Oliver Hoffman and Faiyam Rahman
CS 5460: Parallel and Distributed
Lab 4, part 1a
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Assume a computation domain of size NxN, a stencil of size $(2K-1)x(2K-1)$ as well as a ghost boundary parallelization method. Assume we break the computation domain into subdomains of size XY, where $X|N$ and $Y|N$. We want to find the pair (X,Y) such that the ratio of computation to communication per 1 iteration is optimal. We assert that the ratio of computation to communication is:

$$\frac{\text{Computation}}{\text{Communication}} = \frac{N^2 * (2K-1)^2 * t_c}{\sum_{\mathbb{A}} (\alpha_x + \alpha_y + \alpha_x\alpha_y)t_s + (\alpha_y(K-1)Y + \alpha_x(K-1)X + \alpha_x\alpha_y(K-1)^2)t_u}$$

Let's look at the numerator first. The computation for one iteration is as follows

$$\begin{aligned}\text{Computation} &= \text{Number of Computational Units} * \text{Computation per unit} * \text{Time/Computation Unit}\\ &= N^2 * (2K-1)^2 * t_c\end{aligned}$$

Let's look at the denominator. Let's first establish what all the notation means.

$$\begin{aligned}\mathbb{A} &:= \{\alpha | \alpha \text{ is a computational subdomain of size } XY\}\\ \alpha_x &:= \text{the number of adjacent subdomains on the x axis for subdomain } \alpha\\ \alpha_y &:= \text{the number of adjacent subdomains on the y axis for subdomain } \alpha\\ t_s &:= \text{the latency time to send one message}\\ t_u &:= \text{the amount of time to send one computatational unit}\end{aligned}$$

Now let's breakdown the specifics of the denominator.

$$\text{Communication} = \sum_{\text{All Subdomains}} M_s * t_s + U_s * t_u$$

where M_s is the number of messages sent for the given subdomain and U_s is the number of computational units sent over all messages for the given subdmain. We have:

$$\begin{aligned}M_s &= \text{Messages sent to adjacent subdomains purely on the x axis+}\\ &\quad \text{Messages sent to adjacent subdomains purely on the y axis+}\\ &\quad \text{Messages sent to adjacent diagonal subdomains}\\ &= \alpha_x + \alpha_y + \alpha_x\alpha_y\end{aligned}$$

Note that the number of messages sent to diagonally adjacent subdomains is $\alpha_x\alpha_y$ because its the product of the number of x axis and y axis adjacent subdomains.

$$\begin{aligned}U_s = &\text{Number of units sent to adjacent subdomains purely on the x axis+}\\ &\text{Number of units sent to adjacent subdomains purely on the y axis+}\\ &\text{Number of units sent to adjacent diagonal subdomains}\end{aligned}$$

We have

$$\text{Number of units sent to adjacent subdomains purely on the x axis} = \alpha_x * (K-1) * Y$$

because each such subdomain needs $K-1$ columns from α and α's columns are of size Y. Similarly, we have

$$\text{Number of units sent to adjacent subdomains purely on the y axis} = \alpha_y * (K-1) * X$$

because each such subdomain needs $K-1$ rows from α and α's rows are of size X. Lastly, we have

$$\text{Number of units sent to adjacent diagonal subdomains} = \alpha_y \alpha_x * (K-1)^2$$

because the number of computational units α needs to send is determined by how much of α's computational domain is required for its adjacent subdomains corner computation. For that corner computation, the adjacent subdomain needs $(K-1)$ columns of height $(K-1)$, or equivalently, $(K-1)$ rows of width $(K-1)$ from α.

Now, let's consider how to compute this sum for a given N, X, Y. We have

$$\mathbb{A} = \bigcup_{x,y \in \{0,1,2\}} \mathbb{A}_{xy} \text{ where}$$
$$\mathbb{A}_{xy} = \{\alpha | \alpha_x = x \text{ and } \alpha_y = y\}$$

We now present without further justification formulas for each of the $\mathbb{A}_{xy}$ as they are pretty obvious once you look at an example or two.

$$|\mathbb{A}_{00}| = \begin{cases} 1, & \text{if } N = X, N = Y \\ 0, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{01}| = \begin{cases} 1, & \text{if } \frac{N}{Y} = 2, N = X \\ 0, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{02}| = \begin{cases} 0, & \text{if } N > X, \frac{N}{Y} < 3 \\ \frac{N^2}{XY} - \frac{2N}{X}, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{10}| = \begin{cases} 1, & \text{if } \frac{N}{X} = 2, N = Y \\ `0, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{20}| = \begin{cases} 0, & \text{if } N > Y, \frac{N}{X} < 3 \\ \frac{N^2}{XY} - \frac{2N}{Y}, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{11}| = \begin{cases} 0, & \text{if } \frac{N^2}{XY} < 4 \\ 4, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{12}| = \begin{cases} 0, & \text{if } \frac{N}{X} < 2, \text{ or } \frac{N}{Y} < 3 \\ \frac{2N}{Y} - 4, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{21}| = \begin{cases} 0, & \text{if } \frac{N}{Y} < 2, \text{ or } \frac{N}{X} < 3 \\ \frac{2N}{X} - 4, & \text{otherwise} \end{cases}$$

$$|\mathbb{A}_{22}| = \begin{cases} 0, & \text{if } \frac{N}{X} < 3 \text{ or } \frac{N}{Y} < 3 \\ \frac{N^2}{XY} - \frac{2N}{X} - \frac{2N}{Y} + 4, & \text{otherwise} \end{cases}$$