq min\{m,n\}$
  - $r$ = number of nonzero singular values $\sigma_i$

$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)$

Orthogonal matrix:
$Q^TQ = QQ^T = I$

# Low Rank Approximation with SVD

- How to approximate matrix $A_{m \times n}$ with a rank-$k$ matrix ($k << n$)?
  - Meaning: All the rows are linear combinations of a set of merely $k$ rows (same for columns)
  - Super important: dimensionality reduction (e.g., PCA)
  - Choosing a rank-$k$ matrix boils down to choosing sets of $k$ vectors
  - Do we have any way to do that? Yes, SVD does exactly that!

- Compute SVD of $A = U \Sigma V^T$
  - Keep only the top $k$ left singular vectors of $U$ (first $k$ columns): $U_k$ ($m \times k$ matrix)
  - Keep only the top $k$ right singular vectors of $V^T$ (first $k$ rows): $V_k^T$ ($k \times n$ matrix)
  - Keep only the top $k$ singular values: set $\Sigma_k$ to be the first $k$ rows/columns of $\Sigma$ (a $k \times k$ diagonal matrix) – $k$ largest singular values

$$\mathbf{A}_k = \mathbf{U}_k \; \mathbf{\Sigma}_k \; \mathbf{V}_k^T$$

The Frobenious norm $|| A - A_k ||_F$ is minimized

$$\| \mathbf{A} \|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2}$$

# Basics in
# Spectral Graph Theory

# Spectral Graph Theory

- Spectral graph theory: study the properties of a graph based on matrices that are associated with the graph
  - Spectral properties: eigenvalues and eigenvectors

- Adjacency matrix and Laplacian matrix

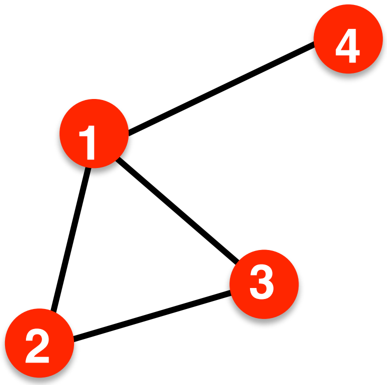
- We have already seen the basic properties of the adjacency matrix of a graph $G$
  - In fact, it's an $n \times n$ matrix $A$
    - Symmetric, if $G$ is undirected
    - Non symmetric, if $G$ is directed

# Laplacian Matrix

- Let $G = (V, E)$ be a graph. Then, the Laplacian matrix is defined as

$$\mathbf{L}_{ij} = \begin{cases} k_i & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Diagonal degree matrix $D$, where $D_{ii} = k_i$ (node degrees)




$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

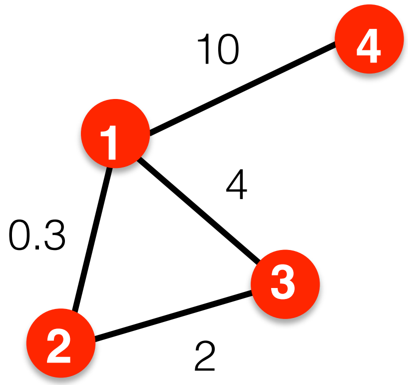
Not so useful spectrum
(largest eigenvalues)

# Weighted Laplacian Matrix

- Let $G = (V, E)$ be a weighted graph

$$\mathbf{L}_{ij} = \begin{cases} k_i = \sum_j w_{ij} & \text{if } i = j \\ -w_{ij} & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Diagonal degree matrix $D$, where $D_{ii} = k_i$ (node degrees)




$$\mathbf{L} = \begin{bmatrix} 14.3 & -0.3 & -4 & -10 \\ -0.3 & 2.3 & -2 & 0 \\ -4 & -2 & 6 & 0 \\ -10 & 0 & 0 & 10 \end{bmatrix}$$

# Normalized Laplacian Matrix

- Let $G = (V, E)$ be a graph. The normalized Laplacian matrix is defined as

$$\mathbf{L^n}_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{\sqrt{k_i k_j}} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

For $L$ : $0 = \lambda_0 \leq \lambda_i \leq 2\, k^{max}$

For $L^n$ : $0 = \lambda_0 \leq \lambda_i \leq 2$

$$\begin{aligned} \mathbf{L^n} &= \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} \\ &= \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2} \end{aligned}$$

# Properties of the Laplacian Matrix $L^n$

- Properties of spectrum $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$ of Laplacian $L^n$
  - Symmetric and positive semi-definite ($\lambda_i \geq 0$, for all $i$). Also, $\lambda_0 = 0$
  - If the second smallest eigenvalue $\lambda_1 \neq 0$, then $G$ is connected
  - If $L^n$ has $K$ zero eigenvalues, $G$ has $K$ connected components
  - The smallest non-zero eigenvalue is called spectral gap

    (the corresponding eigenvector is used in spectral clustering)

More on graph matrices:
Transition matrix
(lecture on random walks and centrality)

# How do we measure and characterize a network?

## More properties

# Key Network Properties

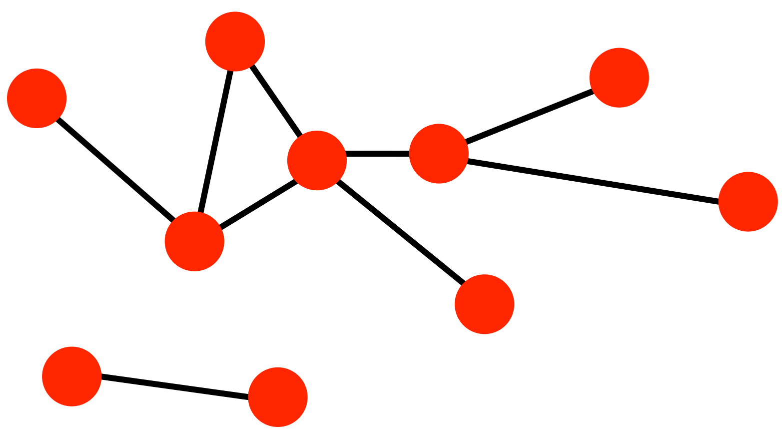
| | |
|---|---|
| Degree distribution: | $P(k)$ |
| Path length: | $h$ |
| Clustering coefficient: | $C$ |

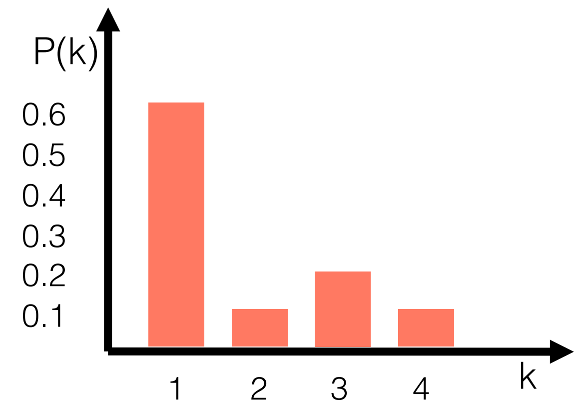
# Degree Distribution

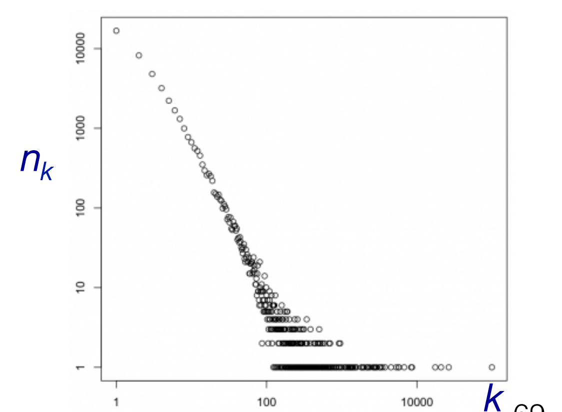
- Degree distribution $P(k)$: Probability that a randomly chosen node has degree $k$

  $n_k$ = # nodes with degree $k$

- Normalized histogram:
  $P(k) = n_k / n$ ➔ plot





# Number of Paths

Property: Let $A^h$ denote the $h$-th power of $A$, with entries $A^h_{uv}$

- Then, element $A^h_{uv}$ captures the number of $u - v$ paths of length $h$ in $G$

  - \# of paths of length $h=1$: If there is a link between $u$ and $v$, $A_{uv}=1$ else $A_{uv}=0$

  - \# of paths of length $h=2$: If there is a path of length two between $u$ and $v$ through $k$, the product $A_{uk} A_{kv}=1$ else $A_{uk} A_{kv}=0$

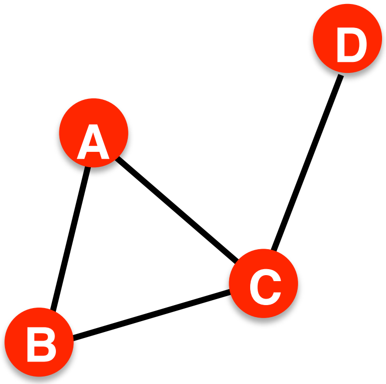
$$H_{uv}^{(2)} = \sum_{k=1}^{n} \mathbf{A}_{uk}\mathbf{A}_{kv} = \mathbf{A}_{uv}^{2}$$

  - \# of paths of length $h$: If there is a path of length $h$ between $u$ and $v$ then $A_{uk}$ .... $A_{kv}=1$ else $A_{uk}$ .... $A_{kv}=0$

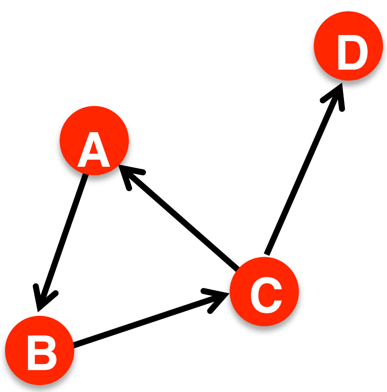
$$H_{uv}^{(h)} = \mathbf{A}_{uv}^{h}$$

(holds for both directed and undirected graphs)

# Distance in a Graph




$h_{B,D} = 2$




$h_{B,C} = 1, h_{C,B} = 2$

- Distance (shortest path, geodesic) between a pair of nodes is defined as the number of edges along the shortest path connecting the nodes
  - If the two nodes are disconnected, the distance is usually defined as infinite
- In directed graphs paths need to follow the direction of the arrows
  - Consequence: Distance is **not symmetric**: $h_{A,C} \neq h_{C,A}$

# Network Diameter

- Diameter: the maximum (shortest path) distance between any pair of nodes in a graph

- Average path length for a connected graph (component) or a strongly connected (component of a) directed graph
  - Many times we compute the average only over the connected pairs of nodes (that is, we ignore “infinite” length paths)

$$\bar{h} = \frac{1}{n(n-1)} \sum_{i,j \neq i} h_{ij}$$

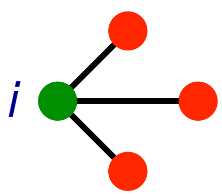
where $h_{ij}$ is the distance from node $i$ to node $j$

# Clustering Coefficient (1/2)

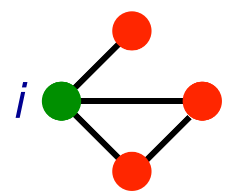
- Clustering coefficient
  - What portion of node i's neighbors are connected?
  - Node *i* with degree $k_i$
  - $C_i \in [0,1]$

$$C_i = \frac{2e_i}{k_i(k_i - 1)}$$

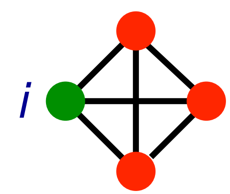
where $e_i$ is the number of edges between the neighbors of node *i*




$C_i$=0




$C_i$=1/3




$C_i$=1

Average clustering coefficient:

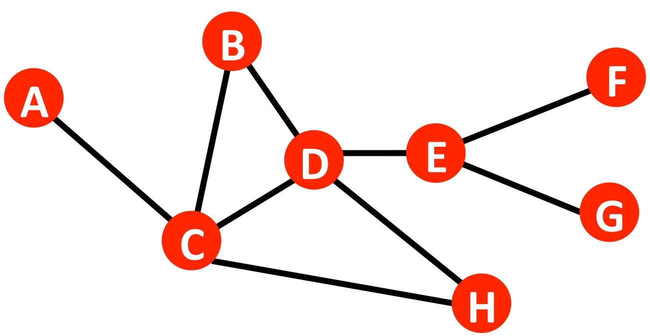
$$C = \frac{1}{|V|} \sum_{i=1}^{|V|} C_i$$

# Clustering Coefficient (2/2)

- Clustering coefficient
  - What portion of node i's neighbors are connected?
  - Node $i$ with degree $k_i$
  - $C_i \in [0,1]$

$$C_i = \frac{2e_i}{k_i(k_i - 1)}$$

where $e_i$ is the number of edges between the neighbors of node $i$




$k_B$=2, $e_B$=1, $C_B$=2/2 = 1

$k_D$=4, $e_D$=2, $C_D$=4/12 = 1/3

# Key Network Properties

Degree distribution: $P(k)$

Path length: $h$

Clustering coefficient: $C$

They can inform us about the structure of a network

What is happening in real networks?

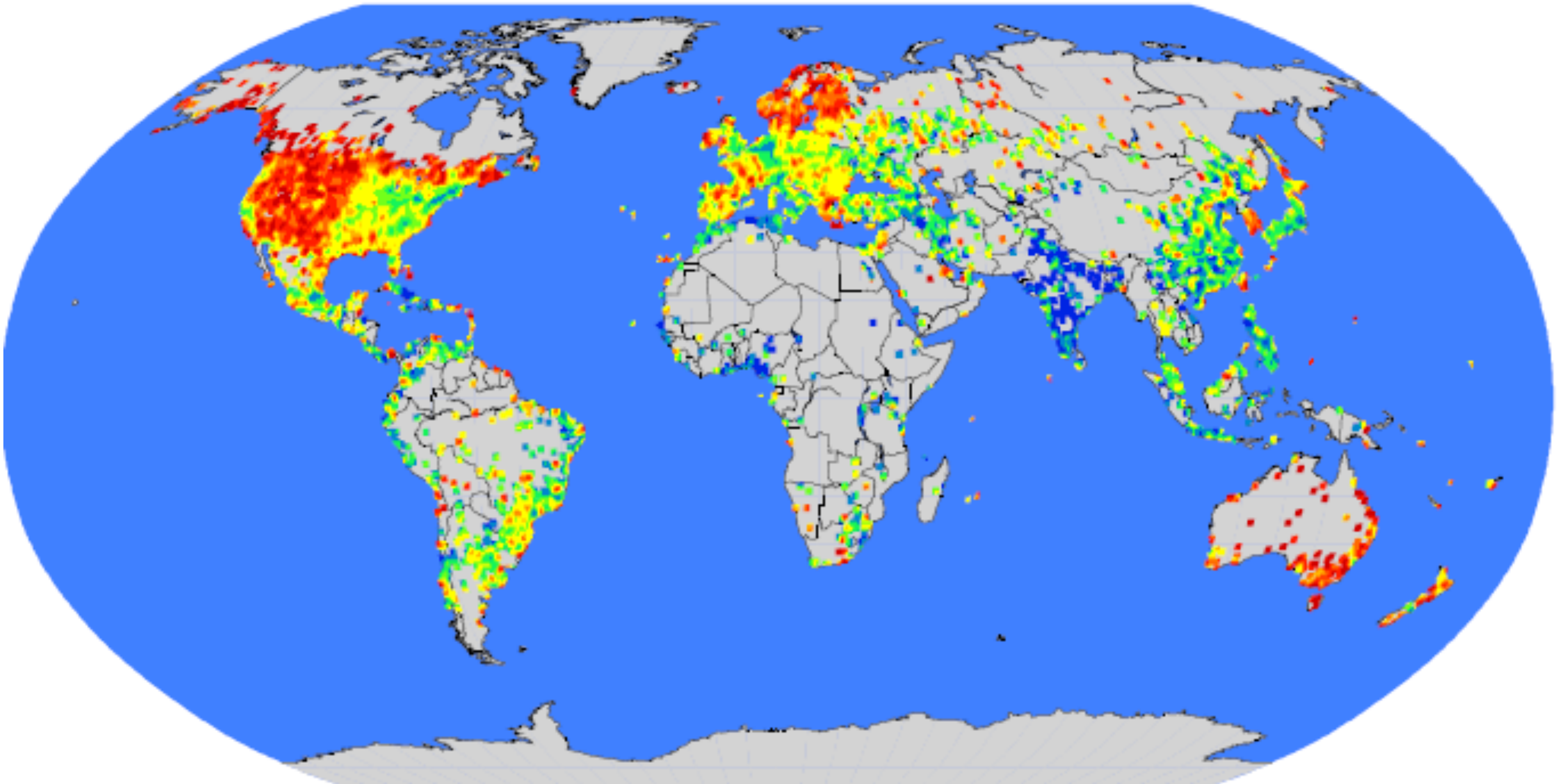
# Let’s see the properties of a real network:

# The MSN messenger case

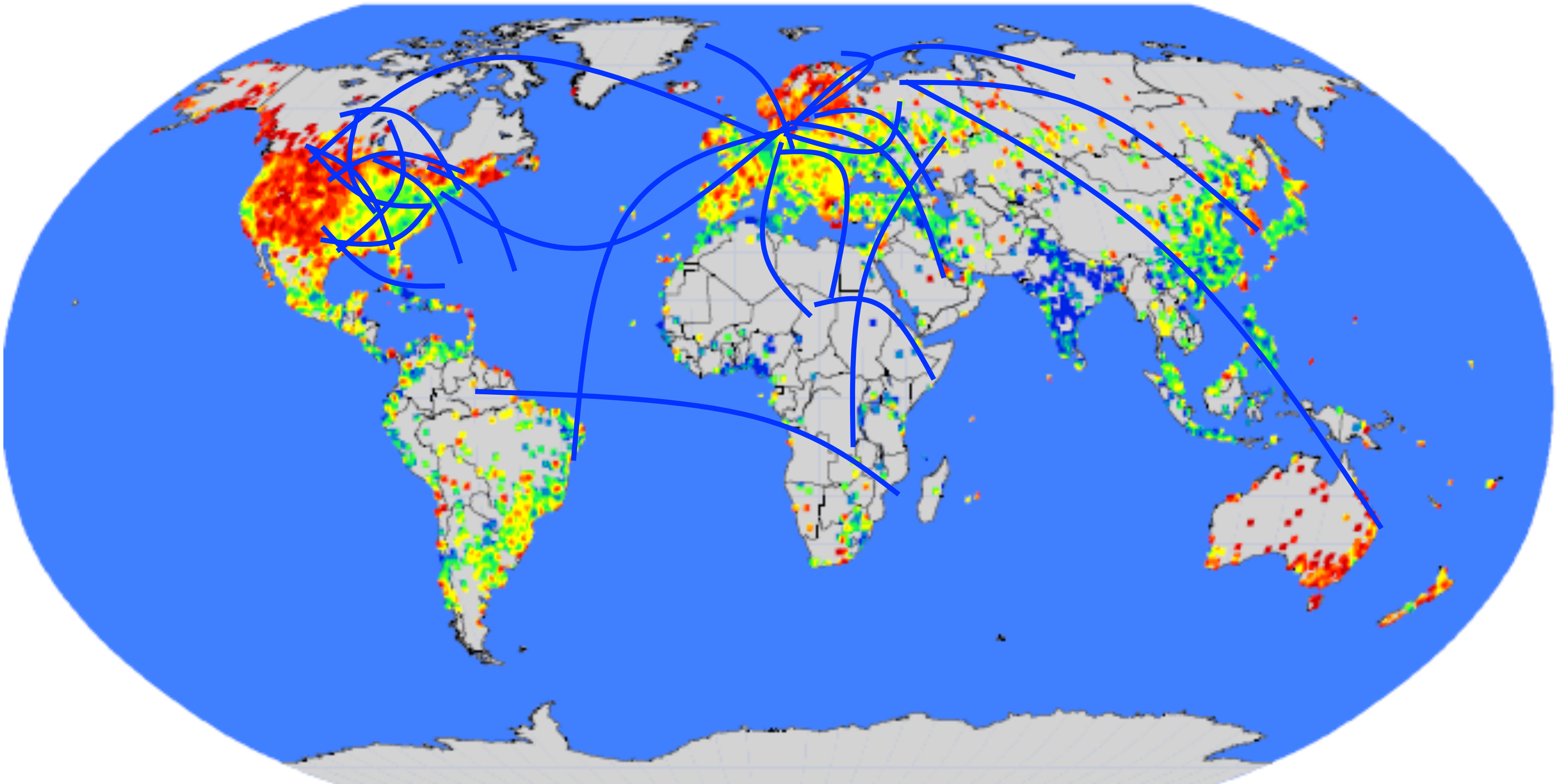
# The MSN Messenger


- MSN Messenger activity in June 2006:
  - 245 million users logged in
  - 180 million users engaged in conversations
  - More than 30 billion conversations
  - More than 255 billion exchanged messages
  - Now called *Windows Live Messenger*

# Communication: Geography

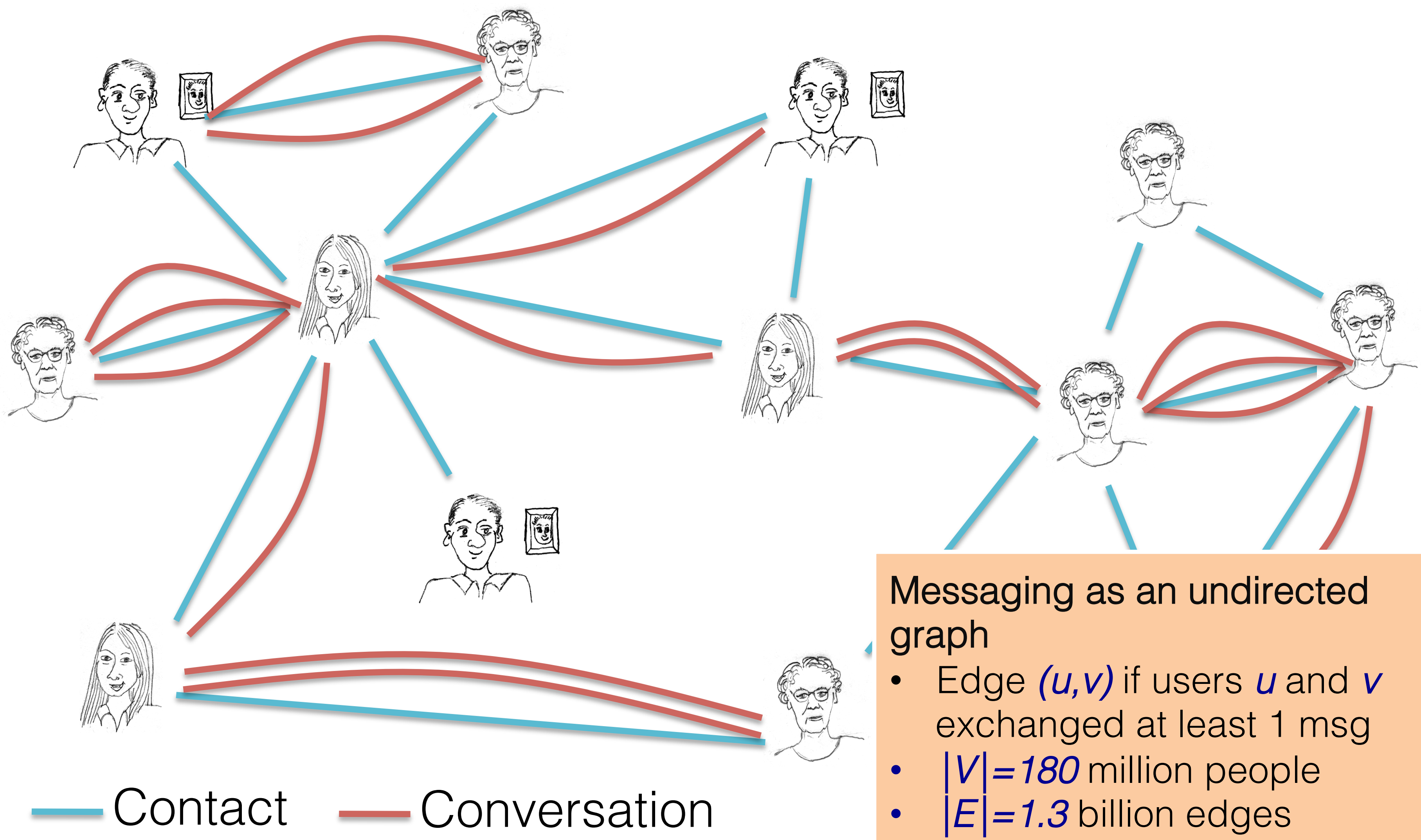


# Communication network



Network: 180M people, 1.3B edges

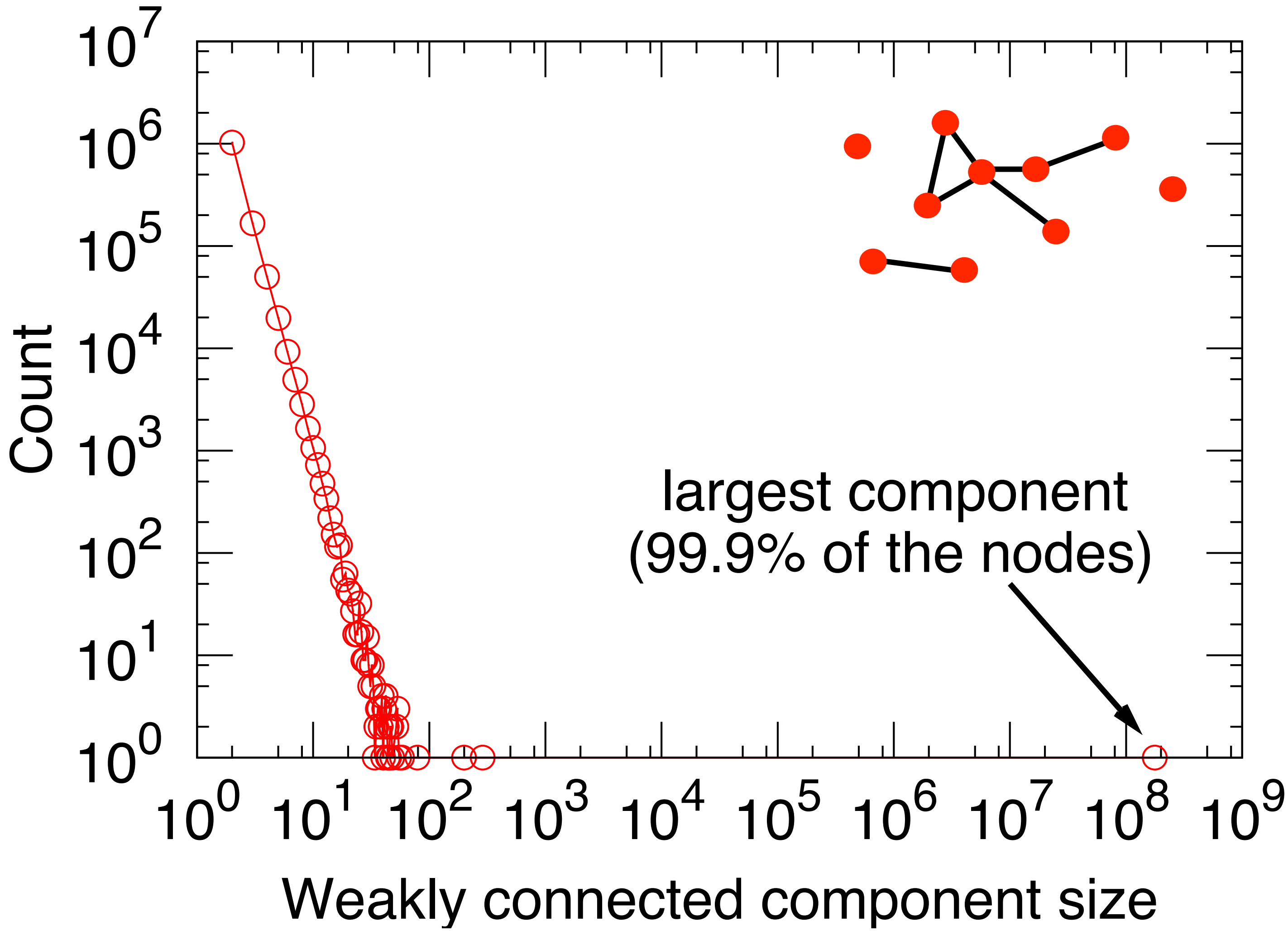
# Messaging as a Multigraph



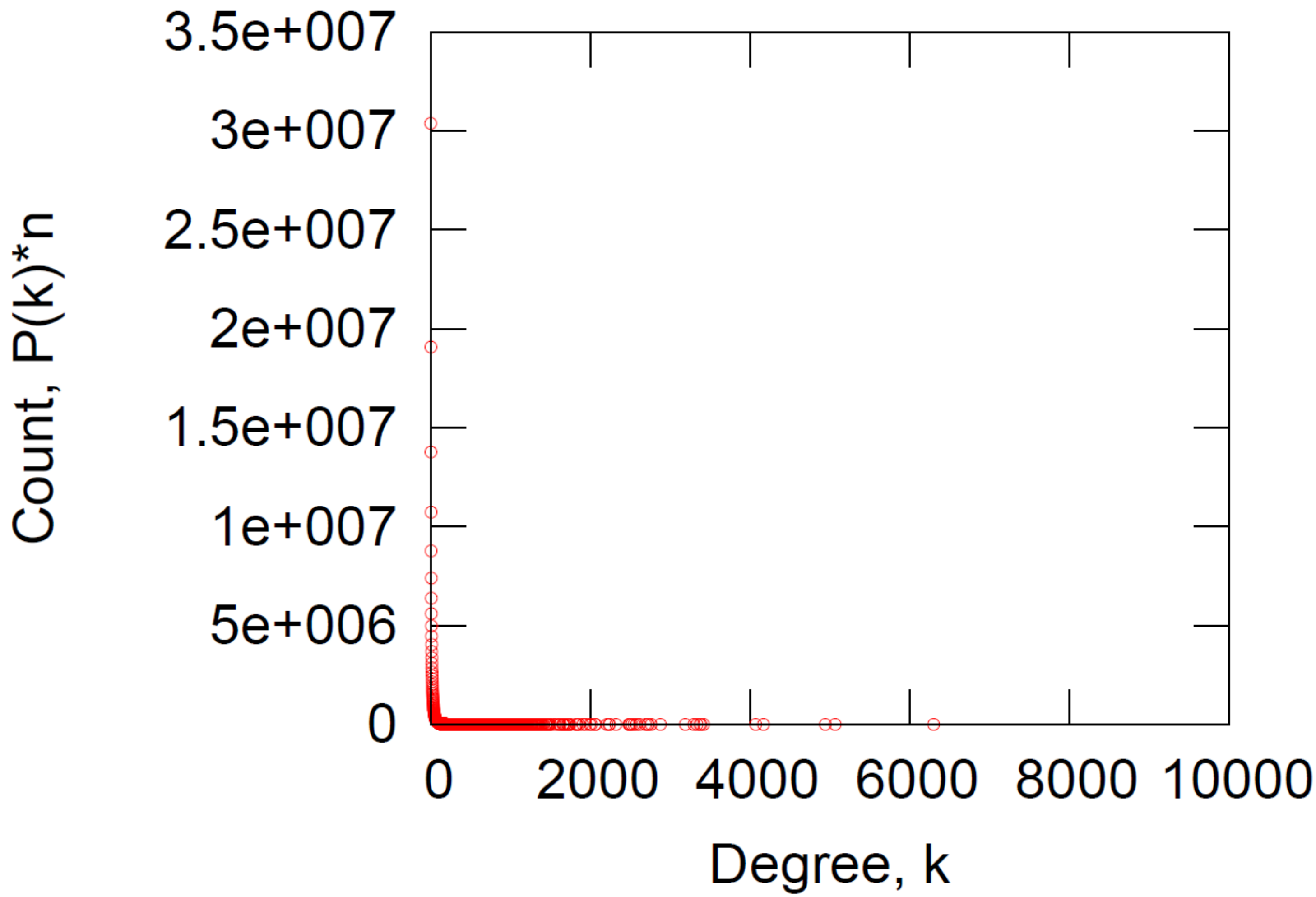

Messaging as an undirected graph

- Edge *(u,v)* if users *u* and *v* exchanged at least 1 msg
- $|V|=180$ million people
- $|E|=1.3$ billion edges

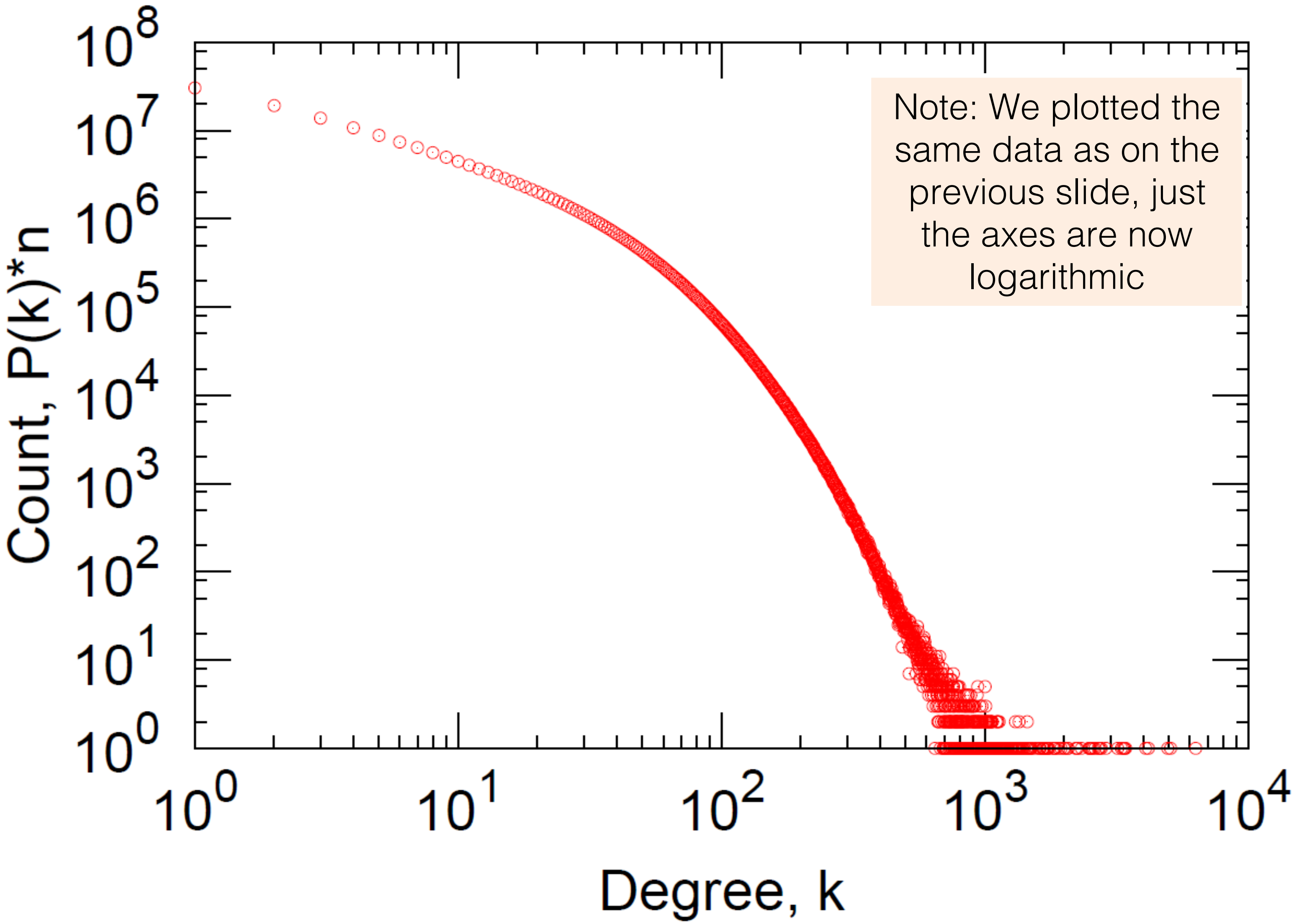
# MSN: Connectivity



# MSN: Degree Distribution

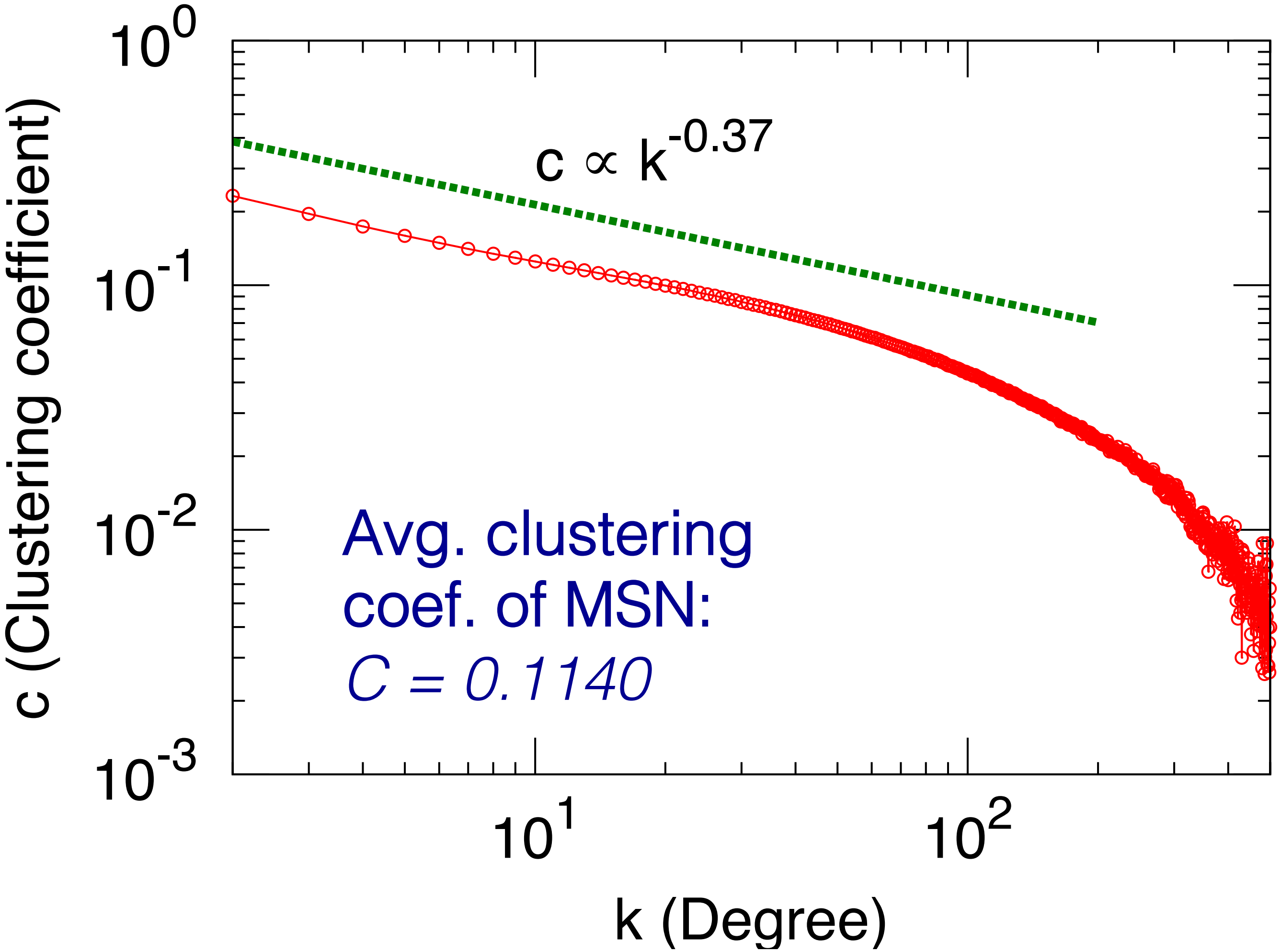


# MSN: Log-Log Degree Distribution




Note: We plotted the same data as on the previous slide, just the axes are now logarithmic

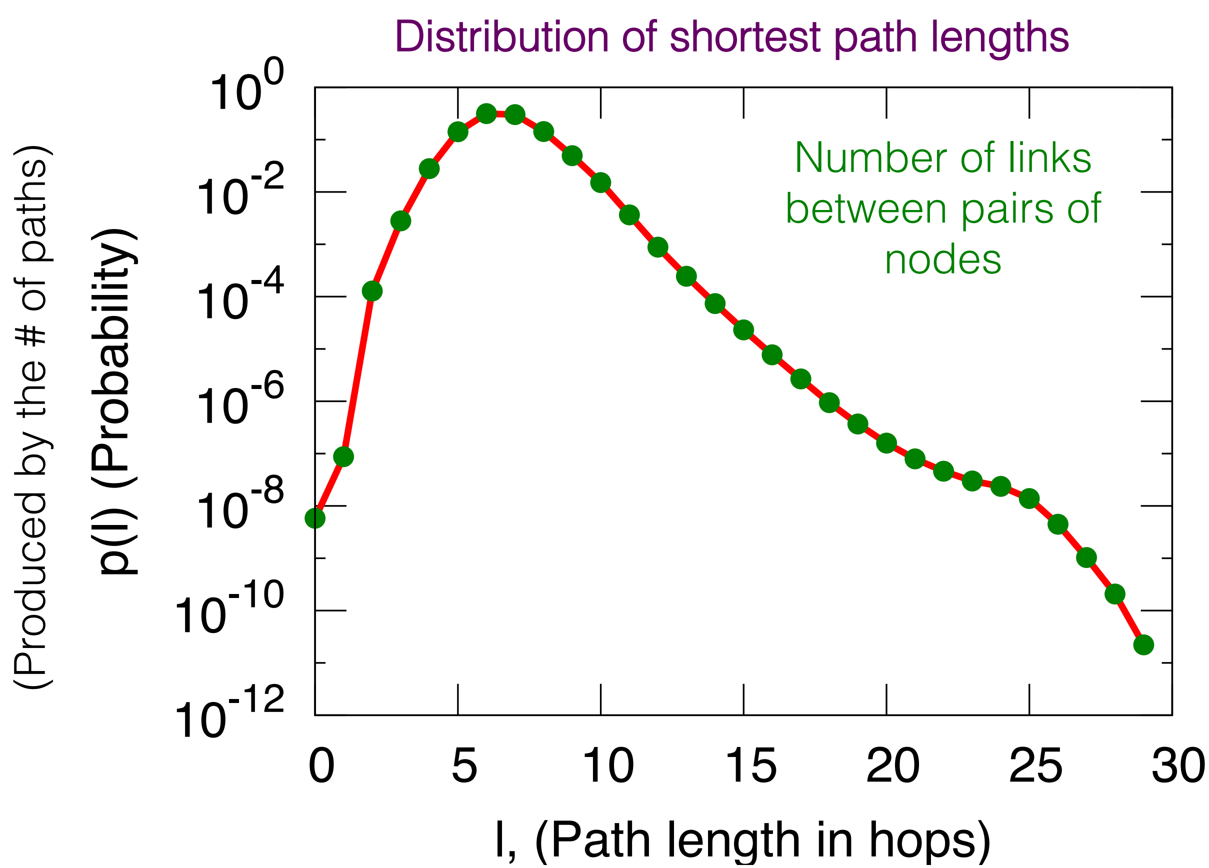
# MSN: Clustering




*$C_k$*: average *$C_i$* of nodes *$i$* of degree *$k$*:

$$C_k = \frac{1}{n_k} \sum_{i:k_i=k}^{n} C_i$$

# MSN: Diameter




Avg. path length **6.6**
90% of the nodes can be reached in < 8 hops

# nodes as we do BFS out of a random node

| Steps | #Nodes |
|---|---|
| 0 | 1 |
| 1 | 10 |
| 2 | 78 |
| 3 | 3,96 |
| 4 | 8,648 |
| 5 | 3,299,252 |
| 6 | 28,395,849 |
| 7 | 79,059,497 |
| 8 | 52,995,778 |
| 9 | 10,321,008 |
| 10 | 1,955,007 |
| 11 | 518,410 |
| 12 | 149,945 |
| 13 | 44,616 |
| 14 | 13,740 |
| 15 | 4,476 |
| 16 | 1,542 |
| 17 | 536 |
| 18 | 167 |
| 19 | 71 |
| 20 | 29 |
| 21 | 16 |
| 22 | 10 |
| 23 | 3 |
| 24 | 2 |
| 25 | 3 |

# MSN: Key Network Properties

| | |
|---|---|
| Degree distribution: | *Heavily skewed* <br> *avg. degree= 14.4* |
| Path length: | *6.6* |
| Clustering coefficient: | *0.11* |

Next lecture

Are these values "expected"?
Are they "surprising"?

To answer this we need a model

# Next Lecture

- Can we explain those properties using network models?
  - The random graph model and its properties
  - Small-world phenomenon
- Power-law degree distribution
  - Preferential Attachment model

# Thank You!